SOBOLEV INEQUALITIES IN MANIFOLDS WITH NONNEGATIVE CURVATURE

SIMON BRENDLE

Abstract. We prove a sharp Sobolev inequality on manifolds with nonnegative Ricci curvature. Moreover, we prove a Michael-Simon inequality for submanifolds in manifolds with nonnegative sectional curvature. Both inequalities depend on the asymptotic volume ratio of the ambient manifold.

1. Introduction

Let M be a complete noncompact manifold of dimension k with nonnegative Ricci curvature. The asymptotic volume ratio of M is defined as

$$\theta := \lim_{r \to \infty} \frac{|\{p \in M : d(p,q) < r\}|}{|B^k|\, r^k},$$

where q is some fixed point in M and B^k denotes the unit ball in $\mathbb{R}^k$. The Bishop-Gromov relative volume comparison theorem implies that the limit exists, and that $\theta \leq 1$. Note that θ does not depend on the choice of the point q.

Our first result gives a sharp Sobolev inequality on manifolds with nonnegative Ricci curvature.

Theorem 1.1. *Let M be a complete noncompact manifold of dimension n with nonnegative Ricci curvature. Let D be a compact domain in M with boundary ∂D, and let f be a positive smooth function on D. Then*

$$\int_D |\nabla f| + \int_{\partial D} f \geq n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}} \Big(\int_D f^{\frac{n}{n-1}} \Big)^{\frac{n-1}{n}},$$

where θ denotes the asymptotic volume ratio of M.

Moreover, we are able to characterize the case of equality in Theorem 1.1:

Theorem 1.2. *Let M be a complete noncompact manifold of dimension n with nonnegative Ricci curvature. Let D be a compact domain in M with boundary ∂D, and let f be a positive smooth function on D. Suppose that*

$$\int_D |\nabla f| + \int_{\partial D} f = n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}} \Big(\int_D f^{\frac{n}{n-1}} \Big)^{\frac{n-1}{n}} > 0,$$

This project was supported by the National Science Foundation under grant DMS-1806190 and by the Simons Foundation.

where θ denotes the asymptotic volume ratio of M. Then f is constant, M is isometric to Euclidean space, and D is a round ball.

Putting $f = 1$ in Theorem 1.1, we obtain a sharp isoperimetric inequality.

Corollary 1.3. *Let M be a complete noncompact manifold of dimension n with nonnegative Ricci curvature. Let D be a compact domain in M with boundary ∂D. Then*

$$|\partial D| \geq n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}}\, |D|^{\frac{n-1}{n}},$$

where θ denotes the asymptotic volume ratio of M.

Corollary 1.3 is similar in spirit to the Lévy-Gromov inequality for manifolds with Ricci curvature at least $n-1$ (cf. [10], Appendix C). The Lévy-Gromov inequality was recently generalized in [13] and [8].

In the three-dimensional case, Corollary 1.3 was proved in a recent work of V. Agostiniani, M. Fogagnolo, and L. Mazzieri (cf. [1], Theorem 6.1). The proof of Theorem 6.1 in [1] builds on an argument due to G. Huisken [12] and uses mean curvature flow.

We will present the proof of Theorem 1.1 in Section 2. The proof of Theorem 1.1 uses the Alexandrov-Bakelman-Pucci method and is inspired in part by an elegant argument due to X. Cabré [5] (see also [4],[6],[15],[16],[17] for related work). The proof of Theorem 1.2 will be discussed in Section 3.

We next turn to Sobolev inequalities for submanifolds. In a recent paper [3], we proved a Michael-Simon-type inequality for submanifolds in Euclidean space. While the classical Michael-Simon inequality (cf. [2], [14]) is not sharp, our inequality is sharp if the codimension is at most 2. In particular, the results in [3] imply a sharp isoperimetric inequality for minimal submanifolds in Euclidean space of codimension at most 2, answering a question first studied by Torsten Carleman [7] in 1921.

The following theorem generalizes the main result in [3] to the Riemannian setting.

Theorem 1.4. *Let M be a complete noncompact manifold of dimension $n+m$ with nonnegative sectional curvature. Let Σ be a compact submanifold of M of dimension n (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. If $m \geq 2$, then*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\, |H|^2} + \int_{\partial\Sigma} f \geq n\, \Big(\frac{(n+m)\,|B^{n+m}|}{m\,|B^m|}\Big)^{\frac{1}{n}}\, \theta^{\frac{1}{n}}\, \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}},$$

where θ denotes the asymptotic volume ratio of M and H denotes the mean curvature vector of Σ.

Note that $(n+2)\,|B^{n+2}| = 2\,|B^2|\,|B^n|$. Hence, we obtain a sharp Sobolev inequality for submanifolds of codimension 2:

Corollary 1.5. *Let M be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let Σ be a compact submanifold* *of M of dimension n (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. Then*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f \geq n\,|B^n|^{\frac{1}{n}}\,\theta^{\frac{1}{n}}\,\Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}},$$

where θ denotes the asymptotic volume ratio of M and H denotes the mean curvature vector of Σ.

Moreover, we can characterize the case of equality in Corollary 1.5:

Theorem 1.6. *Let M be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let Σ be a compact submanifold of M of dimension n (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. Suppose that*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n\,|B^n|^{\frac{1}{n}}\,\theta^{\frac{1}{n}}\,\Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} > 0,$$

where θ denotes the asymptotic volume ratio of M and H denotes the mean curvature vector of Σ. Then f is constant, M is isometric to Euclidean space, and Σ is a flat round ball.

By putting $f = 1$ in Corollary 1.5, we obtain an isoperimetric inequality for minimal submanifolds of codimension 2, generalizing the result in [3].

Corollary 1.7. *Let M be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let Σ be a compact minimal submanifold of M of dimension n with boundary $\partial\Sigma$. Then*

$$|\partial\Sigma| \geq n\,|B^n|^{\frac{1}{n}}\,\theta^{\frac{1}{n}}\,|\Sigma|^{\frac{n-1}{n}},$$

where θ denotes the asymptotic volume ratio of the ambient manifold M.

Finally, the inequalities in Corollary 1.5 and Corollary 1.7 also hold in the codimension 1 setting. Indeed, if Σ is an n-dimensional submanifold of an $(n+1)$-dimensional manifold M, then we can view Σ as a submanifold of the $(n+2)$-dimensional manifold $M \times \mathbb{R}$. Note that the product $M \times \mathbb{R}$ has the same asymptotic volume ratio as M itself.

The proof of Theorem 1.4 will be presented in Section 4. This argument extends our earlier proof in the Euclidean case (cf. [3]), and relies on the Alexandrov-Bakelman-Pucci technique. Moreover, the proof shares some common features with the work of E. Heintze and H. Karcher [11] concerning the volume of a tubular neighborhood of a submanifold. Finally, the proof of Theorem 1.6 will be discussed in Section 5.

2. Proof of Theorem 1.1

Throughout this section, we assume that (M, g) is a complete noncompact manifold of dimension n with nonnegative Ricci curvature. Moreover, we assume that D is a compact domain in M, and f is a positive smooth function on D. Let R denote the Riemann curvature tensor of (M, g).

It suffices to prove the assertion in the special case when D is connected. By scaling, we may assume that

$$\int_D |\nabla f| + \int_{\partial D} f = n \int_D f^{\frac{n}{n-1}}.$$

Since D is connected, we can find a function $u : D \to \mathbb{R}$ with the property that

$$\operatorname{div}(f\,\nabla u) = n\, f^{\frac{n}{n-1}} - |\nabla f|$$

on D and $\langle \nabla u, \eta \rangle = 1$ at each point on ∂D. Here, η denotes the outward-pointing unit normal to ∂D. Standard elliptic regularity theory implies that the function u is of class $C^{2,\gamma}$ for each $0 < \gamma < 1$ (cf. [9], Theorem 6.30).

We define

$$U := \{x \in D \setminus \partial D : |\nabla u(x)| < 1\}.$$

For each $r > 0$, we denote by A_r the set of all points $\bar{x} \in U$ with the property that

$$r\, u(x) + \frac{1}{2}\, d\big(x, \exp_{\bar{x}}(r\, \nabla u(\bar{x}))\big)^2 \geq r\, u(\bar{x}) + \frac{1}{2}\, r^2\, |\nabla u(\bar{x})|^2$$

for all $x \in D$. Moreover, for each $r > 0$, we define a map $\Phi_r : D \to M$ by

$$\Phi_r(x) = \exp_x(r\, \nabla u(x))$$

for all $x \in D$. Note that the map Φ_r is of class $C^{1,\gamma}$ for each $0 < \gamma < 1$.

Lemma 2.1. *Assume that $x \in U$. Then $\Delta u(x) \leq n\, f(x)^{\frac{1}{n-1}}$.*

Proof. Using the inequality $|\nabla u(x)| < 1$ and the Cauchy-Schwarz inequality, we obtain

$$-\langle \nabla f(x), \nabla u(x) \rangle \leq |\nabla f(x)|.$$

Moreover, $\operatorname{div}(f\,\nabla u) = n\, f^{\frac{n}{n-1}} - |\nabla f|$ by definition of u. This implies

$$f(x)\, \Delta u(x) = n\, f(x)^{\frac{n}{n-1}} - |\nabla f(x)| - \langle \nabla f(x), \nabla u(x) \rangle \leq n\, f(x)^{\frac{n}{n-1}}.$$

From this, the assertion follows.

