Convex Floating Bodies of Equilibrium *

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Abstract

We study a long standing open problem by Ulam, which is whether the Euclidean ball is the unique body of uniform density which will float in equilibrium in any direction. We answer this problem in the class of origin symmetric n-dimensional convex bodies whose relative density to water is $\frac{1}{2}$. For $n = 3$, this result is due to Falconer.

1 Introduction and results

1.1 Ulam floating bodies

A long standing open problem asked by Ulam in [16] (see also [9], Problem 19), is whether the Euclidean ball is the unique body of uniform density which floats in a liquid in equilibrium in any direction. We call such a body *Ulam floating body*. The formal definition is given below.

A two-dimensional counterexample was found for relative density $\rho = \frac{1}{2}$ by Auerbach [2] and for densities $\rho \neq \frac{1}{2}$ by Wegner [17]. These counterexamples are not origin symmetric. For higher dimension, Wegner obtained results for non-convex bodies (holes in the body are allowed) in [18]. The problem remains largely open in the class of convex bodies in higher dimension. In order to study Ulam floating bodies, we use the notion of the *convex floating body*, which was introduced independently by Bárány and Larman [3] and by Schütt and Werner [13]. Let K be a convex body in $\mathbb{R}^n$ and let $\delta \in \mathbb{R}$, $0 \leq \delta \leq \frac{1}{2}$. Then the convex floating body K_δ is defined as

$$K_\delta = \bigcap_{u \in S^{n-1}} H^-_{\delta,u}.$$

Here $H^+_{\delta,u}$ is the halfspace with outer unit normal vector u, which "cuts off" a δ proportion of K, i.e. $\mathrm{vol_n}\left(K \cap H^+_{\delta,u}\right) = \delta \ \mathrm{vol_n}(K)$. The convex floating body is a

*Keywords: Ulam Problem, floating bodies, 2020 Mathematics Subject Classification: 52A20
†Partially supported by NSF grant DMS-1811146 and by a Simons Fellowship

natural variation of Dupin's floating body $K_{[\delta]}$ (see [4]). A convex body $K_{[\delta]}$ that is contained in the convex body K is called a *Dupin floating body* of K if every support hyperplane of $K_{[\delta]}$ cuts off a set of volume δ $\mathrm{vol_n}(K)$ exactly. In general $K_{[\delta]}$ need not exist. An example is e.g., the simplex in $\mathbb{R}^n$. Dupin showed that if the Dupin floating body exists, each supporting hyperplane H touches $K_{[\delta]}$ at the centroid $g(K \cap H)$ of $K \cap H$,

$$g(K \cap H) = \int_H x dx. \tag{1}$$

If the floating body $K_{[\delta]}$ exists, it is equal to the convex floating body K_δ. It was shown in [10] that for a symmetric convex body K, one has $K_{[\delta]} = K_\delta$.

We recall the relation between the density ρ and the volume $\delta \mathrm{vol_n}(K)$ that is cut off. If the liquid has density 1 and the body K has unit volume and density ρ, then by Archimedes' law the submerged volume equals the total mass of the body, i.e. ρ, and consequently the floating part has volume $\delta = 1 - \rho$.

In [7], the authors defined the *metronoid* $M_\delta(K)$ of a convex body K to be the body whose boundary consists of centroids of the floating parts of K, i.e. $K \cap H^+_{\delta,u}$. More precisely, denoting $X_{K,\delta}(u) = \frac{1}{\delta \mathrm{vol_n}(K)} \int_{K \cap H^+_{\delta,u}} x\, dx$, they defined $M_\delta(K)$ by

$$\partial M_\delta(K) = \left\{ X_{K,\delta}(u) \,:\, u \in S^{n-1} \right\},$$

and showed that $X_{K,\delta} : S^{n-1} \to \partial M_\delta(K)$ is the Gauss map of $M_\delta(K)$, i.e. the normal to $M_\delta(K)$ at $X_{K,\delta}(u)$ is u. Huang, Slomka and Werner showed the following [8, Section 2.2] for details),

Theorem. [8] *Let K be a convex body in $\mathbb{R}^n$. Then K is an Ulam floating body if and only if $M_\delta(K)$ is a Euclidean ball.*

We utilize this characterization in our proof of Theorem 1.2.

1.2 Main results

We start with the formal definition of the Ulam floating body. In this definition, and elsewhere, we use the notation $g(B)$ for the centroid of a set B and $\mathrm{int}(B)$ for the interior of B.

Definition 1.1. Let K be a convex body in $\mathbb{R}^n$. Let $u \in S^{n-1}$ and H_u a hyperplane with outer normal u such that $H^+_u \cap \mathrm{int}(K) \neq \emptyset$. Then

(i) u is an equilibrium direction for $K \iff g(K) - g(H^+_u \cap K)$ is parallel to u.

(ii) K is an Ulam floating body, if every direction u is an equilibrium direction for K.

We present now two results concerning Ulam's problem. We first establish a relation between Ulam floating bodies and a uniform isotropicity property of sections. Our second result provides a short proof of a known answer to Ulam's problem in the class of symmetric convex bodies with relative density $\rho = 1/2$.

