

On ℓ_p -Gaussian–Grothendieck Problem

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For $p \geq 1$ and $(g_{ij})_{1 \leq i, j \leq n}$ being a matrix of i.i.d. standard Gaussian entries, we study the n -limit of the ℓ_p -Gaussian–Grothendieck problem defined as

$$\max \left\{ \sum_{i,j=1}^n g_{ij} x_i x_j : x \in \mathbb{R}^n, \sum_{i=1}^n |x_i|^p = 1 \right\}.$$

The case $p = 2$ corresponds to the top eigenvalue of the Gaussian orthogonal ensemble; when $p = \infty$, the maximum value is essentially the ground state energy of the Sherrington–Kirkpatrick mean-field spin glass model and its limit can be expressed by the famous Parisi formula. In the present work, we focus on the cases $1 \leq p < 2$ and $2 < p < \infty$. For the former, we compute the limit of the ℓ_p -Gaussian–Grothendieck problem and investigate the structure of the set of all near optimizers along with stability estimates. In the latter case, we show that this problem admits a Parisi-type variational representation and the corresponding optimizer is weakly delocalized in the sense that its entries vanish uniformly in a polynomial order of n^{-1} .

1 Introduction and Main Results

For $1 \leq p \leq \infty$, the ℓ_p -Grothendieck problem (see [37]) seeks to study the following optimization problem

$$\max_{x \in \mathbb{R}^n : \|x\|_p \leq 1} \langle Ax, x \rangle, \tag{1.1}$$

Received January 25, 2021; Revised October 5, 2021; Accepted October 8, 2021
 Communicated by Prof. Assaf Naor

where A is an $n \times n$ input matrix. Here, for $1 \leq p < \infty$, the standard ℓ_p -norm of $x \in \mathbb{R}^n$ is defined as $\|x\|_p := (\sum_{i \in [n]} |x_i|^p)^{1/p}$. Additionally, the ℓ_∞ -norm of x is given by $\|x\|_\infty = \max_{i \in [n]} |x_i|$.

For $p = 2$, the spectral theory tells us that the optimal value of (1.1) is just the maximum eigenvalue of the symmetric matrix $(A + A^T)/2$. For $p = \infty$, under a mild assumption that the diagonal entries of A vanish, the optimal value of (1.1) remains unchanged if the quadratic form is maximized over the discrete hypercube $\{-1, 1\}^n$ instead (see, e.g., [44, Lemma 3.5]). This optimization problem is known to be NP-hard [45]. The optimal value can be approximated within a $O(\log n)$ factor in polynomial time [13, 36, 42, 47]. On the other hand, it is computationally hard to approximate the optimal value within a $O((\log n)^\gamma)$ factor for every $\gamma \in (0, 1/6)$ [38].

The special cases of the ℓ_p -Grothendieck problem appear in clustering algorithms. The cases $p = 2$ and $p = \infty$ are related to the spectral partitioning [27] and the correlation clustering [13], respectively. For $2 < p < \infty$, the ℓ_p -Grothendieck problem can be viewed as an interpolation between these two clustering criteria. In this case, approximating (1.1) within a factor of $\xi_p^2 - \varepsilon$ is NP-hard for any $\varepsilon > 0$, while there exists a polynomial time algorithm that approximates (1.1) within a factor of ξ_p^2 [31, 39], where $\xi_p = (\mathbb{E}|z|^p)^{1/p}$ is the p -th norm of a standard Gaussian random variable z .

The case $p = 1$ is also believed to be computationally hard (see the discussion in [37] and the references therein). Let us mention that if we optimize over a simplex instead, then (1.1) includes, as a special case, finding the cardinality of the maximum independent set of a graph [45], which is a well-known NP-hard problem. The case $1 < p < 2$ seems to remain unexplored in the literature.

While finding an efficient algorithm to compute the optimal value in (1.1) is generally difficult except when $p = 2$, it is natural to study the ℓ_p -Grothendieck problem for random input matrices first. This leads to the following optimization problem that takes a random Gaussian matrix as input, which we will refer to as the ℓ_p -Gaussian-Grothendieck problem or simply the ℓ_p -Grothendieck problem,

$$\max_{x \in \mathbb{R}^n : \|x\|_p \leq 1} \langle G_n x, x \rangle = \max_{x \in \mathbb{R}^n : \|x\|_p = 1} \langle G_n x, x \rangle. \quad (1.2)$$

Here, G_n is an $n \times n$ matrix with entries g_{ij} being i.i.d. standard Gaussians for all $i, j \in [n]$.

Another motivation for investigating the optimization problem (1.2) arises from the study of spin glass models, which, roughly speaking, are disordered spin systems

invented in the ‘70s in order to understand some unusual behaviors of certain alloys, such as CuMn. The quadratic form, $\langle G_n x, x \rangle$ in (1.2), reflecting the pairwise interactions across sites, can be viewed as (the negative of) the Hamiltonian or energy of the Sherrington–Kirkpatrick (SK) mean-field spin glass model on the unit ℓ_p -ball. In the terminology of statistical physics, the maximum value of $\langle G_n x, x \rangle$ over all possible spin configurations $x \in \mathbb{R}^n$ with $\|x\|_p = 1$ is called the ground state energy of the model. When $p = \infty$, this quantity is essentially the ground state energy of the classical SK model [61] with Ising spins. In the case $p = 2$, the spin configurations lie on the standard ℓ_2 -sphere and this model is called the spherical SK model. For physicists’ treatments of mean-field spin glass models, we refer the readers to check [43]. See also the books [51, 64, 65] for the mathematical progress in the past decade.

In the present work, we aim to study the ℓ_p -Gaussian–Grothendieck problems in two major directions: (1) their limits as n tends to infinity and (2) the structure of the set of $(1 - \varepsilon)$ -optimizers, $\mathcal{M}_\varepsilon = \mathcal{M}_\varepsilon(p, n)$, defined as the collection of all $x \in \mathbb{R}^n$ satisfying $\|x\|_p = 1$ and

$$\langle G_n x, x \rangle \geq (1 - \varepsilon) \max_{Y \in \mathbb{R}^n: \|Y\|_p = 1} \langle G_n Y, Y \rangle.$$

We divide our main results into two parts, $1 \leq p < 2$ and $p > 2$, as they exhibit significantly different behaviors. They are presented in the following two subsections. Before we turn to their statements, we introduce some notations that will be used throughout this paper.

General Notation. For $1 \leq p \leq \infty$, let $1 \leq p^* \leq \infty$ be the Hölder conjugate of p , that is, $1/p + 1/p^* = 1$. For $x \in \mathbb{R}^n$, we define the normalized ℓ_p -norm of x as

$$\|x\|_p = \left(\frac{1}{n} \sum_{i \in [n]} |x_i|^p \right)^{1/p} \text{ for } 1 \leq p < \infty \quad \text{and} \quad \|x\|_\infty = \max_{i \in [n]} |x_i|.$$

In addition, we use $\|x\|_0$ to denote the number of nonzero coordinates in x . Let $\bar{G}_n = (G_n + G_n^T)/\sqrt{2}$ be the symmetrized Gaussian matrix, distributed as the Gaussian orthogonal ensemble (GOE). The entries of $\bar{G}_n$ are denoted by $\bar{g}_{ij}$. For $1 \leq p < \infty$, we denote by $\xi_p = (\mathbb{E}|z|^p)^{1/p}$, the p -th norm of a standard Gaussian random variable z .

Let $M_n(\mathbb{R})$ be the space of real-valued matrices of size $n \times n$. For any $A = (a_{ij})_{i,j \in [n]} \in M_n(\mathbb{R})$, denote its p -to- q operator norm of A by $\|A\|_{p \rightarrow q} = \sup_{\|x\|_p = 1} \|Ax\|_q$ for $1 \leq p, q \leq \infty$. In the case $p = q = 2$, we simply denote $\|A\|_{2 \rightarrow 2}$ by $\|A\|_2$. The Frobenius norm of A is defined as $\|A\|_F = \sum_{i,j \in [n]} |a_{ij}|^2$. We use $e_1, \dots, e_n$ to denote the standard basis of $\mathbb{R}^n$.

1.1 Main result: $1 \leq p < 2$

We split our results into two cases, $p = 1$ and $1 < p < 2$, according to their different types of scalings, $\sqrt{\log n}$ and n^{1/p^*} . We start with the easy case of $p = 1$.

Theorem 1.1. Let $p = 1$. The following statements hold:

(i) almost surely,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\log n}} \max_{\|x\|_1=1} \langle G_n x, x \rangle = \sqrt{2}; \quad (1.3)$$

(ii) let $I = \operatorname{argmax}_{i \in [n]} g_{ii}$. For any $\eta > 0$ and $n \geq 1$,

$$\mathbb{P}(\langle G_n e_I, e_I \rangle \geq \sqrt{2(1-\eta) \log n}) \geq 1 - e^{-n^\eta}; \quad (1.4)$$

(iii) there exist constants $c_0, c, C > 0$ such that for any $0 < \delta \leq c_0$ and for all $n \geq 1$,

$$\mathbb{P}\left(\max_{\|x\|_1=1, \|x\|_\infty \leq 1-\delta} \langle G_n x, x \rangle \leq \left(1 - c\delta + \sqrt{\frac{\log 2}{\log n}}\right) \sqrt{2 \log n}\right) \geq 1 - Cn^{-c\delta}. \quad (1.5)$$

Items (i) and (ii) together imply that e_I is a near optimizer of the ℓ_1 -Grothendieck problem. Furthermore, from (iii), if we take ε satisfying $0 < \varepsilon \leq 1/4$ and $\lim_{n \rightarrow \infty} \varepsilon \sqrt{\log n} = \infty$, then for all large n , with high probability, the set $\mathcal{M}_\varepsilon$ is a subset of the disjoint union of ℓ_1 -balls of radius $\delta = O(\varepsilon)$ centered around the $2n$ coordinate vectors $\pm e_1, \pm e_2, \dots, \pm e_n$. Consequently, each near optimizer is localized with a single entry carrying at least $1 - \delta$ of the weight.

Next, we consider the case $1 < p < 2$, which is more interesting. Define $v_i \in \mathbb{R}^n$ as

$$v_i(j) = \operatorname{sgn}(\bar{g}_{ij}) \left(\frac{|\bar{g}_{ij}|^{p^*}}{\|\bar{G}_n e_i\|_{p^*}^{p^*}} \right)^{1/p}, \quad \forall j \in [n], \quad (1.6)$$

where $\operatorname{sgn}(a)$ is 1 if $a \geq 0$ and -1 if $a < 0$. Note that $\|v_i\|_p = 1 = \|e_i\|_p$. Let

$$\mathcal{O} = \left\{ \pm \frac{e_i + v_i}{\|e_i + v_i\|_p} : 1 \leq i \leq n \right\}. \quad (1.7)$$

Note that ignoring the signs of the vectors in $\mathcal{O}$, it can be seen that they are asymptotically orthogonal to each other. The following theorems establish the limit of the ℓ_p -Grothendieck problem and show that any near optimizer lies close to the set $\mathcal{O}$.

Theorem 1.2. Let $1 < p < 2$.

(i) Almost surely,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{1/p^*}} \max_{\|x\|_p=1} \langle G_n x, x \rangle = 2^{\frac{1}{2}-\frac{2}{p}} \xi_{p^*}. \quad (1.8)$$

(ii) Each $x \in \mathcal{O}$ is a near optimizer in the sense that for any $d > 0$, there exists a constant $C > 0$ such that with probability at least $1 - Cn^{-d}$,

$$\min_{x \in \mathcal{O}} \langle G_n x, x \rangle \geq 2^{\frac{1}{2}-\frac{2}{p}} \xi_{p^*} n^{\frac{1}{p^*}} - C(n^{-\frac{1}{p}+\frac{1}{2}+\frac{1}{p^*}} (\log n)^{\frac{p^*}{4}} + \sqrt{\log n}). \quad (1.9)$$

Remark 1.3. The order of the error term on the right-hand side of (1.9) depends on the value of p . When $1 < p < 4/3$, it is of order $\sqrt{\log n}$ and when $4/3 \leq p < 2$, it is of order $n^{-1/p+1/2+1/p^*} (\log n)^{p^*/4}$.

Theorem 1.4 (Stability). $\mathcal{O}$ is essentially the set of all optimizers. More precisely, let $0 < \kappa < 1/3p$ and $d > 0$. There exist positive constants C_1, C_2, C_3 such that for any $n \geq 1$, if $n^{-\kappa} \leq \delta \leq 1$, the event

$$\max_{\text{dist}(x, \mathcal{O}) \geq \delta, \|x\|_p=1} \langle G_n x, x \rangle \leq 2^{\frac{1}{2}-\frac{2}{p}} (1 - C_1 \delta^6) \xi_{p^*} n^{1/p^*} + C_2 n^{p/(2p^*)} \sqrt{\log n} \quad (1.10)$$

occurs with probability at least $1 - C_3 n^{-d}$, where $\text{dist}(x, \mathcal{O}) = \min_{y \in \mathcal{O}} \|x - y\|_p$.

As a consequence of the above two theorems, it can be deduced from (1.10) that if

$$0 < \varepsilon \leq \frac{1}{4} \text{ and } \lim_{n \rightarrow \infty} \frac{\varepsilon}{n^{-\frac{1}{p^*}(1-\frac{p}{2})} \sqrt{\log n}} = \infty,$$

then with high probability, $\mathcal{M}_\varepsilon$ is a subset of the disjoint union of ℓ_p -balls of radius $\delta = O(\varepsilon^{1/6})$ centered around the $2n$ vectors in $\mathcal{O}$. The dependence of δ on ε here is sub-optimal. Note that from the law of large numbers,

$$\frac{e_i + v_i}{\|e_i + v_i\|_p} \approx 2^{-1/p} (e_i + w_i),$$

where

$$w_i(j) = n^{-1/p} \xi_{p^*}^{-p^*/p} \text{sgn}(\bar{g}_{ij}) |\bar{g}_{ij}|^{p^*/p}.$$

Therefore, each near optimizer has a single spike that has magnitude of order 1 while the rest of entries are $O(\delta)$. Hence, in the regime $1 < p < 2$, the near optimizers possess both localized and delocalized components.

1.2 Main result: $2 \leq p \leq \infty$

Consider the normalized Grothendieck problem

$$\text{GP}_{n,p} := \max_{x \in \mathbb{R}^n: \|x\|_p=1} \frac{\langle G_n x, x \rangle}{n^{3/2-2/p}}. \quad (1.11)$$

As we have mentioned before, $p = 2$ is the easy case, since $\text{GP}_{n,2} = 2^{-1/2} n^{-1/2} \max_{\|x\|_2=1} \langle \tilde{G}_n x, x \rangle$ and the optimal value is the largest eigenvalue of the GOE, which converges to 2 as $n \rightarrow \infty$. The case $p = \infty$ corresponds to the ground state energy of the SK on the product space $[-1, 1]^n$ and it is known that $\lim_{n \rightarrow \infty} \text{GP}_{n,p}$ exists and can be expressed as the famous Parisi formula; see [19].

Our main result focuses on the interpolating case $2 < p < \infty$. First, we study the convergence of $\text{GP}_{n,p}$. Note that the optimization problem can be rewritten using the normalized norm as

$$\text{GP}_{n,p} = \max_{x \in \mathbb{R}^n: \|x\|_p=1} \frac{\langle G_n x, x \rangle}{n^{3/2}}.$$

In order to study the above optimization problem, it is natural to remove the ℓ_p -constraint over the maximization set and compensate it by adding a ℓ_p -norm potential, namely, for $t > 0$, we define

$$L_{n,p}(t) = \max_{x \in \mathbb{R}^n} \left(\frac{\langle G_n x, x \rangle}{n^{3/2}} - t \|x\|_p^p \right). \quad (1.12)$$

We also introduce the following restricted version of the above Hamiltonian in (1.12) where the self-overlap $\|x\|^2$ is constrained to be a fixed value,

$$L_{n,p,u}(t) = \max_{x \in \mathbb{R}^n: \|x\|_2^2=u} \left(\frac{\langle G_n x, x \rangle}{n^{3/2}} - t \|x\|_p^p \right), \quad u > 0. \quad (1.13)$$

The following theorem establishes the existence of the deterministic limits of $L_{n,p,u}(t)$ and $L_{n,p}(t)$.

Theorem 1.5. Let $2 < p < \infty$. Almost surely, the following limits exist,

$$L_{p,u}(t) := \lim_{n \rightarrow \infty} L_{n,p,u}(t) = \lim_{n \rightarrow \infty} \mathbb{E} L_{n,p,u}(t), \quad \text{for all } u, t > 0, \quad (1.14)$$

$$L_p(t) := \lim_{n \rightarrow \infty} L_{n,p}(t) = \sup_{u>0} L_{p,u}(t), \quad \text{for all } t > 0. \quad (1.15)$$

Our next result shows the existence of the limit of $\text{GP}_{n,p}$ and its connection to $L_p(t)$.

Theorem 1.6. Let $2 < p < \infty$. Almost surely,

$$\mathrm{GP}_p := \lim_{n \rightarrow \infty} \mathrm{GP}_{n,p} = \frac{p}{2} \left(\frac{p}{2} - 1 \right)^{2/p-1} t^{2/p} L_p(t)^{1-2/p}, \quad \text{for all } t > 0. \quad (1.16)$$

In addition, for $\delta = p/2 - 1$, there exist $C, C' > 0$ such that with probability at least $1 - C'e^{-n/C'}$, if $x_* \in \mathbb{R}^n$ with $\|x_*\|_p = 1$ is an optimizer of (1.11),

$$\|x_*\|_\infty \leq Cn^{\frac{1}{p+\delta} - \frac{1}{p}}. \quad (1.17)$$

Let us point out that the right-hand side of (1.16) is independent of $t > 0$. The inequality (1.17) implies that any optimizer of the ℓ_p -Grothendieck problem is weakly delocalized for $2 < p < \infty$ and its ℓ_p -mass is not concentrated on any set of coordinates with vanishing proportion. This resembles, at least qualitatively, the delocalization behavior in the ℓ_2 -Grothendieck problem, where the supremum norm of the optimizer (which is same as the largest eigenvector of the GOE) is $O((\log n)^{1/2} n^{-1/2})$ with high probability (see, e.g., [48, Section 2]).

Our next result establishes the continuity of the limiting value GP_p in $p \in [2, \infty]$.

Theorem 1.7. GP_p is continuous in $p \in [2, \infty]$.

Remark 1.8. From Theorems 1.2 and 1.6, the ℓ_p -Grothendieck problem scales as n^{1/p^*} for $1 < p < 2$ and as $n^{3/2-2/p}$ for $2 \leq p \leq \infty$. These scaling exponents match at $p = 2$. However, while the limiting value of the ℓ_p -Grothendieck problem is continuous in p in the intervals $(1, 2)$ (as seen from (1.8)) and $[2, \infty]$, it has a jump discontinuity at $p = 2$. Indeed,

$$\lim_{p \rightarrow 2^-} \lim_{n \rightarrow \infty} \frac{1}{n^{1/p^*}} \max_{\|x\|_p=1} \langle G_n x, x \rangle = \lim_{p \rightarrow 2^-} 2^{1/2-2/p} \xi_{p^*} = \frac{1}{\sqrt{2}} \neq \sqrt{2} = \mathrm{GP}_2.$$

Our final result provides a description of the limiting value GP_p by showing that $L_{p,u}(t)$ and $L_p(t)$ admit Parisi-type variational formulas. In physicists' literature, the original Parisi formula was proposed by Parisi [57–59], which gives a variational representation for the limiting free energy of the classical SK model [61] at a positive temperature. It minimizes a functional, which involves the solution to a 2nd-order parabolic PDE (known as the Parisi PDE) and a linear term, with respect to the so-called functional order parameters. This formula was first rigorously verified by Talagrand [63] and later generalized to some variants of the SK model; see [2, 15, 32, 40, 41, 50,

52–55]. Furthermore, the Parisi-type formulas were also established for the ground state energies, corresponding to zero temperature, in a number of related models; see [6, 19, 20, 32, 33].

To state our result, let $u > 0$ and $\mathcal{N}_u^d$ be the collection of all positive measures γ on $[0, u]$ such that γ is atomic with finitely many atoms. For $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$ and $t > 0$, set

$$f_\lambda^t(x) = \sup_{r \in \mathbb{R}} (rx + \lambda r^2 - t|r|^p), \quad x \in \mathbb{R},$$

and denote, with a slight abuse of notation, $\gamma(s) = \gamma([0, s])$. Let $\Psi_{\lambda, \gamma}^{t, u}$ be the solution to the following Parisi PDE:

$$\partial_s \Psi_{\lambda, \gamma}^{t, u}(s, x) = -(\partial_{xx} \Psi_{\lambda, \gamma}^{t, u}(s, x) + \gamma(s)(\partial_x \Psi_{\lambda, \gamma}^{t, u}(s, x))^2), \quad (s, x) \in [0, u] \times \mathbb{R} \quad (1.18)$$

with boundary condition $\Psi_{\lambda, \gamma}^{t, u}(u, x) = f_\lambda^t(x)$.

The fact that γ is atomic with finitely many atoms enables us to solve this Parisi PDE explicitly by using the Hopf–Cole transformation in an iterative fashion. Toward this end, let us express any $\gamma \in \mathcal{N}_u^d$ as

$$\gamma(s) = \sum_{l=0}^{k-1} m_l 1_{[q_l, q_{l+1})}(s) + m_k 1_{[q_k, u]}(s) \quad (1.19)$$

for some

$$q_0 = 0 < q_1 < \cdots < q_k < q_{k+1} = u, \quad 0 \leq m_0 < m_1 < \cdots < m_{k-1} < m_k < \infty.$$

Then the solution to (1.18) can be written iteratively as

$$\Psi_{\lambda, \gamma}^{t, u}(s, x) = \frac{1}{m_l} \log \mathbb{E} \exp m_l \Psi_{\lambda, \gamma}^{t, u}(q_{l+1}, x + \sqrt{2(q_{l+1} - s)z}), \quad \forall (s, x) \in [q_l, q_{l+1}) \times \mathbb{R}, \quad (1.20)$$

for $l = k, k-1, \dots, 0$, where z is standard Gaussian. Here, when $m_0 = 0$, (1.20) should be understood as

$$\Psi_{\lambda, \gamma}^{t, u}(s, x) = \mathbb{E} \Psi_{\lambda, \gamma}^{t, u}(q_1, x + \sqrt{2(q_1 - s)z}), \quad (s, x) \in [0, q_1) \times \mathbb{R}.$$

Note that (1.20) is well defined since there exist some constants $C > 0$ and $a \in (0, 1)$ such that $f_\lambda^t(x) \leq C(1 + |x|^{1+a})$ for all $x \in \mathbb{R}$ and this property is preserved for $\Psi_{\lambda, \gamma}^{t, u}$ by

induction, see Lemma B.2, which also describes the regularity properties of $\Psi_{\lambda,\gamma}^{t,u}$. For any $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$, set

$$\mathcal{P}_{t,u}(\lambda, \gamma) = \Psi_{\lambda,\gamma}^{t,u}(0, 0) - \lambda u - \int_0^u s\gamma(s) \, ds. \quad (1.21)$$

For notational clarity, we ignore the dependence of $\mathcal{P}_{t,u}$ on p . Our result states the following.

Theorem 1.9. Let $2 < p < \infty$. For any $u > 0$ and $t > 0$, we have that

$$L_{p,u}(t) = \inf_{\lambda \in \mathbb{R}, \gamma \in \mathcal{N}_u^d} \mathcal{P}_{t,u}(\lambda, \gamma), \quad (1.22)$$

$$L_p(t) = \sup_{u > 0} \inf_{\lambda \in \mathbb{R}, \gamma \in \mathcal{N}_u^d} \mathcal{P}_{t,u}(\lambda, \gamma). \quad (1.23)$$

Currently, we know very little about the set $\mathcal{M}_\varepsilon$ for $2 < p < \infty$. In contrast to the $1 \leq p < 2$ case, we expect that the landscape of the ℓ_p -Grothendieck problem has a more complex geometric structure, which does not admit a simple description. In fact, as mentioned before, the ℓ_∞ -Grothendieck problem is essentially the ground state energy of the classical SK model and for a fixed but small $\varepsilon > 0$, [18] established, with overwhelming probability, the existence of exponentially many vectors in $\mathcal{M}_\varepsilon$ that are nearly orthogonal to each other; see also [14, 23, 25]. This fact is also true for $p = 2$, which is an easy consequence of the fact that the eigenmatrix for the GOE is Haar-distributed. Therefore, it is natural to believe that in the interpolating case $2 < p < \infty$, the same phenomenon continues to hold. See Section 13 for further discussion.

2 Overview of the Paper

Our approach is based on two major ingredients: (1) the Chevet inequality for bounding the supremum of a Gaussian bilinear form and (2) some tools arising from the study of mean-field spin glasses. To facilitate our arguments, we provide some sketches of the proofs to our main results in two cases, $1 < p < 2$ and $2 < p < \infty$, followed by a description on the organization of the rest of the paper.

2.1 Proof sketch for $1 < p < 2$

We begin by sketching the proof of the upper bound in Theorem 1.2. The key idea is to decompose every ℓ_p -unit vector $x \in \mathbb{R}^n$ as $x = x^o + x^\dagger$, where for a given truncation level

$\varepsilon > 0$, x^o is the delocalized part, $x_i^o = x_i 1(|x_i| \leq \varepsilon)$ for $i \in [n]$, and $x^\dagger$ is the localized part, $x_i^\dagger = x_i 1(|x_i| > \varepsilon)$ for $i \in [n]$. In the proof, we take $\varepsilon = \varepsilon_n \rightarrow 0$ at an appropriate rate. From this decomposition, an application of the bilinearity of the inner product and the triangle inequality readily yields that

$$\left| \max_{\|x\|_p=1} \langle G_n x, x \rangle - \sqrt{2} \max_{\|x\|_p=1} \langle \tilde{G}_n x^o, x^\dagger \rangle \right| \leq \max_{\|x\|_p=1} |\langle G_n x^o, x^o \rangle| + \max_{\|x\|_p=1} |\langle G_n x^\dagger, x^\dagger \rangle|. \quad (2.1)$$

To control the diagonal terms on the right-hand side, we apply the Chevet inequality, which states that for arbitrary bounded sets $S, T \subset \mathbb{R}^n$, $\mathbb{E} \sup_{x \in T, y \in S} \langle y, G_n x \rangle$ can be controlled by the Gaussian width (a measure of “complexity” of a set) and the ℓ_2 -radii of the index sets S and T . See Theorem 3.1 below for the precise statement. It turns out that under the assumption $p < 2$, the set of delocalized vectors in the unit ℓ_p -ball has a vanishing ℓ_2 -radius, which makes the 1st term on the right side of (2.1) smaller order than n^{1/p^*} . On the other hand, each localized vector in the unit ℓ_p -ball has only a few (at most ε^{-p}) nonzero entries, which guarantees that the set $\{x^\dagger : \|x\|_p = 1\}$ has a low Gaussian width. As a result, the 2nd term on the right side of (2.1) is of small order as well. With these observations, we can approximate

$$n^{-1/p^*} \max_{\|x\|_p=1} \langle G_n x, x \rangle \approx n^{-1/p^*} \sqrt{2} \max_{\|x\|_p=1} \langle \tilde{G}_n x^o, x^\dagger \rangle.$$

We now normalize x^o and $x^\dagger$ by their respective ℓ_p -norms and use the duality relation $\sup_{\|y\|_p=1} \langle y, u \rangle = \|u\|_{p^*}$ to obtain the following upper bound:

$$\mathbb{E} \max_{\|x\|_p=1} \langle x^o, \tilde{G}_n x^\dagger \rangle \leq \mathbb{E} \max_{\|x\|_p=1} \left(\|x^\dagger\|_p \|x^o\|_p \left\| \tilde{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\|_{p^*} \right) \leq 2^{-2/p} \mathbb{E} \max_{\|x\|_p=1} \left\| \tilde{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\|_{p^*}, \quad (2.2)$$

where in the last inequality, we used the fact that the product $\|x^\dagger\|_p \|x^o\|_p$ is maximized if $\|x^\dagger\|_p = \|x^o\|_p = 2^{-1/p}$ as $\|x^o\|_p^p + \|x^\dagger\|_p^p = \|x\|_p^p = 1$. Since each $x^\dagger$ is supported on at most ε^{-p} many coordinates, the last expectation in (2.2) can be bounded above by $\mathbb{E} \max_{y \in F} \|\tilde{G}_n y\|_{p^*}$, where F is defined as $F = \{y \in \mathbb{R}^n : \|y\|_p = 1, \|y\|_0 \leq \varepsilon^{-p}\}$. We can exploit the low cardinality of the set F to argue by using concentration and union bound that

$$\mathbb{E} \max_{y \in F} \|\tilde{G}_n y\|_{p^*} \leq (1 + o(1)) \max_{y \in F} \mathbb{E} \|\tilde{G}_n y\|_{p^*}.$$

For any vector y , we have $n^{-1/p^*} \mathbb{E} \|\tilde{G}_n y\|_{p^*} \approx n^{-1/p^*} \|y\|_2 \mathbb{E} \|g\|_{p^*} \approx \|y\|_2 \xi_{p^*}$ for g a standard Gaussian vector, which implies that

$$n^{-1/p^*} \mathbb{E} \max_{y \in F} \|\tilde{G}_n y\|_{p^*} \leq (1 + o(1)) \xi_{p^*} \max_{y \in F} \|y\|_2 = (1 + o(1)) \xi_{p^*}, \quad (2.3)$$

yielding the upper bound (for the expectation) in Theorem 1.2. The above argument also indicates how we can achieve a matching lower bound. A near maximizer x should satisfy $\|x^\dagger\|_p = \|x^o\|_p = 2^{-1/p}$. Moreover, in (2.3), to attain the maximum ℓ_2 -norm, the optimal $y = x^\dagger / \|x^\dagger\|_p \in F$ should be one of the coordinate vectors e_i (up to a sign). Working backwards, the optimal choice of x^o is now dictated by the relation $\langle x^o / \|x^o\|_p, \tilde{G}_n e_i \rangle = \|\tilde{G}_n e_i\|_{p^*}$, which is achieved by taking $x^o / \|x^o\|_p = v_i$, as given in (1.6). Finally, we note that these choices for $x^\dagger$ and x^o are valid since e_i is localized and v_i is delocalized for any i . It leads to the set of near maximizers $\mathcal{O}$, as defined in (1.7).

The stability bound in Theorem 1.4 follows a similar path. However, it is more technically involved as we need to keep track of error incurred in every step of the above argument. Also, we use an appropriate stability bound (in ℓ_p distance) for Hölder's inequality (stated in Lemma A.1).

2.2 Proof sketch for $p > 2$

We continue to sketch the proofs of Theorems 1.5, 1.6, and 1.9. First of all, consider the problem of maximizing a collection of random variables, $(H_n(x))_{x \in S}$, indexed by a measurable subset $S \subset \mathbb{R}^n$. In physics, $H_n(x)$ is called the Hamiltonian or the energy corresponding to the spin configuration $x \in S$ and the quantity $\max_{x \in S} H_n(x)$ is called the ground state energy. To compute this maximum, it is often more convenient to consider the free energy associated with $(H_n(x))_{x \in S}$,

$$F_n(\beta) = \frac{1}{\beta n} \log \int_S e^{\beta H_n(x)} \mu_n(dx)$$

for an appropriately chosen probability measure μ_n supported on S , where $\beta > 0$ is usually called the (inverse) temperature. The key fact here is that in many important models, for example, the classical SK model and its variants, the free energies and the corresponding Gibbs measures exhibit several nice physical and mathematical properties, which allow one to establish the existences of the limiting free energies as $n \rightarrow \infty$ and furthermore derive certain types of variational formulas for these limits; see [51, 64, 65]. Once the limiting free energy is obtained, the limit of the maximum of

H_n can then be recovered by switching the β and n limits as

$$\lim_{\beta \rightarrow \infty} \lim_{n \rightarrow \infty} F_n(\beta) = \lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} F_n(\beta) = \lim_{n \rightarrow \infty} \max_{x \in S} \frac{H_n(x)}{n}.$$

Following this idea, we establish Theorem 1.5 by introducing a free energy (see (7.3)) associated with $L_{n,p,u}(t)$ and adapt the Guerra–Toninelli [30] interpolation to show that this free energy is superadditive in n , which implies the existence of the limiting free energy as $n \rightarrow \infty$. After this, we send the temperature parameter to infinity to obtain the existence of the limit of $L_{n,p,u}(t)$ in (1.14). The convergence of $L_{n,p}(t) = \sup_{u>0} L_{n,p,u}(t)$, claimed in (1.15), now follows by switching the supremum and the n -limit, which can be justified by a standard compactness argument.

