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INVERSE BOUNDARY PROBLEM FOR THE TWO PHOTON ABSORPTION TRANSPORT EQUATION*

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Abstract. We study the inverse boundary problem for the nonlinear two photon absorption radiative transport equation. We show that the absorption coefficients and the scattering coefficient can be uniquely determined from the *albedo* operator. If the scattering is absent, we do not require smallness of the incoming source, and the reconstruction of the absorption coefficients is explicit.

Key words. two photon absorption, inverse boundary problem, nonlinear transport

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1. Introduction.

In this work we study the inverse boundary problem for the two photon absorption radiative transport equation. Two photon absorption happens when it takes two photons to excite a molecule from one state to another [24, 30]. The probability of a two photon absorption at a given point is proportional to the light intensity there regardless of the incoming direction, which makes the corresponding term quadratic. One of the applications of two photon absorption is in medical imaging: the human body is not transparent to optical rays, but it is more transparent to infrared ones. Then fluorescent dyes with good two photon absorption rates can be used successfully with such a large wavelength excitation; see, e.g., [15, 22]. Other applications are pointed out in [22]—for example, microscopy, microfabrication, three-dimensional data storage, etc. For applications to photoacoustic imaging, we refer the reader to [7] and the references therein.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded convex set with a C^1 boundary $\partial\Omega$, and let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $\Gamma_\pm = \{(x,\theta) \in \partial\Omega \times \mathbb{S}^{n-1} \mid \pm n(x)\cdot\theta > 0\}$, where $n(x)$ is the outer normal at $x \in \partial\Omega$. Denote by $u(x,\theta)$ the photon density function at spatial location $x \in \Omega$ in the direction $\theta \in \mathbb{S}^{n-1}$. Then our model is the following equation (see also [23]):

(1)
$$\begin{aligned} \theta\cdot\nabla_x u(x,\theta) + (\sigma_a(x,\theta) + \sigma_b(x,\theta)|\langle u\rangle|)u(x,\theta) - Ku(x,\theta) &= 0 \qquad \text{in } \Omega\times\mathbb{S}^{n-1},\\ u(x,\theta) &= f_-(x,\theta) \qquad \text{on } \Gamma_-, \end{aligned}$$

where $\langle u\rangle$ is the average of $u(x,\theta)$ over the angular variable θ; that is,

$$\langle u\rangle := \int_{\mathbb{S}^{n-1}} u(x,\theta)d\theta, \tag{2}$$

with $d\theta$ being the normalized surface measure on $\mathbb{S}^{n-1}$. When $u \geq 0$, the absolute value in $|\langle u\rangle|$ does not matter, of course, but for general solutions, we include it to

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have a well-posed problem. The linear operator K is defined by

$$Ku(x,\theta) := \int_{\mathbb{S}^{n-1}} k(x,\theta',\theta)u(x,\theta')d\theta'. \tag{3}$$

The coefficients $\sigma_a(x,\theta), k(x,\theta',\theta)$ are the usual total absorption and scattering coefficients, respectively. The coefficient σ_b stands for strength of the nonlinear effect of two photon absorption, and the term $\sigma_a + \sigma_b\langle u\rangle$ can be understood as the effective total absorption coefficient dependent on the solution. They are all assumed to be nonnegative, and we impose smallness assumptions on k, σ_b, and f_-; see Definition 2.1.

If the direct problem (1) is uniquely solvable, one can define the usual *albedo* operator

$$\mathcal{A} : f_- \mapsto f_+, \tag{4}$$

where $f_+(x,\theta) := u(x,\theta)|_{\Gamma_+}$ denotes the exiting photon density. This albedo operator is nonlinear, and we are interested in finding out whether the albedo operator $\mathcal{A}$ determines uniquely the coefficients $\sigma_a(x,\theta), \sigma_b(x,\theta), k(x,\theta',\theta)$.

When $\sigma_b = 0$, the equation (1) is linear. Uniqueness and recovery formulas for σ_a and k, when σ_a depends on x only, were established in [9] for $n \geq 3$ and in [28] for $n = 2$ under a smallness assumption on k. The general case of $\sigma = \sigma(x,\theta)$ for $n \geq 3$ was resolved in [27]. Stability estimates were proved in [2, 3]. Inverse radiative transport in the Riemannian setting was studied in [16, 17, 18, 19, 20, 25], and for a different dynamical system (see [12]), there are also many other works regarding different types of boundary measurement; see [1, 4, 5, 6, 13, 29, 31] and the references therein. References to earlier works can be found in the survey [26].

Inverse problems for nonlinear versions of the transport equation (different from the one we study here) are studied in [11, 14]. In [23], the authors considered the inverse medium problem under the same nonlinear model as (1) and showed the uniqueness and stability of the reconstruction of absorption coefficients from internal data.

The main result is the following. We show that we can recover σ_a, k, and σ_b given the nonlinear operator $\mathcal{A}$. The idea of the proof is the following. If we take f_- small, then we are in the linear regime and can use the result in [9] to recover σ_a if it depends on x only, and k. The latter requires $n \geq 3$; see also [28] for the 2D case. Next, we can take $f_- = f_0 + \delta f_1$ (see (22)) with $0 < \delta \ll 1$ and $f_0 > 0$ smooth but f_1 singular in the θ variable only. Then f_0 would not create singularities in solution at order $\mathcal{O}(\delta)$, but the effective absorption coefficient would involve $\sigma_b\langle u_0\rangle$, where u_0 is the leading $\mathcal{O}(1)$ term of the solution which is determined by f_0; see (23) and (25). This is the reason we require $f_0 > 0$—so that $\langle u_0\rangle > 0$, and we can divide by it eventually to recover σ_b. Then choosing f_1 concentrated near a single θ' (and independent of x) allows us to reconstruct the X-ray transform of σ_b, and therefore σ_b itself; see Theorem 3.3.

Particularly, when $k = 0$, one can solve the equation (1) directly with f_- in the form of $f_- = v_-(x)\delta_{\theta_0}(\theta)$ (a collimated source); see (56), where $v_- > 0$ smooth. Then we are solving a Riccati ODE along each line $s \mapsto (x_0 + s\theta_0, \theta_0)$. This allows us to recover σ_a if it depends on x only, and σ_b through their attenuated X-ray transforms without the smallness assumption on f_- (or of the perturbation of f_- as in (22)); see Theorem 4.1. This way, we may work with signals which are not necessarily small and will be less sensitive to additive background noise.

The rest of the paper is organized as follows. In section 2, we state the preliminary results about the well posedness of the two photon absorption radiative transport model (1). Section 3 consists of the main theorems about the reconstructions of the absorption and scattering coefficients, respectively. The scattering free case $k = 0$ is discussed in section 4.