Theorem 1.2. *Let $\delta \in \left(0, \frac{1}{2}\right]$ and let $K \subset \mathbb{R}^n$ be a convex body such that K_δ is C^1 or $K_\delta = K_{[\delta]}$ reduces to a point. Then K is an Ulam floating body if and only if there exists $R > 0$ such that for all $u \in S^{n-1}$ and $v \in S^{n-1} \cap u^\perp$,*

$$\int_{K \cap H_{\delta,u}} \langle x, v\rangle^2 - \langle g(K \cap H_{\delta,u}), v\rangle^2 \, dx = \delta \ \mathrm{vol_n}(K) R. \tag{2}$$

In that case, $M_\delta(K)$ is a ball of radius R.

Remark. Note that if $K_\delta = K_{[\delta]}$ reduces to a point, which without loss of generality we can assume to be 0, then the condition (2) reduces to

$$\int_{K \cap H_{\delta,u}} \langle x, v\rangle^2 \, dx = \delta \ \mathrm{vol_n}(K) R. \tag{3}$$

We use the characterization given in Theorem 1.2 to give a short proof of the following result which was proved in dimension 3 by Falconer [5]. It also follows from a result in [11].

Theorem 1.3. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body of volume 1 and density $\frac{1}{2}$. If K is an Ulam floating body, then K is a ball.*

2 Background

We collect some definitions and basic results that we use throughout the paper. For further facts in convex geometry we refer the reader to the books by Gardner [6] and Schneider [12].

The radial function $r_{K,p} : S^{n-1} \to \mathbb{R}^+$ of a convex body K about a point $p \in \mathbb{R}^n$ is defined by

$$r_{K,p}(u) = \max\{\lambda \geq 0 : \lambda u \in K - p\}.$$

If $0 \in \mathrm{int}(K)$, the interior of K, we simply write r_K instead of $r_{K,0}$.
Let $K \subset \mathbb{R}^n$ be a convex body containing a strictly convex body D in its interior, and let H be a hyperplane supporting D at a point p. If u is the outer unit normal vector at p, we denote the restriction of the radial function $r_{K \cap H,p}$ to $S^{n-1} \cap u^\perp$ by $r_{K,D}(u, \cdot)$.

We denote by B_2^n the Euclidean unit ball centered at 0 and by $S^{n-1} = \partial B_2^n$ its boundary. The spherical Radon transform $R : C(S^{n-1}) \to C(S^{n-1})$ is defined by

$$Rf(u) = \int_{u^\perp \cap S^{n-1}} f(x) dx \tag{4}$$

for every $f \in C(S^{n-1})$.

2.1 Some results on floating bodies

Since $\delta > \frac{1}{2}$ implies $K_\delta = \emptyset$, we restrict our attention to the range $\delta \in \left[0, \frac{1}{2}\right]$. It was shown in [14] that there is δ_c, $0 < \delta_c \leq \frac{1}{2}$ such that K_{δ_c} reduces to a point. It can happen that $\delta_c < \frac{1}{2}$. An example is the simplex.

In fact, Helly's Theorem (and a simple union bound) implies that $\delta_c > \frac{1}{n+1}$, so we have $\delta_c \in \left(\frac{1}{n+1}, \frac{1}{2}\right]$.

As mentioned above, when Dupin's floating body $K_{[\delta]}$ exists, it coincides with the convex floating body K_δ. The following lemma states that existence of $K_{[\delta]}$ is also guaranteed by smoothness of K_δ. We use this for Theorem 1.2.

Lemma 2.1. *If K_δ is C^1, then $K_{[\delta]}$ exists and $K_\delta = K_{[\delta]}$.*

Proof. Let $x \in \partial K_\delta$. By [14], there is at least one hyperplane H through x that cuts off exactly $\delta \mathrm{vol_n}(K)$ from K and this hyperplane touches ∂K_δ in the barycenter of $H \cap K$. As K_δ is C^1, the hyperplane at every boundary point $x \in K_\delta$ is unique. Thus $K_{[\delta]}$ exists and $K_\delta = K_{[\delta]}$. □

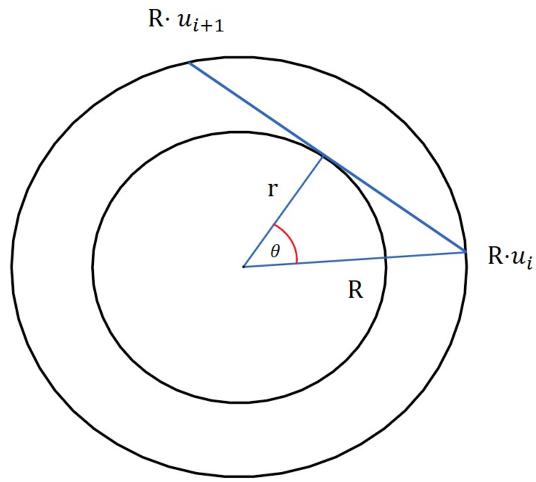
We know that $K_{\frac{1}{2}} = \{0\}$ for every centrally symmetric convex body K. The Homothety Conjecture [14] (see also [15, 19]), states that the homothety $K_\delta = t(\delta)K$ only occurs for ellipsoids. We treat the following conjecture which has a similar flavor.

Conjecture 2.2. *Let $K \subset \mathbb{R}^n$ be a convex body and let $\delta \in \left(0, \frac{1}{2}\right)$. If K_δ is a Euclidean ball, then K is a Euclidean ball.*

We now prove the two dimensional case of the conjecture.