Lemma 2.2. *The set*

$$\{p \in M : d(x,p) < r \text{ for all } x \in D\}$$

is contained in the set

$$\{\Phi_r(x) : x \in A_r\}.$$

Proof. Fix a point $p \in M$ with the property that $d(x,p) < r$ for all $x \in D$. Since $\langle \nabla u, \eta \rangle = 1$ at each point on ∂D, the function $x \mapsto r\, u(x) + \frac{1}{2}\, d(x,p)^2$ cannot attain its minimum on the boundary of D. Let us fix a point $\bar{x} \in D \setminus \partial D$ where the function $x \mapsto r\, u(x) + \frac{1}{2}\, d(x,p)^2$ attains its minimum. Moreover, let $\bar{\gamma} : [0,r] \to M$ be a minimizing geodesic such that $\bar{\gamma}(0) = \bar{x}$ and

$\bar{\gamma}(r) = p$. Clearly, $r\,|\bar{\gamma}'(0)| = d(\bar{x}, p)$. For every smooth path $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in D$ and $\gamma(r) = p$, we obtain

$$\begin{aligned} r\,u(\gamma(0)) + \frac{1}{2}\,r \int_0^r |\gamma'(t)|^2\,dt &\geq r\,u(\gamma(0)) + \frac{1}{2}\,d(\gamma(0), p)^2 \\ &\geq r\,u(\bar{x}) + \frac{1}{2}\,d(\bar{x}, p)^2 \\ &= r\,u(\bar{\gamma}(0)) + \frac{1}{2}\,r^2\,|\bar{\gamma}'(0)|^2 \\ &= r\,u(\bar{\gamma}(0)) + \frac{1}{2}\,r \int_0^r |\bar{\gamma}'(t)|^2\,dt. \end{aligned}$$

In other words, the path $\bar{\gamma}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2}\int_0^r |\gamma'(t)|^2\,dt$ among all smooth paths $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in D$ and $\gamma(r) = p$. Hence, the formula for the first variation of energy implies

$$\nabla u(\bar{x}) = \bar{\gamma}'(0).$$

From this, we deduce that

$$\Phi_r(\bar{x}) = \exp_{\bar{x}}(r\,\nabla u(\bar{x})) = \exp_{\bar{\gamma}(0)}(r\,\bar{\gamma}'(0)) = \bar{\gamma}(r) = p.$$

Moreover,

$$r\,|\nabla u(\bar{x})| = r\,|\bar{\gamma}'(0)| = d(\bar{x}, p).$$

By assumption, $d(\bar{x}, p) < r$. This implies $|\nabla u(\bar{x})| < 1$. Therefore, $\bar{x} \in U$. Finally, for each point $x \in D$, we have

$$\begin{aligned} r\,u(x) + \frac{1}{2}\,d\big(x, \exp_{\bar{x}}(r\,\nabla u(\bar{x}))\big)^2 &= r\,u(x) + \frac{1}{2}\,d(x, p)^2 \\ &\geq r\,u(\bar{x}) + \frac{1}{2}\,d(\bar{x}, p)^2 \\ &= r\,u(\bar{x}) + \frac{1}{2}\,r^2\,|\nabla u(\bar{x})|^2. \end{aligned}$$

Thus, $\bar{x} \in A_r$. This completes the proof of Lemma 2.2.

Lemma 2.3. *Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(t\,\nabla u(\bar{x}))$ for all $t \in [0, r]$. If Z is a smooth vector field along $\bar{\gamma}$ satisfying $Z(r) = 0$, then*

$$(D^2 u)(Z(0), Z(0)) + \int_0^r \big(|D_t Z(t)|^2 - R(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\big)\,dt \geq 0.$$

Proof. Let us consider an arbitrary smooth path $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in D$ and $\gamma(r) = \bar{\gamma}(r)$. Since $\bar{x} \in A_r$, we obtain

$$\begin{aligned} r\, u(\gamma(0)) + \frac{1}{2}\, r \int_0^r |\gamma'(t)|^2\, dt &\geq r\, u(\gamma(0)) + \frac{1}{2}\, d(\gamma(0), \gamma(r))^2 \\ &= r\, u(\gamma(0)) + \frac{1}{2}\, d\big(\gamma(0), \exp_{\bar{x}}(r\, \nabla u(\bar{x}))\big)^2 \\ &\geq r\, u(\bar{x}) + \frac{1}{2}\, r^2\, |\nabla u(\bar{x})|^2 \\ &= r\, u(\bar{\gamma}(0)) + \frac{1}{2}\, r \int_0^r |\bar{\gamma}'(t)|^2\, dt. \end{aligned}$$

In other words, the path $\bar{\gamma}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2} \int_0^r |\gamma'(t)|^2\, dt$ among all smooth paths $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in D$ and $\gamma(r) = \bar{\gamma}(r)$. Hence, the assertion follows from the formula for the second variation of energy.

Lemma 2.4. *Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(t\, \nabla u(\bar{x}))$ for all $t \in [0, r]$. Moreover, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\bar{x}}M$. Suppose that W is a Jacobi field along $\bar{\gamma}$ satisfying $\langle D_t W(0), e_j \rangle = (D^2 u)(W(0), e_j)$ for each $1 \leq j \leq n$. If $W(\tau) = 0$ for some $\tau \in (0, r)$, then W vanishes identically.*

Proof. Suppose that $W(\tau) = 0$ for some $\tau \in (0, r)$. By assumption,

$$\langle D_t W(0), W(0) \rangle = (D^2 u)(W(0), W(0)).$$

Since W is a Jacobi field, we obtain

$$\begin{aligned} &\int_0^\tau \big(|D_t W(t)|^2 - R(\bar{\gamma}'(t), W(t), \bar{\gamma}'(t), W(t))\big)\, dt \\ &= \langle D_t W(\tau), W(\tau) \rangle - \langle D_t W(0), W(0) \rangle \\ &= -(D^2 u)(W(0), W(0)). \end{aligned}$$

Let us define a vector field $\tilde{W}$ along $\bar{\gamma}$ by

$$\tilde{W}(t) = \begin{cases} W(t) & \text{for } t \in [0, \tau] \\ 0 & \text{for } t \in [\tau, r]. \end{cases}$$

Clearly, $\tilde{W}(r) = 0$. Moreover,

$$\begin{aligned} &\int_0^r \big(|D_t \tilde{W}(t)|^2 - R(\bar{\gamma}'(t), \tilde{W}(t), \bar{\gamma}'(t), \tilde{W}(t))\big)\, dt \\ &= -(D^2 u)(\tilde{W}(0), \tilde{W}(0)). \end{aligned}$$

Using Lemma 2.3, we conclude that

$$\begin{aligned}&\int_0^r \big(|D_t Z(t)|^2 - R(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\big)\, dt \\ &\geq \int_0^r \big(|D_t \tilde{W}(t)|^2 - R(\bar{\gamma}'(t), \tilde{W}(t), \bar{\gamma}'(t), \tilde{W}(t))\big)\, dt\end{aligned}$$

for every smooth vector field Z along $\bar{\gamma}$ satisfying $Z(0) = \tilde{W}(0)$ and $Z(r) = \tilde{W}(r)$. By approximation, this inequality holds for every vector field Z which is piecewise C^1 and satisfies $Z(0) = \tilde{W}(0)$ and $Z(r) = \tilde{W}(r)$. In other words, the vector field $\tilde{W}$ minimizes the index form among all vector fields which are piecewise C^1 and have the same boundary values as $\tilde{W}$. Consequently, $\tilde{W}$ must be of class C^1. This implies $D_t W(\tau) = 0$. Since $W(\tau) = 0$ and $D_t W(\tau) = 0$, standard uniqueness results for ODE imply that W vanishes identically.

Proposition 2.5. *Assume that $x \in A_r$. Then the function*

$$t \mapsto (1 + t\, f(x)^{\frac{1}{n-1}})^{-n}\, |\det D\Phi_t(x)|$$

is monotone decreasing for $t \in (0, r)$.

Proof. Fix an arbitrary point $\bar{x} \in A_r$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\bar{x}}M$, and let $(x_1, \ldots, x_n)$ be a system of geodesic normal coordinates around $\bar{x}$ such that $\frac{\partial}{\partial x_i} = e_i$ at $\bar{x}$. Let $\bar{\gamma}(t) := \exp_{\bar{x}}(t\, \nabla u(\bar{x}))$ for all $t \in [0, r]$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of e_i along $\bar{\gamma}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\bar{\gamma}$ satisfying $X_i(0) = e_i$ and

$$\langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j)$$

for all $1 \leq j \leq n$. It follows from Lemma 2.4 that $X_1(t), \ldots, X_n(t)$ are linearly independent for each $t \in (0, r)$.

Let us define an $n \times n$-matrix $P(t)$ by

$$P_{ij}(t) = \langle X_i(t), E_j(t) \rangle$$

for $1 \leq i, j \leq n$. Moreover, we define an $n \times n$-matrix $S(t)$ by

$$S_{ij}(t) = R(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_j(t))$$

for $1 \leq i, j \leq n$. Clearly, $S(t)$ is symmetric. Moreover, since M has nonnegative Ricci curvature, we know that $\mathrm{tr}(S(t)) \geq 0$. Since the vector fields $X_1(t), \ldots, X_n(t)$ are Jacobi fields, we obtain

$$P''(t) = -P(t)S(t).$$

Moreover,

$$P_{ij}(0) = \delta_{ij}$$

and

$$P'_{ij}(0) = (D^2 u)(e_i, e_j).$$

In particular, the matrix $P'(0)P(0)^T$ is symmetric. Moreover, the matrix

$$\frac{d}{dt}(P'(t)P(t)^T) = P''(t)P(t)^T + P'(t)P'(t)^T = -P(t)S(t)P(t)^T + P'(t)P'(t)^T$$

is symmetric for each t. Thus, we conclude that the matrix $P'(t)P(t)^T$ is symmetric for each t.