The proof of Theorem 1.6 is based on the observation that the ℓ_p -Grothendieck problem and $L_{n,p}(t)$ are the Legendre duals of each other (see the proof of Lemma 8.1), that is, at the point of differentiability of $L_{n,p}(t)$,

$$\max_{\|x\|_p^p = -L'_{n,p}(t)} \frac{\langle G_n x, x \rangle}{n^{3/2}} = L_{n,p}(t) - tL'_{n,p}(t).$$

From this, if additionally we have $L'_{n,p}(t) < 0$, the ℓ_p -Grothendieck problem can be written, by a change of variable, as

$$\text{GP}_{n,p} = \frac{1}{(-L'_{n,p}(t))^{2/p}} \left(\max_{\|x\|_p^p = -L'_{n,p}(t)} \frac{\langle G_n x, x \rangle}{n^{3/2}} \right) = \frac{L_{n,p}(t) - tL'_{n,p}(t)}{(-L'_{n,p}(t))^{2/p}}. \quad (2.4)$$

To further simplify this equation, another key observation is that at any point t of differentiability of $L_{n,p}$,

$$L_{n,p}(t) = -(p/2 - 1)tL'_{n,p}(t). \quad (2.5)$$

The above identity can be verified via a straightforward computation. Consequently, if $L_{n,p}$ is differentiable at t and $L'_{n,p}(t) < 0$, plugging (2.5) into (2.4) leads to

$$\text{GP}_{n,p} = \frac{p}{2} \left(\frac{p}{2} - 1 \right)^{2/p-1} t^{2/p} L_{n,p}(t)^{1-2/p}. \quad (2.6)$$

To send $n \rightarrow \infty$, it is crucial to realize that $L_{n,p}$ and L_p are convex on $(0, \infty)$, which readily imply that, with probability one, for almost everywhere $t \in (0, \infty)$, $L'_p(t) = \lim_{n \rightarrow \infty} L'_{n,p}(t)$. In addition, since $L_p(t)$ is strictly positive in t , Equation (2.5) ensures

that whenever $L_{n,p}$ is differentiable at some t , we have $L'_{n,p}(t) < 0$ as long as n is large enough. These allow us to send $n \rightarrow \infty$ in (2.6) to obtain (1.16) almost everywhere for $t \in (0, \infty)$. Using the fact that $L_p(t)$ is continuous in t validates (1.16) on the whole space $(0, \infty)$. The proof for the delocalization of the optimizer in the ℓ_p -Grothendieck problem in (1.17) is established by using the idea of Lagrange multiplier corresponding to the maximizer x_* , which upon combining with the Hölder inequality, leads to the bound

$$\|x_*\|_{p+\delta}^{p+\delta} \leq \frac{n^{1-(p+\delta)/p}}{\sqrt{2}\text{GP}_{n,p}} \frac{\|\bar{G}_n\|_2}{n^{1/2}} \quad \text{for } \delta = p/2 - 1.$$

The assertion (1.17) then follows from the trivial bound $\|x_*\|_\infty \leq \|x_*\|_{p+\delta}$.

For Theorem 1.9, to approximate $L_{n,p,u,\varepsilon}(t)$, we consider the Hamiltonian $H_{n,p,t}(x) = n^{-1/2} \langle G_n x, x \rangle - t \|x\|_p^p$ and introduce two related free energies, $F_{n,t,u,\varepsilon}(\beta)$ and $F_{n,t,u,M,\varepsilon}(\beta)$. The former integrates $H_{n,p,t}$ over a spherical constraint $\|x\|^2 \in [u - \varepsilon, u + \varepsilon]$ and the latter puts an additional constraint $x \in [-M, M]^n$ —see the beginning of Section 10. The free energy $F_{n,t,u,\varepsilon}(\beta)$ can be bounded from above by the so-called Guerra’s replica symmetry breaking bound [29], which in turn implies that $L_{p,u}(t)$ is bounded above by the desired Parisi-type formula (the right-hand side of (1.22)) after sending the temperature parameter to infinity; see Section 12.2.

Deriving a matching lower bound is more difficult. First of all, it is easy to see that $L_{n,p,u,\varepsilon}(t) \geq F_{n,t,u,M,\varepsilon}(\beta)$, where the right-hand side, after a change of variable $x \rightarrow x/M$, is the free energy of the SK model on the product space $[-1, 1]^n$ and its limit can be expressed as a Parisi-type formula [50], similar to (1.22), as stated in (11.6). In order to obtain the matching lower bound, we need to study the β -limit and then the M -limit in this formula. In view of the Parisi PDE term in (11.6), it is a Hamilton–Jacobi–Bellman equation induced by a linear diffusion control problem and is known to admit a stochastic optimal control representation [3, 11, 34], which is described in Proposition 11.7 below. Using this expression, we can successfully handle the β -limit by adapting and refining the approaches in [5, 33]. In contrast, the argument for the M -limit is more intricate mainly due to the fact that the boundary condition of the Parisi PDE is not Lipschitz and as a consequence, the control of the M -limit in the Parisi PDE solution cannot be done directly from the stochastic optimal control representation as the control processes therein lack uniform pointwise controls; see Remarks 11.2 and 12.2. Fortunately, this technical issue can be resolved by showing that the control processes are indeed uniformly square integrable relying on the assumption $p > 2$.

2.3 Organization

The rest of paper consists of two major parts. The 1st is devoted to proving the results for $1 \leq p < 2$. In Section 3, we introduce the Chevet inequality and explain how one can obtain the correct scalings for our ℓ_p -Grothendieck problem from this inequality. Additionally, we list some standard results on the Gaussian concentration and maximal inequalities. These contents will be repeatedly used in the proofs of Theorems 1.1 and 1.2 in Sections 4 and 5, respectively.

The 2nd part establishes our main results for $2 < p < \infty$ based the methodologies generally used in statistical physics and in mean-field spin glasses. The proofs of Theorems 1.5, 1.6, and 1.7 are presented in Sections 7, 8, and 9, respectively. The more intricate part is the proof of Theorem 1.9 that will be handled in Section 12. To this end, Sections 10 and 11 prepare some fundamental results. More precisely, Section 10 derives upper and lower inequalities for $L_{p,u}(t)$ in terms of the free energies $F_{n,t,u,\varepsilon}(\beta)$ and $F_{n,t,u,M,\varepsilon}(\beta)$. In Section 11, we show that the β -limit of the Parisi formula associated to $\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} F_{n,t,u,M,\varepsilon}(\beta)$ can also be expressed as a Parisi-type formula followed by some a priori bounds for the corresponding minimizers in terms of the parameter M .

Some open questions of interest are mentioned in Section 13. Finally, Appendix A provides a new result on the stability of the Hölder inequality that is used in the proof of Theorem 1.4. In addition, analytical properties of the Parisi PDE (1.18) and its stochastic optimal control representation are studied in Appendix B.

3 Gaussian Bounds

3.1 Chevet's inequality and operator norms

For a non-empty subset S of $\mathbb{R}^n$, define its Gaussian width by $w(S) = \mathbb{E} \sup_{x \in S} \langle g, x \rangle$ and radius by $r(S) = \sup_{x \in S} \|x\|_2$. The Chevet inequality states the following.

Theorem 3.1 (Chevet's inequality). Let S and T be non-empty bounded subsets of $\mathbb{R}^n$. We have

$$\max(w(S)r(T), w(T)r(S)) \leq \mathbb{E} \sup_{x \in T, y \in S} \langle y, G_n x \rangle \leq w(S)r(T) + w(T)r(S). \quad (3.1)$$

Remark 3.2. The upper bound of (3.1) goes back to the work of Chevet [21] (see also [28] for sharper constant). It can be derived using Sudakov–Fernique's inequality (see Theorem 8.7.1 and Exercise 8.7.4 in [69]). The lower bound is an easy exercise.

Chevet’s inequality helps us find the correct scaling of the p -to- q operator norms of Gaussian matrices and perhaps surprisingly, it can be used to even pin down the limiting constants for some choices of p and q . In Section 3.2, we use this knowledge to identify the correct scaling for our ℓ_p -Grothendieck problem, which may not be obvious at the 1st glance. To elaborate how Chevet’s inequality provides bounds for the operator norms, for $p, q \in [1, \infty]$, write

$$\|G_n\|_{p \rightarrow q} = \max_{x \in B_p^n} \|G_n x\|_q = \sup_{x \in B_p^n, y \in B_{q^*}^n} \langle G_n x, y \rangle,$$

where $B_r^n := \{x \in \mathbb{R}^n : \|x\|_r \leq 1\}$ denotes the unit ℓ_r -ball in $\mathbb{R}^n$. Note that

$$r(B_r^n) = \begin{cases} 1, & \text{if } 1 \leq r < 2, \\ n^{\frac{1}{r^*} - \frac{1}{2}}, & \text{if } 2 \leq r \leq \infty, \end{cases} = n^{(\frac{1}{r^*} - \frac{1}{2})_+} = n^{(\frac{1}{2} - \frac{1}{r})_+}, \quad (3.2)$$

and from [26, Example 3.5.4] and [8, Proposition 3],

$$w(B_r^n) = \mathbb{E} \|g\|_{r^*} = \begin{cases} \sqrt{2 \log n} (1 + o(1)), & \text{if } r = 1, \\ \xi_{r^*} n^{1/r^*} + O(n^{1/r^* - 1}), & \text{if } 1 < r < \infty, \\ \xi_1 n, & \text{if } r = \infty, \end{cases} \quad (3.3)$$

where g is an n -dimensional standard Gaussian vector. For $1 \leq a, b \leq \infty$, set $\mathfrak{p}(a, b) = (1/a - 1/2)_+ + 1/b$. By plugging $S = B_{q^*}^n$ and $T = B_p^n$ into (3.1), a direct computation using (3.2) and (3.3) readily yields the following proposition.

Proposition 3.3. The following two-sided bounds are valid.

(i) For $p = 1$ and $q = \infty$,

$$(1 + o(1))\sqrt{2 \log n} \leq \mathbb{E} \|G_n\|_{p \rightarrow q} \leq 2(1 + o(1))\sqrt{2 \log n}.$$

(ii) For $1 < p \leq \infty$ and $1 \leq q < \infty$, if $\mathfrak{p}(q, p^*) = \mathfrak{p}(p^*, q)$,

$$(1 + o(1)) \max(\xi_{p^*}, \xi_q) n^{\mathfrak{p}(p^*, q)} \leq \mathbb{E} \|G_n\|_{p \rightarrow q} \leq (1 + o(1)) (\xi_{p^*} + \xi_q) n^{\mathfrak{p}(p^*, q)}.$$

In the complement of the above cases of (p, q) , one of the terms $w(B_p^n)r(B_{q^*}^n)$ and $w(B_{q^*}^n)r(B_p^n)$ strictly dominates the other as n tends to infinity and as a result, we can locate the limiting constants of the scaled $\mathbb{E} \|G_n\|_{p \rightarrow q}$.

Proposition 3.4. The following limits are valid.

- (i) For $p = 1$ and $1 \leq q < \infty$, $\lim_{n \rightarrow \infty} n^{-1/q} \mathbb{E} \|G_n\|_{p \rightarrow q} = \xi_q$.
- (ii) For $1 < p \leq \infty$ and $q = \infty$, $\lim_{n \rightarrow \infty} n^{-1/p^*} \mathbb{E} \|G_n\|_{p \rightarrow q} = \xi_{p^*}$.
- (iii) For $1 < p \leq \infty$ and $1 \leq q < \infty$, if $\mathfrak{p}(q, p^*) \neq \mathfrak{p}(p^*, q)$, then

$$\lim_{n \rightarrow \infty} n^{-\max(\mathfrak{p}(q, p^*), \mathfrak{p}(p^*, q))} \mathbb{E} \|G_n\|_{p \rightarrow q} = \begin{cases} \xi_{p^*}, & \text{if } \mathfrak{p}(q, p^*) > \mathfrak{p}(p^*, q), \\ \xi_q, & \text{if } \mathfrak{p}(q, p^*) < \mathfrak{p}(p^*, q). \end{cases}$$

Remark 3.5. Note that items (i) and (ii) in Proposition 3.4 are equivalent due to the duality relation $\|G_n\|_{p, q} = \|G_n^T\|_{q^*, p^*}$. Moreover, if $1 < p < \infty$ and $q = p^*$, then $\mathfrak{p}(p^*, q) = \mathfrak{p}(q, p^*)$, which is equal to $1/p^*$ if $1 < p < 2$ and $3/2 - 2/p$ if $2 \leq p < \infty$. Therefore, by Proposition 3.3, we have that

$$\begin{aligned} c\sqrt{\log n} &\leq \mathbb{E} \|G_n\|_{p \rightarrow p^*} \leq C\sqrt{\log n}, & \text{if } p = 1, \\ cn^{\frac{1}{p^*}} &\leq \mathbb{E} \|G_n\|_{p \rightarrow p^*} \leq Cn^{\frac{1}{p^*}}, & \text{if } 1 < p < 2, \\ cn^{\frac{3}{2} - \frac{2}{p}} &\leq \mathbb{E} \|G_n\|_{p \rightarrow p^*} \leq Cn^{\frac{3}{2} - \frac{2}{p}}, & \text{if } 2 \leq p < \infty, \end{aligned} \tag{3.4}$$

where $c, C > 0$ are universal constants depending only on p .

3.2 Scalings of the Grothendieck problem

The scalings of the ℓ_p -Grothendieck problem for $1 \leq p < \infty$ are obtained in the following proposition.

Proposition 3.6. We have that

$$\begin{aligned} c\sqrt{\log n} &\leq \mathbb{E} \max_{\|x\|_p=1} \langle G_n x, x \rangle \leq C\sqrt{\log n}, & \text{if } p = 1, \\ cn^{\frac{1}{p^*}} &\leq \mathbb{E} \max_{\|x\|_p=1} \langle G_n x, x \rangle \leq Cn^{\frac{1}{p^*}}, & \text{if } 1 < p < 2, \\ cn^{\frac{3}{2} - \frac{2}{p}} &\leq \mathbb{E} \max_{\|x\|_p=1} \langle G_n x, x \rangle \leq Cn^{\frac{3}{2} - \frac{2}{p}}, & \text{if } 2 \leq p < \infty, \end{aligned}$$

where $c, C > 0$ are universal constants depending only on p .

Proof. First, by Hölder's inequality,

$$\sup_{\|x\|_p=1} \langle G_n x, x \rangle \leq \sup_{\|x\|_p=1} \|G_n x\|_{p^*} \|x\|_p = \sup_{\|x\|_p=1} \|G_n x\|_{p^*} = \|G_n\|_{p \rightarrow p^*}.$$

From Remark 3.5, the Grothendieck problem shares the same upper bounds as those in (3.4). The lower bound requires a bit of work. We argue in three cases.

Case $2 \leq p < \infty$. Let w be the ℓ_2 -normalized eigenvector of the GOE, $\bar{G}_n$, associated with the top eigenvalue $\lambda_1(\bar{G}_n)$. It is well known that w equals $g/\|g\|_2$ in distribution for an n -dimensional standard Gaussian vector g , leading to $n^{1/2-1/p}\|w\|_p \rightarrow \xi_p/\xi_2$ almost surely and in L^2 . Also, $\lambda_1(\bar{G}_n)/\sqrt{n} \rightarrow 2$ almost surely and in L^2 . Therefore, we can take $x = w/\|w\|_p$ to obtain that

$$\mathbb{E} \sup_{\|x\|_p=1} \langle G_n x, x \rangle = \frac{1}{\sqrt{2}} \mathbb{E} \sup_{\|x\|_p=1} \langle \bar{G}_n x, x \rangle \geq \frac{1}{\sqrt{2}} \mathbb{E} \|w\|_p^{-2} \lambda_1(\bar{G}_n) \geq cn^{\frac{3}{2}-\frac{2}{p}},$$

for some constant $c > 0$.

Case $1 < p < 2$. Without loss of generality, assume that n is even. We readily bound that

$$\mathbb{E} \sup_{\|x\|_p=1} \langle G_n x, x \rangle \geq \sup_{\|v\|_p=2^{-1/p}} \mathbb{E} \sup_{\|u\|_p=2^{-1/p}} \left\langle G_n \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle,$$

where $u, v \in \mathbb{R}^{n/2}$. Divide G_n into four smaller blocks of sizes $(n/2) \times (n/2)$,

$$G_n = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Take $v = (2^{-1/p}, 0, \dots, 0)^T$. Clearly, $\mathbb{E} \langle B_{22} v, v \rangle = 0$, and thus,

$$\begin{aligned} \mathbb{E} \sup_{\|u\|_p=2^{-1/p}} \left\langle G_n \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle &= \mathbb{E} \left[\sup_{\|u\|_p=2^{-1/p}} (\langle B_{11} u, u \rangle + \langle (B_{12} + B_{21}^T) v, u \rangle) \right] \\ &= \mathbb{E} \left[\sup_{\|u\|_p=2^{-1/p}} (\langle B_{11} u, u \rangle + 2^{1/2-1/p} \langle \tilde{g}, u \rangle) \right], \end{aligned}$$

where $\tilde{g}$ is a standard Gaussian vector in $\mathbb{R}^{n/2}$. Given $\tilde{g}$, we can choose $u^* = u^*(\tilde{g})$ with $\|u^*\|_p = 2^{-1/p}$ such that $\langle \tilde{g}, u^* \rangle = 2^{-1/p} \|\tilde{g}\|_{p^*}$, which implies that there exists some $c' > 0$ such that

$$\begin{aligned} \mathbb{E} \left[\sup_{\|u\|_p=2^{-1/p}} (\langle B_{11} u, u \rangle + 2^{1/2-1/p} \langle \tilde{g}, u \rangle) \right] &\geq \mathbb{E} [\langle B_{11} u^*, u^* \rangle + 2^{1/2-1/p} \|\tilde{g}\|_{p^*}] \\ &= 2^{1/2-1/p} \mathbb{E} \|\tilde{g}\|_{p^*} \geq c' n^{1/p^*}, \end{aligned}$$

where the equality above follows from the fact that $\mathbb{E}[\langle B_{11} u^*, u^* \rangle | u^*] = 0$ since u^* is independent of B_{11} , whereas the last inequality uses (3.3). Hence, we obtain that $\mathbb{E} \sup_{\|x\|_p=1} \langle G_n x, x \rangle \geq c' n^{1/p^*}$.

Case $p = 1$. A matching lower bound can be easily obtained by optimizing only over the coordinate vectors e_i yielding $\mathbb{E} \max_{\|x\|_1=1} \langle G_n x, x \rangle \geq \mathbb{E} \max_{i \in [n]} g_{ii} = \sqrt{2 \log n} (1 + o(1))$. ■

3.3 Concentration and maximal inequalities

We collect some Gaussian bounds that will be of great use throughout the rest of this paper. We start with the well-known concentration inequality for the Lipschitz function of a standard Gaussian vector states (see, e.g., [9, Theorems 5.5 and 5.6]).

Lemma 3.7 (Gaussian concentration). Let g be a standard Gaussian vector in $\mathbb{R}^n$. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a L -Lipschitz function, that is, $|F(x) - F(y)| \leq L\|x - y\|_2$ for all $x, y \in \mathbb{R}^n$. Then,

$$\mathbb{E} e^{s(F(g) - \mathbb{E}F(g))} \leq e^{\frac{s^2 L^2}{2}}, \quad \forall s \in \mathbb{R} \quad (3.5)$$

and

$$\mathbb{P}(|F(g) - \mathbb{E}F(g)| \geq s) \leq 2e^{-\frac{s^2}{2L^2}}, \quad \forall s > 0. \quad (3.6)$$

Remark 3.8. Let $1 \leq p \leq 2$. From Lemma 3.7, for any non-empty $S, T \subseteq B_p^n$, and any $s > 0$,

$$\mathbb{P}\left(\left|\sup_{x \in S} \langle x, G_n x \rangle - \mathbb{E} \sup_{x \in S} \langle x, G_n x \rangle\right| \geq s\right) \leq 2e^{-s^2/2}, \quad (3.7)$$

$$\mathbb{P}\left(\left|\sup_{x \in S, y \in T} \langle x, G_n y \rangle - \mathbb{E} \sup_{x \in S, y \in T} \langle x, G_n y \rangle\right| \geq s\right) \leq 2e^{-s^2/2}. \quad (3.8)$$

To see this, note that the functions $G_n \mapsto \langle x, G_n x \rangle$ and $G_n \mapsto \sup_{x \in S, y \in T} \langle x, G_n y \rangle$ are Lipschitz continuous with respect to $\|\cdot\|_F$, the Frobenius norm. The former case has the Lipschitz constant, $\sup_{x \in S} \|x\|_2$, whereas the latter case has the Lipschitz constant $\sup_{x \in S, y \in T} \|x\|_2 \|y\|_2$. Both of these Lipschitz constants are bounded above by 1 by the assumption that $p \leq 2$.

Remark 3.9. Let $2 < p < \infty$. It can be shown that $\text{GP}_{n,p}$ and $L_{n,p,u}(t)$ are $n^{-1/2}$ and $un^{-1/2}$ -Lipschitz with respect to $\|\cdot\|_F$, respectively. From Lemma 3.7,

$$\mathbb{P}(|\text{GP}_{n,p} - \mathbb{E}\text{GP}_{n,p}| \geq s) \leq 2e^{-\frac{ns^2}{2}}, \quad \forall s > 0, \quad (3.9)$$

$$\mathbb{P}(|L_{n,p,u}(t) - \mathbb{E}L_{n,p,u}(t)| \geq s) \leq 2e^{-\frac{ns^2}{2u^2}}, \quad \forall s > 0. \quad (3.10)$$

Remark 3.10. We also mention the following concentration bound for the ℓ_∞ -norm of a standard Gaussian vector g in $\mathbb{R}^n$ (see [56, Corollary 4.10] and [10, 60]), which is an improvement upon the one obtained from the preceding Gaussian concentration inequality. There exist absolute constants $c, C > 0$ such that for all $n \geq 1$ and for all $\varepsilon \in (0, 1)$,

$$\mathbb{P}(|\|g\|_\infty - \mathbb{E}\|g\|_\infty| \geq \varepsilon \mathbb{E}\|g\|_\infty) \leq Cn^{-c\varepsilon}. \quad (3.11)$$

Recall that a mean zero random variable X is called σ^2 -subgaussian if $\mathbb{E}e^{tX} \leq e^{t^2\sigma^2/2}$ holds for all $t \in \mathbb{R}$. The following result gives a crude bound of the maxima of a collection of subgaussian random variables (see [68, Lemma 5.1]).

Lemma 3.11 (Maximal inequality). Let $X_1, X_2, \dots, X_m$ be a collection (not necessarily independent) of centered σ^2 -subgaussian random variables. Then,

$$\mathbb{E} \max_{1 \leq i \leq m} X_i \leq \sqrt{2\sigma^2 \log m}.$$

4 Proof of Theorem 1.1

In this section, we provide a proof of Theorem 1.1. We begin with part (i).

Proof of (i). In view of the identity $\langle G_n x, x \rangle = 2^{-1/2} \langle \tilde{G}_n x, x \rangle$, it suffices to prove that

$$\frac{1}{\sqrt{\log n}} \max_{\|x\|_1=1} \langle \tilde{G}_n x, x \rangle \rightarrow 2, \quad \text{almost surely.}$$

Note that the diagonal entries of the symmetric matrix $\bar{G}_n$ are $\bar{g}_{ii} = \sqrt{2}g_{ii}$, whereas the off-diagonal entries are equal to $\bar{g}_{ij} = (g_{ij} + g_{ji})/\sqrt{2}$. We bound

$$\begin{aligned}\langle \bar{G}_n x, x \rangle &= \sqrt{2} \sum_i g_{ii} x_i^2 + 2 \sum_{i < j} \bar{g}_{ij} x_i x_j \leq \max(\sqrt{2} \max_i g_{ii}, \max_{i < j} |\bar{g}_{ij}|) \left(\sum_i x_i^2 + 2 \sum_{i < j} |x_i| |x_j| \right) \\ &= \max(\sqrt{2} \max_i g_{ii}, \max_{i < j} |\bar{g}_{ij}|) \|x\|_1^2,\end{aligned}$$

which leads to that

$$\frac{1}{\sqrt{\log n}} \max_{\|x\|_1=1} \langle \bar{G}_n x, x \rangle \leq \frac{\max(\sqrt{2} \max_i g_{ii}, \max_{i < j} |\bar{g}_{ij}|)}{\sqrt{\log n}}. \quad (4.1)$$

It is a well-known fact (see, e.g., [26, Example 3.5.4]) that if $z_1, z_2, \dots$ is a sequence of i.i.d. $N(0, 1)$, then $(2 \log n)^{-1/2} \max_{i \in [n]} z_i \rightarrow 1$ almost surely. Consequently, the following limits exist almost surely.

$$\lim_n \frac{\sqrt{2} \max_i g_{ii}}{2\sqrt{\log n}} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\max_{i < j} |\bar{g}_{ij}|}{2\sqrt{\log n}} = \lim_{n \rightarrow \infty} \max\left(\frac{\max_{i < j} \bar{g}_{ij}}{\sqrt{2 \log \binom{n}{2}}}, \frac{\max_{i < j} (-\bar{g}_{ij})}{\sqrt{2 \log \binom{n}{2}}}\right) = 1.$$

Plugging the above limits in (4.1), we have the upper bound

$$\limsup_{n \rightarrow \infty} \frac{1}{\sqrt{\log n}} \max_{\|x\|_1=1} \langle \bar{G}_n x, x \rangle \leq 2.$$

For the lower bound, we optimize over just the coordinate vectors $e_i, i \in [n]$ to obtain

$$\liminf_{n \rightarrow \infty} \frac{1}{\sqrt{\log n}} \max_{\|x\|_1=1} \langle \bar{G}_n x, x \rangle \geq \liminf_{n \rightarrow \infty} \frac{1}{\sqrt{\log n}} \max_i \sqrt{2}g_{ii} = 2.$$

This completes the proof of (1.3). ■

Proof of (ii). We realize that $\max_{\|x\|_1=1} \langle \bar{G}_n x, x \rangle \geq \max_{i \in [n]} \sqrt{2}g_{ii}$ and then proceed to prove (1.4) as

$$\begin{aligned}\mathbb{P}\left(\max_{i \in [n]} g_{ii} \leq \sqrt{2(1-\eta) \log n}\right) &= \mathbb{P}(g_{11} \leq \sqrt{2(1-\eta) \log n})^n \\ &\leq (1 - e^{-(1-\eta) \log n})^n = (1 - n^{-(1-\eta)})^n \leq e^{-n^\eta},\end{aligned}$$

where the 1st inequality above follows from the Gaussian tail bound $\mathbb{P}(z > t) \leq e^{-t^2/2}$ for all $t > 0$ and the last inequality uses the bound $1 - t \leq e^{-t}$ for all $t \in \mathbb{R}$. ■

Proof of (iii). For $\mathbf{x}$ with $\|\mathbf{x}\|_1 = 1$ and $\|\mathbf{x}\|_\infty \leq 1 - \delta$, we have

$$|(\bar{G}_n \mathbf{x})_i| \leq \sum_{j \in [n]} |\bar{g}_{ij}| |x_j| \leq (1 - \delta) X_{i,(1)} + \delta X_{i,(2)}, \quad i \in [n],$$

where $X_{i,(1)}$ and $X_{i,(2)}$ denote the largest and the 2nd largest values among the absolute values of the entries of the i -th row of $\bar{G}_n$. Then,

$$\max_{\|\mathbf{x}\|_1=1, \|\mathbf{x}\|_\infty \leq 1-\delta} \langle \bar{G}_n \mathbf{x}, \mathbf{x} \rangle \leq \max_{i \in [n]} ((1 - \delta) X_{i,(1)} + \delta X_{i,(2)}) \leq (1 - \delta) \max_{i \in [n]} X_{i,(1)} + \delta \max_{i \in [n]} X_{i,(2)}. \quad (4.2)$$

Therefore, we need to bound the maxima, $\max_{i \in [n]} X_{i,(1)}$ and $\max_{i \in [n]} X_{i,(2)}$. Note that

$$\max_{i \in [n]} X_{i,(1)} = \max \left(\sqrt{2} \max_i |g_{ii}|, \max_{i < j} |\bar{g}_{ij}| \right).$$

Here, from Lemma 3.11,

$$\mathbb{E} \max_i |g_{ii}| = \mathbb{E} \max(g_{11}, -g_{11}, \dots, g_{nn}, -g_{nn}) \leq \sqrt{2 \log 2n}.$$

Therefore, (3.11) implies that for $\delta \in (0, 1)$,

$$\mathbb{P}(\max_i |g_{ii}| \geq (1 + \delta/22) \sqrt{2 \log(2n)}) \leq C' n^{-c'\delta},$$

where c', C' are positive universal constants. We reapply the above tail bound to the maximum of $\binom{n}{2}$ i.i.d. standard Gaussian random variables $(\bar{g}_{ij})_{1 \leq i < j \leq n}$ and then use the union bound to deduce that

$$\mathbb{P}(\max_{i \in [n]} X_{i,(1)} \geq 2(1 + \delta/22) \sqrt{\log(2n)}) \leq C'' n^{-c''\delta}$$

for some positive constants c'' and C'' . Let $K = 10\sqrt{2}/11 \in (\sqrt{3/2}, \sqrt{2})$. From the usual Gaussian tail bound, denote

$$\theta_n := \mathbb{P}(\sqrt{2}|z| \geq K\sqrt{2 \log n}) = O(n^{-K^2/2}), \quad \phi_n := \mathbb{P}(|z| \geq K\sqrt{2 \log n}) = O(n^{-K^2}).$$

Note that $X_{i,(2)} \geq K\sqrt{2\log n}$ implies that at least two of $(\bar{g}_{ij})_{1 \leq j \leq n}$ are not less than $K\sqrt{2\log n}$. By using the union bound,

$$\begin{aligned} \mathbb{P}(X_{i,(2)} \geq K\sqrt{2\log n}) &\leq \mathbb{P}(\sqrt{2}|g_{ii}| \geq K\sqrt{2\log n} \text{ and } |\bar{g}_{ij}| \geq K\sqrt{2\log n} \text{ for some } j \neq i) \\ &\quad + \mathbb{P}(|\bar{g}_{ij}|, |\bar{g}_{ij'}| \geq K\sqrt{2\log n} \text{ for some distinct } j, j' \neq i) \\ &\leq (n-1)\theta_n\phi_n + \binom{n-1}{2}\phi_n^2 = O(n^{-(3K^2/2-1)}) + O(n^{-(2K^2-2)}), \end{aligned}$$

which, by noting that our choice of K implies $3K^2/2 - 2 > 2K^2 - 3$ and using the union bound again, implies that

$$\mathbb{P}(\max_{i \in [n]} X_{i,(2)} \geq K\sqrt{2\log n}) = O(n^{-(3K^2/2-2)}) + O(n^{-(2K^2-3)}) = O(n^{-(2K^2-3)}).$$

Note that the exponent $2K^2 - 3$ is positive. Therefore, for all $0 < \delta \leq \min((2K^2 - 3)/c'', 1)$, on the event where both the bounds $\max_{i \in [n]} X_{i,(1)} \leq 2(1 + \delta/22)\sqrt{\log(2n)}$ and $\max_{i \in [n]} X_{i,(2)} \leq K\sqrt{2\log n}$ are satisfied, which occurs with probability at least $1 - O(n^{-(2K^2-3)}) - O(n^{-c''\delta}) = 1 - O(n^{-c''\delta})$, we have, by a direction of computation,

$$\begin{aligned} (1 - \delta) \max_{i \in [n]} X_{i,(1)} + \delta \max_{i \in [n]} X_{i,(2)} &\leq 2(1 + \delta/22)\sqrt{\log(2n)}(1 - \delta) + K\delta\sqrt{2\log n} \\ &= \sqrt{2}(1 - \delta/22)\sqrt{2\log(2n)} \\ &\leq \sqrt{2}\left(1 - \delta/22 + \sqrt{\log 2/\log n}\right)\sqrt{2\log n}, \end{aligned}$$

where the last inequality used $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a, b \geq 0$. Finally, this inequality and (4.2), coupled with the fact that $\langle G_n x, x \rangle = 2^{-1/2} \langle \bar{G}_n x, x \rangle$, yield (1.5). $\blacksquare$

5 Proof of Theorem 1.2

In order to prove Theorem 1.2, we first begin with some preparation in the following subsection.

