Kinetic and Related Models
Vol. 15, No. 3, June 2022, pp. 317–340
doi:10.3934/krm.2022004

THE INVISCID LIMIT FOR THE 2D NAVIER-STOKES EQUATIONS IN BOUNDED DOMAINS

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In memory of Robert T. Glassey

Abstract. We prove the inviscid limit for the incompressible Navier-Stokes equations for data that are analytic only near the boundary in a general two-dimensional bounded domain. Our proof is direct, using the vorticity formulation with a nonlocal boundary condition, the explicit semigroup of the linear Stokes problem near the flatten boundary, and the standard wellposedness theory of Navier-Stokes equations in Sobolev spaces away from the boundary.

2020 *Mathematics Subject Classification.* Primary: 35Q30, 35Q35; Secondary: 76D05, 76D10.

Key words and phrases. Inviscid limit, bounded domains, Navier-Stokes equations, boundary vorticity formulation, near boundary analytic spaces, stokes semigroup.

The second author is partly supported by the AMS-Simons Travel Grant Award and the third author is supported by the NSF under grant DMS-2054726.

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1. **Introduction.** We are interested in the inviscid limit of solutions to the incompressible Navier-Stokes equations

$$\begin{aligned} \partial_t u + u\cdot\nabla u + \nabla p &= \nu\Delta u, \\ \nabla\cdot u &= 0, \end{aligned} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^2$, with initial data $u_{|_{t=0}} = u_0(x)$ and with the no-slip boundary condition

$$u|_{\partial\Omega} = 0. \tag{1.2}$$

In the inviscid limit: $\nu\to 0$, one would intuitively expect that the solutions u_ν, of problem (1.1)-(1.2), converge to the corresponding solutions of the Euler equations of ideal incompressible fluids

$$\begin{aligned} \partial_t u + u\cdot\nabla u + \nabla p &= 0, \quad \text{in} \quad \Omega, \\ \nabla\cdot u &= 0, \quad \text{in} \quad \Omega, \\ u\cdot n &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.3}$$

where n denotes the unit normal vector to the boundary pointing inward. However, the inviscid limit for problem (1.1)-(1.2) is strenuous and remains open due to the appearance of boundary layers and strong shear near the boundary that triggers the shedding of unbounded vorticity by the boundary. In their celebrated work [22], Caflisch and Sammartino establish the boundary layer expansion and the inviscid limit for analytic data on the half-plane. Maekawa [20] proved a similar result that allows Sobolev data whose vorticity is supported away from the boundary. The result and its proof was recently simplified [21] and extended in [18, 16], which allow data that are only analytic near the boundary.

In this paper, we prove the inviscid limit of (1.1)-(1.2) for data that are only analytic near the boundary of a general bounded analytic domain in $\mathbb{R}^2$, thus further extending [22, 20, 21, 18] from the case of half-plane to bounded domains with analytic boundaries. Precisely, we assume that

- Ω is a simply-connected bounded domain in $\mathbb{R}^2$ whose boundary $\partial\Omega$ is an analytic curve, defined by an analytic map: $\theta\in\mathbb{T}=\mathbb{R}/(\mathbb{Z}L)\mapsto x(\theta) = (x_1(\theta), x_2(\theta))\in\partial\Omega$.

The analyticity of the boundary naturally extends to an analytic map which maps the near-boundary part of the domain $\{x\in\Omega:\ d(x,\partial\Omega)<\delta\}$ to the case of half-plane $(\theta,z)\in\mathbb{T}\times(0,\delta)$, where z is the distance function from the boundary. Here, for sake of presentation, we have chosen to consider the case of simply-connected domain Ω. The results of this paper apply to the general setting of multi-connected bounded domains whose boundaries consist of closed analytic curves, i.e., including domains with holes. Our analysis near each of the boundaries is close to that on the half-plane. A crucial assumption, however, lies on the analyticity of initial data near the boundary, which appears to be sharp.

The work is dedicated to the memory of Professor Robert T. Glassey, who was a great mathematician, a close friend, and an inspiring teacher.

1.1. **Boundary vorticity formulation.** We shall work with the boundary vorticity formulation [1, 20, 21]. Precisely, let $u=(u_1,u_2)$ be the velocity vector field and $\omega=\nabla^\perp\cdot u=\partial_{x_2}u_1-\partial_{x_1}u_2$ be the corresponding vorticity. Then, the vorticity equation reads

$$\begin{aligned} \partial_t\omega + u\cdot\nabla\omega &= \nu\Delta\omega, \\ u &= \nabla^\perp\Delta^{-1}\omega, \qquad \text{(the Biot-Savart law)}. \end{aligned} \tag{1.4}$$

Here and throughout the paper, Δ^{-1} denotes the inverse of the Laplacian operator in Ω subject to the zero Dirichlet boundary condition. Evidently, this, together with the Biot-Savart law, imply the impermeability boundary condition $u \cdot n = 0$ on $\partial\Omega$. To ensure the full no-slip boundary condition, i.e., that $u \cdot \tau = 0$ on the boundary $\partial\Omega$, where τ in the unit tangent vector to the boundary, we first require that the initial data satisfy the no-slip boundary condition (1.2), and then we impose in addition that $\partial_t u \cdot \tau = 0$ on the boundary, $\partial\Omega$, for all positive time. This leads to the boundary condition

$$0 = \tau \cdot \partial_t u = \tau \cdot \nabla^\perp \Delta^{-1} \partial_t \omega = \partial_n [\Delta^{-1}(\nu \Delta \omega - u \cdot \nabla \omega)] \tag{1.5}$$

on the boundary. Introduce ω^* to be the solution of the nonhomogeneous Dirichlet boundary-value problem

$$\begin{cases} \Delta \omega^* = 0, & \text{in} \quad \Omega \\ \omega^* = \omega, & \text{on} \quad \partial\Omega. \end{cases} \tag{1.6}$$

and define the Dirichlet-Neumann operator by

$$DN\omega = -\partial_n \omega^*, \qquad \text{on} \quad \partial\Omega, \tag{1.7}$$

where ω^* solves (1.6). Observe that $\partial_n[\Delta^{-1}\Delta\omega] = \partial_n[\Delta^{-1}\Delta(\omega - \omega^*)] = (\partial_n + DN)\omega$. Thus, by virtue of the boundary condition (1.5) the boundary condition on vorticity reads

$$\nu(\partial_n + DN)\omega_{|\partial\Omega} = [\partial_n \Delta^{-1}(u \cdot \nabla \omega)]_{|\partial\Omega}, \tag{1.8}$$

together with the Biot-Savart law (1.4).

Throughout this paper, we shall deal with the Navier-Stokes solutions that solve (1.4)-(1.7), or equivalently (1.4) and (1.8). Such a solution will be constructed via the Duhamel's integral representation, treating the nonlinearity as a source term. As we observed earlier the boundary condition $u \cdot n = 0$ on $\partial\Omega$ follows from the Biot-Savart law and the definition of Δ^{-1} with the zero Dirichlet boundary condition.

1.2. Main results.

Our main result reads as follows.

Theorem 1.1. *Let $u_0 \in H^5(\Omega)$ be an initial data that vanishes on the boundary. We assume that the initial vorticity ω_0 is analytic near the boundary $\partial\Omega$ (see Section 3). Then, there is a positive time T, independent of ν, so that the unique solution $u_\nu(t)$ to the Navier-Stokes problem (1.1)-(1.2), for every $\nu > 0$, with initial data u_0, exists on $[0,T]$ and has vorticity $\omega_\nu = \nabla^\perp \cdot u_\nu$ that remains analytic near the boundary, and satisfies*

$$\lim_{\nu \to 0} \sqrt{\nu} \|\omega_\nu\|_{L^\infty([0,T]\times\partial\Omega))} < \infty. \tag{1.9}$$

Moreover, in the inviscid limit as $\nu \to 0$, u_ν converges strongly in $L^\infty([0,T]; L^p(\Omega))$, for any $2 \le p < \infty$, to the corresponding solution u of the Euler equations (1.3) with the same initial data u_0.

The fact that Euler solutions remain analytic near the boundary is a classical result [3, 17], which is a direct consequence of the main theorem. The main difficulty in establishing the inviscid limit is to control the vorticity on the boundary and derive uniform estimates such as (1.9), which is the main contribution of this paper. The inviscid limit then follows easily. In fact, a much weaker bound than (1.9) is sufficient to guarantee the convergence of solutions to the Navier-Stokes to a corresponding solution of the Euler equations. Precisely, we have the following simple Kato's type theorem.

Theorem 1.2. *Let $T > 0$ and u be a weak solution to the Euler equations (1.3) in $[0,T] \times \Omega$ satisfying $\|\nabla u\|_{L^\infty([0,T]\times\Omega)} < \infty$. Suppose that, for every $\nu > 0$, u_ν are Leray weak solutions to the Navier-Stokes problem (1.1)-(1.2) on $[0,T] \times \Omega$, satisfying*

$$\sup_{0<t<T} \|u_\nu(t)\|^2_{L^2(\Omega)} + \nu \int_0^T \|\nabla_x u_\nu(t)\|_{L^2(\Omega)} dt \le C_0, \tag{1.10}$$

uniformly in $\nu \to 0$. Assume that the vorticity $\omega_\nu = \nabla^\perp \cdot u_\nu$ satisfies

$$\limsup_{\nu\to 0} \Big(- \int_0^T \int_{\partial\Omega} \nu \omega_\nu(t,\sigma) u(t,\sigma) \cdot \tau(\sigma) d\sigma dt \Big) = 0, \tag{1.11}$$

then any $\overline{u}_\nu$, which is a **weak$-*$ limit** *in $L^\infty([0,T]; L^2(\Omega))$ of a subsequence u_{ν_j} of the Leray weak solutions, as $\nu_j \to 0$, satisfies the stability estimate:*

$$\|\overline{u_\nu(t)} - u(t)\|^2_{L^2(\Omega)} \le e^{2t\|\nabla u\|_{L^\infty([0,T]\times\Omega)}} \|\overline{u_\nu(0)} - u(0)\|^2_{L^2(\Omega)}. \tag{1.12}$$

In particular, if $u_\nu(0) \to u(0)$ in $L^2(\Omega)$, as $\nu \to 0$, then u_ν converges strongly to u in $L^\infty([0,T]; L^2(\Omega))$.

Proof. An elementary manipulation (e.g., [5]) yields the following energy inequality

$$\begin{aligned} &\|u_\nu(t) - u(t)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla u_\nu(s)\|^2_{L^2(\Omega)} ds \\ &\le \|u_\nu(0) - u(0)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} ds - \int_0^t \int_{\partial\Omega} \nu(\partial_n u_\nu(s,\sigma)) \cdot u(s,\sigma)\, d\sigma ds \\ &+ \int_0^t \int_\Omega |\Big((\nabla u + \nabla^\perp u)(u_\nu - u)\Big) \cdot (u_\nu - u))| dx ds \\ &\le \|u_\nu(0) - u(0)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)} ds - \int_0^t \int_{\partial\Omega} \nu \omega_\nu(s,\sigma)(u(s,\sigma) \cdot \tau(\sigma))\, d\sigma ds \\ &+ 2\|\nabla u\|_{L^\infty([0,T]\times\Omega)} \int_0^t \|u_\nu(s) - u(s)\|^2_{L^2(\Omega)}\, ds, \end{aligned} \tag{1.13}$$

where in the third term in the right-hand side of the last inequality we used the fact that $(\partial_n u_\nu) \cdot u = \omega_\nu (u \cdot \tau)$ on the boundary. Let u_{ν_j} be a subsequence which converges weak$-*$ in $L^\infty([0,T]; L^2(\Omega))$, as $\nu_j \to 0$. We apply the above energy inequality to u_{ν_j} and invoke Gronwall's Lemma. Observe that since the Leray weak solutions belong to $C([0,T]; L^2(\Omega))$ then $\|u_\nu(0)\|^2_{L^2(\Omega)} \le C_0$ by virtue of (1.10). Thanks to the Banach-Alaoglu Theorem and assumption (1.11) we conclude (1.12). The last part of the theorem is an immediate consquence of (1.12). □

1.3. **Remarks.** As mentioned in the introduction, our main results extend the previous works [22, 20, 21, 18] from the case of the half-plane to bounded domains. The analyticity near the boundary is required to control the unbounded vorticity in the inviscid limit. It may be possible to extend the present analysis to include the propagation of boundary layers and the classical Prandtl's boundary layer expansions, whose validity near general boundary layers again requires analyticity.

The first such a result was due to the celebrated work by Asano [2] and Sammartino-Caflisch [22], where the boundary layer expansion was established for data on the half-plane that are analytic in both horizontal and vertical variables. When constructing solutions to the Prandtl equation, the analyticity in the vertical variable can be dropped [19]. It is not known however if such an assumption can be dropped at the level of Navier-Stokes equations. Maekawa [20] established the Prandtl's expansion for data whose vorticity is compactly supported away from the boundary, while recently Kukavica, Nguyen, Vicol and Wang [16] extended the result to include data that are analytic only near the boundary, building upon the vorticity formulation revived by Maekawa [20], the direct proof of the inviscid limit for analytic data developed in Nguyen and Nguyen [21], and the Sobolev-analytic norm developed in Kukavica, Vicol and Wang [18]. All these aforementioned works are on the half-plane. We mention a recent result [23], which to the best of our knowledge was the first to establish a Prandtl asymptotic expansion in a curved domain.