Since $X_1(t), \ldots, X_n(t)$ are linearly independent for each $t \in (0, r)$, the matrix $P(t)$ is invertible for each $t \in (0, r)$. Since $P'(t)P(t)^T$ is symmetric for each $t \in (0, r)$, it follows that the matrix $Q(t) := P(t)^{-1}P'(t)$ is symmetric for each $t \in (0, r)$. The matrix $Q(t)$ satisfies the Riccati equation

$$Q'(t) = P(t)^{-1}P''(t) - P(t)^{-1}P'(t)P(t)^{-1}P'(t) = -S(t) - Q(t)^2$$

for all $t \in (0, r)$. Moreover, since $Q(t)$ is symmetric, we obtain $\operatorname{tr}(Q(t)^2) \geq \frac{1}{n}\operatorname{tr}(Q(t))^2$ for all $t \in (0, r)$. Since $\operatorname{tr}(S(t)) \geq 0$, it follows that

$$\frac{d}{dt}\operatorname{tr}(Q(t)) \leq -\operatorname{tr}(Q(t)^2) \leq -\frac{1}{n}\operatorname{tr}(Q(t))^2$$

for all $t \in (0, r)$. Clearly,

$$\lim_{t \to 0} Q_{ij}(t) = (D^2 u)(e_i, e_j).$$

Using Lemma 2.1, we obtain

$$\lim_{t \to 0} \operatorname{tr}(Q(t)) = \Delta u(\bar{x}) \leq n\, f(\bar{x})^{\frac{1}{n-1}}.$$

Hence, a standard ODE comparison principle implies

$$\operatorname{tr}(Q(t)) \leq \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t \in (0, r)$.

We next consider the determinant of $P(t)$. Clearly, $\det P(t) > 0$ if t is sufficiently small. Since $P(t)$ is invertible for each $t \in (0, r)$, it follows that $\det P(t) > 0$ for all $t \in (0, r)$. Using the estimate for the trace of $Q(t) = P(t)^{-1}P'(t)$, we obtain

$$\frac{d}{dt}\log\det P(t) = \operatorname{tr}(Q(t)) \leq \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t \in (0, r)$. Consequently, the function

$$t \mapsto (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^{-n} \det P(t)$$

is monotone decreasing for $t \in (0, r)$.

Finally, we observe that

$$\frac{\partial \Phi_t}{\partial x_i}(\bar{x}) = X_i(t)$$

for $1 \leq i \leq n$. Consequently, $|\det D\Phi_t(\bar{x})| = \det P(t)$ for all $t \in (0, r)$. Putting these facts together, the assertion follows.

Corollary 2.6. *The Jacobian determinant of* Φ_r *satisfies*

$$|\det D\Phi_r(x)| \leq (1 + r\, f(x)^{\frac{1}{n-1}})^n$$

for all $x \in A_r$.

Proof. Since $\lim_{t \to 0} |\det D\Phi_t(x)| = 1$, the assertion follows from Proposition 2.5.

After these preparations, we now complete the proof of Theorem 1.1. Using Lemma 2.2 and Corollary 2.6, we obtain

$$\begin{aligned} &|\{p \in M : d(x,p) < r \text{ for all } x \in D\}| \\ &\leq \int_{A_r} |\det D\Phi_r(x)|\, d\text{vol}(x) \\ &\leq \int_U (1 + r\, f(x)^{\frac{1}{n-1}})^n\, d\text{vol}(x) \end{aligned}$$

for all $r > 0$. Finally, we divide by r^n and send $r \to \infty$. This gives

$$|B^n|\, \theta \leq \int_U f^{\frac{n}{n-1}} \leq \int_D f^{\frac{n}{n-1}}.$$

Thus,

$$\int_D |\nabla f| + \int_{\partial D} f = n \int_D f^{\frac{n}{n-1}} \geq n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}} \Big(\int_D f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Let (M, g) be a complete noncompact manifold of dimension n with nonnegative Ricci curvature. Let D be a compact domain in M with boundary ∂D, and let f be a positive smooth function on D satisfying

$$\int_D |\nabla f| + \int_{\partial D} f = n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}} \Big(\int_D f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} > 0,$$

where θ denotes the asymptotic volume ratio of M.

If D is disconnected, we may apply Theorem 1.1 to each connected component of D, and take the sum over all connected components. This will lead to a contradiction. Therefore, D must be connected.

By scaling, we may assume that

$$\int_D |\nabla f| + \int_{\partial D} f = n\, |B^n|\, \theta$$

and

$$\int_D f^{\frac{n}{n-1}} = |B^n|\, \theta.$$

In particular,

$$\int_D |\nabla f| + \int_{\partial D} f = n \int_D f^{\frac{n}{n-1}}.$$

Since D is connected, we can find a function $u : D \to \mathbb{R}$ such that

$$\operatorname{div}(f\,\nabla u) = n\,f^{\frac{n}{n-1}} - |\nabla f|$$

on D and $\langle \nabla u, \eta \rangle = 1$ at each point on ∂D. Moreover, u is of class $C^{2,\gamma}$ for each $0 < \gamma < 1$. Let us define U, A_r, and Φ_r as in Section 2.

Lemma 3.1. *Assume that $x \in U$. Then $|\det D\Phi_t(x)| \geq (1 + t\,f(x)^{\frac{1}{n-1}})^n$ for each $t > 0$.*

Proof. Let us fix a point $\bar{x} \in U$. Suppose that $|\det D\Phi_t(\bar{x})| < (1 + t\,f(\bar{x})^{\frac{1}{n-1}})^n$ for some $t > 0$. Let us fix a real number $\varepsilon \in (0,1)$ such that

$$|\det D\Phi_t(\bar{x})| < (1-\varepsilon)\,(1 + t\,f(\bar{x})^{\frac{1}{n-1}})^n.$$

By continuity, we can find an open neighborhood V of the point $\bar{x}$ such that

$$|\det D\Phi_t(x)| \leq (1-\varepsilon)\,(1 + t\,f(x)^{\frac{1}{n-1}})^n$$

for all $x \in V$. Using Proposition 2.5, we conclude that

$$|\det D\Phi_r(x)| \leq (1-\varepsilon)\,(1 + r\,f(x)^{\frac{1}{n-1}})^n$$

for all $r > t$ and all $x \in A_r \cap V$. Using this fact together with Lemma 2.2 and Corollary 2.6, we obtain

$$\begin{aligned} &|\{p \in M : d(x,p) < r \text{ for all } x \in D\}| \\ &\leq \int_{A_r} |\det D\Phi_r(x)|\,d\mathrm{vol}(x) \\ &\leq \int_U (1 - \varepsilon \cdot 1_V(x))\,(1 + r\,f(x)^{\frac{1}{n-1}})^n\,d\mathrm{vol}(x) \end{aligned}$$

for all $r > t$. Finally, we divide by r^n and send $r \to \infty$. This gives

$$|B^n|\,\theta \leq \int_U (1 - \varepsilon \cdot 1_V)\,f^{\frac{n}{n-1}} < \int_D f^{\frac{n}{n-1}} = |B^n|\,\theta.$$

This is a contradiction.

Lemma 3.2. *Assume that $x \in U$. Then $D^2 u(x) = f(x)^{\frac{1}{n-1}}\,g$.*

Proof. Let us fix a point $\bar{x} \in U$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\bar{x}}M$. We define $\bar{\gamma}(t) := \exp_{\bar{x}}(t\,\nabla u(\bar{x}))$ for all $t \geq 0$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of e_i along $\bar{\gamma}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\bar{\gamma}$ satisfying $X_i(0) = e_i$ and

$$\langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j)$$

for all $1 \leq j \leq n$. Finally, we define an $n \times n$-matrix $P(t)$ by

$$P_{ij}(t) = \langle X_i(t), E_j(t) \rangle$$

for $1 \leq i, j \leq n$.

By Lemma 3.1, we know that $|\det P(t)| \geq (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^n$ for all $t > 0$. Since $\det P(t) > 0$ if $t > 0$ is sufficiently small, we conclude that

$$\det P(t) \geq (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^n$$

for all $t > 0$. In particular, $P(t)$ is invertible for each $t > 0$.

We next define $Q(t) := P(t)^{-1} P'(t)$ for all $t > 0$. As in Section 2, we can show that the matrix $Q(t)$ is symmetric for each $t > 0$. The Riccati equation for $Q(t)$ gives

$$\frac{d}{dt}\mathrm{tr}(Q(t)) \leq -\mathrm{tr}(Q(t)^2) \leq -\frac{1}{n}\,\mathrm{tr}(Q(t))^2$$

for all $t > 0$. Moreover,

$$\lim_{t \to 0} \mathrm{tr}(Q(t)) = \Delta u(\bar{x}) \leq n\, f(\bar{x})^{\frac{1}{n-1}}$$

by Lemma 2.1. This implies

$$\mathrm{tr}(Q(t)) \leq \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}},$$

hence

$$\frac{d}{dt} \log \det P(t) \leq \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t > 0$. Integrating this ODE gives

$$\det P(t) \leq (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^n$$

for all $t > 0$.

Putting these facts together, we conclude that $\det P(t) = (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^n$ for all $t > 0$. Differentiating this identity with respect to t, we obtain

$$\mathrm{tr}(Q(t)) = \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t > 0$. Using the Riccati equation for $Q(t)$, we conclude that $\mathrm{tr}(Q(t)^2) = \frac{1}{n}\,\mathrm{tr}(Q(t))^2$ for all $t > 0$. Consequently, the trace-free part of $Q(t)$ vanishes for each $t > 0$. Therefore,

$$Q_{ij}(t) = \frac{f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}\, \delta_{ij}$$

for all $t > 0$. In particular,

$$(D^2 u)(e_i, e_j) = \lim_{t \to 0} Q_{ij}(t) = f(\bar{x})^{\frac{1}{n-1}}\, \delta_{ij}.$$

This completes the proof of Lemma 3.2.

Lemma 3.3. *Assume that $x \in U$. Then $\nabla f(x) = 0$.*

Proof. Let us consider an arbitrary point $x \in U$. Using the definition of u, we obtain

$$f(x)\,\Delta u(x) = n\,f(x)^{\frac{n}{n-1}} - |\nabla f(x)| - \langle \nabla f(x), \nabla u(x) \rangle.$$

On the other hand, Lemma 3.2 implies $\Delta u(x) = n\,f(x)^{\frac{1}{n-1}}$. Putting these facts together, we conclude that $\langle \nabla f(x), \nabla u(x) \rangle = -|\nabla f(x)|$. Since $|\nabla u(x)| < 1$, it follows that $\nabla f(x) = 0$. This completes the proof of Lemma 3.3.