5.1 Preliminary bounds

For a given truncation level $\varepsilon > 0$, set

$$D = B_p^n \cap \{x \in \mathbb{R}^n : \|x\|_\infty \leq \varepsilon\} \quad \text{and} \quad L = B_p^n \cap \{x \in \mathbb{R}^n : |x_i| \neq 0 \Rightarrow |x_i| > \varepsilon \text{ for each } i\}.$$

For $x \in \mathbb{R}^n$, we can uniquely decompose $x = x^o + x^\dagger$ with $x^o \in D$ and $x^\dagger \in L$ by setting

$$x_i^o = x_i 1(|x_i| \leq \varepsilon) \quad \text{and} \quad x_i^\dagger = x_i 1(|x_i| > \varepsilon) \quad \text{for each } i. \quad (5.1)$$

Note that $x^\dagger$ carries the localized part of x , whereas x^o is the remaining delocalized part of x . With these notations in hand, we are ready to state two crucial propositions.

Proposition 5.1. Let $1 < p < 2$. There exists a constant $C > 0$ such that the following statements are valid.

(i) For any $\varepsilon > 0$ and $n \geq 1$,

$$\mathbb{E} \sup_{S \subseteq B_p^n} \left| \max_{x \in S} \langle G_n x, x \rangle - \sqrt{2} \max_{x \in S} \langle \bar{G}_n x^o, x^\dagger \rangle \right| \leq C(\varepsilon^{1-p/2} n^{1/p^*} + \varepsilon^{-p/p^*} \sqrt{\log n}). \quad (5.2)$$

(ii) For any $\varepsilon > 0$, $c > 0$, and $n \geq 1$, with probability at least $1 - 2n^{-c}$,

$$\begin{aligned} & \sup_{S \subseteq B_p^n} \left| \max_{x \in S} \langle G_n x, x \rangle - \sqrt{2} \max_{x \in S} \langle \bar{G}_n x^o, x^\dagger \rangle \right| \\ & \leq C(\varepsilon^{1-p/2} n^{1/p^*} + \varepsilon^{-p/p^*} \sqrt{\log n}) + 2\sqrt{2c \log n}. \end{aligned} \quad (5.3)$$

Since the ℓ_p -Grothendieck problem scales as n^{1/p^*} for $1 < p < 2$, the above proposition says that the main contribution of ℓ_p -Grothendieck problem on an arbitrary subset S of B_p^n comes from the cross-term $\sqrt{2} \max_{x \in S} \langle \bar{G}_n x^o, x^\dagger \rangle$ as long as $\varepsilon = \varepsilon_n$ satisfies that $\limsup_{n \rightarrow \infty} \varepsilon = 0$ and $\liminf_{n \rightarrow \infty} \varepsilon n^{1/p} / (\log n)^{p^*/(2p)} = \infty$. The next result then provides an upper bound on this cross-term in terms of the maximum number of localized coordinates of the vectors in S . For $x \in \mathbb{R}^n$, recall that $\|x\|_0$ denotes the number of nonzero entries of x .

Proposition 5.2. There exists a constant C such that for any $1 \leq s \leq n$ and any subset $L^\# \subseteq \{x : \|x\|_p \leq 1, \|x\|_0 \leq s\}$,

$$\mathbb{E} \sup_{x \in L^\#, y \in B_p^n} \langle y, \bar{G}_n x \rangle \leq r(L^\#) \mathbb{E} \|g\|_{p^*} + C(n^{p/(2p^*)} \sqrt{\log n} + s^{1/p^*} \sqrt{\log n}),$$

where g is an n -dimensional standard Gaussian vector.

Remark 5.3. The bound in Proposition 5.2 will mainly be used in the proof of (1.8) in Theorem 1.2. The key feature here is that the term $r(L^\#)\mathbb{E}\|g\|_{p^*}$ is asymptotically sharp up to an additive error of smaller order for some properly chosen $s = o(n)$ and $L^\#$, yielding the correct limit of the ℓ_p -Grothendieck problem. If we replace $\bar{G}_n$ with G_n , then we could also directly apply Chevet's inequality to obtain

$$\mathbb{E} \sup_{x \in L^\#, y \in B_p^n} \langle y, G_n x \rangle = \mathbb{E} \sup_{x \in L^\#, y \in B_p^n} \langle y, G_n^T x \rangle \leq r(L^\#)\mathbb{E}\|g\|_{p^*} + Cs^{1/p^*} \sqrt{\log n},$$

which implies that, by recalling $\bar{G}_n = (G_n + G_n^T)/\sqrt{2}$,

$$\mathbb{E} \sup_{x \in L^\#, y \in B_p^n} \langle y, \bar{G}_n x \rangle \leq \sqrt{2}r(L^\#)\mathbb{E}\|g\|_{p^*} + Cs^{1/p^*} \sqrt{\log n}.$$

However, this extra prefactor $\sqrt{2}$ in front of $r(L^\#)\mathbb{E}\|g\|_{p^*}$ makes the bound sub-optimal.

We establish the above two propositions in the rest of this subsection. The next lemma bounds the Gaussian width of sparse sets.

Lemma 5.4. Fix $p > 1$ and $s \leq n$. Let $W = \{x \in \mathbb{R}^n : \|x\|_p \leq 1, \|x\|_0 \leq s\}$. Then,

$$w(W) \leq \sqrt{2}s^{1/p^*} \sqrt{\log n}.$$

Proof. Observe that for any subset I of $\{1, 2, \dots, n\}$,

$$\max_{\|x\|_p \leq 1, \text{supp}(x)=I} \langle g, x \rangle = \left(\sum_{i \in I} |g_i|^{p^*} \right)^{1/p^*}.$$

Therefore, by Lemma 3.11, we have

$$\begin{aligned} w(W) &= \mathbb{E} \sup_{x \in W} \langle g, x \rangle \leq \mathbb{E} \max_{|I| \leq s} \left(\sum_{i \in I} |g_i|^{p^*} \right)^{1/p^*} \\ &\leq s^{1/p^*} \mathbb{E} \max_i |g_i| \leq s^{1/p^*} \sqrt{2 \log n}. \end{aligned}$$

■

Proof of Proposition 5.1. Let $\varepsilon > 0$ be fixed. Write, according to (5.1),

$$\langle G_n x, x \rangle = \sqrt{2} \langle \bar{G}_n x^o, x^\dagger \rangle + \langle G_n x^o, x^o \rangle + \langle G_n x^\dagger, x^\dagger \rangle,$$

and then, use the triangle inequality to bound

$$\begin{aligned} \sup_{S \subseteq B_p^n} \left| \max_{x \in S} \langle G_n x, x \rangle - \sqrt{2} \max_{x \in S} \langle \bar{G}_n x^o, x^\dagger \rangle \right| &\leq \max_{\|x\|_p=1} |\langle G_n x^o, x^o \rangle| + \max_{\|x\|_p=1} |\langle G_n x^\dagger, x^\dagger \rangle| \\ &\leq \max_{x \in D} |\langle G_n x, x \rangle| + \max_{x \in L} |\langle G_n x, x \rangle|. \end{aligned} \quad (5.4)$$

Note that as D and L are symmetric, that is, $D = -D$ and $L = -L$, we have

$$\max_{x \in D} |\langle G_n x, x \rangle| \leq \max_{x, y \in D} \langle G_n x, y \rangle, \quad \max_{x \in L} |\langle G_n x, x \rangle| \leq \max_{x, y \in L} \langle G_n x, y \rangle.$$

Applying Chevet's inequality with $S = T = D$, we have $\mathbb{E} \max_{x, y \in D} \langle G_n x, y \rangle \leq 2w(D)r(D)$. We can bound the Gaussian width of D by that of B_p^n to obtain $w(D) \leq w(B_p^n) = \mathbb{E} \|g\|_{p^*} \leq Cn^{1/p^*}$, using (3.3). On the other hand, for $x \in D$, we have $\sum_i |x_i|^2 \leq \varepsilon^{2-p} \sum_i |x_i|^p \leq \varepsilon^{2-p}$. Hence, $r(D) \leq \varepsilon^{1-p/2}$, and therefore,

$$\mathbb{E} \max_{x \in D} |\langle G_n x, x \rangle| \leq 2C\varepsilon^{1-p/2} n^{1/p^*}. \quad (5.5)$$

Similarly, we have $\mathbb{E} \max_{x, y \in L} \langle G_n x, y \rangle \leq 2w(L)r(L)$. Obviously, $r(L) = 1$. Note that if $x \in L$, then $\|x\|_0 \leq \varepsilon^{-p}$. Hence, by Lemma 5.4, we have $w(L) \leq \sqrt{2}\varepsilon^{-p/p^*} \sqrt{\log n}$ and this yields

$$\mathbb{E} \max_{x \in L} |\langle G_n x, x \rangle| \leq 2\sqrt{2}\varepsilon^{-p/p^*} \sqrt{\log n}. \quad (5.6)$$

Now, (5.2) follows from (5.4)–(5.6) after taking C sufficiently large. To prove (5.3), note that $G_n \mapsto \max_{x \in D} |\langle G_n x, x \rangle|$ and $G_n \mapsto \max_{x \in L} |\langle G_n x, x \rangle|$ are 1-Lipschitz with respect to the Frobenius norm. For any $c > 0$, using (3.6) with $s = \sqrt{2c \log n}$, (5.5), and (5.6) yields that with probability at least $1 - 2n^{-c}$,

$$\max_{x \in D} |\langle G_n x, x \rangle| + \max_{x \in L} |\langle G_n x, x \rangle| \leq 2C\varepsilon^{1-p/2} n^{1/p^*} \sqrt{\log n} + 2\sqrt{2}\varepsilon^{-p/p^*} \sqrt{\log n} + 2\sqrt{2c \log n}.$$

Consequently, it follows from (5.4) that (5.3) is valid with probability at least $1 - 2n^{-c}$. $\blacksquare$

Proof of Proposition 5.2. For any $x \in L^\#$, write $x = x^o + x^\dagger$ as in (5.1) by setting $\varepsilon = n^{-1/p^*}$. Then x^o satisfies that $\|x^o\|_p \leq 1$, $\|x^o\|_\infty \leq \varepsilon$, and $\|x^o\|_0 \leq s$. On the other hand, since any non-zero entry of $x^\dagger$ satisfies $|x_i^\dagger| \geq \varepsilon$, it follows that $\|x^\dagger\|_0 \varepsilon^p \leq \|x\|_p^p \leq 1$,

which implies that $\|x^\dagger\|_0 \leq \varepsilon^{-p}$. So, we have

$$L^\# \subseteq U + V := \{x + x' : x \in U, x' \in V\},$$

where

$$U = \{x : \|x\|_p \leq 1, \|x\|_\infty \leq \varepsilon, \|x\|_0 \leq s\}, \quad V = L^\# \cap \{x : \|x\|_0 \leq \varepsilon^{-p}\}.$$

Then, by triangle inequality,

$$\mathbb{E} \sup_{x \in L^\#, y \in B_p^n} \langle y, \bar{G}_n x \rangle \leq \mathbb{E} \sup_{x \in U, y \in B_p^n} \langle y, \bar{G}_n x \rangle + \mathbb{E} \sup_{x \in V, y \in B_p^n} \langle y, \bar{G}_n x \rangle = (I) + (II).$$

Bounding (I): Since $G_n \stackrel{d}{=} G_n^T$, we have $(I) \leq \sqrt{2} \mathbb{E} \sup_{x \in U, y \in B_p^n} \langle y, G_n x \rangle$. By Chevet's inequality,

$$\mathbb{E} \sup_{x \in U, y \in B_p^n} \langle y, G_n x \rangle \leq r(U)w(B_p^n) + r(B_p^n)w(U).$$

Arguing similarly as we did in the proof of Proposition 5.1, $r(U) \leq \varepsilon^{1-p/2}$ and $r(B_p^n) = 1$. Moreover, we have $w(B_p^n) = \mathbb{E} \|g\|_{p^*} \leq C_1 n^{1/p^*}$ and $w(U) \leq \sqrt{2} s^{1/p^*} \sqrt{\log n}$ (by Lemma 5.4). Therefore, we obtain

$$(I) \leq \sqrt{2} C_1 \varepsilon^{1-p/2} n^{1/p^*} + 2s^{1/p^*} \sqrt{\log n} = \sqrt{2} C_1 n^{p/(2p^*)} + 2s^{1/p^*} \sqrt{\log n}.$$

Bounding (II): Note that $(II) = \mathbb{E} \sup_{x \in V} \|\bar{G}_n x\|_{p^*}$. Let $\delta = n^{-1}$. Since each $x \in V$ is supported by at most ε^{-p} coordinates, we can choose a δ -net N_δ of V with respect to $\|\cdot\|_p$ -norm such that the cardinality of N_δ is at most $O((n\delta^{-1})^{\varepsilon^{-p}})$. We now bound

$$\begin{aligned} \mathbb{E} \sup_{x \in V} \|\bar{G}_n x\|_{p^*} &\leq \mathbb{E} \max_{x \in N_\delta} \|\bar{G}_n x\|_{p^*} + \mathbb{E} \sup_{x, x' \in B_p^n : \|x - x'\|_p \leq \delta} \|\bar{G}_n(x - x')\|_{p^*} \\ &\leq \mathbb{E} \max_{x \in N_\delta} \|\bar{G}_n x\|_{p^*} + \delta \mathbb{E} \|\bar{G}_n\|_{p \rightarrow p^*} \leq \mathbb{E} \max_{x \in N_\delta} \|\bar{G}_n x\|_{p^*} + O(1), \end{aligned}$$

where in the last step, we used the bound $\mathbb{E} \|\bar{G}_n\|_{p \rightarrow p^*} \leq \sqrt{2} \mathbb{E} \|G_n\|_{p \rightarrow p^*} \leq C_2 n^{1/p^*}$ from Remark 3.5.

Next, for each $x \in B_p^n$, the function $G_n \mapsto \|\bar{G}_n x\|_{p^*} = \sup_{y \in B_p^n} \langle y, \bar{G}_n x \rangle$ is Lipschitz continuous with respect to the Frobenius norm with Lipschitz constant

$\sqrt{2} \sup_{Y \in B_p^n} \|Y\|_2 \|x\|_2 \leq \sqrt{2}$. Hence, (3.5) yields

$$\mathbb{E} e^{t(\|\tilde{G}_n x\|_{p^*} - \mathbb{E}\|\tilde{G}_n x\|_{p^*})} \leq e^{t^2}, \quad \forall x \in B_p^n, \quad t \in \mathbb{R}.$$

In other words, the random variable $\|\tilde{G}_n x\|_{p^*} - \mathbb{E}\|\tilde{G}_n x\|_{p^*}$ is 2-subgaussian. Therefore, by Lemma 3.11,

$$\mathbb{E} \max_{x \in N_\delta} (\|\tilde{G}_n x\|_{p^*} - \mathbb{E}\|\tilde{G}_n x\|_{p^*}) \leq \sqrt{4 \log |N_\delta|} = O(\varepsilon^{-p/2} \sqrt{\log n}) = O(n^{p/(2p^*)} \sqrt{\log n}).$$

From the above estimates, we deduce that

$$\mathbb{E} \sup_{x \in V} \|\tilde{G}_n x\|_{p^*} \leq \sup_{x \in V} \mathbb{E} \|\tilde{G}_n x\|_{p^*} + O(n^{p/(2p^*)} \sqrt{\log n}). \quad (5.7)$$

It remains to estimate $\mathbb{E}\|\tilde{G}_n x\|_{p^*}$. For any $x \in \mathbb{R}^n$, we have $\tilde{G}_n x \stackrel{d}{=} \|x\|_2 g + zx$, where g , as always, is the standard normal vector on $\mathbb{R}^n$ and z is an independent $N(0, 1)$ random variable. Therefore, for any $x \in B_p^n$,

$$\left| \mathbb{E}\|\tilde{G}_n x\|_{p^*} - \mathbb{E}\|g\|_{p^*} \|x\|_2 \right| \leq \|x\|_{p^*} \mathbb{E}|z| \leq 1.$$

Consequently,

$$\sup_{x \in V} \mathbb{E}\|\tilde{G}_n x\|_{p^*} \leq r(V) \mathbb{E}\|g\|_{p^*} + 1 \leq r(L^\#) \mathbb{E}\|g\|_{p^*} + 1. \quad (5.8)$$

Combining (5.7) and (5.8), we obtain that

$$(II) \leq r(L^\#) \mathbb{E}\|g\|_{p^*} + O(n^{p/(2p^*)} \sqrt{\log n}),$$

completing the proof of the proposition. ■

5.2 Proof of (1.9): the lower bound

For convenience, throughout this proof, for three sequences of random variables $(X_n)_{n \geq 1}$, $(Y_n)_{n \geq 1}$, and $(Z_n)_{n \geq 1}$, we denote

$$X_n = Y_n + O_{\mathbb{P}}(Z_n) \quad (5.9)$$

if for any $d > 0$, there exists some $C > 0$ such that $\mathbb{P}(X_n \leq Y_n + CZ_n) = 1 - O(n^{-d})$ for all $n \geq 1$.

The proof of (1.9) is argued as follows. Expanding

$$\langle v_i + e_i, G_n(v_i + e_i) \rangle = \langle v_i, G_n v_i \rangle + \langle e_i, G_n e_i \rangle + \sqrt{2} \langle v_i, \bar{G}_n e_i \rangle,$$

we have

$$\min_i \langle v_i + e_i, G_n(v_i + e_i) \rangle \geq \min_i \sqrt{2} \langle v_i, \bar{G}_n e_i \rangle - \max_i |\langle v_i, G_n v_i \rangle| - \max_i |\langle e_i, G_n e_i \rangle|. \quad (5.10)$$

We will control each term on the right-hand side. First, we handle the middle one as follows. Since $\|\bar{G}_n e_i\|_{p^*} - \mathbb{E}\|\bar{G}_n e_i\|_{p^*}$ is 2-subgaussian,

$$\mathbb{P}(\max_i \|\bar{G}_n e_i\|_{p^*} - \mathbb{E}\|\bar{G}_n e_i\|_{p^*} > s) \leq 2ne^{-s^2/4}. \quad (5.11)$$

Write, by Jensen's inequality and (3.3),

$$\mathbb{E}\|\bar{G}_n e_i\|_{p^*} = \mathbb{E}\|\bar{G}_n e_1\|_{p^*} \geq \mathbb{E}\|g\|_{p^*} = \xi_{p^*} n^{1/p^*} (1 - a_n), \quad (5.12)$$

for some a_n satisfying $0 < a_n = O(n^{-1})$ and g a standard Gaussian vector. Plugging (5.12) into (5.11) yields

$$\mathbb{P}(\min_i \|\bar{G}_n e_i\|_{p^*} \geq \xi_{p^*} n^{1/p^*} (1 - a_n) + s) \leq 2ne^{-s^2/4}.$$

Using the well-known lower bound of Gaussian tail $\mathbb{P}(|z| \geq t) \geq 2t(t^2 + 1)^{-1}e^{-t^2/2}$ for $t > 0$, we have

$$\mathbb{P}(\max_{i \leq j} |\bar{G}_n(i, j)| \leq s) = \mathbb{P}(|z| \leq s)^{n(n-1)/2} \mathbb{P}(\sqrt{2}|z| \leq s)^n \geq \mathbb{P}(\sqrt{2}|z| \leq s)^{n^2} \geq \left(1 - \frac{2\sqrt{2}s}{s^2 + 2}e^{-s^2/4}\right)^{n^2}.$$

Combining the above bounds and recalling (1.6), we have that for any $s > 0$ and sufficiently large n ,

$$\begin{aligned} \mathbb{P}\left(\max_i \|v_i\|_\infty \leq \frac{2s^{p^*/p}}{\xi_{p^*}^{p^*/p} n^{1/p}}\right) &\geq \mathbb{P}\left(\max_i \|v_i\|_\infty \leq \frac{s^{p^*/p}}{(\xi_{p^*} n^{1/p^*} (1 - a_n) + s)^{p^*/p}}\right) \\ &\geq \left(1 - \frac{2\sqrt{2}s}{s^2 + 2}e^{-s^2/4}\right)^{n^2} - 2ne^{-s^2/4} \\ &\geq \left(1 - \frac{2\sqrt{2}}{s}e^{-s^2/4}\right)^{n^2} - 2ne^{-s^2/4}. \end{aligned}$$

As a result, for any $d > 0$, if we let $s = \xi_{p^*} \sqrt{8(1+d) \log n}$, then for $\varepsilon := 2(8(1+d) \log n)^{p^*/(2p)} n^{-1/p}$,

$$\begin{aligned} \mathbb{P}(\max_i \|v_i\|_\infty \leq \varepsilon) &\geq \left(1 - \frac{2\sqrt{2}}{n^{2(1+d)\xi_{p^*}^2} \xi_{p^*} \sqrt{8(1+d) \log n}}\right)^{n^2} - \frac{2}{n^{(1+d)\xi_{p^*}^2 - 1}} \\ &\geq 1 - \frac{2\sqrt{2}}{n^{2(1+d)\xi_{p^*}^2 - 2} \xi_{p^*} \sqrt{8(1+d) \log n}} - \frac{2}{n^{2(1+d)\xi_{p^*}^2 - 1}} \geq 1 - \frac{C}{n^d} \end{aligned} \quad (5.13)$$

for some universal constant $C > 0$, where the 2nd inequality used the fact that for $m \geq 1$, $(1-x)^m \geq 1 - mx$ for any $0 \leq x \leq 1$ and the 3rd inequality is valid because $\xi_{p^*} \geq 1$. Now, on the event $\max_i \|v_i\|_\infty \leq \varepsilon$, we can bound $\max_i |\langle v_i, G_n v_i \rangle| \leq \sup_{\|x\|_p \leq 1, \|x\|_\infty \leq \varepsilon} |\langle x, G_n x \rangle|$. As argued in the proof of Proposition 5.1,

$$\begin{aligned} \mathbb{E} \sup_{\|x\|_p \leq 1, \|x\|_\infty \leq \varepsilon} |\langle x, G_n x \rangle| &= O(\varepsilon^{1-\frac{p}{2}} n^{\frac{1}{p^*}}) \\ &= O((\log n)^{(1-\frac{p}{2})\frac{p^*}{2p}} n^{-(1-\frac{p}{2})\frac{1}{p} + \frac{1}{p^*}}) = O(n^{-\frac{1}{p} + \frac{1}{2} + \frac{1}{p^*}} (\log n)^{p^*/4}), \end{aligned}$$

where, in handling the logarithmic term to validate the 3rd equality, we used $(1 - p/2)/p \leq 1/2$. Consequently, from (5.13) and the Gaussian concentration (3.7),

$$\max_i |\langle v_i, G_n v_i \rangle| = O_{\mathbb{P}}(n^{-\frac{1}{p} + \frac{1}{2} + \frac{1}{p^*}} (\log n)^{p^*/4} + \sqrt{\log n}). \quad (5.14)$$

To control the 3rd term of (5.10), note that $\max_i |\langle e_i, G_n e_i \rangle| \leq \max_i |g_{ii}|$ and $\mathbb{E} \max_i |g_{ii}| \leq \sqrt{2 \log n}$. Since $\max_i |g_{ii}|$ is 1-Lipschitz with respect to the $\|\cdot\|_2$ -norm, it follows that

$$\max_i |\langle e_i, G_n e_i \rangle| = O_{\mathbb{P}}(\sqrt{\log n}). \quad (5.15)$$

Finally, the 1st term of (5.10) can be handled as follows. Note that $\langle v_i, \bar{G}_n e_i \rangle = \|\bar{G}_n e_i\|_{p^*}$. Write

$$\min_i \|\bar{G}_n e_i\|_{p^*} \geq \min_i \mathbb{E} \|\bar{G}_n e_i\|_{p^*} - \max_i \left| \|\bar{G}_n e_i\|_{p^*} - \mathbb{E} \|\bar{G}_n e_i\|_{p^*} \right|.$$

Here, from (5.11),

$$\max_i \left| \|\bar{G}_n e_i\|_{p^*} - \mathbb{E} \|\bar{G}_n e_i\|_{p^*} \right| = O_{\mathbb{P}}(\sqrt{\log n}),$$

which together with (5.12) implies that

$$\min_i \|\bar{G}_n e_i\|_{p^*} = \xi_{p^*} n^{1/p^*} - O_{\mathbb{P}}(\sqrt{\log n}). \quad (5.16)$$

Finally, combining (5.16), (5.10), (5.14), and (5.15), we have

$$\min_i \langle v_i + e_i, G_n(v_i + e_i) \rangle = \sqrt{2} \xi_{p^*} n^{1/p^*} - O_{\mathbb{P}}(n^{-\frac{1}{p} + \frac{1}{2} + \frac{1}{p^*}} (\log n)^{p^*/4} + \sqrt{\log n}). \quad (5.17)$$

From the definition of v_i and the fact that $||1 + x|^p - 1 - |x|^p| \leq 2^p p |x|$,

$$\max_i |\|v_i + e_i\|_p^p - 2| = \max_i ||1 + v_i(i)|^p - 1 - |v_i(i)|^p| \leq 2^p p \max_i |v_i(i)|. \quad (5.18)$$

Hence, from (5.13),

$$\max_i |\|v_i + e_i\|_p^p - 2| = O_{\mathbb{P}}((\log n)^{p^*/(2p)} n^{-1/p}).$$

The desired lower bound (1.9) follows from (5.17) after dividing both sides by $\|v_i + e_i\|_p^2$. This completes our proof.

5.3 Proof of (1.8): the upper bound

By the virtue of (1.9), it remains to establish the upper bound. Let S be a non-empty subset of B_p^n and $\varepsilon = n^{-2/(pp^*)}$. For this ε and any $x \in S$, write according to (5.1), $x = x^o + x^\dagger$. Applying (5.2) yields

$$\mathbb{E} \max_{x \in S} \langle G_n x, x \rangle \leq \sqrt{2} \mathbb{E} \max_{x \in S} \langle x^o, \bar{G}_n x^\dagger \rangle + O(n^{2/p^*2} \sqrt{\log n}).$$

Since x^o and $x^\dagger$ have disjoint supports, $\|x^o\|_p^p + \|x^\dagger\|_p^p = \|x\|_p^p \leq 1$. After normalizing x^o and $x^\dagger$ by their ℓ_p -norms, we write

$$\begin{aligned} \mathbb{E} \max_{x \in S} \langle x^o, \bar{G}_n x^\dagger \rangle &= \mathbb{E} \max_{x \in S} \|x^o\|_p \|x^\dagger\|_p \left\langle \frac{x^o}{\|x^o\|_p}, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle \\ &\leq 2^{-2/p} \mathbb{E} \max_{x \in S} \left\langle \frac{x^o}{\|x^o\|_p}, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle \leq 2^{-2/p} \mathbb{E} \max_{x \in S, y \in B_p^n} \left\langle y, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle, \end{aligned}$$

where we used the fact that the product $\|x^o\|_p \|x^\dagger\|_p$ is maximized if $\|x^o\|_p^p = \|x^\dagger\|_p^p = 1/2$. Denote $L^\# = \{x^\dagger / \|x^\dagger\|_p : x \in S\}$. Since $\|x^\dagger\|_0 \leq \varepsilon^{-p}$ for all $x \in S$, Proposition 5.2 yields

$$\begin{aligned} \mathbb{E} \max_{x \in S, y \in B_p^n} \left\langle y, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle &\leq r(L^\#) \mathbb{E} \|g\|_{p^*} + O((n^{p/(2p^*)} + n^{2/p^*}) \sqrt{\log n}) \\ &= r(L^\#) \mathbb{E} \|g\|_{p^*} + O(n^{p/(2p^*)} \sqrt{\log n}), \end{aligned}$$

the last equality being a consequence of the fact $pp^* \geq 4$. Combining the above estimates, we arrive at

$$\mathbb{E} \max_{x \in S} \langle G_n x, x \rangle \leq 2^{1/2} 2^{-2/p} r(L^\#) \mathbb{E} \|g\|_{p^*} + O(n^{p/(2p^*)} \sqrt{\log n}). \quad (5.19)$$

Since $\|x^\dagger\|_2 \leq \|x^\dagger\|_p$, we have $r(L^\#) \leq 1$. Consequently, using (3.3) in (5.19), we obtain that

$$\mathbb{E} \max_{\|x\|_p=1} \langle G_n x, x \rangle \leq 2^{1/2-2/p} \xi_{p^*} n^{1/p^*} + O(n^{p/(2p^*)} \sqrt{\log n}).$$

Now, the Gaussian concentration (3.7) ensures that

$$\max_{\|x\|_p=1} \langle G_n x, x \rangle = 2^{1/2-2/p} \xi_{p^*} n^{1/p^*} + O_{\mathbb{P}}(n^{p/(2p^*)} \sqrt{\log n}), \quad (5.20)$$

where the notion $O_{\mathbb{P}}$ is defined in (5.9). Finally, the upper bound (5.20), combined with the lower bound (1.9) and the fact that $p < 2$, implies (1.8).

6 Proof of Theorem 1.4: The Stability Bound

For each $i \in [n]$, define three vectors,

$$o_i = \frac{v_i + e_i}{\|v_i + e_i\|_p}, \quad o_i^o = \frac{v_i - v_i(i)e_i}{\|v_i + e_i\|_p}, \quad o_i^\dagger = \frac{(1 + v_i(i))e_i}{\|v_i + e_i\|_p}.$$

Obviously, $o_i^\dagger / \|o_i^\dagger\|_p = e_i$. Note that a direct computation gives

$$\begin{aligned} \|o_i^\dagger\|_p &= \frac{|1 + v_i(i)|}{\|v_i + e_i\|_p}, \quad \|o_i^o\|_p = \frac{(1 - |v_i(i)|^p)^{1/p}}{\|v_i + e_i\|_p} \quad \text{and} \\ \left\| \frac{o_i^o}{\|o_i^o\|_p} - v_i \right\|_p &= \left\| \frac{v_i - v_i(i)e_i}{\|v_i - v_i(i)e_i\|_p} - v_i \right\|_p = \left((1 - (1 - |v_i(i)|^p)^{1/p})^p + |v_i(i)|^p \right)^{1/p} \\ &\leq 1 - (1 - |v_i(i)|^p)^{1/p} + |v_i(i)| \leq 2|v_i(i)|. \end{aligned}$$

The above results and (5.18) imply that if $2^p p \max_i \|v_i\|_\infty \leq 1$, then there exists a constant $K \geq 1$ such that

$$\max \left(\max_i \left| \|o_i^o\|_p - 2^{-1/p} \right|, \max_i \left| \|o_i^\dagger\|_p - 2^{-1/p} \right|, \max_i \left\| \frac{o_i^o}{\|o_i^o\|_p} - v_i \right\|_p \right) \leq K \max_i \|v_i\|_\infty. \quad (6.1)$$

Next, in view of the proof of (1.8), we continue to work with the same $\varepsilon = n^{-2/(pp^*)}$. Fix $0 < \kappa < 1/(3p)$, and let $n^{-\kappa} \leq \delta \leq 1$. Let $M \geq 1$ set

$$\underline{\delta} = \frac{1}{100M} \min(\delta, 2^{-\frac{1}{2p}}). \quad (6.2)$$

Define $A_n = \{\max_i \|v_i\|_\infty \leq \min(\varepsilon, \underline{\delta}^3/K)\}$. Since $\kappa < 1/(3p)$ and $p^* > 2$, using (5.13) yields that for any $d > 0$, there exists a universal constant $C > 0$ such that

$$\mathbb{P}(A_n) \geq 1 - Cn^{-d}. \quad (6.3)$$

On the event A_n , we readily see that $2^p p \max_i \|v_i\|_\infty \leq 8 \max_i \|v_i\|_\infty \leq 1$, and from (6.1),

$$\max_i \left| \|o_i^o\|_p - 2^{-1/p} \right| \leq \underline{\delta}^3, \quad (6.4)$$

$$\max_i \left| \|o_i^\dagger\|_p - 2^{-1/p} \right| \leq \underline{\delta}^3, \quad (6.5)$$

$$\max_i \left\| \frac{o_i^o}{\|o_i^o\|_p} - v_i \right\|_p \leq \underline{\delta}^3. \quad (6.6)$$

Let $S = \{x : \|x\|_p = 1, \text{dist}(x, \mathcal{O}) \geq \delta\}$. Define $S_1 = \{x : \|x\|_p = 1, \left| \|x^\dagger\|_p - 2^{-1/p} \right| \geq \underline{\delta}^3\}$ and

$$S_2 = \{x : \|x\|_p = 1, \left| \|x^\dagger\|_p - 2^{-1/p} \right| \leq \underline{\delta}^3, \min(\left\| \frac{x^\dagger}{\|x^\dagger\|_p} - e_i \right\|_p, \left\| \frac{x^\dagger}{\|x^\dagger\|_p} + e_i \right\|_p) \geq \underline{\delta}^3 \text{ for all } i\}.$$

Also, let S_3 be the collection of all $x \in S$ satisfying that $\left| \|x^\dagger\|_p - 2^{-1/p} \right| \leq \underline{\delta}^3$ and there exists some i such that either

$$\left(\left\| \frac{x^\dagger}{\|x^\dagger\|_p} - e_i \right\|_p \leq \underline{\delta}^3, \left\| \frac{x^o}{\|x^o\|_p} - v_i \right\|_p \geq \underline{\delta} \right) \text{ or } \left(\left\| \frac{x^\dagger}{\|x^\dagger\|_p} + e_i \right\|_p \leq \underline{\delta}^3, \left\| \frac{x^o}{\|x^o\|_p} + v_i \right\|_p \geq \underline{\delta} \right). \quad (6.7)$$

Lemma 6.1. For any $M \geq 1$, on the event A_n , we have that $S \subseteq S_1 \cup S_2 \cup S_3$.