When background boundary layers have no inflection point, the analyticity can be relaxed to include perturbations in Gevrey-$\frac{3}{2}$ spaces [7, 8], which is sharp in view of the Kelvin-Helmholtz type of instability of generic boundary layers and shear flows [10, 11]. When Sobolev data is allowed, the Prandtl's asymptotic expansion is false due to counter-examples given in [9, 12, 13], where the failure of the convergence from Navier-Stokes to Euler solutions, plus a Prandtl corrector, is due to an emergence of viscous boundary sublayers that reach to order one, independent of viscosity, in L^∞ norm for velocity [12].

2. Navier-Stokes equations near the boundary.

2.1. Global geodesic coordinates.

Following a construction done in [4] we introduce a well adapted representation of $\partial\Omega$,

$$\theta \in \mathbb{T} = \mathbb{R}/(\mathbb{Z}L) \mapsto x(\theta) = (x_1(\theta), x_2(\theta)) \in \partial\Omega$$

which, being global, preserves the analyticity hypothesis. Let $\vec{\tau}(\theta)$ and $\vec{n}(\theta)$ be the unit tangent and interior normal vectors at the boundary:

$$\begin{aligned} &\vec{\tau}(\theta) = \vec{\tau}(x(\theta) = (x_1'(\theta), x_2'(\theta)), \quad \text{and} \quad \vec{n}(\theta) = \vec{n}(x(\theta)) = (-x_2'(\theta), x_1'(\theta)) \\ &\text{with} \quad |x'(\theta)|^2 = (x_1'(\theta))^2 + (x_1'(\theta))^2 = 1. \end{aligned} \tag{2.1}$$

Let $d(x,\partial\Omega)$ denotes the distance of any point $x \in \mathbb{R}^2$ to $\partial\Omega$. Then we have the following classical result.

Proposition 2.1. *There exists a $\delta > 0$ such that for each x on the open set*

$$V_\delta = \{x \in \mathbb{R}^2 \quad \textit{with} \quad d(x,\partial\Omega) < \delta\} \tag{2.2}$$

there is a unique point $\hat{x}(\theta) \in \partial\Omega$ with $d(x,\partial\Omega) = |x - \hat{x}(\theta)|$. The mapping $x \mapsto \hat{x}(\theta)$ is an analytic map from V_δ with value in $\partial\Omega$. In addition, for $x \in V_\delta$, one has the formula

$$\nabla_x d(x,\partial\Omega) = \vec{n}(x(\theta)). \tag{2.3}$$

When no confusion is possible, for $x \in V_\delta$ the notations $\vec{n}(x)$ and $\vec{\tau}(x)$ will be used for $\vec{n}(x(\theta))$ and $\vec{\tau}(x(\theta))$ respectively. Observe that

$$\vec{\tau}'(\theta) \wedge \vec{n}(\theta) = x_1'(\theta)x_1''(\theta) + x_2'(\theta)x_2''(\theta) = \frac{d}{d\theta}|x'(\theta)|^2 = 0\,, \tag{2.4}$$

which implies the relation

$$\vec{n}'(\theta) = \gamma(\theta)\vec{\tau}(\theta) \quad \text{and} \quad \vec{\tau}'(\theta) = \gamma(\theta)\vec{n}(\theta)\,, \tag{2.5}$$

with

$$\gamma(\theta) = x_1''(\theta)x_2'(\theta) - x_1'(\theta)x_2''(\theta), \tag{2.6}$$

being the curvature of the boundary $\partial\Omega$. Therefore the mapping:

$$(\theta, z) \mapsto X(\theta, z) = x(\theta) + z\vec{n}(x(\theta)), \tag{2.7}$$

defines au global C^2 diffeomorphisme of $(\mathbb{R}/(L\mathbb{Z}))\times[-\delta,\delta]$ on $\overline{V_\delta}$. Moreover, for any vector field $x\in\overline{\Omega}\mapsto v(x)$, as soon as $x\in\overline{V}_\delta$, using the above notations, one has:

$$v(x)=(v(x)\cdot\vec{\tau}(x))\vec{\tau}(x)+(v(x)\cdot\vec{n}(x))\vec{n}(x)\,. \tag{2.8}$$

Below, for sake of clarity, the symbol X is used for any $x=X(\theta,z)$. There hold

$$\begin{aligned}&\partial_z X(\theta,z)=\vec{n}(\theta)\,,\quad \partial_\theta X(\theta,z)=J(\theta,z)\vec{\tau}(\theta)\,,\\ &\text{and}\quad J(\theta,z)=1+z\gamma(\theta)>0\quad\text{ for }\quad |z|<\delta\,,\end{aligned} \tag{2.9}$$

provided $\delta>0$ is chosen to be small enough. From the relation

$$\begin{pmatrix}\partial_z X_1 & \partial_\theta X_1\\ \partial_z X_2 & \partial_\theta X_2\end{pmatrix}\begin{pmatrix}\partial_{X_1} z & \partial_{X_2} z\\ \partial_{X_1}\theta & \partial_{X_2}\theta\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{2.10}$$

one deduces the formula:

$$\nabla_X\theta=\frac{\vec{\tau}(\theta,z))}{J(\theta,z)}\quad\text{and}\quad\nabla_X z=\vec{n}(\theta)\,. \tag{2.11}$$

We collect the following useful relations whose derivations are classical. For any vector field u, we have

$$\begin{aligned}\nabla\cdot u&=\frac{1}{J}(\partial_z(J(u\cdot\vec{n}))+\partial_\theta(u\cdot\vec{\tau}))=\partial_z(u\cdot\vec{n})+\frac{1}{J}\partial_\theta(u\cdot\vec{\tau}))+\frac{\gamma}{J}u\cdot\vec{n}\,,\\ \nabla\wedge u&=\frac{1}{J}(\partial_z(Ju\cdot\vec{\tau})-\partial_\theta(u\cdot\vec{\tau}))=\partial_z(u\cdot\vec{\tau})-\frac{1}{J}\partial_\theta(u\cdot\vec{\tau}))+\frac{\gamma}{J}(u\cdot\vec{\tau})\,.\end{aligned} \tag{2.12}$$

For any scalar function Ψ, we have

$$\nabla\wedge\Psi=\frac{1}{J}\begin{pmatrix}\partial_z(J\Psi)\\ -\partial_\theta\Psi\end{pmatrix}=\begin{pmatrix}\partial_z\Psi\\ -\dfrac{1}{J}\partial_\theta\Psi\end{pmatrix}+\begin{pmatrix}\dfrac{\gamma}{J}\Psi\\ 0\end{pmatrix}, \tag{2.13}$$

and

$$\Delta\Psi=\frac{1}{J}\partial_z(J\partial_z\Psi)+\frac{1}{J}\partial_\theta(\frac{1}{J}\partial_\theta\Psi)=\Delta_{\theta,z}\Psi+R_\Delta\Psi\,, \tag{2.14}$$

in which we denote

$$\Delta_{\theta,z}=\partial_\theta^2+\partial_z^2$$

$$R_\Delta=m(\theta,z)\partial_\theta^2+\frac{\gamma}{1+z\gamma}\partial_z-\frac{z\gamma'}{(1+z\gamma)^3}\partial_\theta\quad\text{and}\quad m(\theta,z)=-\frac{2z\gamma+(z\gamma)^2}{(1+z\gamma)^2}.$$

2.2. **Scaled coordinates.** In view of (2.14), we observe that the Laplacian Δ is nearly the flat Laplacian $\Delta_{\theta,z}$, in the (θ,z) coordinates, near the boundary. To make use of this fact, we introduce the following scaled variables

$$(\widetilde{\theta},\widetilde{z})=(\lambda\theta,\lambda z) \tag{2.15}$$

for sufficiently small $\lambda\in(0,1)$. By construction, we compute

$$\Delta=\lambda^2\Big(\Delta_{\widetilde{\theta},\widetilde{z}}+\lambda^2\widetilde{R}_\Delta\Big)\,, \tag{2.16}$$

in which $\Delta_{\widetilde{\theta},\widetilde{z}}=(\partial_{\widetilde{z}}^2+\partial_{\widetilde{\theta}}^2)$ and

$$\begin{aligned}\widetilde{R}_\Delta&=\widetilde{m}(\widetilde{\theta},\widetilde{z})\partial_{\widetilde{\theta}}^2+\frac{\widetilde{\gamma}}{1+\lambda^2\widetilde{z}\widetilde{\gamma}}\partial_{\widetilde{z}}-\frac{\widetilde{z}\widetilde{\gamma}'}{(1+\lambda^2\widetilde{z}\widetilde{\gamma})^3}\partial_{\widetilde{\theta}}\\ \widetilde{m}(\widetilde{\theta},\widetilde{z})&=-\frac{2\widetilde{z}\widetilde{\gamma}+\lambda^2(\widetilde{z}\widetilde{\gamma})^2}{(1+\lambda^2\widetilde{z}\widetilde{\gamma})^2}\,,\end{aligned} \tag{2.17}$$

where $\gamma = \lambda^3\widetilde{\gamma}(\widetilde{\theta})$. In the analysis, λ will be taken sufficiently small, and so Δ is indeed approximated by $\lambda^2\Delta_{\widetilde{\theta},\widetilde{z}}$, treating $\lambda^2\widetilde{R}_\Delta$ as a perturbation.

2.3. **Vorticity equations near the boundary.** In this section, we derive vorticity equations in the geodesic coordinates near the boundary in the region V_δ defined as in Proposition 2.1. Introduce a smooth cutoff function $\phi^b(x)$ so that

$$\phi^b(x) = \begin{cases} 1, & \text{if } \quad \lambda d(x,\partial\Omega) \le \delta_0 + \rho_0 \\ 0, & \text{if } \quad \lambda d(x,\partial\Omega) \ge \delta_0 + 2\rho_0 \end{cases} \tag{2.18}$$

for small positive constants δ_0, ρ_0 so that $\delta_0 + 2\rho_0 < \lambda\delta$ to guarantee that $\mathrm{supp}(\phi^b) \subset V_\delta$ as in Proposition 2.1. Define

$$\omega^b = \phi^b(x)\omega(t,x). \tag{2.19}$$

It follows from (1.4) that

$$\partial_t\omega^b - \nu\Delta\omega^b = N^b, \tag{2.20}$$

where

$$N^b := -u\cdot\nabla\omega^b + (u\cdot\nabla\phi^b)\omega - \nu(\Delta\phi^b)\omega - 2\nu\nabla\phi^b\cdot\nabla\omega.$$

Observe that $N^b(u,\omega) = 0$ on $\{\lambda d(x,\partial\Omega) \ge \delta_0 + 2\rho_0\}$ where the cutoff function ϕ^b vanishes. We then introduce the following scaled vorticity

$$\omega^b(t,x) = \widetilde{\omega}(\lambda^2 t, \lambda\theta, \lambda z), \qquad (\widetilde{t},\widetilde{\theta},\widetilde{z}) = (\lambda^2 t, \lambda\theta, \lambda z), \tag{2.21}$$

for small $\lambda > 0$. Using (2.16), we rewrite the vorticity equation as

$$\left(\partial_{\widetilde{t}} - \nu\Delta_{\widetilde{\theta},\widetilde{z}}\right)\widetilde{\omega} = -\nu\lambda^2\widetilde{R}_\Delta\widetilde{\omega} + \lambda^{-2}N^b. \tag{2.22}$$

Equation (2.22) is defined on $(\widetilde{\theta},\widetilde{z}) \in \mathbb{T}\times\mathbb{R}_+$ (in fact, the equation vanishes for $\widetilde{z} \ge \delta_0 + 2\rho_0$). We shall solve (2.22) together with the boundary condition (1.8), which now reads

$$\nu(\partial_{\widetilde{z}} + \widetilde{DN})\widetilde{\omega}_{|_{\widetilde{z}=0}} = \lambda^{-1}[\partial_n\Delta^{-1}(u\cdot\nabla\omega)]_{|\partial\Omega}. \tag{2.23}$$

System (2.22)-(2.23) will be our main equation for the scaled vorticity near the boundary. Away from the boundary, we construct vorticity using the original system as derived in Section 1.1.

2.4. **Dirichlet-Neumann operator.** Let us precise the Dirichlet-Neumann operator defined as in (1.6)-(1.7).

Lemma 2.2. *For $\omega \in H^{1/2}(\partial\Omega)$, let $DN\omega$ be the Dirichlet-Neumann operator defined as in (1.6)-(1.7). In the scaled variables, there holds*

$$\widetilde{DN}\widetilde{\omega} = |\partial_{\widetilde{\theta}}|\widetilde{\omega} + \widetilde{B}\widetilde{\omega} \tag{2.24}$$

for some linear bounded operator $\widetilde{B}$ from $L^2(\partial\Omega)$ to itself: namely,

$$\|\widetilde{B}\widetilde{\omega}\|_{L^2(\partial\Omega)} \le C_0\|\widetilde{\omega}\|_{L^2(\partial\Omega)}$$

for some positive constant C_0.