Lemma 3.4. *The set U is dense in D.*

Proof. Suppose that U is not dense in D. Arguing as in Section 2, we obtain

$$|B^n|\,\theta \leq \int_U f^{\frac{n}{n-1}} < \int_D f^{\frac{n}{n-1}} = |B^n|\,\theta.$$

This is a contradiction. This completes the proof of Lemma 3.4.

Since U is a dense subset of D, we conclude that $\nabla f = 0$ and $D^2 u = f^{\frac{1}{n-1}}\,g$ at each point on D. Since D is connected, it follows that f is constant. This implies $|\partial D| = n\,|B^n|^{\frac{1}{n}}\,\theta^{\frac{1}{n}}\,|D|^{\frac{n-1}{n}}$.

Note that u is a smooth function on D. Each critical point of u lies in the interior of D and is nondegenerate with Morse index 0. In particular, the function u has at most finitely many critical points.

We next consider the flow on D generated by the vector field $-\nabla u$. Since the vector field $-\nabla u$ points inward along the boundary ∂D, the flow is defined for all nonnegative times. This gives a one-parameter family of smooth maps $\psi_s : D \to D$, where $s \geq 0$. Since D is connected, standard arguments from Morse theory imply that the function u has exactly one critical point, and u attains its global minimum at that point. It follows that the diameter of $\psi_s(D)$ converges to 0 as $s \to \infty$.

Since $D^2 u$ is a constant multiple of the metric, the isoperimetric ratio is unchanged under the flow ψ_s. This implies

$$|\psi_s(\partial D)| = n\,|B^n|^{\frac{1}{n}}\,\theta^{\frac{1}{n}}\,|\psi_s(D)|^{\frac{n-1}{n}}$$

for each $s \geq 0$. If $\theta < 1$, this contradicts the Euclidean isoperimetric inequality when s is sufficiently large. Thus, we conclude that $\theta = 1$. Consequently, M is isometric to Euclidean space.

Once we know that M is isometric to Euclidean space, it follows that D is a round ball. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.4

Throughout this section, we assume that $(M, \bar{g})$ is a complete noncompact manifold of dimension $n + m$ with nonnegative sectional curvature. Moreover, we assume that Σ is a compact submanifold of M of dimension n (possibly with boundary $\partial\Sigma$), and f is a positive smooth function on Σ. Let $\bar{D}$ denote the Levi-Civita connection on the ambient manifold $(M, \bar{g})$, and let $\bar{R}$ denote the Riemann curvature tensor of $(M, \bar{g})$. We denote by $I\!I$ the second fundamental form of Σ. For each point $x \in \Sigma$, $I\!I$ is a symmetric bilinear form on $T_x\Sigma$ which takes values in the normal space $T_x^\perp\Sigma$. If X and Y are tangent vector fields on Σ and V is a normal vector field along Σ, then $\langle I\!I(X,Y), V\rangle = \langle \bar{D}_X Y, V\rangle = -\langle \bar{D}_X V, Y\rangle$.

It suffices to prove the assertion in the special case when Σ is connected. By scaling, we may assume that

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n \int_\Sigma f^{\frac{n}{n-1}}.$$

Since Σ is connected, we can find a function $u : \Sigma \to \mathbb{R}$ with the property that

$$\operatorname{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$$

on Σ and $\langle \nabla^\Sigma u, \eta\rangle = 1$ at each point on $\partial\Sigma$. Here, η denotes the co-normal to $\partial\Sigma$. Standard elliptic regularity theory implies that the function u is of class $C^{2,\gamma}$ for each $0 < \gamma < 1$ (cf. [9], Theorem 6.30).

We define

$$\begin{aligned} &\Omega := \{x \in \Sigma \setminus \partial\Sigma : |\nabla^\Sigma u(x)| < 1\}, \\ &U := \{(x,y) : x \in \Sigma \setminus \partial\Sigma,\ y \in T_x^\perp\Sigma,\ |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}. \end{aligned}$$

For each $r > 0$, we denote by A_r the set of all points $(\bar{x}, \bar{y}) \in U$ with the property that

$$r\,u(x) + \frac{1}{2}\,d\big(x, \exp_{\bar{x}}(r\,\nabla^\Sigma u(\bar{x}) + r\,\bar{y})\big)^2 \geq r\,u(\bar{x}) + \frac{1}{2}\,r^2\,(|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2)$$

for all $x \in \Sigma$. Moreover, for each $r > 0$, we define a map $\Phi_r : T^\perp\Sigma \to M$ by

$$\Phi_r(x,y) = \exp_x(r\,\nabla^\Sigma u(x) + r\,y)$$

for all $x \in \Sigma$ and $y \in T_x^\perp\Sigma$. Note that the map Φ_r is of class $C^{1,\gamma}$ for each $0 < \gamma < 1$.

Lemma 4.1. *Assume that $x \in \Omega$ and $y \in T_x^\perp\Sigma$ satisfy $|\nabla^\Sigma u(x)|^2 + |y|^2 \leq 1$. Then $\Delta_\Sigma u(x) - \langle H(x), y\rangle \leq n\,f(x)^{\frac{1}{n-1}}$.*

Proof. Using the inequality $|\nabla^\Sigma u(x)|^2 + |y|^2 \leq 1$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &-\langle \nabla^\Sigma f(x), \nabla^\Sigma u(x)\rangle - f(x)\,\langle H(x), y\rangle \\ &\leq \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2\,|H(x)|^2}\,\sqrt{|\nabla^\Sigma u(x)|^2 + |y|^2} \\ &\leq \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2\,|H(x)|^2}. \end{aligned}$$

Moreover, $\mathrm{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$ by definition of u. Consequently,

$$\begin{aligned} f(x)\,\Delta_\Sigma u(x) - f(x)\,\langle H(x), y\rangle &= n\,f(x)^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2\,|H(x)|^2} \\ &\quad - \langle \nabla^\Sigma f(x), \nabla^\Sigma u(x)\rangle - f(x)\,\langle H(x), y\rangle \\ &\leq n\,f(x)^{\frac{n}{n-1}}. \end{aligned}$$

From this, the assertion follows.

Lemma 4.2. *For each $0 \leq \sigma < 1$, the set*

$$\{p \in M : \sigma r < d(x,p) < r \text{ for all } x \in \Sigma\}$$

is contained in the set

$$\{\Phi_r(x,y) : (x,y) \in A_r,\ |\nabla^\Sigma u(x)|^2 + |y|^2 > \sigma^2\}.$$

Proof. Let us fix a real number $0 \leq \sigma < 1$ and a point $p \in M$ with the property that $\sigma r < d(x,p) < r$ for all $x \in \Sigma$. Since $\langle \nabla^\Sigma u, \eta\rangle = 1$ at each point on $\partial\Sigma$, the function $x \mapsto r\,u(x) + \frac{1}{2}\,d(x,p)^2$ cannot attain its minimum on the boundary of Σ. Let us fix a point $\bar{x} \in \Sigma \setminus \partial\Sigma$ where the function $x \mapsto r\,u(x) + \frac{1}{2}\,d(x,p)^2$ attains its minimum. Moreover, let $\bar{\gamma} : [0,r] \to M$ be a minimizing geodesic such that $\bar{\gamma}(0) = \bar{x}$ and $\bar{\gamma}(r) = p$. Clearly, $r\,|\bar{\gamma}'(0)| = d(\bar{x},p)$. For every smooth path $\gamma : [0,r] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(r) = p$, we obtain

$$\begin{aligned} r\,u(\gamma(0)) + \frac{1}{2}\,r\int_0^r |\gamma'(t)|^2\,dt &\geq r\,u(\gamma(0)) + \frac{1}{2}\,d(\gamma(0),p)^2 \\ &\geq r\,u(\bar{x}) + \frac{1}{2}\,d(\bar{x},p)^2 \\ &= r\,u(\bar{\gamma}(0)) + \frac{1}{2}\,r^2\,|\bar{\gamma}'(0)|^2 \\ &= r\,u(\bar{\gamma}(0)) + \frac{1}{2}\,r\int_0^r |\bar{\gamma}'(t)|^2\,dt. \end{aligned}$$

In other words, the path $\bar{\gamma}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2}\int_0^r |\gamma'(t)|^2\,dt$ among all smooth paths $\gamma : [0,r] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(r) = p$. Hence, the formula for the first variation of energy implies

$$\nabla^\Sigma u(\bar{x}) - \bar{\gamma}'(0) \in T_{\bar{x}}^\perp\Sigma.$$

Consequently, we can find a vector $\bar{y} \in T_{\bar{x}}^\perp\Sigma$ such that

$$\nabla^\Sigma u(\bar{x}) + \bar{y} = \bar{\gamma}'(0).$$

From this, we deduce that

$$\Phi_r(\bar{x},\bar{y}) = \exp_{\bar{x}}(r\,\nabla^\Sigma u(\bar{x}) + r\,\bar{y}) = \exp_{\bar{\gamma}(0)}(r\,\bar{\gamma}'(0)) = \bar{\gamma}(r) = p.$$

Moreover,

$$r^2\,(|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2) = r^2\,|\nabla^\Sigma u(\bar{x}) + \bar{y}|^2 = r^2\,|\bar{\gamma}'(0)|^2 = d(\bar{x},p)^2.$$

By assumption, $\sigma r < d(\bar{x}, p) < r$. This implies $\sigma^2 < |\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 < 1$. In particular, $(\bar{x}, \bar{y}) \in U$. Finally, for each point $x \in \Sigma$, we have

$$\begin{aligned} r\, u(x) + \frac{1}{2}\, d\big(x, \exp_{\bar{x}}(r\, \nabla^\Sigma u(\bar{x}) + r\, \bar{y})\big)^2 &= r\, u(x) + \frac{1}{2}\, d(x, p)^2 \\ &\geq r\, u(\bar{x}) + \frac{1}{2}\, d(\bar{x}, p)^2 \\ &= r\, u(\bar{x}) + \frac{1}{2}\, r^2\, (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2). \end{aligned}$$

Thus, $(\bar{x}, \bar{y}) \in A_r$. This completes the proof of Lemma 4.2.