Proof. Recall that, on A_n , Inequalities (6.4)–(6.6) are valid. Take $x \in S \setminus (S_1 \cup S_2)$. Then,

$$|\|x^\dagger\|_p - 2^{-1/p}| \leq \underline{\delta}^3 \quad \text{and} \quad (6.8)$$

$$\text{either } \left\| \frac{x^\dagger}{\|x^\dagger\|_p} - e_i \right\|_p \leq \underline{\delta}^3 \quad \text{or} \quad \left\| \frac{x^\dagger}{\|x^\dagger\|_p} + e_i \right\|_p \leq \underline{\delta}^3 \quad \text{for some } i. \quad (6.9)$$

Assume, for definiteness, that the former condition of (6.9) is valid for some i . To show that $x \in S_3$, we verify that $\|x^o/\|x^o\|_p - v_i\|_p \geq \underline{\delta}$ as follows. First, note that from (6.5) and (6.8),

$$|\|x^\dagger\|_p - \|o_i^\dagger\|_p| \leq 2\underline{\delta}^3.$$

We claim that $\|x^\dagger - o_i^\dagger\|_p \leq 3\underline{\delta}^3$. If $\|x^\dagger - o_i^\dagger\|_p > 3\underline{\delta}^3$, we use the fact that $\|x^\dagger\|_p \leq 1$ and the triangle inequality to deduce

$$\begin{aligned} \left\| \frac{x^\dagger}{\|x^\dagger\|_p} - e_i \right\|_p &= \left\| \frac{x^\dagger}{\|x^\dagger\|_p} - \frac{o_i^\dagger}{\|o_i^\dagger\|_p} \right\|_p \geq \frac{\|x^\dagger - o_i^\dagger\|_p}{\|x^\dagger\|_p} - \|o_i^\dagger\|_p \left| \frac{1}{\|x^\dagger\|_p} - \frac{1}{\|o_i^\dagger\|_p} \right| \\ &= \frac{1}{\|x^\dagger\|_p} (\|x^\dagger - o_i^\dagger\|_p - \|x^\dagger\|_p + \|o_i^\dagger\|_p) > 3\underline{\delta}^3 - 2\underline{\delta}^3 = \underline{\delta}^3, \end{aligned}$$

which contradicts our assumption that $\|x^\dagger/\|x^\dagger\|_p - e_i\|_p \leq \underline{\delta}^3$. This validates our claim. Now, since $x \in S$, we have that $\|x - o_i\|_p \geq \underline{\delta}$, which, coupled with the fact $\|x^\dagger - o_i^\dagger\|_p \leq 3\underline{\delta}^3$, leads to

$$\|x^o - o_i^o\|_p \geq \|x - o_i\|_p - \|x^\dagger - o_i^\dagger\|_p \geq \underline{\delta} - 3\underline{\delta}^3. \quad (6.10)$$

On the other hand, it follows from (6.8) that

$$|\|x^o\|_p - 2^{-1/p}| \leq 16\underline{\delta}^3. \quad (6.11)$$

Indeed, a straightforward computation using (6.8), (6.2), and the fact $1 = \|x\|_p^p = \|x^o\|_p^p + \|x^\dagger\|_p^p$ shows $\|x^o\|_p \geq 4^{-1}$. Now, by applying the mean value theorem on the function $t \mapsto t^p$ twice and using (6.8), the assertion (6.11) follows since

$$4^{-1} |\|x^o\|_p - 2^{-1/p}| \leq |\|x^o\|_p^p - 2^{-1}| = |\|x^\dagger\|_p^p - 2^{-1}| \leq 2|\|x^\dagger\|_p - 2^{-1/p}| \leq 2\underline{\delta}^3.$$

Now, from (6.4), (6.10), (6.11), and the fact that $\|x^o\|_p \leq 1$, we obtain that

$$\begin{aligned} \left\| \frac{x^o}{\|x^o\|_p} - \frac{o_i^o}{\|o_i^o\|_p} \right\|_p &\geq \frac{1}{\|x^o\|_p} (\|x^o - o_i^o\|_p - |\|x^o\|_p - \|o_i^o\|_p|) \\ &\geq \frac{1}{\|x^o\|_p} (\|x^o - o_i^o\|_p - |\|x^o\|_p - 2^{-1/p}| - |2^{-1/p} - \|o_i^o\|_p|) \geq \delta - 20\underline{\delta}^3. \end{aligned} \quad (6.12)$$

Moreover, another application of triangle inequality on (6.6) and (6.12) and the fact that $\delta \geq 100\underline{\delta}$ yield

$$\left\| \frac{x^o}{\|x^o\|_p} - v_i \right\|_p \geq \delta - 21\underline{\delta}^3 \geq \underline{\delta}^3.$$

In the case that the 2nd condition of (6.9) holds, we can similarly show, by using (6.5) and (6.6), that $\|x^o/\|x^o\|_p + v_i\|_p \geq \underline{\delta}$. Therefore, $x \in S_3$ and the assertion is established. $\blacksquare$

Lemma 6.2. For any $d > 0$, there exist constants $M \geq 1$ and $C, C' > 0$ such for all sufficiently large n , with probability at least $1 - 8n^{-d}$, we have that whenever $n^{-\kappa} < \delta \leq 1$,

$$\max_{x \in S_1 \cup S_2 \cup S_3} \langle G_n x, x \rangle \leq 2^{1/2-2/p} (1 - C\underline{\delta}^6) \xi_{p^*} n^{1/p^*} + C' n^{p/(2p^*)} \sqrt{\log n}. \quad (6.13)$$

Proof. Throughout this proof, the constants C_1, C_2 , etc. are independent of δ and n . Let $d > 0$. Below, we handle each of the cases $x \in S_i$ for $i = 1, 2, 3$ separately. Together, they will complete the proof of the lemma.

Maximization over S_1 . Note that $|1 - a^p| \geq |1 - a|$ for all $a > 0$. For $x \in S_1$, since $|\|x^\dagger\|_p - 2^{-1/p}| \geq \underline{\delta}^3$, we see that

$$|\|x^\dagger\|_p^p - 2^{-1}| \geq 2^{-1+1/p} |\|x\|_p - 2^{-1/p}| \geq 2^{-1/2} |\|x\|_p - 2^{-1/p}| \geq \frac{\underline{\delta}^3}{2}.$$

Hence,

$$\|x^o\|_p \|x^\dagger\|_p = (\|x^\dagger\|_p^p (1 - \|x^\dagger\|_p^p))^{1/p} \leq \left(\frac{1}{4} - \frac{\underline{\delta}^6}{4}\right)^{1/p} \leq 2^{-2/p} (1 - \underline{\delta}^6).$$

By following the argument used in the derivation in (5.19) and noting that $r(\{x^\dagger/\|x^\dagger\|_p : x \in S_1\}) \leq 1$, we obtain

$$\mathbb{E} \max_{x \in S_1} \langle G_n x, x \rangle \leq 2^{1/2-2/p} (1 - \underline{\delta}^6) \xi_{p^*} n^{1/p^*} + O(n^{p/(2p^*)} \sqrt{\log n}).$$

Note that S_1 is nonrandom. We can apply (3.7) with $t = \sqrt{2d \log n}$ to get that there exists some $C_1 > 0$ such that the following event is valid with probability at least $1 - 2n^{-d}$,

$$\max_{x \in S_1} \langle G_n x, x \rangle \leq 2^{1/2-2/p} (1 - \underline{\delta}^6) \xi_{p^*} n^{1/p^*} + C_1 n^{p/(2p^*)} \sqrt{\log n}.$$

Maximization over S_2 . For $x \in S_2$, $\|x^\dagger/\|x^\dagger\|_p \mp e_i\|_p \geq \underline{\delta}^3$ for all i . By Lemma A.2, there exists a constant $C_2 > 0$ such that

$$r(\{x^\dagger/\|x^\dagger\|_p : x \in S_2\}) \leq 1 - C_2 \underline{\delta}^{3p} \leq 1 - C_2 \underline{\delta}^6.$$

On the other hand, the product $\|x^o\|_p \|x^\dagger\|_p$ can be trivially bounded above by $1/4$. Hence, (5.19) yields

$$\mathbb{E} \max_{x \in S_2} \langle G_n x, x \rangle \leq 2^{1/2-2/p} (1 - C_2 \underline{\delta}^6) \xi_{p^*} n^{1/p^*} + O(n^{p/(2p^*)} \sqrt{\log n}).$$

Similar to the 1st case, as S_2 is also nonrandom, we can apply (3.7) with $t = \sqrt{2d \log n}$ to get that there exists some $C_3 > 0$ such that the following event is valid with probability at least $1 - 2n^{-d}$:

$$\max_{x \in S_2} \langle G_n x, x \rangle \leq 2^{1/2-2/p} (1 - C_2 \underline{\delta}^6) \xi_{p^*} n^{1/p^*} + C_3 n^{p/(2p^*)} \sqrt{\log n}.$$

Maximization over S_3 . This case requires some extra treatments since S_3 is a random set. If (6.7) is satisfied, then

$$\begin{aligned} \max_{x \in S_3} \left\langle \frac{x^o}{\|x^o\|_p}, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle &\leq \max_i \max_{\substack{\|u\|_p=1, \|u-e_i\|_p \leq \underline{\delta}^3, \\ \|v\|_p=1, \|v-v_i\|_p \geq \underline{\delta}}} \langle v, \bar{G}_n u \rangle \\ &\leq \max_i \max_{\|v\|_p=1, \|v-v_i\|_p \geq \underline{\delta}} \langle v, \bar{G}_n e_i \rangle + \underline{\delta}^3 \|\bar{G}_n\|_{p \rightarrow p^*} \\ &\leq \left(1 - \frac{p^2 \underline{\delta}^2}{16p^*}\right) \max_i \|\bar{G}_n e_i\|_{p^*} + \underline{\delta}^3 \|\bar{G}_n\|_{p \rightarrow p^*}, \end{aligned}$$

where the 3rd inequality used Lemma A.1. Now, the fact that $\sup_{x \in B_n^p} \|x^o\|_p \|x^\dagger\|_p \leq 2^{-2/p}$ implies that

$$\begin{aligned} \max_{x \in S_3} \langle x^o, \bar{G}_n x^\dagger \rangle &= \max_{x \in S_3} \|x^o\|_p \|x^\dagger\|_p \left\langle \frac{x^o}{\|x^o\|_p}, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle \\ &\leq 2^{-2/p} \left(\left(1 - \frac{p^2 \underline{\delta}^2}{16p^*}\right) \max_i \|\bar{G}_n e_i\|_{p^*} + \underline{\delta}^3 \|\bar{G}_n\|_{p \rightarrow p^*} \right). \end{aligned}$$

To control the right-hand side, note that $\|\bar{G}_n e_i\|_{p^*} - \mathbb{E} \|\bar{G}_n e_i\|_{p^*}$ is 2-subgaussian. Lemma 3.11 yields that

$$\mathbb{E} \max_i \|\bar{G}_n e_i\|_{p^*} \leq \max_i \mathbb{E} \|\bar{G}_n e_i\|_{p^*} + 2\sqrt{\log n} \leq \xi_{p^*} n^{1/p^*} + O(\sqrt{\log n}).$$

Moreover, $\mathbb{E} \|\bar{G}_n\|_{p \rightarrow p^*} = O(n^{1/p^*})$ by (3.4). Using (3.7) with $t = \sqrt{2(d+1) \log n}$ and (3.8) with $t = \sqrt{2d \log n}$, we see that there exist universal positive constants C_4 and C_5 such that with probability at least $1 - 4n^{-d}$,

$$\max_i \|\bar{G}_n e_i\|_{p^*} \leq \xi_{p^*} n^{1/p^*} + C_4 \sqrt{\log n} \quad \text{and} \quad \|\bar{G}_n\|_{p \rightarrow p^*} \leq C_5 n^{1/p^*}.$$

From these estimates, we can choose $M \geq 1$ large enough such that the following statement holds: there exist constants $C_6, C_7 > 0$ such that with probability at least $1 - 4n^{-d}$, we have that whenever $n^{-\kappa} < \delta \leq 1$,

$$\max_{x \in S_3} \left\langle \frac{x^o}{\|x^o\|_p}, \bar{G}_n \frac{x^\dagger}{\|x^\dagger\|_p} \right\rangle \leq 2^{-2/p} (1 - C_6 \underline{\delta}^2) \xi_{p^*} n^{1/p^*} + C_7 n^{p/(2p^*)} \sqrt{\log n}.$$

Plugging this into (5.3), we arrive at that with probability at least $1 - 4n^{-d}$,

$$\begin{aligned} \max_{x \in S_3} \langle G_n x, x \rangle &\leq 2^{1/2-2/p} (1 - C_6 \underline{\delta}^2) \xi_{p^*} n^{1/p^*} + C_7 n^{2/p^*} \sqrt{\log n} \\ &\leq 2^{1/2-2/p} (1 - C_6 \underline{\delta}^6) \xi_{p^*} n^{1/p^*} + C_7 n^{p/(2p^*)} \sqrt{\log n}, \end{aligned}$$

where the 2nd inequality used $\underline{\delta} \leq 1$ and $pp^* \geq 4$. ■

We now proceed to complete the proof of Theorem 1.4. Let $d > 0$. Denote by A'_n the event on which (6.13) is valid. By Lemmas 6.1 and 6.2, we have that on the event

$A_n \cap A'_n$, for all $n^{-\kappa} < \delta \leq 1$,

$$\max_{x \in S} \langle G_n x, x \rangle \leq \max_{x \in S_1 \cup S_2 \cup S_3} \langle G_n x, x \rangle \leq 2^{1/2-2/p} (1 - C\delta^6) \xi_{p^*} n^{1/p^*} + C' n^{p/(2p^*)} \sqrt{\log n}.$$

Here, from (6.3) and Lemma 6.2, $\mathbb{P}(A_n \cap A'_n) \geq 1 - \mathbb{P}(A_n^c) - \mathbb{P}(A_n'^c) \geq 1 - (8 + C)n^{-d}$. This completes our proof.

7 Proof of Theorem 1.5

7.1 Proof of (1.14)

For $t > 0$ and $x \in \mathbb{R}^n$, set

$$H_{n,p,t}(x) = \frac{\langle G_n x, x \rangle}{n^{1/2}} - t \|x\|_p^p. \quad (7.1)$$

Let $u, t > 0$ be fixed. Note that $L_{n,p,u}(t) = n^{-1} \max_{\|x\|_2^2 = u} H_{n,p,t}(x)$. We claim that $n\mathbb{E}L_{n,p,u}(t)$ is superadditive, that is,

$$n\mathbb{E}L_{n,p,u}(t) \geq n_1\mathbb{E}L_{n_1,p,u}(t) + n_2\mathbb{E}L_{n_2,p,u}(t), \quad \forall n_1, n_2 \in \mathbb{N} \text{ and } n = n_1 + n_2. \quad (7.2)$$

To prove this, we use Guerra–Toninelli’s interpolation [30]. Let $n_1, n_2, n \in \mathbb{N}$ satisfy $n = n_1 + n_2$. For $0 \leq s \leq 1$, consider the interpolating Hamiltonian,

$$H_{n,s}(z) = \sqrt{\frac{s}{n}} \langle G_n z, z \rangle + \sqrt{\frac{1-s}{n_1}} \langle G_{n_1}^1 x, x \rangle + \sqrt{\frac{1-s}{n_2}} \langle G_{n_2}^2 y, y \rangle - t \|x\|_p^p - t \|y\|_p^p$$

for any $z = (x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $G_{n_1}^1 = (g_{ij}^1)_{i,j \in [n_1]}$ and $G_{n_2}^2 = (g_{ij}^2)_{i,j \in [n_2]}$ are i.i.d. standard Gaussian. Let μ_{n_1} and μ_{n_2} be probability measures fully supported on the sphere $S_{n_1} = \{x \in \mathbb{R}^{n_1} : \|x\|_2^2 = u\}$ and $S_{n_2} = \{y \in \mathbb{R}^{n_2} : \|y\|_2^2 = u\}$, respectively. Here, note that the $\|x\|_2$ and $\|y\|_2$ are normalized by n_1 and n_2 , respectively. Define

$$F_{n_1, n_2, \beta}(s) = \frac{1}{\beta n} \mathbb{E} \log \int_{S_{n_1} \times S_{n_2}} e^{\beta H_{n,s}(z)} \mu_{n_1}(dx) \otimes \mu_{n_2}(dy). \quad (7.3)$$

Note that

$$F_{n_1, n_2, \beta}(1) = \frac{1}{\beta n} \mathbb{E} \log \int_{S_{n_1} \times S_{n_2}} e^{\frac{\beta}{\sqrt{n}} \langle G_n z, z \rangle} \mu_{n_1}(dx) \otimes \mu_{n_2}(dy),$$

$$F_{n_1, n_2, \beta}(0) = \frac{n_1}{\beta n} \mathbb{E} \log \int_{S_{n_1}} e^{\frac{\beta}{\sqrt{n_1}} \langle G_{n_1}^1 x, x \rangle} \mu_{n_1}(dx) + \frac{n_2}{\beta n} \mathbb{E} \log \int_{S_{n_2}} e^{\frac{\beta}{\sqrt{n_2}} \langle G_{n_2}^2 y, y \rangle} \mu_{n_2}(dy).$$

Denote by $R_k(a, b) = k^{-1} \sum_{i \in [k]} a_i b_i$ for any $k \geq 1$ and $a, b \in \mathbb{R}^k$. Note that $R_n(z, z) = R_{n_1}(x, x) = R_{n_2}(y, y) = u$ for $z = (x, y) \in S_{n_1} \times S_{n_2}$. Computing the derivative of (7.3) in s directly by using the Gaussian integration by parts and noting that $\|x\|_2^2 = un_1$ and $\|y\|_2^2 = un_2$, we obtain that

$$\begin{aligned} nF'_{n_1, n_2, \beta}(s) &= \frac{1}{2} \mathbb{E} \left\langle \frac{\langle G_n z, z \rangle}{\sqrt{s}} - \frac{\langle G_{n_1}^1 x, x \rangle}{\sqrt{1-s}} - \frac{\langle G_{n_2}^2 y, y \rangle}{\sqrt{1-s}} \right\rangle \\ &= \frac{\beta^2 n}{2} \mathbb{E} \langle R_n(z, z)^2 - R_n(z^1, z^2)^2 \rangle - \frac{\beta^2 n_1}{2} \mathbb{E} \langle R_{n_1}(x, x)^2 - R_{n_1}(x^1, x^2)^2 \rangle \\ &\quad - \frac{\beta^2 n_2}{2} \mathbb{E} \langle R_{n_2}(y, y)^2 - R_{n_2}(y^1, y^2)^2 \rangle \\ &= -\frac{\beta^2}{2} \mathbb{E} \langle n R_n(z^1, z^2)^2 - n_1 R_{n_1}(x^1, x^2)^2 - n_2 R_{n_2}(y^1, y^2)^2 \rangle, \end{aligned}$$

where the angle bracket outside is the expectation with respect to the Gibbs measure

$$G_{n_1, n_2, \beta}(dx, dy) = \frac{e^{\beta H_{n, s}(z)} \mu_{n_1}(dx) \otimes \mu_{n_2}(dy)}{\int e^{\beta H_{n, s}(z)} \mu_{n_1}(dx) \otimes \mu_{n_2}(dy)}.$$

and $z^1 = (x^1, y^1)$ and $z^2 = (x^2, y^2)$ are two independent samples from this measure. Now, note that from Jensen's inequality,

$$\begin{aligned} R_n(z^1, z^2)^2 &= \left(\frac{n_1}{n} \frac{1}{n_1} R_{n_1}(x^1, x^2) + \frac{n_2}{n} R_{n_2}(y^1, y^2) \right)^2 \\ &\leq \frac{n_1}{n} R_{n_1}(x^1, x^2)^2 + \frac{n_2}{n} R_{n_2}(y^1, y^2)^2, \end{aligned}$$

leading to $F'_{n_1, n_2, \beta}(s) \geq 0$ for all $s \in (0, 1)$. Hence, $F_{n_1, n_2, \beta}(1) \geq F_{n_1, n_2, \beta}(0)$ and sending β to infinity imply that

$$n \mathbb{E} L_{n, p, u}(t) \geq \mathbb{E} \max_{S_{n_1} \times S_{n_2}} H_{n, p, t}(z) \geq n_1 \mathbb{E} L_{n_1, p, u}(t) + n_2 \mathbb{E} L_{n_2, p, u}(t),$$

establishing our claim (7.2). So, $L_{p,u}(t) := \lim_{n \rightarrow \infty} \mathbb{E}[L_{n,p,u}(t)]$ exists by superadditivity. Hence, from the concentration bound (3.10), it follows that $\lim_{n \rightarrow \infty} L_{n,p,u}(t) = \lim_{n \rightarrow \infty} \mathbb{E}[L_{n,p,u}(t)] = L_{p,u}(t)$ exists almost surely for each $u > 0$ and $t > 0$. It remains to show that the statement that almost surely this limit exists simultaneously for all $u > 0$ and $t > 0$. We need the following lemma.

Lemma 7.1. For any $0 < K_1 < K_2$, there exist positive constants C and D such that with probability at least $1 - Ce^{-n/C}$,

$$|L_{n,p,u}(t) - L_{n,p,u'}(t')| \leq D(|u - u'| + |t - t'|), \quad \forall u, u', t, t' \in [K_1, K_2]. \quad (7.4)$$

From the above inequality and the Borel–Cantelli lemma, it follows that for any $0 < K_1 < K_2$, there exists some $D = D(K_1, K_2) > 0$ such that almost surely

$$\limsup_n |L_{n,p,u}(t) - L_{n,p,u'}(t')| \leq D(|u - u'| + |t - t'|), \quad \forall u, u', t, t' \in [K_1, K_2]. \quad (7.5)$$

The above inequality also implies that

$$|L_{p,u}(t) - L_{p,u'}(t')| \leq D(|u - u'| + |t - t'|), \quad \forall u, u', t, t' \in [K_1, K_2]. \quad (7.6)$$

For any $u, t \in \mathbb{Q} \cap (0, \infty)$, let $\Omega(u, t)$ be the event on which $\lim_{n \rightarrow \infty} L_{n,p,u}(t)$ exists. Also, for any $K_1, K_2 \in \mathbb{Q} \cap (0, \infty)$ with $K_1 < K_2$, let $\Omega(K_1, K_2)$ be the event on which Inequality (7.5) holds. Let Ω be the intersection of all $\Omega(u, t)$ and $\Omega(K_1, K_2)$. Obviously, Ω is of probability one. Moreover, on Ω , for any $t > 0$ and $u > 0$ with $u, t \in [K^{-1}, K]$ for some rational $K > 0$, we have, using (7.5) and (7.6), that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} |L_{n,p,u}(t) - L_{p,u}(t)| \\ & \leq \limsup_{n \rightarrow \infty} |L_{n,p,u}(t) - L_{n,p,u'}(t')| + \limsup_{n \rightarrow \infty} |L_{n,p,u'}(t') - L_{p,u'}(t')| + |L_{p,u'}(t') - L_{p,u}(t)| \\ & \leq 2D'(|u - u'| + |t - t'|) \end{aligned}$$

for any $u', t' \in \mathbb{Q} \cap [K^{-1}, K]$, where D' is a constant depending only on K . Sending $u' \rightarrow u$ and $t' \rightarrow t$ implies that $\lim_{n \rightarrow \infty} L_{n,p,u}(t) = L_{p,u}(t)$ on Ω . This completes the proof of (1.14).

Proof of Lemma 7.1. Let $0 < K_1 < K_2$ be fixed. Note that a change of variable, $x \rightarrow x/\sqrt{u}$, yields that

$$L_{n,p,u}(t) = \max_{\|x\|_2=1} \left(\frac{u}{n^{3/2}} \langle G_n x, x \rangle - tu^{p/2} \|x\|_p^p \right). \quad (7.7)$$

Let $\mathbb{1} = (1, \dots, 1) \in \mathbb{R}^n$. Since

$$L_{n,p,u}(t) \geq \frac{u}{n^{3/2}} \langle G_n \mathbb{1}, \mathbb{1} \rangle - tu^{p/2} \|\mathbb{1}\|_p^p \geq -\frac{u \|G_n\|_2}{n^{1/2}} - tu^{p/2},$$

it follows that if x is an optimizer of (7.7), then

$$\|x\|_p^p \leq \frac{u^{1-p/2}}{tn^{3/2}} \langle G_n x, x \rangle + \frac{u^{1-p/2} \|G_n\|_2}{tn^{1/2}} + 1 \leq \frac{2u^{1-p/2} \|G_n\|_2}{tn^{1/2}} + 1 \leq M_n \quad (7.8)$$

for $M_n := 2K_1^{-p/2} n^{-1/2} \|G_n\|_2 + 1$. Consequently, $L_{n,p,u}(t)$ can be written as the same supremum as (7.7), but with the extra constraint (7.8). This implies that for any $u, u', t, t' \in [K_1, K_2]$,

$$\begin{aligned} |L_{n,p,u}(t) - L_{n,p,u'}(t')| &\leq \frac{\|G_n\|_2}{\sqrt{n}} |u - u'| + M_n |tu^{p/2} - t'u'^{p/2}| \\ &\leq \frac{\|G_n\|_2}{\sqrt{n}} |u - u'| + M_n |(t - t')u^{p/2} + t'(u^{p/2} - u'^{p/2})| \\ &\leq \frac{\|G_n\|_2}{\sqrt{n}} |u - u'| + M_n \left(K_2^{p/2} |t - t'| + \frac{pK_2^{p/2}}{2} |u - u'| \right). \end{aligned} \quad (7.9)$$

Since $G_n \mapsto \|G_n\|_2$ is 1-Lipschitz with respect to $\|\cdot\|_F$, It follows from the concentration Inequality (3.6) for $\|G_n\|_2$ and (3.4) that there exist some $C, C_* > 0$ such that

$$\mathbb{P}(n^{-1/2} \|G_n\|_2 \geq C_*) \leq Ce^{-n/C}. \quad (7.10)$$

The bound (7.4) is now a consequence of (7.9) and (11.21). ■

7.2 Proof of (1.15)

Lemma 7.2. There exist absolute constants $c_0, C > 0$ such that with probability at least $1 - Ce^{-n/C}$, for any $0 < K < t < \infty$,

$$L_{n,p}(t) = \sup_{u \in [0, c_0/K]} L_{n,p,u}(t).$$

Proof. Note that $L_{n,p}(t) \geq 0$. Recall the definition of $H_{n,p,t}(x)$ from (7.1). Whenever $H_{n,p,t}(x) \geq 0$ for some $x \in \mathbb{R}^n$, we have, by Jensen's inequality, that

$$t \|x\|_p^p \leq \frac{\langle G_n x, x \rangle}{n^{3/2}} \leq \frac{\|G_n\|_2}{\sqrt{n}} \|x\|_2^2 \leq \frac{\|G_n\|_2}{\sqrt{n}} \|x\|_p^2.$$

The above inequality, together with another application of Jensen's inequality, yields that

$$\|x\|_2 \leq \|x\|_p \leq \left(\frac{\|G_n\|_2}{t\sqrt{n}} \right)^{1/(p-2)} \leq \left(\frac{\|G_n\|_2}{K\sqrt{n}} \right)^{1/(p-2)}. \quad (7.11)$$

Consequently, we can write $L_{n,p}(t)$ as the supremum of $L_{n,p,u}(t)$ for all u between zero and the above upper bound. Using (11.21) completes our proof. ■

The proof of (1.15) is now argued as follows. From Lemma 7.2, there exist $c_0, C > 0$ such that with probability at least $1 - Ce^{-n/C}$, $L_{n,p}(t) = \sup_{0 \leq u \leq c_0/K} L_{n,p,u}(t)$ for any $0 < K \leq t < \infty$. Now, consider an arbitrary $0 < K < \min(\sqrt{c_0}, 1)$. From Lemma 7.1, there exist $C', D > 0$ such that with probability at least $1 - C'e^{-n/C'}$,

$$|L_{n,p,u}(t) - L_{n,p,u'}(t)| \leq D|u - u'|, \quad \forall u, u', t \in [K, c_0/K]. \quad (7.12)$$

Consequently, for any $k \geq 1$, if $u_0 < \dots < u_k$ form an equidistant partition of $[K, c_0/K]$, then with probability at least $1 - C'e^{-n/C'}$,

$$\left| \sup_{u \in [K, c_0/K]} L_{n,p,u}(t) - \sup_{0 \leq j \leq k} L_{n,p,u_j}(t) \right| \leq \frac{D}{k}(c_0/K - K), \quad \forall t \in [K, c_0/K].$$

This implies that with probability at least $1 - Ce^{-n/C} - C'e^{-n/C'}$,

$$\begin{aligned} \left| L_{n,p}(t) - \sup_{0 \leq j \leq k} L_{n,p,u_j}(t) \right| &\leq \sup_{u \in [0, K]} L_{n,p,u}(t) + \frac{D}{k}(c_0/K - K) \\ &\leq \frac{\|G_n\|_2 K}{\sqrt{n}} + \frac{D}{k}(c_0/K - K), \quad \forall t \in [K, c_0/K], \end{aligned} \quad (7.13)$$

where the last inequality used the fact that $L_{n,p,u}(t) \leq \|G_n\|_2 u / \sqrt{n}$. Combining the fact that $|L_{p,u}(t) - L_{p,u'}(t)| \leq D|u - u'|$ for all $u, u' \in [K, c_0/K]$, which follows from (7.12), and the bound (11.21) on $\|G\|_2$, we can now pass to the limit in (7.13) by first letting $n \rightarrow \infty$

and then letting $k \rightarrow \infty$ to obtain that almost surely, for all $t \in [K, c_0/K]$,

$$-C_*K + \sup_{u \in [K, c_0/K]} L_{p,u}(t) \leq \liminf_{n \rightarrow \infty} L_{n,p}(t) \leq \limsup_{n \rightarrow \infty} L_{n,p}(t) \leq \sup_{u \in [K, c_0/K]} L_{p,u}(t) + C_*K.$$

Since this is valid for all $0 < K < \sqrt{c_0}$, we can further send $K \downarrow 0$ to get that almost surely,

$$\lim_{K \downarrow 0} \sup_{u \in [K, c_0/K]} L_{p,u}(t) \leq \liminf_{n \rightarrow \infty} L_{n,p}(t) \leq \limsup_{n \rightarrow \infty} L_{n,p}(t) \leq \sup_{u \in [0, \infty)} L_{p,u}(t), \quad \forall t > 0. \quad (7.14)$$

Finally, note that for any $u \geq 0$ and $t > 0$, $L_{n,p}(t) \geq L_{n,p,u}(t)$. Also, almost surely, $\lim_{n \rightarrow \infty} L_{n,p,u}(t) = L_{p,u}(t)$ for all $t > 0$ and $u > 0$. From (7.13),

$$L_{p,u}(t) \leq \sup_{0 \leq j \leq k} L_{p,u_j}(t) + C_*K + \frac{D}{k}(c_0/K - K) \leq \sup_{[K, c_0/K]} L_{p,u}(t) + C_*K + \frac{D}{k}(c_0/K - K)$$

for all $t \in [K, c_0/K]$, $u \geq 0$, and $k \geq 1$. Hence,

$$\sup_{u \in [0, \infty)} L_{p,u}(t) \leq \sup_{[K, c_0/K]} L_{p,u}(t) + C_*K,$$

which, together with (7.14), completes our proof after sending $K \downarrow 0$.

