Partial Differential Equations in Applied Mathematics 5 (2022) 100346

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

Overview of the Kadomtsev–Petviashvili-hierarchy reduction method for solitons

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ARTICLE INFO

This paper is written for the occasion of Prof. Wen-Xiu Ma's 60th birthday

Keywords:
Bilinear method
KP hierarchy reduction
Solitons

ABSTRACT

The Kadomtsev–Petviashvili (KP) hierarchy reduction method is a prominent direct method for deriving explicit solutions to integrable equations. This method is based on Hirota's bilinear formulation of integrable systems, as well as the observation that bilinear forms of integrable systems belong to the KP hierarchy or its extensions. Thus, solutions to the KP hierarchy or its extensions, under proper reductions, would yield solutions to the underlying integrable system. In this article, we give a brief overview of this KP-hierarchy reduction method for the derivation of solitons, dark solitons, lumps and rogue waves in well-known integrable equations, such as the KP equation, the Korteweg–de Vries equation, the Davey–Stewartson equations, and the nonlinear Schrödinger equation.

1. Introduction

The bilinear method for integrable systems started with R. Hirota in the early 1970s. Hirota noticed that many integrable equations could be converted into bilinear forms through variable transformations. In addition, series solutions to these bilinear equations would terminate, resulting in exact explicit solutions to the underlying integrable system. Hirota's original solutions were expressed as a summation, which would quickly become complicated when the order of the solution gets large. It was realized later from the Sato theory that bilinear forms of integrable equations belong to the Kadomtsev–Petviashvili (KP) hierarchy or its extension. Thus, solutions to the KP hierarchy or its extension, under proper reductions, would result in solutions to integrable systems. Since KP hierarchies admit large classes of determinant solutions, these determinants could yield very compact solutions for integrable systems. This KP-hierarchy reduction method is a higher version of the bilinear theory, and will be described in this article.

The purpose of this article is to introduce this KP-hierarchy reduction method to a non-expert. Thus, we will keep it as simple as possible. We will show how to use this method to quickly derive sophisticated explicit solutions to well-known integrable equations, such as the KP equation, the Korteweg–de Vries (KdV) equation, the Davey–Stewartson (DS) equations, and the nonlinear Schrödinger (NLS) equation. Solutions we will derive include solitons, dark solitons, lumps and rogue waves. Most of the solutions we will present have been reported before, but some of them seem to be new. Since this paper is meant to be a simple introduction to the KP-reduction method, we will not attempt to give a comprehensive review of literature on this subject. Thus, our references will be brief.

This paper is organized as follows. In Section 2, we introduce bilinear forms for integrable equations, such as KP, KdV, DS and NLS. In Section 3, we introduce the KP hierarchy and some of its extensions. In Sections 4 and 5, solutions to the KP hierarchy and its extensions, in the form of Wronskian determinants and Gram determinants, are presented respectively. In Section 6, KPII solitons are derived. In Section 7, KPI solitons are derived. In Section 8, KPI lumps are derived. In Section 9, KdV solitons are derived. In Section 10, DSI and DSII dark solitons are derived. In Section 11, DSI and DSII rogue waves are derived. In Section 12, dark solitons in the defocusing NLS equation are derived. In Section 13, rogue waves in the focusing NLS equation are derived. Section 14 is the summary.

2. Bilinear forms of integrable equations

2.1. The bilinear form of the KP equation

Consider the KP equation

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3\sigma u_{yy} = 0. \tag{2.1}$$

This equation is called KPI when $\sigma = -1$ and KPII when $\sigma = 1$.

Under the transformation

$$u = 2(\log \tau)_{xx}, \tag{2.2}$$

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https://doi.org/10.1016/j.padiff.2022.100346
Received 3 January 2022; Received in revised form 8 March 2022; Accepted 24 March 2022

this KP equation is reduced to the bilinear form

$$(D_x^4 - 4D_x D_t + 3\sigma D_y^2)\,\tau\cdot\tau = 0, \tag{2.3}$$

where D is Hirota's bilinear differential operator defined by

$$\begin{aligned} &P\left(D_x, D_y, D_t\right) F(x,y,t)\cdot G(x,y,t) \\ &\equiv P\left(\partial_x - \partial_{x'}, \partial_y - \partial_{y'}, \partial_t - \partial_{t'}\right)\times \\ &\quad F(x,y,t)G(x',y',t')\big|_{x'=x,\ y'=y,\ t'=t}, \end{aligned} \tag{2.4}$$

and P is a polynomial of (D_x, D_y, D_t).

2.2. The bilinear form of the KdV equation

The KdV equation is

$$-4u_t + u_{xxx} + 6uu_x = 0. \tag{2.5}$$

Under the variable transformation (2.2), i.e.,

$$u = 2(\log\tau)_{xx}, \tag{2.6}$$

this KdV is reduced to the bilinear form

$$(D_x^4 - 4D_x D_t)\,\tau\cdot\tau = 0. \tag{2.7}$$

2.3. The bilinear form of the DS system

The Davey–Stewartson (DS) equations are

$$\begin{aligned} &\mathrm{i}A_t = -\sigma A_{xx} + A_{yy} + \epsilon(|A|^2 + 2\sigma Q)A, \\ &\sigma Q_{xx} + Q_{yy} = -(|A|^2)_{xx}. \end{aligned} \tag{2.8}$$

When $\sigma = -1$, it is called DSI, and when $\sigma = 1$, it is called DSII. The parameter $\epsilon = \pm 1$ is the sign of nonlinearity.

Under the variable transformation

$$A = \sqrt{2}\,\frac{g}{f}, \quad Q = -\sigma - \epsilon\,(2\log f)_{xx}, \tag{2.9}$$

where f is a real variable and g a complex one, DS equations (2.8) are reduced to the following bilinear forms

$$\begin{aligned} &(\mathrm{i}D_t + \sigma D_x^2 - D_y^2)g\cdot f = 0, \\ &(\sigma D_x^2 + D_y^2)f\cdot f = 2\epsilon(gg^* - f^2), \end{aligned} \tag{2.10}$$

where the asterisk '*' represents complex conjugation. The transformation (2.9) and the resulting bilinear system (2.10) are suitable for deriving DS solutions on non-zero constant backgrounds, such as dark solitons and rogue waves.

With the invertible coordinate transform

$$\begin{aligned} &x_1 = \tfrac{1}{2}(x + \sigma' y), \quad x_{-1} = -\tfrac{1}{2}\epsilon\sigma(x - \sigma' y), \\ &x_2 = \tfrac{1}{2}\mathrm{i}\sigma t, \qquad\quad\ x_{-2} = -\tfrac{1}{2}\mathrm{i}\sigma t, \end{aligned} \tag{2.11}$$

where $\sigma' = \sqrt{-\sigma}$, bilinear equations (2.10) can be split into the following system

$$\begin{aligned} &(D_{x_1}^2 + D_{x_2})f\cdot g = 0, \\ &(D_{x_{-1}}^2 - D_{x_{-2}})f\cdot g = 0, \\ &D_{x_1}D_{x_{-1}}f\cdot f = 2(f^2 - gg^*). \end{aligned} \tag{2.12}$$

2.4. The bilinear form of the NLS equation

The NLS equation is

$$iu_t + u_{xx} + 2\epsilon u|u|^2 = 0, \tag{2.13}$$

where $\epsilon = \pm 1$ is the sign of nonlinearity ($\epsilon = 1$ for focusing and $\epsilon = -1$ for defocusing). Under the variable transformation

$$u = \frac{g}{f}e^{2\mathrm{i}\epsilon t}, \tag{2.14}$$

where f is a real function and g a complex one, the bilinear form of this NLS is

$$\begin{aligned} &(D_x^2 - \mathrm{i}D_t)f\cdot g = 0, \\ &D_x^2 f\cdot f = 2\epsilon(gg^* - f^2). \end{aligned} \tag{2.15}$$

This bilinear system is suitable for deriving NLS solutions on non-zero constant backgrounds.

3. The KP hierarchy and its extensions

Let $\mathbf{x} = (\cdots, x_{-2}, x_{-1}, x_1, x_2, \ldots)$ be a sequence of independent variables. Then, the KP hierarchy is[1]

$$\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)\tau\cdot\tau = 0, \tag{3.1}$$

$$\left[(D_1^3 + 2D_3)D_2 - 3D_1D_4\right]\tau\cdot\tau = 0, \tag{3.2}$$

$$\left(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 - 45D_1^2D_2^2\right)\tau\cdot\tau = 0, \tag{3.3}$$

.........,

where $D_j \equiv D_{x_j}$. Notice that the first member of this hierarchy, Eq. (3.1), is the same as KP's bilinear form (2.3) if we set $(x_1, x_2, x_3) = (x, \sqrt{\sigma}\,y, t)$. This is why the above hierarchy is called the KP hierarchy.