2. Preliminaries. We first study the well posedness of (1) and of the albedo operator $\mathcal{A}$. Define $\tau_\pm(x,\theta) := \min\{t \geq 0 \mid x \pm t\theta \in \partial\Omega\}$, which stands for the distance between x and the boundary $\partial\Omega$ along $\pm\theta$. Set $\tau(x,\theta) = \tau_-(x,\theta) + \tau_+(\theta)$, and define the boundary measure $d\xi = |n(x)\cdot\theta| d\mu(x) d\theta$, where $d\mu(x)$ is the Lebesgue measure on $\partial\Omega$. Define the function space

$$\mathcal{H}^1(\Omega \times \mathbb{S}^{n-1}) := \left\{ f \mid f \in L^1(\Omega \times \mathbb{S}^{n-1}) \text{ and } \theta \cdot \nabla_x f \in L^1(\Omega \times \mathbb{S}^{n-1}) \right\}.$$

We further denote the function subspaces $L^1_S(\Gamma_-, d\xi) \subset L^1(\Gamma_-, d\xi)$ by

$$L^1_S(\Gamma_-, d\xi) := \Big\{ f \mid f \in L^1(\Gamma_-, d\xi) \text{ and } \|f\|_* < \infty \Big\}, \tag{5}$$

where the $\|\cdot\|_*$ norm is defined by

$$\|f\|_* := \Big\| \int_{\mathbb{S}^{n-1}} |f(x - \tau_-(x,\theta)\theta, \theta)| d\theta \Big\|_{L^\infty(\Omega)}. \tag{6}$$

Definition 2.1. *We call the tuple of functions* $(\sigma_a, \sigma_b, k, f_-)$ admissible *if*

1. $\sigma_a, \sigma_b \in L^\infty(\Omega \times \mathbb{S}^{n-1})$, $\sigma_a \geq 0$, *and* $\sigma_b \geq 0$*;*
2. $0 \leq k(x, \theta', \theta) \in L^\infty(\Omega \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ *and there exists a constant* $\mu \in [0,1)$ *such that*
$$\|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} \|k\|_{L^\infty(\Omega\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})} \leq \mu;$$
3. $f_- \in L^1_S(\Gamma_-, d\xi)$ *and there exists* $\nu \in [0,1)$ *such that*
$$\|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} \|\sigma_b\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} \|f_-\|_* \leq \nu(1-\mu)^2.$$

Definition 2.2. *We define the following operators:*

$$Tu := -\theta \cdot \nabla_x u, \quad Su := \langle u \rangle, \quad \Sigma(m)u := -(\sigma_a + \sigma_b m)u.$$

Then $\Sigma(m) = -(\sigma_a + \sigma_b m)$*. Let* $J(m) : L^1_S(\Gamma_-, d\xi) \mapsto L^1(\Omega \times \mathbb{S}^{n-1})$ *be defined by*

$$J(m)f_-(x,\theta) = f_-(x - \tau_-(x,\theta)\theta, \theta) \exp\left(\int_0^{\tau_-(x,\theta)} \Sigma(m)(x - l\theta, \theta) dl \right), \tag{7}$$

and let $H(m) : L^1(\Omega \times \mathbb{S}^{n-1}) \to L^1(\Omega \times \mathbb{S}^{n-1})$ *be defined by*

$$H(m)u(x,\theta) = \int_0^{\tau_-(x,\theta)} \exp\left(\int_0^l \Sigma(m)(x - s\theta, \theta) ds \right) Ku(x - l\theta, \theta) dl. \tag{8}$$

Lemma 2.3. *If the coefficients are admissible, then for any* $m \in L^\infty(\Omega)$*,*

$$|H(|m|)u(x,\theta)| \leq \mu \|\langle |u| \rangle\|_{L^\infty(\Omega)}.$$

Proof. Since $K|u|(x,\theta) \leq \|k\|_{L^\infty(\Omega\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}\langle|u|\rangle(x)$, we can derive

$$\begin{aligned}|H(|m|)u(x,\theta)| &= \left|\int_0^{\tau_-(x,\theta)} \exp\left(\int_0^l \Sigma(|m|)(x-s\theta,\theta)ds\right) Ku(x-l\theta,\theta)dl\right| \\ &\leq \int_0^{\tau_-(x,\theta)} \exp\left(\int_0^l \Sigma(|m|)(x-s\theta,\theta)ds\right) K|u|(x-l\theta,\theta)dl \\ &\leq \int_0^{\tau_-(x,\theta)} \|k\|_{L^\infty(\Omega\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}\langle|u|\rangle(x-l\theta)dl \\ &\leq \|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\|k\|_{L^\infty(\Omega\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}\|\langle|u|\rangle\|_{L^\infty(\Omega)} \\ &\leq \mu\|\langle|u|\rangle\|_{L^\infty(\Omega)}. \end{aligned}$$ ∎

In particular, this shows that operator $H(|m|)$ is a contraction in $L^\infty(\Omega, L^1(\mathbb{S}^{n-1}))$.

LEMMA 2.4. *If $(\sigma_a,\sigma_b,k,f_-)$ is admissible and if $m \in L^\infty(\Omega)$, then the linear initial value problem*

$$\begin{aligned}(T+\Sigma(|m|)+K)\,u &= 0 \qquad \text{in } \Omega\times\mathbb{S}^{n-1}, \\ u(x,\theta) &= f_-(x,\theta) \qquad \text{on } \Gamma_-\end{aligned} \tag{9}$$

has a unique solution $u \in L^\infty(\Omega, L^1(\mathbb{S}^{n-1}))\cap\mathcal{H}^1(\Omega\times\mathbb{S}^{n-1})$, and this solution satisfies

$$\|\langle|u|\rangle\|_{L^\infty(\Omega)} \leq \frac{1}{1-\mu}\|f_-\|_*.$$

Proof. The solution u to (9) satisfies

$$u(x,\theta) = H(|m|)u(x,\theta) + J(|m|)f_-(x,\theta), \tag{10}$$

and vice versa, every solution to (10) solves (9) (in a weak sense). Take the absolute value on both sides of (10), and apply the operator S to get, for every $x\in\Omega$,

$$\begin{aligned}\langle|u|\rangle(x) &\leq \int_{\mathbb{S}^{n-1}} |H(|m|)u(x,\theta)|d\theta + \int_{\mathbb{S}^{n-1}} |J(|m|)f_-(x,\theta)|d\theta \\ &\leq \mu\|\langle|u|\rangle\|_{L^\infty(\Omega)} + \int_{\mathbb{S}^{n-1}} |J(|0|)f_-(x,\theta)|d\theta \\ &\leq \mu\|\langle|u|\rangle\|_{L^\infty(\Omega)} + \|f_-\|_*,\end{aligned} \tag{11}$$

where we used Lemma 2.3. The supremum on the left-hand side satisfies

$$\|\langle|u|\rangle\|_{L^\infty(\Omega)} \leq \mu\|\langle|u|\rangle\|_{L^\infty(\Omega)} + \|f_-\|_*. \tag{12}$$

In particular, this shows that the operator $H(|m|)$ is a contraction in $L^\infty(\Omega, L^1(\mathbb{S}^{n-1}))$, and that $J(|m|)f_-(x,\theta)$ belongs to that space; thus (10) is solvable in $L^\infty(\Omega, L^1(\mathbb{S}^{n-1}))$. Moreover, it satisfies the estimate in the lemma by (12). Then we can apply T to (10) to conclude that $u\in\mathcal{H}^1$ and solves (1) in the strong sense. ∎

COROLLARY 2.5. *Under the assumptions of Lemma* 2.4*, the solution u to* (9) *also satisfies*

$$\left\|\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(x,\theta)} |u(x-s\theta,\theta)|dsd\theta\right\|_{L^\infty(\Omega)} \leq \frac{\|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}}{1-\mu}\|f_-\|_*.$$