8 Proof of Theorem 1.6

We establish the proof of Theorem 1.6 in this section. First, we need the following important lemma that connects $L_{n,p}$ and $\text{GP}_{n,p}$.

Lemma 8.1. Let $2 < p < \infty$ and $n \geq 1$. Then the following statements hold for any realization of G_n : if $L_{n,p}$ is differentiable at some $t \in (0, \infty)$ with $L'_{n,p}(t) < 0$, then

$$L_{n,p}(t) = -\left(\frac{p}{2} - 1\right)tL'_{n,p}(t) \quad (8.1)$$

and

$$\text{GP}_{n,p} = \frac{p}{2} \left(\frac{p}{2} - 1\right)^{2/p-1} t^{2/p} L_{n,p}(t)^{1-2/p}. \quad (8.2)$$

Remark 8.2. Though the differential Equation (8.1) has a simple explicit solution, it does not yield an expression for $L_{n,p}(t)$ in absence of a boundary condition.

Proof. Let t be a point of differentiability of $L_{n,p}$ with $L'_{n,p}(t) < 0$. Let $\varepsilon > 0$ be fixed. Then,

$$\begin{aligned} \max_{\|x\|_p^p \geq -L'_{n,p}(t) + \varepsilon} \frac{H_{n,p,t}(x)}{n} &\leq \max_{\|x\|_p^p \geq -L'_{n,p}(t) + \varepsilon} \left(\frac{H_{n,p,t}(x)}{n} + \lambda (\|x\|_p^p + L'_{n,p}(t) - \varepsilon) \right) \\ &\leq L_{n,p}(t - \lambda) + \lambda L'_{n,p}(t) - \lambda \varepsilon \\ &= \lambda \left(L'_{n,p}(t) - \frac{L_{n,p}(t) - L_{n,p}(t - \lambda)}{\lambda} \right) - \lambda \varepsilon + L_{n,p}(t) \end{aligned}$$

and

$$\begin{aligned} \max_{\|x\|_p^p \leq -L'_{n,p}(t) - \varepsilon} \frac{H_{n,p,t}(x)}{n} &\leq \max_{\|x\|_p^p \leq -L'_{n,p}(t) - \varepsilon} \left(\frac{H_{n,p,t}(x)}{n} + \lambda (-\|x\|_p^p - L'_{n,p}(t) - \varepsilon) \right) \\ &\leq L_{n,p}(t + \lambda) - \lambda L'_{n,p}(t) - \lambda \varepsilon \\ &= \lambda \left(\frac{L_{n,p}(t + \lambda) - L_{n,p}(t)}{\lambda} - L'_{n,p}(t) \right) - \lambda \varepsilon + L_{n,p}(t). \end{aligned}$$

From the above two inequalities, since $L_{n,p}$ is differentiable at t , we can choose λ small enough such that for some $c > 0$ (depending on ε)

$$\max_{\|x\|_p^p + L'_{n,p}(t) \geq \varepsilon} \frac{H_{n,p,t}(x)}{n} \leq L_{n,p}(t) - c\varepsilon. \quad (8.3)$$

In other words, if x satisfies $\|x\|_p^p + L'_{n,p}(t) \geq \varepsilon$, then it cannot be an optimizer of $L_{n,p}(t)$.

Now, to show (8.1), note that $\|x\|_p^p$ is differentiable on $\mathbb{R}^n$ since $p > 2$. If x is a critical point of $H_{n,p,t}$, then

$$\nabla H_{n,p,t}(x) = \nabla \left(\frac{1}{\sqrt{2}n^{3/2}} \langle \bar{G}_n x, x \rangle - t \|x\|_p^p \right) = \frac{2}{\sqrt{2}n^{3/2}} \bar{G}_n x - \left(\frac{pt}{n} |x_i|^{p-1} \text{sgn}(x_i) \right)_i = 0.$$

So, we have $\langle x, \nabla H_{n,p,t}(x) \rangle = 0$, which yields

$$\frac{2}{n^{3/2}} \langle G_n x, x \rangle = \frac{\sqrt{2}}{n^{3/2}} \langle \bar{G}_n x, x \rangle = pt \|x\|_p^p.$$

It follows that if $x(t)$ is an optimizer of $H_{n,p,t}$, then

$$L_{n,p}(t) = \left(\frac{p}{2} - 1 \right) t \|x(t)\|_p^p.$$

On the other hand, from (8.3), we also see that $|||x(t)|||_p^p + L'_{n,p}(t) < \varepsilon$ for all $\varepsilon > 0$, which implies that $|||x(t)|||_p^p = -L'_{n,p}(t)$. Plugging this into the above display results in (8.1).

Next, we establish (8.2). First of all, from (8.3),

$$\max_{|||x|||_p^p + L'_{n,p}(t) \leq \varepsilon} \frac{H_{n,p,t}(x)}{n} = L_{n,p}(t).$$

Since this is valid for all $\varepsilon > 0$, sending it to zero yields that

$$L_{n,p}(t) = \max_{|||x|||_p^p = -L'_{n,p}(t)} \frac{H_{n,p,t}(x)}{n} = \max_{|||x|||_p^p = -L'_{n,p}(t)} \frac{\langle G_n x, x \rangle}{n^{3/2}} + tL'_{n,p}(t).$$

Now, performing a change of variable $x \rightarrow x/(-L'_{n,p}(t))^{1/p}$ yields

$$\text{GP}_{n,p} = \frac{L_{n,p}(t) - tL'_{n,p}(t)}{(-L'_{n,p}(t))^{2/p}}.$$

Finally, by substituting $L'_{n,p}(t)$ using (8.1) implies (8.2). ■

Lemma 8.3. Let $2 < p < \infty$. We have that $L_p(t)$ is differentiable in $t \in (0, \infty)$. In addition, for all $t \in (0, \infty)$,

$$L_p(t) = -\left(\frac{p}{2} - 1\right)tL'_p(t) \quad (8.4)$$

and

$$L'_p(t) < 0. \quad (8.5)$$

Proof. To show differentiability of $L_p(t)$ and (8.4), we first fix a realization of $(G_n)_{n \geq 1}$ for which $L_{n,p}(t)$ converges to $L(t)$ for all $t \in (0, \infty)$. Note that $L_{n,p}$ and L_p are convex. Hence, both of them are differentiable almost everywhere on $(0, \infty)$. For any $0 < s' < s < s''$, define

$$D_{n,p}^-(s; s') = \frac{L_{n,p}(s) - L_{n,p}(s')}{s - s'}, \quad D_{n,p}^+(s; s'') = \frac{L_{n,p}(s'') - L_{n,p}(s)}{s'' - s}.$$

Note that for any $0 < t_1 < s < t_2$, if $L_{n,p}$ is differentiable at s , then

$$D_{n,p}^-(s; t_1) \leq L'_{n,p}(s) \leq D_{n,p}^+(s; t_2),$$

which combining with (8.1) implies that

$$-\left(\frac{p}{2} - 1\right) s D_{n,p}^+(s; t_2) \leq L_{n,p}(s) \leq -\left(\frac{p}{2} - 1\right) s D_{n,p}^-(s; t_1).$$

Since the set of points of differentiability of $L_{n,p}$ is dense in $(0, \infty)$, this inequality is indeed valid for all $0 < t_1 < t < t_2 < \infty$,

$$-\left(\frac{p}{2} - 1\right) t D_{n,p}^+(t; t_2) \leq L_{n,p}(t) \leq -\left(\frac{p}{2} - 1\right) t D_{n,p}^-(t; t_1).$$

By sending $n \rightarrow \infty$ and then letting $t_1 \uparrow t$ and $t_2 \downarrow t$, it follows from the above inequality that if L_p is differentiable at t , then (8.4) is valid at this point as well. To show that (8.4) holds for all $t > 0$, fix an arbitrary $t > 0$, and pick two monotonic sequences t_k and s_k belonging to the set of differentiable points of L_p such that $t_k \uparrow t$ and $s_k \downarrow t$. From the continuity and convexity of L_p and the validity of (8.4) on (t_k) and (s_k) ,

$$-(p/2 - 1)t \lim_{k \rightarrow \infty} L'_p(t_k) = L_p(t) = -(p/2 - 1)t \lim_{k \rightarrow \infty} L'_p(s_k),$$

which implies that $\lim_{k \rightarrow \infty} L'_p(t_k) = \lim_{k \rightarrow \infty} L'_p(s_k)$. From the convexity of L_p again, for any a, b satisfying $t_k < a < t < b < s_k$,

$$L'_p(t_k) \leq \frac{L_p(t) - L_p(a)}{t - a} \leq \frac{L_p(b) - L_p(t)}{b - t} \leq L'_p(s_k).$$

Consequently, we obtain the differentiability of L_p at t and this completes the proof of (8.4).

To show (8.5), note that $L'_p(t) \leq 0$ since L_p is nonincreasing. If $L'_p(t) = 0$, (8.4) implies that $L_p(t) = 0$. Let λ_1 be the top eigenvalue of $\bar{G}_n$ associated with the eigenvector x with $\|x\|_2 = \sqrt{n\delta}$ for some $0 < \delta < 1$. We can express x as $x/\|x\|_2 = g/\|g\|_2$ for a n -dimensional standard Gaussian vector g . Thus, using the law of large number and the fact that $\lambda_1/\sqrt{n} \rightarrow 2$ almost surely that we have almost sure lower bound,

$$L_{n,p}(t) \geq \frac{1}{n^{3/2}} \langle G_n x, x \rangle - t \|x\|_p^p = \frac{1}{\sqrt{2}n^{3/2}} \langle \bar{G}_n x, x \rangle - t \|x\|_p^p \rightarrow \sqrt{2}\delta - t\delta^{p/2} \mathbb{E}|z|^p,$$

where $z \sim N(0, 1)$. The lower bound is strictly positive if δ is taken to be small enough, contradicting $L_p(t) = 0$. Hence, $L'_p(t) < 0$. ■

Now, we turn to the proof of Theorem 1.6. For a given realization of $(G_n)_{n \geq 1}$ for which $\lim_{n \rightarrow \infty} L_{n,p}(t)$ converges to $L_p(t)$ for all $t \in (0, \infty)$, let $\Omega \subset (0, \infty)$ be the collection

of all points t , on which $L_{n,p}(t)$ is differentiable for all $n \geq 1$. Since L_p is differentiable everywhere, by Griffiths' lemma, we have that $\lim_{n \rightarrow \infty} L'_{n,p}(t) = L'_p(t)$ at an arbitrary $t \in \Omega$, which, together with (8.5), implies that $L'_{n,p}(t) < 0$ as long as n is large enough. Thus, we can pass to the limit in (8.2) to get that

$$\lim_{n \rightarrow \infty} \text{GP}_{n,p} = \frac{p}{2} \left(\frac{p}{2} - 1 \right)^{2/p-1} t^{2/p} L_p(t)^{1-2/p}.$$

Since this is valid for all $t \in \Omega$ and this set is dense in $(0, \infty)$ since $L_{n,p}$ is convex for all $n \geq 1$, this equation must also be valid on $(0, \infty)$ by using the continuity of L_p and this completes the proof of (1.16).

Next, we are going to prove (1.17). Assume that x_* is an optimization of $\sup_{\|x\|_p=1} \langle G_n x, x \rangle$. If we let $x_0 = n^{1/p} x_*$, then x_0 is an optimizer of $\sup_{\|x\|_p=1} \langle G_n x, x \rangle$. Note that as will be argued later in the proof of Lemma 9.2, x_0 must satisfy Inequality (9.3) with $\delta = p/2 - 1$. As a result,

$$\frac{1}{n} \|x_0\|_\infty^{p+\delta} \leq \frac{1}{n} \sum_{i \in [n]} |x_{0,i}|^{p+\delta} = \|x_0\|_{p+\delta}^{p+\delta} \leq \frac{1}{\sqrt{2} \text{GP}_{n,p}} \frac{\|\bar{G}_n\|_2}{n^{1/2}}.$$

From (3.9) and (11.21), we see that there exist constants $C, C' > 0$ depending only on p such that with probability at least $1 - C'e^{-n/C'}$, $\|x_0\|_\infty \leq Cn^{1/(p+\delta)}$. This implies that $\|x_*\|_\infty \leq Cn^{1/(p+\delta)-1/p}$.

9 Proof of Theorem 1.7

In this section, we establish the continuity of GP_p in $p \in [2, \infty]$. First, we do a change of variable to obtain

$$\text{GP}_{n,p} = \max_{\|x\|_p=1} \frac{1}{n^{3/2}} \langle G_n x, x \rangle = \max_{\|x\|_p \leq 1} \frac{1}{n^{3/2}} \langle G_n x, x \rangle. \quad (9.1)$$

The following lemma establishes the monotonicity of GP_p .

Lemma 9.1. For any $2 \leq p \leq p' \leq \infty$, we have that $\text{GP}_p \geq \text{GP}_{p'}$.

Proof. In view of (1.16) and (9.1), the assertion follows immediately from the monotonicity of the averaged norm $\|x\|_p$ in p due to Jensen's inequality. ■

We divide our discussion into three cases: $p = 2$, $2 < p < \infty$, and $p = \infty$.

9.1 Case $p = 2$

Note that $\text{GP}_2 \geq \text{GP}_p$ for all $p \geq 2$. It suffices to show that $\text{GP}_2 \leq \lim_{p \downarrow 2} \text{GP}_p$. Write

$$\text{GP}_{n,2} = \max_{\|x\|_2=1} \frac{1}{n^{3/2}} \langle G_n x, x \rangle = \frac{1}{\sqrt{2}} \max_{\|x\|_2=1} \frac{1}{n^{3/2}} \langle \bar{G}_n x, x \rangle.$$

Here, the maximum on the right-hand side is the largest eigenvalue of the GOE, $\bar{G}_n/\sqrt{n}$, and the corresponding eigenvector y with $\|y\|_2 = 1$ has the distribution $g/\|g\|_2$ for g an n -dimensional standard Gaussian vector. For all $p \geq 2$, since $\lim_{n \rightarrow \infty} \|y\|_p^p = \mathbb{E}|z|^p$ almost surely for $z \sim N(0, 1)$ by the strong law of large number. Therefore, for any $\varepsilon > 0$,

$$\begin{aligned} \text{GP}_2 &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2}} \frac{1}{n^{3/2}} \langle \bar{G}_n y, y \rangle \\ &\leq \frac{1}{\sqrt{2}} \limsup_{n \rightarrow \infty} \max_{\|x\|_p^p - \mathbb{E}|z|^p < \varepsilon} \frac{1}{n^{3/2}} \left\langle \bar{G}_n \left(\frac{x}{\|x\|_p} \right), \left(\frac{x}{\|x\|_p} \right) \right\rangle \|x\|_p^2 \\ &\leq \frac{1}{\sqrt{2}} (\mathbb{E}|z|^p + \varepsilon)^{2/p} \limsup_{n \rightarrow \infty} \max_{\|x\|_p^p - \mathbb{E}|z|^p < \varepsilon} \frac{1}{n^{3/2}} \left\langle \bar{G}_n \left(\frac{x}{\|x\|_p} \right), \left(\frac{x}{\|x\|_p} \right) \right\rangle \\ &\leq (\mathbb{E}|z|^p + \varepsilon)^{2/p} \text{GP}_p. \end{aligned}$$

Hence, $\text{GP}_2 \leq \lim_{p \downarrow 2} \text{GP}_p$ and this completes our proof.

9.2 Case $2 < p < \infty$

To discuss this case, we need a key lemma, which states that for all $\delta > 0$ with $2(1 + \delta) < p$, we can essentially rewrite $\text{GP}_{n,p}$ as a maximization problem restricted to $x \in \mathbb{R}^n$ satisfying $\|x\|_p = 1$ and $\|x\|_{p+\delta}^{p+\delta} \leq \text{GP}_2/\text{GP}_\infty$. In other words, this means that the optimizer of $\text{GP}_{n,p}$ has a bounded norm of slightly higher order than p .

Lemma 9.2. Let $p \in (2, \infty)$. For any $\delta > 0$ satisfying $2(1 + \delta) \leq p$, we have that

$$\text{GP}_p = \lim_{n \rightarrow \infty} \max_{x \in A_{n,p,\delta}} \frac{1}{n^{3/2}} \langle G_n x, x \rangle,$$

where $A_{n,p,\delta}$ is the collection of all $x \in \mathbb{R}^n$ satisfying that $\|x\|_p = 1$ and $\|x\|_{p+\delta}^{p+\delta} \leq \text{GP}_2/\text{GP}_\infty$.

Proof. Assume that $x_0 = (x_{0,1}, \dots, x_{0,n}) \in \mathbb{R}^n$ with $\|x_0\|_p = 1$ is an optimizer of the 1st maximization problem in (9.1). Notice that $\sum_{i \in [n]} |x_i|^p$ is twice differentiable since $p > 2$.

Using the Lagrange multiplier, there exists some $c \in \mathbb{R}$ such that for all $i \in [n]$,

$$\frac{cp}{n} |x_{0,i}|^{p-1} \text{sgn}(x_{0,i}) = \frac{1}{n^{3/2}} (G_n x_0)_i + \frac{1}{n^{3/2}} (G_n^T x_0)_i = \frac{\sqrt{2}}{n^{3/2}} (\bar{G}_n x_0)_i. \quad (9.2)$$

Multiplying $x_{0,i}$ to both sides and adding them together give

$$cp = cp \|x_0\|_p^p = \frac{\sqrt{2}}{n^{3/2}} \langle \bar{G}_n x_0, x_0 \rangle = 2 \max_{\|x\|_p=1} \frac{1}{n^{3/2}} \langle G_n x, x \rangle = 2\text{GP}_{n,p}.$$

Hence, $c = 2p^{-1}\text{GP}_{n,p}$. On the other hand, for any $\delta > 0$, we can multiply (9.2) by $|x_{0,i}|^{1+\delta} \text{sgn}(x_{0,i})$ and use the Cauchy–Schwarz inequality to get

$$\begin{aligned} cp \|x_0\|_{p+\delta}^{p+\delta} &= \frac{\sqrt{2}}{n^{1/2}} \frac{1}{n} \sum_{i \in [n]} (\bar{G}_n x_0)_i |x_{0,i}|^{1+\delta} \text{sgn}(x_{0,i}) \\ &\leq \frac{\sqrt{2}}{n^{1/2}} \|\bar{G}_n x_0\|_2 \|x_0\|_{2(1+\delta)}^{1+\delta} \leq \sqrt{2} \frac{\|\bar{G}_n\|_2}{n^{1/2}} \|x_0\|_2 \|x_0\|_{2(1+\delta)}^{1+\delta}. \end{aligned}$$

If now δ satisfies $2(1+\delta) \leq p$, then Jensen's inequality implies

$$cp \|x_0\|_{p+\delta}^{p+\delta} \leq \sqrt{2} \frac{\|\bar{G}_n\|_2}{n^{1/2}} \|x_0\|_p \|x_0\|_p^{1+\delta} = \sqrt{2} \frac{\|\bar{G}_n\|_2}{n^{1/2}}.$$

Hence,

$$\|x_0\|_{p+\delta}^{p+\delta} \leq \frac{1}{\sqrt{2}\text{GP}_{n,p}} \frac{\|\bar{G}_n\|_2}{n^{1/2}}. \quad (9.3)$$

Since $\text{GP}_{n,p}$ converges to GP_p almost surely and $\|\bar{G}_n\|_2/\sqrt{n}$ converges to 2 almost surely, it follows that

$$\text{GP}_p = \lim_{n \rightarrow \infty} \text{GP}_{n,p} = \lim_{n \rightarrow \infty} \max \frac{1}{n^{3/2}} \langle G_n x, x \rangle,$$

where the maximum is taken over all $x \in \mathbb{R}^n$ satisfying that $\|x\|_p = 1$ and $\|x\|_{p+\delta}^{p+\delta} \leq \sqrt{2}/\text{GP}_p$. Our proof is then completed by noting that $\text{GP}_2 = \sqrt{2}$ and $\text{GP}_p \geq \text{GP}_\infty$. ■

We now proceed to verify the continuity of GP_p on $(2, \infty)$. For any $2 < p_1 < p_2 < p_3 < \infty$, set $\theta = (p_2 - p_1)/(p_3 - p_1) \in (0, 1)$, which yields $p_2 = (1 - \theta)p_1 + \theta p_3$. From the Hölder inequality, $\|x\|_{p_2}^{p_2} \leq \|x\|_{p_1}^{(1-\theta)p_1} \|x\|_{p_3}^{\theta p_3}$. Note that if $x \in \mathbb{R}^n$ satisfies that

$\|x\|_{p_1} = 1$ and $\|x\|_{p_3}^{p_3} \leq \text{GP}_2/\text{GP}_\infty$, then $\|x\|_{p_2}^{p_2} \leq (\text{GP}_2/\text{GP}_\infty)^\theta$. Consequently, if p_1, p_2, p_3 also satisfy $2(1+p_3-p_1) < p_1$, then by applying Lemma 9.2 with $p = p_1$ and $\delta = p_3 - p_1$,

$$\begin{aligned} \text{GP}_{p_1} &\leq \limsup_{n \rightarrow \infty} \max_{\|x\|_{p_2}^{p_2} \leq \left(\frac{\text{GP}_2}{\text{GP}_\infty}\right)^\theta} \frac{1}{n^{3/2}} \langle G_n x, x \rangle \\ &= \limsup_{n \rightarrow \infty} \max_{\|x\|_{p_2}^{p_2} \leq \left(\frac{\text{GP}_2}{\text{GP}_\infty}\right)^\theta} \frac{1}{n^{3/2}} \left\langle G_n \frac{x}{\|x\|_{p_2}}, \frac{x}{\|x\|_{p_2}} \right\rangle \|x\|_{p_2}^2 \\ &\leq \left(\frac{\text{GP}_2}{\text{GP}_\infty}\right)^{2\theta/p_2} \text{GP}_{p_2} = \left(\frac{\text{GP}_2}{\text{GP}_\infty}\right)^{\frac{2(p_2-p_1)}{p_2(p_3-p_1)}} \text{GP}_{p_2}. \end{aligned}$$

This and Lemma 9.1 imply

$$0 \leq \text{GP}_{p_1} - \text{GP}_{p_2} \leq \left(\left(\frac{\text{GP}_2}{\text{GP}_\infty} \right)^{\frac{2(p_2-p_1)}{p_2(p_3-p_1)}} - 1 \right) \text{GP}_{p_2}.$$

This clearly validates the continuity of GP_p on $(2, \infty)$.

9.3 Case $p = \infty$.

Note that $\text{GP}_\infty \leq \text{GP}_p$ for all $p \geq 2$. From Lemma 9.1, it suffices to show that $\lim_{p \rightarrow \infty} \text{GP}_p \leq \text{GP}_\infty$. For any $\varepsilon > 0$ and $x \in \mathbb{R}^n$ satisfying $\|x\|_p = 1$, denote $x_i^\varepsilon = x_i 1(|x_i| \geq 1 + \varepsilon)$ and $\tilde{x}_i^\varepsilon = x_i 1(|x_i| < 1 + \varepsilon)$. An application of the Hölder inequality followed by the Markov inequality yields

$$\begin{aligned} \|x^\varepsilon\|_2^2 &\leq \|x\|_p^2 \left(\frac{1}{n} \sum_{i \in [n]} 1(|x_i| \geq 1 + \varepsilon) \right)^{1-2/p} = \left(\frac{1}{n} \sum_{i \in [n]} 1(|x_i| \geq 1 + \varepsilon) \right)^{1-2/p} \\ &\leq \left(\frac{1}{(1 + \varepsilon)^p} \|x\|_p^p \right)^{1-2/p} = \frac{1}{(1 + \varepsilon)^{p-2}}. \end{aligned}$$

Therefore,

$$\left| \frac{1}{n^{3/2}} \langle G_n x^\varepsilon, x^\varepsilon \rangle \right| \leq \frac{\|G_n\|_2}{\sqrt{n}} \|x^\varepsilon\|_2^2 \leq \frac{\|G_n\|_2}{\sqrt{n}(1 + \varepsilon)^{p-2}}$$

and

$$\left| \frac{1}{n^{3/2}} \langle G_n x^\varepsilon, \tilde{x}^\varepsilon \rangle \right| \leq \frac{\|G_n\|_2}{\sqrt{n}} \|x^\varepsilon\|_2 \|\tilde{x}^\varepsilon\|_2 \leq \frac{\|G_n\|_2}{\sqrt{n}(1 + \varepsilon)^{p/2-1}}.$$

From (11.21), it follows that

$$\begin{aligned}
\lim_{p \rightarrow \infty} \text{GP}_p &= \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} \max_{\|x\|_p=1} \frac{1}{n^{3/2}} (\langle G_n x^\varepsilon, x^\varepsilon \rangle + 2\langle G_n x^\varepsilon, \tilde{x}^\varepsilon \rangle + \langle G_n \tilde{x}^\varepsilon, \tilde{x}^\varepsilon \rangle) \\
&= \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} \max_{\|x\|_p=1} \frac{1}{n^{3/2}} \langle G_n \tilde{x}^\varepsilon, \tilde{x}^\varepsilon \rangle \\
&\leq \lim_{n \rightarrow \infty} \max_{\|x\|_\infty \leq 1+\varepsilon} \frac{(1+\varepsilon)^2}{n^{3/2}} \langle G_n x(1+\varepsilon)^{-1}, x(1+\varepsilon)^{-1} \rangle \leq (1+\varepsilon)^2 \text{GP}_\infty.
\end{aligned}$$

Letting $\varepsilon \rightarrow 0+$, we obtain that $\lim_{p \rightarrow \infty} \text{GP}_p \leq \text{GP}_\infty$. This completes our proof.

10 Bounding $L_{p,u}(t)$ via Free Energies

For $2 < p < \infty$ and $u > 0$, recall $L_{n,p,u}(t)$ from (1.13) and that from Theorem 1.5, its n -limit exists and is denoted by $L_{p,u}(t)$. This section is a preparation for the proof of Theorem 1.9. We establish upper and lower bounds for $L_{p,u}(t)$ in terms of two free energies that we define now. Let $\varepsilon > 0$. For $\beta > 0$, define the free energy associated with $H_{n,p,t}$ with ℓ_2 -band constraint by

$$F_{n,t,u,\varepsilon}(\beta) = \frac{1}{\beta n} \log \int_{x \in \mathbb{R}^n: \|x\|_2^2 \in [u-\varepsilon, u+\varepsilon]} \exp \beta H_{n,p,t}(x) dx,$$

where dx is the Lebesgue measure on $\mathbb{R}^n$. Similarly, we shall also consider an analogous version of $F_{n,t,u,\varepsilon}(\beta)$ with an additional box confinement, namely, for $u^{1/2} < M < \infty$ and $\beta > 0$, define

$$F_{n,t,u,M,\varepsilon}(\beta) = \frac{1}{\beta n} \log \int_{x \in [-M,M]^n: \|x\|_2^2 \in [u-\varepsilon, u+\varepsilon]} \exp \beta H_{n,p,t}(x) dx. \quad (10.1)$$

The parameter β in the above two free energies is usually called the (inverse) temperature in statistical physics. The following proposition gives upper and lower bounds for $L_{p,u}(t)$.

Proposition 10.1. Let $2 < p < \infty$. For any $u > 0$ and $t > 0$,

$$\lim_{M \rightarrow \infty} \liminf_{\beta \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E} F_{n,t,u,M,\varepsilon}(\beta) \leq L_{p,u}(t) \leq \lim_{\beta \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} F_{n,t,u,\beta^{-2}}(\beta).$$

Remark 10.2. We emphasize that the outer limit in the upper bound is along $\varepsilon = \beta^{-2}$. To use these inequalities, the highly nontrivial part is to find the tight limits on both

sides. In Section 11, we will show that the lower bound can be handled by a Parisi-type variational formula similar to the one in Theorem 1.9. The upper bound will be controlled by the so-called Guerra’s replica symmetry breaking bound in Section 12.2. Ultimately, we show that these match each other, leading to the desired formula in (1.22); see Section 12.

For the rest of this section, we establish the proof of Proposition 10.1. Consider a modified version of $L_{n,p,u}(t)$ on a ℓ_2 -band, namely, for $\varepsilon > 0$ and $t > 0$, define

$$L_{n,p,u,\varepsilon}(t) := \max_{x \in \mathbb{R}^n: \|x\|_2^2 \in [u-\varepsilon, u+\varepsilon]} \frac{H_{n,p,t}(x)}{n} = \max_{x \in \mathbb{R}^n: \|x\|_2^2 \in [u-\varepsilon, u+\varepsilon]} \left(\frac{\langle G_n x, x \rangle}{n^{3/2}} - t \|x\|_p^p \right).$$

When ε is small, this quantity is essentially the same as $L_{n,p,u}(t)$. The following lemma establishes upper and lower bounds for $L_{n,p,u,\varepsilon}(t)$ in terms of the free energies introduced above.

Lemma 10.3. Let $2 < p < \infty$. For any $0 < u < \sqrt{M}$ and $t > 0$,

$$\liminf_{\beta \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} \mathbb{E} F_{n,t,u,M,\varepsilon}(\beta) \leq \lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E} L_{n,p,u,\varepsilon}(t). \quad (10.2)$$

In addition, there exist some constants $C, \kappa > 0$ (depending on t, u) such that for any $\beta > 0$, $0 < \varepsilon < 1$, and $n \geq 1$,

$$\mathbb{E} F_{n,t,u,\varepsilon}(\beta) \geq \mathbb{E} L_{n,p,u,\varepsilon/\kappa}(t) - \varepsilon C + \frac{1}{\beta} \log \frac{2\varepsilon}{\kappa} - C e^{-n/C}. \quad (10.3)$$

Proof. It is easy to see that $F_{n,t,u,M,\varepsilon}(\beta) \leq L_{n,p,u,\varepsilon}(t) + \beta^{-1} \log 2M$, which immediately implies the upper bound (10.2). So, let us focus on the lower bound. Let $0 < \varepsilon < 1$. Note that we can choose $\kappa > \max(1/u, 1)$ sufficiently large such that for any x^* satisfying $|||x^*|||_2^2 - u| < \varepsilon/\kappa$, we have

$$B(x^*) := x^* + [-\varepsilon/\kappa, \varepsilon/\kappa]^n \subset \{x \in \mathbb{R}^n : |||x|||_2^2 - u| < \varepsilon\}.$$

Note that $\varepsilon/\kappa < 1$. For any $x \in B(x^*)$, a direct computation using triangle inequalities shows that

$$\begin{aligned}
H_{n,p,t}(x) - H_{n,p,t}(x^*) &\geq -n^{-1/2} \|G_n\|_2 \|x - x^*\|_2 (\|x^*\|_2 + \|x\|_2) - t(\|x - x^*\|_p + \|x^*\|_p)^p - \|x^*\|_p^p \\
&\geq -n^{1/2} \|G_n\|_2 \|x - x^*\|_2 (\|x^*\|_2 + \|x\|_2) \\
&\quad - t p n \|x - x^*\|_p (\|x - x^*\|_p + \|x^*\|_p)^{p-1} \\
&\geq -\frac{n\varepsilon}{\kappa} \left(2n^{-1/2} \|G_n\|_2 \sqrt{u+1} + t p (1 + \|x^*\|_p)^{p-1} \right), \tag{10.4}
\end{aligned}$$

where we have used the inequality $(a+b)^p - a^p = \int_a^{a+b} p x^{p-1} \leq p b (a+b)^{p-1}$ for any $a, b \geq 0$ in the 2nd step above. From now on, we take x^* as a maximizer of $H_{n,p,t}$ on $\{x \in \mathbb{R}^n : |||x|||_2^2 - u| < \varepsilon/\kappa\}$. By imitating the proof as that of (7.8), it can be verified that there exist constants $C, D > 0$ such that

$$\mathbb{P}(||x^*|||_p \leq D) \geq 1 - C e^{-n/C}. \tag{10.5}$$

From (11.21), (10.4), and (10.5), there exists a constant $C(t, u)$ such that with probability at least $1 - C' e^{-n/C'}$, we have for any $x \in B(x^*)$,

$$H_{n,p,t}(x) - H_{n,p,t}(x^*) \geq -\frac{n\varepsilon}{\kappa} C(t, u).$$

Let A be the event such that this inequality is valid. From the above inequality and noting that $\text{Vol}(B(x^*)) = (2\varepsilon/\kappa)^n$, we have that

$$\begin{aligned}
\mathbb{E} F_{n,t,u,\varepsilon}(\beta) &\geq \frac{1}{n\beta} \mathbb{E} \left[\log \int_{B(x^*)} e^{n\beta H_{n,p,t}(x)} dx; A \right] \\
&\geq \mathbb{E} [L_{n,p,u,\varepsilon/\kappa}(t); A] - \frac{\varepsilon}{\kappa} C(t, u) + \frac{1}{\beta} \log \frac{2\varepsilon}{\kappa} \\
&= \mathbb{E} L_{n,p,\varepsilon/\kappa}(t) - \mathbb{E} [L_{n,p,u,\varepsilon/\kappa}(t); A^c] - \frac{\varepsilon}{\kappa} C(t, u) + \frac{1}{\beta} \log \frac{2\varepsilon}{\kappa}. \tag{10.6}
\end{aligned}$$

To control $\mathbb{E} [L_{n,p,u,\varepsilon/\kappa}(t); A^c]$, we argue as follows. Recall that if x is an optimizer of $L_{n,p}(t)$, then it must satisfy the 2nd inequality of (7.11). Using this and dropping the negative term in $L_{n,p}(t)$ yield that

$$L_{n,p}(t) \leq \frac{1}{\sqrt{n}} \|G_n\|_2 \|x\|_2^2 \leq t^{-2/(p-2)} \left(\frac{\|G_n\|_2}{\sqrt{n}} \right)^{1+2/(p-2)}.$$

Here, applying the concentration Inequality (3.6) to $\|G_n\|_2$ and using (3.4) lead to $\mathbb{E}(n^{-1/2}\|G_n\|_2)^k \leq C_k$ for all $k \geq 1$. Thus, there exists $C(t)$ independent of n such that

$$\mathbb{E}L_{n,p}(t)^2 \leq C(t), \quad (10.7)$$

which implies that

$$\begin{aligned} |\mathbb{E}[L_{n,p,u,\varepsilon/\kappa}(t); A^c]| &\leq (\mathbb{E}L_{n,p,u,\varepsilon/\kappa}(t)^2)^{1/2} \mathbb{P}(A^c)^{1/2} \\ &\leq (\mathbb{E}L_{n,p}(t)^2)^{1/2} (C'e^{-n/C'})^{1/2} \leq \sqrt{C(t)}(C'e^{-n/C'})^{1/2}. \end{aligned}$$

The above inequality and (10.6) yield the 2nd assertion. $\blacksquare$

Next, we show that $L_{n,p,u}(t)$ and $L_{n,p,u,\varepsilon}(t)$ are asymptotically the same.

