

ALGEBRAIC GROUPS AS DIFFERENCE GALOIS GROUPS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the inverse problem in the difference Galois theory of linear differential equations over the difference-differential field $\mathbb{C}(x)$ with derivation $\frac{d}{dx}$ and endomorphism $f(x) \mapsto f(x+1)$. Our main result is that every linear algebraic group, considered as a difference algebraic group, occurs as the difference Galois group of some linear differential equation over $\mathbb{C}(x)$.

INTRODUCTION

The Galois group of a polynomial over a field is a finite group. The inverse problem in the Galois theory of polynomials asks to determine, for a given field, which finite groups occur. For example, for the field $\mathbb{C}(x)$ of rational functions over $\mathbb{C}$, it is known that every finite group occurs (see e.g., [Sza09, Cor. 3.4.4]).

The Galois group of a linear differential equation over a differential field is a linear algebraic group. The inverse problem in the Galois theory of linear differential equations asks to determine, for a given differential field, which linear algebraic groups occur. For example, for the field $\mathbb{C}(x)$ with derivation $\frac{d}{dx}$, it is known that every linear algebraic group occurs. This was first proved in [TT79], based on the solution of Hilbert's 21st problem.

A difference-differential field is a field equipped with two commuting operators, a derivation and an endomorphism, usually denoted with σ . The σ -Galois group of a linear differential equation over a difference-differential field is a linear difference algebraic group, i.e., a subgroup of a general linear group defined by algebraic difference equations in the matrix entries. The problem we are concerned with in this article is the inverse problem in the σ -Galois theory of linear differential equations. It asks to determine, for a given difference-differential field, which difference algebraic groups occur. We are mainly interested in the difference-differential field $\mathbb{C}(x)$ with derivation $\frac{d}{dx}$ and endomorphism $\sigma: \mathbb{C}(x) \rightarrow \mathbb{C}(x)$, $f(x) \mapsto f(x+1)$.

As we will show, not every difference algebraic group occurs as a σ -Galois group of a linear differential equation over $\mathbb{C}(x)$. In fact, constant subgroups of unipotent linear algebraic

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groups do not occur (Corollary 6.16) and moreover, we isolate two properties that any σ -Galois group over $\mathbb{C}(x)$ must have (Theorem 6.1). On the positive side, our main result is the following:

Theorem (Theorem 5.2). *Every linear algebraic group, considered as a difference algebraic group, occurs as a σ -Galois group over $\mathbb{C}(x)$.*

If a linear algebraic group G , considered as a difference algebraic group, occurs as a σ -Galois group of a linear differential equation $y' = Ay$, then the Galois group of the linear differential equation $y' = Ay$ is the linear algebraic group G . Thus the above theorem generalizes the solution of the inverse problem in the Galois theory of linear differential equations over $\mathbb{C}(x)$.

Our main tool for the proof of the above theorem is patching. In fact, we establish a general patching result (Theorem 3.12) for σ -Picard-Vessiot rings over difference-differential fields that we deem of independent interest. Here a σ -Picard-Vessiot ring is the analog of the splitting field in the Galois theory of polynomials. This patching result is analogous to known patching results in the Galois theories of linear differential equations ([BHH16, Theorem 2.4]) and parameterized linear differential equations ([Mai15, Theorem 2.2]) that turned out to be very useful in the study of the corresponding inverse problems. We therefore expect Theorem 3.12 to have further applications in the study of the inverse problem in the σ -Galois theory of linear differential equations.

To put our results into perspective, let us review the state of the art of the inverse problem in the various Galois theories. The three most relevant Galois theories for us are the following:

- (i) The Galois theory of linear differential equations, where the Galois groups are linear algebraic groups. See e.g. [vdPS03].
- (ii) The Galois theory of parameterized linear differential equations, where the Galois groups are differential algebraic groups. See [CS07] and [Lan08].
- (iii) The σ -Galois theory of linear differential equations, where the Galois groups are difference algebraic groups. See [DVHW14].

Galois theory of linear differential equations. Building on work of several authors, the inverse problem in the Galois theory of linear differential equations over $k(x)$, where k is an algebraically closed field of characteristic zero, was eventually solved in [Har05]: All linear algebraic groups over k occur as Galois groups. For non-algebraically closed fields k , there are only partial results. For example, if k is a Laurent series field, all linear algebraic groups over k occur as Galois groups ([BHH16, Theorem 4.14]) and the same holds for $k = \mathbb{Q}_p$ ([BHHP20]).

Going beyond the solution of the inverse problem, there has been recent progress in the study of differential embedding problems and the structure of the absolute differential Galois group of $k(x)$ using patching techniques. See [BHHW18, BHH18, BHHP20, Wib20, BHHW].

Galois theory of parameterized linear differential equations. If the coefficients of a linear differential equation depend on an auxiliary parameter, one can differentiate the solutions with respect to this parameter. The Galois group of a parameterized linear differential equation is a differential algebraic group that measures the algebraic relations among the

solutions and their derivatives with respect to the auxiliary parameter. From an algebraic perspective, this setup is modeled by considering a field K with two commuting derivations δ and ∂ . A typical example is $K = \mathbb{C}(t, x)$, where x is the main variable and t the parameter, i.e., we are interested in linear differential equations with respect to $\delta = \frac{d}{dx}$ and the $\partial = \frac{d}{dt}$ -derivatives of their solutions. More generally, if k is a field equipped with a derivation ∂ , then $K = k(x)$ is naturally equipped with two commuting derivations, $\delta = \frac{d}{dx}$ and ∂ , where $\partial(x) = 0$. This Galois theory and its variants have proven to be very useful in questions of hypertranscendence. See [HS08], [Arr13], [HMO17], [DHR18], [DV12].

The inverse problem in this Galois theory is not well-understood. For example, it is not known which differential algebraic groups occur as Galois groups over $\mathbb{C}(t, x)$. The most comprehensive result is only available under strong assumptions on k : Building on [MS12], it was shown in [Dre14] that if k is a universal ∂ -field, then a differential algebraic group over k is a Galois group over $k(x)$ if and only if it is the Kolchin closure of a finitely generated subgroup. For certain differential algebraic groups, including linear algebraic groups, the latter condition was translated into group theoretic conditions in [Sin13] and [MOS14].

For k a Laurent series field, it was shown in [Mai15] that a large class of linear algebraic groups, considered as differential algebraic groups, occur as Galois groups. Also certain differential algebraic groups occur in this situation ([Bac18]). On the other hand, it is also shown in [Bac18] that many differential algebraic subgroups of the additive or the multiplicative group do not occur over $k(x)$, unless the ∂ -field k is fairly big.

σ -Galois theory of linear differential equations. The σ -Galois theory of linear differential equations is similar to the Galois theory of parameterized linear differential equations. Again one considers a linear differential equation depending on a parameter t . But instead of deriving the solutions with respect to t , one applies a discrete transformation to t and the solutions, e.g., $t \mapsto t + 1$. The Galois groups are difference algebraic groups and they measure the algebraic relations among the solutions and their transforms under a discrete transformation usually denoted with σ . From an algebraic perspective, this setup is modeled by considering a field K , with two commuting operators, a derivation δ and an endomorphism σ . The inverse problem in this Galois theory is wide open. It appears that beyond some initial observations in [DVHW17] nothing is known. In this paper, for the first time, a significant class of difference algebraic groups is shown to occur as Galois groups.

It should also be noted that in [AS17], the authors consider δ -Galois groups of linear *difference* equations over $\mathbb{C}(x)$ with respect to the difference operator either the shift $x \mapsto x + 1$, a q -dilation $x \mapsto qx$ or a Mahler operator $x \mapsto x^q$. For certain classes of solvable differential algebraic groups, they characterize which groups occur as δ -Galois groups. It seems plausible that using similar methods as in [AS17] it could also be shown that certain solvable difference algebraic groups (that are not algebraic groups) occur as σ -Galois groups.

The direct problem in the above Galois theories is to compute the Galois group of a given (parameterized) linear differential equation. We note that progress in the inverse problem can be helpful for the direct problem. For example, if it is already known that the Galois group of a given (parameterized) differential equation is non-trivial and contained in a certain group G , the information, that no non-trivial subgroup of G is a Galois group would already

imply that the searched for Galois group equals G . As we show (Prop. 6.14) this occurs, e.g. for the additive group $G = \mathbb{G}_a$.

We conclude the introduction with an overview of the article. In the first section we recall and establish some basic preparatory definitions and results concerning difference algebraic groups and the σ -Galois theory of linear differential equations. The main patching result (Theorem 3.12) is established in Section 3. Roughly speaking, it states that if two difference algebraic subgroups H_1 and H_2 of some difference algebraic group occur as Galois groups in a certain compatible fashion, then also the difference algebraic group generated by H_1 and H_2 occurs. It follows from this result that to realize all linear algebraic groups (considered as difference algebraic groups) as Galois groups it suffices to realize certain building blocks, namely, the multiplicative group, the additive group and finite cyclic groups. Building on work in the second section, these building blocks are then dealt with in Section 4. Our main result, that all linear algebraic groups (considered as difference algebraic groups) occur as Galois groups over $\mathbb{C}(x)$, where $\sigma(f(x)) = f(x+1)$, is then established in Section 5. In the final section we show that not all difference algebraic groups occur as Galois groups over $\mathbb{C}(x)$. In fact, we isolate two properties that any Galois group over $\mathbb{C}(x)$ must have. The first property is that it has to be σ -reduced. This follows rather directly from the fact that $\sigma: \mathbb{C}(x) \rightarrow \mathbb{C}(x)$ is bijective. The second property is σ -connectedness. This boils down to the fact that $\mathbb{C}(x)$ does not have any finite difference field extensions. We also show that no proper non-trivial difference algebraic subgroup of the additive group $\mathbb{G}_a$ is a σ -Galois group over $\mathbb{C}(x)$ and deduce from this that also the constant points of unipotent linear algebraic groups do not occur.