Lemma 4.3. *Assume that $(\bar{x}, \bar{y}) \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(t\, \nabla^\Sigma u(\bar{x}) + t\, \bar{y})$ for all $t \in [0, r]$. If Z is a smooth vector field along $\bar{\gamma}$ satisfying $Z(0) \in T_{\bar{x}}\Sigma$ and $Z(r) = 0$, then*

$$\begin{aligned} &(D^2_\Sigma u)(Z(0), Z(0)) - \langle I\!I(Z(0), Z(0)), \bar{y} \rangle \\ &+ \int_0^r \big(|\bar{D}_t Z(t)|^2 - \bar{R}(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\big)\, dt \geq 0. \end{aligned}$$

Proof. Let us consider an arbitrary smooth path $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(r) = \bar{\gamma}(r)$. Since $(\bar{x}, \bar{y}) \in A_r$, we obtain

$$\begin{aligned} r\, u(\gamma(0)) + \frac{1}{2}\, r \int_0^r |\gamma'(t)|^2\, dt &\geq r\, u(\gamma(0)) + \frac{1}{2}\, d(\gamma(0), \gamma(r))^2 \\ &= r\, u(\gamma(0)) + \frac{1}{2}\, d\big(\gamma(0), \exp_{\bar{x}}(r\, \nabla^\Sigma u(\bar{x}) + r\, \bar{y})\big)^2 \\ &\geq r\, u(\bar{x}) + \frac{1}{2}\, r^2\, (|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2) \\ &= r\, u(\bar{\gamma}(0)) + \frac{1}{2}\, r \int_0^r |\bar{\gamma}'(t)|^2\, dt. \end{aligned}$$

In other words, the path $\bar{\gamma}$ minimizes the functional $u(\gamma(0)) + \frac{1}{2} \int_0^r |\gamma'(t)|^2\, dt$ among all smooth paths $\gamma : [0, r] \to M$ satisfying $\gamma(0) \in \Sigma$ and $\gamma(r) = \bar{\gamma}(r)$. Using the formula for the second variation of energy, we obtain

$$\begin{aligned} &(D^2_\Sigma u)(Z(0), Z(0)) - \langle I\!I(Z(0), Z(0)), \bar{\gamma}'(0) \rangle \\ &+ \int_0^r \big(|\bar{D}_t Z(t)|^2 - \bar{R}(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\big)\, dt \geq 0. \end{aligned}$$

On the other hand, the identity $\bar{\gamma}'(0) = \nabla^\Sigma u(\bar{x}) + \bar{y}$ implies

$$\langle I\!I(Z(0), Z(0)), \bar{\gamma}'(0) \rangle = \langle I\!I(Z(0), Z(0)), \bar{y} \rangle.$$

Putting these facts together, the assertion follows.

Lemma 4.4. *Assume that $(\bar{x}, \bar{y}) \in A_r$. Then $g + r\, D^2_\Sigma u(\bar{x}) - r\, \langle I\!I(\bar{x}), \bar{y} \rangle \geq 0$.*

Proof. As above, we define $\bar{\gamma}(t) := \exp_{\bar{x}}(t\,\nabla u(\bar{x}) + t\,\bar{y})$ for all $t \in [0,r]$. Let us fix an arbitrary vector $w \in T_{\bar{x}}\Sigma$, and let $W(t)$ denote the parallel transport of w along $\bar{\gamma}$. Applying Lemma 4.3 to the vector field $Z(t) := (r-t)\,W(t)$ gives

$$\begin{aligned}&r\,g(w,w) + r^2\,(D_\Sigma^2 u)(w,w) - r^2\,\langle I\!I(w,w),\bar{y}\rangle \\ &- \int_0^r (r-t)^2\,\bar{R}(\bar{\gamma}'(t),W(t),\bar{\gamma}'(t),W(t))\,dt \geq 0.\end{aligned}$$

Since M has nonnegative sectional curvature, it follows that

$$r\,g(w,w) + r^2\,(D_\Sigma^2 u)(w,w) - r^2\,\langle I\!I(w,w),\bar{y}\rangle \geq 0,$$

as claimed.

Lemma 4.5. *Assume that $(\bar{x},\bar{y}) \in A_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}(t\,\nabla^\Sigma u(\bar{x}) + t\,\bar{y})$ for all $t \in [0,r]$. Moreover, let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of $T_{\bar{x}}\Sigma$. Suppose that W is a Jacobi field along $\bar{\gamma}$ satisfying $W(0) \in T_{\bar{x}}\Sigma$ and $\langle \bar{D}_t W(0), e_j\rangle = (D_\Sigma^2 u)(W(0),e_j) - \langle I\!I(W(0),e_j),\bar{y}\rangle$ for each $1 \leq j \leq n$. If $W(\tau) = 0$ for some $\tau \in (0,r)$, then W vanishes identically.*

Proof. Suppose that $W(\tau) = 0$ for some $\tau \in (0,r)$. By assumption,

$$\langle \bar{D}_t W(0), W(0)\rangle = (D_\Sigma^2 u)(W(0),W(0)) - \langle I\!I(W(0),W(0)),\bar{y}\rangle.$$

Since W is a Jacobi field, we obtain

$$\begin{aligned}&\int_0^\tau \big(|\bar{D}_t W(t)|^2 - \bar{R}(\bar{\gamma}'(t),W(t),\bar{\gamma}'(t),W(t))\big)\,dt \\ &= \langle \bar{D}_t W(\tau), W(\tau)\rangle - \langle \bar{D}_t W(0), W(0)\rangle \\ &= -(D_\Sigma^2 u)(W(0),W(0)) + \langle I\!I(W(0),W(0)),\bar{y}\rangle.\end{aligned}$$

Let us define a vector field $\tilde{W}$ along $\bar{\gamma}$ by

$$\tilde{W}(t) = \begin{cases} W(t) & \text{for } t \in [0,\tau] \\ 0 & \text{for } t \in [\tau,r]. \end{cases}$$

Clearly, $\tilde{W}(0) = W(0) \in T_{\bar{x}}\Sigma$ and $\tilde{W}(r) = 0$. Moreover,

$$\begin{aligned}&\int_0^r \big(|\bar{D}_t \tilde{W}(t)|^2 - \bar{R}(\bar{\gamma}'(t),\tilde{W}(t),\bar{\gamma}'(t),\tilde{W}(t))\big)\,dt \\ &= -(D_\Sigma^2 u)(\tilde{W}(0),\tilde{W}(0)) + \langle I\!I(\tilde{W}(0),\tilde{W}(0)),\bar{y}\rangle.\end{aligned}$$

Using Lemma 4.3, we conclude that

$$\begin{aligned}&\int_0^r \big(|\bar{D}_t Z(t)|^2 - \bar{R}(\bar{\gamma}'(t),Z(t),\bar{\gamma}'(t),Z(t))\big)\,dt \\ &\geq \int_0^r \big(|\bar{D}_t \tilde{W}(t)|^2 - \bar{R}(\bar{\gamma}'(t),\tilde{W}(t),\bar{\gamma}'(t),\tilde{W}(t))\big)\,dt\end{aligned}$$

for every smooth vector field Z along $\bar{\gamma}$ satisfying $Z(0) = \tilde{W}(0)$ and $Z(r) = \tilde{W}(r)$. By approximation, this inequality holds for every vector field Z which is piecewise C^1 and satisfies $Z(0) = \tilde{W}(0)$ and $Z(r) = \tilde{W}(r)$. In other words, the vector field $\tilde{W}$ minimizes the index form among all vector fields which are piecewise C^1 and have the same boundary values as $\tilde{W}$. Consequently, $\tilde{W}$ must be of class C^1. This implies $\bar{D}_t W(\tau) = 0$. Since $W(\tau) = 0$ and $\bar{D}_t W(\tau) = 0$, standard uniqueness results for ODE imply that W vanishes identically.

Proposition 4.6. *Assume that $(x, y) \in A_r$. Then the function*

$$t \mapsto t^{-m} \, (1 + t \, f(x)^{\frac{1}{n-1}})^{-n} \, |\det D\Phi_t(x, y)|$$

is monotone decreasing for $t \in (0, r)$.

Proof. Fix an arbitrary point $(\bar{x}, \bar{y}) \in A_r$. Let us choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_{\bar{x}} M$ such that the $n \times n$-matrix

$$(D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle$$

is diagonal. Let $(x_1, \ldots, x_n)$ be a system of geodesic normal coordinates on Σ around the point $\bar{x}$. We can arrange that $\frac{\partial}{\partial x_i} = e_i$ at $\bar{x}$. Let $\{\nu_{n+1}, \ldots, \nu_{n+m}\}$ be a local orthonormal frame for the normal bundle, chosen so that $\langle \bar{D}_{e_i} \nu_\alpha, \nu_\beta \rangle = 0$ at $\bar{x}$. We write a normal vector y as $y = \sum_{\alpha=n+1}^{n+m} y_\alpha \nu_\alpha$. With this understood, $(x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+m})$ is a local coordinate system on the total space of the normal bundle $T^\perp \Sigma$.

Let $\bar{\gamma}(t) := \exp_{\bar{x}}(t \, \nabla^\Sigma u(\bar{x}) + t \, \bar{y})$ for all $t \in [0, r]$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of e_i along $\bar{\gamma}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\bar{\gamma}$ satisfying $X_i(0) = e_i$ and

$$\begin{aligned} \langle \bar{D}_t X_i(0), e_j \rangle &= (D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle, \\ \langle \bar{D}_t X_i(0), \nu_\beta \rangle &= \langle I\!I(e_i, \nabla^\Sigma u), \nu_\beta \rangle \end{aligned}$$

for all $1 \leq j \leq n$ and all $n + 1 \leq \beta \leq n + m$. For each $n + 1 \leq \alpha \leq n + m$, we denote by $N_\alpha(t)$ the parallel transport of ν_α along $\bar{\gamma}$. Moreover, for each $n + 1 \leq \alpha \leq n + m$, we denote by $Y_\alpha(t)$ the unique Jacobi field along $\bar{\gamma}$ satisfying $Y_\alpha(0) = 0$ and $\bar{D}_t Y_\alpha(0) = \nu_\alpha$. It follows from Lemma 4.5 that $X_1(t), \ldots, X_n(t), Y_{n+1}, \ldots, Y_{n+m}(t)$ are linearly independent for each $t \in (0, r)$.