Lemma 10.4. Let $2 < p < \infty$. For any $t, u > 0$, we have that almost surely,

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} L_{n,p,u,\varepsilon}(t) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{E}L_{n,p,u,\varepsilon}(t) = L_{p,u}(t).$$

Proof. The 1st limit follows directly from the Gaussian concentration inequality (3.6) as $L_{n,p,u,\varepsilon}(t)$ is $(u + \varepsilon)n^{-1/2}$ -Lipschitz in G_n with respect to the $\|\cdot\|_F$ -norm. Thus, it remains to prove the second equality. Write

$$\begin{aligned} L_{n,p,u,\varepsilon}(t) &= \max_{v: |v-u| \leq \varepsilon} \max_{\|x\|_2^2=1} \left(\frac{v}{n^{3/2}} \langle G_n x, x \rangle - tv^{p/2} \|x\|_p^p \right), \\ L_{n,p,u}(t) &= \max_{\|x\|_2^2=1} \left(\frac{u}{n^{3/2}} \langle G_n x, x \rangle - tu^{p/2} \|x\|_p^p \right). \end{aligned}$$

If x_v is an optimizer to the inner optimization problem of $L_{n,p,u,\varepsilon}(t)$ associated with v , then

$$t \|x_v\|_p^p \leq v^{1-p/2} n^{-1/2} \|G_n\|_2 \|x_v\|_2^2 = v^{1-p/2} n^{-1/2} \|G_n\|_2$$

since $L_{n,p,u,\varepsilon}(t) \geq 0$. Consequently, for any $0 < \varepsilon < \min(1, u)$,

$$\begin{aligned} L_{n,p,u,\varepsilon}(t) &\leq L_{n,p,u}(t) + \varepsilon (n^{-1/2} \|G_n\|_2 + 2^{-1} tp(u + \varepsilon)^{p/2-1} \max_{v: |v-u| \leq \varepsilon} \|x_v\|_p^p) \\ &\leq L_{n,p,u}(t) + \varepsilon (n^{-1/2} \|G_n\|_2 + 2^{-1} pn^{-1/2} \|G_n\|_2 (u + \varepsilon)^{p/2-1} (u - \varepsilon)^{1-p/2}). \end{aligned}$$

By passing to limit, we obtain that $\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} L_{n,p,u,\varepsilon}(t) \leq L_{p,u}(t)$. Since obviously we also have $L_{n,p,u}(t) \leq L_{n,p,u,\varepsilon}(t)$, it follows that $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} L_{n,p,u,\varepsilon}(t) = \lim_{n \rightarrow \infty} L_{n,p,u}(t) = L_{p,u}(t)$ and this completes our proof. ■

To complete the proof of Proposition 10.1, the lower bound follows directly from (10.2) and Lemma 11.10. As for the upper bound, it can be obtained by sending $n \rightarrow \infty$ and then $\varepsilon = \beta^{-2} \rightarrow 0$ in (10.3) and using again Lemma 11.10.

11 Low-Temperature Limit of the Parisi Formula

By Proposition 10.1, the validity of Theorem 1.9 relies on showing that $\lim_{\beta \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E}F_{n,t,u,\beta^{-2}}(\beta)$ and $\lim_{M \rightarrow \infty} \lim_{\beta \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E}F_{n,t,u,M,\varepsilon}(\beta)$ are equal, together with establishing the desired formula in (1.22). In this section, we shall focus on handling the limit

$$\lim_{\beta \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} F_{n,t,u,M,\varepsilon}(\beta). \quad (11.1)$$

To this end, we shall need several Parisi-type variational formulas associated with a variety of the free energies and the ground state energies. For the convenience of the reader, we collect all of them together in Section 11.1, including those that will be used later on. Our main result, Proposition 11.3, establishing a Parisi-type formula for (11.1), is stated in Section 11.2 followed by its proof in the rest of this section.

11.1 Parisi PDEs and functionals

Fix $u > 0$. Let $\mathcal{N}_u$ be the collection of all positive measures γ on $[0, u]$ with $\int_0^u \gamma(s) ds < \infty$. Recall that $\mathcal{N}_u^d \subset \mathcal{N}_u$ is the collection of all positive measures on $[0, u]$ with finitely many atoms. Moreover, let $\mathcal{M}_u$ denote the collection of probability measures on $[0, u]$, and let $\mathcal{M}_u^d \subset \mathcal{M}_u$ be the collection of probability measures on $[0, u]$ with finitely many atoms. For each $\lambda \in \mathbb{R}$, let $h_\lambda : \mathbb{R} \rightarrow \mathbb{R}_+$ be a continuous function.

For $\beta < \infty$, we take $(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u^d$ or $\mathbb{R} \times \mathcal{M}_u$ and consider the solution $\Theta_{\lambda,\alpha}^\beta$ to the following PDE:

$$\begin{aligned} \partial_s \Theta_{\lambda,\alpha}^\beta(s, x) &= -(\partial_{xx} \Theta_{\lambda,\alpha}^\beta(s, x) + \beta \alpha(s) (\partial_x \Theta_{\lambda,\alpha}^\beta(s, x))^2), \quad (s, x) \in [0, u] \times \mathbb{R}, \\ \Theta_{\lambda,\alpha}^\beta(u, x) &= h_\lambda(x). \end{aligned} \quad (11.2)$$

Here, $\Theta_{\lambda,\alpha}^\beta$ may exist in the classical or weak sense (see Remarks 11.1 and 11.2 below) depending on whether the measure α is atomic and on the regularity of the boundary condition h_λ . The associated Parisi functional $\mathcal{P}_\beta$ on $\mathbb{R} \times \mathcal{M}_u^d$ or $\mathbb{R} \times \mathcal{M}_u$ is defined as

$$\mathcal{P}_\beta(\lambda, \alpha) = \Theta_{\lambda,\alpha}^\beta(0, 0) - \lambda u - \int_0^u s \beta \alpha(s) \, ds, \quad (11.3)$$

where $\alpha(s) := \alpha([0, s])$ is the cumulative distribution function of the measure α .

In a similar manner, the functional (11.3) has a natural analogy in the case of $\beta = \infty$. For $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$ or $\mathbb{R} \times \mathcal{N}_u$, consider the (classical or weak) solution $\Theta_{\lambda,\gamma}$ to the following PDE:

$$\begin{aligned} \partial_s \Theta_{\lambda,\gamma}(s, x) &= -(\partial_{xx} \Theta_{\lambda,\gamma}(s, x) + \gamma(s) (\partial_x \Theta_{\lambda,\gamma}(s, x))^2), \quad (s, x) \in [0, u] \times \mathbb{R}, \\ \Theta_{\lambda,\gamma}(u, x) &= h_\lambda(x), \end{aligned} \quad (11.4)$$

where $\gamma(s) := \gamma([0, s])$. Define the associated Parisi functional $\mathcal{P}$ on $\mathbb{R} \times \mathcal{N}_u$ or $\mathbb{R} \times \mathcal{N}_u^d$ as

$$\mathcal{P}(\lambda, \gamma) = \Theta_{\lambda,\gamma}(0, 0) - \lambda u - \int_0^u s \gamma(s) \, ds. \quad (11.5)$$

The different boundary conditions h_λ that we will use throughout this paper are all dependent on a collection of functions $(\omega_\lambda^t)_{(\lambda,t) \in \mathbb{R} \times (0,\infty)}$ defined as $\omega_\lambda^t(x, r) = rx + \lambda r^2 - t|r|^p$ for $(x, r) \in \mathbb{R} \times \mathbb{R}$. The following table collects the relevant cases of the Parisi PDEs and functionals that will be needed for the rest of the paper.

Remark 11.1. Lemma B.1 implies that $f_\lambda^{\beta,t}(x)$ and $f_\lambda^t(x)$ are bounded by $C(1 + |x|^{a+1})$ for some $0 < a < 1$ and $C > 0$. From these, when $\alpha \in \mathcal{M}_u^d$ and $\gamma \in \mathcal{N}_u^d$, we can apply the Hopf–Cole transformation (see Lemma B.2) to solve the above four PDEs in the classical sense.

Remark 11.2. The functional $\mathcal{P}_{t,u}$ here is of course the same as (1.21), defined in the introduction. In Table 1, we define the functionals $\mathcal{P}_{t,u,M}$ and $\mathcal{P}_{t,u}$ on different measure spaces $\mathcal{N}_u$ and $\mathcal{N}_u^d$, respectively, mainly due to technical purposes. Indeed, the fact that $f_\lambda^{t,M}$ is M -Lipschitz allows us to show that for any $0 < s_0 < u$ and $k \geq 1$, $\partial_x^k \Psi_{\lambda,\gamma}^{t,u,M}(s, \cdot)$ is also Lipschitz uniformly over all $s \in [0, s_0]$, $\lambda \in \mathbb{R}$, and $\gamma \in \mathcal{N}_u^d$. Together with a compactness argument, this enables us to show the existence and uniqueness of the weak solution $\Psi_{\lambda,\gamma}^{t,u,M}$ for any $\gamma \in \mathcal{N}_u$. We refer the reader to check

TABLE 1 Various Parisi functionals

β	Measure space	Boundary condition h_λ	PDE solution	Type	Parisi functional
$< \infty$	$\alpha \in \mathcal{M}_u^d$	$f_\lambda^{\beta,t}(x) = \frac{1}{\beta} \log \int_{-\infty}^{\infty} e^{\beta \omega_\lambda^t(x,r)} dr$	$\Phi_{\lambda,\alpha}^{\beta,t,u}(s,x)$	Classical	$\mathcal{P}_{\beta,t,u}(\lambda, \alpha)$
$< \infty$	$\alpha \in \mathcal{M}_u$	$f_\lambda^{\beta,t,M}(x) = \frac{1}{\beta} \log \int_{-M}^M e^{\beta \omega_\lambda^t(x,r)} dr$	$\Phi_{\lambda,\alpha}^{\beta,t,u,M}(s,x)$	Weak	$\mathcal{P}_{\beta,t,u,M}(\lambda, \alpha)$
$= \infty$	$\gamma \in \mathcal{N}_u^d$	$f_\lambda^t(x) = \sup_{r \in \mathbb{R}} \omega_\lambda^t(r, x)$	$\Psi_{\lambda,\gamma}^{t,u}(s,x)$	Classical	$\mathcal{P}_{t,u}(\lambda, \gamma)$
$= \infty$	$\gamma \in \mathcal{N}_u$	$f_\lambda^{t,M}(x) = \sup_{r \in [-M,M]} \omega_\lambda^t(r, x)$	$\Psi_{\lambda,\gamma}^{t,u,M}(s,x)$	Weak	$\mathcal{P}_{t,u,M}(\lambda, \gamma)$

[18, 33] for details. It might seem plausible that one can also construct unique weak solutions to $\Psi_{\lambda,\gamma}^{t,u}$ for arbitrary $\gamma \in \mathcal{N}_u$ by similar lines of arguments in [18, 33]. However, as the boundary condition f_λ^t is no longer Lipschitz (see Lemma B.1), this makes the compactness argument in [18, 33] technically more tedious. As this part of the analysis is not quite relevant to this work, we do not pursue this extension in this paper. For the same reason, $\mathcal{P}_{\beta,t,u,M}$ and $\mathcal{P}_{\beta,t,u}$ in Table 1 are also defined over different measure spaces $\mathcal{M}_u$ and $\mathcal{M}_u^d$.

11.2 Parisi-type formula

Recall from [50] that $\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} F_{n,t,u,M,\varepsilon}(\beta)$ exists and it can be expressed as the following Parisi-type formula associated with the Parisi functional given in the 2nd row of Table 1:

$$\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} F_{n,t,u,M,\varepsilon}(\beta) = \inf_{(\lambda,\alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha). \quad (11.6)$$

The following proposition shows that the variational problem in (11.6) converges to the Parisi-type formula associated with the functional $\mathcal{P}_{t,u,M}$ (refer to the 4th row of Table 1) and gives the existence of a minimizer along with quantitative controls uniformly in M .

Proposition 11.3. Let $2 < p < \infty$ and $t, u > 0$. We have that

$$\lim_{\beta \rightarrow \infty} \lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} F_{n,t,u,M,\varepsilon}(\beta) = \lim_{\beta \rightarrow \infty} \inf_{(\lambda,\alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) = \inf_{(\lambda,\gamma) \in \mathbb{R} \times \mathcal{N}_u} \mathcal{P}_{t,u,M}(\lambda, \gamma). \quad (11.7)$$

In addition, there exists a constant $C > 0$ depending only on t, u such that for any $M > 2u^{1/2}$, there exists some $(\lambda_M, \gamma_M) \in \mathbb{R} \times \mathcal{N}_u$ satisfying

$$\inf_{(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u} \mathcal{P}_{t,u,M}(\lambda, \gamma) = \mathcal{P}_{t,u,M}(\lambda_M, \gamma_M) \quad (11.8)$$

such that

$$|\lambda_M| \leq C \text{ and } \int_0^u \gamma_M(s) \, ds \leq C. \quad (11.9)$$

Remark 11.4. Though we do not need this here, it can be checked by the same argument in Lemma 10.3 that $\lim_{n \rightarrow \infty} \max_{x \in [-M, M]^n: \|x\|_2^2 = u} H_{n,p,t}(x)/n$ is equal to the left-hand side of (11.7) and the right-hand side of (11.7) established a Parisi-type formula for this ground state energy.

Remark 11.5. The assertion (11.7) is indeed a special case of [19, Theorem 5], but it does not include the existence of a minimizer and the bound (11.9), which are the most crucial results we need in the proof of Theorem 1.9.

The rest of this section is devoted to establishing Proposition 11.3.

11.3 Properties of the Parisi PDEs

We state a number of fundamental properties for $\Phi_{\lambda,\alpha}^{\beta,t,u,M}$ and $\Psi_{\lambda,\gamma}^{t,u,M}$ that will be of great use throughout this section. Their proofs will be omitted as they follow directly from the same arguments from [3, 34] with no essential changes. First of all, their regularities are summarized in the following proposition.

Proposition 11.6 (Regularity). Let $\alpha \in \mathcal{M}_u$ and $\gamma \in \mathcal{N}_u$. For any $s \in [0, u]$, $\Phi_{\lambda,\alpha}^{\beta,t,u,M}(s, \cdot)$ and $\Psi_{\lambda,\gamma}^{t,u,M}(s, \cdot)$ are twice partially differentiable in x . Their 1st partial derivatives in x are uniformly bounded over $[0, u] \times \mathbb{R}$ by M . In addition, $\partial_x \Phi_{\lambda,\alpha}^{\beta,t,u,M}(s, \cdot)$ is uniformly Lipschitz over all $s \in [0, u]$, while for any $0 < u_0 < u$, $\partial_x \Psi_{\lambda,\gamma}^{t,u,M}(s, \cdot)$ is also uniformly Lipschitz over all $s \in [0, u_0]$.

Note that $\Phi_{\lambda,\alpha}^{\beta,t,u,M}$ and $\Psi_{\lambda,\gamma}^{t,u,M}$ are special cases of the Hamilton–Jacobi–Bellman equation, induced by a linear problem of diffusion control. In this case, they can be expressed in terms of stochastic optimal control problems. More precisely, let $0 < u < \sqrt{M}$ and $\mathcal{D}_u^M$ be the collection of all progressively measurable processes

$v = (v(s))_{0 \leq s \leq u}$ with respect to a standard Brownian motion $W = (W(s))_{0 \leq s \leq u}$ and satisfying $\sup_{0 \leq s \leq u} |v(s)| \leq M$.

Proposition 11.7 (Stochastic optimal control representation). We have

$$\Phi_{\lambda,\alpha}^{\beta,t,u,M}(0,0) = \sup_{v \in \mathcal{D}_u^M} \mathbb{E} \left[f_{\lambda}^{\beta,t,M} \left(2 \int_0^u \beta \alpha(s) v(s) ds + \sqrt{2} W(u) \right) - \int_0^u \beta \alpha(s) v(s)^2 ds \right], \quad (11.10)$$

$$\Psi_{\lambda,\gamma}^{t,u,M}(0,0) = \sup_{v \in \mathcal{D}_u^M} \mathbb{E} \left[f_{\lambda}^{t,M} \left(2 \int_0^u \gamma(s) v(s) ds + \sqrt{2} W(u) \right) - \int_0^u \gamma(s) v(s)^2 ds \right]. \quad (11.11)$$

Here, the 1st supremum is achievable by $v(s) = \partial_x \Phi_{\lambda,\alpha}^{\beta,t,u,M}(s, X(s))$ and the 2nd supremum is attained by $v(s) = \partial_x \Psi_{\lambda,\gamma}^{t,u,M}(s, X'(s))$, where $X = (X(s))_{0 \leq s \leq u}$ and $X' = (X'(s))_{0 \leq s \leq u}$ are the strong solutions to the following SDEs:

$$\begin{aligned} dX(s) &= 2\beta\alpha(s) \partial_x \Phi_{\lambda,\alpha}^{\beta,t,u,M}(s, X(s)) ds + \sqrt{2} dW(s), \\ dX'(s) &= 2\gamma(s) \partial_x \Psi_{\lambda,\gamma}^{t,u,M}(s, X'(s)) ds + \sqrt{2} dW(s) \end{aligned}$$

for $0 \leq s \leq u$ with $X(0) = X'(0) = 0$.

Next, note that the boundary condition $f_{\lambda}^{\beta,t,M}$ and $f_{\lambda}^{t,M}$ are Lipschitz and convex in $(\lambda, x) \in \mathbb{R}^2$. Using this together with Proposition 11.7, it can be shown that these properties propagate throughout the entire solutions.

Proposition 11.8 (Lipschitz property and convexity). $(\lambda, \alpha) \mapsto \Phi_{\lambda,\alpha}^{\beta,t,u,M}(0,0)$ and $(\lambda, \gamma) \mapsto \Psi_{\lambda,\gamma}^{t,u,M}(0,0)$ are Lipschitz and convex respectively with respect to the norms

$$|\lambda - \lambda'| + \int_0^u |\alpha(s) - \alpha'(s)| ds \quad \text{and} \quad |\lambda - \lambda'| + \int_0^u |\gamma(s) - \gamma'(s)| ds. \quad (11.12)$$

11.4 Optimality of the Parisi functional

We investigate some properties of the variational formula of $\mathcal{P}_{\beta,t,u,M}$.

Lemma 11.9. Let $t, u > 0$. For any $M > u^{1/2}$ and $\beta > 0$, the functional $\mathcal{P}_{\beta,t,u,M}$ has a unique minimizer in $\mathbb{R} \times \mathcal{M}_u$.

Proof. Let $\nu_{t,\beta,M}(dr) = e^{-t|r|^p} dr$ and $\Delta = \int_{[-M,-\sqrt{u}] \cup [\sqrt{u},M]} dv_{t,\beta,M}(r)$. Note that

$$\begin{aligned} f_{\lambda}^{\beta,t,M}(x) &\geq \lambda u + \frac{1}{\beta} \log \frac{1}{\Delta} \int_{[-M,-\sqrt{u}] \cup [\sqrt{u},M]} \exp \beta(rx + \lambda(r^2 - u)) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta \\ &\geq \lambda u + \frac{1}{\Delta} \int_{[-M,-\sqrt{u}] \cup [\sqrt{u},M]} (rx + \lambda(r^2 - u)) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta \\ &= \lambda u + \frac{\lambda}{\Delta} \int_{[-M,-\sqrt{u}] \cup [\sqrt{u},M]} (r^2 - u) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta, \end{aligned}$$

where the 2nd inequality used Jensen's inequality. Since the right-hand side of this inequality is independent of x , it follows from (11.10) that $\Phi_{\lambda,\alpha}^{\beta,t,u,M}(0,0)$ is bounded by this lower bound, and hence,

$$\inf_{\alpha \in \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) \geq \frac{\lambda}{\Delta} \int_{[-M,-\sqrt{u}] \cup [\sqrt{u},M]} (r^2 - u) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta - \int_0^u s\beta ds \xrightarrow{\lambda \rightarrow \infty} \infty.$$

Similarly, let $\Delta' = \int_{[-\sqrt{u},\sqrt{u}]} dv_{t,\beta,M}(r)$. We also have

$$\begin{aligned} f_{\lambda}^{\beta,t,M}(x) &= \lambda u + \frac{1}{\beta} \log \frac{1}{\Delta'} \int_{[-\sqrt{u},\sqrt{u}]} \exp \beta(rx + \lambda(r^2 - u)) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta' \\ &\geq \lambda u + \frac{1}{\Delta'} \int_{[-\sqrt{u},\sqrt{u}]} (rx + \lambda(r^2 - u)) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta' \\ &= \lambda u + \frac{\lambda}{\Delta'} \int_{[-\sqrt{u},\sqrt{u}]} (r^2 - u) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta'. \end{aligned}$$

Again, since this lower bound is independent of x , it follows that by using (11.10),

$$\inf_{\alpha \in \mathcal{M}} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) \geq \frac{\lambda}{\Delta'} \int_{[-\sqrt{u},\sqrt{u}]} (r^2 - u) dv_{t,\beta,M}(r) + \frac{1}{\beta} \log \Delta' - \int_0^u s\beta ds \xrightarrow{\lambda \rightarrow \infty} \infty.$$

In addition, note that from Proposition 11.8, $\lambda \mapsto \inf_{\alpha \in \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha)$ is continuous. From these, we see that there exists some $\lambda_0 \in \mathbb{R}$ such that

$$\inf_{(\lambda,\alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) = \inf_{\lambda \in \mathbb{R}} \inf_{\alpha \in \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) = \inf_{\alpha \in \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda_0, \alpha).$$

Finally, by using Proposition 11.8 and noting that $\mathcal{M}_u$ is a compact space with respect to the metric $\int_0^u |\alpha(s) - \alpha'(s)| ds$, we see that $\mathcal{P}_{\beta,t,u,M}(\lambda_0, \cdot)$ is minimized by some $\alpha_0 \in \mathcal{M}_u$. Hence, (λ_0, α_0) is a minimizer of $\mathcal{P}_{\beta,t,u,M}$. This completes our proof. ■

For the rest of this section, we denote the minimizer of $\mathcal{P}_{\beta,t,u,M}$ by $(\lambda_{M,\beta}, \alpha_{M,\beta})$. In next two lemmas, we show that $\beta\alpha_{M,\beta}$ is uniformly integrable and $\lambda_{M,\beta}$ is uniformly bounded in (M, β) .

Lemma 11.10. Let $t, u > 0$. For any $M > u^{1/2}$ and $\beta > 0$, we have

$$\int_0^u s\beta\alpha_{M,\beta}(s) ds \leq \frac{u}{\sqrt{2}}. \quad (11.13)$$

Proof. Our proof consists of three major steps. First of all, we introduce an auxiliary free energy. For $b > 0$, define

$$F_n(b) = \frac{1}{n} \log \int_{x \in [-M, M]^n : \|x\|_2^2 \in [u-\varepsilon, u+\varepsilon]} \exp\left(\frac{b}{n^{1/2}} \langle G_n x, x \rangle - \beta t \|x\|_p^p\right) dx.$$

Here, to keep the notation light, we ignore the dependence on β, t, u, M , and ε . In a similar manner as (11.6), the general result in [50] ensures that this free energy can also be expressed as a similar Parisi-type formula,

$$\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} F_n(b) = \inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_b(\lambda, \alpha) \quad (11.14)$$

for

$$\mathcal{P}_b(\lambda, \alpha) := \Phi_{\lambda, \alpha}^b(0, 0) - \lambda u - b^2 \int_0^u s\alpha(s) ds,$$

where $\Phi_{\lambda, \alpha}^b(0, 0)$ is defined through the weak solution (see [34]) to the following PDE:

$$\partial_s \Phi_{\lambda, \alpha}^b(s, x) = -b^2 (\partial_{xx} \Phi_{\lambda, \alpha}^b(s, x) + \alpha(s) (\partial_x \Phi_{\lambda, \alpha}^b(s, x))^2) \quad (11.15)$$

for $(s, x) \in [0, u] \times \mathbb{R}$ with boundary condition

$$\Phi_{\lambda, \alpha}^b(u, x) = f_\lambda(x) := \log \int_{-M}^M \exp(sx + \lambda s^2 - \beta t |s|^p) ds.$$

Note that $F_{n,t,u,M,\varepsilon}(b) = b^{-1} F_n(b)$, and consequently, (11.26) is equal to (11.6) modulo a multiplicative factor b^{-1} . Here, it can be checked that the two solutions, $\Phi_{\lambda, \alpha}^b$ and $\Phi_{\lambda, \alpha}^{\beta, t, u, M}$ for $b = \beta$, are essentially the same up to a transformation,

$$\Phi_{b^{-1}\lambda, \alpha}^{b, t, u, M}(s, \beta^{-1}x) = b^{-1} \Phi_{\lambda, \alpha}^b(s, x), \quad (s, x) \in [0, u] \times \mathbb{R},$$

and therefore,

$$\mathcal{P}_{b,t,u,M}(\lambda, \alpha) = b^{-1} \mathcal{P}_b(b\lambda, \alpha), \quad \forall (\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u. \quad (11.16)$$

From these, Proposition 11.8 and Lemma 11.9, we see that for ever $b > 0$, $\mathcal{P}_b$ is convex with respect to the metric (11.12) and it has a minimizer, (λ_b, α_b) , satisfying $(\lambda_b, \alpha_b) = (b^{-1}\lambda_{M,b}, \alpha_{M,b})$.