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1. BASICS ON DIFFERENCE GALOIS THEORY

In this section we recall the necessary definitions concerning difference algebraic groups and σ -Picard-Vessiot theory. We also establish some results of a preparatory nature.

All rings are assumed to be commutative and unital.

1.1. Difference algebraic groups. We begin by recalling some basic notions from difference algebra. Standard references for difference algebra are [Coh65] and [Lev08]. For more background on difference algebraic groups see [DVHW14, Appendix A] or [Wib].

A *difference ring*, or σ -ring for short, is a ring R together with an endomorphism $\sigma: R \rightarrow R$. A morphism $\psi: R \rightarrow S$ of σ -rings is a morphism of rings such that

$$\begin{array}{ccc} R & \xrightarrow{\psi} & S \\ \sigma \downarrow & & \downarrow \sigma \\ R & \xrightarrow{\psi} & S \end{array}$$

commutes. A σ -ring is a σ -field if the underlying ring is a field.

Let k be a σ -ring. A k - σ -algebra is a σ -ring R together with a morphism $k \rightarrow R$ of σ -rings. A morphism of k - σ -algebras is a morphism of k -algebras that is also a morphism of σ -rings.

For a subset B of R , the smallest k - σ -subalgebra of R that contains B is denoted by $k\{B\}$. Note that $k\{B\}$ is generated by $B, \sigma(B), \sigma^2(B), \dots$ as a k -algebra. If $R = k\{B\}$ for a finite subset B of R then R is called *finitely σ -generated*. If R and S are k - σ -algebras, then $R \otimes_k S$ is a k - σ -algebra with σ defined by $\sigma(r \otimes s) = \sigma(r) \otimes \sigma(s)$ for $r \in R$ and $s \in S$.

An ideal $\mathfrak{a}$ of a σ -ring R is a σ -ideal if $\sigma(\mathfrak{a}) \subseteq \mathfrak{a}$. In this case $R/\mathfrak{a}$ is naturally a σ -ring.

The σ -polynomial ring $k\{y\} = k\{y_1, \dots, y_n\}$ over k in the σ -variables $y_1, \dots, y_n$ is the polynomial ring over k in the variables $\sigma^i(y_j)$ ($1 \leq j \leq n$, $0 \leq i$) with $\sigma: k\{y\} \rightarrow k\{y\}$ extending $\sigma: k \rightarrow k$ as suggested by the naming of the variables. If $f \in k\{y_1, \dots, y_n\}$ is a σ -polynomial and $x = (x_1, \dots, x_n) \in R^n$ for some k - σ -algebra R , then the element $f(x) \in R$ is obtained from f by substituting $\sigma^i(y_j)$ with $\sigma^i(x_j)$. For a k - σ -algebra R and $F \subseteq k\{y_1, \dots, y_n\}$ we set $\mathbb{V}_R(F) = \{x \in R^n \mid \text{for all } f \in F : f(x) = 0\}$. Note that $R \rightsquigarrow \mathbb{V}_R(F)$ is naturally a functor from the category of k - σ -algebras to the category of sets.

Let k be a σ -field. A σ -variety X over k is a functor $R \rightsquigarrow X(R)$ from the category of k - σ -algebras to the category of sets that is isomorphic (as a functor) to a functor of the form $R \rightsquigarrow \mathbb{V}_R(F)$ for some $n \geq 1$ and $F \subseteq k\{y_1, \dots, y_n\}$. Thus a functor X from the category of k - σ -algebras to the category of sets is a σ -variety if and only if it is representable by a finitely σ -generated k - σ -algebra, i.e., there exists a finitely σ -generated k - σ -algebra S such that $X \simeq \text{Hom}(S, -)$. By the Yoneda Lemma, the k - σ -algebra S is uniquely determined (up to an isomorphism) by X . We therefore denote it with $k\{X\}$ and call it the *coordinate ring* of X . A morphism $\phi: X \rightarrow Y$ of σ -varieties is a morphism of functors (i.e., a natural transformation). Again, by the Yoneda Lemma, the category of σ -varieties over k is anti-equivalent to the category of finitely σ -generated k - σ -algebras. The morphism dual to $\phi: X \rightarrow Y$ is denoted by $\phi^*: k\{Y\} \rightarrow k\{X\}$.

A σ -closed σ -subvariety Y of a σ -variety X is a subfunctor Y of X defined by a σ -ideal $\mathfrak{a}$ of $k\{X\}$, i.e., $Y(R) = \{\psi \in \text{Hom}(k\{X\}, R) \mid \mathfrak{a} \subseteq \ker(\psi)\} \subseteq \text{Hom}(k\{X\}, R) = X(R)$ for every k - σ -algebra R . Then Y is a σ -variety with $k\{Y\} = k\{X\}/\mathfrak{a}$. In terms of equations, if $F \subseteq G \subseteq k\{y\}$, then $R \rightsquigarrow \mathbb{V}_R(G)$ is a σ -closed σ -subvariety of $R \rightsquigarrow \mathbb{V}_R(F)$.

If $\phi: X \rightarrow Y$ is a morphism of σ -varieties, there exists a unique σ -closed σ -subvariety $\phi(X)$ of Y such that ϕ factors through $\phi(X)$ and for every other σ -closed σ -subvariety Z of Y such that ϕ factors through Z we have $\phi(X) \subseteq Z$ ([Wib, Lemma 1.5]). Indeed, $\phi(X)$ is the σ -closed σ -subvariety of Y defined by the kernel of $\phi^*: k\{Y\} \rightarrow k\{X\}$. A morphism $\phi: X \rightarrow Y$ of σ -varieties is a σ -closed embedding if it induces an isomorphism between X and a σ -closed σ -subvariety of Y , i.e., the morphism $X \rightarrow \phi(X)$ is an isomorphism. This is equivalent to $\phi^*: k\{Y\} \rightarrow k\{X\}$ being surjective.

The category of σ -varieties over k has products. Indeed, if X and Y are σ -varieties, the functor $R \rightsquigarrow X(R) \times Y(R)$ is a product of X and Y with coordinate ring $k\{X \times Y\} = k\{X\} \otimes_k k\{Y\}$.

If X is a σ -variety over k and k'/k is an extension of σ -fields, then $X_{k'}$ denotes the σ -variety over k' obtained from X by base change from k to k' , i.e., $X_{k'}(R') = X(R')$ for every k' - σ -algebra R' and $k'\{X_{k'}\} = k\{X\} \otimes_k k'$.

For every k -algebra R , there exists a k - σ -algebra $[\sigma]_k R$ together with a morphism $R \rightarrow [\sigma]_k R$ of k -algebras such that for every k - σ -algebra S and k -algebra morphism $R \rightarrow S$ there

exists a unique morphism $[\sigma]_k R \rightarrow S$ of k - σ -algebra such that

$$\begin{array}{ccc} R & \xrightarrow{\quad} & [\sigma]_k R \\ & \searrow & \swarrow \\ & S & \end{array}$$

commutes. Explicitly, $[\sigma]_k R$ can be described as follows: For $i \geq 0$ let $\sigma^i R = R \otimes_k k$ denote the k -algebra obtained from the k -algebra R by base change via $\sigma^i: k \rightarrow k$. Set $R[i] = R \otimes_k {}^\sigma R \otimes_k \dots \otimes_k \sigma^i R$ and let $[\sigma]_k R$ denote the union of the $R[i]$'s. We turn $[\sigma]_k R$ into a k - σ -algebra by setting

$$\sigma((r_0 \otimes \lambda_0) \otimes \dots \otimes (r_i \otimes \lambda_i)) = (1 \otimes 1) \otimes (r_0 \otimes \sigma(\lambda_0)) \otimes \dots \otimes (r_i \otimes \sigma(\lambda_i)) \in R[i+1]$$

for $(r_0 \otimes \lambda_0) \otimes \dots \otimes (r_i \otimes \lambda_i) \in R[i]$. If $\psi: R \rightarrow S$ is a morphism of k -algebras, then so is $R \rightarrow S \rightarrow [\sigma]_k S$ and we obtain a morphism $[\sigma]_k \psi: [\sigma]_k R \rightarrow [\sigma]_k S$ of k - σ -algebras. For later use, we record a lemma:

Lemma 1.1. *Let k be a σ -field and let $\psi: R \rightarrow S$ be an injective morphism of k -algebras. Then also $[\sigma]_k \psi: [\sigma]_k R \rightarrow [\sigma]_k S$ is injective.*

Proof. The restriction of $[\sigma]_k \psi$ to $R[i]$ is $\psi \otimes {}^\sigma \psi \otimes \dots \otimes \sigma^i \psi$ which is injective. Here, $\sigma^i \psi: \sigma^i R \rightarrow \sigma^i S$ is obtained from $\psi: R \rightarrow S$ by base change via $\sigma^i: k \rightarrow k$. $\square$