Proof. Using the estimate from Lemma 2.3, and (10), $\forall s \in [0, \tau_-(x,\theta)]$,

$$\begin{aligned} |u(x-s\theta,\theta)| &\le \mu \|\langle |u| \rangle\|_{L^\infty(\Omega)} + |f_-(x-s\theta-\tau_-(x-s\theta,\theta)\theta,\theta| \\ &= \mu \|\langle |u| \rangle\|_{L^\infty(\Omega)} + |f_-(x-\tau_-(x,\theta)\theta,\theta)|. \end{aligned} \tag{13}$$

Apply the integrals with respect to s and θ; we obtain

$$\int_{\mathbb{S}^{n-1}} \int_0^{\tau_-(x,\theta)} |u(x-s\theta,\theta)| ds d\theta \le \tau_-(x,\theta) \left(\mu \|\langle |u| \rangle\|_{L^\infty(\Omega)} + \int_{\mathbb{S}^{n-1}} |f_-(x-\tau_-(x,\theta)\theta,\theta)| d\theta \right). \tag{14}$$

Then take the supremum on both sides and use the conclusion of Lemma 2.4 to get

$$\begin{aligned} \Big\| \int_{\mathbb{S}^{n-1}} \int_0^{\tau_-(x,\theta)} |u(x-s\theta,\theta)| ds d\theta \Big\|_{L^\infty(\Omega)} &\le \|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} \left(\mu \|\langle |u| \rangle\|_{L^\infty(\Omega)} + \|f_-\|_* \right) \\ &\le \|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})} \frac{1}{1-\mu} \left(\|f_-\|_* \right). \end{aligned} \tag{15}$$

□

LEMMA 2.6. *If $(\sigma_a, \sigma_b, k, f_-)$ is admissible, then the radiative transport equation (1) permits a unique solution $u(x,\theta) \in \mathcal{H}^1(\Omega \times \mathbb{S}^{n-1})$, and*

$$\|\langle |u| \rangle\|_{L^\infty(\Omega)} \le \frac{1}{1-\mu} \|f_-\|_*. \tag{16}$$

In addition, if there exists a constant $c_0 \ge 0$ such that $f_- \ge c_0$, then there is a constant $C = C(\Omega, \sigma_a, \sigma_b, k, f_-) > 0$ such that $u(x,\theta) \ge Cc_0$.

Proof. The proof is based on the Banach fixed point theorem. Define the mapping $\mathcal{C} : L^\infty(\Omega) \mapsto L^\infty(\Omega)$ by $\mathcal{C}(m) := \langle u \rangle$, where $u(x,\theta)$ solves (9). Define the sets of functions $\mathcal{M}$ and $\mathcal{M}_+$ by

$$\mathcal{M} := \left\{ m \in L^\infty(\Omega) : |m(x)| \le \frac{1}{1-\mu} \|f_-\|_* \right\}$$

and

$$\mathcal{M}_+ := \left\{ m \in L^\infty(\Omega) : 0 \le m(x) \le \frac{1}{1-\mu} \|f_-\|_* \right\}.$$

We prove that $\mathcal{C}$ is a contraction mapping on $\mathcal{M}$ (resp., $\mathcal{M}_+$) with the $L^\infty(\Omega)$ metric. First we show that $\mathcal{C} : \mathcal{M} \to \mathcal{M}$. If $m \in \mathcal{M}$, the solution to (9) will satisfy (10). Take the absolute value on both sides of (10) and apply the operator S. By (16), $\langle |u| \rangle \in \mathcal{M}$; hence $\langle u \rangle \in \mathcal{M}$. When $f_- \ge 0$, from the theory of linear transport [10], the solution $u(x,\theta)$ to (10) is nonnegative as well through a fixed point iteration; thus we have the mapping $\mathcal{C} : \mathcal{M}_+ \to \mathcal{M}_+$. Next, we show $\mathcal{C}$ is indeed a contraction mapping on both sets. Let $m_1, m_2 \in \mathcal{M}$ (resp., $\mathcal{M}_+$) and u_1, u_2 be the solutions to (9), respectively. Denote $w = u_1 - u_2$; then

$$\begin{aligned} (T + \Sigma(|m_1|) + K) w &= \sigma_b u_2 (|m_1| - |m_2|), &&\text{in } \Omega \times \mathbb{S}^{n-1}, \\ w(x,\theta) &= 0 &&\text{on } \Gamma_-. \end{aligned} \tag{17}$$

Let $q(x,\theta) := \sigma_b u_2 (|m_1| - |m_2|)$; then the solution $w(x,\theta)$ solves

$$w(x,\theta) = H(|m_1|) w(x,\theta) + \int_0^{\tau_-(x,\theta)} \exp\left(\int_0^l \Sigma(|m_1|)(x-s\theta) ds \right) q(x-l\theta,\theta) dl. \tag{18}$$

Apply the integral operator S on the second term on the right-hand side to get

$$\begin{aligned}\left|\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(x,\theta)}\exp\left(\int_0^l \Sigma(|m_1|)(x-s\theta)ds\right)q(x-l\theta,\theta)dld\theta\right|\\ \leq \|\sigma_b(||m_1|-|m_2||)\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\left|\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(x,\theta)}|u_2(x-l\theta,\theta)|dld\theta\right|\\ \leq \|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\|\sigma_b(||m_1|-|m_2||)\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\frac{1}{1-\mu}\|f_-\|_*.\end{aligned}$$

The last inequality comes directly from Corollary 2.5. As in the proof of Lemma 2.4,

$$\langle|w|\rangle(x)\leq\mu\|\langle|w|\rangle\|_{L^\infty(\Omega)}+\frac{\|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\|\sigma_b|m_1-m_2|\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}}{1-\mu}\|f_-\|_*, \tag{19}$$

where we have used the triangle inequality $||m_1|-|m_2||\leq|m_1-m_2|$. Then use $|\langle w\rangle(x)|\leq\langle|w|\rangle(x)$ and $\langle u_2\rangle\in\mathcal{M}$ (resp., $\mathcal{M}_+$ when $f_-(x,\theta)\geq 0$) to get

$$|\langle w\rangle(x)|\leq\frac{\|\tau\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\|\sigma_b|m_1-m_2|\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}}{(1-\mu)^2}\|f_-\|_*. \tag{20}$$

By condition (3) in Definition 2.1, $\mathcal{C}$ is a contraction mapping on both $\mathcal{M}$ and $\mathcal{M}_+$ with the $L^\infty(\Omega)$ metric. Then by the Banach fixed point theorem, $\mathcal{C}$ has a unique fixed point in $\mathcal{M}$ (resp., $\mathcal{M}_+$ when $f_-(x,\theta)\geq 0$). Then (16) follows from Lemma 2.4. In particular, when $f_-(x,\theta)\geq c_0>0$, then $u(x,\theta)\geq 0$ and $0\leq\langle u\rangle\leq\frac{1}{1-\mu}\|f_-\|_*$; therefore,

$$\begin{aligned}u(x,\theta)=H(\langle u\rangle)u(x,\theta)+J(\langle u\rangle)f_-(x,\theta)\geq J\left(\frac{1}{1-\mu}\|f_-\|_*\right)f_-(x,\theta)\\ \geq c_0\exp\left(-\mathrm{diam}(\Omega)\left(\|\sigma_a\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}+\frac{1}{1-\mu}\|\sigma_b\|_{L^\infty(\Omega\times\mathbb{S}^{n-1})}\|f_-\|_*\right)\right).\end{aligned} \tag{21}$$ ∎