Next, we continue to bound $b \int_0^u s \alpha_b(s) ds$ through the b -derivative of the Parisi formula in (11.26). First, it can be shown that $\Phi_{\lambda,\alpha}^b(0,0)$ also admits an analogous stochastic optimal control representation as that for $\Phi_{\lambda,\alpha}^{\beta,t,u,M}$ in Proposition 11.7. More precisely, let W and $\mathcal{D}_u^M$ be defined as in Proposition 11.7. We can write

$$\Phi_{\lambda,\alpha}^b(0,0) = \sup_{v \in \mathcal{D}_u^M} \mathbb{E} \left[f_\lambda \left(2 \int_0^u b^2 \alpha(s) v(s) ds + \sqrt{2b} W(u) \right) - \int_0^u b^2 \alpha(s) v(s)^2 ds \right].$$

Here, the optimal process attaining the maximum is given by $v_{\lambda,\alpha}(s) = \partial_x \Phi_{\lambda,\alpha}^b(s, X_{\lambda,\alpha}(s))$, where $X_{\lambda,\alpha}$ is the strong solution to the following SDE:

$$dX_{\lambda,\alpha}(s) = 2b^2 \alpha(s) \partial_x \Phi_{\lambda,\alpha}^b(s, X_{\lambda,\alpha}(s)) ds + b\sqrt{2} dW(s), \quad 0 \leq s \leq u, \quad \text{and} \quad X_{\lambda,\alpha}(0) = 0.$$

Using this representation, the same computation as [4, Proposition 4] yields that for any (λ, α) ,

$$\frac{d}{db} \Phi_{\lambda,\alpha}^b(0,0) = 2b \left(u \mathbb{E} v_{\lambda,\alpha}(u)^2 + u \mathbb{E} \partial_{xx} \Phi_{\lambda,\alpha}^b(u, X_{\lambda,\alpha}(u)) - \int_0^u s \mathbb{E} v_{\lambda,\alpha}(s)^2 \alpha(ds) \right). \quad (11.17)$$

To handle this derivative, note that $\mathcal{P}_b$ is also a convex functional. It can be argued (see, e.g., [16]) that the directional derivative of $\mathcal{P}_b$ at (λ_b, α_b) can be explicitly computed. In particular, the optimality of (λ_b, α_b) ensures that

$$\mathbb{E}(\partial_x \Phi_{\lambda_b, \alpha_b}^b(s, X_{\lambda_b, \alpha_b}(s)))^2 = \mathbb{E} v_{\lambda_b, \alpha_b}(s)^2 = s \quad (11.18)$$

for any point s in the support of α_b and

$$\mathbb{E} \partial_\lambda \Phi_{\lambda_b, \alpha_b}^b(u, X_{\lambda_b, \alpha_b}(u)) = u. \quad (11.19)$$

From (11.19) and noting that

$$\partial_{xx} \Phi_{\lambda, \alpha}^b(u, x) = \partial_{xx} f_{\lambda}(x) = \partial_{\lambda} f_{\lambda}(x) - (\partial_x f_{\lambda}(x))^2 = \partial_{\lambda} \Phi_{\lambda, \alpha}^b(u, x) - (\partial_x \Phi_{\lambda, \alpha}^b(u, x))^2,$$

it follows that

$$\begin{aligned} \mathbb{E} \partial_{xx} \Phi_{\lambda_b, \alpha_b}^b(u, X_{\lambda_b, \alpha_b}(u)) &= \mathbb{E} \partial_{\lambda} \Phi_{\lambda_b, \alpha_b}^b(u, X_{\lambda_b, \alpha_b}(u)) - \mathbb{E} V_{\lambda_b, \alpha_b}(u)^2 \\ &= u - \mathbb{E} V_{\lambda_b, \alpha_b}(u)^2. \end{aligned}$$

Plugging this equation and (11.18) into (12.9) leads to

$$\frac{d}{db} \Phi_{\lambda, \alpha}^b(0, 0) \Big|_{(\lambda, \alpha) = (\lambda_b, \alpha_b)} = 2b \left(u^2 - \int_0^u s^2 \alpha_b(ds) \right) = 4b \int_0^u s \alpha_b(s) ds,$$

where the last equality used the integration by parts. As a result, we arrive at

$$\frac{d}{db} \mathcal{P}_b(\lambda, \alpha) \Big|_{(\lambda, \alpha) = (\lambda_b, \alpha_b)} = 4b \int_0^u s \alpha_b(s) ds - 2b \int_0^u s \alpha_b(s) ds = 2b \int_0^u s \alpha_b(s) ds, \quad (11.20)$$

Now, from (11.26), we have that for any $0 < b' < b < \infty$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} \frac{F_n(b) - F_n(b')}{b - b'} &= \frac{1}{b - b'} \left(\inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_b(\lambda, \alpha) - \inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{b'}(\lambda, \alpha) \right) \\ &\geq \frac{1}{b - b'} (\mathcal{P}_b(\lambda_b, \alpha_b) - \mathcal{P}_{b'}(\lambda_b, \alpha_b)), \end{aligned}$$

which combining with the convexity of $F_n(b)$ and (11.20), after sending $b' \uparrow b$,

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} F'_n(b) \geq 2b \int_0^u s \alpha_b(s) ds. \quad (11.21)$$

In the last step, note that a direct computation gives

$$F'_n(b) = \mathbb{E} \left\langle \frac{G_n X, X}{n^{3/2}} \right\rangle_b \leq \mathbb{E} \sup_{x \in [-M, M]^n : \|x\|_2^2 \in [u - \varepsilon, u + \varepsilon]} \frac{\langle G_n X, x \rangle}{n^{3/2}} \leq \frac{(u + \varepsilon)}{\sqrt{2}} \frac{\mathbb{E} \|\bar{G}_n\|_2}{n^{1/2}}, \quad (11.22)$$

where $\langle \cdot \rangle_b$ is the Gibbs expectation associated with the partition function,

$$\int_{x \in [-M, M]^n : \|x\|_2^2 \in [u - \varepsilon, u + \varepsilon]} \exp \left(\frac{b}{n^{1/2}} \langle G_n X, x \rangle - \beta t \|x\|_p^p \right) dx.$$

It follows from (11.21), (12.5), and the convexity of F_n that

$$2b \int_0^u s \alpha_b(s) \, ds \leq \lim_{\varepsilon \downarrow 0} \liminf_{n \rightarrow \infty} F'_n(b) \leq \sqrt{2}u.$$

To translate this inequality back to that for $\alpha_{M,\beta}$, we recall that $\alpha_b = \alpha_{M,\beta}$ for $b = \beta$. This completes our proof of (11.13). ■

Lemma 11.11. Let $t, u > 0$. There exists a constant $K > 0$ depending only on t, u such that

$$|\lambda_{M,\beta}| \leq K \quad \text{for any } \beta > M > 2u^{1/2}.$$

Proof. Since $f_\lambda^{\beta,t,M}$ is convex, we can take $v \equiv 0$ in (11.10) and then apply Jensen's inequality to get

$$\Phi_{\lambda,\alpha}^{\beta,t,u,M}(0,0) \geq \mathbb{E} f_\lambda^{\beta,t,M}(\sqrt{2}W(u)) \geq f_\lambda^{\beta,t,M}(0). \quad (11.23)$$

Fix any $M > 2u^{1/2}$. For $0 < D \leq M$, we have

$$\begin{aligned} f_\lambda^{\beta,t,M}(0) &\geq \frac{1}{\beta} \log \int_{-D}^D e^{\beta(\lambda r^2 - t|r|^p)} \, dr \geq \frac{1}{\beta} \log \frac{1}{2D} \int_{-D}^D e^{\beta(\lambda r^2 - t|r|^p)} \, dr + \frac{1}{\beta} \log 2D \\ &\geq \frac{1}{2D} \int_{-D}^D (\lambda r^2 - t|r|^p) \, dr + \frac{1}{\beta} \log 2D, \end{aligned}$$

where the last inequality used Jensen's inequality. Now, from (B5),

$$\mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) \geq \lambda \left(\frac{1}{2D} \int_{-D}^D r^2 \, dr - u \right) - \frac{t}{2D} \int_{-D}^D |r|^p \, dr + \frac{\log 2D}{\beta} - \int_0^u s \beta \alpha(s) \, ds,$$

and therefore, from Lemma 11.10,

$$\lambda \left(\frac{1}{2D} \int_{-D}^D r^2 \, dr - u \right) \leq \inf_{\lambda,\alpha} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) + \frac{t}{2D} \int_{-D}^D |r|^p \, dr - \frac{\log 2D}{\beta} + \frac{u}{\sqrt{2}}.$$

Now, take $\lambda = \lambda_{M,\beta}$. If $\lambda_{M,\beta} > 0$, we set $D = \sqrt{u}$ and if $\lambda_{M,\beta} < 0$, set $D = 2\sqrt{u}$ so that $(2D)^{-1} \int_{-D}^D r^2 \, dr - u$ shares the same sign with $\lambda_{M,\beta}$. Therefore,

$$|\lambda_{M,\beta}| \leq \frac{1}{\left| \frac{1}{2D} \int_{-D}^D r^2 \, dr - u \right|} \left(\inf_{\lambda,\alpha} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) + \frac{t}{2D} \int_{-D}^D |r|^p \, dr - \frac{\log 2D}{\beta} + \frac{u}{\sqrt{2}} \right).$$

Finally, from (10.7),

$$\inf_{\lambda, \alpha} \mathcal{P}_{\beta, t, u, M}(\lambda, \alpha) = \lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} \mathbb{E} F_{n, t, u, M, \varepsilon}(\beta) \leq \limsup_{n \rightarrow \infty} \mathbb{E} L_{n, p}(t) + \frac{\log 2M}{\beta} \leq C(t)^{1/2} + \frac{\log 2M}{\beta}.$$

These yield the desired bound as long as $\beta > M > 2u^{1/2}$. $\blacksquare$

11.5 Proof of Proposition 11.3

Recall that $(\lambda_{M, \beta}, \alpha_{M, \beta})$ is the unique minimizer of $\mathcal{P}_{\beta, t, u, M}$. From Lemma 11.10 and the monotonicity of $\beta \alpha_{M, \beta}(s)$ in s ,

$$\begin{aligned} \int_0^u \beta \alpha_{M, \beta}(s) ds &= \int_0^{u/2} \beta \alpha_{M, \beta}(s) ds + \int_{u/2}^u \beta \alpha_{M, \beta}(s) ds \\ &\leq 2 \int_{u/2}^u \beta \alpha_{M, \beta}(s) ds \leq 2 \cdot \frac{2}{u} \int_{u/2}^u s \beta \alpha_{M, \beta}(s) ds \leq 2\sqrt{2}. \end{aligned} \quad (11.24)$$

Define a positive measure $\nu_{M, \beta}$ on $[0, u]$ by $\nu_{M, \beta}(A) = \int_A \beta \alpha_{M, \beta}(s) ds$. From the inequality (B12) and noting that $\alpha_{M, \beta}(s)$ is nondecreasing in s , we see that for any $0 < r < u$, $\sup_{\beta > 0, s \in [0, r]} \beta \alpha_{M, \beta}(s) < \infty$ and hence, we can pass to a subsequence $(\beta_n)_{n \geq 1}$ such that $(\beta_n \alpha_{M, \beta_n})_{n \geq 1}$ converges to some $\gamma_{M, \infty} \in \mathcal{N}_u$ almost everywhere on $[0, u]$. In addition, we can assume that along the same subsequence, $L_M := \lim_{n \rightarrow \infty} \int_0^u \beta \alpha_{M, \beta_n}(s) ds$ exists. Consequently, we conclude that ν_{M, β_n} converges vaguely on $[0, u]$ to some $\nu_{M, \infty}$, defined as $\nu_{M, \infty}(A) = \int_A \gamma_{M, \infty}(s) ds + \Gamma_{M, \infty} \delta_u$, where $0 \leq \Gamma_{M, \infty} := L_M - \int_0^u \gamma_{M, \infty}(s) ds < \infty$. On the other hand, Lemma 11.11 also allows us to pass to a subsequence of $\lambda_{M, \beta}$, along which it is convergent. From now on, without loss of generality, we assume that these are convergent without passing to a subsequence. To proceed, from the above discussion, combined with Lemma 11.11 and Inequality (B12), we note that there exists a constant K depending only on t, u such that for any $M > 2u^{1/2}$,

$$\max\left(|\lambda_{M, \infty}|, \Gamma_{M, \infty}, \int_0^u \gamma_{M, \infty}(s) ds, \sup_{s \in [0, u]} (u - s) \gamma_{M, \infty}(s)\right) \leq K. \quad (11.25)$$

Lemma 11.12. For any $0 < u < M^2$, we have that

$$\liminf_{\beta \rightarrow \infty} \Phi_{\lambda_{M, \beta}, \alpha_{M, \beta}}^{\beta, t, u, M}(0, 0) \geq \Psi_{\lambda_M, \gamma_{M, \infty}}^{t, u, M}(0, 0),$$

where $\lambda_M := \lambda_{M, \infty} + \Gamma_{M, \infty}$.

Proof. Recall the representations in Proposition 11.7. Consider any progressively measurable process $v \in \mathcal{D}_u^M$ on $[0, u]$, which is left-continuous at u , that is, $\lim_{s \rightarrow u-} v(s) = v(u)$. Note that $\max_{s \in [0, u]} |v(s)| \leq M$. From the vague convergence of $\nu_{M, \beta}$ to $\nu_{M, \infty}$ on $[0, u]$, we have that

$$\begin{aligned} & \liminf_{\beta \rightarrow \infty} \Phi_{\lambda_{M, \beta}, \alpha_{M, \beta}}^{\beta, t, u, M}(0, 0) \\ & \geq \liminf_{\beta \rightarrow \infty} \mathbb{E} \left[f_{\lambda_{M, \beta}}^{\beta, t, M} \left(2 \int_0^u v(s) \beta \alpha_{M, \beta}(s) \, ds + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \beta \alpha_{M, \beta}(s) \, ds \right] \\ & = \mathbb{E} \left[f_{\lambda_{M, \beta}}^{t, M} \left(2 \int_0^u v(s) \nu_{M, \infty}(ds) + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \nu_{M, \infty}(ds) \right]. \end{aligned} \quad (11.26)$$

From the definition of $\nu_{M, \infty}$, the above lower bound can be written as

$$\begin{aligned} & \mathbb{E} \left[f_{\lambda_{M, \infty}}^{t, M} \left(2 \int_0^u v(s) \nu_{M, \infty}(ds) + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \nu_{M, \infty}(ds) \right] \\ & = \mathbb{E} \left[f_{\lambda_{M, \infty}}^{t, M} \left(2 \int_0^u v(s) \gamma_{M, \infty}(s) \, ds + 2v(u) \Gamma_{M, \infty} + \sqrt{2} W(u) \right) \right. \\ & \quad \left. - \int_0^u v(s)^2 \gamma_{M, \infty}(s) \, ds - \Gamma_{M, \infty} v(u)^2 \right]. \end{aligned} \quad (11.27)$$

On the other hand, recall from (11.11),

$$\Psi_{\lambda_M, \gamma_{M, \infty}}^{t, u, M}(0, 0) = \sup_{v \in \mathcal{D}_u^M} \mathbb{E} \left[f_{\lambda_M}^{t, M} \left(2 \int_0^u v(s) \gamma_{M, \infty}(s) \, ds + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \gamma_{M, \infty}(s) \, ds \right]. \quad (11.28)$$

For $0 \leq s < u$, denote by $r^*(s) \in [-M, M]$ the largest (random) maximizer in the definition of

$$f_{\lambda_M}^{t, M} \left(2 \int_0^s v(l) \gamma_{M, \infty}(l) \, dl + \sqrt{2} W(s) \right).$$

Using the compactness of $[-M, M]$, we can pick a sequence $(s_k) \subset (0, u)$ with $\lim_{k \rightarrow \infty} s_k = u$ such that $r^*(u) := \lim_{k \rightarrow \infty} r^*(s_k)$ is a maximizer to

$$f_{\lambda_M}^{t, M} \left(2 \int_0^u v(l) \gamma_{M, \infty}(l) \, dl + \sqrt{2} W(u) \right).$$

Next, fix an arbitrary $v \in \mathcal{D}_u^M$. Define a progressively measurable process $v_k \in \mathcal{D}_u^M$ by

$$v_k(s) = v(s)1_{[0,s_k)}(s) + r^*(s_k)1_{[s_k,u]}(s).$$

Note that v_k is left-continuous at u . Consequently, we can plug this v_k into (11.26) and (11.27) to obtain that

$$\begin{aligned} & \liminf_{\beta \rightarrow \infty} \Phi_{\lambda_{M,\beta}, \alpha_{M,\beta}}^{\beta, t, u, M}(0, 0) \\ & \geq \mathbb{E} \left[f_{\lambda_{M,\infty}}^{t, M} \left(2 \int_0^{s_k} v(s) \gamma_{M,\infty}(s) ds + 2r^*(s_k) \int_{s_k}^u \gamma_{M,\infty}(s) ds + 2r^*(s_k) \Gamma_{M,\infty} + \sqrt{2}W(u) \right) \right. \\ & \quad \left. - \int_0^{s_k} v(s)^2 \gamma_{M,\infty}(s) ds - r^*(s_k)^2 \int_{s_k}^u \gamma_{M,\infty}(s) ds - r^*(s_k)^2 \Gamma_{M,\infty} \right]. \end{aligned}$$

Sending $k \rightarrow \infty$ and noting that $\int_0^u \gamma_{M,\infty}(s) ds < \infty$ imply that

$$\begin{aligned} \liminf_{\beta \rightarrow \infty} \Phi_{\lambda_{M,\beta}, \alpha_{M,\beta}}^{\beta, t, u, M}(0, 0) & \geq \mathbb{E} \left[f_{\lambda_{M,\infty}}^{t, M} \left(2 \int_0^u v(s) \gamma_{M,\infty}(s) ds + 2r^*(u) \Gamma_{M,\infty} + \sqrt{2}W(u) \right) \right. \\ & \quad \left. - \int_0^u v(s)^2 \gamma_{M,\infty}(s) ds - r^*(u)^2 \Gamma_{M,\infty} \right]. \end{aligned} \quad (11.29)$$

Here, from the definition of $f_{\lambda_{M,\infty}}^{t, M}$,

$$\begin{aligned} & f_{\lambda_{M,\infty}}^{t, M} \left(2 \int_0^u v(s) \gamma_{M,\infty}(s) ds + 2r^*(u) \Gamma_{M,\infty} + \sqrt{2}W(u) \right) \\ & \geq \left(2 \int_0^u v(s) \gamma_{M,\infty}(s) ds + 2r^*(u) \Gamma_{M,\infty} + \sqrt{2}W(u) \right) r^*(u) + \lambda_{M,\infty} r^*(u)^2 - t |r^*(u)|^p \\ & = \left(2 \int_0^u v(s) \gamma_{M,\infty}(s) ds + \sqrt{2}W(u) \right) r^*(u) + (\lambda_{M,\infty} + 2\Gamma_{M,\infty}) r^*(u)^2 - t |r^*(u)|^p. \end{aligned}$$

Plugging this lower bound into the right-hand side of (12.6) yields that

$$\begin{aligned} & \liminf_{\beta \rightarrow \infty} \Phi_{\lambda_{M,\beta}, \alpha_{M,\beta}}^{\beta, t, u, M}(0, 0) \\ & \geq \mathbb{E} \left[\left(2 \int_0^u v(s) \gamma_{M,\infty}(ds) + \sqrt{2}W(u) \right) r^*(u) + (\lambda_{M,\infty} + \Gamma_{M,\infty}) r^*(u)^2 \right. \\ & \quad \left. - t |r^*(u)|^p - \int_0^u v^2 \gamma_{M,\infty}(s) ds \right] \\ & = \mathbb{E} \left[f_{\lambda_M}^{t, M} \left(2 \int_0^u v(s) \gamma_{M,\infty}(s) ds + \sqrt{2}W(u) \right) - \int_0^u v(s)^2 \gamma_{M,\infty}(s) ds \right]. \end{aligned}$$

Since this is valid for all $v \in \mathcal{D}_u^M$, taking supremum over this set completes our proof by using (11.28). ■

We now proceed to establish the proof of Proposition 11.3. Assume that $\gamma \in \mathcal{N}_u$ satisfies $\gamma(u-) < \infty$ and $\lambda \in \mathbb{R}$. Define $\alpha_\beta(s) = \beta^{-1} \gamma(s) 1_{[0,u)}(s) + 1_{\{u\}}(s)$. Here, $\alpha_\beta \in \mathcal{M}_u$ as long as β is large enough. Since

$$f_\lambda^{\beta,u,M}(x) \leq f_\lambda^{t,M}(x) + \beta^{-1} \log 2M,$$

we have

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \Phi_{\lambda, \alpha_\beta}^{\beta,t,u,M}(0,0) &= \lim_{\beta \rightarrow \infty} \sup_{v \in \mathcal{D}_u^M} \mathbb{E} \left[f_\lambda^{\beta,t,M} \left(2 \int_0^u v(s) \gamma(s) ds + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \gamma(s) ds \right] \\ &\leq \sup_{v \in \mathcal{D}_u^M} \mathbb{E} \left[f_\lambda^{t,M} \left(2 \int_0^u v(s) \gamma(s) ds + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \gamma(s) ds \right]. \end{aligned}$$

Also note $\int_0^u s \beta \alpha_\beta(s) ds = \int_0^u s \gamma(s) ds$. From these,

$$\limsup_{\beta \rightarrow \infty} \inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) \leq \mathcal{P}_{t,u,M}(\lambda, \gamma).$$

The continuity of $\mathcal{P}_{t,u,M}$ in Proposition 11.8 then implies that

$$\limsup_{\beta \rightarrow \infty} \inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) \leq \inf_{(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u} \mathcal{P}_{t,u,M}(\lambda, \gamma).$$

To prove the reverse inequality, we use Lemma 11.12 to obtain

$$\begin{aligned} \liminf_{\beta \rightarrow \infty} \inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) &= \liminf_{\beta \rightarrow \infty} \mathcal{P}_{\beta,t,u,M}(\lambda_{M,\beta}, \alpha_{M,\beta}) \\ &\geq \Psi_{\lambda_M, \gamma_{M,\infty}}^{t,u,M}(0,0) - \lambda_{M,\infty} u - \int_0^u s v_{M,\infty}(ds) \\ &= \Psi_{\lambda_M, \gamma_{M,\infty}}^{t,u,M}(0,0) - \lambda_M u - \int_0^u s \gamma_{M,\infty}(s) ds = \mathcal{P}_{t,u,M}(\lambda_M, \gamma_{M,\infty}). \end{aligned}$$

Let $\gamma_M := \gamma_{M,\infty}$. Combining this lower bound with the upper bound obtained above, we deduce

$$\liminf_{\beta \rightarrow \infty} \inf_{(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u} \mathcal{P}_{\beta,t,u,M}(\lambda, \alpha) = \inf_{(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u} \mathcal{P}_{t,u,M}(\lambda, \gamma) = \mathcal{P}_{t,u,M}(\lambda_M, \gamma_M),$$

which gives (11.8). Consequently, we have (11.7) by using (11.6). Finally, (B11) and the fact $\lambda_M = \lambda_{M,\infty} + \Gamma_{M,\infty}$ validate (11.9). This completes our proof.

12 Proof of Theorem 1.9

We present the proof of Theorem 1.9 in this section. With the help of (1.15), the assertion (1.23) follows immediately from (1.22), so we only need to establish (1.22). Our argument consists of matching the upper and lower bounds.

12.1 Lower bound

We verify that

$$L_{p,u}(t) \geq \inf_{(\lambda,\gamma) \in \mathbb{R} \times \mathcal{N}_u^d} \mathcal{P}_{t,u}(\lambda, \gamma).$$

From Propositions 10.1 and 11.3, it remains to show that

$$\limsup_{M \rightarrow \infty} \inf_{(\lambda,\gamma) \in \mathbb{R} \times \mathcal{N}_u} \mathcal{P}_{t,u,M}(\lambda, \gamma) \geq \inf_{(\lambda,\gamma) \in \mathbb{R} \times \mathcal{N}_u^d} \mathcal{P}_{t,u}(\lambda, \gamma). \quad (12.1)$$

Our argument relies on a stochastic optimal control representation for $\Psi_{\lambda,\gamma}^{t,u}(0,0)$ in the same spirit as that of Proposition 11.7, whose proof is deferred to the appendix.

Proposition 12.1. Let $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$. There exist a probability space $(\Omega, \mathbb{P}, \mathcal{F})$, a filtration $(\mathcal{F}_s)_{0 \leq s \leq u}$, a continuous adapted process $X = (X(s))_{0 \leq s \leq u}$, and a standard Brownian motion $W = (W(s))_{0 \leq s \leq u}$ such that together they form a weak solution to the following SDE:

$$dX(s) = 2\gamma(s) \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, X(s)) ds + \sqrt{2} dW(s), \quad 0 \leq s \leq u, \quad X(0) = 0. \quad (12.2)$$

Furthermore, we have

$$\Psi_{\lambda,\gamma}^{t,u}(0,0) = \sup_{v \in \mathcal{D}_u} \mathbb{E} \left[f_{\lambda}^{t,u} \left(2 \int_0^u v(s) \gamma(s) ds + \sqrt{2} W(u) \right) - \int_0^u v(s)^2 \gamma(s) ds \right], \quad (12.3)$$

where $\mathcal{D}_u$ is the collection of all progressively measurable processes $v = (v(s))_{0 \leq s \leq u}$ with respect to the filtration $(\mathcal{F}_s)_{0 \leq s \leq u}$ with $\mathbb{E} \int_0^u |v(s)|^2 ds < \infty$. Here, the supremum is attained by the process $v \in \mathcal{D}_u$ defined by $v(s) = \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, X(s))$ for $0 \leq s \leq u$.

Remark 12.2. Recall from Proposition 11.6 that $\partial_x \Psi_{\lambda,\gamma}^{t,u,M}(s, \cdot)$ and $\partial_x \Phi_{\lambda,\alpha}^{\beta,t,u,M}(s, \cdot)$ are uniformly Lipschitz over all $x \in [0, u]$. This ensures the existence of the strong solutions of X and X' in Proposition 11.7. In (12.2), $\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, \cdot)$ is not Lipschitz, but it can be shown from the Hopf–Cole transformation (1.20) that it is of moderate growth, namely there exists some $0 < a < 1$ such that $|\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, x)| \leq C(1 + |x|^a)$ for any $(s, x) \in ([0, u] \times \mathbb{R}) \setminus \{(u, 0)\}$ (see Lemma B.3), where $C > 0$ depends on γ . Hence,

$$|\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, x)| \leq 2C(1 + |x|), \quad \forall (s, x) \in ([0, u] \times \mathbb{R}) \setminus \{(u, 0)\}.$$

This ensures the existence of the weak solution of (12.2); see [35, Proposition 5.3.6]. Additionally, from this upper bound, it is a standard application of the Gronwall inequality (see, e.g., [35, Problem 5.3.15]) that

$$\mathbb{E} \max_{0 \leq s \leq u} |X(s)|^2 \leq 4u(C_0^2 + 1)e^{4u^2 C_0^2}, \quad C_0 := 2\gamma(u)C.$$

However, this upper bound is not good enough for us since in the Parisi formula we take infimum over all γ and this bound might diverge.

The following technical lemma controls $\mathbb{E}|X(u)|^2$ uniformly over bounded λ and $\int_0^u \gamma(s)ds$.

Lemma 12.3. Assume that $t, u > 0$. There exists a nonnegative continuous function K on $(2, \infty) \times (0, \infty)^2 \times \mathbb{R} \times [0, \infty) \rightarrow [0, \infty)$ such that for any $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$, we have that

$$\mathbb{E}|X(u)|^2 \leq K\left(p, t, u, \lambda, \int_0^u \gamma(s) ds\right).$$

Proof. From (B2),

$$\begin{aligned} \Psi_{\lambda,\gamma}^{t,u}(u, X(u)) &\leq \left(\frac{2}{t}\right)^{1/(p-1)} |X(u)|^{1+1/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)} |X(u)| \\ &\quad + \left(\frac{2}{t}\right)^{2/(p-1)} |\lambda| |X(u)|^{2/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{2/(p-2)} |\lambda|. \end{aligned} \quad (12.4)$$

Note that the assumption $p > 2$ ensures that

$$\max\left(1 + \frac{1}{p-1}, \frac{2}{p-1}\right) < 2. \quad (12.5)$$

Using Jensen's inequality yields

$$\mathbb{E}\Psi_{\lambda,\gamma}^{t,u}(u, X(u)) \leq K_1 \left((\mathbb{E}|X(u)|^2)^{1/2+1/2(p-1)} + (\mathbb{E}|X(u)|^2)^{1/2} + (\mathbb{E}|X(u)|^2)^{1/(p-1)} + 1 \right) \quad (12.6)$$

for

$$K_1 := \left(\frac{2}{t}\right)^{1/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)} + \left(\frac{2}{t}\right)^{2/(p-1)}|\lambda| + \left(\frac{2|\lambda|}{t}\right)^{2/(p-2)}|\lambda|.$$

Let $v(s) := \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, X(s))$ and $L := \int_0^u \gamma(s)v(s)ds$. Since

$$\mathbb{E}|X(u)|^2 \leq 16(\mathbb{E}L^2 + \mathbb{E}|W(u)|^2) = 16(\mathbb{E}L^2 + u),$$

it follows that by using the trivial bound $(x+y)^a \leq 2^a(x^a + y^a)$ for all $a, x, y \geq 0$, $(\mathbb{E}|X(u)|^2)^a \leq 32^a(\mathbb{E}L^2)^a + 32^a u^a$. From this and (12.6),

$$\mathbb{E}\Psi_{\lambda,\gamma}^{t,u}(u, X(u)) \leq K_2 \left((\mathbb{E}L^2)^{1/2+1/2(p-1)} + (\mathbb{E}L^2)^{1/2} + (\mathbb{E}L^2)^{1/(p-1)} + 1 \right), \quad (12.7)$$

where K_2 depends on t, u, p, λ . By comparing the representation (12.3) of $\Psi_{\lambda,\gamma}^{t,u}(0, 0)$ with the optimal control process $v(s) = \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, X(s))$ against the zero control process, we obtain that

$$\mathbb{E}\Psi_{\lambda,\gamma}^{t,u}(u, X(u)) - \int_0^u \gamma(s)\mathbb{E}v(s)^2 ds = \Psi_{\lambda,\gamma}^{t,u}(0, 0) \geq \mathbb{E}f_{\lambda}^{t,u}(\sqrt{2}W(u)) \geq 0,$$

which implies that, by the Cauchy-Schwarz inequality,

$$\mathbb{E}\Psi_{\lambda,\gamma}^{t,u}(u, X(u)) \geq \int_0^u \gamma(s)\mathbb{E}v(s)^2 ds \geq \frac{\mathbb{E}L^2}{\int_0^u \gamma(s) ds}.$$

From this and (12.7), we arrive at

$$\mathbb{E}L^2 \leq K_2 \left(\int_0^u \gamma(s) ds \right) \left((\mathbb{E}L^2)^{1/2+1/2(p-1)} + (\mathbb{E}L^2)^{1/2} + (\mathbb{E}L^2)^{1/(p-1)} + 1 \right).$$

From (12.5), $1/2+1/2(p-1)$ and $1/(p-1)$ are strictly less than 1. Now, if $\mathbb{E}L^2 \geq 1$, dividing $(\mathbb{E}L^2)^\delta$ on the both sides of the above inequality deduces that

$$(\mathbb{E}L^2)^{1-\delta} \leq 4K_2 \int_0^u \gamma(s) ds$$

for $\delta := \max(1/2 + 1/2(p-1), 1/(p-1))$. Hence, we arrive at the bound,

$$\mathbb{E}L^2 \leq K_3 := \max\left(1, \left(4K_2 \int_0^u \gamma(s) ds\right)^{1/(1-\delta)}\right).$$

Our proof then follows by using $\mathbb{E}|X(u)|^2 \leq 16(\mathbb{E}L^2 + u)$. ■

We now establish (12.7). Recall the optimizer (λ_M, γ_M) of $\mathcal{P}_{t,u,M}$ from Proposition 11.3. Note that from the continuity of $\mathcal{P}_{t,u,M}$ in Proposition 11.8, we can approximate $\mathcal{P}_{t,u,M}(\lambda_M, \gamma_M)$ by considering $\mathcal{P}_{t,u,M}(\lambda_M, \gamma)$ for $\gamma \in \mathcal{N}_u^d$ and by letting γ to be as close as we want to γ_M under the norm $\int_0^u |\gamma(s) - \gamma_M(s)| ds$. Hence, without loss of generality, we may assume that $\gamma_M \in \mathcal{N}_u^d$. Recall from (11.9) that λ_M and $\int_0^u \gamma_M ds$ are bounded by a constant C independent of M .

As $p > 2$, in the definition of f_λ^t , the term $t|r|^p$ dominates the linear term xr and the quadratic term λr^2 . So, for any $L > 0$, there exists some $M_L > 0$ such that

$$f_\lambda^t(x) = f_\lambda^{t,M}(x), \quad \text{for all } |x| \leq L, |\lambda| \leq C, M \geq M_L.$$

Hence, as long as $M \geq M_L$, we can write that for any $x \in \mathbb{R}$,

$$f_\lambda^t(x) = f_\lambda^{t,M}(x)1_{\{|x| \leq L\}} + f_\lambda^t(x)1_{\{|x| > L\}} \leq f_\lambda^{t,M}(x) + f_\lambda^t(x)1_{\{|x| > L\}}.$$

Let X_M be the stochastic process X defined in Proposition 12.1 associated with $\gamma = \gamma_M$. From this inequality, letting $v_M(s) := \partial_x \Psi_{\lambda_M, \gamma_M}^{t,u}(s, X_M(s))$, and using Propositions 11.7 and 12.1, we have

$$\begin{aligned} \Psi_{\lambda_M, \gamma_M}^{t,u}(0, 0) &= \mathbb{E} \Psi_{\lambda_M, \gamma_M}^{t,u}(u, X_M(u)) - \int_0^u \gamma_M(s) \mathbb{E} v_M(s)^2 ds \\ &\leq \Psi_{\lambda_M, \gamma_M}^{t,u,M}(0, 0) + \mathbb{E} \Psi_{\lambda_M, \gamma_M}^{t,u}(u, X_M(u)) 1_{\{|X_M(u)| > L\}}. \end{aligned} \tag{12.8}$$

To handle the 2nd term, we need some moment controls. Note that $p > 2$ implies

$$\zeta := \frac{2}{\max(1 + \frac{1}{p-1}, \frac{2}{p-1})} \in (1, 2).$$

From (12.4) and Lemma 12.3, we can apply Jensen's inequality along with the bounds $|\lambda_M| \leq C$ and $\int_0^u \gamma_M ds \leq C$ to get

$$\mathbb{E} |\Psi_{\lambda_M, \gamma_M}^{t,u}(u, X_M(u))|^\zeta \leq K((\mathbb{E}|X_M(u)|^2)^{\frac{\zeta}{2}(1+\frac{1}{p-1})} + (\mathbb{E}|X_M(u)|^2)^{\frac{\zeta}{2}} + (\mathbb{E}|X_M(u)|^2)^{\frac{\zeta}{p-1}} + 1) \leq K',$$

where K and K' are constants depending only on p, t, u . On the other hand, from the Markov inequality and Lemma 12.3,

$$\mathbb{P}(|X_M(u)| > L) \leq \frac{\mathbb{E}|X_M(u)|^2}{L^2} \leq \frac{K''}{L^2},$$

where again K'' depends only on p, t, u . Applying the Hölder inequality to (12.8) and using the above two inequalities yield that

$$\begin{aligned} \Psi_{\lambda_M, \gamma_M}^{t,u}(0, 0) &\leq \Psi_{\lambda_M, \gamma_M}^{t,u,M}(0, 0) + (\mathbb{E}|\Psi_{\lambda_M, \gamma_M}^{t,u}(u, X_M(u))|^\zeta)^{1/\zeta} \mathbb{P}(|X_M(u)| > L)^{1/\zeta'} \\ &\leq \Psi_{\lambda_M, \gamma_M}^{t,u,M}(0, 0) + \frac{K'^{1/\zeta} K''^{1/\zeta'}}{L^{2/\zeta'}}, \end{aligned}$$

where ζ' is the Hölder conjugate exponent of ζ . As a result, for any $M \geq M_L$,

$$\begin{aligned} \inf_{(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d} \mathcal{P}_{t,u}(\lambda, \gamma) &\leq \mathcal{P}_{t,u}(\lambda_M, \gamma_M) \\ &\leq \mathcal{P}_{t,u,M}(\lambda_M, \gamma_M) + \frac{K'^{1/\zeta} K''^{1/\zeta'}}{L^{2/\zeta'}} = \inf_{(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u} \mathcal{P}_{t,u,M}(\lambda, \gamma) + \frac{K'^{1/\zeta} K''^{1/\zeta'}}{L^{2/\zeta'}}. \end{aligned}$$

Since this is valid for any $L > 0$ and K', K'' are independent of L , sending $M \rightarrow \infty$ and then $L \rightarrow \infty$ completes the proof of (12.7).