Any affine scheme $\mathcal{X}$ of finite type over k can be interpreted as a σ -variety. Indeed, the functor $[\sigma]_k \mathcal{X}$ from the category of k - σ -algebras to the category of sets defined by $R \rightsquigarrow \mathcal{X}(R)$ is a σ -variety. If $\mathcal{X} = \text{Spec}(k[\mathcal{X}])$, then $k\{[\sigma]_k \mathcal{X}\} = [\sigma]_k k[\mathcal{X}]$. If $\phi: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of affine schemes of finite type over k , then $[\sigma]_k \phi: [\sigma]_k \mathcal{X} \rightarrow [\sigma]_k \mathcal{Y}$ defined by $([\sigma]_k \phi)_R = \phi_R: \mathcal{X}(R) \rightarrow \mathcal{Y}(R)$ for any k - σ -algebra R is a morphism of σ -varieties. A σ -closed σ -subvariety Y of $\mathcal{X}$ is a σ -closed σ -subvariety of $[\sigma]_k \mathcal{X}$. Such a Y is defined by a σ -ideal $\mathfrak{a}$ of $[\sigma]_k k[\mathcal{X}] = \cup_{i \geq 0} k[\mathcal{X}][i]$. The closed subscheme $Y[i]$ of $\mathcal{X} \times {}^\sigma \mathcal{X} \times \dots \times \sigma^i \mathcal{X}$ defined by $\mathfrak{a} \cap k[\mathcal{X}][i]$ is called the i -th order Zariski closure of Y in $\mathcal{X}$. Here $\sigma^i \mathcal{X}$ denotes the affine scheme obtained from $\mathcal{X}$ by base change via $\sigma^i: k \rightarrow k$. Note that the i -th order Zariski closure of $[\sigma]_k \mathcal{X}$, considered as a σ -closed σ -subvariety of $\mathcal{X}$, is $\mathcal{X} \times {}^\sigma \mathcal{X} \times \dots \times \sigma^i \mathcal{X}$.

A σ -algebraic group G (over k) is a group object in the category of σ -varieties over k . For example, if $\mathcal{G}$ is an affine group scheme of finite over k , then $[\sigma]_k \mathcal{G}$ is a σ -algebraic group over k . A σ -closed subgroup H of a σ -algebraic group G is a σ -subvariety such that $H(R)$ is a subgroup of $G(R)$ for every k - σ -algebra R . A σ -closed subgroup of an affine group scheme $\mathcal{G}$ of finite type over k is a σ -closed subgroup of $[\sigma]_k \mathcal{G}$. If $\mathcal{H}$ is a closed subgroup of $\mathcal{G}$, then $[\sigma]_k \mathcal{H}$ is a σ -closed subgroup of $[\sigma]_k \mathcal{G}$.

Lemma 1.2 ([Wib, Prop. 2.16]). *Every σ -algebraic group is isomorphic to a σ -closed subgroup of GL_n for some $n \geq 1$.*

Let H_i , $i \in I$ be a family of σ -closed subgroups of a σ -algebraic group G . Since the intersection of a family of σ -closed subgroups of G is a σ -closed subgroup of G , we see that there exists a smallest σ -closed subgroup $\langle H_i \mid i \in I \rangle$ such that H_i is contained in $\langle H_i \mid i \in I \rangle$ for every $i \in I$. If $G = \langle H_i \mid i \in I \rangle$, then G is generated by the H_i 's.

1.2. σ -Picard-Vessiot theory. We first recall some basic definitions and results from [DVHW14]. A δ -ring is a ring R together with a derivation $\delta: R \rightarrow R$. An ideal $\mathfrak{a} \subseteq R$ such that $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ is called a δ -ideal. If every δ -ideal of R is trivial, R is called δ -simple. The δ -constants of R are $R^\delta = \{r \in R \mid \delta(r) = 0\}$.

A $\delta\sigma$ -ring is a ring R together with a derivation $\delta: R \rightarrow R$ and a ring endomorphism $\sigma: R \rightarrow R$ such that $\delta(\sigma(r)) = \hbar\sigma(\delta(r))$ for all $r \in R$ for some fixed unit $\hbar \in R^\delta$. If R is a field, we speak of a $\delta\sigma$ -field. There are the obvious notions of a morphism of $\delta\sigma$ -rings, of $\delta\sigma$ -algebras etc. A typical example of a $\delta\sigma$ -field is the field $k(x)$ of rational functions over a σ -field k , considered as a $\delta\sigma$ -field with $\delta = \frac{d}{dx}$ and $\sigma: k(x) \rightarrow k(x)$ extending $\sigma: k \rightarrow k$ by $\sigma(x) = x$. The $\delta\sigma$ -field we are primarily interested in is the field $\mathbb{C}(x)$ with derivation $\delta = \frac{d}{dx}$ and endomorphism σ given by $\sigma(f(x)) = f(x+1)$.

From now on let F denote a $\delta\sigma$ -field of characteristic zero and let $k = F^\delta$ be the σ -field of δ -constants of F . We consider a linear differential equation $\delta(y) = Ay$ with a matrix $A \in F^{n \times n}$.

Definition 1.3. A σ -Picard-Vessiot ring¹ for $\delta(y) = Ay$ is an F - $\delta\sigma$ -algebra R such that

- (i) there exists a matrix $Y \in \mathrm{GL}_n(R)$ with $\delta(Y) = AY$ and $R = F\{Y, 1/\det(Y)\}$,
- (ii) R is δ -simple and
- (iii) $R^\delta = k$.

The definition of a (classical) Picard-Vessiot ring for $\delta(y) = Ay$ is identical to the above definition other than $R = F\{Y, 1/\det(Y)\}$ replaced with $R = F[Y, 1/\det(Y)]$. In practice, σ -Picard-Vessiot rings often arise as in the following lemma:

Lemma 1.4. Let E/F be an extension of $\delta\sigma$ -fields such that $E^\delta = F^\delta$. If $A \in F^{n \times n}$ and $Y \in \mathrm{GL}_n(E)$ are such that $\delta(Y) = AY$, then $R = F\{Y, 1/\det(Y)\}$ is a σ -Picard-Vessiot ring for $\delta(y) = Ay$.

Proof. The field of fractions of R is a σ -Picard-Vessiot extension for $\delta(y) = Ay$ in the sense of [DVHW14, Def. 1.2]. It thus follows from [DVHW14, Prop. 1.5] that R is a σ -Picard-Vessiot ring for $\delta(y) = Ay$. $\square$

A σ -Picard-Vessiot ring is an integral domain and σ and δ extend uniquely to the field of fractions of R . The field of fractions E of a σ -Picard-Vessiot ring is a σ -Picard-Vessiot extension. The σ -Galois group G of a σ -Picard-Vessiot ring R/F is the functor from the category of k - σ -algebras to the category of groups given by

$$G(S) = \mathrm{Aut}^{\delta\sigma}(R \otimes_k S / F \otimes_k S)$$

for every k - σ -algebra S . Here $R \otimes_k S$ is considered as a $\delta\sigma$ -ring with δ being the trivial derivation on S , i.e., $\delta(s) = 0$ for $s \in S$. The choice of a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$ for $\delta(y) = Ay$ determines a σ -closed embedding of G into GL_n . Indeed, for every k - σ -algebra S and $g \in G(S)$ there exists a matrix $\phi_S(g) \in \mathrm{GL}_n(S)$ such that $g(Y) = Y\phi_S(g)$. Then $\phi: G \rightarrow \mathrm{GL}_n$ is a σ -closed embedding. We let $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$ denote

¹This definition differs from Definition [DVHW14, Def. 1.2] where the condition $R^\delta = k$ is dropped. Definition 1.3 is more convenient for us and [DVHW14, Prop. 1.5] shows that with our definition, σ -Picard-Vessiot rings correspond to σ -Picard-Vessiot extensions.

the image $\phi(G)$ of G in GL_n and call it the σ -Galois group of R/F with respect to Y . The coordinate ring of G is $k\{G\} = (R \otimes_F R)^\delta$ and the canonical map

$$(1) \quad R \otimes_k k\{G\} \rightarrow R \otimes_F R$$

is an isomorphism.

Example 1.5. Consider the differential equation $\delta(y) = y$ over $\mathbb{C}(x)$ equipped with $\delta = d/dx$ and the shift operator σ , i.e., $\sigma(f(x)) = f(x+1)$. Then $R = \mathbb{C}(x)\{e^x, e^{-x}\}$ is a σ -Picard-Vessiot ring for this equation by Lemma 1.4 applied to the field E of meromorphic functions. We compute $\sigma(e^x) = e^{x+1} = e \cdot e^x$ and conclude $R = \mathbb{C}(x)[e^x, e^{-x}]$. Let S be a k - σ -algebra and let γ be an automorphism on $R \otimes_{\mathbb{C}} S$ of $(F \otimes_{\mathbb{C}} S)$ - $\delta\sigma$ -algebras. Then γ is uniquely determined by $\gamma(e^x)$ and as γ commutes with δ , there exists an $\alpha \in S^\times$ with $\gamma(e^x) = e^x \alpha$. Moreover, γ commutes with σ and hence $\sigma(\alpha) = \alpha$. Conversely, every $\alpha \in S$ with $\sigma(\alpha) = \alpha$ gives rise to such an automorphism (here we use that e^x is transcendental over $\mathbb{C}(x)$). We conclude that the σ -Galois group of R is the constant subgroup of the multiplicative group: $G(S) = \{\alpha \in S^\times \mid \sigma(\alpha) = \alpha\}$ for all k - σ -algebras S .