The following bilinear equations

$$(D_1^2 + D_2)\,\tau_n\cdot\tau_{n+1} = 0, \tag{3.4}$$

$$(D_{-1}^2 - D_{-2})\,\tau_n\cdot\tau_{n+1} = 0, \tag{3.5}$$

$$D_1D_{-1}\,\tau_n\cdot\tau_n = 2(\tau_n^2 - \tau_{n+1}\tau_{n-1}), \tag{3.6}$$

belong to an extension of the KP hierarchy.[1,2] Specifically, the first two Eqs. (3.4)–(3.5) belong to the first modified KP hierarchy, and the last equation is the bilinear form of the two-dimensional Toda lattice.

4. Wronskian solutions of KP hierarchies

The original KP hierarchy (3.1)–(3.3) admits the following Wronskian solutions

$$\tau = \begin{vmatrix} \phi_1 & \partial_{x_1}\phi_1 & \cdots & \partial_{x_1}^{N-1}\phi_1 \\ \phi_2 & \partial_{x_1}\phi_2 & \cdots & \partial_{x_1}^{N-1}\phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ \phi_N & \partial_{x_1}\phi_N & \cdots & \partial_{x_1}^{N-1}\phi_N \end{vmatrix}, \tag{4.1}$$

where N is an arbitrary positive integer, and functions $\{\phi_1(\mathbf{x}), \ldots \phi_N(\mathbf{x})\}$ satisfy the linear partial differential equations

$$\partial_{x_j}\phi_i = \partial_{x_1}^j\phi_i, \quad j = 1, 2, \ldots. \tag{4.2}$$

Indeed, substituting the above τ function into the KP hierarchy, the hierarchy equations reduce to the Plücker relation for determinants.[3]

The extended KP hierarchy equations (3.4)–(3.6) admit the following Wronskian solutions[2]

$$\tau_n = \begin{vmatrix} \partial_{x_1}^n\phi_1 & \partial_{x_1}^{n+1}\phi_1 & \cdots & \partial_{x_1}^{n+N-1}\phi_1 \\ \partial_{x_1}^n\phi_2 & \partial_{x_1}^{n+1}\phi_2 & \cdots & \partial_{x_1}^{n+N-1}\phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{x_1}^n\phi_N & \partial_{x_1}^{n+1}\phi_N & \cdots & \partial_{x_1}^{n+N-1}\phi_N \end{vmatrix}, \tag{4.3}$$

where

$$\partial_{x_j}\phi_i = \partial_{x_1}^j\phi_i, \quad j = -2, -1, 1, 2. \tag{4.4}$$

5. Gram solutions of KP hierarchies

Gram solutions to the original KP hierarchy (3.1)–(3.3) are[3]:

$$\begin{aligned} &\tau = \det_{1\le i,j\le N}\left(m_{ij}\right), \\ &m_{ij} = c_{ij} + \int^{x_1} f_i g_j\, dx_1, \\ &\partial_{x_n} f_i = \partial_{x_1}^n f_i, \\ &\partial_{x_n} g_j = (-1)^{n-1}\partial_{x_1}^n g_j, \end{aligned} \tag{5.1}$$

where c_{ij} are constants. Substituting these Gram solutions into the KP hierarchy, this hierarchy reduces to the Jacobi identity of determinants.

Gram solutions to the extended KP hierarchy equations (3.4)–(3.6) are

$$\begin{aligned} &\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right), \\ &m_{ij}^{(n)} = c_{ij} + \int^{x_1} \varphi_i^{(n)}\psi_j^{(n)}\, dx_1, \\ &\partial_{x_k}\varphi_i^{(n)} = \varphi_i^{(n+k)}, \quad \partial_{x_k}\psi_j^{(n)} = -\psi_j^{(n-k)}, \end{aligned} \tag{5.2}$$

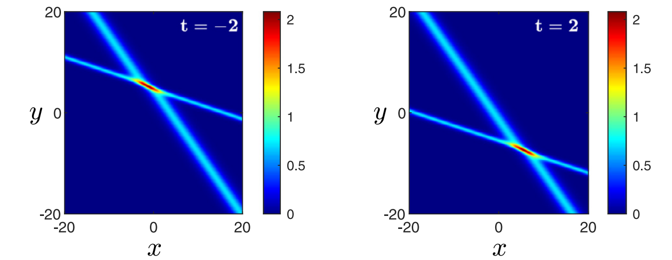

Fig. 1. A KPII two-soliton from Wronskian formulae (4.1) and (6.4) with parameters (6.5).

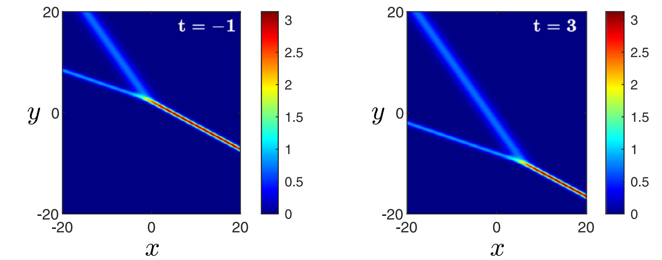

Fig. 2. A degenerate KPII two-soliton from the Wronskian formulae (4.1) and (6.4) with parameters (6.6).

where $k = -2, -1, 1, 2$, and c_{ij} are constants. These solutions satisfying the Toda-lattice equation (3.6) can be found in Ref. 3, while their satisfying (3.4) can be found in Ref. 4. Their satisfying (3.5) can be inferred directly from Ref. 4.

6. KPII solitons

In this section, we derive KPII solitons. For KPII, $\sigma = 1$. When we set $(x_1, x_2, x_3) = (x, y, t)$, the bilinear KP equation (2.3) becomes the first member (3.1) of the KP hierarchy. Thus, solutions to the KP hierarchy, with all the higher coordinates $(x_4, x_5, \ldots)$ set as zero, will give KPII solutions. Since KP-hierarchy solutions come in both Wronskian and Gram forms, we can also derive KPII solitons in different forms.

6.1. KPII solitons in Wronskian form

First, we derive KPII solitons in Wronskian form. The corresponding Wronskian solution of the τ function is given in Eq. (4.1). To get explicit solutions, the simplest choice of functions ϕ_i is

$$\phi_i(\mathbf{x}) = \alpha_i \exp(\xi(\mathbf{x}, p_i)), \tag{6.1}$$

where

$$\xi(\mathbf{x}, k) \equiv \sum_{j=1}^{3} k^j x_j = kx + k^2 y + k^3 t, \tag{6.2}$$

and α_i, p_i are real constants. Obviously, these ϕ_i functions satisfy the linear partial differential equations (4.2). However, since the KPII solution given through the variable transformation (2.2) is invariant under the τ-transformation

$$\tau \to e^{c_1 x_1 + c_2 x_2 + c_3 x_3} \tau, \tag{6.3}$$

where (c_1, c_2, c_3) are arbitrary real constants, it is easy to see that this simplest choice (6.1) gives the trivial zero solution $u(x, y, t) = 0$.

The simplest nontrivial choice of ϕ_i is

$$\begin{aligned}\phi_i(\mathbf{x}) &= \alpha_i \exp(\xi(\mathbf{x}, p_i)) + \beta_i \exp(\xi(\mathbf{x}, q_i)) \\ &= \alpha_i\, e^{p_i x + p_i^2 y + p_i^3 t} + \beta_i\, e^{q_i x + q_i^2 y + q_i^3 t},\end{aligned} \tag{6.4}$$

where α_i, β_i, p_i and q_i are arbitrary real constants. Obviously, these new ϕ_i functions also satisfy the linear partial differential equations (4.2). The corresponding Wronskian solution (4.1), through the bilinear transformation (2.2), then gives soliton solutions to the KPII equation.

To demonstrate these soliton solutions, we choose $N = 2$ and

$$p_1 = 1, p_2 = 2.3, q_1 = -0.2, q_2 = 1.1, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1. \tag{6.5}$$

The corresponding two-soliton is displayed in Fig. 1. This solution is X-shaped and describes the interaction between KPII's two fundamental (line) solitons.