Theorem 2.3. *Let $K \subset \mathbb{R}^2$ and suppose there is $\delta \in \left(0, \frac{1}{2}\right)$ such that $K_\delta = r\ B_2^2$. Then $K = R\ B_2^2$, for some $R > 0$.*

Proof. We shall prove that the radial function $r_K : S^1 \to \mathbb{R}$ is constant. If the continuous function r_K is not constant, it must attain some value $R > r$ such that the angle $\theta = \arccos\left(\frac{r}{R}\right) \in \left(0, \frac{\pi}{2}\right)$ is not a rational multiple of π. Let $u_1 \in S^1$ be the point with $r_K(u_1) = R$, and let $\{u_i\}_{i=1}^\infty$ be the arithmetic progression on S^1 with difference 2θ. We claim that r_K is constant on $\{u_i\}$. Indeed, assuming $Ru_i \in \partial K$, we consider the triangle with vertices O, Ru_i, Ru_{i+1} (see the figure below). The edge $[Ru_i, Ru_{i+1}]$ is tangent to $K_\delta = rB_2^2$ at its midpoint m_i, and since K_δ is smooth, the chord on ∂K containing Ru_i and m_i is bisected by m_i, which implies $Ru_{i+1} \in \partial K$, i.e., $r_K(u_{i+1}) = R$. Since θ is not a rational multiple of π, the sequence $\{u_i\}$ is dense in S^1. Since r_K is constant on a dense set and continuous, it is constant on S^1, as required.



$\square$

3 Proof of the main theorems

3.1 Proof of Theorem 1.2

Proof. We first treat the case $n = 2$. Also, we first consider when $K_\delta = K_{[\delta]}$ reduces to a point. Without loss of generality we can assume that this point is 0. Then we have for all $u \in S^1$ that $g(K \cap H_{\delta,u}) = 0$ by (1) and thus $\langle g(K \cap H_{\delta,u}), v\rangle = 0$, for all $v \in u^\perp \cap S^1$ and the condition reduces to $\int_{K\cap H_{\delta,u}} \langle x, v\rangle^2 dx = C$. This observation is true in all dimensions.
Let $u \in S^1$. Let $v \in u^\perp \cap S^1 = H_{\delta,u} \cap S^1$. Then, as $H_{\delta,u} = \text{span}\{v\}$, we get for all $v \in S^1$,

$$\int_{K\cap H_{\delta,u}} \langle x, v\rangle^2 dx = \int_{-r_K(v)}^{r_K(v)} x^2 dx = \frac{2}{3} r_K(v)^3, \tag{5}$$

as $r_K(v) = r_K(-v)$. Hence if $\int_{K\cap H_{\delta,u}} \langle x, v\rangle^2 dx = C$, then we get that for all $v \in S^1$ that $r_K(v) = C_1$, and hence K is a Euclidean ball and therefore also $M_\delta(K)$ is a Euclidean ball.

For the other direction, we fix $u \in S^1$. We may assume that $u = (1, 0)$, i.e., in polar coordinates u_0 corresponds to $\theta = 0$ and $r(\theta) = 1$. For ϕ small, let $w = (\cos\phi, \sin\phi)$

and define the sets

$$E_1 = H^+_{\delta,u} \cap H^+_{\delta,w} \cap K, \quad E_2 = H^+_{\delta,u} \cap H^-_{\delta,w} \cap K, \quad E_3 = H^-_{\delta,u} \cap H^+_{\delta,w} \cap K.$$

In order to compute the derivative of the boundary curve of $M_\delta(K)$ we write

$$\begin{aligned}
&\delta \mathrm{vol}_2(K) \cdot \big[X_{K,\delta}(w) - X_{K,\delta}(u)\big] \\
&= \int_{E_1 \cup E_3} x\, dx - \int_{E_1 \cup E_2} x\, dx = \int_{E_3} x\, dx - \int_{E_2} x\, dx \\
&= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\phi} \int_0^{r_K(\theta)} (\cos\theta, \sin\theta)\ r^2\, dr\, d\theta - \int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+\phi} \int_0^{r_K(\theta)} (\cos\theta, \sin\theta)\ r^2\, dr\, d\theta \\
&= 2 \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\phi} \int_0^{r_K(\theta)} (\cos\theta, \sin\theta)\ r^2\, dr\, d\theta = \frac{2}{3} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\phi} r_K(\theta)^3 (\cos\theta, \sin\theta)\ d\theta.
\end{aligned}$$

Thus

$$\frac{d}{d\phi}\left[X_{K,\delta}(w) - X_{K,\delta}(u)\right] = \frac{2}{3\ \delta\ \mathrm{vol}_2(K)}\ r_K\left(\phi + \frac{\pi}{2}\right)^3 (-\sin\phi, \cos\phi)$$

and hence

$$\left|\frac{d}{d\phi}\left[X_{K,\delta}(w) - X_{K,\delta}(u)\right]\right| = \frac{2}{3\ \delta\ \mathrm{vol}_2(K)}\ r_K\left(\phi + \frac{\pi}{2}\right)^3, \tag{6}$$

where $|\cdot|$ denotes the Euclidean norm. With $z = w^\perp$, we get from (5) and (6) that

$$\left|\frac{d}{d\phi}\left[X_{K,\delta}(w) - X_{K,\delta}(u)\right]\right| = \frac{1}{\delta\ \mathrm{vol}_2(K)} \int_{K \cap H_{\delta,w}} \langle x, z\rangle^2 dx. \tag{7}$$

