Fields of locally compact quantum groups: continuity and pushouts

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Abstract

We prove that discrete compact quantum groups (or more generally locally compact, under additional hypotheses) with coamenable dual are continuous fields over their central closed quantum subgroups, and (b) the same holds for free products of discrete quantum groups with coamenable dual amalgamated over a common central subgroup. Along the way we also show that free products of continuous fields of C^*-algebras are again free via a Fell-topology characterization for C^*-field continuity, recovering a result of Blanchard's in a somewhat more general setting.

Key words: C^*-algebra; continuous field; weak containment; Fell topology; locally compact quantum group; discrete quantum group; pushout; free product with amalgamation

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Introduction

The initial motivation for the present note was the desire to extend one of the main results of [10] (Theorem 3.2 therein) in two ways: from plain ("classical") to *quantum* groups, and from discrete to locally compact. The result appears below as Theorem 2.1:

Theorem 0.1 *For*

- *a locally compact quantum group G with coamenable dual*
- *with a central closed quantum subgroup $H \leqslant G$*
- *such that G/H has coamenable dual (automatic if G is discrete)*

the group C^*-algebra $C_0^r(\widehat{G})$ forms a continuous field over the group algebra $C_0^r(\widehat{H})$.

We recall the terminology and notation below in §1.1 (e.g. $\widehat{G}$ for the *dual* $\widehat{G}$ of G), pausing here only to remind the reader that a *reduced locally compact quantum group* G in the sense of [20] (see also [26, Definition 8.1.17]) consists of a generally non-unital C^*-algebra $C_0^r(G)$ (thought of as a algebra of continuous functions vanishing at infinity on G) equipped with

- a coassociative *comultiplication* morphism $\Delta : C_0^r(G) \to C_0^r(G)^{\otimes 2}$ (minimal C^* tensor product) in the sense of Definition 1.1 (i.e. landing in the multiplier algebra);
- left and right-invariant *Haar weights* (on which we do not elaborate);

- and hence a von Neumann algebra $L^\infty(G)$ ([26, Definition 8.1.4] or [21, Definition 1.1]) attached to one of these invariant weights via the GNS construction; here, the quantum-group structure is captured by a comultiplication

$$L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$$

into the von-Neumann-algebraic (or *spatial*) tensor product, sometimes denoted by '$\overline{\otimes}$' (e.g. [25, §IV.5]).

Since furthermore [10, Theorem 3.2] handles *pushouts* of amenable groups G_i, $i = 1, 2$ over a common central subgroup H, it seemed desirable to have an analogous extension here (Theorem 2.2):

Theorem 0.2 *Let G_i, $i \in I$ be a family of discrete quantum groups with coamenable duals and a common central closed quantum subgroup $H \leqslant G_i$. Then, the C^* pushout*

$$\mathop{\ast}_{C^r(\hat{H})} C^r\left(\widehat{G_i}\right)$$

is a continuous field over the commutative C^*-algebra $C^r\left(\widehat{H}\right)$.

Note that as opposed to Theorem 0.1, where G is locally compact, here the G_i are *discrete* (equivalently, $C_0^r(G)$ are unital, hence the missing '0' subscript in C_0^r): this is to avoid the unpleasantness of working with non-unital pushouts.

One natural path to Theorem 0.2 (or something like it, perhaps covering pushouts of only *two* quantum groups) would be to start with Theorem 0.1 and apply [7, Theorem 3.7], to the effect that a pushout $A *_C B$ of fields of C^*-algebras continuous over a central C^*-algebra C is again continuous over C. This was the initial intention, but in the process of unwinding that cited result the proof appeared to contain a gap. For that reason, it seemed worthwhile to try to recover [7, Theorem 3.7] here via a different approach (Theorem 3.5):

Theorem 0.3 *Let X be a compact Hausdorff space and A_i, $i \in I$ a family of unital $C(X)$-algebras. If all A_i are continuous then so is the pushout*

$$A := \mathop{\ast}_{C(X)} A_i.$$

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1 Preliminaries

C^*-algebras are not assumed unital unless we do so explicitly, and morphisms are defined as is customary for generally-non-unital C^*-algebras (e.g. [20, Notations and conventions]):