REMARK 2.7. *The mapping $\mathcal{C}$ may not be compact when $f_-\in L^1_S(\Gamma_-,d\xi)$; therefore, the Schauder fixed point theorem does not apply.*

3. Main theorems. In this section, we show that the nonlinear albedo operator determines the three coefficients σ_a, σ_b, k, under the conditions $\sigma_a(x,\theta)=\sigma_a(x)$ and $\sigma_b(x,\theta)=\sigma_b(x)$. In the following, we consider a source function $f_-(x,\theta)$ in the form of

$$f_-(x,\theta)=f_0(x,\theta)+\delta f_1(x,\theta) \tag{22}$$

with $\delta\to 0$ a scaling parameter, with $f_i\in L^1_S(\Gamma_-,d\xi)$ nonnegative, $i=1,2$. Formally, the nonnegative solution u expands as

$$u(x,\theta)=u_0(x,\theta)+\delta u_1(x,\theta)+\delta^2u_2(x,\theta)+\cdots. \tag{23}$$

Then u_0 and u_1 will satisfy the equations

$$\begin{aligned}(T+\Sigma(\langle u_0\rangle)+K)u_0=0 &\quad \text{in } \Omega\times\mathbb{S}^{n-1},\\ u_0(x,\theta)=f_0(x,\theta) &\quad \text{on } \Gamma_-,\end{aligned} \tag{24}$$

and

$$
\begin{aligned}
(T+\Sigma(\langle u_0\rangle)+K)u_1 &= -\sigma_b\langle u_1\rangle u_0 \qquad \text{in } \Omega\times\mathbb{S}^{n-1},\\
u_1(x,\theta) &= f_1(x,\theta) \qquad \text{on } \Gamma_-.
\end{aligned} \tag{25}
$$

When the coefficients are admissible and $f_0=0$, then (24) has unique solution $u_0=0$, and (25) becomes the linear transport equation. Then one can follow the method in [9] to decompose the singularities, which leads to the reconstruction of σ_a and k; the latter requires dimension $n\geq 3$. After the coefficients σ_a and k are recovered, we can select arbitrary nonzero $f_0\in L^1_S(\Gamma_-,d\xi)$ such that u_0 is nonsingular. Then in (25), the most singular part in the solution will come from the source f_1 if we select it to be singular in angular variable θ. Therefore, $\Sigma(\langle u_0\rangle)$ can be recovered, and then u_0 can be solved from (24), which finally reconstructs σ_b. In the following, we rigorously prove these claims.

3.1. Reconstruction of $\boldsymbol{\sigma_a}$. In the next theorem, we show that we can recover the X-ray transform of $\sigma_a(x,\theta)$. As a corollary, if σ_a is θ-independent, one recovers it through the inverse X-ray transform [21].

Here and below, we take sources approximating singular ones in the spirit of [9]. Let B_1 be the unit ball centered at origin in $\mathbb{R}^n$, $h\in C_0^\infty(B_1)$ with $0\leq h\leq 1$, and $h\equiv 1$ near origin be a cut-off function. Given $\theta'\in\mathbb{S}^{n-1}$, define the source function

$$
f_-^{\varepsilon,\delta}(x,\theta;\theta')=\frac{\delta}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right), \tag{26}
$$

where $\delta,\varepsilon>0$ are small parameters such that $f_-^{\varepsilon,\delta}\in L_-^S(\Gamma_-,d\xi)$ and ω_{n-1} is the constant defined by

$$
\omega_{n-1}:=\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}\frac{1}{\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)d\theta. \tag{27}
$$

We view $f_-^{\varepsilon,\delta}$ as δ times an approximation (a Friedrichs mollifier) of the delta function $\delta_{\theta'}(\theta)$ on the sphere. Then $f_-^{\varepsilon,\delta}$ plays the role of δf_1 in (22) with $f_0=0$ there.

THEOREM 3.1. *Let $f_-=f_-^{\varepsilon,\delta}$, and assume the tuple $(\sigma_a,\sigma_b,k,f_-)$ is* admissible*; then*

$$
\lim_{\gamma\to 0}\lim_{\varepsilon,\delta\to 0}\int_{\mathbb{S}^{n-1}}\frac{u^{\varepsilon,\delta}(x,\theta)}{\delta}h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta=\exp\left(-\int_0^{\tau_-(x,\theta')}\sigma_a(x-s\theta',\theta')ds\right),
$$

where $u^{\varepsilon,\delta}$ is the unique solution to (1) *with boundary condition $f_-^{\varepsilon,\delta}$.*

Proof. Let w^ε be the unique solution to the following radiative transport equation:

$$
\begin{aligned}
(T+\Sigma(0)+K)w^\varepsilon &= 0 \qquad \text{in } \Omega\times\mathbb{S}^{n-1},\\
w^\varepsilon(x,\theta) &= \frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right) \qquad \text{on } \Gamma_-.
\end{aligned} \tag{28}
$$

The solution w^ε then satisfies

$$
w^\varepsilon(x,\theta)=\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)\exp\left(-\int_0^{\tau_-(x,\theta)}\sigma_a(x-s\theta,\theta)ds\right)+H(0)w^\varepsilon, \tag{29}
$$

where $|H(0)w^\varepsilon| \leq \mu\|\langle w^\varepsilon\rangle\|_{L^\infty(\Omega)}$, which is uniformly bounded from Lemma 2.3. Therefore, the following iterated limit holds:

$$\begin{aligned}&\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}} w^\varepsilon(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\\ &=\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)\exp\left(-\int_0^{\tau_-(x,\theta)}\sigma_a(x-s\theta,\theta)ds\right)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\\ &\qquad+\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}H(0)w^\varepsilon(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\\ &=\exp\left(-\int_0^{\tau_-(x,\theta')}\sigma_a(x-s\theta',\theta')ds\right).\end{aligned}\tag{30}$$

The term containing $H(0)$ vanishes because when $\gamma\to 0$,

$$\left|\int_{\mathbb{S}^{n-1}}H(0)w^\varepsilon(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\right|\leq\mu\|\langle w^\varepsilon\rangle\|_{L^\infty(\Omega)}\int_{\mathbb{S}^{n-1}}h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\to 0.\tag{31}$$

Denote $\phi=\frac{1}{\delta}u^{\varepsilon,\delta}-w^\varepsilon$; then

$$\begin{aligned}(T+\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)+K)\phi&=\sigma_b|\langle u^{\varepsilon,\delta}\rangle|w^\varepsilon &&\text{in }\Omega\times\mathbb{S}^{n-1},\\ \phi(x,\theta)&=0&&\text{on }\Gamma_-.\end{aligned}\tag{32}$$