If $M_\delta(K)$ is a Euclidean ball with radius R, we write $X_{K,\delta}\,(\cos\phi, \sin\phi) = R\,(\cos\phi, \sin\phi)$ in polar coordinates and get from (7)

$$\begin{aligned}
\frac{1}{\delta\ \mathrm{vol}_2(K)} \int_{K \cap H_{\delta,w}} \langle x, z\rangle^2 dx &= \left|\frac{d}{d\phi}\left[X_{K,\delta}(w) - X_{K,\delta}(u)\right]\right| \\
&= \left|\frac{d}{d\phi}\left[X_{K,\delta}(w)\right]\right| = R.
\end{aligned}$$

To treat the case when K_δ does not consist of just one point, we introduce the following coordinate system for the complement of an open, strictly convex body $D \subset \mathbb{R}^2$ with smooth boundary (see also e.g., [20]). Let $\gamma : [0, 2\pi] \to \partial D$ be the inverse Gauss map, and $T : [0, 2\pi] \to S^1$ be the unit tangent vector to the curve at $\gamma(\theta)$, oriented counterclockwise, i.e.,

$$n(\theta) := n_D(\gamma(\theta)) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad T(\theta) = \frac{\gamma'(\theta)}{|\gamma'(\theta)|} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

The coordinate system $F : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}^2 \setminus D$ is defined by

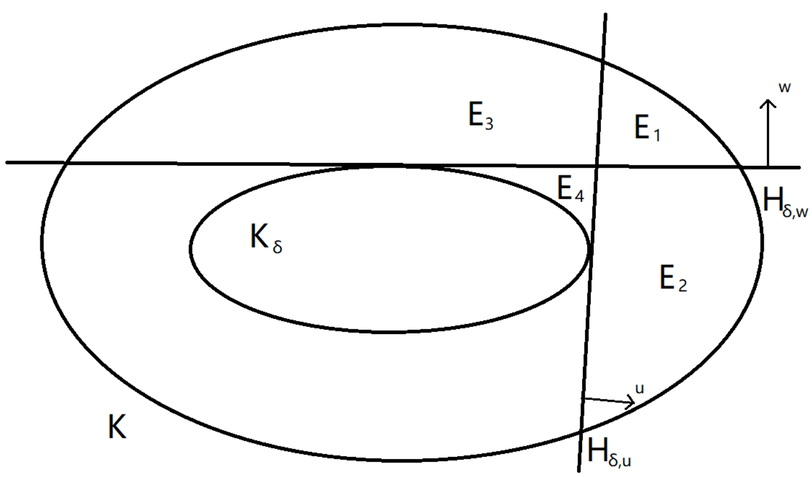
$$F(r, \theta) = \gamma(\theta) + rT(\theta). \tag{8}$$

Since $\frac{\partial F}{\partial r} = T$ and $\frac{\partial F}{\partial \theta} = \gamma' - rn$, the Jacobian of F is given by $|r|$.

Now we fix $0 < \delta < 1/2$ and assume that K_δ does not just consist of one point. We then set $D = \mathrm{int}(K_\delta)$, which has smooth boundary by assumption. Without loss of generality, we can assume that $0 \in \mathrm{int}(K_\delta)$. It was shown in [14] that K_δ is strictly convex. Let $u \in S^1$, and assume without loss of generality that $u = n(0) = (1, 0)$. Let $w = n(\phi)$ for an angle $\phi > 0$ small enough, such that the lines $H_{\delta,u}$ and $H_{\delta,w}$ intersect in the interior of K. Define the sets

$$E_1 = H^+_{\delta,u} \cap H^+_{\delta,w} \cap K, \quad E_2 = H^+_{\delta,u} \cap H^-_{\delta,w} \cap K, \quad E_3 = H^-_{\delta,u} \cap H^+_{\delta,w} \cap K,$$

and let E_4 be the bounded connected component of $\left(H^-_{\delta,u} \cap H^-_{\delta,w}\right) \setminus K_\delta$, see Figure.



Figure

Again, in order to compute the derivative of the boundary curve of $M_\delta(K)$ we write

$$\begin{aligned} \delta \ \mathrm{vol}_2(K) \cdot [X_{K,\delta}(w) - X_{K,\delta}(u)] &= \int_{E_1 \cup E_3} x \, dx - \int_{E_1 \cup E_2} x \, dx \\ &= \int_{E_3 \cup E_4} x \, dx - \int_{E_2 \cup E_4} x \, dx. \end{aligned}$$

Now we use the above introduced coordinate system. For $n = n(\theta)$ and $T = T(\theta)$, let $r_{K,K_\delta}(\theta)$ be such that $\gamma(\theta) + r_{K,K_\delta}(\theta)\ T \in \partial K$. As $K_\delta = K_{[\delta]}$, $\gamma(\theta)$ is the midpoint of

$n(\theta)^\perp \cap K$ by Dupin's characterization of $K_{[\delta]}$. Therefore

$$\begin{aligned}
&\delta \ \mathrm{vol}_2(K) \cdot [X_{K,\delta}(w) - X_{K,\delta}(u)] \\
= \ & \int_0^\phi \int_0^{r_{K,K_\delta}(\theta)} F(r,\theta)|r| \, dr \, d\theta - \int_0^\phi \int_{-r_{K,K_\delta}(\theta)}^0 F(r,\theta)|r| \, dr \, d\theta \\
= \ & \int_0^\phi \int_0^{r_{K,K_\delta}(\theta)} F(r,\theta) r \, dr \, d\theta + \int_0^\phi \int_{-r_{K,K_\delta}(\theta)}^0 F(r,\theta) r \, dr \, d\theta \\
= \ & \int_0^\phi \int_{-r_{K,K_\delta}(\theta)}^{r_{K,K_\delta}(\theta)} F(r,\theta) r \, dr \, d\theta.
\end{aligned}$$