Let S be a k - σ -algebra and $g \in G(S)$. Then $g: R \otimes_k S \rightarrow R \otimes_k S$ extends to an automorphism $\tilde{g}: \mathrm{Frac}(R \otimes_k S) \rightarrow \mathrm{Frac}(R \otimes_k S)$ on the total ring of fractions of $R \otimes_k S$, which includes $E = \mathrm{Frac}(R)$. An element $a \in E$ is *invariant* under g if $\tilde{g}(a) = a$. For a σ -closed subgroup H of G , the set of all elements in E that are invariant under $H(S)$ for every k - σ -algebra S is denoted by E^H .

Theorem 1.6 (σ -Galois correspondence, [DVHW14, Theorem 3.2]). *The map $H \mapsto E^H$ is an inclusion reversing bijection between the σ -closed subgroups of G and the intermediate $\delta\sigma$ -field of E/F . In particular, $E^H = F$ if and only if $H = G$.*

The following proposition will be helpful for constructing explicit examples.

Proposition 1.7 ([DVHW14, Prop. 2.15]). *Let $R = F\{Y, 1/\det Y\}$ be a σ -Picard-Vessiot ring for $\delta(y) = Ay$, where $A \in F^{n \times n}$. Then, $F[Y, \sigma(Y), \dots, \sigma^i(Y), 1/\det(Y \dots \sigma^i(Y))]$ is a (classical) Picard-Vessiot ring with (classical) Galois group isomorphic to the i -th order Zariski closure of $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$ in GL_n for every $i \geq 1$.*

Corollary 1.8. *Let $R = F\{Y, 1/\det Y\}$ be a σ -Picard-Vessiot ring for $\delta(y) = Ay$ and define $R_i = F[Y, \sigma(Y), \dots, \sigma^i(Y), 1/\det(Y \dots \sigma^i(Y))]$ for $i \geq 1$. Let further $\mathcal{H} \leq \mathrm{GL}_n$ be a connected algebraic group with $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F) \leq [\sigma]_k \mathcal{H}$. Then $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F) = [\sigma]_k \mathcal{H}$ if and only if $\dim(R_i) = (i+1) \dim(\mathcal{H})$ for all $i \geq 1$.*

Proof. Set $G = \underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$. By Proposition 1.7, R_i/F is a (classical) Picard-Vessiot ring with (classical) Galois group $G[i]$. Hence $\dim(R_i) = \dim(G[i])$ for all $i \geq 1$. If $G = [\sigma]_k \mathcal{H}$, then $G[i] = \mathcal{H} \times {}^\sigma \mathcal{H} \times \dots \times {}^{\sigma^i} \mathcal{H} \simeq \mathcal{H}^{i+1}$ and thus $\dim(R_i) = \dim(G[i]) = (i+1) \dim(\mathcal{H})$ for all $i \geq 1$. Conversely, assume that G is properly contained in $[\sigma]_k \mathcal{H}$. Then the defining ideal $\mathfrak{a} \subseteq [\sigma]_k k[\mathcal{H}]$ of G as a σ -closed subgroup of $[\sigma]_k \mathcal{H}$ is non-zero and so $\mathfrak{a} \cap k[\mathcal{H}][i]$ is non-zero for some $i \geq 1$. Hence $G[i]$ is properly contained in $\mathcal{H} \times {}^\sigma \mathcal{H} \times \dots \times {}^{\sigma^i} \mathcal{H} \simeq \mathcal{H}^{i+1}$ and as $\mathcal{H}^{i+1}$ is connected, we conclude $\dim(R_i) = \dim(G[i]) < \dim(\mathcal{H}^{i+1}) = (i+1) \dim(\mathcal{H})$. $\square$

For later use we record a lemma about the σ -Galois group with respect to a fundamental solution matrix.

Lemma 1.9. *Let $R = F\{Y, 1/\det(Y)\}$ be a σ -Picard-Vessiot ring for $\delta(y) = Ay$. Then the following holds.*

- (i) *For every $B \in \mathrm{GL}_n(F)$, R is a σ -Picard-Vessiot ring for $\delta(y) = (BAB^{-1} + \delta(B)B^{-1})y$ with fundamental solution matrix BY and we have $\underline{\mathrm{Gal}}_{BY}^{\delta\sigma}(R/F) = \underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$ as σ -closed subgroups of GL_n .*
- (ii) *Let $F_0 \supseteq F_1 \supseteq F$ be $\delta\sigma$ -field extensions with $F_0^\delta = F$ and suppose that $R \subseteq F_0$ as an F - $\delta\sigma$ -subalgebra. Then $R_1 = F_1\{Y, 1/\det(Y)\} \subseteq F_0$ is a σ -Picard-Vessiot ring over F_1 and $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R_1/F_1) \leq \underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$ as σ -closed subgroups of GL_n .*

Proof. For (i), define $\tilde{Y} = BY$. Then $\delta(\tilde{Y})\tilde{Y}^{-1} = \delta(BY)Y^{-1}B^{-1} = BAB^{-1} + \delta(B)B^{-1}$, since $\delta(Y) = AY$. Also $R = F\{Y, 1/\det(Y)\} = F\{\tilde{Y}, 1/\det(\tilde{Y})\}$. Hence R is a σ -Picard-Vessiot ring for $\delta(y) = (BAB^{-1} + \delta(B)B^{-1})y$ with fundamental solution matrix BY . Let S be a k - σ -algebra and let $g \in \mathrm{Aut}^{\delta\sigma}(R \otimes_k S / F \otimes_k S)$. Then $\tilde{Y}^{-1}g(\tilde{Y}) = Y^{-1}B^{-1}Bg(Y) = Y^{-1}g(Y)$ and thus the S -rational points of $\underline{\mathrm{Gal}}_{BY}^{\delta\sigma}(R/F)$ and $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$ coincide as subsets of $\mathrm{GL}_n(S)$ and (i) follows.

For (ii), first note that R_1 is a σ -Picard-Vessiot ring over F_1 by Lemma 1.4, since $F_0^\delta = F_1^\delta$. Let S be a k - σ -algebra and let $g_1 \in \mathrm{Aut}^{\delta\sigma}(R_1 \otimes_k S / F_1 \otimes_k S)$. Then $g_1(Y) = Y\phi_S(g_1)$ with $\phi_S(g_1) \in \mathrm{GL}_n(S)$. Hence $g_1(Y)$ has entries in $R \otimes_k S$ and thus g_1 restricts to an injective homomorphism g on $R \otimes_k S$. As $R \otimes_k S$ is generated by the entries of $Y = g(Y\phi_S(g_1)^{-1})$ and $1/\det(Y) = g(1/\det(Y\phi_S(g_1)^{-1}))$, g is surjective and thus an element of $\mathrm{Aut}^{\delta\sigma}(R \otimes_k S / F \otimes_k S)$. As g_1 is uniquely determined by $g_1(Y) = g(Y)$, we obtain an inclusion $\mathrm{Aut}^{\delta\sigma}(R_1 \otimes_k S / F_1 \otimes_k S) \hookrightarrow \mathrm{Aut}^{\delta\sigma}(R \otimes_k S / F \otimes_k S)$ that corresponds to a containment $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R_1/F_1)(S) \subseteq \underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)(S)$ as subsets of $\mathrm{GL}_n(S)$, since $Y^{-1}g_1(Y) = Y^{-1}g(Y)$. $\square$

2. MULTIPLICATIVE, ADDITIVE, AND FINITE CYCLIC GROUPS AS σ -GALOIS GROUPS

The patching method will allow to break down the task of realizing a given group G as a σ -Galois group to the task of realizing generating subgroups as σ -Galois groups over certain overfields (see Theorem 3.12). As our task is to realize all algebraic groups as σ -Galois groups, we need to find generating subgroups that we can explicitly realize as σ -Galois groups. In this section, we first prove that every algebraic group can be generated (as a σ -algebraic group) by finitely many subgroups that are each either isomorphic to the multiplicative, or the additive, or a finite cyclic group. Then we show how these three types of groups can be realized as σ -Galois groups.

2.1. Generating algebraic groups as σ -algebraic groups. It is known (see e.g., [BHH16, Prop. 3.1]) that every linear algebraic group over an algebraically closed field of characteristic zero can be generated by finitely many closed subgroups isomorphic to the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$ or a finite cyclic group. We will need a slightly refined version of this result.