However, if we choose a slightly different set of parameters,

$$p_1 = 1, p_2 = 2.3, q_1 = -0.2, q_2 = 1, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1, \tag{6.6}$$

where only the value of q_2 is changed from 1.1 to 1, the solution's graph would look quite different, see Fig. 2. This solution is Y-shaped, and can be called a degenerate KPII two-soliton. This Y-shaped solution is very common on beaches, see Ref. 5.

In Ref. 6, Kodama investigated a wider class of KPII solutions, and identified different types of solitons with cobweb structures. This class of solutions of KPII comes from choosing

$$\phi_i(\mathbf{x}) = \sum_{j=1}^{M} \alpha_{ij} \exp(\xi(\mathbf{x}, p_{ij})), \tag{6.7}$$

where M is an arbitrary positive integer, and α_{ij}, p_{ij} are free real constants. The reduced echelon form of the $N \times M$ matrix $A = (\alpha_{ij})$ is used to classify the type of cobweb structures in the solution.

6.2. KPII solitons in gram form

The Gram form of the τ solution to the KP hierarchy (3.1) is given in Eq. (5.1). Setting $(x_1, x_2, x_3) = (x, y, t)$ and all the higher coordinates $(x_4, x_5, \ldots)$ zero, this Gram τ function gives a KPII solution. KPII solitons will result when we choose

$$\begin{aligned} f_i &= e^{\xi_i}, \quad g_j = e^{\eta_j}, \\ \xi_i &= p_i x + p_i^2 y + p_i^3 t + \xi_{i0}, \\ \eta_j &= q_j x - q_j^2 y + q_j^3 t + \eta_{j0}, \end{aligned} \tag{6.8}$$

and p_i, q_j, ξ_{i0} and η_{j0} are arbitrary real constants. These functions obviously satisfy the linear partial differential equations in (5.1). The corresponding τ function is then

$$\begin{aligned} \tau &= \det_{1 \le i,j \le N} \left(m_{ij} \right), \\ m_{ij} &= c_{ij} + \frac{1}{p_i + q_j} e^{\xi_i + \eta_j}, \end{aligned} \tag{6.9}$$

where c_{ij} are also arbitrary real constants. This τ solution is real-valued as required. To demonstrate these KPII solitons in Gram form, we choose $N = 2$,

$$\begin{aligned} & p_1 = 1, p_2 = 2.3, q_1 = 0.2, q_2 = -1.1, \\ & \xi_{10} = \xi_{20} = \eta_{10} = \eta_{20} = 0, \quad (c_{ij}) = \begin{pmatrix} 1 & -1 \\ 1 & 0.01 \end{pmatrix}. \end{aligned} \tag{6.10}$$

The graph of the corresponding solution is shown in Fig. 3. This is a KPII two-soliton with a long stem at the center. Such solutions have been observed on beaches in Ref. 5. By choosing other c_{ij} values, we can also get KPII solitons such as those shown in Figs. 1–2.

7. KPI solitons

For KPI, $\sigma = -1$ in Eq. (2.1). In this case, when setting $(x_1, x_2, x_3) = (x, \mathrm{i}y, t)$, the bilinear KPI equation (2.3) becomes the KP-hierarchy equation (3.1), whose Gram solutions are (5.1). A simple choice of these Gram solutions is still (6.9), but with

$$\begin{aligned} \xi_i &= p_i x_1 + p_i^2 x_2 + p_i^3 x_3 + \xi_{i0} = p_i x + \mathrm{i} p_i^2 y + p_i^3 t + \xi_{i0}, \\ \eta_j &= q_j x_1 - q_j^2 x_2 + q_j^3 x_3 + \eta_{j0} = q_j x - \mathrm{i} q_j^2 y + q_j^3 t + \eta_{j0}, \end{aligned}$$

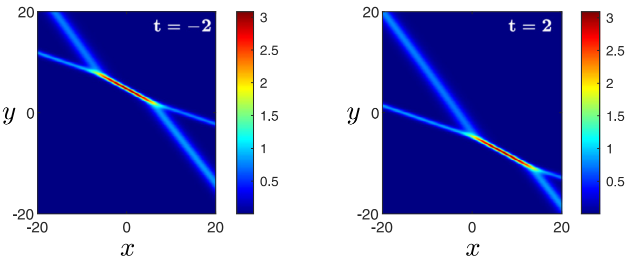

Fig. 3. A KPII two-soliton from the Gram formula (6.9) with parameters (6.10).

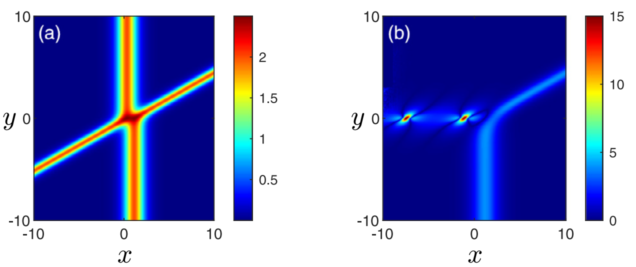

Fig. 4. Two KPI solitons from the Gram formula (7.2) with parameters (7.3). The c_{ij} values for (a) and (b) are the first and second sets in Eq. (7.4) respectively.

and c_{ij}, p_i, q_j, ξ_{i0}, η_{j0} are now complex constants. To ensure this τ function is real-valued, we impose parameter conditions

$$c_{ij}^* = c_{ji},\ \ p_i = q_i^*,\ \ \xi_{i0} = \eta_{i0}^*. \tag{7.1}$$

Under these conditions, $m_{ij}^* = m_{ji}$, and thus τ is real. Putting these results together, we find that KPI admits the following Gram solutions

$$\begin{aligned} \tau &= \det_{1\le i,j\le N}\left(m_{ij}\right), \\ m_{ij} &= c_{ij} + \frac{1}{p_i+p_j^*} e^{\xi_i+\xi_j^*}, \\ \xi_i &= p_i x + \mathrm{i} p_i^2 y + p_i^3 t + \xi_{i0}, \end{aligned} \tag{7.2}$$

where $c_{ij}^* = c_{ji}$, and p_i, ξ_{i0} are free complex constants. These solutions give KPI solitons. To illustrate, we choose

$$N = 2, p_1 = 1, p_2 = 1 + \mathrm{i}, \xi_{10} = \xi_{20} = 0, \tag{7.3}$$

and two sets of c_{ij} values,

$$(c_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (c_{ij}) = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}. \tag{7.4}$$

The corresponding two solutions at $t = 0$ are displayed in Fig. 4(a,b) respectively. Fig. 4(a) describes the interaction between two KPI line solitons. Fig. 4(b) is a degenerate and more interesting solution, where lumps are periodically emitted from the intersection of the two semi-line solitons. These lumps, after emission, would move along the negative-x axis.

8. KPI lumps

KPI admits stable fundamental lump solutions that are bounded rational functions decaying in all spatial directions. In addition, it admits rational solutions that describe the interaction between these lumps. These rational solutions can be written as

$$\begin{aligned} \tau &= \det_{1\le i,j\le N}\left(m_{ij}\right), \\ m_{ij} &= A_i B_j \left.\frac{1}{p_i+q_j} e^{\xi_i+\eta_j}\right|_{q_j=p_j^*}, \\ \xi_i &= p_i x + \mathrm{i} p_i^2 y + p_i^3 t, \\ \eta_j &= q_j x - \mathrm{i} q_j^2 y + q_j^3 t, \end{aligned} \tag{8.1}$$

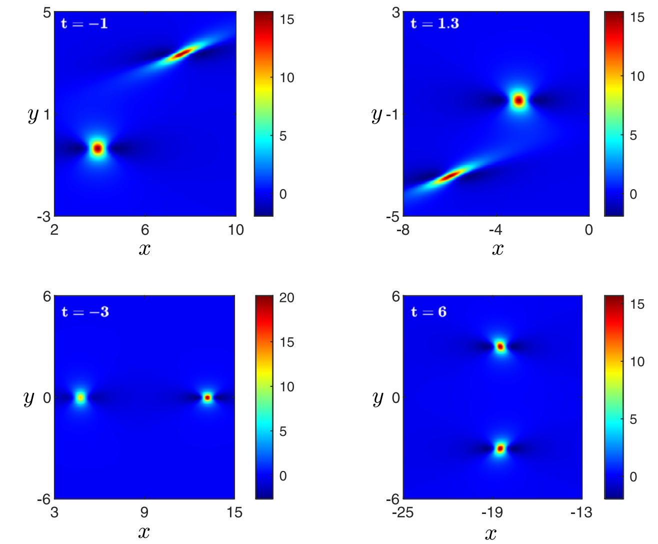

Fig. 5. Lump solutions of KPI. Upper row: a two-lump with parameters (8.3) at two time values of $t = -1$ and 1.3. Lower row: a second-order lump with parameters (8.4) at two time values of $t = -3$ and 6.

where

$$A_i = \sum_{k=0}^{n_i} c_{ik} (p_i \partial_{p_i})^{n_i-k}, \quad B_j = \sum_{k=0}^{n_j} c_{jk}^* (q_j \partial_{q_j})^{n_j-k}, \tag{8.2}$$

n_i is an arbitrary positive integer, p_i is an arbitrary complex constant with $p_i + p_j^* \neq 0$, and c_{ik} is also an arbitrary complex constant. It is easy to see, from the previous KPI solutions (7.2), that the above new solutions (8.1) are also special cases of the general Gram solutions (5.1), since differential operators A_i, B_j and ∂_{x_n} commute with each other. In addition, the above τ is real since $m_{ij}^* = m_{ji}$. Thus, the above solutions also satisfy the KPI equation. The exponential terms $e^{\xi_i+\eta_j}$ in these solutions will eventually factor out of the τ determinant and disappear from the u solution (2.2). Thus, these u solutions are rational functions of (x, y, t).

