

# Finite chemical potential equation of state for QCD from an alternative expansion scheme

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**Abstract.** The Taylor expansion approach to the equation of state of QCD at finite chemical potential struggles to reach large chemical potential  $\mu_B$ . This is primarily due to the intrinsic difficulty in precisely determining higher order Taylor coefficients, as well as the structure of the temperature dependence of such observables. In these proceedings, we illustrate a novel scheme [1] that allows us to extrapolate the equation of state of QCD without suffering from the poor convergence typical of the Taylor expansion approach. We continuum extrapolate the coefficients of our new expansion scheme and show the thermodynamic observables up to  $\mu_B/T \leq 3.5$ .

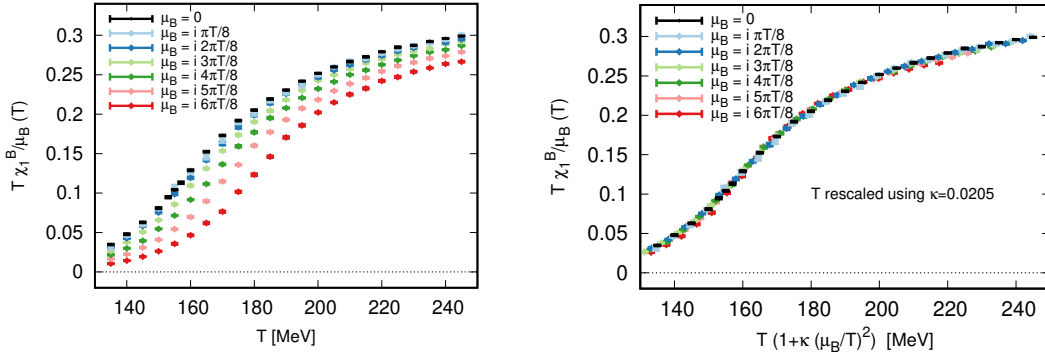
## 1 Introduction

The phase diagram of Quantum Chromodynamics (QCD) is a field of major interest from both a theoretical and experimental standpoint. At vanishing baryon density, it is known from lattice QCD [2] that the chiral/deconfinement transition is a smooth crossover. At finite baryon density lattice QCD faces a sign problem, due to the introduction of a real chemical potential that makes the action complex. Although new techniques that allow to directly simulate the theory at finite chemical potential exist [3, 4], they cannot be applied to large scale QCD simulations yet.

Several results have been published that exploit simulations at zero or imaginary chemical potential, and rely on an analytical continuation to extrapolate to real chemical potential. The most straightforward method is the Taylor expansion, whereby the first few  $\mu_B$ -derivatives of thermodynamic quantities are calculated [5, 6]. Notable successes of analytic continuation methods include the determination of the QCD transition line, whose  $\mu_B$ -dependence is now known up to the next-to-leading order [7, 8].

At large temperatures, resummed perturbation theory has provided a quantitative description of the chemical potential dependence of several observables, which agree with lattice

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**Figure 1.** (Left panel): Normalized baryon density at imaginary baryon chemical potentials. The points at  $\mu_B = 0$  (black) show the second baryon susceptibility  $\chi_2^B(T)$ . (Right panel): Same curves as in the upper panel, with a temperature rescaled in accordance to Eq. (1) with  $\kappa = 0.0205$ .

calculations [9, 10]. Similarly, functional methods provide an alternative approach to the study of the phase diagram [11]. However, lattice QCD remains the major *first-principle* tool of investigation, especially in the transition region of QCD.

From an experimental standpoint, heavy-ion collisions are capable of exploring the phase structure of strongly interacting matter. Systems created in such experiments are modeled through hydrodynamic simulations, for which the equation of state (EoS) of QCD is a crucial input. At vanishing chemical potential it has been known for several years [12–14]. Despite enormous computational efforts aimed at extending these results to higher  $\mu_B$  by adding more terms in the Taylor series, at the moment even the sixth  $\mu_B$ -derivative of the QCD pressure is only available with modest precision from lattice simulations [5, 6].

In this contribution, we describe a new [1], alternative scheme for the extrapolation of the QCD equation of state to finite density, which displays improved convergence over the Taylor series, leading to wider coverage in  $\mu_B$  and to more precise results for the thermodynamic observables.

## 2 The alternative scheme

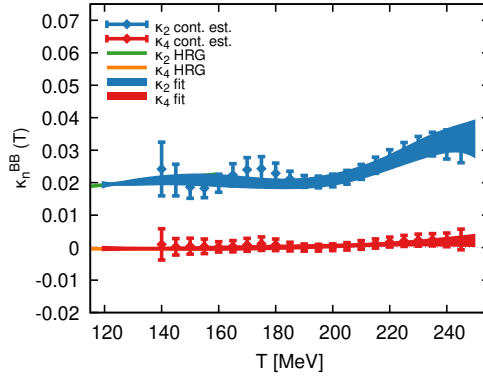
We start by *observing* that, at imaginary chemical potentials, the normalized baryon density  $\chi_1^B/\hat{\mu}_B$  presents the same structure at all chemical potentials (including  $\hat{\mu}_B = 0$ , where the limit  $\lim_{\mu_B \rightarrow 0} \chi_1^B/\hat{\mu}_B = \chi_2^B$ ), as can be seen in the left panel of Fig.1. This is reflected in the fact that a  $\mu_B$ -dependent rescaling of the temperatures makes all the curves collapse onto each other, as shown in the right panel of Fig. 1. In particular, a single  $T$ -independent parameter can encode such rescaling quite well according to the following equality:

$$\frac{\chi_1^B(T, \hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T', 0), \quad T' = (1 + \kappa \hat{\mu}_B^2). \quad (1)$$