Lemma 2.1. *Let $\mathcal{G}$ be a linear algebraic group over an algebraically closed field of characteristic zero. Then there exist closed subgroups $\mathcal{H}_1, \dots, \mathcal{H}_r$ of $\mathcal{G}$ such that*

- (i) *each $\mathcal{H}_i$ is isomorphic to either $\mathbb{G}_a$, $\mathbb{G}_m$ or a finite cyclic group and*
- (ii) *for some $n \geq 1$ and $(i_1, \dots, i_n) \in \{1, \dots, r\}^n$ the multiplication map $\phi: \mathcal{X} = \mathcal{H}_{i_1} \times \dots \times \mathcal{H}_{i_n} \rightarrow \mathcal{G}$ is surjective.*

Proof. It is shown in the proof of [BHH16, Prop. 3.1] that the identity component $\mathcal{G}^o$ of $\mathcal{G}$ can be generated by finitely many copies of $\mathbb{G}_a$'s and $\mathbb{G}_m$'s. According to [Spr09, Cor. 2.2.7], Condition (ii) above is automatically satisfied for generating subgroups that are connected. So we can find finitely many closed subgroups $\mathcal{H}_1, \dots, \mathcal{H}_s$ of $\mathcal{G}^o$ isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$ such that the multiplication map from $\mathcal{H}_1 \times \dots \times \mathcal{H}_s$ to $\mathcal{G}^o$ is surjective. According to a result of Borel and Serre ([BS64, Lemma 5.11]), there exists a finite subgroup $\mathcal{W}$ of $\mathcal{G}$ that meets every connected component of $\mathcal{G}$. Clearly $\mathcal{W}$, is generated by finite cyclic groups $\mathcal{H}_{s+1}, \dots, \mathcal{H}_r$, even in the strong sense of (ii). Thus the subgroups $\mathcal{H}_1, \dots, \mathcal{H}_r$ satisfy condition (ii). $\square$

We now transfer the above result to σ -algebraic groups.

Proposition 2.2. *Let k be an algebraically closed σ -field of characteristic zero and let $\mathcal{G}$ be a linear algebraic group over k . Then the σ -algebraic group $G = [\sigma]_k \mathcal{G}$ can be generated by finitely many σ -closed subgroups isomorphic to $[\sigma]_k \mathcal{H}$, where $\mathcal{H}$ is $\mathbb{G}_m$, $\mathbb{G}_a$ or a finite cyclic group.*

Proof. Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ and $\phi: \mathcal{X} \rightarrow \mathcal{G}$ be as in Lemma 2.1 and consider the morphism $[\sigma]_k \phi: X \rightarrow G$ of σ -varieties, where $X = [\sigma]_k \mathcal{X} = [\sigma]_k \mathcal{H}_{i_1} \times \dots \times [\sigma]_k \mathcal{H}_{i_r}$. Then $[\sigma]_k \phi$ is the multiplication map, hence $([\sigma]_k \phi)(X) \subseteq \langle [\sigma]_k \mathcal{H}_i \mid i = 1, \dots, r \rangle \subseteq G$. On the other hand, $([\sigma]_k \phi)(X)$ is the σ -closed σ -subvariety of G defined by the kernel of $([\sigma]_k \phi)^* = [\sigma]_k \phi^*: k\{G\} \rightarrow k\{X\}$. Since $\phi^*: k\{\mathcal{G}\} \rightarrow k\{\mathcal{X}\}$ is injective, also $[\sigma]_k \phi^*: k\{G\} \rightarrow k\{X\}$ is injective (Lemma 1.1) and so $([\sigma]_k \phi)(X) = G$. Thus $G = \langle [\sigma]_k \mathcal{H}_i \mid i = 1, \dots, r \rangle$ as desired. $\square$

2.2. Realization of $\mathbb{G}_m$, $\mathbb{G}_a$ and finite cyclic groups as σ -Galois groups. In this subsection, we present criteria for when a σ -Picard-Vessiot ring has a σ -Galois group isomorphic to the additive group $\mathbb{G}_a$, the multiplicative group $\mathbb{G}_m$, or a finite cyclic group (all interpreted as σ -algebraic groups). We let F denote a $\delta\sigma$ -field of characteristic zero and we let $k = F^\delta$ be its σ -field of δ -constants.

We begin with the multiplicative case.

Proposition 2.3. *Let L/F be a $\delta\sigma$ -field extension with $L^\delta = F^\delta$. Let $y \in L$ be an element such that $\delta(y)y^{-1} \in F$. Then $F\{y, y^{-1}\} \subseteq L$ is a σ -Picard-Vessiot ring with σ -Galois group G isomorphic to a σ -closed subgroup of the multiplicative group $\mathbb{G}_m$. We have $G \simeq [\sigma]_k \mathbb{G}_m$ if and only if $y, \sigma(y), \sigma^2(y), \dots$ are algebraically independent over F .*

Proof. Define $a = \delta(y)y^{-1} \in F$. By Lemma 1.4, $R = F\{y, y^{-1}\}$ is a σ -Picard Vessiot ring for the differential equation $\delta(y) = ay$. Thus $\underline{\text{Gal}}_y^{\delta\sigma}(R/F)$ is a σ -closed subgroup of $\text{GL}_1 = \mathbb{G}_m$. Set $G = \underline{\text{Gal}}_y^{\delta\sigma}(R/F)$.

For every $i \geq 1$, define $R_i = F[y, \sigma(y), \dots, \sigma^i(y), 1/(y \dots \sigma^i(y))]$. By Corollary 1.8, $G = [\sigma]_k \mathbb{G}_m$ if and only if $\dim(R_i) = i + 1$ for all $i \geq 1$. Clearly, $\dim(R_i) = i + 1$ holds if and only if $y, \sigma(y), \sigma^2(y), \dots, \sigma^i(y)$ are algebraically independent and the claim follows. $\square$

We next treat the additive case.

Proposition 2.4. *Let L/F be a $\delta\sigma$ -field extension with $L^\delta = F^\delta$. Let $y \in L$ be an element such that $\delta(y) \in F$. Then $F\{y\} \subseteq L$ is a σ -Picard-Vessiot ring with σ -Galois group G*

isomorphic to a σ -closed subgroup of the additive group $\mathbb{G}_a$. We have $G \simeq [\sigma]_k \mathbb{G}_a$ if and only if $y, \sigma(y), \sigma^2(y), \dots$ are algebraically independent over F .

Proof. Define $a = \delta(y) \in F$ and define matrices

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in F^{2 \times 2}, \quad Y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(R).$$

Then $R = F\{Y, 1/\det(Y)\} = F\{y\}$ is a σ -Picard-Vessiot ring for the differential equation $\delta(y) = Ay$ by Lemma 1.4. The σ -Galois group of R is isomorphic to a σ -closed subgroup of $\mathbb{G}_a$. Indeed, if S is a k - σ -algebra, $g \in \mathrm{Aut}^{\delta\sigma}(R \otimes_k S / F \otimes_k S)$ and $\phi: G \rightarrow \mathrm{GL}_2$ is the σ -closed embedding associated with the choice of Y , then

$$\begin{pmatrix} 1 & g(y) \\ 0 & 1 \end{pmatrix} = g(Y) = Y\phi_S(g) = \begin{pmatrix} g_{11} + yg_{21} & g_{12} + yg_{22} \\ g_{21} & g_{22} \end{pmatrix},$$

where

$$\phi_S(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \mathrm{GL}_2(S).$$

Thus, $g_{21} = 0$ and $g_{11} = g_{22} = 1$. This shows that $\phi(G) = \underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$ is contained in the σ -closed subgroup $H \simeq [\sigma]_k \mathbb{G}_a$ of GL_2 defined by

$$H(S) = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in S \right\}$$

for any k - σ -algebra S .

For every $i \geq 1$, consider the F - δ -algebra

$$R_i = F[y, \sigma(y), \dots, \sigma^i(y)] = F[Y, \sigma(Y), \dots, \sigma^i(Y), 1/\det(Y \dots \sigma^i(Y))].$$

By Corollary 1.8, $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F) = H$ holds if and only if $\dim(R_i) = i + 1$ for all $i \geq 1$. Clearly, $\dim(R_i) = i + 1$ holds if and only if $y, \sigma(y), \sigma^2(y), \dots, \sigma^i(y)$ are algebraically independent and the claim follows. $\square$

Finally, we treat the case of a finite cyclic group.

Proposition 2.5. *Let L/F be a $\delta\sigma$ -field extension with $L^\delta = F^\delta$ algebraically closed. Let $y \in L$ be such that $y^d \in F$ for some $d \in \mathbb{N}$. Then $F\{y, y^{-1}\} \subseteq L$ is a σ -Picard-Vessiot ring with σ -Galois group G isomorphic to a σ -closed subgroup of $\mathcal{H}$, where $\mathcal{H}$ is the finite cyclic group of order d (considered as an algebraic group over k). We have $G \simeq [\sigma]_k \mathcal{H}$ if and only if for every $i \in \mathbb{N}$, $\sigma^i(y)$ has degree d over $F(y, \sigma(y), \dots, \sigma^{i-1}(y))$.*

Proof. Let $b = y^d \in F$. Then $\delta(y) = \frac{\delta(b)}{bd}y$ and it follows from Lemma 1.4 that $R = F\{y, y^{-1}\} = F(y, \sigma(y), \dots)$ is a σ -Picard-Vessiot ring. For every k - σ -algebra S and $g \in \underline{\mathrm{Gal}}_y^{\delta\sigma}(R/F)(S) \leq \mathrm{GL}_1(S)$ we have $b = g(b) = g(y^d) = y^d g^d = b g^d$ and it follows that $g^d = 1$. Thus $\underline{\mathrm{Gal}}_y^{\delta\sigma}(R/F)$ is a σ -closed subgroup of μ_d , the algebraic group of d -th roots of unity over k . Since k is algebraically closed, μ_d is isomorphic to $\mathcal{H}$. So G is isomorphic to a σ -closed subgroup of $\mathcal{H}$.