Then one can show that $\phi(x,\theta)=\mathcal{L}_1(x,\theta)+\mathcal{L}_2(x,\theta)$, where

$$\begin{aligned}\mathcal{L}_1&=\int_0^{\tau_-(x,\theta)}\exp\left(\int_0^l\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)(x-s\theta,\theta)ds\right)K\phi(x-l\theta,\theta)dl,\\ \mathcal{L}_2&=-\int_0^{\tau_-(x,\theta)}\exp\left(\int_0^l\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)(x-s\theta,\theta)ds\right)\sigma_b|\langle u^{\varepsilon,\delta}\rangle|w^\varepsilon(x-l\theta,\theta)dl.\end{aligned}\tag{33}$$

The first term $\mathcal{L}_1$ is uniformly bounded in the L^∞ norm; this could be derived from Lemmas 2.3 and 2.6 by observing that

$$\frac{1}{\varepsilon^{n-1}}\int_{\mathbb{S}^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)d\theta=\int_{\frac{1}{\varepsilon}\mathbb{S}^{n-1}}h(\theta-\theta')d\theta\leq c|\partial B_1|\tag{34}$$

for some absolute constant $c>0$. Therefore,

$$\int_{\mathbb{S}^{n-1}}\mathcal{L}_1(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta=\mathcal{O}(\gamma^{n-1})\to 0,\ \text{as }\gamma\to 0.\tag{35}$$

For the second term $\mathcal{L}_2$ we have

$$\begin{aligned}\left|\int_{\mathbb{S}^{n-1}}\mathcal{L}_2(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\right|&\leq\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(x,\theta)}\sigma_b|\langle u^{\varepsilon,\delta}\rangle|w^\varepsilon(x-l\theta,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta dl\\ &\leq\|\sigma_b\langle u^{\varepsilon,\delta}\rangle\|_{L^\infty(\Omega)}\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(x,\theta)}w^\varepsilon(x-l\theta,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta dl.\end{aligned}$$

Note that $\|\sigma_b\langle u^{\varepsilon,\delta}\rangle\|_{L^\infty(\Omega)}=\mathcal{O}(\delta)$ by Lemma 2.6, and the integral part is uniformly bounded by the decomposition for w^ε in (29); therefore

$$\lim_{\gamma\to 0}\lim_{\varepsilon,\delta\to 0}\int_{\mathbb{S}^{n-1}}\mathcal{L}_2(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta=0.\tag{36}$$

Combining (30), (35), and (36), we arrive at our conclusion. ∎

3.2. Reconstruction of *k*. We show next that once σ_a is known, one can recover k pointwise.

When $n \geq 3$, we let $\theta, \theta' \in \mathbb{S}^{n-1}$ such that $\theta \nparallel \theta'$ and denote by $\pi_{\theta,\theta'}(x)$ the projection of x onto the subspace Θ spanned by θ, θ'. Let $\theta'_\perp \in \Theta = \mathrm{span}(\theta, \theta')$ be the unit vector such that $\theta'_\perp \cdot \theta' = 0$. Take any $\varphi \in C_0^\infty(-1, 1)$ such that $0 \leq \varphi \leq 1$ and $\int_{\mathbb{R}} \varphi(t)dt = 1$. We then define the test function

$$\phi_{\gamma_1,\gamma_2}(x, \theta, \theta') = \frac{1}{\gamma_1}\varphi\left(\frac{x \cdot \theta'_\perp}{\gamma_1 \theta \cdot \theta'_\perp}\right) h\left(\frac{x - \pi_{\theta,\theta'}(x)}{\gamma_2}\right). \tag{37}$$

We also define the source function $f_-^{\varepsilon,\varepsilon',\delta}$ in the form of

$$f_-^{\varepsilon,\varepsilon',\delta}(x, \theta; x', \theta') = \frac{\delta}{\omega_{n-1}^2 \varepsilon^{n-1}} h\left(\frac{x - x'}{\varepsilon'}\right) h\left(\frac{\theta - \theta'}{\varepsilon}\right) \tag{38}$$

such that $f_-^{\varepsilon,\varepsilon',\delta} \in L_1^S(\Gamma_-, d\xi)$; the constant ω_{n-1} is defined by (27).

THEOREM 3.2. *Let $n \geq 3$, set $f_- = f_-^{\varepsilon,\varepsilon',\delta}$, and assume the tuple $(\sigma_a, \sigma_b, k, f_-)$ is admissible. Then*

$$\begin{aligned}
&\lim_{\gamma_1\to 0}\lim_{\gamma_2\to 0}\lim_{\varepsilon'\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\partial\Omega} \frac{u^{\varepsilon,\varepsilon',\delta}(x + \tau_+(x,\theta)\theta, \theta; x', \theta')}{\varepsilon'^{n-1}\delta}\phi_{\gamma_1,\gamma_2}(x' - x + \tau_-(x,\theta')\theta', \theta, \theta')d\mu(x')\\
&= \exp\left(-\int_0^{\tau^+(x,\theta)} \sigma_a(x + s\theta)ds\right)\exp\left(-\int_0^{\tau^-(x,\theta')} \sigma_a(x - s\theta')ds\right)k(x, \theta', \theta),
\end{aligned}$$

where $u^{\varepsilon,\varepsilon',\delta}(x, \theta; x', \theta')$ is the unique solution to (1) *with boundary condition $f_-^{\varepsilon,\varepsilon',\delta}$. The limit holds in $L^1_{loc}(\Omega \times (\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}\backslash D))$ where $D = \{(\theta, \theta') \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \mid \theta \nparallel \theta'\}$.*

Proof. Similarly to section 3 of [9], we can write the decomposed solution as

$$\begin{aligned}
u^{\varepsilon,\varepsilon',\delta}(x, \theta) &= J(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)f_- + H(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)J(|\langle u^{\varepsilon,\delta}\rangle|)f_-\\
&\quad + (I - H(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|))^{-1}H^2(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)J(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)f_-\\
&= \mathcal{L}_1(x, \theta) + \mathcal{L}_2(x, \theta) + \mathcal{L}_3(x, \theta),
\end{aligned} \tag{39}$$

with the terms there corresponding to the ballistic, the single-scattering, and the multiple-scattering components. First, it is simple to see that when ε is small enough so that $|\theta - \theta'| > \varepsilon$, then $h(\frac{\theta-\theta'}{\varepsilon}) = 0$; hence

$$\begin{aligned}
&\int_{\partial\Omega} \frac{\mathcal{L}_1(x + \tau_+(x,\theta)\theta, \theta)}{\varepsilon'^{n-1}\delta}\phi_{\gamma_1,\gamma_2}(x' - x + \tau_-(x,\theta')\theta', \theta, \theta')d\mu(x')\\
&\quad= \int_{\partial\Omega} \frac{1}{\omega_{n-1}^2\varepsilon'^{n-1}\varepsilon^{n-1}} h\left(\frac{x - \tau_-(x,\theta)\theta - x'}{\varepsilon}\right) h\left(\frac{\theta - \theta'}{\varepsilon}\right)\\
&\qquad\times \exp\left(-\int_0^{\tau(x,\theta)} \Sigma(|\langle u^{\varepsilon,\delta}\rangle|)(x - s\theta, \theta)ds\right)\phi_{\gamma_1,\gamma_2}(x' - x + \tau_-(x,\theta')\theta', \theta, \theta')d\mu(x')\\
&\quad= 0.
\end{aligned} \tag{40}$$