Proof. Let ϕ^b be the cutoff function defined as in (2.18), and set $\omega^{*b} = \phi^b\omega^*$, where ϕ^* solves (1.6). It follows that

$$\begin{cases} \Delta\omega^{*b} = (\Delta\phi^b)\omega^* - 2\nabla\phi^b\cdot\nabla\omega^*, & \text{in} \quad \Omega \\ \omega^{*b} = \omega, \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.25}$$

Since ϕ^p vanishes away from the boundary, we can work in the scaled variables, which reads $\widetilde{DN}\widetilde{\omega} = -\partial_{\widetilde{z}}\widetilde{\omega}^*_{|\widetilde{z}=0}$. Recalling (2.16), the scaled function $\widetilde{\omega}^*(\widetilde{t},\widetilde{\theta},\widetilde{z})$ of ω^{*b} solves

$$\Delta_{\widetilde{\theta},\widetilde{z}}\widetilde{\omega}^* = -\lambda^2\widetilde{R}_\Delta\widetilde{\omega}^* + \lambda^{-2}[(\Delta\phi^b)\omega^* - 2\nabla\phi^b\cdot\nabla\omega^*], \qquad \widetilde{\omega}^*|_{\widetilde{z}=0} = \widetilde{\omega}|_{\widetilde{z}=0}\,,$$

on $\mathbb{T}\times\mathbb{R}_+$, which can be solved explicitly. Indeed, let $\widetilde{\omega}_\alpha$ be the Fourier coefficient of $\widetilde{\omega}(\widetilde{\theta},\widetilde{z})$ in variable $\widetilde{\theta}$. Note that $\widetilde{\omega}_\alpha$ vanishes for $\alpha=0$, and thus we focus on the case when $\alpha\neq 0$. Let $K_\alpha(\widetilde{y},\widetilde{z}) = \frac{1}{2|\alpha|}(e^{-|\alpha(\widetilde{y}-\widetilde{z})|} - e^{-|\alpha(\widetilde{y}+\widetilde{z})|})$ be the Green function of the Laplacian $\partial^2_{\widetilde{z}} - \alpha^2$ with the Dirichlet boundary condition. It follows that

$$\begin{aligned}\widetilde{\omega}^*_\alpha(\widetilde{z}) &= e^{-|\alpha|\widetilde{z}}\widetilde{\omega}_\alpha(0) + \lambda^2\int_0^\infty K_\alpha(\widetilde{y},\widetilde{z})(\widetilde{R}_\Delta\widetilde{\omega}^*)_\alpha(\widetilde{y})\,d\widetilde{y}\\ &\quad+\lambda^{-2}\int_0^\infty K_\alpha(\widetilde{y},\widetilde{z})\Big[(\Delta\phi^b)\omega^* - 2\nabla\phi^b\cdot\nabla\omega^*\Big]_\alpha(\widetilde{y})\,d\widetilde{y}\end{aligned} \tag{2.26}$$

for $\widetilde{z}\geq 0$. The Dirichlet-Neumann operator is thus computed by

$$\begin{aligned}(\widetilde{DN}\widetilde{\omega})_\alpha &= -\partial_{\widetilde{z}}\widetilde{\omega}^*_\alpha(0)\\ &= |\alpha|\widetilde{\omega}_\alpha(0) + \int_0^\infty e^{-|\alpha|\widetilde{y}}\Big[\lambda^2(\widetilde{R}_\Delta\widetilde{\omega}^*)_\alpha + \lambda^{-2}((\Delta\phi^b)\omega^* - 2\nabla\phi^b\cdot\nabla\omega^*)_\alpha\Big](\widetilde{y})\,d\widetilde{y}.\end{aligned}$$

The decomposition (2.24) thus follows, upon defining $\widetilde{B}$ as the integral term

$$(\widetilde{B}\widetilde{\omega})_\alpha := \int_0^\infty e^{-|\alpha|\widetilde{y}}\Big[\lambda^2(\widetilde{R}_\Delta\widetilde{\omega}^*)_\alpha + \lambda^{-2}((\Delta\phi^b)\omega^* - 2\nabla\phi^b\cdot\nabla\omega^*)_\alpha\Big](\widetilde{y})\,d\widetilde{y}\,, \tag{2.27}$$

for each Fourier variable $\alpha\in\mathbb{Z}$. It remains to prove the boundedness of $\widetilde{B}$. Note that by definition, the last two terms are defined on the region $\widetilde{y}\geq\delta_0+\rho_0$ where the cutoff function $\phi^b=1$. Therefore,

$$\Big|\int_0^\infty e^{-|\alpha|\widetilde{y}}((\Delta\phi^b)\omega^* - 2\nabla\phi^b\cdot\nabla\omega^*)_\alpha(\widetilde{y})\,d\widetilde{y}\Big| \lesssim \|\omega^\star\|_{H^1(\lambda d(x,\partial\Omega)\geq\delta_0+\rho_0)}.$$

It remains to bound the first integral term in (2.27). In view of (2.17), we write

$$\begin{aligned}\widetilde{R}_\Delta\widetilde{\omega}^* &= \partial^2_{\widetilde{\theta}}[\widetilde{m}\widetilde{\omega}^*] - \partial_{\widetilde{\theta}}\Big[2\partial_{\widetilde{\theta}}\widetilde{m}\widetilde{\omega}^* + \frac{\widetilde{z}\widetilde{\gamma}'}{(1+\lambda^2\widetilde{z}\widetilde{\gamma})^3}\widetilde{\omega}^*\Big] + \partial_{\widetilde{z}}\Big(\frac{\widetilde{\gamma}}{1+\lambda^2\widetilde{z}\widetilde{\gamma}}\widetilde{\omega}^*\Big)\\ &\quad+\Big[(\partial^2_{\widetilde{\theta}}\widetilde{m}) - \partial_{\widetilde{z}}\Big(\frac{\widetilde{\gamma}}{1+\lambda^2\widetilde{z}\widetilde{\gamma}}\Big) + \partial_{\widetilde{\theta}}\Big(\frac{\widetilde{z}\widetilde{\gamma}'}{(1+\lambda^2\widetilde{z}\widetilde{\gamma})^3}\Big)\Big]\widetilde{\omega}^*,\end{aligned}$$

noting the coefficients are analytic near the boundary. We note in particular that there is no growth in large $\widetilde{z}$: for instance, $m(\widetilde{\theta},\widetilde{z})\lesssim\lambda^{-2}$ uniformly in large $\widetilde{z}$. In addition, we note that $\widetilde{m} = \widetilde{z}\widetilde{m}_1$ for some bounded function $\widetilde{m}_1$. Thus, using the fact that $|\alpha|\widetilde{y}e^{-\frac12|\alpha|\widetilde{y}}\lesssim 1$, the second-order derivative term $\partial^2_{\widetilde{\theta}}[\widetilde{m}\widetilde{\omega}^*]$ thus can be treated as the first order derivative term. Precisely, we can treat the first integral in (2.27) systematically as follows: for some smooth and bounded coefficients $b(\widetilde{\theta},\widetilde{z})$,

$$\lambda^2\int_0^\infty e^{-\frac12|\alpha|\widetilde{y}}|(\alpha,\partial_{\widetilde{y}})(b\widetilde{\omega}^*)_\alpha|(\widetilde{y})\,d\widetilde{y}\lesssim\lambda^2|\alpha|^{-1/2}\|(\alpha,\partial_{\widetilde{y}})(b\widetilde{\omega}^*)_\alpha\|_{L^2_{\widetilde{y}}}.$$

This yields

$$|(\widetilde{B}\widetilde{\omega})_\alpha|\lesssim\lambda^2|\alpha|^{-1/2}\|(\alpha,\partial_{\widetilde{y}})(b\widetilde{\omega}^*)_\alpha\|_{L^2_{\widetilde{y}}} + \|\omega^\star\|_{H^1(\lambda d(x,\partial\Omega)\geq\delta_0+\rho_0)}\,.$$

Taking L^2_α, we thus obtain

$$\sum_\alpha |(\widetilde{B}\widetilde{\omega})_\alpha|^2 \lesssim \lambda^2 \sum_\alpha |\alpha|^{-1} \|(\alpha, \partial_{\widetilde{y}})\widetilde{\omega}^*_\alpha\|^2_{L^2_{\widetilde{y}}}, \tag{2.28}$$

upon noting that the coefficients $b(\widetilde{\theta}, \widetilde{z})$, which in particular have $\|b_\alpha(\widetilde{z})\|_{L^1_\alpha L^\infty_{\widetilde{z}}} < \infty$. It remains to bound the right-hand side of (2.28). Directly from (2.26), we compute

$$|(\alpha, \partial_{\widetilde{z}})\widetilde{\omega}^*_\alpha(\widetilde{z})| \lesssim |\alpha| e^{-|\alpha|\widetilde{z}} |\widetilde{\omega}_\alpha(0)| + \lambda^2 \int_0^\infty e^{-|\alpha(\widetilde{z}-\widetilde{z}')|} |(\widetilde{R}_\Delta \widetilde{\omega}^*)_\alpha(\widetilde{z}')| \, d\widetilde{z}'$$
$$+ |\alpha|^{-1/2} \|\omega^\star\|_{H^1(\lambda d(x,\partial\Omega)\geq \delta_0+\rho_0)}.$$

Therefore, together with the standard Hausdorff-Young's inequality, we bound

$$\|(\alpha, \partial_{\widetilde{z}})\widetilde{\omega}^*_\alpha\|_{L^2_{\widetilde{z}}} \lesssim |\alpha|^{1/2} |\widetilde{\omega}_\alpha(0)| + \lambda^2 |\alpha|^{-1} \|(\widetilde{R}_\Delta \widetilde{\omega}^*)_\alpha\|_{L^2_{\widetilde{z}}} + |\alpha|^{-1/2} \|\omega^\star\|_{H^1(\lambda d(x,\partial\Omega)\geq \delta_0+\rho_0)}$$

which yields

$$\begin{aligned}
&\sum_\alpha |\alpha|^{-1} \|(\alpha, \partial_{\widetilde{z}})\widetilde{\omega}^*_\alpha\|^2_{L^2_{\widetilde{z}}} \\
&\lesssim \sum_\alpha |\widetilde{\omega}_\alpha(0)|^2 + \lambda^2 \sum_\alpha |\alpha|^{-3} \|(\widetilde{R}_\Delta \widetilde{\omega}^*)_\alpha\|^2_{L^2_{\widetilde{z}}} + \|\omega^\star\|^2_{H^1(\lambda d(x,\partial\Omega)\geq \delta_0+\rho_0)} \\
&\lesssim \sum_\alpha |\widetilde{\omega}_\alpha(0)|^2 + \lambda^2 \sum_\alpha |\alpha|^{-1} \|(\alpha, \partial_{\widetilde{z}})\widetilde{\omega}^*_\alpha\|^2_{L^2_{\widetilde{z}}} \\
&\quad + \sum_\alpha |\alpha|^{-1} \|(\alpha, \partial_{\widetilde{z}})\widetilde{\omega}^*_\alpha\|^2_{L^2_{\{\widetilde{z}\geq\delta_0+\rho_0\}}} + \|\omega^\star\|_{H^1(\lambda d(x,\partial\Omega)\geq \delta_0+\rho_0)}.
\end{aligned}$$

Taking λ sufficiently small so that the second term on the right can be absorbed into the left. On the other hand, using the standard elliptic theory, the last term is bounded by

$$\sum_\alpha |\alpha|^{-1} \|(\alpha, \partial_{\widetilde{z}})\widetilde{\omega}^\star_\alpha\|^2_{L^2_{\{\widetilde{z}\geq\delta_0+\rho_0\}}} \lesssim \|\omega^\star\|^2_{H^1(\lambda d(x,\partial\Omega)\geq\delta_0)} \lesssim \|\omega\|^2_{L^2(\partial\Omega)}.$$

Putting these back into (2.28), we obtain the lemma. □

3. Near boundary analytic spaces.

3. Near boundary analytic spaces. In this section, we introduce the near boundary analytic norm used to control the vorticity that is analytic near the boundary, but however only has Sobolev regularity away from the boundary. We then derive sufficient elliptic estimates, bilinear estimates, as well as the semigroup estimates in these analytic spaces.

3.1. Analytic norms. Let $\delta > 0$ be small and so that Proposition 2.1 applies for $\bar{V}_\delta = \{d(x, \partial\Omega) \leq \delta\}$. In particular, δ is small so that the statement of 2.1 still holds for $V_{2\delta}$. Now for any constant $\lambda \in (0, 1)$, we have

$$\lambda d(x, \partial\Omega) \leq \lambda\delta$$

for all $x \in \bar{V}_\delta$. Let $\delta_0 = \lambda\delta$, which will the size of the analytic domain for our solution near the boundary. We fix $\rho_0 \in (0, 1/10)$, and assume that $\rho \in (0, \rho_0)$. Then

$$\begin{aligned}
\Omega_\rho = &\{\widetilde{z} \in \mathbb{C} : 0 \leq \Re\widetilde{z} \leq \delta_0, |\Im\widetilde{z}| \leq \rho\Re\widetilde{z}\} \\
&\bigcup \{\widetilde{z} \in \mathbb{C} : \delta_0 \leq \Re\widetilde{z} \leq \delta_0 + \rho, |\Im\widetilde{z}| \leq \delta_0 + \rho - \Re\widetilde{z}\}
\end{aligned} \tag{3.1}$$

denotes the complex domain for functions of the $\widetilde{z}$ variable. We note that the domain Ω_ρ only contains $\widetilde{z}$ with $0 \le \Re\widetilde{z} \le \delta_0 + \rho$. For a complex valued function f defined on Ω_ρ, let

$$\|f\|_{L^1_\rho} = \sup_{0\le\eta<\rho} \|f\|_{L^1(\partial\Omega_\eta)}, \qquad \|f\|_{L^\infty_\rho} = \sup_{0\le\eta<\rho} \|f\|_{L^\infty(\partial\Omega_\eta)}$$

where the integration is taken over the two directed paths along the boundary of the domain Ω_η. Now for an analytic function $f(\widetilde{\theta}, \widetilde{z})$ defined on $(\widetilde{\theta}, \widetilde{z}) \in \mathbb{T} \times \Omega_\rho$, we define

$$\begin{aligned} \|f\|_{\mathcal{L}^1_\rho} &= \sum_{\alpha\in\mathbb{Z}} \|e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha|} f_\alpha\|_{L^1_\rho}, \\ \|f\|_{\mathcal{L}^\infty_\rho} &= \sum_{\alpha\in\mathbb{Z}} \|e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha|} f_\alpha\|_{L^\infty_\rho}, \end{aligned} \tag{3.2}$$

where f_α denotes the Fourier transform of f with respect to variable $\widetilde{\theta}$. The function spaces $\mathcal{L}^1_\rho$ and $\mathcal{L}^\infty_\rho$ are to control the scaled vorticity and velocity, respectively. We stress that the analyticity weight vanishes on $\Re\widetilde{z} \ge \delta_0 + \rho$. For convenience, we also introduce the following analytic norms

$$\|f\|_{\mathcal{W}^{k,p}_\rho} = \sum_{i+j\le k} \|\partial^i_{\widetilde{\theta}} (\widetilde{z}\partial_{\widetilde{z}})^j f\|_{\mathcal{L}^p_\rho} \tag{3.3}$$

for $k \ge 0$ and $p = 1, \infty$. We observe the following simple algebra.