Assume that $\underline{\mathrm{Gal}}_y^{\delta\sigma}(R/F) = [\sigma]_k \mu_d$. As $(\sigma^i(y))^d = \sigma^i(b)$, $\sigma^i(y)$ has degree at most d over $F(y, \sigma(y), \dots, \sigma^{i-1}(y))$. For $i \geq 0$, $R_i = F[y, \dots, \sigma^i(y)] = F(y, \dots, \sigma^i(y))$ is a (classical) Picard-Vessiot ring with (classical) Galois group isomorphic to μ_d^{i+1} by Proposition 1.7. By

the (classical) torsor theorem, $R_i \otimes_F R_i \cong R_i \otimes_k k[\mu_d^{i+1}]$. Hence the dimension of R_i as an F -vector space agrees with the dimension of $k[\mu_d^{i+1}]$ as a k -vector space, and the latter is d^{i+1} . Thus $[R_1 : R_0][R_2 : R_1] \cdots [R_i : R_{i-1}] = d^{i+1}$ and clearly, the field extension R_j/R_{j-1} has degree at most d for every $j \geq 0$, since $(\sigma^j(y))^d = \sigma^j(b)$. Therefore $[R_i : R_{i-1}] = d$ and $\sigma^i(y)$ has degree d over $F(y, \sigma(y), \dots, \sigma^{i-1}(y))$.

Conversely, if $F[y, \dots, \sigma^i(y)]$ has F -dimension d^{i+1} for every $i \geq 1$, then the i -th order Zariski closure of $\underline{\text{Gal}}_y^{\delta\sigma}(R/F)$ in GL_1 must equal $\mu_d \times \dots \times \sigma^i \mu_d$ and so $\underline{\text{Gal}}_y^{\delta\sigma}(R/F)$ must equal $[\sigma]_k \mu_d$. $\square$

3. PATCHING AND σ -PICARD-VESSIOT THEORY

In this section, we prove a patching result for σ -Picard-Vessiot rings that allows us to glue together two σ -Picard-Vessiot rings under certain assumptions. The proof relies on factorization of the fundamental solution matrices, so it is crucial that the σ -Galois groups of the given σ -Picard-Vessiot rings are embedded in GL_n for the same n . Therefore, we start the section by showing that the representation of a σ -Galois group can be changed without changing the σ -Picard-Vessiot ring.

3.1. Changing the representation of a σ -Galois group. Let F be a $\delta\sigma$ -field of characteristic zero and R a σ -Picard-Vessiot ring over F with σ -Galois group G . The choice of matrices $A \in F^{n \times n}$ and $Y \in \text{GL}_n(R)$ such that R is a σ -Picard-Vessiot ring for $\delta(y) = Ay$ with fundamental solution matrix Y , determines a σ -closed embedding of G into GL_n via $G \rightarrow \underline{\text{Gal}}_Y^{\delta\sigma}(R/F)$. The following proposition shows that, conversely, if we start with a σ -closed embedding of G into GL_n , then we can find appropriate matrices A and Y .

Proposition 3.1. *Let R/F be a σ -Picard Vessiot ring with σ -Galois group G and let G' be a σ -closed subgroup of GL_n isomorphic to G . Then there exist $A \in F^{n \times n}$ and $Y \in \text{GL}_n(R)$ with $\delta(Y) = AY$ such that R is a σ -Picard-Vessiot ring for $\delta(y) = Ay$ and $\underline{\text{Gal}}_Y^{\delta\sigma}(R/F) = G'$.*

Proof. Let X denote the σ -variety over the σ -field F represented by the F - σ -algebra R . Then X is naturally equipped with a right action $\phi: X \times G_F \rightarrow X$ of the σ -algebraic group G_F over F : For $x \in X(S)$, i.e., $x: R \rightarrow S$ is a morphism of F - σ -algebras, and $g \in G_F(S) = G(S)$ the element $\phi(x, g) \in X(S)$ is the composition

$$R \rightarrow R \otimes_k S \xrightarrow{g} R \otimes_k S \rightarrow S,$$

where the first map is the inclusion into the first factor and the last map sends $r \otimes s$ to $x(r)s$. The morphism $X \times G_F \rightarrow X \times X$, given by $(x, g) \mapsto (x, \phi(x, g))$ for $x \in X(S)$, $g \in G_F(S)$ for any F - σ -algebra S , is an isomorphism, since the dual map $R \otimes_F R \simeq R \otimes_F F\{G_F\} = R \otimes_k k\{G\}$ is the isomorphism (1). Thus X is a right G_F -torsor. In [BW, Lemma 4.4] it is shown that every left torsor for a σ -closed subgroup of GL_n is isomorphic to a σ -closed σ -subvariety of GL_n , with action given by matrix multiplication. If $\tilde{X}$ is a right $\tilde{G}$ -torsor with action $\tilde{\phi}: \tilde{X} \times \tilde{G} \rightarrow \tilde{X}$, then $\tilde{X}$ is a left torsor with action $\tilde{G} \times \tilde{X} \rightarrow \tilde{X}$, $(g, x) \mapsto \tilde{\phi}(x, g^{-1})$. Therefore, it follows that there exists a σ -closed embedding $\varphi: X \rightarrow \text{GL}_n$ such that the right action of G_F on X is given by matrix multiplication, where G is identified with the σ -closed subgroup G' of GL_n . The σ -closed embedding $\varphi: X \rightarrow \text{GL}_n$ corresponds to a

surjective morphism $\varphi^*: F\{Z, 1/\det Z\} \rightarrow R$ of F - σ -algebras, where Z is an $n \times n$ matrix of σ -indeterminates over F . Let $Y \in \mathrm{GL}_n(R)$ denote the image of Z .

For a k - σ -algebra S , the action of an element $g \in G(S)$ on Y is given by $g(Y) = Yg$, where g is considered as an element of $\mathrm{GL}_n(S)$ via the isomorphism $G \rightarrow G'$. Therefore,

$$g(\delta(Y)Y^{-1}) = \delta(g(Y))g(Y)^{-1} = \delta(Yg)(Yg)^{-1} = \delta(Y)gg^{-1}Y = \delta(Y)Y.$$

By Theorem 1.6 we have $A = \delta(Y)Y^{-1} \in F^{n \times n}$ and we see that R is a σ -Picard-Vessiot ring for $\delta(y) = Ay$ with fundamental solution matrix Y and $\underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F) = G'$. $\square$

3.2. A $\delta\sigma$ -setup for applying patching methods. The method of patching over fields was introduced by Harbater and Hartmann in [HH10]. In the most basic setup, they consider fields $F \subseteq F_1, F_2$ with a common overfield F_0 and show that finite dimensional vector spaces V_1 over F_1 and V_2 over F_2 together with an isomorphism $\psi: V_1 \otimes_{F_1} F_0 \rightarrow V_2 \otimes_{F_2} F_0$ can be glued together to an F -vector space V with isomorphisms $\phi_1: V \otimes_F F_1 \simeq V_1$ and $\phi_2: V \otimes_F F_2 \simeq V_2$ such that $(\phi_2 \otimes_{F_2} F_0) \circ (\phi_1 \otimes_{F_1} F_0)^{-1} = \psi$ if (F, F_1, F_2, F_0) satisfies two properties called *intersection* and *factorization*. We call such quadruples *diamonds with the factorization property*. In [BHHW18], we considered *differential diamonds* with the factorization property.

Definition 3.2. A diamond with the factorization property is a quadruple (F, F_1, F_2, F_0) of fields with inclusions $F \subseteq F_1, F_2$ and $F_1, F_2 \subseteq F_0$ such that $F_1 \cap F_2 = F$ (intersection) and such that for every $n \in \mathbb{N}$ and every matrix $A \in \mathrm{GL}_n(F_0)$, there exist matrices $B \in \mathrm{GL}_n(F_1)$ and $C \in \mathrm{GL}_n(F_2)$ with $A = B \cdot C$ (factorization). If in addition $\mathrm{char}(F) = 0$ and all of the fields are equipped with a derivation that is compatible with the inclusions, then (F, F_1, F_2, F_0) is called a *differential diamond with the factorization property*. Similarly, if all four fields are $\delta\sigma$ -fields of characteristic zero such that both δ and σ are compatible with the inclusions, then we call (F, F_1, F_2, F_0) a *$\delta\sigma$ -diamond with the factorization property*.

Over $F = \mathbb{C}(x)$, diamonds with the factorization property arise as fields of meromorphic functions on suitable open subsets of $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$, as the following lemma states.

Lemma 3.3 ([BHHW18], Lemma 3.4). *Let F be a one-variable function field over $\mathbb{C}$, or equivalently the field of meromorphic functions on a compact Riemann surface $\mathcal{X}$. Let O_1, O_2 be connected metric open subsets of $\mathcal{X}$ such that $O_i \neq \mathcal{X}$, $O_1 \cup O_2 = \mathcal{X}$, and $O_0 := O_1 \cap O_2$ is connected. Let F_i be the field of meromorphic functions on O_i . Then (F, F_1, F_2, F_0) is a diamond with the factorization property.*

Example 3.4. Let $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$ be the Riemann sphere and $F = \mathbb{C}(x)$. Consider the open subsets $O_1 = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid |x| < 1\}$ and $O_2 = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid |x| > \frac{1}{2}\}$. Then $O_1 \cup O_2 = \mathcal{X}$ and the annulus $O_0 = O_1 \cap O_2 = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid \frac{1}{2} < |x| < 1\}$ is connected. Let F_i be the field of meromorphic functions on O_i . Then (F, F_1, F_2, F_0) is a diamond with the factorization property by Lemma 3.3.