Although quite suggestive, Eq.(1) cannot serve as an alternative expansion scheme as is. Thus, we include higher order terms in the expansion, and allow the expansion parameters to depend on the temperature:

$$T' = T \left( 1 + \kappa_2(T) \hat{\mu}_B^2 + \kappa_4(T) \hat{\mu}_B^4 + O(\hat{\mu}_B^6) \right). \quad (2)$$

In essence, the expansion scheme we present amounts to an expansion in  $\Delta T = T' - T$  inside the argument of  $\chi_2^B$ , and corresponds to a re-organization of the different terms in the



**Figure 2.** Continuum extrapolated result for the expansion parameters  $\kappa_2(T)$  and  $\kappa_4(T)$ . HRG results are shown up to  $T = 160$  MeV (in green for  $\kappa_2$ , orange for  $\kappa_4$ , respectively). The bands show correlated polynomial fits.

Taylor series. One can visualize this as an expansion along lines of constant (normalized) baryon density  $\chi_1^B \hat{\mu}_B$ , while a Taylor expansion is carried out at constant temperature. Working out such expansions, and equating equal order terms, one finds:

$$\kappa_2(T) = \frac{1}{6T} \frac{\chi_4^B(T)}{\chi_2^{B'}(T)}, \quad \kappa_4(T) = \frac{1}{360 \chi_2^{B'}(T)^3} \left( 3 \chi_2^{B'}(T)^2 \chi_6^B(T) - 5 \chi_2^{B''}(T) \chi_4^B(T)^2 \right). \quad (3)$$

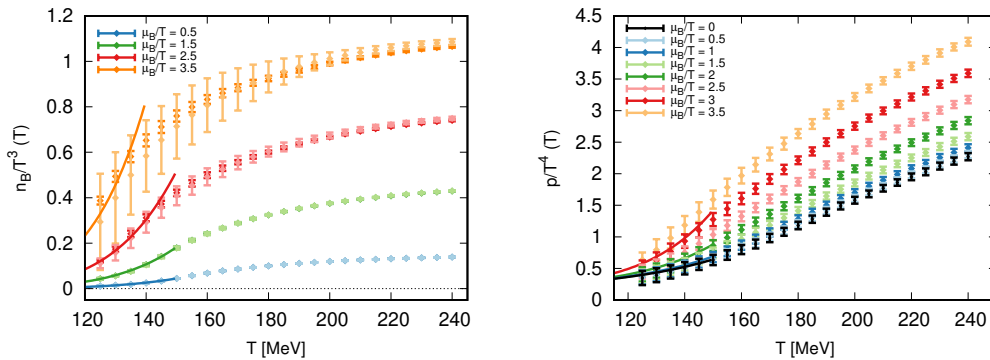
These relations can be straightforwardly extended to higher orders, and could in principle utilized to determine the coefficients  $\kappa_n(T)$ . However, for  $n > 2$ , such coefficients would still suffer from severe cancellations between competing terms. Thus, exploiting simulations at imaginary chemical potentials  $\hat{\mu}_B = i n \pi/8$ , with  $n = 0, \dots, 8$ , we calculate the proxy quantity:

$$\Pi(T, \hat{\mu}_B^2) = \frac{T' - T}{T \hat{\mu}_B^2} = \kappa_2(T) + \kappa_4(T) \hat{\mu}_B^2 + O(\hat{\mu}_B^4), \quad (4)$$

at different temperatures, and on three different lattices with  $N_\tau = 10, 12, 16$ . We then perform a combined fit in  $\hat{\mu}_B^2$  and  $1/N_\tau^2$ , from which we extract  $\kappa_2$  and  $\kappa_4$  at each temperature. The results for these coefficients are shown in Fig. 2, along with polynomial fits of both  $\kappa_n(T)$  which take into account the full statistical and systematic correlations between different temperatures. At low temperature, we also show results from the hadron resonance gas (HRG) model, which is in good agreement with our results. Notably, we find a large separation in the values of  $\kappa_2(T)$  and  $\kappa_4(T)$ , which hints at improved convergence of the expansion.

We reconstruct the baryon density at finite real chemical potential by employing Eq. (1), then obtain the pressure from it by simply integrating over the chemical potential. We show the baryon density and the pressure in the left and right panels of Fig. 3, respectively. We also compare with HRG results at low temperature, finding good agreement at all chemical potentials.

Crucially, the errors are well under control up to  $\hat{\mu}_B = 3.5$ , and no unphysical, nonmonotonic behavior is observed. For the baryon density, we compare the results with a simplified scenario in which we set  $\kappa_4 \equiv 0$ . We find that the inclusion of a non-zero  $\kappa_4$  increases the error on the final results, but does not change its prediction. This hints at a better convergence of our series expansion, compared to the Taylor method.



**Figure 3.** Baryon density (left) and pressure (right) at increasing values of  $\hat{\mu}_B$ . With solid lines we show the results from the HRG model. For the baryon density we also show in darker shades the results obtained by setting  $\kappa_4 \equiv 0$ .

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