Lemma 3.1. *There hold*

$$\|fg\|_{\mathcal{L}^1_\rho} \le \|f\|_{\mathcal{L}^\infty_\rho} \|g\|_{\mathcal{L}^1_\rho} \tag{3.4}$$

and for any $0 < \rho' < \rho$,

$$\|\partial_{\widetilde{\theta}} f\|_{\mathcal{L}^1_{\rho'}} + \|\widetilde{z}\partial_{\widetilde{z}} f\|_{\mathcal{L}^1_{\rho'}} \lesssim \frac{1}{\rho - \rho'} \|f\|_{\mathcal{L}^1_\rho}. \tag{3.5}$$

Proof. By definition, we compute

$$\begin{aligned} e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha|} |(fg)_\alpha(\widetilde{z})| &\le \sum_{\alpha'} |f_{\alpha-\alpha'}(\widetilde{z}) g_{\alpha'}(\widetilde{z})| e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha|} \\ &\le \sum_{\alpha'} |e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha-\alpha'|} f_{\alpha-\alpha'}(\widetilde{z}) e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha'|} g_{\alpha'}(\widetilde{z})| \end{aligned}$$

which gives

$$\|e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha|}(fg)_\alpha(\widetilde{z})\|_{\mathcal{L}^1_\rho} \le \sum_{\alpha'} \|e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha-\alpha'|} f_{\alpha-\alpha'}\|_{\mathcal{L}^\infty_\rho} \|e^{\varepsilon_0(\delta_0+\rho-\Re\widetilde{z})|\alpha'|} g_{\alpha'}\|_{\mathcal{L}^1_\rho}.$$

The estimate (3.4) follows from taking the summation in α over $\mathbb{Z}$. The stated bounds on derivatives are classical (e.g., [22, 21]), making use of the fact that $(\rho - \rho')|\alpha| e^{(\rho'-\rho)|\alpha|}$ is bounded. □

3.2. **Elliptic estimates in the half-plane.** In this section, we derive some basic elliptic estimates in the analytic spaces $\mathcal{W}^{k,p}_\rho$. Precisely, we consider

$$\begin{cases} \Delta_{\theta,z}\phi = f, & \text{in} \quad \mathbb{T} \times \mathbb{R}_+ \\ \phi_{|z=0} = 0 \end{cases} \tag{3.6}$$

in which we drop titles for sake of presentation. The $\mathcal{W}^{k,p}_\rho$ analytic norm is defined on $\Re z \le \delta_0 + \rho$ as introduced in the previous section. We obtain the following proposition.

Proposition 3.2. *Let ϕ be the solution of (3.6). Then, the velocity field $u = \nabla^\perp \phi$ satisfies*

$$
\begin{aligned}
\|u\|_{\mathcal{W}^{k,\infty}_\rho} &\lesssim \|f\|_{\mathcal{W}^{k,1}_\rho} + \|f\|_{H^{k+1}(\{z\ge\delta_0+\rho\})}\\
\|(\frac{1}{z}\partial_\theta\phi)\|_{\mathcal{W}^{k,\infty}_\rho} &\lesssim \|f\|_{\mathcal{W}^{k,1}_\rho} + \|\partial_\theta f\|_{\mathcal{W}^{k,1}_\rho} + \|f\|_{H^{k+1}(\{z\ge\delta_0+\rho\})} \qquad (3.7)\\
\|\nabla_{\theta,z} u\|_{\mathcal{W}^{k,\infty}_\rho} &\lesssim \|f\|_{\mathcal{W}^{k,\infty}_\rho} + \|f\|_{H^{k+2}(\{z\ge\delta_0+\rho\})}
\end{aligned}
$$

for $k \ge 0$.

Proof. The elliptic problem (3.6) can be solved explicitly in Fourier space. Indeed, taking the Fourier transform in θ, we get the elliptic equation

$$(\partial_z^2 - \alpha^2)\phi_\alpha = f_\alpha$$

for the Fourier transform ϕ_α. We focus on the case $\alpha > 0$; the other case is similar. The solution is given by

$$\phi_\alpha(z) = \int_0^z K_-(y,z) f_\alpha(y)dy + \int_z^\infty K_+(y,z) f_\alpha(y)dy$$

with the Green function defined by

$$K_\pm(y,z) = -\frac{1}{2\alpha}\Big(e^{\pm\alpha(z-y)} - e^{-\alpha(y+z)}\Big).$$

This expression may be extended to complex values of z. Indeed, for $z \in \Omega_\sigma$, there is a positive θ so that $z \in \partial\Omega_\theta$. We then write $\partial\Omega_\theta = \gamma_-(z) \cup \gamma_+(z)$, consisting of complex numbers $y \in \partial\Omega_\theta$ so that $\Re y < \Re z$ and $\Re y > \Re z$, respectively. Then, the integral is taken over $\gamma_-(z)$ and $\gamma_+(z)$, respectively. We note in particular that for $y \in \gamma_\pm(z)$, there hold the same bounds on the Green function

$$|K_\pm(y,z)| \le \alpha^{-1} e^{-\alpha|y-z|}.$$

This proves that

$$|\phi_\alpha(z)| \le \int_{\partial\Omega_\theta} \alpha^{-1} e^{-\alpha|y-z|} |f_\alpha(y)||dy|. \qquad (3.8)$$

By definition of $\mathcal{L}^1_\rho$ norm, we only need to consider the case when $0 \le \Re z \le \delta_0 + \rho$. Now, for $0 \le \Re y \le \delta_0 + \rho$, we bound

$$e^{-\alpha|\Re y - \Re z|} e^{-\varepsilon_0(\delta_0+\rho-\Re y)\alpha} \le e^{-\varepsilon_0(\delta_0+\rho-\Re z)\alpha} e^{-(1-\epsilon_0)\alpha|\Re y-\Re z|}$$

noting $\epsilon_0 \le 1/2$. On the other hand, for $\Re y \ge \delta_0 + \rho$ (recalling $\delta_0 + \rho \ge \Re z$), we bound

$$e^{-\alpha|\Re y-\Re z|} \le e^{-\epsilon_0(\delta_0+\rho-\Re z)\alpha} e^{-(1-\epsilon_0)\alpha|\Re y-\Re z|}.$$

Therefore, we bound

$$
\begin{aligned}
\int_{\Re y\le\delta_0+\rho} \alpha^{-1} e^{-\alpha|y-z|}|f_\alpha(y)||dy| &\lesssim \alpha^{-1} e^{-\varepsilon_0(\delta_0+\rho-\Re z)\alpha} \|e^{\varepsilon_0(\delta_0+\rho-\Re y)\alpha} f_\alpha\|_{L^1_\rho},\\
\int_{\Re y\ge\delta_0+\rho} \alpha^{-1} e^{-\alpha|y-z|}|f_\alpha(y)||dy| &\lesssim \alpha^{-3/2} e^{-\varepsilon_0(\delta_0+\rho-\Re z)\alpha} \|f_\alpha\|_{L^2(y\ge\delta_0+\rho)}.
\end{aligned}
$$

Similarly, we also have

$$\int_{\Re y\le\delta_0+\rho} \alpha^{-1} e^{-\alpha|y-z|}|f_\alpha(y)||dy| \lesssim \alpha^{-2} e^{-\varepsilon_0(\delta_0+\rho-\Re z)\alpha} \|e^{\varepsilon_0(\delta_0+\rho-\Re y)\alpha} f_\alpha\|_{L^\infty_\rho},$$

which gains an extra factor of α. This proves

$$\|e^{\varepsilon_0(\delta_0+\rho-\Re z)\alpha}(\alpha,\partial_z)\phi_\alpha\|_{L^\infty_\rho} \le \|e^{\varepsilon_0(\delta_0+\rho-\Re y)\alpha} f_\alpha\|_{L^1_\rho} + \alpha^{-1/2}\|f_\alpha\|_{L^2(y\ge\delta_0+\rho)}$$
$$\|e^{\varepsilon_0(\delta_0+\rho-\Re z)\alpha}(\alpha,\partial_z)^2\phi_\alpha\|_{L^\infty_\rho} \le \|e^{\varepsilon_0(\delta_0+\rho-\Re y)\alpha} f_\alpha\|_{L^\infty_\rho} + \alpha^{1/2}\|f_\alpha\|_{L^2(y\ge\delta_0+\rho)}.$$

Taking the summation in $\alpha\in\mathbb{Z}$ yields the first and last estimates in (3.7) for $k=0$. For $k\ge 0$, the estimates follow similarly. For the estimates involving the weight z^{-1}, we use the fact that the Green function vanishes on the boundary $z=0$, and so $|G_\pm(y,z)|\le ze^{-\alpha|y-z|}$. □

3.3. **Biot-Savart law in Ω.** In this section, we bound the velocity through the Biot-Savart law: namely, $u=\nabla^\perp\phi$, where

$$\begin{cases}\Delta\phi=\omega, & \text{in} \quad \Omega\\ \phi=0, & \text{on} \quad \partial\Omega.\end{cases} \tag{3.9}$$

Without loss of generality, we will work with the cut-off vorticity ω^b (see Section 4.1) near the boundary where the rescaled coordinates introduced in Section 2.3 apply. We obtain the following proposition.

Proposition 3.3. *Let ϕ be the solution of (3.9). Then, the velocity field $u=\nabla^\perp\phi$ satisfies*

$$\begin{aligned}\|u\|_{\mathcal{W}^{k,\infty}_\rho} &\lesssim \|\omega\|_{\mathcal{W}^{k,1}_\rho} + \|\omega\|_{H^{k+1}(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})}\\ \|(\frac{1}{\widetilde z}\partial_{\widetilde\theta}\phi)\|_{\mathcal{W}^{k,\infty}_\rho} &\lesssim \|\omega\|_{\mathcal{W}^{k,1}_\rho} + \|\partial_{\widetilde\theta}\omega\|_{\mathcal{W}^{k,1}_\rho} + \|\omega\|_{H^{k+1}(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})}\end{aligned} \tag{3.10}$$

for $k\ge 0$.

Proof. Using (2.16) and (3.9), the scaled stream function $\widetilde\phi(\widetilde t,\widetilde\theta,\widetilde z)$ solves

$$\Delta_{\widetilde\theta,\widetilde z}\widetilde\phi = \lambda^{-2}\widetilde\omega - \lambda^2\widetilde R_\Delta\widetilde\phi, \qquad \widetilde\phi_{|\widetilde z=0}=0$$

on $\mathbb{T}\times\mathbb{R}_+$, and so the elliptic theory, Proposition 3.2, developed in the previous section can be applied, yielding

$$\begin{aligned}\|u\|_{\mathcal{W}^{k,\infty}_\rho} &\lesssim \|\omega\|_{\mathcal{W}^{k,1}_\rho} + \|\omega\|_{H^{k+1}(\{\lambda d(x,\partial\Omega)\ge\delta_0+\rho\})}\\ &\quad + \lambda^2\|\partial_{\widetilde\theta}^{-1}\widetilde R_\Delta\widetilde\phi\|_{\mathcal{W}^{k,\infty}_\rho} + \lambda^2\|\widetilde R_\Delta\widetilde\phi\|_{H^{k+1}(\{\widetilde z\ge\delta_0+\rho\})}.\end{aligned} \tag{3.11}$$

It thus remains to bound $\widetilde R_\Delta\widetilde\phi$. Recall from (2.17) that

$$\widetilde R_\Delta = \widetilde m(\widetilde\theta,\widetilde z)\partial^2_{\widetilde\theta} + \frac{\widetilde\gamma}{1+\lambda^2\widetilde z\widetilde\gamma}\partial_{\widetilde z} - \frac{\widetilde z\widetilde\gamma'}{(1+\lambda^2\widetilde z\widetilde\gamma)^3}\partial_{\widetilde\theta}, \qquad \widetilde m(\widetilde\theta,\widetilde z) = -\frac{2\widetilde z\widetilde\gamma+\lambda^2(\widetilde z\widetilde\gamma)^2}{(1+\lambda^2\widetilde z\widetilde\gamma)^2}.$$

Thanks to the analyticity of the boundary, the coefficients are clearly bounded in $\mathcal{W}^{k,\infty}_\rho$. Therefore, using a similar algebra as in (3.4), we bound

$$\lambda^2\|\partial_{\widetilde\theta}^{-1}\widetilde R_\Delta\widetilde\phi\|_{\mathcal{W}^{k,\infty}_\rho} \lesssim \lambda^2\|\partial_{\widetilde\theta}\widetilde\phi\|_{\mathcal{W}^{k,\infty}_\rho} + \lambda^2\|\partial_{\widetilde z}\widetilde\phi\|_{\mathcal{W}^{k,\infty}_\rho} \tag{3.12}$$

That is, this term can be absorbed into the left hand side of (3.11), upon taking λ sufficiently small. As for the last term in (3.11), we note that for large $\widetilde z$, $|\widetilde m(\widetilde\theta,\widetilde z)|\lesssim\lambda^{-2}$, which in particular proves that there is no growth in $\widetilde z$. This gives

$$\begin{aligned}\lambda^2\|\widetilde R_\Delta\widetilde\phi\|_{H^{k+1}(\{\widetilde z\ge\delta_0+\rho\})} &\lesssim \|\phi\|_{H^{k+3}(\{\lambda d(x,\partial\Omega)\ge\delta_0+\rho\})}\\ &\lesssim \lambda^2\|\widetilde\phi\|_{\mathcal{W}^{k,\infty}_\rho} + \|\omega\|_{H^{k+1}(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})},\end{aligned} \tag{3.13}$$

in which the last estimate follows from the standard elliptic theory in Sobolev spaces. The proposition follows. □

3.4. **Bilinear estimates.** In this section, we show that the Sobolev-analytic norm is well adapted to treat the nonlinear $u \cdot \nabla \omega$. We have the following lemma.