Corollary 3.5. *Let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ and $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ be chains of proper connected metric open subsets of the Riemann sphere $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$ such that $U_r \cap V_r$ is connected and $U_r \cup V_r = \mathcal{X}$ for all $r \in \mathbb{N}$. Let F_{U_r}, F_{V_r} and $F_{U_r \cap V_r}$ denote the fields of meromorphic functions on U_r, V_r and $U_r \cap V_r$, respectively, and let $F = \mathbb{C}(x)$ (the field of meromorphic functions on $\mathcal{X}$). Define $F_1 = \varinjlim F_{U_r}$ (with respect to the natural inclusions $F_{U_r} \rightarrow F_{U_s}$ for*

$r \leq s$) and similarly $F_2 = \varinjlim F_{V_r}$ and $F_0 = \varinjlim F_{U_r \cap V_r}$. Then there are natural inclusions $F_1, F_2 \subseteq F_0$ and (F, F_1, F_2, F_0) is a diamond with the factorization property.

Proof. By Lemma 3.3, $(F, F_{U_r}, F_{V_r}, F_{U_r \cap V_r})$ is a diamond with the factorization property for every $r \in \mathbb{N}$. Let $f \in F_0$ be contained in the intersection $F_1 \cap F_2$. Then there exists an $r \in \mathbb{N}$ with $f \in F_{U_r} \cap F_{V_r} = F$. Thus $F_1 \cap F_2 = F$. Similarly, for $n \in \mathbb{N}$ and $A \in \text{GL}_n(F_0)$ there exists an $r \in \mathbb{N}$ with $A \in \text{GL}_n(F_{U_r \cap V_r})$. Hence there exist matrices $B \in \text{GL}_n(F_{U_r}) \subseteq \text{GL}_n(F_1)$ and $C \in \text{GL}_n(F_{V_r}) \subseteq \text{GL}_n(F_2)$ with $A = B \cdot C$. $\square$

Example 3.6. Let $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$ be the Riemann sphere and $F = \mathbb{C}(x)$. Consider the open subsets $U_r = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid |x| < 1\}$ and $V_r = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid |x| > 1 - \frac{1}{r}\}$. Then for all $r \in \mathbb{N}$, $U_r \cup V_r = \mathcal{X}$ and the annulus $U_r \cap V_r = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid 1 - \frac{1}{r} < |x| < 1\}$ is connected. Define $F_1 = \varinjlim F_{U_r}$, $F_2 = \varinjlim F_{V_r}$ and $F_0 = \varinjlim F_{U_r \cap V_r}$. Then $F_1 = F_{U_1}$ is the field of meromorphic functions on the open set $\{x \in \mathbb{P}_{\mathbb{C}}^1 \mid |x| < 1\}$, F_2 is the field of meromorphic functions on the open set $\{x \in \mathbb{P}_{\mathbb{C}}^1 \mid |x| > 1\}$ that are meromorphic in $x = 1$ and F_0 is the field of functions that are meromorphic in $x = 1$.

We proceed with a lemma that equips fields occurring as direct limits as in Corollary 3.5 with a structure of $\delta\sigma$ -fields.

Lemma 3.7. Let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ be a chain of connected metric open subsets of the Riemann sphere $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$ and define $F_1 = \varinjlim F_{U_r}$. Then the natural derivations δ on F_{U_r} induce a derivation δ on F_1 . If in addition there are homomorphisms $\sigma_r: F_{U_r} \rightarrow F_{U_{r+1}}$ for all $r \in \mathbb{N}$ such that

$$\begin{array}{ccc} F_{U_r} & \xrightarrow{\sigma_r} & F_{U_{r+1}} \\ \delta \downarrow & & \downarrow \delta \\ F_{U_r} & \xrightarrow{\sigma_r} & F_{U_{r+1}} \end{array} \quad \text{and} \quad \begin{array}{ccc} F_{U_{r+1}} & \xrightarrow{\sigma_{r+1}} & F_{U_{r+2}} \\ \uparrow & & \uparrow \\ F_{U_r} & \xrightarrow{\sigma_r} & F_{U_{r+1}} \end{array}$$

commute, then F_1 is a $\delta\sigma$ -field with $F_1^\delta = \mathbb{C}$.

Proof. It is immediate that the derivations on F_{U_r} induce a well-defined derivation on the direct limit and also that the morphisms $\{\sigma_r\}_{r \geq 1}$ induce a well-defined endomorphism $\sigma: F_1 \rightarrow F_1$ that commutes with δ . Finally, $F_{U_r}^\delta = \mathbb{C}$ for all r and thus $F_1^\delta = \mathbb{C}$. $\square$

In the proof of our main theorem, we will work with the following kind of fields.

Example 3.8. Fix a natural number $m \in \mathbb{N}$.

- (i) For $r \in \mathbb{N}$, let $V_{r,m} = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid x \notin [-r, 0] + mi\}$ be the connected metric open subset of the Riemann sphere $\mathbb{P}_{\mathbb{C}}^1$ obtained by deleting the translate $[-r, 0] + mi$ of the real interval $[-r, 0]$. Note that $V_{1,m} \supseteq V_{2,m} \supseteq V_{3,m} \supseteq \dots$. We let $F_{V_{r,m}}$ denote the field of meromorphic functions on $V_{r,m}$. Then $x \mapsto x + 1$ defines a holomorphic function $V_{r+1,m} \rightarrow V_{r,m}$ that induces a homomorphism $\sigma_r: F_{V_{r,m}} \rightarrow F_{V_{r+1,m}}$ for all $r \in \mathbb{N}$. Moreover, σ_r is compatible with the natural derivations δ on $F_{V_{r,m}}$ and $F_{V_{r+1,m}}$ and $\sigma_{r+1}: F_{V_{r+1,m}} \rightarrow F_{V_{r+2,m}}$ restricts to $\sigma_r: F_{V_{r,m}} \rightarrow F_{V_{r+1,m}}$. Hence the direct limit

$$L(m) = \varinjlim_{r \in \mathbb{N}} F_{V_{r,m}}$$

is a $\delta\sigma$ -field with $F(m)^\delta = \mathbb{C}$ by Lemma 3.7.

(ii) For $r \in \mathbb{N}$, consider the subset

$$U_{r,m} = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid x \notin \bigcup_{l=1}^m ([-r, 0] + li)\}$$

of the Riemann sphere $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$, where we delete the translates $[-r, 0] + i, [-r, 0] + 2i, \dots, [-r, 0] + mi$ of the real interval $[-r, 0]$. Note that $U_{r,m}$ is a connected metric open subset of $\mathcal{X}$ and $U_{1,m} \supseteq U_{2,m} \supseteq U_{3,m} \supseteq \dots$. Let $F_{U_{r,m}}$ denote the field of meromorphic functions on $U_{r,m}$. Then $x \mapsto x + 1$ defines a holomorphic function $U_{r+1,m} \rightarrow U_{r,m}$ that induces a homomorphism $\sigma_r: F_{U_{r,m}} \rightarrow F_{U_{r+1,m}}$ for all $r \in \mathbb{N}$. Moreover, σ_r is compatible with the natural derivations δ on $F_{U_{r,m}}$ and $F_{U_{r+1,m}}$ and $\sigma_{r+1}: F_{U_{r+1,m}} \rightarrow F_{U_{r+2,m}}$ restricts to $\sigma_r: F_{U_{r,m}} \rightarrow F_{U_{r+1,m}}$. Hence the direct limit

$$F(m) = \varinjlim_{r \in \mathbb{N}} F_{U_{r,m}}$$

is a $\delta\sigma$ -field with $F(m)^\delta = \mathbb{C}$ by Lemma 3.7.

Remark 3.9. Note that the σ -fields $L(m)$ and $F(m)$ are not inversive, i.e., σ is not surjective. For example, the image of σ contains no functions with a singularity at the point $(-1/2 + mi)$. Assume to the contrary that there exists an $f \in L(m)$ or $f \in F(m)$ such that $f(x + 1)$ has a singularity at $(-1/2 + mi)$. Then f has a singularity at $1/2 + mi$, a contradiction.

Using Lemma 3.7, we can now equip diamonds with the factorization property as in Corollary 3.5 with a $\delta\sigma$ -structure.

Corollary 3.10. Let $U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$ and $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots$ be chains of proper connected metric open subsets of the Riemann sphere $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$ such that $U_r \cap V_r$ is connected and $U_r \cup V_r = \mathcal{X}$ for all $r \in \mathbb{N}$. Let F_{U_r} , F_{V_r} and $F_{U_r \cap V_r}$ denote the fields of meromorphic functions on U_r , V_r and $U_r \cap V_r$, respectively, and let $F = \mathbb{C}(x)$ denote the field of meromorphic functions on $\mathcal{X}$. Define $F_1 = \varinjlim F_{U_r}$ (with respect to the natural inclusions $F_{U_r} \rightarrow F_{U_s}$ for $r \leq s$) and similarly $F_2 = \varinjlim F_{V_r}$ and $F_0 = \varinjlim F_{U_r \cap V_r}$. For every $r \geq 1$ let $\sigma_r: F_{U_r \cap V_r} \rightarrow F_{U_{r+1} \cap V_{r+1}}$ be a morphism such that $\sigma_r(F_{U_r}) \subseteq F_{U_{r+1}}$, $\sigma_r(F_{V_r}) \subseteq F_{V_{r+1}}$ and

$$\begin{array}{ccc} F_{U_r \cap V_r} & \xrightarrow{\sigma_r} & F_{U_{r+1} \cap V_{r+1}} \\ \delta \downarrow & & \downarrow \delta \\ F_{U_r \cap V_r} & \xrightarrow{\sigma_r} & F_{U_{r+1} \cap V_{r+1}} \end{array} \quad \text{and} \quad \begin{array}{ccc} F_{U_{r+1} \cap V_{r+1}} & \xrightarrow{\sigma_{r+1}} & F_{U_{r+2} \cap V_{r+2}} \\ \uparrow & & \uparrow \\ F_{U_r \cap V_r} & \xrightarrow{\sigma_r} & F_{U_{r+1} \cap V_{r+1}} \end{array}$$

commute. Then (F, F_1, F_2, F_0) is a $\delta\sigma$ -diamond with the factorization property.