Next, we compute the contribution of the single-scattering term. Let $E(x, y, m)$ denote

$$E(x, y, m) = \exp\left(|x - y|\int_0^1 \Sigma(m)(x + s(y - x))ds\right).$$

In order to make the derivation concise, we also introduce the following notation:

$$
\begin{aligned}
x_{\pm,\theta} &= x \pm \tau_{\pm}(x,\theta)\theta,\\
y_{l,\theta} &= x_{+,\theta} - l\theta,\\
z_{l,\theta,\theta''} &= y_{l,\theta} - \tau_{-}(y_l,\theta'')\theta''.
\end{aligned}
\tag{41}
$$

Then we can write

$$
\begin{aligned}
&\lim_{\varepsilon'\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\partial\Omega}\frac{\mathcal{L}_2(x+\tau_+(x,\theta)\theta,\theta)}{\varepsilon'^{n-1}\delta}\phi_{\gamma_1,\gamma_2}(x'-x+\tau_-(x,\theta')\theta',\theta,\theta')d\mu(x')\\
&= \lim_{\varepsilon'\to 0}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{\partial\Omega}\int_0^{\tau(x,\theta)}\int_{\mathbb{S}^{n-1}} E(x_{+,\theta},y_{l,\theta},|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)E(y_{l,\theta},z_{l,\theta,\theta''},|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)\\
&\quad\times\frac{1}{\omega_{n-1}\varepsilon'^{n-1}}h\left(\frac{z_{\theta''}-x'}{\varepsilon'}\right)\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta''-\theta'}{\varepsilon}\right)k(y_{l,\theta},\theta'',\theta)\\
&\quad\times\phi_{\gamma_1,\gamma_2}(x'-x+\tau_-(x,\theta')\theta',\theta,\theta')d\theta''dld\mu(x')\\
&= \lim_{\varepsilon'\to 0}\lim_{\varepsilon\to 0}\int_{\partial\Omega}\int_0^{\tau(x,\theta)}\int_{\mathbb{S}^{n-1}} E(x_{+,\theta},y_{l,\theta},0)E(y_{l,\theta},z_{l,\theta,\theta''},0)\\
&\quad\times\frac{1}{\omega_{n-1}\varepsilon'^{n-1}}h\left(\frac{z_{\theta''}-x'}{\varepsilon'}\right)\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta''-\theta'}{\varepsilon}\right)k(y_{l,\theta},\theta'',\theta)\\
&\quad\times\phi_{\gamma_1,\gamma_2}(x'-x+\tau_-(x,\theta')\theta',\theta,\theta')d\theta''dld\mu(x')\\
&= \lim_{\varepsilon'\to 0}\int_{\partial\Omega}\int_0^{\tau(x,\theta)} E(x_{+,\theta},y_{l,\theta},0)E(y_{l,\theta},z_{l,\theta,\theta'},0)\\
&\quad\times\frac{1}{\omega_{n-1}\varepsilon'^{n-1}}h\left(\frac{z_{\theta'}-x'}{\varepsilon'}\right)k(y_{l,\theta},\theta',\theta)\\
&\quad\times\phi_{\gamma_1,\gamma_2}(x'-x+\tau_-(x,\theta')\theta',\theta,\theta')dld\mu(x')\\
&= \int_0^{\tau(x,\theta)} E(x_{+,\theta},y_{l,\theta},0)E(y_{l,\theta},z_{l,\theta,\theta'},0)k(y_{l,\theta},\theta',\theta)\\
&\quad\times\phi_{\gamma_1,\gamma_2}(y_{l,\theta}-x+\tau_-(x,\theta')\theta',\theta,\theta')dl.
\end{aligned}
$$

The right-hand side has the limit

$$
\begin{aligned}
&\lim_{\gamma_1\to 0}\lim_{\gamma_2\to 0}\int_0^{\tau(x,\theta)} E(x_{+,\theta},y_{l,\theta},0)E(y_{l,\theta},z_{l,\theta,\theta'},0)k(y_{l,\theta},\theta',\theta)\\
&\qquad\times\phi_{\gamma_1,\gamma_2}(y_{l,\theta}-x+\tau_-(x,\theta')\theta',\theta,\theta')dl\\
&\qquad= \lim_{\gamma_1\to 0}\int_0^{\tau(x,\theta)} E(x_{+,\theta},y_{l,\theta},0)E(y_{l,\theta},z_{l,\theta,\theta'},0)k(y_{l,\theta},\theta',\theta)\frac{1}{\gamma_1}\varphi\left(\frac{\tau^+(x,\theta)-l}{\gamma_1}\right)dl\\
&\qquad= E(x_{+,\theta},x,0)E(x,x_{-,\theta'},0)k(x,\theta',\theta).
\end{aligned}
\tag{42}
$$

To show that the multiscattering contribution is zero, we only need to show that $\frac{1}{(\varepsilon')^{n-1}\delta}\mathcal{L}_3(x,\theta)\in L^1(\Omega\times\mathbb{S}^{n-1})$ uniformly and hence is uniformly bounded in $L^1(\Gamma_\pm,d\xi)$. Given any $\chi\in C_0^\infty(\Omega\times(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}\backslash D))$, we have

$$
\begin{aligned}
&\int_{\Omega\times\mathbb{S}^{n-1}\times\Gamma_-}\frac{\mathcal{L}_3(x+\tau_+(x,\theta),\theta;x',\theta')}{\varepsilon'^{n-1}\delta}\phi_{\gamma_1,\gamma_2}(x'-x+\tau_-(x,\theta')\theta',\theta,\theta')\chi d\xi(x',\theta')d\mu(x)d\theta\\
&\le\frac{1}{\gamma_1}\int_{T_{\gamma_2}}\frac{\mathcal{L}_3(x+\tau_+(x,\theta),\theta;x',\theta')}{\varepsilon'^{n-1}\delta}\chi d\mu(x)d\theta d\xi(x',\theta'),
\end{aligned}
\tag{43}
$$

where $T_{\gamma_2} = \{(x,\theta,x',\theta') \in \Omega\times\mathbb{S}^{n-1}\times\Gamma_-\cap \operatorname{supp}\chi$ and $|x-x'-\pi_{\theta,\theta'}(x-x')| \le c\gamma_2\}$. When $\frac{1}{(\varepsilon')^{n-1}\delta}\mathcal{L}_3(x,\theta)$ is uniformly bounded in $L^1(\Gamma_+,d\xi)$, the integrand of (43) is an L^1 function. On the other hand, $\operatorname{meas}(T_{\gamma_2}) \to 0$ as $\gamma_2 \to 0$; therefore the integral vanishes as $\gamma_2 \to 0$. In the following, we prove $\frac{1}{(\varepsilon')^{n-1}\delta}\mathcal{L}_3(x,\theta) \in L^1(\Omega\times\mathbb{S}^{n-1})$ with a uniform bound there with respect to $\varepsilon' \ll 1$ and $\delta \ll 1$.