Now we use the definition of F. Thus

$$\begin{aligned}
X_{K,\delta}(w) - X_{K,\delta}(u) &= \frac{1}{\delta \ \mathrm{vol}_2(K)} \int_0^\phi \int_{-r_{K,K_\delta}(\theta)}^{r_{K,K_\delta}(\theta)} r\gamma(\theta) + r^2 T(\theta) \, dr \, d\theta \\
&= \frac{1}{\delta \ \mathrm{vol}_2(K)} \int_0^\phi \frac{2}{3} \ r^3_{K,K_\delta}(\theta) \ T(\theta) \, d\theta.
\end{aligned}$$

As $M_\delta(K)$ is strictly convex and C^1 by [8], we can differentiate with respect to ϕ and get,

$$\frac{dX_{K,\delta}(n(\phi))}{d\phi} = \frac{2 \ r^3_{K,K_\delta}(\phi)}{3 \ \delta \ \mathrm{vol}_2(K)} \ T(\phi). \tag{9}$$

On the other hand, for any $\theta \in [0, 2\pi]$,

$$\begin{aligned}
&\int_{K \cap H_{\delta,n(\theta)}} \langle x, T(\theta) \rangle^2 \, dx = \int_{-r_{K,K_\delta}(\theta)}^{r_{K,K_\delta}(\theta)} \langle \gamma(\theta) + rT(\theta), T(\theta) \rangle^2 \ dr \\
&= 2 r_{K,K_\delta}(\theta) \ \langle \gamma(\theta), T(\theta) \rangle^2 + 2 \langle \gamma(\theta), T(\theta) \rangle \int_{-r_{K,K_\delta}(\theta)}^{r_{K,K_\delta}(\theta)} r dr + \int_{-r_{K,K_\delta}(\theta)}^{r_{K,K_\delta}(\theta)} r^2 dr \\
&= \langle g(K \cap H_{\delta,n(\theta)}), T(\theta) \rangle^2 \ \mathrm{vol}_1(K \cap H_{\delta,n(\theta)}) + \frac{2 \ r^3_{K,K_\delta}(\theta)}{3},
\end{aligned} \tag{10}$$

since $\gamma(\theta)$ is the centroid $g(K \cap H_{\delta,n(\theta)})$ of $K \cap H_{\delta,n(\theta)}$. Combining (9) and (10), we get that for $\theta = \phi$,

$$\left| \frac{dX_{K,\delta}(n(\phi))}{d\phi} \right| = \frac{1}{\delta \ \mathrm{vol}_2(K)} \ \int_{K \cap H_{\delta,n(\phi)}} \langle x, T(\phi) \rangle^2 - \langle g(K \cap H_{\delta,n(\phi)}), T(\phi) \rangle^2 \, dx.$$

By [7, 8], the normal to $\partial M_\delta(K)$ at $X_{K,\delta}(n(\phi))$ is $n(\phi) = (\cos\phi, \sin\phi)$ and as $M_\delta(K)$ is strictly convex and C^1 by [8], $\xi(\phi) = X_{K,\delta}(n(\phi))$ is a parametrization of $\partial M_\delta(K)$ with respect to the angle of the normal. The curvature is given by $\frac{d\phi}{ds}$ where s is the arc length along the curve. Since $\frac{d\xi}{ds}$ is a unit vector, we get by the chain rule $\frac{d\xi}{ds} = \frac{d\xi}{d\phi}\frac{d\phi}{ds}$ that the radius of curvature is given by

$$R(\phi) = \left| \frac{dX_{K,\delta}(n(\phi))}{d\phi} \right| = \frac{1}{\delta \ \mathrm{vol}_2(K)} \int_{K \cap H_{\delta,n(\phi)}} \langle x, T(\phi) \rangle^2 - \langle g(K \cap H_{\delta,n(\phi)}), T(\phi) \rangle^2 \, dx.$$

Since $M_\delta(K)$ is a disk if and only if its radius of curvature is constant, the theorem follows.

Let now $n \geq 3$.

Let $u \in S^{n-1}$ be arbitrary, but fixed, and let $v \in S^{n-1} \cap u^\perp$. We denote by $W = \text{span}\{u, v\}$ the span of u and v and by $W^\perp$ the $(n-2)$-dimensional subspace that is the orthogonal complement of W. $\bar{K} = K|W$ is the orthogonal projection of the convex body K onto the 2-dimensional subspace W. For a small ϕ, let $w = \cos\phi \, u + \sin\phi \, v$. We define $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ as follows,

$$\bar{E}_1 = (H^+_{\delta,u} \cap H^+_{\delta,w})\big|W, \quad \bar{E}_2 = (H^+_{\delta,u} \cap H^-_{\delta,\eta})\big|W, \quad \bar{E}_3 = (H^-_{\delta,u} \cap H^+_{\delta,w})\big|W$$

and $\bar{E}_4$ is the curvilinear triangle enclosed by $H_{\delta,u}|W$, $H_{\delta,w}\big|W$, and the boundary of $\bar{K}_\delta = K_\delta|W$. Then the picture is identical to the previous Figure. We let

$$E_i = \bar{E}_i \times W^\perp, \quad \text{for} \;\; i = 1, 2, 3, 4.$$

When K_δ reduces to a point, we can assume without loss of generality that $K_\delta = \{0\}$. As noted before, the condition then reduces to $\int_{K\cap H_{\delta,u}} \langle x, v\rangle^2 dx = C$. In this case $\bar{E}_4 = \emptyset$ and the proof continues along the same lines as below. Alternatively, one can replace the coordinate system (8) by the usual polar coordinates in W as it was done in the case $n = 2$.