These solutions (8.1) contain many types of lump solutions, such as multi-lumps, higher-order lumps, and their combinations. Multi-lumps are solutions where $N > 1$, all p_i's are distinct, and $n_1 = n_2 = \cdots = n_N = 1$, which describe the elastic interaction between multiple fundamental lumps with different terminal velocities. Higher-order lumps, on the other hand, are solutions with $N \ge 1$ and all p_i's the same, which describe the anomalous scattering between multiple fundamental lumps with identical terminal velocities. As a demonstration of multi-lump solutions, we choose

$$N = 2, p_1 = 1, p_2 = 1 + \mathrm{i}, c_{10} = c_{20} = 1, c_{11} = c_{21} = 0. \tag{8.3}$$

The corresponding two-lump is shown in the upper row of Fig. 5. To demonstrate a higher-order lump, we choose

$$N = 1, p_1 = 1, n_1 = 2, c_{10} = 1, c_{11} = c_{12} = 0. \tag{8.4}$$

The corresponding second-order lump is shown in the lower row of Fig. 5. These two solutions have very different behaviors. In the two-lump solution, the two individual lumps simply pass through each other after collision without change of velocity or phase. But in the second-order lump, the two individual lumps first move toward each other along the x axis, then collide, and then separate away from each other along the y direction.

More explicit expressions for higher-order KPI lumps through Schur polynomials, as well as analysis of their large-time solution patterns, can be found in Ref. 7.

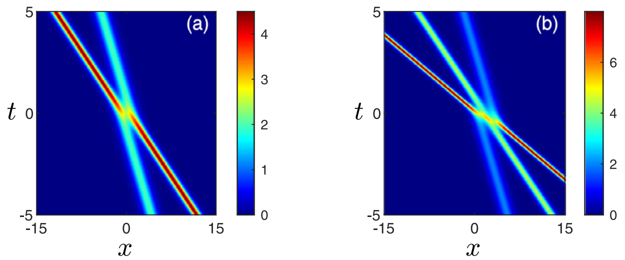

Fig. 6. (a) A KdV two-soliton from the Wronskian formulae (9.4) and (9.6) with parameters (9.7). (b) A KdV three-soliton from the Gram formula (9.12) with parameters (9.13).

9. KdV solitons

The bilinear form (2.7) of KdV is a special case of the KP hierarchy equation (3.1) when we set $(x_1, x_3) = (x, t)$ and the τ function is independent of x_2. Thus, to derive KdV solutions, we need to do a dimension reduction and reduce the KP-hierarchy's solution τ to one without x_2 dependence. To do so, one cannot simply set $x_2 = 0$ in the KP-hierarchy's $\tau(\mathbf{x})$ solution. The correct dimension reduction process depends on the Wronskian or Gram form of the hierarchy solution τ, and will be done separately below.

9.1. KdV solitons in Wronskian form

For the Wronskian solution (4.1) of the KP hierarchy, we choose ϕ_i functions as in (6.4), i.e.,

$$\phi_i(\mathbf{x}) = \alpha_i\, e^{p_i x_1 + p_i^2 x_2 + p_i^3 x_3} + \beta_i\, e^{q_i x_1 + q_i^2 x_2 + q_i^3 x_3}. \tag{9.1}$$

To achieve dimension reduction, we constrain parameters p_i and q_i by

$$q_i = -p_i. \tag{9.2}$$

Under this constraint, the above ϕ_i becomes

$$\phi_i(\mathbf{x}) = e^{p_i^2 x_2}\, \hat{\phi}_i(x_1, x_3), \tag{9.3}$$

where

$$\hat{\phi}_i(x_1, x_3) = \alpha_i\, e^{p_i x_1 + p_i^3 x_3} + \beta_i\, e^{-p_i x_1 - p_i^3 x_3}. \tag{9.4}$$

Inserting this $\phi_i(\mathbf{x})$ into the Wronskian (4.1), it is easy to see that the corresponding τ function is of the form

$$\tau(\mathbf{x}) = e^{(p_1^2 + p_2^2 + \cdots + p_N^2) x_2}\, \hat{\tau}(x_1, x_3), \tag{9.5}$$

where

$$\hat{\tau}(x_1, x_3) = \begin{vmatrix} \hat{\phi}_1 & \partial_{x_1}\hat{\phi}_1 & \cdots & \partial_{x_1}^{N-1}\hat{\phi}_1 \\ \hat{\phi}_2 & \partial_{x_1}\hat{\phi}_2 & \cdots & \partial_{x_1}^{N-1}\hat{\phi}_2 \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\phi}_N & \partial_{x_1}\hat{\phi}_N & \cdots & \partial_{x_1}^{N-1}\hat{\phi}_N \end{vmatrix}. \tag{9.6}$$

Since the KP hierarchy equation (3.1) is invariant when τ is multiplied by an exponential of a linear function of x_2, this $\hat{\tau}(x_1, x_3)$ then also satisfies (3.1). But it is independent of x_2, and thus satisfies KdV's bilinear form (2.7), where $(x_1, x_3) = (x, t)$. This $\hat{\tau}(x, t)$ gives KdV solitons through the bilinear transformation (2.6).

To demonstrate, we choose $N = 2$,

$$p_1 = 1, p_2 = 1.5, \alpha_1 = \alpha_2 = 1, \beta_1 = 1, \beta_2 = -1. \tag{9.7}$$

The corresponding two-soliton of KdV is displayed in Fig. 6(a).

Extending the above bilinear approach, one can also obtain other types of KdV solutions. For example, by choosing $\phi_i(\mathbf{x})$ to be $e^{p^2 x_2}$ multiplying the p^i's coefficient of the power expansion of $\alpha\, e^{p x_1 + p^3 x_3} + \beta\, e^{-p x_1 - p^3 x_3}$ in p, where α, β and p are arbitrary real constants, such $\phi_i(\mathbf{x})$ would still satisfy the linear partial differential equations (4.2). Then, applying the same KP-hierarchy invariance when τ is multiplied by an exponential of a linear function of x_2, we see that the $\hat{\tau}(x_1, x_3)$ function, whose elements $\hat{\phi}_i(x_1, x_3)$ are polynomial functions of (x_1, x_3) from the power expansion of $\alpha\, e^{p x_1 + p^3 x_3} + \beta\, e^{-p x_1 - p^3 x_3}$ in p, would satisfy KdV's bilinear form (2.7) with $(x_1, x_3) = (x, t)$. Such $\hat{\tau}(x, t)$ gives rational solutions of KdV, which are singular. Other ways to derive these rational KdV solutions can be found in Ref. 8.

9.2. KdV solitons in Gram form

For the Gram solution (5.1) of the KP hierarchy, we choose m_{ij} as in (6.9). To achieve dimension reduction, we choose

$$c_{ij} = \delta_{ij}, \tag{9.8}$$

where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise. Then, by factoring out e^{ξ_i} from the ith row and e^{η_j} from the jth column of τ, and redistributing $e^{\xi_i + \eta_i}$ to the ith row of the resulting determinant, we find that elements m_{ij} of this τ can be rewritten as

$$m_{ij} = \delta_{ij} + \frac{1}{p_i + q_j} e^{\xi_i + \eta_i}. \tag{9.9}$$

Since

$$\xi_i + \eta_i = (p_i + q_i)x_1 + (p_i^2 - q_i^2)x_2 + (p_i^3 + q_i^3)x_3 + (\xi_{i0} + \eta_{i0}), \tag{9.10}$$

if we constrain real parameters (p_i, q_i) as

$$q_i = p_i, \tag{9.11}$$

then this τ would be independent of x_2, and thus satisfies KdV's bilinear form (2.7), where $(x_1, x_3) = (x, t)$. This τ gives the N-soliton solution of KdV. Its expression can be written out more explicitly as

$$\tau(x, t) = \det_{1 \le i,j \le N} \left(\delta_{ij} + \frac{1}{p_i + p_j} e^{2 p_i x + 2 p_i^3 t + \gamma_i} \right), \tag{9.12}$$

where $\gamma_i \equiv \xi_{i0} + \eta_{i0}$, and γ_i, p_i are arbitrary real constants.