Since $(I - H(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|))^{-1}$ is a uniformly bounded operator in $L^1(\Omega\times\mathbb{S}^{n-1})$, we merely have to show that $\frac{1}{(\varepsilon')^{n-1}\delta}H^2(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)J(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)f_-$ is also uniformly bounded; see (39). Let $y_{l,\theta} = x - l\theta$, $z_{s,\theta''} = y_{l,\theta} - s\theta''$, and $w_{\theta'''} = z_{s,\theta''} - \tau_-(z_{s,\theta''},\theta''')\theta'''$. Then

(44)
$$\begin{aligned}
&\left|\frac{H^2(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)J(|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)f_-(x,\theta)}{\varepsilon'^{n-1}\delta}\right| \\
&\le \int_0^{\tau_-(x,\theta)}\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(y_{l,\theta},\theta'')}\int_{\mathbb{S}^{n-1}} E(x,y_{l,\theta},|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)E(y_{l,\theta},z_{s,\theta''},|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|) \\
&\quad\times E(z_{s,\theta''},w_{\theta'''},|\langle u^{\varepsilon,\varepsilon',\delta}\rangle|)k(y_{l,\theta},\theta'',\theta)k(z_{s,\theta''},\theta''',\theta'')|f_-(w_{\theta'''},\theta''')|d\theta''' ds d\theta'' dl \\
&\le \int_0^{\tau_-(x,\theta)}\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(y_{l,\theta},\theta'')}\int_{\mathbb{S}^{n-1}} k(y_{l,\theta},\theta'',\theta)k(z_{s,\theta''},\theta''',\theta'')|f_-(w_{\theta'''},\theta''')|d\theta''' ds d\theta'' dl.
\end{aligned}$$

Since $z_{s,\theta''} = y_{l,\theta} - s\theta''$, we change the variable such that $dz_{s,\theta''} = s^{n-1}dsd\theta''$, and we recall the formula

$$\int_{\Omega\times\mathbb{S}^{n-1}} g(x,\theta)dxd\theta = \int_{\Gamma_-}\int_0^{\tau_+(x',\theta)} g(x'+t\theta,\theta)dtd\xi(x',\theta) \tag{45}$$

(see [9]) with $x' = x - \tau_-(x,\theta)\theta$. We obtain

(46)
$$\begin{aligned}
&\int_0^{\tau_-(x,\theta)}\int_{\mathbb{S}^{n-1}}\int_0^{\tau_-(y_{l,\theta},\theta'')}\int_{\mathbb{S}^{n-1}} k(y_{l,\theta},\theta'',\theta)k(z_{s,\theta''},\theta''',\theta'')|f_-(w_{\theta'''},\theta''')|d\theta''' ds d\theta'' dl \\
&= \int_0^{\tau_-(x,\theta)}\int_{\Gamma_-}\int_0^{\tau_+(w_{\theta'''},\theta''')} k(y_{l,\theta},\theta'',\theta)k(w_{\theta'''}+t\theta''',\theta''',\theta'') \\
&\quad\times \frac{1}{\varepsilon'^{n-1}}|f_-(w_{\theta'''},\theta''')|s^{1-n}dtd\xi(w_{\theta'''},\theta''')dl \\
&\le C\left\|\frac{1}{\varepsilon'^{n-1}}f_-\right\|_{L^1(\Gamma_-,d\xi)}\int_0^{\tau_-(x,\theta)}\int_0^{\tau_+(x',\theta')} s^{1-n}dtdl \in L^1(\Omega\times\mathbb{S}^{n-1}),
\end{aligned}$$

which is uniformly bounded in $L^1(\Omega\times\mathbb{S}^{n-1})$ with respect to ε', where $s = |y_{l,\theta} - (x'+t\theta')|$ and $C = \|k\|^2_{L^\infty(\Omega\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}$. ∎

3.3. Reconstruction of $\boldsymbol{\sigma_b}$. Let the source function f_- be chosen in the following form:

$$f_-^{\varepsilon,\delta}(x,\theta;\theta') = c_0 + \frac{\delta}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right), \tag{47}$$

where c_0 is a positive constant and δ,ε are positive small parameters. Compared with (26), here we have added $f_0 = c_0$ in (22).

Theorem 3.3. *Let $f_- = f_-^{\varepsilon,\delta}$, and assume the tuple $(\sigma_a,\sigma_b,k,f_-)$ is admissible; then*

(48)
$$\lim_{\delta\to0}\lim_{\gamma\to0}\lim_{\varepsilon\to0}\int_{\mathbb{S}^{n-1}}\frac{u^{\varepsilon,\delta}(x,\theta)}{\delta}h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta = \exp\left(\int_0^{\tau_-(x,\theta')}\Sigma(|\langle w\rangle|)(x-s\theta')ds\right),$$

where $u^{\varepsilon,\delta}$ is the unique solution to (1) *with boundary condition f_-, and w is the unique solution to* (1) *with the boundary condition $f_- = c_0$.*

Proof. Let $w(x,\theta)$ be the solution to the following equation:

$$\begin{aligned}(T+\Sigma(|\langle w\rangle|)+K)w &= 0 \qquad \text{in } \Omega\times\mathbb{S}^{n-1},\\ w(x,\theta) &= c_0 \qquad \text{on } \Gamma_-.\end{aligned} \tag{49}$$

Then $w\in L^\infty(\Omega\times\mathbb{S}^{n-1})$, which implies

$$\lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}\frac{w(x,\theta)}{\delta}h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta = 0. \tag{50}$$

We denote $\phi = \frac{1}{\delta}\left(u^{\varepsilon,\delta}-w\right)$. It satisfies

$$\begin{aligned}(T+\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)+K)\phi &= \sigma_b\left(\frac{|\langle u^{\varepsilon,\delta}\rangle|-|\langle w\rangle|}{\delta}\right)w \qquad \text{in } \Omega\times\mathbb{S}^{n-1},\\ \phi(x,\theta) &= \frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right) \qquad \text{on } \Gamma_-.\end{aligned} \tag{51}$$

Therefore, the solution ϕ can be written in the following form:

$$\begin{aligned}\phi(x,\theta) &= \exp\left(\int_0^{\tau_-(x,\theta)}\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)(x-s\theta)ds\right)\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)\\ &\quad+\int_0^{\tau_-(x,\theta)}\exp\left(\int_0^l\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)(x-s\theta)ds\right)K\phi(x-l\theta,\theta)dl\\ &\quad-\int_0^{\tau_-(x,\theta)}\exp\left(\int_0^l\Sigma(|\langle u^{\varepsilon,\delta}\rangle|)(x-s\theta)ds\right)\left[\sigma_b\left(\frac{|\langle u^{\varepsilon,\delta}\rangle|-|\langle w\rangle|}{\delta}\right)w(x-l\theta,\theta)\right]dl\\ &= \mathcal{L}_1(x,\theta)+\mathcal{L}_2(x,\theta)+\mathcal{L}_3(x,\theta).\end{aligned}$$

Integrate $\phi(x,\theta)$ over $\mathbb{S}^{n-1}$ and note that $||\langle u^{\varepsilon,\delta}\rangle|-|\langle w\rangle||\le\delta|\langle\phi\rangle|$ to obtain

$$\|\langle\phi\rangle\|_{L^\infty(\Omega)}\le\frac{1}{(1-\mu)(1-\nu)}\left|\int_{\mathbb{S}^{n-1}}\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)d\theta\right|. \tag{52}$$

This implies that $\mathcal{L}_2,\mathcal{L}_3$ are both uniformly bounded in $L^\infty(\Omega\times\mathbb{S}^{n-1})$; hence