In the general case we thus have that

$$\begin{aligned}
&\delta \ \text{vol}_\text{n}(K)\,[X_{K,\delta}(w) - X_{K,\delta}(u)] = \int_{K\cap H^+_{\delta,w}} x\,dx - \int_{K\cap H^+_{\delta,u}} x\,dx \\
&= \int_{K\cap(E_1\cup E_3)} x\,dx - \int_{K\cap(E_1\cup E_2)} x\,dx = \int_{K\cap E_3} x\,dx - \int_{K\cap E_2} x\,dx \\
&= \int_{K\cap(E_3\cup E_4)} x\,dx - \int_{K\cap(E_2\cup E_4)} x\,dx.
\end{aligned}$$

For $x \in W$, we consider the following parallel section function,

$$A_{K,W}(x) = \text{vol}_{n-2}\left(K \cap \{W^\perp + x\}\right) \tag{11}$$

and observe that by Fubini,

$$\delta \ \text{vol}_\text{n}(K)\,[X_{K,\delta}(w)] = \int_{K\cap H^+_{\delta,w}} z\;dz = \int_{\bar{K}} \left(\int_{(x+W^\perp)\cap K\cap H^+_{\delta,w}} y\;dy \right) dx.$$

We denote by $g(B)=\frac{1}{\mathrm{vol_n}(B)}\int_B y\ dy$ the centroid of the set B. Then we get

$$\begin{aligned}
\delta(X_{K,\delta}(w))\big|W &= \left(\int_{K\cap(E_3\cup E_4)} x\,dx\right)\Bigg|W = \left(\int_{\bar K\cap(\bar E_3\cup\bar E_4)}\left(\int_{(x+W^\perp)\cap K} y\ dy\right)\ dx\right)\Bigg|W\\
&= \left(\int_{\bar K\cap(\bar E_3\cup\bar E_4)} A_{K,W}(x)\ g((x+W^\perp)\cap K)\ dx\right)\Bigg|W\\
&= \int_{\bar K\cap(\bar E_3\cup\bar E_4)} A_{K,W}(x)\ \left(g((x+W^\perp)\cap K)\right)\big|W\ dx\\
&= \int_{\bar K\cap(\bar E_3\cup\bar E_4)} A_{K,W}(x)\ x\ dx,
\end{aligned}$$

and similarly for $\delta\ \mathrm{vol_n}(K)(X_{K,\delta}(u))\big|W$. Now we will use the coordinate system $F:\mathbb{R}\times[0,2\pi]\to\mathbb{R}\setminus\mathrm{int}\left(\bar K_\delta\right)$, introduced earlier in (8),

$$F(r,\theta)=\gamma(\theta)+rT(\theta).$$

We can assume that $n(0)=u$. Then $T(0)=v$. We recall that the Jacobian of F is given by $|r|$. We abbreviate $n=n(\theta)$ and $T=T(\theta)$. We let $r_{\bar K,\bar K_\delta}(n(\theta)),T(\theta))=r_{\bar K,\bar K_\delta}(n,T)>0$ be such that $\gamma(\theta)+r_{\bar K,\bar K_\delta}(n,T)\ T(\theta)\in\partial\bar K$, and $r_{\bar K,\bar K_\delta}(n,-T)>0$ be such that $\gamma(\theta)+r_{\bar K,\bar K_\delta}(n,T)\ (-T(\theta))\in\partial\bar K$. We get

$$\begin{aligned}
&\delta\ \mathrm{vol_n}(K)\left(X_{K,\delta}(w)-X_{K,\delta}(u)\right)\big|W\\
&=\int_0^\phi\int_0^{r_{\bar K,\bar K_\delta}(n,T)} F(r,\theta)\ A_{K,W}(F(r,\theta))\ |r|\,dr\,d\theta\\
&\qquad\qquad -\int_0^\phi\int_{-r_{\bar K,\bar K_\delta}(n,-T)}^0 F(r,\theta)\ A_{K,W}(F(r,\theta))\ |r|\,dr\,d\theta\\
&=\int_0^\phi\int_0^{r_{\bar K,\bar K_\delta}(n,T)} F(r,w)\ A_{K,W}(F(r,\theta))\ r\,dr\,d\theta\\
&\qquad\qquad +\int_0^\phi\int_{-r_{\bar K,\bar K_\delta}(n,-T)}^0 F(r,\theta)\ A_{K,W}(F(r,\theta))\ r\,dr\,d\theta\\
&=\int_0^\phi\int_{-r_{\bar K,\bar K_\delta}(n,-T)}^{r_{\bar K,\bar K_\delta}(n,T)} F(r,\theta)\ A_{K,W}(F(r,\theta))\ r\,dr\,d\theta.
\end{aligned}$$