Lemma 3.4. *For any ω and ω', denoting by u the velocity related to ω, we have*

$$\|u \cdot \nabla \omega'\|_{\mathcal{L}^1_\rho} \le C\Big(\|\omega\|_{\mathcal{L}^1_\rho} + \|\omega\|_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\Big)\|\partial_{\widetilde{\theta}}\omega'\|_{\mathcal{L}^1_\rho} + C\Big(\|\omega\|_{\mathcal{L}^1_\rho} + \|\partial_{\widetilde{\theta}}\omega\|_{\mathcal{L}^1_\rho} + \|\omega\|_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\Big)\|\widetilde{z}\partial_{\widetilde{z}}\omega'\|_{\mathcal{L}^1_\rho}.$$

Proof. By definition, the $\mathcal{L}^1_\rho$ norm is defined near the boundary $\{\lambda d(x,\partial\Omega) \le \delta_0+\rho\}$, on which we can write

$$u\cdot\nabla\omega' = \frac{1}{1+z\gamma(\theta)}\partial_\theta\phi\partial_z\omega' - \frac{1}{(1+z\gamma(\theta))^2}\partial_z\phi\partial_\theta\omega'$$

with $\Delta\phi = \omega$. In the rescaled variable $(\widetilde{\theta},\widetilde{z})$, we get

$$u\cdot\nabla\omega' = \frac{\lambda^2}{1+\lambda^2\widetilde{z}\widetilde{\gamma}(\widetilde{\theta})}(\partial_{\widetilde{\theta}}\widetilde{\phi})(\partial_{\widetilde{z}}\widetilde{\omega}') - \frac{\lambda^2}{(1+\lambda^2\widetilde{z}\widetilde{\gamma}(\widetilde{\theta}))^2}(\partial_{\widetilde{z}}\widetilde{\phi})(\partial_{\widetilde{\theta}}\widetilde{\omega}')$$

Note that thanks to the analyticity of $\partial\Omega$, the coefficient $(1+\lambda^2\widetilde{z}\widetilde{\gamma}(\widetilde{\theta}))^{-1}$ is bounded in $\mathcal{L}^\infty_\rho$. Using (3.4) and Proposition 3.3, we bound

$$\begin{aligned}\|(\partial_{\widetilde{z}}\widetilde{\phi})(\partial_{\widetilde{\theta}}\widetilde{\omega}')\|_{\mathcal{L}^1_\rho} &\lesssim \|\partial_{\widetilde{z}}\widetilde{\phi}\|_{\mathcal{L}^\infty_\rho}\|\partial_{\widetilde{\theta}}\widetilde{\omega}'\|_{\mathcal{L}^1_\rho}\\ &\lesssim \Big(\|\omega\|_{\mathcal{L}^1_\rho} + \|\omega\|_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\Big)\|\partial_{\widetilde{\theta}}\omega'\|_{\mathcal{L}^1_\rho}\\ \|(\partial_{\widetilde{\theta}}\widetilde{\phi})(\partial_{\widetilde{z}}\widetilde{\omega}')\|_{\mathcal{L}^1_\rho} &\lesssim \|\frac{1}{\widetilde{z}}\partial_{\widetilde{\theta}}\widetilde{\phi}\|_{\mathcal{L}^\infty_\rho}\|\widetilde{z}\partial_{\widetilde{z}}\widetilde{\omega}'\|_{\mathcal{L}^1_\rho}\\ &\lesssim \Big(\|\omega\|_{\mathcal{L}^1_\rho} + \|\partial_{\widetilde{\theta}}\omega\|_{\mathcal{L}^1_\rho} + \|\omega\|_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\Big)\|\widetilde{z}\partial_{\widetilde{z}}\omega'\|_{\mathcal{L}^1_\rho}\end{aligned}$$

giving the lemma. □

3.5. **Semigroup estimates in the half-plane.** In this section, we give bounds on the Stokes semigroup $e^{\nu tS}$ in the analytic spaces $\mathcal{W}^{k,1}_\rho$ on the half-plane $\mathbb{T}\times\mathbb{R}_+$. We also denote by $\Gamma(\nu t) = e^{\nu tS}(\mathcal{H}^1_{\mathbb{T}\times\{\widetilde{z}=0\}})$ the trace of the semigroup on the boundary, with $\mathcal{H}^1_{\mathbb{T}\times\{\widetilde{z}=0\}}$ being the one-dimensional Hausdorff measure restricted on the boundary. The results in this section are an easy adaptation from those obtained in [21], where the analytic spaces contained no cutoff in z. Precisely, we consider

$$\begin{aligned}(\partial_t - \nu\Delta_{\theta,z})\omega &= 0\\ \nu(\partial_z + |\partial_\theta|)\omega_{|_{z=0}} &= 0\end{aligned} \tag{3.14}$$

on $\mathbb{T}\times\mathbb{R}_+$ (where we drop titles for sake of presentation). We obtain the following proposition.

Proposition 3.5. *Let $e^{\nu tS}$ be the semigroup of the linear Stokes problem (3.14), and let $\Gamma(\nu t)g$ be its trace on the boundary. Then, for any $t\ge0$, $\rho>0$, and $k\ge0$, there hold*

$$\begin{aligned}\|e^{\nu tS}f\|_{\mathcal{W}^{k,1}_\rho} &\le C_0\|f\|_{\mathcal{W}^{k,1}_\rho} + \|zf\|_{H^{k+1}(z\ge\delta_0+\rho)}\\ \|\Gamma(\nu t)g\|_{\mathcal{W}^{k,1}_\rho} &\le C_0\sum_{\alpha\in\mathbb{Z}}|\alpha^k g_\alpha|e^{\epsilon_0(\delta_0+\rho)|\alpha|}\end{aligned} \tag{3.15}$$

uniformly in the inviscid limit.

Proof. The proof follows closely from that in [21]. Indeed, taking the Fourier transform of the semigroup $e^{\nu tS}$ in variable θ, we obtain

$$(e^{\nu tS}f)_\alpha(z) = \int_0^\infty G_\alpha(t,y;z)f_\alpha(y)\,dy, \qquad (\Gamma(\nu t)g)_\alpha(z) = G_\alpha(t,0;z)g_\alpha, \tag{3.16}$$

for each Fourier variable $\alpha \in \mathbb{Z}$, where $G_\alpha(t,y;z)$ is the corresponding Green function. We recall the following result of Proposition 3.3 from [21] that

$$G_\alpha(t,y;z) = H_\alpha(t,y;z) + R_\alpha(t,y;z), \tag{3.17}$$

where

$$H_\alpha(t,y;z) = \frac{1}{\sqrt{\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\alpha^2\nu t},$$

$$|\partial_z^k R_\alpha(t,y;z)| \lesssim \mu_f^{k+1}e^{-\theta_0\mu_f|y+z|} + (\nu t)^{-\frac{k+1}{2}}e^{-\theta_0\frac{|y+z|^2}{\nu t}}e^{-\frac{1}{8}\alpha^2\nu t},$$

for $y,z \ge 0$, $k \ge 0$, and for some $\theta_0 > 0$ and for $\mu_f = |\alpha| + \frac{1}{\sqrt{\nu}}$. In particular, $\|G_\alpha(t,y;\cdot)\|_{L^1_\rho} \lesssim 1$, for each fixed y,t.

Now, for $z, y \le \delta_0 + \rho$, we note that

$$\begin{aligned} e^{-a|y\pm z|}e^{-\epsilon_0(\delta_0+\rho-y)|\alpha|} &= e^{-a|y\pm z|+\epsilon_0|\alpha|(y-z)}e^{-\epsilon_0(\delta_0+\rho-z)|\alpha|} \\ &\le e^{-(a-\epsilon_0|\alpha|)|y\pm z|}e^{-\epsilon_0(\delta_0+\rho-z)|\alpha|} \end{aligned} \tag{3.18}$$

for any real number a and for ϵ_0 sufficiently small. Taking $a = \frac{1}{2}\theta_0\mu_f$, we have $a \ge \epsilon_0|\alpha|$ and so

$$e^{-\theta_0\mu_f|y+z|}e^{-\epsilon_0(\delta_0+\rho-y)|\alpha|} \le e^{-\epsilon_0(\delta_0+\rho-z)|\alpha|}e^{-\frac{1}{2}\theta_0\mu_f|y+z|}$$

On the other hand, taking $a = \frac{1}{2}\theta_0\frac{|y\pm z|}{\nu t}$ in (3.18), we have either $a \ge \epsilon_0|\alpha|$ or $\frac{1}{2}\theta_0\alpha^2\nu t \ge \epsilon_0|\alpha||y\pm z|$. Therefore, we have

$$e^{-\theta_0\frac{|y+z|^2}{\nu t}}e^{-\theta_0\alpha^2\nu t}e^{-\epsilon_0(\delta_0+\rho-y)|\alpha|} \le e^{-\frac{1}{2}\theta_0\frac{|y+z|^2}{\nu t}}e^{-\epsilon_0(\delta_0+\rho-z)|\alpha|}.$$

This proves that for $z \le \delta_0 + \rho$,

$$\begin{aligned} &e^{\epsilon_0(\delta_0+\rho-z)|\alpha|}\int_0^{\delta_0+\rho}|G_\alpha(t,y;z)f_\alpha(y)|\,dy \\ &\le \int_0^{\delta_0+\rho}\Big[(\nu t)^{-\frac{1}{2}}e^{-\frac{1}{2}\theta_0\frac{|y\pm z|^2}{\nu t}} + \mu_f e^{-\frac{1}{2}\theta_0\mu_f|y+z|}\Big]|e^{\epsilon_0(\delta_0+\rho-y)|\alpha|}f_\alpha(y)|\,dy. \end{aligned}$$

Since the term in the bracket is bounded in L^1_z norm, we have

$$\Big\|e^{\epsilon_0(\delta_0+\rho-z)|\alpha|}\int_0^{\delta_0+\rho}G_\alpha(t,y;z)f_\alpha(y)\,dy\Big\|_{\mathcal{L}^1_\rho} \lesssim \|e^{\epsilon_0(\delta_0+\rho-y)|\alpha|}f_\alpha\|_{\mathcal{L}^1_\rho}.$$

Taking the summation in α yields the stated bounds for this term.

Next, consider the case when $y \ge \delta_0 + \rho \ge z$. In this case, we simply use

$$e^{-\epsilon_0|\alpha||y-z|} \le e^{-\epsilon_0|\alpha|(\delta_0+\rho-z)},$$

giving the right analyticity weight in z. The control of the weight $e^{\epsilon_0|\alpha||y-z|}$ is done exactly as above, yielding

$$\begin{aligned} &e^{\epsilon_0(\delta_0+\rho-z)|\alpha|}\int_{\delta_0+\rho}^\infty|G_\alpha(t,y;z)f_\alpha(y)|\,dy \\ &\le \int_{\delta_0+\rho}^\infty\Big[(\nu t)^{-\frac{1}{2}}e^{-\frac{1}{2}\theta_0\frac{|y\pm z|^2}{\nu t}} + \mu_f e^{-\frac{1}{2}\theta_0\mu_f|y+z|}\Big]|f_\alpha(y)|\,dy. \end{aligned}$$

Therefore,

$$\sum_\alpha \|e^{\epsilon_0(\delta_0+\rho-z)|\alpha|}\int_{\delta_0+\rho}^\infty |G_\alpha(t,y;z)f_\alpha(y)|\,dy\|_{\mathcal{L}^1_\rho} \lesssim \sum_\alpha \|f_\alpha\|_{L^1(z\ge\delta_0+\rho)} \lesssim \|zf\|_{H^1(z\ge\delta_0+\rho)}.$$

Similarly, from (3.16), the Fourier transform of the trace operator $\Gamma(\nu t)g$ is estimated by

$$\begin{aligned}|(\Gamma(\nu t)g)_\alpha(z)| &\le |G_\alpha(t,0;z)g_\alpha|\\ &\le \Big[\mu_f e^{-\theta_0\mu_f|z|} + (\nu t)^{-\frac12}e^{-\theta_0\frac{|z|^2}{\nu t}}e^{-\frac18\alpha^2\nu t}\Big]|g_\alpha|\\ &\le \Big[\mu_f e^{-\frac12\theta_0\mu_f|z|} + (\nu t)^{-\frac12}e^{-\frac12\theta_0\frac{|z|^2}{\nu t}}\Big]e^{-\epsilon_0(\delta_0+\rho-z)|\alpha|}|g_\alpha|e^{\epsilon_0(\delta_0+\rho)|\alpha|}\end{aligned}$$

in which the last inequality is a special case of the previous calculations for $y=0$ and $z\le\delta_0+\rho$. The bounds $\Gamma(\nu t)g$ are thus direct. Finally, the bounds on derivatives follow from the similar adaptation of derivatives bounds provided in [21]. We skip repeating the details. □

3.6. **Semigroup estimates near $\partial\Omega$.** In this section, we provide bounds on the Stokes semigroup $e^{\nu tS}$, which will be used to estimate the vorticity ω^b (see Section 4.1) near the boundary in the analytic spaces $\mathcal{W}^{k,1}_\rho$. Precisely, we consider

$$\begin{cases}\partial_t\omega - \nu\Delta\omega = 0\\ \nu(\partial_n + DN)\omega_{|\partial\Omega} = 0\end{cases} \tag{3.19}$$

in Ω. We obtain the following proposition.