Let us define an $(n + m) \times (n + m)$-matrix $P(t)$ by

$$\begin{aligned} P_{ij}(t) &= \langle X_i(t), E_j(t) \rangle, \quad P_{i\beta}(t) = \langle X_i(t), N_\beta(t) \rangle, \\ P_{\alpha j}(t) &= \langle Y_\alpha(t), E_i(t) \rangle, \quad P_{\alpha\beta}(t) = \langle Y_\alpha(t), N_\beta(t) \rangle \end{aligned}$$

for $1 \leq i, j \leq n$ and $n + 1 \leq \alpha, \beta \leq n + m$. Moreover, we define an $(n + m) \times (n + m)$-matrix $S(t)$ by

$$\begin{aligned} S_{ij}(t) &= \bar{R}(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_j(t)), \quad S_{i\beta}(t) = \bar{R}(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), N_\beta(t)), \\ S_{\alpha j}(t) &= \bar{R}(\bar{\gamma}'(t), N_\alpha(t), \bar{\gamma}'(t), E_j(t)), \quad S_{\alpha\beta}(t) = \bar{R}(\bar{\gamma}'(t), N_\alpha(t), \bar{\gamma}'(t), N_\beta(t)) \end{aligned}$$

for $1 \leq i, j \leq n$ and $n + 1 \leq \alpha, \beta \leq n + m$. Clearly, $S(t)$ is symmetric. Moreover, $S(t) \geq 0$ since M has nonnegative sectional curvature. Since the vector fields $X_1(t), \ldots, X_n(t), Y_{n+1}(t), \ldots, Y_{n+m}(t)$ are Jacobi fields, we obtain

$$P''(t) = -P(t)S(t).$$

Moreover,

$$P(0) = \begin{bmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$P'(0) = \begin{bmatrix} (D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle & \langle I\!I(e_i, \nabla^\Sigma u), \nu_\beta \rangle \\ 0 & \delta_{\alpha\beta} \end{bmatrix}.$$

In particular, the matrix $P'(0)P(0)^T$ is symmetric. Moreover, the matrix

$$\frac{d}{dt}(P'(t)P(t)^T) = P''(t)P(t)^T + P'(t)P'(t)^T = -P(t)S(t)P(t)^T + P'(t)P'(t)^T$$

is symmetric for each t. Thus, we conclude that the matrix $P'(t)P(t)^T$ is symmetric for each t.

Since $X_1(t), \ldots, X_n(t), Y_{n+1}, \ldots, Y_{n+m}(t)$ are linearly independent for each $t \in (0, r)$, the matrix $P(t)$ is invertible for each $t \in (0, r)$. Since $P'(t)P(t)^T$ is symmetric for each $t \in (0, r)$, it follows that the matrix $Q(t) := P(t)^{-1}P'(t)$ is symmetric for each $t \in (0, r)$. The matrix $Q(t)$ satisfies the Riccati equation

$$Q'(t) = P(t)^{-1}P''(t) - P(t)^{-1}P'(t)P(t)^{-1}P'(t) = -S(t) - Q(t)^2$$

for all $t \in (0, r)$. Since $S(t) \geq 0$, it follows that

$$Q'(t) \leq -Q(t)^2$$

for all $t \in (0, r)$. Using the asymptotic expansion

$$P(t) = \begin{bmatrix} \delta_{ij} + O(t) & O(t) \\ O(t) & t\,\delta_{\alpha\beta} + O(t^2) \end{bmatrix},$$

we obtain

$$P(t)^{-1} = \begin{bmatrix} \delta_{ij} + O(t) & O(1) \\ O(1) & t^{-1}\,\delta_{\alpha\beta} + O(1) \end{bmatrix}$$

as $t \to 0$. Moreover,

$$P'(t) = \begin{bmatrix} (D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle + O(t) & O(1) \\ O(t) & \delta_{\alpha\beta} + O(t) \end{bmatrix}$$

as $t \to 0$. Consequently, the matrix $Q(t) = P(t)^{-1}P'(t)$ satisfies the asymptotic expansion

$$Q(t) = \begin{bmatrix} (D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle + O(t) & O(1) \\ O(1) & t^{-1}\,\delta_{\alpha\beta} + O(1) \end{bmatrix}$$

as $t \to 0$.

By our choice of $\{e_1, \dots, e_n\}$, the matrix $(D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle$ is diagonal. Let us write

$$(D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y} \rangle = \lambda_i \, \delta_{ij}$$

for $1 \leq i, j \leq n$. It follows from Lemma 4.4 that $1 + r\lambda_i \geq 0$ for each $1 \leq i \leq n$. Since

$$Q(\tau) = \begin{bmatrix} \lambda_i \, \delta_{ij} + O(\tau) & O(1) \\ O(1) & \tau^{-1} \, \delta_{\alpha\beta} + O(1) \end{bmatrix}$$

as $\tau \to 0$, we can find a small number $\tau_0 \in (0, r)$ such that

$$Q(\tau) < \begin{bmatrix} (\lambda_i + \sqrt{\tau}) \, \delta_{ij} & 0 \\ 0 & 2\tau^{-1} \, \delta_{\alpha\beta} \end{bmatrix}$$

for all $\tau \in (0, \tau_0)$. A standard ODE comparison principle implies

$$Q(t) \leq \begin{bmatrix} \frac{(\lambda_i + \sqrt{\tau})}{1 + (t - \tau)(\lambda_i + \sqrt{\tau})} \, \delta_{ij} & 0 \\ 0 & (t - \frac{\tau}{2})^{-1} \, \delta_{\alpha\beta} \end{bmatrix}$$

for all $\tau \in (0, \tau_0)$ and all $t \in (\tau, r)$. Passing to the limit as $\tau \to 0$, we conclude that

$$Q(t) \leq \begin{bmatrix} \frac{\lambda_i}{1 + t\lambda_i} \, \delta_{ij} & 0 \\ 0 & t^{-1} \, \delta_{\alpha\beta} \end{bmatrix}$$

for all $t \in (0, r)$. In particular, the trace of $Q(t)$ satisfies

$$\operatorname{tr}(Q(t)) \leq \frac{m}{t} + \sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i}$$

for all $t \in (0, r)$. It follows from Lemma 4.1 that

$$\sum_{i=1}^{n} \lambda_i = \Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle \leq n \, f(\bar{x})^{\frac{1}{n-1}}.$$

Using the arithmetic-harmonic mean inequality, we obtain

$$\sum_{i=1}^{n} \frac{1}{1 + t\lambda_i} \geq \frac{n^2}{\sum_{i=1}^{n} (1 + t\lambda_i)} \geq \frac{n}{1 + t \, f(\bar{x})^{\frac{1}{n-1}}},$$

hence

$$\sum_{i=1}^{n} \frac{\lambda_i}{1 + t\lambda_i} = \frac{1}{t} \Big(n - \sum_{i=1}^{n} \frac{1}{1 + t\lambda_i} \Big) \leq \frac{n \, f(\bar{x})^{\frac{1}{n-1}}}{1 + t \, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t \in (0, r)$. Putting these facts together, we conclude that

$$\operatorname{tr}(Q(t)) \leq \frac{m}{t} + \frac{n \, f(\bar{x})^{\frac{1}{n-1}}}{1 + t \, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t \in (0, r)$.

We next consider the determinant of $P(t)$. Clearly, $\lim_{t \to 0} t^{-m} \det P(t) = 1$. In particular, $\det P(t) > 0$ if $t > 0$ is sufficiently small. Since $P(t)$ is invertible for each $t \in (0,r)$, it follows that $\det P(t) > 0$ for all $t \in (0,r)$. Using the estimate for the trace of $Q(t) = P(t)^{-1} P'(t)$, we obtain

$$\frac{d}{dt} \log\det P(t) = \operatorname{tr}(Q(t)) \leq \frac{m}{t} + \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t \in (0,r)$. Consequently, the function

$$t \mapsto t^{-m}\, (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^{-n}\, \det P(t)$$

is monotone decreasing for $t \in (0,r)$.

Finally, we observe that

$$\frac{\partial \Phi_t}{\partial x_i}(\bar{x},\bar{y}) = X_i(t), \qquad \frac{\partial \Phi_t}{\partial y_\alpha}(\bar{x},\bar{y}) = Y_\alpha(t)$$

for $1 \leq i \leq n$ and $n+1 \leq \alpha \leq n+m$. Consequently, $|\det D\Phi_t(\bar{x},\bar{y})| = \det P(t)$ for all $t \in (0,r)$. Putting these facts together, the assertion follows.

Corollary 4.7. *The Jacobian determinant of Φ_r satisfies*

$$|\det D\Phi_r(x,y)| \leq r^m\, (1 + r\, f(x)^{\frac{1}{n-1}})^n$$

for all $(x,y) \in A_r$.

Proof. Since $\lim_{t \to 0} t^{-m}\, |\det D\Phi_t(x,y)| = 1$, the assertion follows from Proposition 4.6.