Definition 1.1 Let A and B be two possibly-non-unital C^*-algebras. A *morphism* $A \to B$ is a linear, bounded, multiplicative and $*$-preserving map $f : A \to M(B)$ (the *multiplier algebra* of B; [29, §2.2] or [4, §II.7.3]) that is *non-degenerate* in the sense that

$$f(A)B := \mathrm{span}\{f(a)b \mid a \in A,\ b \in B\}$$

is dense in B. ♦

As noted in [20, Notations and conventions], morphisms $A \to B$ in this sense extend uniquely to unital morphisms $M(A) \to M(B)$ that are *strictly continuous* on bounded subsets. Recall that strict continuity means continuity with respect to the seminorms

$$M(A) \ni x \mapsto \|xa\| + \|ax\|,\ a \in A.$$

1.1 Locally compact quantum groups

We will need some background on these (abbreviated as LCQGs) as introduced in [20]. Additional sources include the excellent textbook [26] as well as various other papers cited in the process of (very briefly) recalling some of the relevant notions.

In addition to the structure reviewed briefly in the introduction, on can define, for an LCQG G,

- the *universal* version $C_0^u(G)$ [19, §11] that is again a C^*-algebra equipped with a coassociative morphism $\Delta : C_0^u(G) \to C_0^u(G)^{\otimes 2}$ and a surjection $C_0^u(G) \to C_0^r(G)$ intertwining the comultiplications;
- the *Pontryagin dual* $\widehat{G}$ of G ([26, Definition 8.3.14]), whose underlying universal C^*-algebra $C_0^u(\widehat{G})$ analogizes the universal group algebra of G (in particular, *representations* of G on Hilbert spaces are precisely representations of $C_0^u(\widehat{G})$ as a C^*-algebra [19, §5]).

The following version of the notion of discreteness will be most directly applicable below (see also [26, §3.3] or [28]).

Definition 1.2 G is *discrete* if $C_0^r(\widehat{G})$ is unital. ♦

Regarding the relationship between C_0^u and C_0^r, recall [1, Definition 3.1, Theorem 3.1]:

Definition 1.3 A locally compact quantum group G is *coamenable* if either of the two following equivalent conditions holds:

- there is a character $\varepsilon : C_0^r(G) \to \mathbb{C}$ such that $(\mathrm{id}\otimes\varepsilon)\Delta = \mathrm{id}_{C_0^r(G)}$;
- the surjection $C_0^u(G) \to C_0^r(G)$ is an isomorphism. ♦

As in the classical setting, we can talk about closed quantum subgroups ([27, Definition 2.5] and [12, Definitions 3.1 and 3.2]):

Definition 1.4 Let G be a locally compact quantum group.

(a) A *(Vaes-)closed quantum subgroup* $H \leqslant G$ is a locally compact quantum group H equipped with a normal embedding

$$L^\infty(\widehat{H}) \to L^\infty(\widehat{G})$$

intertwining the comultiplications.

(b) This then induces a surjection $C_0^u(G) \to C_0^u(H)$, thus realizing H as a *Woronowicz-closed quantum subgroup of* G ([12, Definition 3.2, Theorems 3.5 and 3.6]).

(c) The (Vaes-)closed $H \leqslant G$ is *central* [17, §1.1] if

$$L^\infty(\widehat{H}) \subseteq L^\infty(\widehat{G})$$

is contained in the center. ♦

There is also a notion of *normality* for closed quantum subgroups $H \leqslant G$, for which [18, Section 4] gives a good, pithy account.