We differentiate with respect to ϕ,

$$\begin{aligned}
&\delta\ \mathrm{vol_n}(K)\frac{d}{d\phi}\left((X_{K,\delta}(w)-X_{K,\delta}(u))\big|W\right)=\delta\ \mathrm{vol_n}(K)\frac{d}{d\phi}\left((X_{K,\delta}(w))\big|W\right)\\
&=\int_{-r_{\bar K,\bar K_\delta}(n(\phi),-T(\phi))}^{r_{\bar K,\bar K_\delta}(n(\phi),T(\phi))} F(r,\phi)A_{K,W}(F(r,\phi))\ r\,dr.
\end{aligned}$$

Putting $\phi = 0$, results in

$$\delta \ \mathrm{vol_n}(K) \frac{d}{d\phi} \left((X_{K,\delta}(w)) \big| W \right) \Bigg|_{\phi=0} = \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} F(r,0) A_{K,W}(F(r,0)) \ r \, dr$$

$$= \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} [\gamma(0) + rv] \ A_{K,W}(F(r,0)) \ r \, dr$$

$$= \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} \gamma(0) \ A_{K,W}(F(r,0)) \ r \, dr + \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} r^2 \ v \ A_{K,W}(F(r,0)) \, dr$$

$$= v \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} r^2 \ A_{K,W}(F(r,0)) \, dr, \tag{12}$$

where the last equality holds by Dupin since $H_{\delta,u} \cap K_\delta$ is the centroid of $H_{\delta,u} \cap K$, i.e.

$$\int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} \gamma(0) \ A_{K,W}(F(r,0)) \ r \, dr = 0. \tag{13}$$

Indeed, in the coordinate system (8), the centroid of $H_{\delta,u} \cap K$ is $\gamma(0)$. Thus, with coordinate system (8) we get as above

$$\mathrm{vol}_{n-1} \left(K \cap H_{\delta,u} \right) \langle v, \gamma(0) \rangle = \int_{K \cap H_{\delta,u}} \langle v, x \rangle \ dx$$

$$= \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} \langle \gamma(0) + rv, v \rangle \ A_{K,W}(F(r,0)) \ dr$$

$$= \langle \gamma(0), v \rangle \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} A_{K,W}(F(r,0)) \ dr + \langle v, v \rangle \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} r \ A_{K,W}(F(r,0)) \ dr$$

$$= \langle \gamma(0), v \rangle \ \mathrm{vol}_{n-1} \left(K \cap H_{\delta,u} \right) + \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} r \ A_{K,W}(F(r,0)) \ dr.$$

On the other hand, again in the coordinate system (8), and also using (13),

$$\int_{K \cap H_{\delta,u}} \langle x, v \rangle^2 \ dx = \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} \langle \gamma(0) + rv, v \rangle^2 \ A_{K,W}(F(r,0)) dr$$

$$= \langle \gamma(0), v \rangle^2 \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} A_{K,W}(F(r,0)) dr + \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} r^2 \ A_{K,W}(F(r,0)) dr$$

$$= \langle \gamma(0), v \rangle^2 \ \mathrm{vol}_{n-1} \left(K \cap H_{\delta,u} \right) + \int_{-r_{\bar{K},\bar{K}_\delta}(u,-v)}^{r_{\bar{K},\bar{K}_\delta}(u,v)} r^2 \ A_{K,W}(F(r,0)) dr. \tag{14}$$

As $w = \cos\phi \ u + \sin\phi \ v$, it follows from (12) and (14) that

$$\left| \frac{d}{d\phi} \Big(X_{K,\delta}(\cos\phi \ u + \sin\phi \ v) \Big| W \Big) \Big|_{\phi=0} \right| = \frac{1}{\delta \mathrm{vol_n}(K)} \int_{K \cap H_{\delta,u}} \left(\langle x, v \rangle^2 - \langle \gamma(0), v \rangle^2 \right) dx$$

$$= \frac{1}{\delta \ \mathrm{vol_n}(K)} \int_{K \cap H_{\delta,u}} \left(\langle x, v \rangle^2 - \langle g(K \cap H_{\delta,u}), v \rangle^2 \right) \ dx.$$

Observe that in the case when $K_\delta = \{0\}$, $\langle g(K \cap H_{\delta,u}), v\rangle = 0$. We have that $w = n(\phi) = \cos\phi\ u + \sin\phi\ v \in W$ is the outer unit normal to $M_\delta(K)$ in $X_{\delta,K}(w)$. Therefore, w is the outer unit normal to $M_\delta(K)\big|W$ in $X_{\delta,K}(w)\big|W$. Again, as $M_\delta(K)$ and therefore $M_\delta(K)\big|W$ is strictly convex and C^1 by [8], $X_{\delta,K}(n(\phi))$ is a parametrization of the boundary of $M_\delta(K)\big|W$ with respect to the angle of the normal. Thus the curvature of $M_\delta(K)\big|W$ is constant, which implies that $M_\delta(K)\big|W$ is a disk. Since W is arbitrary, we get that every two dimensional projection of M_δ is a disk, and it follows that $M_\delta(K)$ is a Euclidean ball ([6], Corollary 3.1.6).