After these preparations, we now complete the proof of Theorem 1.4. Using Lemma 4.2 and Corollary 4.7, we obtain

$$\begin{aligned}
&|\{p \in M : \sigma r < d(x,p) < r \text{ for all } x \in \Sigma\}| \\
&\leq \int_\Omega \bigg(\int_{\{y \in T_x^\perp \Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} |\det D\Phi_r(x,y)|\, 1_{A_r}(x,y)\, dy \bigg)\, d\mathrm{vol}(x) \\
&\leq \int_\Omega \bigg(\int_{\{y \in T_x^\perp \Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} r^m\, (1 + r\, f(x)^{\frac{1}{n-1}})^n\, dy \bigg)\, d\mathrm{vol}(x) \\
&= |B^m| \int_\Omega \Big[(1 - |\nabla^\Sigma u(x)|^2)^{\frac{m}{2}} - (\sigma^2 - |\nabla^\Sigma u(x)|^2)_+^{\frac{m}{2}} \Big] \\
&\qquad\qquad\qquad\qquad \cdot r^m\, (1 + r\, f(x)^{\frac{1}{n-1}})^n\, d\mathrm{vol}(x)
\end{aligned}$$

for all $r > 0$ and all $0 \leq \sigma < 1$. Since $m \geq 2$, the mean value theorem implies $b^{\frac{m}{2}} - a^{\frac{m}{2}} \leq \frac{m}{2}\,(b-a)$ for $0 \leq a \leq b \leq 1$. Hence, we have the pointwise inequality

$$\begin{aligned}
&(1 - |\nabla^\Sigma u(x)|^2)^{\frac{m}{2}} - (\sigma^2 - |\nabla^\Sigma u(x)|^2)_+^{\frac{m}{2}} \\
&\leq \frac{m}{2} \Big[(1 - |\nabla^\Sigma u(x)|^2) - (\sigma^2 - |\nabla^\Sigma u(x)|^2)_+ \Big] \leq \frac{m}{2}\,(1 - \sigma^2)
\end{aligned}$$

for all $x \in \Omega$ and all $0 \leq \sigma < 1$. Therefore,

$$\begin{aligned} &|\{p \in M : \sigma r < d(x,p) < r \text{ for all } x \in \Sigma\}| \\ &\leq \frac{m}{2}\,|B^m|\,(1-\sigma^2) \int_\Omega r^m\,(1 + r\,f(x)^{\frac{1}{n-1}})^n\,d\mathrm{vol}(x) \end{aligned}$$

for all $r > 0$ and all $0 \leq \sigma < 1$. In the next step, we divide by r^{n+m} and send $r \to \infty$ while keeping σ fixed. This gives

$$|B^{n+m}|\,(1-\sigma^{n+m})\,\theta \leq \frac{m}{2}\,|B^m|\,(1-\sigma^2) \int_\Omega f^{\frac{n}{n-1}}$$

for all $0 \leq \sigma < 1$. Finally, if we divide by $1-\sigma$ and send $\sigma \to 1$, we obtain

$$(n+m)\,|B^{n+m}|\,\theta \leq m\,|B^m| \int_\Omega f^{\frac{n}{n-1}} \leq m\,|B^m| \int_\Sigma f^{\frac{n}{n-1}}.$$

Thus,

$$\begin{aligned} &\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f \\ &= n \int_\Sigma f^{\frac{n}{n-1}} \geq n \left(\frac{(n+m)\,|B^{n+m}|}{m\,|B^m|}\right)^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\int_\Sigma f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}. \end{aligned}$$

This completes the proof of Theorem 1.4.

5. Proof of Theorem 1.6

Let $(M, \bar{g})$ be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let Σ be a compact submanifold of M of dimension n (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ satisfying

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n\,|B^n|^{\frac{1}{n}}\,\theta^{\frac{1}{n}} \left(\int_\Sigma f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} > 0,$$

where θ denotes the asymptotic volume ratio of M.

If Σ is disconnected, we may apply Corollary 1.5 to each connected component of Σ, and take the sum over all connected components. This will lead to a contradiction. Therefore, Σ must be connected.

By scaling, we may assume that

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n\,|B^n|\,\theta$$

and

$$\int_\Sigma f^{\frac{n}{n-1}} = |B^n|\,\theta.$$

In particular,

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n \int_\Sigma f^{\frac{n}{n-1}}.$$

Since Σ is connected, we can find a function $u : \Sigma \to \mathbb{R}$ such that

$$\operatorname{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$$

on Σ and $\langle \nabla^\Sigma u, \eta\rangle = 1$ at each point on $\partial\Sigma$. Moreover, u is of class $C^{2,\gamma}$ for each $0 < \gamma < 1$. Let us define Ω, U, A_r, and Φ_r as in Section 4.

Lemma 5.1. *Assume that $x \in \Omega$ and $y \in T_x^\perp\Sigma$ satisfy $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Then $|\det D\Phi_t(x,y)| \geq t^2\,(1 + t\,f(x)^{\frac{1}{n-1}})^n$ for each $t > 0$.*

Proof. Let us fix a point $\bar{x} \in \Omega$ and a vector $\bar{y} \in T_{\bar{x}}^\perp\Sigma$ satisfying $|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 = 1$. Suppose that $|\det D\Phi_t(\bar{x},\bar{y})| < t^2\,(1 + t\,f(\bar{x})^{\frac{1}{n-1}})^n$ for some $t > 0$. Let us fix a real number $\varepsilon \in (0,1)$ such that

$$|\det D\Phi_t(\bar{x},\bar{y})| < (1-\varepsilon)\,t^2\,(1 + t\,f(\bar{x})^{\frac{1}{n-1}})^n.$$

By continuity, we can find an open neighborhood V of the point $(\bar{x},\bar{y})$ such that

$$|\det D\Phi_t(x,y)| \leq (1-\varepsilon)\,t^2\,(1 + t\,f(x)^{\frac{1}{n-1}})^n$$

for all $(x,y) \in V$. Using Proposition 4.6, we conclude that

$$|\det D\Phi_r(x,y)| \leq (1-\varepsilon)\,r^2\,(1 + r\,f(x)^{\frac{1}{n-1}})^n$$

for all $r > t$ and all $(x,y) \in A_r \cap V$. Using this fact together with Lemma 4.2 and Corollary 4.7, we obtain

$$\begin{aligned}
&|\{p \in M : \sigma r < d(x,p) < r \text{ for all } x \in \Sigma\}| \\
&\leq \int_\Omega \bigg(\int_{\{y \in T_x^\perp\Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} |\det D\Phi_r(x,y)|\,1_{A_r}(x,y)\,dy\bigg)\,d\mathrm{vol}(x) \\
&\leq \int_\Omega \bigg(\int_{\{y \in T_x^\perp\Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} (1 - \varepsilon \cdot 1_V(x,y)) \\
&\qquad\qquad \cdot r^2\,(1 + r\,f(x)^{\frac{1}{n-1}})^n\,dy\bigg)\,d\mathrm{vol}(x) \\
&\leq |B^2|\,(1-\sigma^2) \int_\Omega r^2\,(1 + r\,f(x)^{\frac{1}{n-1}})^n\,d\mathrm{vol}(x) \\
&- \varepsilon \int_\Omega \bigg(\int_{\{y \in T_x^\perp\Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} 1_V(x,y)\,r^2\,(1 + r\,f(x)^{\frac{1}{n-1}})^n\,dy\bigg)\,d\mathrm{vol}(x)
\end{aligned}$$

for all $r > t$ and all $0 \leq \sigma < 1$. We now divide by r^{n+2} and send $r \to \infty$, while keeping σ fixed. This implies

$$\begin{aligned}
&|B^{n+2}|\,(1-\sigma^{n+2})\,\theta \\
&\leq |B^2|\,(1-\sigma^2) \int_\Omega f(x)^{\frac{n}{n-1}}\,d\mathrm{vol}(x) \\
&- \varepsilon \int_\Omega \bigg(\int_{\{y \in T_x^\perp\Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} 1_V(x,y)\,f(x)^{\frac{n}{n-1}}\,dy\bigg)\,d\mathrm{vol}(x)
\end{aligned}$$

for all $0 \leq \sigma < 1$. Dividing by $1 - \sigma$ and taking the limit as $\sigma \to 1$ gives

$$(n+2)\,|B^{n+2}|\,\theta < 2\,|B^2| \int_\Omega f^{\frac{n}{n-1}} \leq 2\,|B^2|\,|B^n|\,\theta.$$

This contradicts the fact that $(n+2)\,|B^{n+2}| = 2\,|B^2|\,|B^n|$.

Lemma 5.2. *Assume that $x \in \Omega$ and $y \in T_x^\perp \Sigma$ satisfy $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Then $D_\Sigma^2 u(x) - \langle I\!I(x), y\rangle = f(x)^{\frac{1}{n-1}}\, g$.*

Proof. Let us fix a point $\bar{x} \in \Omega$ and a vector $\bar{y} \in T_{\bar{x}}^\perp \Sigma$ satisfying $|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 = 1$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\bar{x}} M$ with the property that the $n \times n$-matrix

$$(D_\Sigma^2 u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y}\rangle$$

is diagonal, and let $\{\nu_{n+1}, \nu_{n+2}\}$ be an orthonormal basis of $T_{\bar{x}}^\perp \Sigma$. We define $\bar{\gamma}(t) := \exp_{\bar{x}}(t\,\nabla u(\bar{x}) + t\,\bar{y})$ for all $t \geq 0$. For each $1 \leq i \leq n$, we denote by $E_i(t)$ the parallel transport of e_i along $\bar{\gamma}$. Moreover, for each $1 \leq i \leq n$, we denote by $X_i(t)$ the unique Jacobi field along $\bar{\gamma}$ satisfying $X_i(0) = e_i$ and

$$\begin{aligned}
&\langle \bar{D}_t X_i(0), e_j\rangle = (D_\Sigma^2 u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y}\rangle, \\
&\langle \bar{D}_t X_i(0), \nu_\beta\rangle = \langle I\!I(e_i, \nabla^\Sigma u), \nu_\beta\rangle
\end{aligned}$$

for all $1 \leq j \leq n$ and all $n+1 \leq \beta \leq n+2$. For each $n+1 \leq \alpha \leq n+2$, we denote by $N_\alpha(t)$ the parallel transport of ν_α along $\bar{\gamma}$. Moreover, for each $n+1 \leq \alpha \leq n+2$, we denote by $Y_\alpha(t)$ the unique Jacobi field along $\bar{\gamma}$ satisfying $Y_\alpha(0) = 0$ and $\bar{D}_t Y_\alpha(0) = \nu_\alpha$.