Definition 1.5 Let $H \leqslant G$ be a closed quantum subgroup of a locally compact quantum group. It then induces, as in [18, §2.2], left and right actions

$$\rho_l : L^\infty(G) \to L^\infty(H) \otimes L^\infty(G) \text{ and } \rho_r : L^\infty(G) \to L^\infty(G) \otimes L^\infty(H).$$

The fixed-point subalgebras

$$L^\infty(H\backslash G) := \{x \in L^\infty(G) \mid \rho_l(x) = 1 \otimes x\}$$
$$L^\infty(G/H) := \{x \in L^\infty(G) \mid \rho_r(x) = x \otimes 1\}$$

can be regarded as algebras of functions on the corresponding quantum homogeneous spaces.

H is said to be *normal* if $L^\infty(H\backslash G) = L^\infty(G/H)$. In that case the embedding

$$L^\infty(G/H) \subset L^\infty(G)$$

equips $L^\infty(G/H)$ with its own comultiplication, making G/H into a quantum group. Moreover, that embedding realizes $\widehat{G/H}$ as a closed quantum subgroup of $\widehat{G}$ in the sense of Definition 1.4. ♦

1.2 Fields of C^*-algebras

Denoting, as usual, by $C_0(X)$ the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space X, we work with $C_0(X)$-algebras in the sense of [9, Introduction] or [8, Definition 2.1] (see also [5, Definition 1.1] and [7, Definition 2.2] for the case of compact X):

Definition 1.6 A $C_0(X)$-*algebra* is a (possibly non-unital) C^*-algebra A equipped with a non-degenerate morphism from $C_0(X)$ to the center of the multiplier algebra $M(A)$. ♦

One can form, for every point $x \in X$, the *fiber* A_x of A at x as

$$A_x = A/\{\text{ideal generated by } ma = am\}$$

where $a \in A$ and $m \in M(A)$ ranges over the image through $C_0(X) \to M(A)$ of the ideal of functions vanishing at x.

For $a \in A$ we follow [7] in denoting by a_x the image of a through $A \to A_x$, and when the need arises to distinguish between the norms of the various A_x we write $\|\cdot\|_x$ for the latter. Recall [6, Définition 3.1] (see also [8, Definition 2.2] or [7, discussion following Definition 2.2]):

Definition 1.7 The $C_0(X)$-algebra A is *continuous* as such, or *a continuous field* over X if

$$X \ni x \mapsto \|a_x\| \tag{1-1}$$

is continuous for every $a \in A$. ♦

Remark 1.8 Note that the map (1-1) is always *upper* semicontinuous [23, Proposition 1.2], so it is lower semicontinuity that is the core issue motivating Definition 1.7. ♦

When working with *pushouts* (e.g. [7, 22]) we specialize to the unital case: A_i will typically be unital algebras equipped with unital morphisms $C(X) \to A_i$, thus allowing the formation of the pushout

$$\ast_{C(X)} A_i. \tag{1-2}$$

To make sense of Lemma 1.9 below we need some background on the *Fell topology* on (unitary isomorphism classes of) representations and on *weak containment*. We refer to [15, 14], [13, Chapter 3] and [3, Appendix F] for material on these topics.

In the statement of Lemma 1.9, for a point $x \in X$

$$p_x : C_0(X) \to \mathbb{C}$$

denotes the evaluation character, regarded as a $C_0(X)$-representation.

Lemma 1.9 *A $C_0(X)$-algebra A is a continuous field if and only if for every convergent net*

$$x_\alpha \to x \in X \tag{1-3}$$

every representation of A that factors through $A \to A_x$ is a Fell limit of a net of representations that factor through $A \to A_{x_\alpha}$.

Proof We prove the two implications separately.

($\Rightarrow$) Suppose A is continuous over X. To prove the desired conclusion we have to show ([13, Theorem 3.4.4]) that for a net (1-3) and a representation

$$\rho : A \to A_x \to B(H) \tag{1-4}$$

where $C_0(X)$ acts via $p := p_x$ we can find representations

$$\rho_\alpha : A \to B(H_\alpha)$$

such that

- ρ_α restricts to $C_0(X)$ as the character p_{x_α}, and
- the intersection of the kernels of ρ_α is contained in $\ker \rho$.