□

3.2 Proof of Theorem 1.3

Proof. Since K is symmetric and has volume 1 and density $\rho = \frac{1}{2}$, we have that $\delta = \frac{1}{2}$, as noted above. Therefore, $K_{\frac{1}{2}} = K_{[\frac{1}{2}]} = \{0\}$. Since K is an Ulam floating body, the remark after Theorem 1.2 implies that for any $u \in S^{n-1}$ and $v \in u^\perp \cap S^{n-1}$

$$\int_{u^\perp \cap K} \langle x, v\rangle^2\, dx = C, \tag{15}$$

for some constant C. Let $u \in S^{n-1}$ be arbitrary, but fixed. We pass to polar coordinates in $u^\perp$ and get for all $v \in u^\perp \cap S^{n-1}$,

$$C = \int_{u^\perp \cap S^{n-1}} \int_{t=0}^{r_K(\xi)} t^n \langle \xi, v\rangle^2 dt\ d\sigma(\xi) = \frac{1}{n+1} \int_{u^\perp \cap S^{n-1}} r_K(\xi)^{n+1} \langle \xi, v\rangle^2 d\sigma(\xi).$$

Now we integrate over all $v \in u^\perp \cap S^{n-1} = S^{n-2}$ w.r. to the normalized Haar measure μ on S^{n-2}. We use that $\int_{S^{n-2}} \langle \xi, v\rangle^2 d\mu(v) = c_n \|\xi\| = c_n$, where $c_n = 2\frac{\mathrm{vol}_{n-2}(B^{n-2})}{\mathrm{vol}_{n-2}(S^{n-2})}$ and get that

$$\frac{(n+1)C}{c_n} = \int_{u^\perp \cap S^{n-1}} r_K(\xi)^{n+1} d\sigma(\xi) = R\ r_K^{n+1}(u),$$

where R is the spherical Radon transform (4). We rewrite this equation as

$$\int_{u^\perp \cap S^{n-1}} d\sigma(\xi) = \int_{u^\perp \cap S^{n-1}} \frac{2\mathrm{vol}_{n-2}\,(B^{n-2})}{(n+1)C} r_K(\xi)^{n+1} d\sigma(\xi),$$

or

$$0 = \int_{u^\perp \cap S^{n-1}} \left(\frac{2\mathrm{vol}_{n-1}\,(B^{n-2})}{(n+1)C} r_K(\xi)^{n+1} - 1 \right) d\sigma(\xi).$$

As u was arbitrary and as r_K is even, it then follows from e.g., Theorem C.2.4 of [6] that $r_K = \text{const.}$ for σ almost all u and as r_K is continuous, $r_K = \text{const.}$ on S^{n-1}. Thus K is a ball. □

References

[1] D. Amir, *Characterizations of Inner Product Spaces*. Birkhäuser Verlag (1986).

[2] H. Auerbach, *Sur un Problème de M. Ulam Concernant l'Équilibre des Corps Flottants*. Studia Mathematica **7**, 121–142 (1938).

[3] I. Bárány and D.G. Larman, *Convex bodies, economic cap coverings, random polytopes*. Mathematika **35**, 274–291 (1988).

[4] C. Dupin, *Application de géométrie et de méchanique á la marine, aux ponts et chaussées*. Bachelier, Paris (1822).

[5] K. J. Falconer, *Applications of a result on spherical integration to the theory of convex sets*. The American Mathematical Monthly **90**, issue 10, 690–693 (1983).

[6] R. J. Gardner, *Geometric Tomography. Second edition*. Cambridge University Press, New York (2006).

[7] H. Huang and B. A. Slomka, *Approximations of Convex Bodies by Measure-Generated Sets*. Geometriae Dedicata **200**, issue 1, 173–196 (2019).

[8] H. Huang, B. A. Slomka and E. M. Werner, *Ulam Floating Bodies*. Journal of the London Mathematical Society **100**, 425–446 (2019).

[9] R. Mauldin (ed.), *The Scottish Book*. Birkhäuser Boston (1981).

[10] M. Meyer and S. Reisner, *A Geometric Property of the Boundary of Symmetric Convex Bodies and Convexity of Flotation Surfaces*. Geometriae Dedicata **37**, 327–337 (1991).

[11] R. Schneider, *Functional Equations Connected with Rotations and their Geometric Applications*. L'Enseignement Mathématique **16**, 297–305 (1970).

[12] R. Schneider, *Convex Bodies: The Brunn Minkowski Theory. Second expanded edition*. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press (2014).

[13] C. Schütt and E. M. Werner, *The Convex Floating Body*. Mathematica Scandinavia **66**, 275–290 (1990).

[14] C. Schütt and E. M. Werner, *Homothetic Floating Bodies*. Geometriae Dedicata **49**, 335–348 (1994).

[15] A. Stancu, *Two volume product inequalities and their applications.* Canadian Math. Bulletin **52** 464–472 (2009).

[16] R. S. Ulam, *A collection of mathematical problems*, Interscience, New York (1960).

[17] F. Wegner, *Floating Bodies of Equilibrium.* Studies of Applied mathematics **111**, 167–183 (2003).

[18] F. Wegner, *Floating Bodies of Equilibrium in Three Dimensions. The Central Symmetric Case.* arxiv:0803.1043.

[19] E. M. Werner and D. Ye, *On the Homothety Conjecture.* Indiana Univ. Math. J. **60** No. 1, 1–20 (2011).

[20] V. Yaskin and N. Zhang, *Non-central Sections of Convex Bodies.* Israel J. Math. **220**, 763–790 (2017).

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