Finally, we define an $(n+2) \times (n+2)$-matrix $P(t)$ by

$$\begin{aligned}
P_{ij}(t) &= \langle X_i(t), E_j(t)\rangle, \quad P_{i\beta}(t) = \langle X_i(t), N_\beta(t)\rangle, \\
P_{\alpha j}(t) &= \langle Y_\alpha(t), E_i(t)\rangle, \quad P_{\alpha\beta}(t) = \langle Y_\alpha(t), N_\beta(t)\rangle
\end{aligned}$$

for $1 \leq i, j \leq n$ and $n+1 \leq \alpha, \beta \leq n+2$.

By Lemma 5.1, we know that $|\det P(t)| \geq t^2\,(1 + t\,f(\bar{x})^{\frac{1}{n-1}})^n$ for all $t > 0$. Since $\det P(t) > 0$ if $t > 0$ is sufficiently small, we conclude that

$$\det P(t) \geq t^2\,(1 + t\,f(\bar{x})^{\frac{1}{n-1}})^n$$

for all $t > 0$. In particular, $P(t)$ is invertible for each $t > 0$.

We next define $Q(t) := P(t)^{-1} P'(t)$ for all $t > 0$. Moreover, we write

$$(D_\Sigma^2 u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y}\rangle = \lambda_i\,\delta_{ij}$$

for $1 \leq i, j \leq n$. Arguing as in Section 4, we obtain

$$\operatorname{tr}(Q(t)) \leq \frac{2}{t} + \sum_{i=1}^n \frac{\lambda_i}{1 + t\lambda_i}$$

for all $t > 0$ satisfying $\min_{1 \leq i \leq n}(1 + t\lambda_i) > 0$. Moreover, $\sum_{i=1}^n \lambda_i \leq n\, f(\bar{x})^{\frac{1}{n-1}}$ by Lemma 4.1. As above, the arithmetic-harmonic mean inequality implies

$$\sum_{i=1}^n \frac{\lambda_i}{1 + t\lambda_i} \leq \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t > 0$ satisfying $\min_{1 \leq i \leq n}(1 + t\lambda_i) > 0$. Therefore,

$$\operatorname{tr}(Q(t)) \leq \frac{2}{t} + \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}},$$

hence

$$\frac{d}{dt} \log \det P(t) \leq \frac{2}{t} + \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t > 0$ satisfying $\min_{1 \leq i \leq n}(1 + t\lambda_i) > 0$. Integrating this ODE gives

$$\det P(t) \leq t^2\, (1 + t\, f(\bar{x})^{\frac{1}{n-1}})^n$$

for all $t > 0$ satisfying $\min_{1 \leq i \leq n}(1 + t\lambda_i) > 0$.

Putting these facts together, we conclude that $\det P(t) = t^2\, (1+t\, f(\bar{x})^{\frac{1}{n-1}})^n$ for all $t > 0$ satisfying $\min_{1 \leq i \leq n}(1 + t\lambda_i) > 0$. Differentiating this identity with respect to t, we obtain

$$\operatorname{tr}(Q(t)) = \frac{2}{t} + \frac{n\, f(\bar{x})^{\frac{1}{n-1}}}{1 + t\, f(\bar{x})^{\frac{1}{n-1}}}$$

for all $t > 0$ satisfying $\min_{1 \leq i \leq n}(1 + t\lambda_i) > 0$. Consequently, we must have equality in the arithmetic-harmonic mean equality, and furthermore $\sum_{i=1}^n \lambda_i = n\, f(\bar{x})^{\frac{1}{n-1}}$. Therefore, $\lambda_i = f(\bar{x})^{\frac{1}{n-1}}$ for each $1 \leq i \leq n$. This completes the proof of Lemma 5.2.

Lemma 5.3. *Assume that $x \in \Omega$. Then $D^2_\Sigma u(x) = f(x)^{\frac{1}{n-1}}\, g$ and $I\!I(x) = 0$.*

Proof. By Lemma 5.2, $D^2_\Sigma u(x) - \langle I\!I(x), y\rangle = f(x)^{\frac{1}{n-1}}\, g$ for all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Replacing y by $-y$ gives $D^2_\Sigma u(x) + \langle I\!I(x), y\rangle = f(x)^{\frac{1}{n-1}}\, g$ for all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Therefore, $D^2_\Sigma u(x) = f(x)^{\frac{1}{n-1}}\, g$ and $\langle I\!I(x), y\rangle = 0$ for all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. From this, the assertion follows easily.

Lemma 5.4. *Assume that $x \in \Omega$. Then $\nabla^\Sigma f(x) = 0$.*

Proof. Let us consider an arbitrary point $x \in \Omega$. Using the definition of u, we obtain

$$\begin{aligned} f(x)\, \Delta_\Sigma u(x) &= n\, f(x)^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2\, |H(x)|^2} \\ &\quad - \langle \nabla^\Sigma f(x), \nabla^\Sigma u(x)\rangle. \end{aligned}$$

On the other hand, Lemma 5.3 implies $\Delta_\Sigma u(x) = n\, f(x)^{\frac{1}{n-1}}$ and $H(x) = 0$. Putting these facts together, we conclude that $\langle \nabla^\Sigma f(x), \nabla^\Sigma u(x) \rangle = -|\nabla^\Sigma f(x)|$. Since $|\nabla^\Sigma u(x)| < 1$, it follows that $\nabla^\Sigma f(x) = 0$. This completes the proof of Lemma 5.4.

Lemma 5.5. *The set Ω is dense in Σ.*

Proof. Suppose that Ω is not dense in Σ. Arguing as in Section 4, we obtain

$$(n+2)\, |B^{n+2}|\, \theta \leq 2\, |B^2| \int_\Omega f^{\frac{n}{n-1}} < 2\, |B^2| \int_\Sigma f^{\frac{n}{n-1}} = 2\, |B^2|\, |B^n|\, \theta.$$

This contradicts the fact that $(n+2)\, |B^{n+2}| = 2\, |B^2|\, |B^n|$. This completes the proof of Lemma 5.5.

Since Ω is a dense subset of Σ, we conclude that $\nabla^\Sigma f = 0$, $D^2_\Sigma u = f^{\frac{1}{n-1}}\, g$, and $II = 0$ at each point on Σ. Since Σ is connected, it follows that f is constant. This implies $|\partial\Sigma| = n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}}\, |\Sigma|^{\frac{n-1}{n}}$.

Note that u is a smooth function on Σ. Each critical point of u lies in the interior of Σ and is nondegenerate with Morse index 0. In particular, the function u has at most finitely many critical points.

We next consider the flow on Σ generated by the vector field $-\nabla^\Sigma u$. Since the vector field $-\nabla^\Sigma u$ points inward along the boundary $\partial\Sigma$, the flow is defined for all nonnegative times. This gives a one-parameter family of smooth maps $\psi_s : \Sigma \to \Sigma$, where $s \geq 0$. Since Σ is connected, standard arguments from Morse theory imply that the function u has exactly one critical point, and u attains its global minimum at that point. It follows that the diameter of $\psi_s(\Sigma)$ converges to 0 as $s \to \infty$.

Since $D^2_\Sigma u$ is a constant multiple of the metric, the isoperimetric ratio is unchanged under the flow ψ_s. This implies

$$|\psi_s(\partial\Sigma)| = n\, |B^n|^{\frac{1}{n}}\, \theta^{\frac{1}{n}}\, |\psi_s(\Sigma)|^{\frac{n-1}{n}}$$

for each $s \geq 0$. If $\theta < 1$, this contradicts the Euclidean isoperimetric inequality when s is sufficiently large. Thus, we conclude that $\theta = 1$. Consequently, M is isometric to Euclidean space.

Once we know that M is isometric to Euclidean space, the arguments in [3] imply that Σ is a flat round ball. This completes the proof of Theorem 1.6.

References

[1] V. Agostiniani, M. Fogagnolo, and L. Mazzieri, *Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature,* Invent. Math. 222, 1033–1101 (2020)

[2] W. Allard, *On the first variation of a varifold,* Ann. of Math. 95, 417–491 (1972)

[3] S. Brendle, *The isoperimetric inequality for a minimal submanifold in Euclidean space,* J. Amer. Math. Soc. 34, 595–603 (2021)

[4] X. Cabré, *Nondivergent elliptic equations on manifolds with nonnegative curvature,* Comm. Pure Appl. Math. 50, 623–665 (1997)
[5] X. Cabré, *Elliptic PDEs in probability and geometry. Symmetry and regularity of solutions,* Discrete Cont. Dyn. Systems A 20, 425–457 (2008)
[6] X. Cabré, X. Ros-Oton, and J. Serra, *Sharp isoperimetric inequalities via the ABP method,* J. Eur. Math. Soc. 18, 2971–2998 (2016)
[7] T. Carleman, *Zur Theorie der Minimalflächen,* Math. Z. 9, 154–160 (1921)
[8] F. Cavalletti and A. Mondino, *Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds,* Invent. Math. 208, 803–849 (2017)
[9] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, 2001
[10] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces,* Progress in Mathematics vol. 152, Birkhäuser, Boston, 1999.
[11] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds,* Ann. Sci. École Norm. Sup. 11, 451–470 (1978)
[12] G. Huisken, *An isoperimetric concept for the mass in general relativity,* Lecture given at the Institute for Advanced Study on March 20, 2009
`https://www.ias.edu/video/marston-morse-isoperimetric-concept-mass-general-relativity`
[13] B. Klartag, *Needle decompositions in Riemannian geometry,* Mem. Amer. Math. Soc. 249, no. 1180 (2017)
[14] J.H. Michael and L.M. Simon, *Sobolev and mean value inequalities on generalized submanifolds of* $\mathbb{R}^n$*,* Comm. Pure Appl. Math. 26, 361–379 (1973)
[15] N. Trudinger, *Isoperimetric inequalities for quermassintegrals,* Ann. Inst. H. Poincaré Anal. Non Linéaire 11, 411–425 (1994)
[16] Y. Wang and X.W. Zhang, *Alexandrov-Bakelman-Pucci estimate on Riemannian manifolds,* Adv. Math. 232, 499–512 (2013)
[17] C. Xia and X. Zhang, *ABP estimate and geometric inequalities,* Comm. Anal. Geom. 25, 685–708 (2017)

Department of Mathematics, Columbia University, New York NY 10027