Now choose a net of representations

$$\rho_\alpha : A \to A_{x_\alpha} \to B(H_\alpha), \tag{1-5}$$

faithful on A_{x_α} respectively. By the continuity of the field A over X, any element annihilated by all ρ_α must also be annihilated by $A \to A_x$, and hence by the map ρ from (1-4).

($\Leftarrow$) Conversely, to prove that A is continuous over X we have to argue that for every convergent net (1-3) and every $a \in A$ we have

$$\|a_x\| \leqslant \limsup_\alpha \|a_{x_\alpha}\|. \tag{1-6}$$

The hypothesis says that we can find a net of representations (1-5) Fell-converging to some representation (1-5) faithful on A_x. Since the Fell topology does not distinguish between sums of copies of a given representation (regardless of the cardinality of the set of summands), we may as well assume that

$$\rho \cong \rho^{\oplus \aleph_0}$$

and similarly for all ρ_α. But in that case [15, Lemma 2.4] shows that

- we can realize ρ concretely and non-degenerately on some large Hilbert space H
- which also houses (possibly degenerate) copies of ρ_α
- so that for every $a \in A$ and every $\xi \in H$ we have

$$\lim_\alpha \|\rho(a)\xi - \rho_\alpha(a)\xi\| = 0.$$

Since ρ is assumed faithful on A_x, (1-6) follows. ■

2 Fields over central locally compact quantum groups

We begin with

Theorem 2.1 *Let G be an LCQG with coamenable dual and $H \leqslant G$ a central closed quantum subgroup.*

(a) If G/H also has coamenable dual then $C_0^u\left(\widehat{H}\right) \to C_0^u\left(\widehat{G}\right)$ is a continuous field.

(b) The hypothesis in (a) is automatic if G is discrete.

Proof We treat the two clauses separately.

Part (a) We mimic the proof of [10, Theorem 3.2]. Having chosen a convergent net

$$x_\alpha \to x \in \widehat{H}$$

of characters, we have to argue that an arbitrary unitary representation ρ of G on which H acts via p_x is in the Fell closure of a family of unitary representations where H acts by p_{x_α}. The coamenability condition ensures that G/H has coamenable dual and hence its regular representation weakly contains its trivial representation $\mathbf{1}_{G/H}$. We thus similarly have

$$\mathbf{1}_G \preceq \mathrm{Ind}_H^G(\mathbf{1}_H),$$

and it follows that

$$\rho \cong \rho \otimes \mathbf{1}_G \preceq \rho \otimes \mathrm{Ind}_H^G(\mathbf{1}_H).$$

Because induction preserves weak containment [16, Theorem A.1], the latter representation is a Fell-topology limit of

$$\rho \otimes \mathrm{Ind}_H^G(p_{x_\alpha x^{-1}})$$

where H acts respectively by p_{x_α}, hence the conclusion.

Part (b) According to [11, Theorem 3.2] the amenability of G entails that of G/H ,whereas by [2, Corollary 9.6] amenability and dual coamenability are equivalent for discrete quantum groups. ■

Next, Theorem 2.1 extends to arbitrary pushouts.

Theorem 2.2 *Let G_i, $i \in I$ be a family of discrete quantum groups with coamenable duals and a common central closed quantum subgroup $H \leqslant G_i$. Then, the C^* pushout*

$$*_{C^u(\widehat{H})} C^u\left(\widehat{G_i}\right)$$

is a continuous field over the commutative C^*-algebra $C^u\left(\widehat{H}\right)$.