Proposition 3.6. *Let $e^{\nu tS}$ be the semigroup of the linear Stokes problem (3.19), and let $\Gamma(\nu t)$ be its trace on the boundary. Fix any finite time T. Then, for sufficiently small λ, and for any $0\le t\le T$, $\rho>0$, and $k\ge0$, there hold*

$$\begin{aligned}\|e^{\nu tS}f\|_{\mathcal{W}^{k,1}_\rho} &\le C_0\|f\|_{\mathcal{W}^{k,1}_\rho} + \|f\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\\ \|\Gamma(\nu t)g\|_{\mathcal{W}^{k,1}_\rho} &\le C_0\sum_{\alpha\in\mathbb{Z}}|\alpha^k g_\alpha|e^{\epsilon_0(\delta_0+\rho)|\alpha|}\end{aligned} \tag{3.20}$$

uniformly in the inviscid limit.

Proof. In the scaled variables, the Stokes problem for near boundary vorticity ω becomes

$$\begin{cases}(\partial_{\widetilde t} - \nu\Delta_{\widetilde\theta,\widetilde z})\widetilde\omega = -\lambda^2\nu\widetilde R_\Delta\widetilde\omega\\ \nu(\partial_{\widetilde z} + |\partial_{\widetilde\theta}|)\widetilde\omega|_{\widetilde s=0} = -\nu\widetilde B\widetilde\omega\end{cases}$$

where $\widetilde R_\Delta$ and $\widetilde B$ are defined as in (2.17) and (2.27). Using the Duhamel, the solution with initial data ω_0 can be written as

$$\widetilde\omega(\widetilde t) = e^{\nu\widetilde tS}\widetilde\omega_0 - \nu\lambda^2\int_0^{\widetilde t} e^{\nu(\widetilde t-\widetilde t')S}\widetilde R_\Delta\widetilde\omega(\widetilde t')\,d\widetilde t' - \nu\int_0^{\widetilde t}\Gamma(\nu(\widetilde t-\widetilde t'))\widetilde B\widetilde\omega(\widetilde t')\,d\widetilde t'. \tag{3.21}$$

We shall bound the integral terms on the right in term of the initial data. Recall from (2.17) that

$$\widetilde R_\Delta = \widetilde m(\widetilde\theta,\widetilde z)\partial^2_{\widetilde\theta} + \frac{\widetilde\gamma}{1+\lambda^2\widetilde z\widetilde\gamma}\partial_{\widetilde z} - \frac{\widetilde z\widetilde\gamma'}{(1+\lambda^2\widetilde z\widetilde\gamma)^3}\partial_{\widetilde\theta}, \qquad \widetilde m(\widetilde\theta,\widetilde z) = -\frac{2\widetilde z\widetilde\gamma + \lambda^2(\widetilde z\widetilde\gamma)^2}{(1+\lambda^2\widetilde z\widetilde\gamma)^2}.$$

We rewrite the operator in the following form

$$\widetilde{R}_\Delta \widetilde{\omega} = \partial^2_{\widetilde{\theta}}[\widetilde{m}\widetilde{\omega}] - \partial_{\widetilde{\theta}}\Big[2\partial_{\widetilde{\theta}}\widetilde{m}\widetilde{\omega} + \frac{\widetilde{z}\widetilde{\gamma}'}{(1+\lambda^2\widetilde{z}\widetilde{\gamma})^3}\widetilde{\omega}\Big] + \partial_{\widetilde{z}}\Big(\frac{\widetilde{\gamma}}{1+\lambda^2\widetilde{z}\widetilde{\gamma}}\widetilde{\omega}\Big)$$
$$+ \Big[(\partial^2_{\widetilde{\theta}}\widetilde{m}) - \partial_{\widetilde{z}}\Big(\frac{\widetilde{\gamma}}{1+\lambda^2\widetilde{z}\widetilde{\gamma}}\Big) + \partial_{\widetilde{\theta}}\Big(\frac{\widetilde{z}\widetilde{\gamma}'}{(1+\lambda^2\widetilde{z}\widetilde{\gamma})^3}\Big)\Big]\widetilde{\omega}.$$

We now bound each term appearing in the Duhamel formula (3.21). Thanks to the analyticity of the boundary, the coefficients are bounded in $\mathcal{W}^{k,\infty}_\rho$. Now, recall from (3.17) that the Green function has two components:

$$e^{\nu\widetilde{t}S} = e^{\nu\widetilde{t}S_H} + e^{\nu\widetilde{t}S_R}$$

which corresponds to the Green kernel H_α (i.e., the heat kernel) and the other from the stationary Stokes kernel R_α.

We first claim that

$$\Big\|\nu\lambda^2\int_0^{\widetilde{t}} e^{\nu(\widetilde{t}-\widetilde{t}')S_H}\widetilde{R}_\Delta\widetilde{\omega}(\widetilde{t}')\,d\widetilde{t}'\Big\|_{\mathcal{W}^{k,1}_\rho} \lesssim \lambda^2 \sup_{0\le \widetilde{t}'\le\widetilde{t}}\|\omega\|_{\mathcal{W}^{k,1}_\rho} + \|\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)}. \tag{3.22}$$

For the heat semigroup, we may integrate by parts in $\widetilde{\theta}$ or $\widetilde{z}$. It follows directly from the representation of the Green function that derivatives of the semigroup $\nabla_{\widetilde{\theta},\widetilde{z}}e^{\nu\widetilde{t}S_H}$ are of order $(\nu\widetilde{t})^{-1/2}$ of the semigroup itself. Therefore, the first-order derivative term in $\widetilde{R}_\Delta$ can be treated systematically as follows:

$$\nu\lambda^2\Big\|\int_0^{\widetilde{t}} e^{\nu(\widetilde{t}-\widetilde{t}')S_H}\nabla_{\widetilde{\theta},\widetilde{z}}h(\widetilde{t}')\,d\widetilde{t}'\Big\|_{\mathcal{W}^{k,1}_\rho} \lesssim \nu\lambda^2\int_0^{\widetilde{t}}(\nu(\widetilde{t}-\widetilde{t}'))^{-1/2}\|h(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho}\,d\widetilde{t}'$$
$$\lesssim \sqrt{\nu}\lambda^2\sup_{0\le\widetilde{t}'\le\widetilde{t}}\|h\|_{\mathcal{W}^{k,1}_\rho}.$$

The zero-order term is treated similarly. The analysis doesn't apply directly to the second-order derivative term $\partial^2_{\widetilde{\theta}}[\widetilde{m}\widetilde{\omega}]$ due to the singularity in time $(\nu t)^{-1}$, if integration by parts was to perform twice. However, in the Fourier variable α, we compute

$$\nu\lambda^2\int_0^{\widetilde{t}}(e^{\nu(\widetilde{t}-\widetilde{t}')S_H}\partial^2_{\widetilde{\theta}}[\widetilde{m}\widetilde{\omega}])_\alpha(\widetilde{t}')\,d\widetilde{t}' = \nu\alpha^2\lambda^2\int_0^{\widetilde{t}}\int_0^\infty H_\alpha(t,\widetilde{y};\widetilde{z})[\widetilde{m}\widetilde{\omega}]_\alpha(\widetilde{t}')\,d\widetilde{y}d\widetilde{t}'.$$

Observe that the Green kernel H_α has the diffusion term $e^{-\nu\alpha^2\widetilde{t}}$, for which we use

$$\nu\alpha^2\lambda^2\int_0^{\widetilde{t}} e^{-\nu\alpha^2(\widetilde{t}-\widetilde{t}')}d\widetilde{t}' \lesssim \lambda^2$$

yielding the claim (3.22).

Next, we claim that

$$\Big\|\nu\lambda^2\int_0^{\widetilde{t}} e^{\nu(\widetilde{t}-\widetilde{t}')S_R}\widetilde{R}_\Delta\widetilde{\omega}(\widetilde{t}')\,d\widetilde{t}'\Big\|_{\mathcal{W}^{k,1}_\rho}$$
$$\lesssim \nu\lambda^2\int_0^{\widetilde{t}}\|\partial_{\widetilde{\theta}}\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho}\,d\widetilde{t} + \|\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)}. \tag{3.23}$$

It suffices to check for the stationary Green kernel $\mu_f e^{-\theta_0\mu_f(\widetilde{y}+\widetilde{z})}$ and for the second-order derivative term $\partial^2_{\widetilde{\theta}}[\widetilde{m}\widetilde{\omega}]$ appearing in $\widetilde{R}_\Delta\widetilde{\omega}(\widetilde{t}')$. For this term, we make use of the fact that $\widetilde{m}$ vanishes at $\widetilde{z}=0$; namely, we can write $\widetilde{m}=\widetilde{z}\widetilde{m}_1$ and use

$\mu_f e^{-\theta_0\mu_f\widetilde{z}}\widetilde{z} \lesssim 1$, which controls one spatial derivative, since $\mu_f = |\alpha| + \nu^{-1/2}$. This proves the claim (3.23).

Finally, putting the previous bounds together into the Duhamel representation (3.21), we have obtained

$$\begin{aligned}\|\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho} &\lesssim \|\omega_0\|_{\mathcal{W}^{k,1}_\rho} + \|\omega_0\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)}\\ &\quad + \lambda^2 \sup_{0\le\widetilde{t}'\le\widetilde{t}} \|\omega(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho} + \nu\lambda^2\int_0^{\widetilde{t}} \|\partial_{\widetilde{\theta}}\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho}\, d\widetilde{t}\\ &\quad + \|\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)}\end{aligned} \tag{3.24}$$

for any $k\ge 0$. The standard energy estimates for the heat equation (away from the boundary) yield

$$\|\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)} \lesssim \|\omega_0\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0/2)}. \tag{3.25}$$

It remains to treat the third and forth terms on the right hand side of (3.24). We bound these terms by iteration, introducing

$$A_0(\beta) := \sup_{0\le k\le 4}\Big(\sup_{0<\beta\widetilde{t}<\rho_0}\ \sup_{0<\rho<\rho_0-\beta\widetilde{t}} \Big\{\|\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho} + \|\partial_{\widetilde{\theta}}\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho}(\rho_0-\rho-\beta\widetilde{t})^\zeta\Big\}\Big)$$

for some $\zeta\in(0,1)$. We bound

$$\begin{aligned}\nu\lambda^2\int_0^{\widetilde{t}} \|\partial_{\widetilde{\theta}}\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho}\, d\widetilde{t} &\le C_0\nu\lambda^2 A_0(\beta)\int_0^{\widetilde{t}} (\rho_0-\rho-\beta\widetilde{s})^{-\zeta}\, d\widetilde{s}\\ &\le C_0\nu\lambda^2\beta^{-1}A_0(\beta).\end{aligned}$$

Next, we check the bound on $\|\partial_{\widetilde{\theta}}\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_\rho}$. We focus only the worst term as in (3.23). Note that $\rho<\rho_0-\beta\widetilde{t}\le\rho_0-\beta\widetilde{s}$. Thus, we take $\rho' = \frac{\rho+\rho_0-\beta s}{2}$ and bound

$$\begin{aligned}&\Big\|\nu\lambda^2\partial_{\widetilde{\theta}}\int_0^{\widetilde{t}} e^{\nu(\widetilde{t}-\widetilde{t}')S_R}\widetilde{R}_\Delta\widetilde{\omega}(\widetilde{t}')\, d\widetilde{t}'\Big\|_{\mathcal{W}^{k,1}_\rho}\\ &\lesssim \nu\lambda^2\int_0^{\widetilde{t}} \frac{1}{\rho'-\rho}\|\partial_{\widetilde{\theta}}\omega(\widetilde{t})\|_{\mathcal{W}^{k,1}_{\rho'}}\, d\widetilde{t} + \|\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)}\\ &\le C_0\nu\lambda^2\int_0^{\widetilde{t}} (\rho_0-\rho-\beta s)^{-1-\zeta}\, ds + \|\omega_0\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\\ &\le C_0\nu\lambda^2\beta^{-1}A_0(\beta)(\rho_0-\rho-\beta\widetilde{t})^{-\zeta} + \|\omega_0\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0/2)}.\end{aligned}$$

This proves that

$$A_0(\beta) \lesssim \|\omega_0\|_{\mathcal{W}^{k,1}_\rho} + \|\omega_0\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0/2)} + \Big(\lambda^2+\nu\lambda^2\beta^{-1}\Big)A_0(\beta).$$

Taking λ and ν small, the last term can be absorbed into the left hand side, completing the bounds on $A_0(\beta)$ or the $\mathcal{W}^{k,1}_\rho$ norm for the vorticity. Note that we do not require β to be sufficiently large (compared with the nonlinear iteration provided in the next section). As a consequence, the proposition holds for any given finite time. □

4. **Nonlinear analysis.** As already mentioned in the introduction, we construct the solutions to the Navier-Stokes equation via the vorticity formulation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega \tag{4.1}$$

together with the nonlocal boundary condition (1.8) and with initial data $\omega_{|t=0} = \omega_0$ satisfying

$$\|\omega_0\|_{\mathcal{W}^{2,1}_\rho} + \|\omega_0\|_{H^4(\{\lambda d(x,\partial\Omega)\geq \delta_0/2\})} < \infty. \tag{4.2}$$

Introduce the smooth cutoff function ϕ^b as in (2.18), and write

$$\omega = \omega^b + \omega^i, \qquad \omega^b = \phi^b \omega, \qquad \omega^i = (1-\phi^b)\omega. \tag{4.3}$$

We also define the corresponding velocity field through the Biot-Savart law

$$u = u^b + u^i, \qquad u^b = \nabla^\perp \Delta^{-1}\omega^b, \qquad u^i = \nabla^\perp \Delta^{-1} \omega^i. \tag{4.4}$$

This yields

$$\begin{cases} \partial_t \omega^b + u \cdot \nabla \omega^b = \nu \Delta \omega^b \\ \nu(\partial_n + DN)\omega^b_{|\partial\Omega} = [\partial_n \Delta^{-1}(u\cdot\nabla\omega)]_{|\partial\Omega} \end{cases} \tag{4.5}$$

for the vorticity near the boundary, and

$$\begin{cases} \partial_t \omega^i + u \cdot \nabla \omega^i = \nu \Delta \omega^i \\ \omega^i_{|\partial\Omega} = 0 \end{cases} \tag{4.6}$$

for the vorticity away from the boundary. Here, we note that the boundary condition on ω^i follows directly from the definition (4.3), while the boundary condition on ω^b was due to the fact that $DN\omega^i = 0$ by Lemma 2.2. We also note that the velocity field u that appears in both the systems is the full velocity, which is the summation of u^b and u^i generated by ω^b and ω^i, respectively.