To demonstrate these KdV solitons in Gram form, we choose

$$N = 3, p_1 = 1, p_2 = 1.5, p_3 = 2, \gamma_1 = \gamma_2 = \gamma_3 = 0. \tag{9.13}$$

The corresponding three-soliton is displayed in Fig. 6(b).

10. DS dark solitons

10.1. DSI and DSII dark solitons in Gram form

DS' bilinear system (2.12) belongs to the extended KP hierarchy (3.4)–(3.6). Thus, its Gram solutions are

$$\begin{aligned} &A = \sqrt{2}\tfrac{g}{f}, \quad Q = -\sigma - \epsilon\,(2\log f)_{xx}, \\ &f = \tau_0, \quad g = \tau_1, \quad g^* = \tau_{-1}, \end{aligned} \tag{10.1}$$

where τ_n is given in Eq. (5.2). The simplest τ_n solutions are obtained when we choose

$$\begin{aligned} &\varphi_i^{(n)} = p_i^n e^{\xi_i}, \quad \psi_j^{(n)} = (-q_j)^{-n} e^{\eta_j}, \\ &\xi_i = \tfrac{1}{p_i^2} x_{-2} + \tfrac{1}{p_i} x_{-1} + p_i x_1 + p_i^2 x_2 + \xi_{i0}, \\ &\eta_j = -\tfrac{1}{q_j^2} x_{-2} + \tfrac{1}{q_j} x_{-1} + q_j x_1 - q_j^2 x_2 + \eta_{j0}, \end{aligned} \tag{10.2}$$

where p_i, q_j, ξ_{i0} and η_{j0} are complex constants. Obviously, these $\varphi_i^{(n)}, \psi_j^{(n)}$ functions satisfy the linear partial differential equations in (5.2). The resulting τ_n solutions are then

$$\begin{aligned} &\tau_n = \det_{1 \le i,j \le N} \left(m_{ij}^{(n)} \right), \\ &m_{ij}^{(n)} = c_{ij} + \tfrac{1}{p_i + q_j} (-\tfrac{p_i}{q_j})^n e^{\xi_i + \eta_j}, \end{aligned} \tag{10.3}$$

where c_{ij} are complex constants.

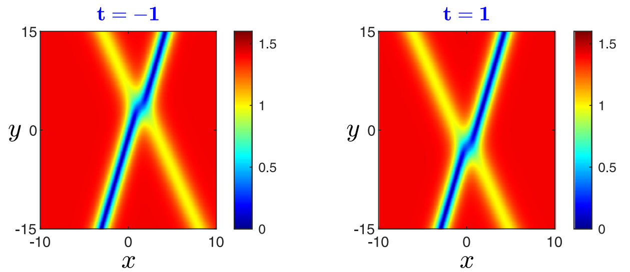

Fig. 7. A DSI two-dark soliton $|A|$ from the Gram solution (10.1)–(10.3) and (10.6) with parameters (10.8) at times $t = \pm 1$.

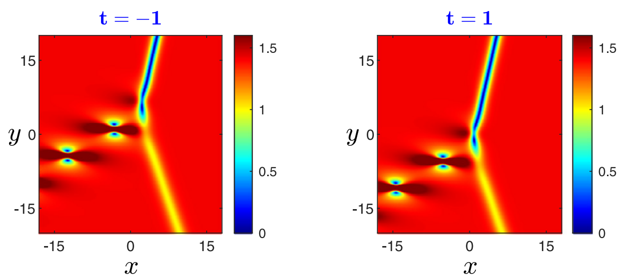

Fig. 8. DSI degenerate two-dark soliton $|A|$ from the Gram solution (10.1)–(10.3) and (10.6) with parameters (10.9) at times $t = \pm 1$.

In order for Eq. (10.3) to give true DS solutions, we must meet the complex-conjugation conditions

$$\tau_0 = \tau_0^*, \quad \tau_{-1} = \tau_1^*, \tag{10.4}$$

since f needs to be real and (τ_1, τ_{-1}) need to be (g, g^*) from Eq. (10.1). These conjugation conditions can be met in different ways for DSI and DSII, which will be treated separately below.

10.1.1. DSI dark solitons

For DSI, the coordinate transform (2.11) reduces to

$$\begin{aligned} x_1 &= \tfrac{1}{2}(x+y), \quad & x_{-1} &= \tfrac{1}{2}\epsilon(x-y), \\ x_2 &= -\tfrac{1}{2}\mathrm{i}t, \quad & x_{-2} &= \tfrac{1}{2}\mathrm{i}t, \end{aligned} \tag{10.5}$$

where (x_1, x_{-1}) are real and (x_2, x_{-2}) purely imaginary. To ensure the conjugation conditions (10.4), we constrain parameters as

$$q_j = p_j^*, \quad \eta_{j0} = \xi_{j0}^*, \quad c_{ij} = c_{ji}^*. \tag{10.6}$$

Then

$$\xi_j^* = \eta_j, \quad \left(m_{ij}^{(n)}\right)^* = m_{ji}^{(-n)}, \quad \tau_n^* = \tau_{-n}, \tag{10.7}$$

and thus the conjugation conditions (10.4) are satisfied.

To illustrate these DSI solutions, we choose

$$N = 2, \epsilon = 1, p_1 = 1 + \mathrm{i}, p_2 = 0.8, \xi_{j0} = 0, c_{ij} = \delta_{ij}. \tag{10.8}$$

Then we get a two-dark-soliton which is shown in Fig. 7.

With other choices of the c_{ij} values, these DSI solutions could look different. For example, when we choose the same parameters as in (10.8) but with all c_{ij} taken as 1, i.e.,

$$N = 2, \epsilon = 1, p_1 = 1 + \mathrm{i}, p_2 = 0.8, \xi_{j0} = 0, c_{ij} = 1, \tag{10.9}$$

the corresponding solution is plotted in Fig. 8. This solution is triple-branched, with two of the branches dark solitons, but the third branch comprising periodically spaced dipoles. This kind of DSI dark soliton can be called a degenerate two-dark soliton.

10.1.2. DSII dark solitons

For DSII, the coordinate transform (2.11) becomes

$$\begin{aligned} x_1 &= \tfrac{1}{2}(x+\mathrm{i}y), \quad & x_{-1} &= -\tfrac{1}{2}\epsilon(x-\mathrm{i}y), \\ x_2 &= \tfrac{1}{2}\mathrm{i}t, \quad & x_{-2} &= -\tfrac{1}{2}\mathrm{i}t, \end{aligned} \tag{10.10}$$

where (x_1, x_{-1}) become complex. To satisfy conjugation conditions (10.4) for DSII, we need to constrain parameters differently from (10.6) of DSI. Indeed, in this case, we need to split parameters $p_i, q_i, \xi_{i0}, \eta_{i0}, c_{ij}$ into two blocks and constrain them analogously to what was done in Ref. 9, so that the resulting τ_n determinant becomes a 2×2 block determinant with certain symmetries. Details will be omitted.

10.2. DSII dark solitons in Wronskian form

Since DS' bilinear system (2.12) belongs to the extended KP hierarchy (3.4)–(3.6), its Wronskian solutions are

$$f = \tau_0, \quad g = \tau_1, \quad g^* = \tau_{-1}, \tag{10.11}$$

where τ_n is given in Eqs. (4.3)–(4.4), i.e.,

$$\tau_n = \begin{vmatrix} \partial_{x_1}^n \phi_1 & \partial_{x_1}^{n+1} \phi_1 & \cdots & \partial_{x_1}^{n+N-1} \phi_1 \\ \partial_{x_1}^n \phi_2 & \partial_{x_1}^{n+1} \phi_2 & \cdots & \partial_{x_1}^{n+N-1} \phi_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{x_1}^n \phi_N & \partial_{x_1}^{n+1} \phi_N & \cdots & \partial_{x_1}^{n+N-1} \phi_N \end{vmatrix}, \tag{10.12}$$

and ϕ_i satisfies the linear partial differential equations (4.4). For this τ_n to be true solutions, it must also meet the complex conjugation conditions (10.4).