Proof Once we have Theorem 2.1 we can conclude

- for pushouts of *two* CQGs by Theorem 3.5 (or [7, Theorem 3.7], but see Section 3 for an aside on its proof), stating that
$$A_1 *_{C(X)} A_2$$
is continuous whenever A_i are;

- for finite families $\{G_i\}$ by induction;

- in general, by taking inductive limits (or *filtered colimits* [24, Tag 04AX]) over the finite subsets of the index set I, via Proposition 2.3: in the present setting the latter requires that for an inclusion $F \subset F'$ of finite sets we have embeddings
$$\mathop{\ast}_{C^u(\widehat{H}),F} C^u\left(\widehat{G_i}\right) \subseteq \mathop{\ast}_{C^u(\widehat{H}),F'} C^u\left(\widehat{G_i}\right)$$
and similarly for x-fibers, with x ranging over the spectrum of $C^u\left(\widehat{H}\right)$. The former is an inclusion e.g. by [22, Theorem 4.2], whereas in the latter case the fiber of a pushout is simply the free product of the fibers, and hence we can invoke the selfsame [22, Theorem 4.2].

This finishes the proof, modulo the last item regarding colimits. ■

It remains to address the filtered-colimit claim that the proof of Theorem 2.2 punts on:

Proposition 2.3 *Let X be a locally compact Hausdorff space, $(I, \leqslant)$ a filtered poset, and*
$$\iota_{ji} : A_i \to A_j,\ \forall i \leqslant j \in I$$
a functor from I to the category of $C_0(X)$-algebras with

- *ι_{ji} injective*

- *and inducing fiber-level injections $\iota_{ji,x} : A_{i,x} \to A_{j,x}$ for all $x \in X$.*

If all A_i are continuous then so is
$$A := \varinjlim_{i \in I} A_i$$

Proof As noted before (Remark 1.8) we need lower semicontinuity, since upper semicontinuity is automatic. Concretely, having fixed an element $a \in A$, a point $x \in X$ and $\varepsilon > 0$, we have to argue that for y ranging over some neighborhood of x we have
$$\|a_y\| > \|a_x\| - \varepsilon. \tag{2-1}$$
Denote by $\iota_i : A_i \to A$ the structure map of the colimit.

The injectivity assumptions will allow us to suppress both the inclusions $\iota_i : A_i \to A$ and their induced fiber-level maps $\iota_{i,x} : A_{i,x} \to A_x$ in estimating norms. Choose some $i \in I$ and an element $a_i \in A_i$ such that
$$\|a - a_i\| < \frac{\varepsilon}{3} \Rightarrow \|a_p - a_{i,p}\| < \frac{\varepsilon}{3},\ \forall p \in X.$$
By the continuity of A_i, we can find a neighborhood $U \ni x$ such that
$$\|a_{i,y}\| > \|a_{i,x}\| - \frac{\varepsilon}{3},\ \forall y \in U.$$

But then, for $y \in U$ we have

$$\begin{aligned}\|a_y\| &\geqslant \|a_{i,y}\| - \|a_y - a_{i,y}\| > \|a_{i,y}\| - \frac{\varepsilon}{3}\\ &> \|a_{i,x}\| - \frac{2\varepsilon}{3} \geqslant \|a_x\| - \|a_x - a_{i,x}\| - \frac{2\varepsilon}{3}\\ &> \|a_x\| - \varepsilon.\end{aligned}$$

This concludes the proof. ■

3 A comment on the literature

The present side-note is devoted to an issue I believe is present in the proof of [7, Theorem 3.7]. Though apparently the problem is fixable, the proof as-is posed some difficulties (for this reader, at least). The setup is as follows: one considers an arbitrary element a in the dense purely algebraic pushout

$$A_1 *_{C(X)}^{\mathrm{alg}} A_2 \subset A_1 *_{C(X)} A_2$$

and seeks to show that

$$X \ni x \mapsto \|a_x\|$$

is lower semicontinuous. This is done for separable C^*-algebras first, in [7, Lemma 3.5], and then generalized to the present setting by