We shall construct the near boundary vorticity solving (4.5) through the semigroup of the Stokes problem. Indeed, we have the following standard Duhamel's integral representation, written in the scaled variables,

$$\widetilde{\omega}(\widetilde{t}) = e^{\nu \widetilde{t} S}\widetilde{\omega}_0 + \int_0^{\widetilde{t}} e^{\nu(\widetilde{t}-\widetilde{t}')S} f(\widetilde{t}')\, d\widetilde{t}' + \int_0^{\widetilde{t}} \Gamma(\nu(\widetilde{t}-\widetilde{t}'))g(\widetilde{t}')\, d\widetilde{t}' \tag{4.7}$$

where

$$f(\widetilde{t}) = -\lambda^{-2} u\cdot\nabla\omega^b, \qquad g(\widetilde{t}) = \lambda^{-1}[\partial_n \Delta^{-1}(u\cdot\nabla\omega)]_{|\partial\Omega}. \tag{4.8}$$

Here, $e^{\nu\widetilde{t}S}$ denotes the semigroup of the corresponding Stokes problem and $\Gamma(\nu\widetilde{t})$ being its trace on the boundary; see Section 3.6.

4.1. **Global Sobolev-analytic norm.** We now introduce Sobolev-analytic norms to control global vorticity. Let us fix positive numbers ρ_0, δ_0, and $\zeta \in (0,1)$. Introduce the following family of nonlinear iterative norms for vorticity:

$$\begin{aligned} A(\beta) := \sup_{0<\lambda^2\beta t<\rho_0} \Big[& \sup_{0<\rho<\rho_0-\beta\lambda^2 t} \Big\{ \|\omega(t)\|_{\mathcal{W}^{1,1}_\rho} + \|\omega(t)\|_{\mathcal{W}^{2,1}_\rho}(\rho_0-\rho-\lambda^2\beta t)^\zeta \Big\} \\ & + \|\omega(t)\|_{H^4(\{\lambda d(x,\partial\Omega)\geq\delta_0/2\})}\Big] \end{aligned} \tag{4.9}$$

for a parameter $\beta > 0$, with recalling

$$\|\omega(t)\|_{\mathcal{W}^{k,1}_\rho} = \sum_{j+\ell\leq k} \|\partial^j_{\widetilde{\theta}}(\widetilde{z}\partial_{\widetilde{z}})^\ell \omega(t)\|_{\mathcal{L}^1_\rho}.$$

Note that by definition the norm $\|\cdot\|_{\mathcal{W}^{k,1}_\rho}$ controls the analyticity of the vorticity near the boundary, precisely in the region $\lambda d(x,\partial\Omega)\le \delta_0+\rho$, while the H^4 norm is to control the Sobolev regularity away from the boundary. We shall show that the vorticity norm remains finite for sufficiently large β. The weight $(\rho_0-\rho-\lambda^2\beta t)^\zeta$, with a small $\zeta>0$, is standard in the literature to avoid time singularity when recovering the loss of derivatives ([2, 6]). See also [14] for an alternative framework to construct analytic solutions through generator functions.

Our goal is to prove the following key proposition.

Proposition 4.1. *For $\beta>0$, there holds*

$$A(\beta)\le C_0\|\omega_0\|_{\mathcal{W}^{2,1}_\rho}+C_0\|\omega_0\|_{H^4(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})}+C_0\beta^{-1}A(\beta)^2.$$

In Section 4.4, we will show that our main theorem, Theorem 1.1, follows straightforwardly from Proposition 4.1.

4.2. **Analytic bounds near the boundary.** In this section, we bound the vorticity near the boundary $\lambda d(x,\partial\Omega)\le\delta_0+\rho_0$, on which by definition $\omega=\omega^b$ and therefore the Duhamel representation (4.7) holds. Let $\rho<\rho_0-\lambda^2\beta t$. Recalling the notation $\widetilde t=\lambda^2 t$ and using (4.7), we bound

$$\begin{aligned}\|\widetilde\omega(\widetilde t)\|_{\mathcal{W}^{k,1}_\rho}\le \|e^{\nu\widetilde tS}\widetilde\omega_0\|_{\mathcal{W}^{k,1}_\rho}&+\int_0^{\widetilde t}\|e^{\nu(\widetilde t-\widetilde t')S}f(\widetilde t')\|_{\mathcal{W}^{k,1}_\rho}\,d\widetilde t'\\&+\int_0^{\widetilde t}\|\Gamma(\nu(\widetilde t-\widetilde t'))g(\widetilde t')\|_{\mathcal{W}^{k,1}_\rho}\,d\widetilde t'\end{aligned}\tag{4.10}$$

for $0<k\le 4$ and for f,g defined as in (4.8). Let us bound each term on the right. Using the semigroup estimates, Proposition 3.5, we have

$$\begin{aligned}\|e^{\nu\widetilde tS}\widetilde\omega_0\|_{\mathcal{W}^{k,1}_\rho}&\le C_0\|\widetilde\omega_0\|_{\mathcal{W}^{k,1}_\rho}+\|\widetilde z\widetilde\omega_0\|_{H^{k+1}(\widetilde z\ge\delta_0+\rho)}\\&\le C_0\|\widetilde\omega_0\|_{\mathcal{W}^{k,1}_\rho}+\|\omega_0\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\rho)}.\end{aligned}$$

While for the second integral term in (4.10), we have

$$\int_0^{\widetilde t}\|e^{\nu(\widetilde t-\widetilde t')S}f(\widetilde t')\|_{\mathcal{W}^{k,1}_\rho}\,d\widetilde t'\lesssim\int_0^{\widetilde t}\Big[\|f(\widetilde t')\|_{\mathcal{W}^{k,1}_\rho}+\|\widetilde z f(\widetilde t')\|_{H^{k+1}(\widetilde z\ge\delta_0+\rho)}\Big]\,d\widetilde t'.$$

Then, we use (4.8), in the above formula with $f(\widetilde t)$ replaced by $-\lambda^{-2}u\cdot\nabla\omega^b$. First, using the standard elliptic theory for $k=0,1,2$, we bound

$$\|\widetilde z(u\cdot\nabla\omega^b)(\widetilde t')\|_{H^{k+1}(\widetilde z\ge\delta_0+\rho)}\lesssim\|\omega\|^2_{H^4(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})}\lesssim A(\beta)^2.$$

Next, for the analytic norm, with the bilinear estimates from Lemma 3.4, we have:

$$\begin{aligned}\|u\cdot\nabla\omega^b\|_{\mathcal{L}^1_\rho}&\le C\Big(\|\omega\|_{\mathcal{L}^1_\rho}+\|\omega\|_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\Big)\|\partial_{\widetilde\theta}\omega^b\|_{\mathcal{L}^1_\rho}\\&\quad+C\Big(\|\omega\|_{\mathcal{L}^1_\rho}+\|\partial_{\widetilde\theta}\omega\|_{\mathcal{L}^1_\rho}+\|\omega\|_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\Big)\|\widetilde z\partial_{\widetilde z}\omega^b\|_{\mathcal{L}^1_\rho}\\&\lesssim\|\omega\|^2_{\mathcal{W}^{1,1}_\rho}+\|\omega\|^2_{H^1(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\\&\lesssim A(\beta)^2\\\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{1,1}_\rho}&\lesssim\|\omega\|_{\mathcal{W}^{1,1}_\rho}\|\omega\|_{\mathcal{W}^{1,2}_\rho}+\|\omega\|^2_{H^2(\{\lambda d(x,\partial\Omega)\ge\delta_0\})}\\&\lesssim A(\beta)^2(\rho_0-\rho-\beta\widetilde t)^{-\zeta}.\end{aligned}$$

Therefore,

$$\int_0^{\widetilde{t}} \|u\cdot\nabla\omega^b\|_{\mathcal{W}^{1,1}_\rho}\, d\widetilde{s} \le C_0 A(\beta)^2 \int_0^{\widetilde{t}} (\rho_0-\rho-\beta\widetilde{s})^{-\zeta}\, d\widetilde{s} \\ \le C_0\beta^{-1}A(\beta)^2.$$

Similarly, we consider the case when $k=2$. Noting $\rho<\rho_0-\beta t\le\rho_0-\beta s$, we take $\rho'=\frac{\rho+\rho_0-\beta s}{2}$ and compute

$$\begin{aligned}\int_0^t \|u\cdot\nabla\omega^b\|_{\mathcal{W}^{2,1}_\rho}\, ds &\le C_0\int_0^t \frac{1}{\rho'-\rho}\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{1,1}_{\rho'}}\, ds\\ &\le C_0A(\beta)^2\int_0^t(\rho_0-\rho-\beta s)^{-1-\zeta}\, ds\\ &\le C_0\beta^{-1}A(\beta)^2(\rho_0-\rho-\beta t)^{-\zeta}.\end{aligned}$$

Finally, we treat the last integral term in (4.10). Precisely, we will show that, for $k\le 2$:

$$\begin{aligned}\|\Gamma(\nu(\widetilde{t}-\widetilde{t}'))g(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho}\le\ & C_0\|u\cdot\nabla\omega^b(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho}+C_0\|\omega(\widetilde{t}')\|^2_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\\ &+C_0\|\omega(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho}\|\omega(\widetilde{t}')\|_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\end{aligned}\tag{4.11}$$

which would then imply

$$\int_0^{\widetilde{t}}\|\Gamma(\nu(\widetilde{t}-\widetilde{t}'))g(\widetilde{t}')\|_{\mathcal{W}^{2,1}_\rho}d\widetilde{t}'\le C_0\left(A(\beta)^2+\beta^{-1}A(\beta)^2(\rho_0-\rho-\beta t)^{-\zeta}\right).$$

Here the constant C_0 may change from line to line. It remains to give the proof for the inequality (4.11). First, by Proposition 3.5, we have

$$\|\Gamma(\nu(\widetilde{t}-\widetilde{t}'))g(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho}\le C_0\sum_\alpha|\alpha|^k|g_\alpha|e^{\varepsilon_0(\delta_0+\rho)|\alpha|},$$

where g_α is given by

$$g_\alpha=\lambda^{-1}\partial_n\Delta^{-1}(u\cdot\nabla\omega)_\alpha|_{\partial\Omega}.$$

Let $\Phi=\Delta^{-1}(u\cdot\nabla\omega)$. By definition, Φ solves

$$\begin{cases}\Delta\Phi=u\cdot\nabla\omega, & x\in\Omega\\ \Phi|_{\partial\Omega}=0.\end{cases}$$

In the rescaled geodesic coordinates, we have $g_\alpha=\partial_{\widetilde{z}}\Phi_\alpha(0)$. Let $\Phi^b=\Phi(x)\phi^b(x)$, we have

$$\begin{cases}\Delta\Phi^b=2\nabla_x\phi^b\cdot\nabla_x\Phi^b+\Delta\phi^b\Phi+\phi^bu\cdot\nabla\omega\\ \Phi^b|_{z=0}=0.\end{cases}$$

By a direct calculation, we have

$$\begin{aligned}&e^{\varepsilon_0(\delta_0+\rho)|\alpha|}g_\alpha(\widetilde{t}')=\partial_z\Phi^b_\alpha|_{\widetilde{z}=0}\\ &=\int_0^\infty e^{|\alpha|(\varepsilon_0(\delta_0+\rho)-\widetilde{z})}\left\{\lambda^2\left(\widetilde{R}_\Delta\widetilde{\Phi}_b\right)_\alpha(\widetilde{z})-\lambda^{-2}\left(2\nabla_x\phi^b\cdot\nabla_x\Phi^b-\Phi\Delta\phi^b-\phi^bu\cdot\nabla\omega\right)_\alpha\right\}d\widetilde{z}\\ &=I_{1,\alpha}+I_{2,\alpha}+I_{3,\alpha}+I_{4,\alpha}.\end{aligned}$$