Here, we consider defocusing DSII, where $\epsilon = -1$. In this case, the coordinate transform (2.11) is

$$\begin{aligned} x_1 &= \tfrac{1}{2}(x+\mathrm{i}y), \quad & x_{-1} &= \tfrac{1}{2}(x-\mathrm{i}y), \\ x_2 &= \tfrac{1}{2}\mathrm{i}t, \quad & x_{-2} &= -\tfrac{1}{2}\mathrm{i}t, \end{aligned} \tag{10.13}$$

where $x_{-1} = x_1^*$ and $x_{-2} = x_2^*$. If we choose ϕ_i functions as

$$\begin{aligned} \phi_1 &= \sum_{j=1}^{M} \alpha_j \, e^{p_j x_1 + p_j^2 x_2 + p_j^{-1} x_{-1} + p_j^{-2} x_{-2}}, \\ \phi_i &= \partial_{x_{-1}}^{i-1} \phi_1, \qquad i = 2, \ldots, N, \end{aligned} \tag{10.14}$$

with M and N being arbitrary positive integers, then these ϕ_i satisfy the linear partial differential equations (4.4). In addition, if we impose parameter constraints

$$|p_j| = 1, \quad \alpha_j \text{ are all real}, \tag{10.15}$$

then the complex conjugation conditions (10.4) are also satisfied. Thus, the corresponding (10.12) gives DSII solutions. To illustrate these solutions, we choose

$$\begin{aligned} &N = 1, M = 4, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, \\ &(p_1, p_2, p_3, p_4) = (e^{-\mathrm{i}}, e^{2\mathrm{i}}, e^{0.45\mathrm{i}}, e^{-2.6\mathrm{i}}). \end{aligned} \tag{10.16}$$

The corresponding DSII solution at $t = 0$ is shown in Fig. 9(a). This solution is a X-shaped dark soliton.

If we keep the same p_j values but change α_j to

$$\alpha_1 = 1, \alpha_2 = 0.1, \alpha_3 = 30, \alpha_4 = 300, \tag{10.17}$$

then the solution at $t = 0$ becomes two Y-shaped dark solitons joined by a long dark stem, as is shown in Fig. 9(b).

Under other choices of ϕ_i functions, different DSII solutions can also be obtained. For instance, it has been pointed out by G. Biondini and K. Maruno (private communications) that if one takes

$$\phi_i = \sum_{j=1}^{M} \alpha_{ij} \, p_j^{-(N-1)/2} e^{p_j x_1 + p_j^2 x_2 + p_j^{-1} x_{-1} + p_j^{-2} x_{-2}}, \quad i = 1, \ldots N,$$

and impose parameter constraints of

$$|p_j| = 1, \quad (\alpha_{ij})_{N \times M} \text{ are all real},$$

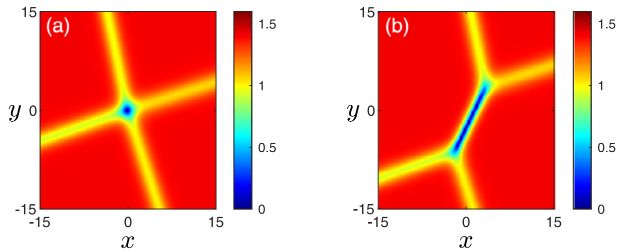

Fig. 9. DSII dark solitons $|A|$ from the Wronskian solution (10.11)–(10.15) at time $t = 0$: (a) with parameters (10.16); (b) with p_j values as in (10.16) but α_j values as in (10.17).

then the τ_n function in (10.12) would also give DSII solutions through (10.11). In this case, if the integer part of $N/2$ is odd, then this τ_n function would be such that

$$\tau_0^* = -\tau_0, \quad \tau_{-1}^* = -\tau_1,$$

and thus does not satisfy the conjugation conditions (10.4). However, when we define

$$f = \mathrm{i}\tau_0, \quad g = \mathrm{i}\tau_1, \quad g^* = \mathrm{i}\tau_{-1},$$

these (f, g, g^*) functions would satisfy DSII's bilinear forms (2.12) and meet the proper complex conjugation conditions. Graphs of these solutions exhibit cobweb structures of dark solitons.

11. DS rogue waves

Since DS' bilinear system (2.12) is in the extended KP hierarchy (3.4)–(3.6), to obtain DS rogue waves, we choose Gram solutions (5.2) to this extended KP hierarchy (3.4)–(3.6) as

$$\begin{aligned}
&\tau_n = \det_{1\le i,j\le N}\left(\tilde{m}_{ij}^{(n)}\right),\\
&\tilde{m}_{ij}^{(n)} = \int^{x_1} \tilde{\varphi}_i^{(n)}\tilde{\psi}_j^{(n)} dx_1,\\
&\tilde{\varphi}_i^{(n)} = A_i\varphi_i^{(n)}, \quad \tilde{\psi}_j^{(n)} = B_j\psi_j^{(n)},\\
&\varphi_i^{(n)} = p_i^n e^{\xi_i}, \quad \psi_j^{(n)} = (-q_j)^{-n}e^{\eta_j},\\
&\xi_i = \tfrac{1}{p_i^2}x_{-2} + \tfrac{1}{p_i}x_{-1} + p_i x_1 + p_i^2 x_2,\\
&\eta_j = -\tfrac{1}{q_j^2}x_{-2} + \tfrac{1}{q_j}x_{-1} + q_j x_1 - q_j^2 x_2,
\end{aligned} \tag{11.1}$$

where A_i and B_j are differential operators of order n_i and m_j with respect to p_i and q_j,

$$A_i = \sum_{k=0}^{n_i} c_{ik}(p_i\partial_{p_i})^{n_i-k}, \quad B_j = \sum_{k=0}^{m_j} d_{jk}(q_j\partial_{q_j})^{m_j-k}, \tag{11.2}$$

and p_i, q_j, c_{ik}, d_{jk} are complex constants. Obviously, the above $\tilde{\varphi}_i^{(n)}$ and $\tilde{\psi}_j^{(n)}$ satisfy the linear partial differential equations in (5.2) since operators A_i and B_j commute with differentials ∂_{x_k}. Thus, this τ_n satisfies the extended KP hierarchy (3.4)–(3.6), and hence DS' bilinear system (2.12) under the (f, g, g^*) choices of (10.11). A more explicit expression for the above τ_n solution is taken as

$$\begin{aligned}
&\tau_n = \det_{1\le i,j\le N}\left(\tilde{m}_{ij}^{(n)}\right),\\
&\tilde{m}_{ij}^{(n)} = A_i B_j \tfrac{1}{p_i+q_j}(-\tfrac{p_i}{q_j})^n e^{\xi_i+\eta_j}.
\end{aligned} \tag{11.3}$$

Here, we intentionally do not include constants of integration c_{ij} into $\tilde{m}_{ij}^{(n)}$, because this way, this τ_n would be a polynomial function of $\mathbf{x} = (x_{-2}, x_{-1}, x_1, x_2)$ multiplying a n-independent exponential of a linear function of $\mathbf{x}$. Thus, the resulting (A, Q) solutions from the transformation (2.9) would be rational functions of $\mathbf{x}$. These rational functions give rogue waves in the DS system (2.8) through (2.9) and (10.11), i.e.,

$$\begin{aligned}
&A = \sqrt{2}\tfrac{g}{f}, \quad Q = -\sigma - \epsilon\,(2\log f)_{xx},\\
&f = \tau_0, \quad g = \tau_1, \quad g^* = \tau_{-1},
\end{aligned} \tag{11.4}$$

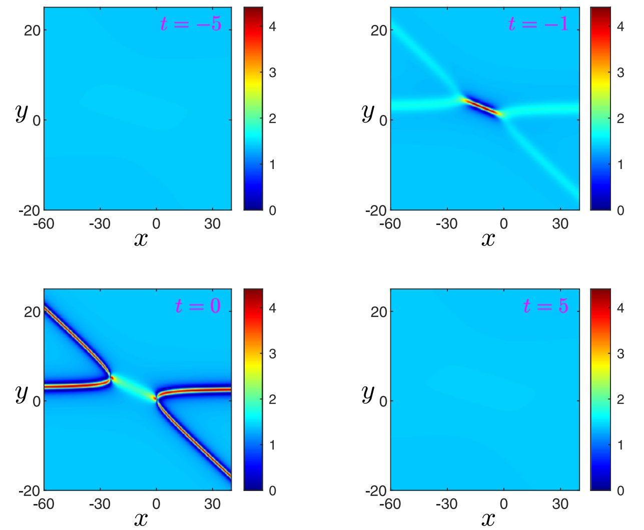

Fig. 10. A DSI rogue wave $|A|$ in Eqs. (11.3)–(11.5) with parameters (11.7).

if the above τ_n also satisfies the conjugation conditions (10.4), and parameters p_i, q_j meet some restrictions.