- first embedding a into a pushout $D_1 *_{C(Y)} D_2$ where $C(Y) \subseteq C(X)$ and $D_i \subseteq A_i$ are separable C^*-subalgebras;
- citing [22, Theorem 4.2] to conclude that there is an embedding

$$D_1 *_{C(Y)} D_2 \subseteq A_1 *_{C(X)} A_2. \tag{3-1}$$

The problem is with this last step: [22, Theorem 4.2] proves inclusions of the form

$$D_1 *_C D_2 \subseteq A_1 *_C A_2$$

given inclusions

$$C \subseteq D_i \subseteq A_i,\ \ i = 1, 2.$$

Note that the amalgam C is the same on both sides. On the other hand, given that on the right-hand side of (3-1) one amalgamates over a (generally-speaking) *larger* algebra $C(X) \supset C(Y)$, it is not at all obvious that the natural map (3-1) is indeed an inclusion. Indeed, it certainly will not be in full generality:

Remark 3.1 The proof of [7, Theorem 3.7] is unproblematic in the separable case, as in that case one does run into the difficulties related to the (non-)embedding (3-1). ♦

Example 3.2 Consider the case where $D_i = A_i$, but $C(Y)$ is trivial (i.e. $\mathbb{C}$) whereas $C(X)$ is not. (3-1) is then a *sur*jection but not an injection. ♦

Note also that one cannot exhaust A_i and $C(X)$ respectively by separable D_i and $C(Y)$ and naively hope for a continuity permanence property under filtered colimits

$$A_1 *_{C(X)} A_2 = \varinjlim_{D_i, C(Y)} D_1 *_{C(Y)} D_2 :$$

Example 3.3 Consider the space $X = \mathbb{Z}_p$ (the p-adic integers), expressed as a limit

$$X = \varprojlim_n (X_n := \mathbb{Z}/p^n).$$

This affords us a filtered-colimit description

$$C(X) = \varinjlim_n C(X_n),$$

but if $A \supset C(X)$ denotes the algebra of *all* (possibly discontinuous) bounded functions on X, then

- all $C(X_n) \subset A$ are continuous simply because X_n are discrete, while
- $C(X) \subset A$ isn't, despite the fact that the inclusion $C(X) \subset A$ is the inductive limit of the inclusions
$$C(X_n) \subset A.$$ ♦

Remark 3.4 Contrast Example 3.3 with Proposition 2.3, where the base space for the fields A_i whose colimit is being considered is fixed. Such problems arise precisely when the base space changes, much as in the initial observation that (3-1) need not be an embedding. ♦

For all of these reasons, it would seem worthwhile to have a proof of field-continuity permanence under pushouts that is independent of the separable case. We sketch such a proof here.

Theorem 3.5 *Let X be a compact Hausdorff space and A_i, $i \in I$ a family of unital C^*-algebras equipped with central embeddings of $C(X)$. If all A_i are continuous then so is the pushout*

$$A := \circledast_{C(X)} A_i.$$

Proof As in the proof of Theorem 2.2, once we have the statement for *pairs* of algebras A_1 and A_2 the general conclusion follows via Proposition 2.3 by taking a filtered colimit. We thus focus, for the duration of the proof, on the case of just two algebras A_i, $i = 1, 2$.

The proof will mimic that of [10, Theorem 3.2]. Fix a convergent net (1-3) in X and consider a representation ρ of $A = A_1 *_{C(X)} A_2$ which

- factors through A_x for some $x \in X$ fixed throughout the proof, and
- induces a faithful representation of A_x.

Then, by Lemma 1.9 (and as in its proof), the continuity of A_i, $i = 1, 2$ implies that the restrictions ρ_i of ρ to A_i can be Fell-approximated by representations $\rho_{i,\alpha}$ factoring respectively through $(A_i)_{x_\alpha}$, in the sense that

$$\lim_\alpha \|\rho_i(a_i)\xi - \rho_{i,\alpha}(a_i)\xi\| = 0, \ i = 1, 2 \tag{3-2}$$

for $a_i \in A_i$. Since $C(X)$ acts via the character p_{x_α} in both both $\rho_{i,\alpha}$, one can form the amalgamated free product of the $\rho_{i,\alpha}$, $i = 1, 2$; by (3-2) that free product will Fell-converge to ρ, hence the conclusion by Lemma 1.9. ■

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