Treating $I_{1,\alpha}$. As in the proof of Proposition 3.20 for $\widetilde{R}_\Delta$, we have

$$|I_{1,\alpha}| \le C_0|\alpha|^2\lambda^2 \int_0^\infty e^{|\alpha|(\varepsilon_0(\delta_0+\rho)-\widetilde{z})}|\widetilde{z}\Phi^b_\alpha(\widetilde{z})|d\widetilde{z} + C_0\lambda^2\int_0^\infty e^{|\alpha|(\varepsilon_0(\delta_0+\rho)-\widetilde{z})}\left(|\alpha||\Phi^b_\alpha(\widetilde{z})| + |\partial_{\widetilde{z}}\Phi^b_\alpha|\right)d\widetilde{z}.$$

First, we use the inequality $\widetilde{z}|\alpha|e^{-|\alpha|\widetilde{z}} \le e^{-\frac{1}{2}|\alpha|\widetilde{z}}$ to get

$$\begin{aligned}|I_{1,\alpha}| &\le C_0\lambda^2\int_0^\infty e^{|\alpha|\left(\varepsilon_0(\delta_0+\rho)-\frac{1}{2}\widetilde{z}\right)}\left(|\alpha||\Phi^b_\alpha(\widetilde{z})| + |\partial_{\widetilde{z}}\Phi^b_\alpha|\right)d\widetilde{z}\\ &\le C_0\lambda^2\int_0^{\delta_0+\rho} e^{|\alpha|\varepsilon_0(\delta_0+\rho-\widetilde{z})}\left(|\alpha||\Phi^b_\alpha(\widetilde{z})| + |\partial_{\widetilde{z}}\Phi^b_\alpha|\right)d\widetilde{z}\\ &\quad + C_0\lambda^2\int_{\delta_0+\rho}^\infty \left(|\alpha||\Phi^b_\alpha(\widetilde{z})| + |\partial_{\widetilde{z}}\Phi^b_\alpha|\right)d\widetilde{z}.\end{aligned}$$

For the first term, we use the L^1_μ elliptic estimate for the velocity (since the kernel $K_\alpha \in L^1$), to get

$$\sum_\alpha |\alpha|^k\int_0^{\delta_0+\rho} e^{|\alpha|\varepsilon_0(\delta_0+\rho-\widetilde{z})}\left(|\alpha||\Phi^b_\alpha(\widetilde{z})| + |\partial_{\widetilde{z}}\Phi^b_\alpha|\right)d\widetilde{z} \le C\|\phi^b u\cdot\nabla\omega\|_{\mathcal{W}^{k,1}_\rho} + C\|\Phi\|_{H^{k+2}(\lambda d(x,\partial\Omega)\ge\delta_0+\rho_0)}. \tag{4.12}$$

Now we have

$$\begin{aligned}\|\phi^b u\cdot\nabla\omega\|_{\mathcal{W}^{k,1}_\rho} &= \|u\cdot\nabla\omega^b - (u\cdot\nabla\phi^b)\omega\|_{\mathcal{W}^{k,1}_\rho}\\ &\le C\left(\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{k,1}_\rho} + \|u\omega\|_{H^k(\lambda d(x,\partial\Omega)\ge\delta_0)}\right)\\ &\le C\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{k,1}_\rho}\\ &\quad + C\|\omega\|_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\left(\|\omega\|_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)} + \|\omega\|_{\mathcal{W}^{k,1}_\rho}\right).\end{aligned} \tag{4.13}$$

By standard elliptic estimate, we have

$$\begin{aligned}\|\Phi\|_{H^{k+2}(\lambda d(x,\partial\Omega)\ge\delta_0+\rho_0)} &\le C\|\phi^b u\cdot\nabla\omega\|_{\mathcal{W}^{k,1}_\rho} + C\|u\cdot\nabla\omega\|_{H^k(\lambda d(x,\partial\Omega)\ge\delta_0)}\\ &\le C\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{k,1}_\rho} + \|\omega\|^2_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\\ &\quad + \|\omega\|_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\|\omega\|_{\mathcal{W}^{k,1}_\rho}.\end{aligned} \tag{4.14}$$

Combining (4.12),(4.14) and (4.13), we have

$$\sum_\alpha |\alpha|^k|I_{1,\alpha}| \lesssim \|u\cdot\nabla\omega^b(\widetilde{t}')\|_{\mathcal{W}^{k,1}_\rho} + \|\omega\|^2_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)} + \|\omega\|_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\|\omega\|_{\mathcal{W}^{k,1}_\rho}$$

as claimed in (4.11). The proof for $I_{1,\alpha}$ is complete.

Treating $I_{2,\alpha}$. For $I_{2,\alpha}$, we note that the domain of integration is $\widetilde{z} \ge \delta_0+\rho_0 > \delta_0+\rho$, we have

$$|\alpha|^k e^{|\alpha|(\varepsilon_0(\delta_0+\rho)-\widetilde{z})} \le C.$$

Thus we have

$$\begin{aligned}\sum_\alpha |\alpha|^k |I_{2,\alpha}| &\le C \sum_\alpha \|\nabla_x \Phi^b_\alpha\|_{L^1(\widetilde{z}\ge\delta_0+\rho_0)} \le C\|d(x,\partial\Omega)\nabla_x\Phi\|_{H^1(\lambda d(x,\partial\Omega)\ge\delta_0+\rho_0)}\\ &\le C\|d(x,\partial\Omega)\Phi\|_{H^2(\lambda d(x,\partial\Omega)\ge\delta_0+\rho_0)}\\ &\le C\|\phi^b u\cdot\nabla\omega\|_{\mathcal{W}^{k,1}_\rho} + C\|u\cdot\nabla\omega\|_{L^2(\lambda d(x,\partial\Omega)\ge\delta_0)},\end{aligned}$$

which is bounded by the right hand side of (4.11). The proof for $I_{2,\alpha}$ is complete.

Treating $I_{3,\alpha}$**.** Similarly, for $I_{3,\alpha}$, we get

$$\begin{aligned}\sum_\alpha |\alpha|^k |I_{3,\alpha}| &\le C\|d(x,\partial\Omega)\Phi\|_{H^1(\lambda d(x,\partial\Omega)\ge\delta_0+\rho_0)}\\ &\le C\|\phi^b u\cdot\nabla\omega\|_{\mathcal{W}^{k,1}_\rho} + C\|u\cdot\nabla\omega\|_{L^2(\lambda d(x,\partial\Omega)\ge\delta_0)}.\end{aligned}$$

This is also bounded by the right hand side of (4.11). The proof for $I_{3,\alpha}$ is complete.

Treating $I_{4,\alpha}$**.** For $I_{4,\alpha}$ we have

$$\sum_\alpha |\alpha|^k |I_{4,\alpha}| \le \|\phi^b u\cdot\nabla\omega\|_{\mathcal{W}^{k,1}_\rho}.$$

We rewrite $\phi^b u\cdot\nabla\omega = u\cdot\nabla(\phi^b\omega) - u\cdot\nabla\phi^b\omega = u\cdot\nabla\omega^b - (u\cdot\nabla\phi^b)\omega$. Hence we obtain

$$\begin{aligned}\sum_\alpha |\alpha|^k |I_{4,\alpha}| &\le C\left(\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{k,1}_\rho} + \|u\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\rho_0)}\right)\\ &\le C\|u\cdot\nabla\omega^b\|_{\mathcal{W}^{k,1}_\rho} + C\|\omega\|^2_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}\\ &\quad + C\|\omega\|_{\mathcal{W}^{k,1}_\rho}\|\omega\|_{H^4(\lambda d(x,\partial\Omega)\ge\delta_0/2)}.\end{aligned}$$

This completes the bound for $I_{4,\alpha}$.

Combining all of the above, we obtain bounds on $A(\beta)$ in the analytic norm.

4.3. **Sobolev bounds away from the boundary.** Finally, we bound the vorticity away from the boundary. Recall that

$$\begin{cases}\partial_t\omega^i + u\cdot\nabla\omega^i = \nu\Delta\omega^i\\ \omega^i_{|\partial\Omega} = 0\end{cases}\tag{4.15}$$

Note that by definition, ω^i vanishes in the region when $\lambda d(x,\partial\Omega)\le\delta_0$. We perform the standard energy estimates, for $k\ge 3$ so that the standard Sobolev embedding applies, yielding

$$\begin{aligned}\frac{d}{dt}\|\omega^i\|^2_{H^k} + \nu\|\nabla\omega^i\|^2_{H^k} &\lesssim \|u\|_{H^k}\|\omega^i\|^2_{H^k}\\ &\lesssim \|\omega^i\|^3_{H^k} + \|u^b\|^3_{H^k(\lambda d(x,\partial\Omega)\ge\delta_0)}.\end{aligned}$$

Using the elliptic theory for the Biot-Savart law $u^b = \nabla^\perp\Delta^{-1}\omega^b$, we have

$$\|u^b\|_{H^k(\lambda d(x,\partial\Omega)\ge\delta_0)} \lesssim \|\omega^b\|_{\mathcal{W}^{k,1}_\rho} + \|\omega^b\|_{H^k(\lambda d(x,\partial\Omega)\ge\delta_0)}.$$

This proves that

$$\frac{d}{dt}\|\omega^i\|^2_{H^k} \lesssim \|\omega^b\|^3_{\mathcal{W}^{k,1}_\rho} + \|\omega^b\|^3_{H^k(\lambda d(x,\partial\Omega)\ge\delta_0)}.$$

Integrating in time and recalling the iterative norm $A(\beta)$, we arrive at

$$\|\omega^i\|^2_{H^4} \lesssim \|\omega_0\|^2_{H^4} + TA(\beta)^2.$$

This bounds the Sobolev norm in $A(\beta)$, completing the proof of Proposition 4.1.

4.4. **Proof of Theorem 1.1.** Finally, we show that our main theorem, Theorem 1.1, follows from Proposition 4.1. Indeed, taking β sufficiently large in Proposition 4.1, we obtain uniform bounds on the iterative norm (4.9) in term of initial data, which gives the local solution in $\mathcal{W}^{1,1}_\rho + H^4(\{\lambda d(x,\partial\Omega) \ge \delta_0/2\})$ for $t \in [0,T]$, with $T = \beta^{-1}\lambda^{-2}\rho_0$. In particular, by definition of the iterative norm $A(\beta)$, we have

$$\|\omega(t)\|_{\mathcal{W}^{1,1}_\rho} + \|\omega(t)\|_{H^4(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})} \le C_0$$

for $t \in [0,T]$. To prove the stated bound (1.9) on vorticity, we note that

$$\|\omega\|_{L^\infty(\partial\Omega)} \lesssim \|\partial_{\widetilde{z}}\omega\|_{\mathcal{L}^1_\rho} + \|\omega(t)\|_{H^2(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})}.$$

It thus suffices to prove that $\|\partial_{\widetilde{z}}\omega\|_{\mathcal{L}^1_\rho} \lesssim \nu^{-1/2}$. Indeed, similar to (4.10), we bound

$$\|\partial_{\widetilde{z}}\omega(\widetilde{t})\|_{\mathcal{L}^1_\rho} \le \|\partial_{\widetilde{z}} e^{\nu\widetilde{t}S}\omega_0\|_{\mathcal{L}^1_\rho} + \int_0^{\widetilde{t}} \|\partial_{\widetilde{z}} e^{\nu(\widetilde{t}-\widetilde{t}')S} f(\widetilde{t}')\|_{\mathcal{L}^1_\rho}\, d\widetilde{t}' + \int_0^{\widetilde{t}} \|\partial_{\widetilde{z}}\Gamma(\nu(\widetilde{t}-\widetilde{t}'))g(\widetilde{t}')\|_{\mathcal{L}^1_\rho}\, d\widetilde{t}'$$

for the same f, g defined as in (4.8). It follows directly from the construction, see Section 3.6, that the $\widetilde{z}$-derivative of the semigroup $\partial_{\widetilde{z}} e^{\nu\widetilde{t}S}$ satisfies the same bounds as does $e^{\nu\widetilde{t}S}$, up to an extra factor of $(\nu\widetilde{t})^{-1/2}$ or $|\partial_{\widetilde{\theta}}| + \nu^{-1/2}$. Therefore, using the previous bounds on $f(\widetilde{t})$, we have

$$\begin{aligned}\int_0^{\widetilde{t}} \|\partial_{\widetilde{z}} e^{\nu(\widetilde{t}-\widetilde{t}')S} f(\widetilde{t}')\|_{\mathcal{L}^1_\rho}\, d\widetilde{t}' &\lesssim \int_0^{\widetilde{t}} (\nu(\widetilde{t}-\widetilde{t}'))^{-1/2}\Big[\|f(\widetilde{t}')\|_{\mathcal{W}^{1,1}_\rho} + \|\widetilde{z} f(\widetilde{t}')\|_{H^1(\widetilde{z}\ge\delta_0+\rho)}\Big]\, d\widetilde{t}' \\ &\lesssim \int_0^{\widetilde{t}} (\nu(\widetilde{t}-\widetilde{t}'))^{-1/2}\, d\widetilde{t}' \\ &\lesssim \nu^{-1/2}.\end{aligned}$$

Other terms are estimated similarly, giving $\|\partial_{\widetilde{z}}\omega\|_{\mathcal{L}^1_\rho} \lesssim \nu^{-1/2}$ as claimed.

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Received for publication November 2021; early access January 2022.

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