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}\phi(x,\theta)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\\ &=\lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}\mathcal{L}_1(x,\theta)\frac{1}{\omega_{n-1}\varepsilon^{n-1}}h\left(\frac{\theta-\theta'}{\varepsilon}\right)h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\\ &=\lim_{\delta\to 0}\exp\left(\int_0^{\tau_-(x,\theta')}\Sigma(|\langle u^{\varepsilon,\theta'}\rangle|)(x-s\theta')ds\right)\\ &=\exp\left(\int_0^{\tau_-(x,\theta')}\Sigma(|\langle w\rangle|)(x-s\theta')ds\right).\end{aligned} \tag{53}$$

Combine this with (50) to obtain

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^{n-1}}\frac{u^{\varepsilon,\delta}(x,\theta)}{\delta}h\left(\frac{\theta-\theta'}{\gamma}\right)d\theta\\ &\qquad=\exp\left(\int_0^{\tau_-(x,\theta')}\Sigma(|\langle w\rangle|)(x-s\theta')ds\right).\end{aligned} \tag{54}$$

□

Theorem 3.3 implies that $\Sigma(|w|)$ can be reconstructed from the albedo operator. Therefore, the solution w of (49) can be uniquely determined, and there exists a constant $C > 0$ such that $w(x, \theta) \geq Cc_0$ by Lemma 2.6; and when σ_a is known, one can find $\sigma_b = (\Sigma(|w|) - \sigma_a)/|\langle w\rangle|$.

4. Scattering-free media. For media with $k = 0$, there exists a more direct explicit reconstruction method. Moreover, no smallness assumptions on the boundary source are needed. Equation (1) reduces to

$$\theta \cdot \nabla u + \sigma_a u + \sigma_b \langle u\rangle u = 0. \tag{55}$$

Choose the boundary condition

$$f_- = v_-(x)\delta_{\theta_0}(\theta) \tag{56}$$

in (1) with some $v_-(x) \geq 0$ in C^1. We are going to look for a nonnegative weak solution, i.e., for a solution of the integrated equation

$$u(x, \theta) = v_-(x - \tau_-(x, \theta)\theta)\delta_{\theta_0}(\theta) \exp\left(-\int_0^{\tau_-(x,\theta)} (\sigma_a + \sigma_b\langle u\rangle)(x - s\theta)ds\right) \tag{57}$$

in the following class: $u(x, \theta)$ is a measure-valued function in θ, $C^1(\Omega) \cap C(\bar{\Omega})$ in the x variable. Then $\langle u\rangle(x)$ is in the latter space. By (57), $u = \delta_{\theta_0}(\theta)v$ with $v \in C(\bar{\Omega} \times \mathbb{S}^{n-1})$; also, v is C^1 except for (x, θ) such that $x \in \partial\Omega$ and θ is tangent to $\partial\Omega$ (which is $\partial\Gamma_0$). Clearly, only the value of v at $\theta = \theta_0$ matters for u. With some abuse of notation, we denote $v(x, \theta_0)$ by $v(x)$. Then by (56), v must satisfy the boundary condition $v = v_-$ on $\partial\Omega$.

In view of the C^1 regularity of v as stated above, we can differentiate (57) to get back to the differential form (55), which in this case reduces to

$$\left(\theta_0 \cdot \nabla v + \sigma_a v + \sigma_b v^2\right) = 0, \tag{58}$$

since $\langle u\rangle = v$. Here, σ_a and σ_b can depend on θ as well; then $\theta = \theta_0$ above. Therefore, on each line $s \mapsto (x_0 + s\theta_0, \theta_0)$, the equation reduces to

$$v' + \sigma_a v + \sigma_b v^2 = 0. \tag{59}$$

This is a homogeneous Riccati equation. For each initial condition $v(0) = v_-(x_0)$, we measure $v(\tau_+(x, \theta_0))$.

Let $\mu(t) = \exp\left(-\int_0^t \sigma_a(s)\,ds\right)$; then $1/\mu$ is the integrating factor. Multiply (59) by $1/\mu$ to get

$$(v/\mu)' + \sigma_b v^2/\mu = 0. \tag{60}$$

This is a separable ODE for v/μ, and the solution satisfies

$$\frac{\mu}{v} = \frac{1}{v_-(x_0)} + \int_0^s \mu(t)\sigma_b(t)\,dt; \tag{61}$$

therefore,

$$v(s) = \mu(s)\left(\frac{1}{v_-(x_0)} + \int_0^s \mu(t)\sigma_b(t)\,dt\right)^{-1}. \tag{62}$$

Hence, at $s = \tau_+(x_0, \theta_0)$ we recover the attenuated X-ray transform of σ_b with attenuation σ_a, assuming σ_a is known. One way to recover σ_a is to replace $v_-(x_0)$ by $\delta v_-(x_0)$ as in the previous section with $\delta \to 0$; then we get the X-ray transform $-\log \mu(\tau_+(x, \theta_0))$ of σ_a; and by varying θ, we can recover σ_a. Then we recover σ_b by inverting the attenuated X-ray transform of σ_b; see [8, 21].

If we do not want to deal with small signals which may be corrupted by background noise, we can proceed as follows. To reconstruct σ_a, we choose two distinct boundary sources $f_{-,j} = v_{-,j}(x)\delta_{\theta_0}(\theta)$, $j = 1, 2$, such that $\forall x \in \partial\Omega$, $v_{-,1}(x) > v_{-,2}(x)$. Let v_1, v_2 be the solutions to (59) with $v_j(0) = v_{-,j}(x_0)$; then from (62) we observe that

$$\frac{1}{v_j(s)} = \frac{1}{\mu(s)} \left(\frac{1}{v_{-,j}(x_0)} + \int_0^s \mu(t)\sigma_b(t)\, dt \right), \quad j = 1, 2. \tag{63}$$

Subtracting the above formulas with $j = 1, 2$, we obtain

$$\frac{1}{v_1(s)} - \frac{1}{v_2(s)} = \frac{1}{\mu(s)} \left(\frac{1}{v_{-,1}(x_0)} - \frac{1}{v_{-,2}(x_0)} \right), \tag{64}$$

which implies

$$\mu(s) = \left(\frac{1}{v_1(s)} - \frac{1}{v_2(s)} \right)^{-1} \left(\frac{1}{v_{-,1}(x_0)} - \frac{1}{v_{-,2}(x_0)} \right). \tag{65}$$

Take $s = \tau_+(x_0, \theta_0)$ to get $\mu(\tau_+(x_0, \theta_0)) = \exp(-X\sigma_a(x_0, \theta_0))$, where X is the X-ray transform, which can be determined by (65). Therefore, we can recover σ_a first by varying θ_0 and inverting the X-ray transform of σ_a as above. After that, we recover σ_b as above.

Also, one can take $v_-(x_0)$ approximating $\delta_{x_0}(x)$; this corresponds to a single beam.

Therefore, we proved the following.

THEOREM 4.1. *Assume $k = 0$. Let σ_a and σ_b depend only on x and be in $C^0(\overline{\Omega})$. Then $\mathcal{A}$ acting on f_- as in* (56) *determines σ_a, σ_b uniquely by inverting their attenuated, respectively, nonattenuated, X-ray transforms, which can be determined by* (62) *and* (65).

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