12.2 Upper bound

Next, we verify that

$$L_{p,u}(t) \leq \inf_{\lambda \in \mathbb{R}, \gamma \in \mathcal{N}_u^d} \mathcal{P}_{t,u}(\lambda, \gamma). \quad (12.9)$$

From Proposition 10.1, we have seen that

$$L_{p,u}(t) \leq \lim_{\beta \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} F_{n,t,u,\beta^{-2}}(\beta). \quad (12.10)$$

To control the right-hand side, recall $\mathcal{P}_{\beta,t,u}(\lambda, \alpha)$ for $(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u^d$ from Table 1. The replica symmetry breaking bound, due to Guerra (see [29] and [50]), states that for any $\beta > 0$, $\varepsilon > 0$, and $(\lambda, \alpha) \in \mathbb{R} \times \mathcal{M}_u^d$, we have that

$$\mathbb{E} F_{n,t,u,\varepsilon}(\beta) \leq \mathcal{P}_{\beta,t,u}(\lambda, \alpha) + C_0 \beta \varepsilon, \quad (12.11)$$

where $C_0 > 0$ is a universal constant independent of n, β , and ε . Recall $f_\lambda^{\beta,t}(x)$ and $f_\lambda^t(x)$ from Table 1. To relate $\mathcal{P}_{\beta,t,u}(\lambda, \alpha)$ to $\mathcal{P}_{t,u}(\lambda, \gamma)$, fix $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$, and let $\alpha(s) = \beta^{-1}\gamma(s)\mathbf{1}_{[0,u)}(s) + \mathbf{1}_{\{u\}}(s)$. Note that $\alpha \in \mathcal{M}_u^d$ as long as β is large enough. For any $0 < \delta < t$, since

$$\begin{aligned} f_\lambda^{\beta,t}(x) &= \frac{1}{\beta} \log \frac{\int_{-\infty}^{\infty} e^{\beta(rx + \lambda r^2 - (t-\delta)|r|^p)} e^{-\beta\delta|r|^p} dr}{\int_{-\infty}^{\infty} e^{-\beta\delta|r|^p} dr} + \frac{1}{\beta} \log \int_{-\infty}^{\infty} e^{-\beta\delta|r|^p} dr \\ &\leq f_\lambda^{t-\delta}(x) + \frac{1}{\beta} \log \int_{-\infty}^{\infty} e^{-\beta\delta|r|^p} dr \end{aligned}$$

and $\int_{-\infty}^{\infty} e^{-\beta\delta|r|^p} dr = (\beta\delta)^{-1/p} \int_{-\infty}^{\infty} e^{-|s|^p} ds$, it can be argued by using the Hopf–Cole transformation in Lemma B.2 iteratively that

$$\Phi_{\lambda,\alpha}^{\beta,t,u}(0,0) \leq \Psi_{\lambda,\gamma}^{t-\delta,u}(0,0) - \frac{1}{p\beta} \log \beta\delta + \frac{1}{\beta} \log \int_{-\infty}^{\infty} e^{-|s|^p} ds.$$

Note $\int_0^u s\beta\alpha(s) ds = \int_0^u s\gamma(s) ds$. From (12.10) and (12.11), after sending β to infinity, $L_{p,u}(t) \leq \mathcal{P}_{t-\delta,u}(\lambda, \gamma)$. Finally, using (B12), (B14), and the dominated convergence theorem yields $\lim_{\delta \downarrow 0} \mathcal{P}_{t-\delta,u}(\lambda, \gamma) = \mathcal{P}_{t,u}(\lambda, \gamma)$ and, consequently, $L_{p,u}(t) \leq \mathcal{P}_{t,u}(\lambda, \gamma)$. Taking infimum in this inequality validates (12.9).

13 Open Questions

In this section, we mention a few open problems and further directions.

1. Compute the order and the limiting distribution of the fluctuation of the ℓ_p -Grothendieck problem. The asymptotic fluctuation for $p = 2$ (the largest eigenvalue of the GOE) is known to follow the Tracy–Widom distribution with the usual $n^{-1/6}$ scaling [66, 67]. The proof of Theorem 1.1 suggests that the limiting fluctuation in the case $p = 1$ could possibly be Gumbel. For the case $2 < p < \infty$, since $L_{n,p}(t)$ is essentially a SK model with ℓ_p external field, it seems reasonable to believe that the limiting fluctuation of $L_{n,p}$ is Gaussian (see [17]) and in view of (8.2), $G_{n,p}$ should also be Gaussian up to a transformation. When $p = \infty$, the ℓ_p -Grothendieck problem becomes the ground state energy of the SK model and the numerical studies, see, for example, [49], indicate that the limiting fluctuation should follow the Gumbel distribution, even though a rigorous proof remains elusive. We do not have a prediction in the case $1 < p < 2$. To analyze this case, one

would naturally use the fact that the global optimizer lies close to one of the approximate optimizers in $\mathcal{O}$, described in (1.7). However, it may be possible that determining the limiting distribution of fluctuation requires finer approximation for the global optimizer (with better error bound).

2. For the ℓ_p -Grothendieck problem for $2 < p < \infty$, show that with high probability, there exist exponentially many near optimizers that are nearly mutually orthogonal to each other.
3. For the ℓ_p -Grothendieck problem for $2 < p < \infty$, devise a polynomial-time algorithm that outputs an approximate optimizer with high probability. For $p = \infty$ (SK model), [24, 44] used the approximate message passing algorithm to find a near ground state in polynomial time with high probability (see also [62]).
4. Improve the delocalization bound (1.17) for $2 < p < \infty$. In this case, we expect that an optimizer x_* in the unit ℓ_p -ball should satisfy $\|x_*\|_\infty = O((\log n)^C n^{-1/p})$ for some constant $C > 0$ with high probability.
5. Study the ℓ_p -Grothendieck problem for random i.i.d. matrices with more general entry distribution (e.g., with subgaussian distribution). Do we have universality if $2 < p < \infty$?
6. Find the limit of the operator norm $\|G_n\|_{p \rightarrow q}$ for all $1 \leq p, q \leq \infty$. In view of Proposition 3.4, this problem is yet to be resolved for pairs (p, q) satisfying $(p, q) \in (1, \infty) \times [1, \infty)$ and $\mathfrak{p}(q, p^*) = \mathfrak{p}(p^*, q)$, which is the set of pairs satisfying that

$$(i) (p, q) \in (1, 2] \times [2, \infty) \text{ with } q = p^* \text{ or } (ii) (p, q) \in [2, \infty] \times [1, 2].$$

A special case of (i) and (ii) is when $1 < p \leq \infty$ and $q = p^*$, which corresponds to the p -to- p^* norm and it can be expressed as a bipartite ℓ_p -Grothendieck problem via

$$\|G_n\|_{p \rightarrow p^*} = \max_{x, y \in \mathbb{R}^n: \|x\|_p \leq 1, \|y\|_{p^*} \leq 1} \langle G_n x, y \rangle.$$

For $1 < p < 2$, an argument similar to the one presented in Theorem 1.2 might be useful to find the limit of $\|G_n\|_{p \rightarrow p^*}$. For $p = \infty$, the limit of $\|G_n\|_{p \rightarrow p^*} = \|G_n\|_{\infty \rightarrow 1}$ is known as the ground state energy of the bipartite SK model and is conjectured to admit a minmax-type Parisi formula (see [7, 46]). For $2 < p < \infty$, we expect that the same formulation should still be valid.

Remark 13.1. In a recent work, Dhara–Mukherjee–Ramanan [22] studied the p -to- q -norm for $n \times n$ symmetric random matrix A_n with i.i.d. nonnegative entries, including the adjacency matrices of Erdős–Rényi random graphs. The p -to- q -norm is related to the ℓ_p -Grothendieck problem in the following way (see [22, Proposition 2.14]): for $p \geq 2$ and for any $n \times n$ matrix A_n such that $A_n^T A_n$ is an irreducible matrix with nonnegative entries, it holds that

$$\|A_n\|_{p \rightarrow p^*} = \max_{x \in \mathbb{R}^n: \|x\|_p \leq 1} \langle A_n x, x \rangle.$$

In [22], Gaussian fluctuation for $\|A_n\|_{p \rightarrow q}$ if $1 \leq q \leq p < \infty$ was established, which includes, as a special case, the ℓ_p -Grothendieck problem for A_n for $p \geq 2$. However, the proof uses a nonlinear power iteration method, originally introduced by Boyd [12], to approximate the p -to- q -norm. It relies heavily on the nonnegativity of the entries of A_n and cannot be directly applied to the Gaussian case. Also, for $1 \leq q \leq p < \infty$, the optimizer for $\|A_n\|_{p \rightarrow q}$ is close to the scaled constant vector. Obviously, this does not hold in the Gaussian case.

A Stability of Hölder’s Inequality

In this appendix, we gather some quantitative results on the inequalities used in the proof of Theorem 1.4. The 1st lemma provides a stability bound for Hölder’s inequality.

Lemma A.1. Let $1 < p \leq 2$, and let w be a nonzero vector in $\mathbb{R}^n$. Let v be the vector with unit ℓ_p -norm such that $\langle v, w \rangle = \|w\|_{p^*}$, that is, v satisfies

$$\operatorname{sgn}(v_i) |v_i|^p = \frac{\operatorname{sgn}(w_i) |w_i|^{p^*}}{\|w\|_{p^*}^{p^*}}, \quad \forall i.$$

Then, for any $u \in \mathbb{R}^n$ with $\|u\|_p = 1$,

$$\langle u, w \rangle \leq \|w\|_{p^*} \left(1 - \frac{p^2}{16p^*} \|u - v\|_p^2 \right).$$

Proof. First of all, we claim that for $1 < p \leq 2$ and $a, b \in \mathbb{R}$, we have the following stability bound for Young’s inequality:

$$\frac{|a|^p}{p} + \frac{|b|^{p^*}}{p^*} \geq ab + \frac{1}{2p^*} (|a|^{p/2} \operatorname{sgn}(a) - |b|^{p^*/2} \operatorname{sgn}(b))^2.$$

The case when both a and b have the same sign follows from [1, Lemma 2.1]. If a and b have opposite signs, then we need to show, after ignoring the negative term ab , that

$$\frac{|a|^p}{p} + \frac{|b|^{p^*}}{p^*} \geq \frac{1}{2p^*} (|a|^{p/2} + |b|^{p^*/2})^2,$$

which follows from the inequality $(x + y)^2 \leq 2(x^2 + y^2)$ and the fact that $p \leq p^*$. This completes the proof of our claim.

To prove our main result, without loss of generality, assume that $\|w\|_{p^*} = 1$. Applying our claim to u_i and w_i and then summing over i , we obtain

$$\begin{aligned} \langle u, w \rangle &\leq 1 - \frac{1}{2p^*} \sum_i (|u_i|^{p/2} \operatorname{sgn}(u_i) - |w_i|^{p^*/2} \operatorname{sgn}(w_i))^2 \\ &= 1 - \frac{1}{2p^*} \sum_i (|u_i|^{p/2} \operatorname{sgn}(u_i) - |v_i|^{p/2} \operatorname{sgn}(v_i))^2. \end{aligned}$$

For $0 \leq \theta \leq 1$, we have

$$1 - \theta^{p/2} = \int_{\theta}^1 \frac{p}{2} x^{p/2-1} dx \geq \frac{p}{2} (1 - \theta) \quad \text{and} \quad 1 + \theta^{p/2} \geq 1 + \theta \geq \frac{p}{2} (1 + \theta). \quad (\text{A.1})$$

Set $a_i = \min(|u_i|, |v_i|)$ and $b_i = \max(|u_i|, |v_i|)$. Also, let $K = \sum_i b_i^p \in [1, 2]$. We now write

$$\begin{aligned} &\sum_i (|u_i|^{p/2} \operatorname{sgn}(u_i) - |v_i|^{p/2} \operatorname{sgn}(v_i))^2 \\ &= K \sum_{i: b_i \neq 0} \frac{b_i^p}{K} \left(1 - \operatorname{sgn}(u_i v_i) \left(\frac{a_i}{b_i}\right)^{p/2}\right)^2 \geq \frac{Kp^2}{4} \sum_{i: b_i \neq 0} \frac{b_i^p}{K} \left(1 - \operatorname{sgn}(u_i v_i) \frac{a_i}{b_i}\right)^2 \\ &\geq \frac{Kp^2}{4} \left(\sum_{i: b_i \neq 0} \frac{b_i^p}{K} \left|1 - \operatorname{sgn}(u_i v_i) \frac{a_i}{b_i}\right|^p\right)^{2/p} = \frac{p^2}{4K^{2/p-1}} \left(\sum_{i: b_i \neq 0} \left|b_i - \operatorname{sgn}(u_i v_i) a_i\right|^p\right)^{2/p} \\ &= \frac{p^2}{4K^{2/p-1}} \left(\sum_{i: b_i \neq 0} |u_i - v_i|^p\right)^{2/p} \geq \frac{p^2}{8} \|u - v\|_p^2, \end{aligned}$$

where the 1st inequality follows from (A.1) and the 2nd one uses Jensen's inequality. This finishes our proof. $\blacksquare$

Next, we show that if a ℓ_p -unit vector ($1 < p < 2$) stays away from the coordinate vectors, then its ℓ^2 -norm must be strictly less than one.

Lemma A.2. Fix $1 < p < 2$. There exists constant $c > 0$ such that if $x \in \mathbb{R}^n$ satisfies $\|x\|_p = 1$ and $\|x \pm e_i\|_p \geq \delta$ for all i and for some $\delta > 0$, then $\|x\|_2 \leq 1 - c\delta^p$.

Proof. It is enough to prove the bound for $0 < \delta \leq \delta_0$ for sufficiently small constant δ_0 . We first claim that $\|x\|_\infty \leq 1 - \delta^p/3$. Suppose, if possible, that $|x_i| \geq 1 - \delta^p/3$ for some i . For definiteness, assume that $x_i \geq 1 - \delta^p/3$ as the case $x_i \leq -(1 - \delta^p/3)$ can be dealt similarly. Using the identity $\sum_{j \neq i} |x_j|^p = 1 - |x_i|^p$, we obtain that

$$\|x - e_i\|_p^p = |1 - x_i|^p + 1 - |x_i|^p < |1 - x_i| + 1 - |x_i|^2 = (1 - x_i)(2 + x_i) \leq \frac{\delta^p}{3} \cdot 3 = \delta^p,$$

contradicting the assumption that $\|x - e_i\|_p \geq \delta$. Under the constraints $\|x\|_\infty \leq 1 - \delta^p/3$ and $\|x\|_p = 1$, the ℓ_2 -norm of x is maximized if x is supported only on two coordinates, one of them being equal to $1 - \delta^p/3$ in magnitude and the other nonzero component of x must have absolute value $(1 - (1 - \delta^p/3)^p)^{1/p}$. Hence,

$$\begin{aligned} \|x\|_2^2 &\leq (1 - \delta^p/3)^2 + (1 - (1 - \delta^p/3)^p)^{2/p} \leq (1 - \delta^p/3)^2 + (1 - (1 - \delta^p/3)^2)^{2/p} \\ &\leq 1 - 2\delta^p/3 + (\delta^p/3)^2 + (2\delta^p/3)^{2/p} \leq 1 - \delta^p/3, \end{aligned}$$

by choosing δ_0 sufficiently small. This implies that $\|x\|_2 \leq (1 - \delta^p/3)^{1/2} \leq 1 - \delta^p/6$. ■

B Analytic Results of the Parisi PDE

For $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$, recall the PDE solution $\Psi_{\lambda, \gamma}^{t, u}$ defined in (1.20). In this appendix, we will gather some regularity properties of this PDE solution and provide the proof of Proposition 12.1. First of all, the boundary condition f_λ^t has the following regularity.

Lemma B.1. If $\lambda = 0$, then for all $x \neq 0$,

$$\begin{aligned} f_\lambda^t(x) &= \frac{(p-1)}{p} \frac{1}{p^{\frac{1}{p-1}} t^{\frac{1}{p-1}}} |x|^{1+\frac{1}{p-1}}, \\ \left| \frac{d}{dx} f_\lambda^t(x) \right| &= \frac{1}{p^{\frac{1}{p-1}} t^{\frac{1}{p-1}}} |x|^{\frac{1}{p-1}}. \end{aligned} \tag{B.1}$$

If $\lambda \neq 0$, then f_λ^t is twice differentiable on $\mathbb{R} \setminus \{0\}$ and for $x \neq 0$,

$$\begin{aligned} 0 < f_\lambda^t(x) &\leq \left(\frac{2}{t}\right)^{1/(p-1)} |x|^{1+1/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)} |x| \\ &\quad + \left(\frac{2}{t}\right)^{2/(p-1)} |\lambda| |x|^{2/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{2/(p-2)} |\lambda| \end{aligned} \tag{B.2}$$

and

$$\left| \frac{d}{dx} f_\lambda^t(x) \right| \leq \max\left(\left(\frac{2|x|}{t}\right)^{1/(p-1)}, \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)}\right). \tag{B.3}$$

Proof. When $\lambda = 0$, the assertion (B.1) can be checked by a straightforward computation. For the remainder of the proof, we will only focus on the case $\lambda \neq 0$. Note that f_λ^t

is an even function. To prove our assertion, it suffices to assume that $x > 0$. Evidently, $f_\lambda^t(x) > 0$. Note that for any $r > 0$,

$$rx + \lambda r^2 - t|r|^p > (-r)x + \lambda(-r)^2 - t|-r|^p,$$

which implies that $f_\lambda^t(x) = \sup_{r>0} (xr + \lambda r^2 - tr^p)$. Here, for $r > 0$,

$$\frac{d}{dr}(xr + \lambda r^2 - tr^p) = x + 2\lambda r - ptr^{p-1}. \quad (\text{B.4})$$

Since $x > 0$, $0 < r \mapsto x + 2\lambda r$ is a linear function, and $0 < r \mapsto ptr^{p-1}$ is strictly convex, we see that (B.4) has a unique zero, $r(x) > 0$. In addition, since (B.4) equals $x > 0$ at $r \rightarrow 0^+$ and it is negative when r is sufficient large, these imply that $r(x)$ is an unique optimizer of $f_\lambda^t(x)$. This also implies that $0 < x \mapsto r(x)$ must be continuous. Now, if

$$r(x) \geq \max\left(\left(\frac{2x}{t}\right)^{1/(p-1)}, \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)}\right),$$

then

$$xr(x) - t\frac{r(x)^p}{2} < 0 \text{ and } \lambda r(x)^2 - t\frac{r(x)^p}{2} < 0.$$

Together, they imply that $f_\lambda^t(x) = xr(x) + \lambda r(x)^2 - tr(x)^p < 0$, a contradiction. Hence, we must have that

$$r(x) < \max\left(\left(\frac{2x}{t}\right)^{1/(p-1)}, \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)}\right). \quad (\text{B.5})$$

Consequently, dropping $t|r|^p$ and using this inequality, we arrive at (B.2) since

$$\begin{aligned} f_\lambda^t(x) &\leq x \max\left(\left(\frac{2x}{t}\right)^{1/(p-1)}, \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)}\right) + |\lambda| \max\left(\left(\frac{2x}{t}\right)^{2/(p-1)}, \left(\frac{2|\lambda|}{t}\right)^{2/(p-2)}\right) \\ &\leq \left(\frac{2}{t}\right)^{1/(p-1)} x^{1+1/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{1/(p-2)} x + \left(\frac{2}{t}\right)^{2/(p-1)} |\lambda| x^{2/(p-1)} + \left(\frac{2|\lambda|}{t}\right)^{2/(p-2)} |\lambda|. \end{aligned}$$

To show (B.3), note that

$$\frac{d}{dr}(x + 2\lambda r - ptr^{p-1}) \Big|_{r=r(x)} = 2\lambda - pt(p-1)r(x)^{p-2}.$$

If $\lambda < 0$, then this derivative is strictly less than $2\lambda < 0$; if $\lambda > 0$, plugging

$$x + 2\lambda r(x) = ptr(x)^{p-1}$$

into the above equation gives that

$$\frac{d}{dr}(x + 2\lambda r - ptr^{p-1}) \Big|_{r=r(x)} = -2(p-2)\lambda - \frac{x}{r(x)}(p-1) < 0.$$

From these and the implicit function theorem, $r(x)$ is differentiable on $(0, \infty)$, and consequently,

$$\frac{d}{dx}f_\lambda^t(x) = \frac{d}{dx}(xr(x) + \lambda r(x)^2 - tr(x)^p) = r(x) + r'(x)(x + 2\lambda r(x) - ptr(x)^{p-1}) = r(x).$$

The inequality (B.3) then follows from (B.5). $\blacksquare$

Lemma B.2. Let $a \in (0, 1)$ and $b \in (0, \infty)$. Let f be a nonnegative, even, and convex function on $\mathbb{R}$. Assume that there exists a constant $C > 0$ such that

$$f(x) \leq C(1 + |x|^{a+1}), \quad x \in \mathbb{R}. \quad (\text{B.6})$$

Let $m > 0$. For any $(s, x) \in [0, b] \times \mathbb{R}$, set

$$F(s, x) = \frac{1}{m} \log \mathbb{E} \exp mf(x + \sqrt{s}z), \quad (\text{B.7})$$

where z is standard normal. The following statements hold.

- (i) For any $s \in [0, b]$, $F(s, \cdot)$ is nonnegative, even, and convex on $\mathbb{R}$.
- (ii) $F \in C^{1,2}((0, b] \times \mathbb{R})$.
- (iii) F satisfies

$$\partial_s F(s, x) = \frac{1}{2} \left(\partial_{xx} F(s, x) + m(\partial_x F(s, x))^2 \right), \quad (s, x) \in (0, b] \times \mathbb{R},$$

$$\lim_{s \downarrow 0, x \rightarrow x_0} F(s, x) = f(x), \quad \forall x_0 \in \mathbb{R}.$$

- (iv) There exists a constant $C' > 0$ depending on a, b, m, C such that for any $(s, x) \in (0, b] \times \mathbb{R}$,

$$|F(s, x)| \leq C'(1 + |x|^{a+1}). \quad (\text{B.8})$$

- (v) If additionally, f is differentiable on $\mathbb{R} \setminus \{0\}$ and there exists a constant C such that

$$|f'(x)| \leq C(1 + |x|^a), \quad x \in \mathbb{R} \setminus \{0\}, \quad (\text{B.9})$$

then there exists a constant C' depending on a, b, m, C such that for any $(s, x) \in (0, b] \times \mathbb{R}$,

$$|\partial_x F(s, x)| \leq C'(1 + |x|^a). \quad (\text{B.10})$$

Proof. First of all, note that F is well defined due to (B.6) and $a < 1$. To prove (i), the nonnegativity and the evenness of F follow directly from those of f ; using the Hölder inequality and the convexity of f yield that for all $0 \leq s \leq b$, $0 \leq \ell \leq 1$, and $x, y \in \mathbb{R}$,

$$\begin{aligned} F(s, \ell x + (1 - \ell)y) &= \frac{1}{m} \log \mathbb{E} \exp mf(\ell x + (1 - \ell)y + \sqrt{s}z) \\ &\leq \frac{1}{m} \log \mathbb{E} \exp m(\ell f(x + \sqrt{s}z) + (1 - \ell)f(y + \sqrt{s}z)) \leq \ell F(s, x) + (1 - \ell)F(s, y), \end{aligned}$$

establishing the convexity of $F(s, \cdot)$. To show (ii) and (iv), write

$$F(s, x) = \frac{1}{m} \log \frac{1}{\sqrt{2\pi s}} \int e^{mf(y) - \frac{(y-x)^2}{2s}} dy.$$

From this expression, (B.6), and the dominated convergence theorem, we see that F lies in the class $C^{1,2}((0, b] \times \mathbb{R})$. Furthermore, for any $(s, x) \in (0, b] \times \mathbb{R}$,

$$\begin{aligned} \partial_x F(s, x) &= \frac{1}{sm} \frac{\int_{\mathbb{R}} (y-x) \varphi(y) dy}{\int_{\mathbb{R}} \varphi(y) dy}, \\ \partial_{xx} F(s, x) &= -\frac{1}{sm} + \frac{1}{s^2 m} \frac{\int_{\mathbb{R}} (y-x)^2 \varphi(y) dy}{\int_{\mathbb{R}} \varphi(y) dy} - \frac{1}{s^2 m} \left(\frac{\int_{\mathbb{R}} (y-x) \varphi(y) dy}{\int_{\mathbb{R}} \varphi(y) dy} \right)^2, \\ \partial_s F(s, x) &= -\frac{1}{2sm} + \frac{1}{2s^2 m} \frac{\int_{\mathbb{R}} (y-x)^2 \varphi(y) dy}{\int_{\mathbb{R}} \varphi(y) dy} \end{aligned}$$

for $\varphi(y) := e^{mf(y) - (y-x)^2/(2s)}$. This validates (ii) as well as the PDE in (iii). The continuity of F at the boundary in (iii) is also guaranteed by using (B.6), (B.7), and the dominated convergence theorem. The assertion (iv) can follow directly from (B.6) and noting $a < 1$.

Finally, we prove (v). Without loss of generality, we assume that $x \geq 0$. Note that we can also use (B.6), (B.9), and the dominated convergence theorem to get that

$$\partial_x F(s, x) = \frac{\mathbb{E} f'(x + \sqrt{s}Z) e^{mf(x + \sqrt{s}Z)}}{\mathbb{E} e^{mf(x + \sqrt{s}Z)}}.$$

From this, write

$$|\partial_x F(s, x)| \leq \left| \frac{\mathbb{E}[f'(x + \sqrt{s}Z) e^{mf(x + \sqrt{s}Z)}; |Z| \geq x]}{\mathbb{E} e^{mf(x + \sqrt{s}Z)}} \right| + \left| \frac{\mathbb{E}[f'(x + \sqrt{s}Z) e^{mf(x + \sqrt{s}Z)}; |Z| < x]}{\mathbb{E} e^{mf(x + \sqrt{s}Z)}} \right|. \quad (\text{B.11})$$

To bound the 1st term, note that since $e^{mf(\cdot)}$ is convex, Jensen's inequality and the assumption $f \geq 0$ imply that

$$1 \leq e^{mf(x)} = e^{mf(x + \sqrt{s}\mathbb{E}Z)} \leq \mathbb{E} e^{mf(x + \sqrt{s}Z)}.$$

On the other hand, from (B.6) and (B.9),

$$\mathbb{E}[|f'(x + \sqrt{s}Z)| e^{mf(x + \sqrt{s}Z)}; |Z| \geq x] \leq C \mathbb{E}[(1 + (1 + \sqrt{b})^a |Z|^a) e^{mC(1 + (1 + \sqrt{b})^{a+1} |Z|^{a+1})}] =: C',$$

where C' is finite due to the assumption that $a < 1$. Putting these two inequalities together implies that the 1st term of (B.11) is bounded above by C' . For the 2nd term in (B.11), note that

$$\begin{aligned} \mathbb{E}[|f'(x + \sqrt{s}Z)| e^{mf(x + \sqrt{s}Z)}; |Z| < x] &\leq C(1 + (1 + \sqrt{s})^a |x|^a) \mathbb{E}[e^{mf(x + \sqrt{s}Z)}; |Z| < x] \\ &\leq C(1 + (1 + \sqrt{b})^a |x|^a) \mathbb{E}[e^{mf(x + \sqrt{s}Z)}]. \end{aligned}$$

From this, the 2nd term in (B.11) is bounded above by $C''|x|^a$ for some constant $C'' > 0$. These validate (B.10). ■

Lemma B.3. Let $p > 2, t > 0, u > 0$. Assume that $\lambda \in \mathbb{R}$ and $\gamma \in \mathcal{N}_u^d$ satisfies (1.19). Define $\Psi_{\lambda,\gamma}^{t,u}$ from $[0, u] \times \mathbb{R}$ to $\mathbb{R}$ by letting $\Psi_{\lambda,\gamma}^{t,u}(u, x) = f_\lambda^t(x)$ and iteratively for $l = k, k-1, \dots, 0$

$$\Psi_{\lambda,\gamma}^{t,u}(s, x) = \frac{1}{m_l} \log \mathbb{E} \exp m_l \Psi_{\lambda,\gamma}^{t,u}(q_{l+1}, x + \sqrt{2(q_{l+1} - s)z}), \quad s \in [q_l, q_{l+1}) \times \mathbb{R}, \quad (\text{B.12})$$

where $z \sim N(0, 1)$. Then, for any $0 \leq l \leq k$, $\Psi_{\lambda,\gamma}^{t,u} \in C^{1,2}([q_l, q_{l+1}) \times \mathbb{R})$ and it satisfies the following PDE:

$$\partial_s \Psi_{\lambda,\gamma}^{t,u}(s, x) = -(\partial_{xx} \Psi_{\lambda,\gamma}^{t,u}(s, x) + \gamma(s)(\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, x))^2), \quad (s, x) \in (q_l, q_{l+1}) \times \mathbb{R} \quad (\text{B.13})$$

and

$$\lim_{s \uparrow q_{l+1}, x \rightarrow x_0} \Psi_{\lambda,\gamma}^{t,u}(s, x) = \Psi_{\lambda,\gamma}^{t,u}(q_{l+1}, x_0), \quad \forall x_0 \in \mathbb{R}.$$

Furthermore, there exists a constant $C > 0$ such that

$$|\Psi_{\lambda,\gamma}^{t,u}(s, x)| \leq C(1 + |x|^{1/(p-1)}), \quad \forall (s, x) \in [0, u] \times \mathbb{R}, \quad (\text{B.14})$$

$$|\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, x)| \leq C(1 + |x|^{1/(p-1)}), \quad \forall (s, x) \in [0, u] \times \mathbb{R}. \quad (\text{B.15})$$

Proof. Note that $\Psi_{\lambda,\gamma}^{t,u}(u, x) = f_\lambda^t(x)$ satisfies (B.6) and (B.9) with $a = (p-1)^{-1} < 1$. Our proof follows by applying Lemma B.2 iteratively. ■

Proof of Proposition 12.1. Fix $(\lambda, \gamma) \in \mathbb{R} \times \mathcal{N}_u^d$. Since $\gamma(u) < \infty$, from (B.15), there exists some constant $C > 0$ such that

$$|\gamma(s) \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, x)| \leq C(1 + |x|^{1/(p-1)}), \quad \forall (s, x) \in ([0, u] \times \mathbb{R}) \setminus \{(u, 0)\}.$$

Consequently, there exists a probability space $(\Omega, \mathbb{P}, \mathcal{F})$, a filtration $(\mathcal{F}_s)_{0 \leq s \leq u}$, a continuous process X , and a standard Brownian motion W such that they together form a weak solution of (12.2); see, for example, Proposition 5.3.6 in [35]. Next, for any $v \in \mathcal{D}_u$, set $Y(s) = 2 \int_0^s \gamma(r) v(r) dr + \sqrt{2} W(r)$. Using Itô's formula and the PDE (B.13) leads to

$$d\Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) = -\gamma(s) [(\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s)))^2 - 2v(s) \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s))] ds + \sqrt{2} \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) dW(s).$$

Here, we can further rewrite this equation as

$$d\Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) = -\gamma(s) (\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) - v(s))^2 ds + \sqrt{2} \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) dW(s) + \gamma(s) v(s)^2 ds.$$

Consequently,

$$\mathbb{E}\Psi_{\lambda,\gamma}^{t,u}(u, Y(u)) = \Psi_{\lambda,\gamma}^{t,u}(0, 0) - \int_0^u \gamma(s) \mathbb{E}(\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) - v(s))^2 ds + \int_0^u \gamma(s) \mathbb{E}v(s)^2 ds,$$

which implies that

$$\Psi_{\lambda,\gamma}^{t,u}(0, 0) = \mathbb{E}\Psi_{\lambda,\gamma}^{t,u}(u, Y(u)) - \int_0^u \gamma(s) \mathbb{E}v(s)^2 ds + \int_0^u \gamma(s) \mathbb{E}(\partial_x \Psi_{\lambda,\gamma}^{t,u}(s, Y(s)) - v(s))^2 ds.$$

Clearly, this implies that

$$\Psi_{\lambda,\gamma}^{t,u}(0, 0) \geq \max_{v \in \mathcal{D}_u} \left[f_{\lambda}^t \left(2 \int_0^u \gamma(s) v(s) ds + \sqrt{2} W(u) \right) - \int_0^u \gamma(s) v(s)^2 ds \right]$$

and the equality is achieved if $v(s) \equiv \partial_x \Psi_{\lambda,\gamma}^{t,u}(s, X(s))$, where X is defined through (12.2).

This completes our proof. ■

Funding

This work was supported by the National Science Foundation [DMS-17-52184].

Acknowledgments

WKC and AS would like to thank Souvik Dhara for explaining the results of [22] and for a helpful discussion.

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