11.1. DSI rogue waves

For DSI, the coordinate transform (2.11) is (10.5), where (x_1, x_{-1}) are real and (x_2, x_{-2}) purely imaginary. In this case, for τ_n in (11.3) to satisfy conjugacy conditions (10.4), we constrain the parameters as

$$q_j = p_j^*, \quad m_j = n_j, \quad d_{jl} = c_{jl}^*. \tag{11.5}$$

Then,

$$\xi_j^* = \eta_j, \quad \left(\tilde{m}_{ij}^{(n)}\right)^* = \tilde{m}_{ji}^{(-n)}, \quad \tau_n^* = \tau_{-n}. \tag{11.6}$$

Thus, conjugation conditions (10.4) are met. The corresponding solutions (11.3)–(11.4) give DSI rogue waves when all p_j's are real.[10] As an example, we choose

$$\begin{aligned}
&\epsilon = 1, N = 2, n_1 = n_2 = 1, p_1 = 1, p_2 = 1.5,\\
&c_{10} = c_{20} = 1, c_{11} = c_{21} = 0,
\end{aligned} \tag{11.7}$$

and the corresponding rogue wave is shown in Fig. 10.

11.2. DSII rogue waves

For DSII, the coordinate transform (2.11) is (10.10), where (x_1, x_{-1}) are complex. To derive rogue waves for DSII, we need to constrain parameters in the τ_n solution (11.3) blockwise, so that τ_n becomes a 2×2 block determinant with certain symmetry. See Ref. 9 for details.

12. Dark solitons in the defocusing NLS equation

The bilinear forms (2.15) of the NLS equation (2.13) belong to the extended KP hierarchy [see (3.4)–(3.6)]. From Section 5, we know that Gram solutions (5.2), i.e.,

$$\begin{aligned}
&\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right),\\
&m_{ij}^{(n)} = c_{ij} + \int^{x_1} \varphi_i^{(n)}\psi_j^{(n)} dx_1,\\
&\partial_{x_k}\varphi_i^{(n)} = \varphi_i^{(n+k)}, \quad \partial_{x_k}\psi_j^{(n)} = -\psi_j^{(n-k)},
\end{aligned} \tag{12.1}$$

with $k = -1, 1$ and 2, satisfy the extended KP hierarchy equations (3.4) and (3.6), i.e.,

$$\begin{aligned}
&(D_{x_1}^2 + D_{x_2})\,\tau_n \cdot \tau_{n+1} = 0,\\
&D_{x_1}D_{x_{-1}}\,\tau_n \cdot \tau_n = 2(\tau_n\tau_n - \tau_{n+1}\tau_{n-1}).
\end{aligned} \tag{12.2}$$

Here, we have deleted the x_{-2} dependence in the above τ_n function since the two bilinear equations in (12.2) do not involve x_{-2}. The simplest Gram solutions (12.1) are

$$\begin{aligned} &\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right), \\ &m_{ij}^{(n)} = c_{ij} + \tfrac{1}{p_i+q_j}\left(-\tfrac{p_i}{q_j}\right)^n e^{\xi_i+\eta_j}, \\ &\xi_i = \tfrac{1}{p_i}x_{-1} + p_i x_1 + p_i^2 x_2 + \xi_{i0}, \\ &\eta_j = \tfrac{1}{q_j}x_{-1} + q_j x_1 - q_j^2 x_2 + \eta_{j0}, \end{aligned} \tag{12.3}$$

which are the same solutions (10.2)–(10.3) but with $x_{-2} = 0$. Here, p_i, q_j, $c_{i,j}$, ξ_{i0} and η_{j0} are all arbitrary complex parameters.

To reduce the three-dimensional bilinear equations (12.2) to the two-dimensional NLS bilinear equations (2.15), we need to perform a dimension reduction. To do so, we impose the condition

$$\partial_{x_1}\tau_n = \partial_{x_{-1}}\tau_n. \tag{12.4}$$

This way, we will have

$$D_{x_1}D_{x_{-1}}\ \tau_n\cdot\tau_n = D_{x_1}^2\ \tau_n\cdot\tau_n. \tag{12.5}$$

Then, Eqs. (12.2) would reduce to the NLS bilinear equations (2.15) when we set $x_1 = x$, $x_2 = \mathrm{i}t$, $\epsilon = -1$ (defocusing NLS), and

$$f = \tau_0, \quad g = \tau_1, \quad g^* = \tau_{-1}. \tag{12.6}$$

To achieve the dimensional reduction (12.4), we choose

$$c_{ij} = \delta_{ij}. \tag{12.7}$$

Then, by factoring out e^{ξ_i} from the ith row and e^{η_j} from the jth column of τ_n in (12.3), and then redistributing $e^{\xi_i+\eta_i}$ to the ith row of the resulting determinant, we find that elements $m_{ij}^{(n)}$ of this τ_n can be rewritten as

$$m_{ij}^{(n)} = \delta_{ij} + \frac{1}{p_i+q_j}\left(-\frac{p_i}{q_j}\right)^n e^{\xi_i+\eta_i}. \tag{12.8}$$

Since

$$\xi_i+\eta_i = \left(\frac{1}{p_i}+\frac{1}{q_i}\right)x_{-1} + (p_i+q_i)x_1 + (p_i^2-q_i^2)x_2 + \gamma_i, \tag{12.9}$$

where $\gamma_i = \xi_{i0}+\eta_{i0}$, when we insert the τ_n determinant with matrix elements (12.8) into the dimensional reduction Eq. (12.4), we see that this equation would hold if parameters p_i and q_i satisfy the relation

$$p_i + q_i = \frac{1}{p_i} + \frac{1}{q_i}. \tag{12.10}$$

One simple way to meet this relation is to constrain (p_i, q_i) by

$$q_i = \frac{1}{p_i}. \tag{12.11}$$

After this dimension reduction, x_{-1} becomes irrelevant and can be set to zero in the τ_n solutions (12.3).

From Eq. (12.6), we see that to obtain valid NLS solutions, we still need to meet the complex conjugation conditions

$$\tau_0^* = \tau_0, \quad \tau_{-1} = \tau_1^*. \tag{12.12}$$

Since $x_1 = x$ is real and $x_2 = \mathrm{i}t$ pure imaginary, when we further impose parameter conditions

$$q_i = p_i^*, \quad \eta_{i0} = \xi_{i0}^*, \tag{12.13}$$

we would get relations (10.7). Thus, complex conjugation conditions (12.12) are satisfied.

Combining the complex-conjugation constraints (12.13) with the dimensional-reduction constraint (12.11), we get

$$|p_i| = 1. \tag{12.14}$$

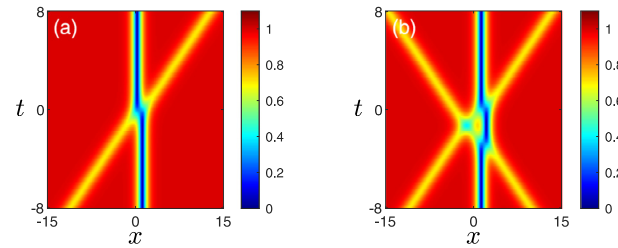

Fig. 11. (a) A NLS two-dark soliton $|u|$ in Eq. (12.15) with parameters (12.16). (b) A NLS three-dark soliton $|u|$ in Eq. (12.15) with parameters (12.17).

To summarize, the defocusing NLS equation (2.13) (with $\epsilon = -1$) admits the following solutions

$$\begin{aligned} &u_N(x,t) = \tfrac{\tau_1}{\tau_0}e^{-2\mathrm{i}t}, \\ &\tau_n = \det_{1\le i,j\le N}\left(m_{i,j}^{(n)}\right), \\ &m_{ij}^{(n)} = \delta_{ij} + \tfrac{1}{p_i+p_j^*}\left(-p_ip_j\right)^n e^{(p_i+p_i^*)x+(p_i^2-p_i^{*2})\mathrm{i}t+\gamma_i}, \end{aligned} \tag{12.15}$$

where $|p_i| = 1$, and γ_i are free real constants. These solutions are dark solitons.

As an example, we set

$$N = 2, p_1 = 1, p_2 = e^{\mathrm{i}\pi/4}, \gamma_1 = \gamma_2 = 0. \tag{12.16}$$

The resulting two-dark soliton is plotted in Fig. 11(a). If we choose $N = 3$ and

$$p_1 = 1, p_2 = e^{\mathrm{i}\pi/4}, p_3 = e^{-\mathrm{i}\pi/4}, \gamma_1 = \gamma_2 = 0, \gamma_3 = 5, \tag{12.17}$$

we would get a three-dark soliton as is shown in Fig. 11(b).

Following similar methods, one can derive dark solitons in the Manakov system (a two-component generalization of the NLS equation). In this case, the extended KP hierarchy equations (3.4) and (3.6) need to be further generalized to include one more independent variable as well as its associated index for the τ function. See Ref. 11 for details.

13. Rogue waves in the focusing NLS equation

To derive rogue waves in the focusing NLS equation (2.13) (with $\epsilon = 1$), we choose Gram solutions (5.2) as

$$\begin{aligned} &\tau_n = \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right), \\ &m_{ij}^{(n)} = \int^{x_1}\varphi_i^{(n)}\psi_j^{(n)}dx_1, \\ &\varphi_i^{(n)} = A_i\varphi^{(n)}, \quad \psi_j^{(n)} = B_j\psi^{(n)}, \\ &\varphi^{(n)} = p^n e^{\xi}, \quad \psi^{(n)} = (-q)^{-n}e^{\eta}, \\ &\xi = \tfrac{1}{p}x_{-1} + px_1 + p^2x_2, \\ &\eta = \tfrac{1}{q}x_{-1} + qx_1 - q^2x_2, \end{aligned} \tag{13.1}$$

where A_i and B_j are differential operators of order n_i and m_j as

$$A_i = \sum_{k=0}^{n_i}\frac{a_k}{(n_i-k)!}(p\partial_p)^{n_i-k}, \quad B_j = \sum_{k=0}^{m_j}\frac{b_k}{(m_j-k)!}(q\partial_q)^{m_j-k}, \tag{13.2}$$

and a_k, b_k are complex constants. It is easy to see that these $\varphi_i^{(n)}$ and $\psi_j^{(n)}$ satisfy the linear partial differential equations in (5.2) for $k = -1, 1$ and 2. Thus, the above τ_n function satisfies the extended KP hierarchy equations (3.4) and (3.6), i.e.,

$$\begin{aligned} &(D_{x_1}^2 + D_{x_2})\ \tau_n\cdot\tau_{n+1} = 0, \\ &D_{x_1}D_{x_{-1}}\ \tau_n\cdot\tau_n = 2(\tau_n\tau_n - \tau_{n+1}\tau_{n-1}). \end{aligned} \tag{13.3}$$

The elements $m_{ij}^{(n)}$ in the above τ_n function can be written out more explicitly as

$$m_{ij}^{(n)} = A_iB_j\left[\frac{1}{p+q}\left(-\frac{p}{q}\right)^n e^{\xi+\eta}\right]. \tag{13.4}$$

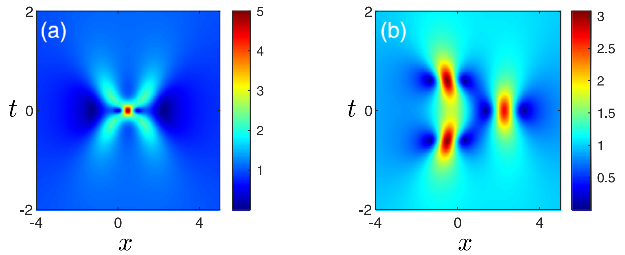

Fig. 12. Two NLS rogue waves $|u|$ in Eq. (13.11) with parameters (13.12) and (a) $a_3 = -1/12$; (b) $a_3 = 5/3$.

Here, we do not introduce the c_{ij} constant into $m_{ij}^{(n)}$ for the same reason as in Eq. (11.3) for DS rogue waves.

To reduce the three-dimensional bilinear equations (13.3) to the two-dimensional NLS bilinear equations (2.15) with $\epsilon = 1$, we also need to perform a dimension reduction. In this case, we impose the condition

$$\left(\partial_{x_1} + \partial_{x_{-1}}\right) \tau_n = C\tau_n, \tag{13.5}$$

where C is a constant. Under this condition, we would have

$$D_{x_1} D_{x_{-1}}\ \tau_n \cdot \tau_n = -D_{x_1}^2\ \tau_n \cdot \tau_n. \tag{13.6}$$

Thus, Eqs. (13.3) would reduce to the NLS bilinear equations (2.15) when we set $x_1 = x$, $x_2 = \mathrm{i}t$, $\epsilon = 1$ (focusing NLS), and

$$f = \tau_0, \quad g = \tau_1, \quad g^* = \tau_{-1}. \tag{13.7}$$

To satisfy the dimension reduction condition (13.5), we set

$$n_i = 2i - 1, m_j = 2j - 1, p = q = 1 \tag{13.8}$$

in the τ_n function with elements (13.4). Under these parameter choices, we can show after some algebra that[4]

$$\left(\partial_{x_1} + \partial_{x_{-1}}\right) \tau_n = 4N\tau_n. \tag{13.9}$$

Hence, the dimension-reduction condition (13.5) is met. After that, x_{-1} becomes irrelevant and can be set to zero.

Due to Eq. (13.7) and the requirement of f being real in the bilinear transformation (2.14), we also need to satisfy the complex conjugation condition

$$\tau_0^* = \tau_0, \quad \tau_{-1} = \tau_1^*. \tag{13.10}$$

These conjugation conditions can be met by constraining parameters (a_k, b_k) as $b_k = a_k^*$ in differential operators A_i and B_j.

To summarize, the focusing NLS equation (2.13) (with $\epsilon = 1$) admits the following rational solutions[4]

$$\begin{aligned} u &= \frac{\tau_1}{\tau_0} e^{2\mathrm{i}t}, \\ \tau_n &= \det_{1\le i,j\le N}\left(m_{ij}^{(n)}\right), \\ m_{ij}^{(n)} &= A_i B_j \left[\frac{1}{p+q}\left(-\frac{p}{q}\right)^n e^{(p+q)x+(p^2-q^2)\mathrm{i}t}\right]_{p=q=1}, \\ A_i &= \sum_{k=0}^{2i-1} \frac{a_k}{(2i-1-k)!}(p\partial_p)^{2i-1-k}, \\ B_j &= \sum_{k=0}^{2j-1} \frac{a_k^*}{(2j-1-k)!}(q\partial_q)^{2j-1-k}, \end{aligned} \tag{13.11}$$

where a_k are free complex constants. These solutions are rogue waves. For example, when we choose

$$N = 2, a_0 = 1, a_1 = a_2 = 0, \tag{13.12}$$

then two 2nd-order rogue waves with $a_3 = -1/12$ and $5/3$ are shown in Fig. 12(a, b) respectively.

The rogue wave solutions in Eq. (13.11) are expressed through differential operators. More explicit expressions through elementary Schur polynomials can be derived by introducing the generator $\mathcal{G}$ of differential operators $(p\partial_p)^k(q\partial_q)^l$ as

$$\mathcal{G} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\kappa^k}{k!}\frac{\lambda^l}{l!}(p\partial_p)^k(q\partial_q)^l,$$

and then using the identity

$$\mathcal{G}F(p,q) = F(e^{\kappa}p, e^{\lambda}q), \tag{13.13}$$

where $F(p,q)$ is an arbitrary function. See Ref. 4 for details.

In rogue waves of Eq. (13.11), free parameters were introduced through summations in differential operators A_i and B_j. A better way of introducing free parameters was presented in Ref. 12, where A_i and B_j contained a single differential term, and free parameters appeared in exponentials. Under this latter parameterization, rogue wave expressions through Schur polynomials are much simpler.

14. Summary

In this article, we have introduced the KP-hierarchy reduction method for solitons. Starting from bilinear forms of integrable equations and matching them to the KP hierarchy or its extension, then using Wronskian/Gram solutions of these hierarchies and performing proper reductions such as dimension reduction and complex-conjugation reduction, we have derived solitons, dark solitons, lumps and rogue waves in the KP, KdV, DS and NLS equations. Graphs of the derived solutions are also illustrated. This KP-hierarchy reduction method is a powerful technique and can be applied to any integrable equation in principle.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This material is based upon work supported by the National Science Foundation (U.S.) under award number DMS-1910282.

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