Proof. By Corollary 3.5, (F, F_1, F_2, F_0) is a diamond with the factorization property. For all $r \in \mathbb{N}$, the fields F_{U_r} are differential fields and the derivation is compatible with the inclusions $F_{U_r} \subseteq F_{U_s}$ for $r \leq s$, hence we can equip F_1 with a derivation and similarly for F_2 and F_0 . These derivations are compatible with the natural inclusions $F \subseteq F_1, F_2$ and $F_1, F_2 \subseteq F_0$ and thus (F, F_1, F_2, F_0) is a differential diamond with the factorization property. The morphisms $\{\sigma_r\}_{r \geq 1}$ induce a well-defined endomorphism $\sigma: F_0 \rightarrow F_0$ that restricts to

endomorphisms on F_1 and F_2 . Moreover, $\delta(\sigma(f)) = \sigma(\delta(f))$ for all $f \in F_0$ and σ restricts to an endomorphism of F because $F_{U_r} \cap F_{V_r} = F$. Hence (F, F_1, F_2, F_0) is a $\delta\sigma$ -diamond with the factorization property. $\square$

Using Corollary 3.10, we can now use the fields from Example 3.8 to obtain $\delta\sigma$ -diamonds $(F, F(m-1), L(m), F(m))$ for all $m \in \mathbb{N}$:

Example 3.11. Consider $F = \mathbb{C}(x)$, $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^1$ and fix a natural number $m \in \mathbb{N}$. For $r \in \mathbb{N}$, let $U_{r,m}, V_{r,m}$ be as defined in Example 3.8. Hence $U_{r,m-1} \cap V_{r,m} = U_{r,m}$ is connected and $U_{r,m-1} \cup V_{r,m} = \mathcal{X}$ for all $r \in \mathbb{N}$.

Define

$$\begin{aligned} F_1 &= \varinjlim_{r \in \mathbb{N}} F_{U_{r,m-1}} = F(m-1) \\ F_2 &= \varinjlim_{r \in \mathbb{N}} F_{V_{r,m}} = L(m) \\ F_0 &= \varinjlim_{r \in \mathbb{N}} F_{U_{r,m-1} \cap V_{r,m}} = F(m). \end{aligned}$$

By Example 3.8 together with Corollary 3.10, (F, F_1, F_2, F_0) is a $\delta\sigma$ -diamond with the factorization property with $F_0^\delta = \mathbb{C} = F^\delta$.

3.3. A patching result for σ -Picard-Vessiot rings. The next theorem allows us to glue together two σ -Picard-Vessiot rings under certain assumptions. The proof is similar to related statements for Picard-Vessiot rings ([BHH16, Thm. 2.4]) and parameterized Picard-Vessiot rings ([Mai15, Thm. 2.2]).

Theorem 3.12. *Let (F, F_1, F_2, F_0) be a $\delta\sigma$ -diamond with the factorization property such that $k := F^\delta = F_0^\delta$. Let G be a σ -algebraic group over k generated by two σ -closed subgroups H_1 and H_2 . For $i = 1, 2$, let R_i/F_i be a σ -Picard-Vessiot ring with σ -Galois group isomorphic to H_i such that $R_i \subseteq F_0$ as an F_i - $\delta\sigma$ -subalgebra. Then there exists a σ -Picard-Vessiot ring R/F with σ -Galois group isomorphic to G and $R \subseteq F_0$ as an F - $\delta\sigma$ -subalgebra.*

Proof. According to Lemma 1.2, we can identify G with a σ -closed subgroup of GL_n for a suitable $n \in \mathbb{N}$ and thus also view H_1 and H_2 as σ -closed subgroups of GL_n . By Proposition 3.1, for $i = 1, 2$, there exists a differential equation $\delta(y) = A_i y$ with $A_i \in F_i^{n \times n}$ together with a fundamental solution matrix $Y_i \in \mathrm{GL}_n(R_i)$ such that $\underline{\mathrm{Gal}}_{Y_i}^{\delta\sigma}(R_i/F_i) = H_i$. Since $R_i \subseteq F_0$, we can consider the matrix $Y_1 Y_2^{-1} \in \mathrm{GL}_n(F_0)$. The factorization property implies that there exist matrices $B_1 \in \mathrm{GL}_n(F_1)$, $B_2 \in \mathrm{GL}_n(F_2)$ with $Y_1 Y_2^{-1} = B_1^{-1} B_2$. Define $Y = B_1 Y_1 = B_2 Y_2 \in \mathrm{GL}_n(F_0)$ and $A = \delta(Y) Y^{-1} \in F_0^{n \times n}$. Then for both $i = 1, 2$,

$$\begin{aligned} A &= \delta(B_i Y_i) Y_i^{-1} B_i^{-1} \\ &= \delta(B_i) B_i^{-1} + B_i A_i B_i^{-1} \in F_i^{n \times n}. \end{aligned}$$

As $F_1 \cap F_2 = F_0$, we conclude that A has entries in F . Consider the differential equation $\delta(y) = Ay$ over F and define $R = F\{Y, 1/\det(Y)\} \subseteq F_0$. Since $F_0^\delta = F^\delta$, R is a σ -Picard-Vessiot ring over F for $\delta(y) = Ay$ by Lemma 1.4. Define $H = \underline{\mathrm{Gal}}_Y^{\delta\sigma}(R/F)$. We claim that $H = G$ as σ -closed subgroups of GL_n . For $i = 1, 2$, $R_i = F_i\{Y_i, 1/\det(Y_i)\} =$

$F_i\{Y, 1/\det(Y)\}$ and

$$H_i = \underline{\text{Gal}}_{Y_i}^{\delta\sigma}(R_i/F_i) = \underline{\text{Gal}}_{B_i Y_i}^{\delta\sigma}(R_i/F_i) = \underline{\text{Gal}}_Y^{\delta\sigma}(R_i/F_i)$$

by Lemma 1.9.(i). Furthermore, $\underline{\text{Gal}}_Y^{\delta\sigma}(R_i/F_i) \leq \underline{\text{Gal}}_Y^{\delta\sigma}(R/F)$ by Lemma 1.9.(ii) as σ -closed subgroups of GL_n . Hence $H_1, H_2 \leq H$ inside GL_n and thus $G \leq H$.

For both $i = 1, 2$,

$$\text{Frac}(R)^G \subseteq \text{Frac}(R)^{H_i} \subseteq \text{Frac}(R_i)^{H_i} = F_i,$$

where the last equality follows from Theorem 1.6. Hence $\text{Frac}(R)^G \subseteq F_1 \cap F_2 = F$ and therefore $G = H$ by Theorem 1.6. $\square$

4. BUILDING BLOCKS

Theorem 3.12 breaks down the problem of realizing a given group as σ -Galois group over $F = \mathbb{C}(x)$ to the task of realizing suitable *building blocks*, i.e., realizing suitable, generating subgroups as σ -Galois groups over certain overfields of F . In this section, we explicitly construct these building blocks.

4.1. Independence of translates. In this subsection, we provide results on the algebraic or linear independence of translates of certain exponential, logarithmic and root functions as a preparation for constructing σ -Picard-Vessiot rings with σ -Galois groups isomorphic to $[\sigma]_{\mathbb{C}}\mathcal{H}$ for $\mathcal{H} = \mathbb{G}_m, \mathbb{G}_a$, or a finite cyclic group.

The following lemma corresponds to the case $\mathcal{H} = \mathbb{G}_m$.

Lemma 4.1. *For $\gamma \in \mathbb{C}$, the (infinite) set of functions*

$$\exp\left(\frac{1}{x-\gamma}\right), \exp\left(\frac{1}{x+1-\gamma}\right), \exp\left(\frac{1}{x+2-\gamma}\right), \dots$$

is algebraically independent over $\mathbb{C}(x)$.

Proof. Assume to the contrary that the set is algebraically dependent. Then there exists an $m \in \mathbb{N}$ such that $\exp\left(\frac{1}{x-\gamma}\right), \exp\left(\frac{1}{x+1-\gamma}\right), \dots, \exp\left(\frac{1}{x+m-\gamma}\right)$ are algebraically dependent. By the theorem of Kolchin-Ostrowski (see e.g. Section 2 in [Kol68]), this can only happen if there exist a non-zero vector $(e_0, \dots, e_m) \in \mathbb{Z}^{m+1}$ with

$$f := \prod_{j=0}^m \left(\exp\left(\frac{1}{x+j-\gamma}\right) \right)^{e_j} \in \mathbb{C}(x).$$

We compute

$$\frac{\delta f}{f} = \sum_{j=0}^m \frac{-e_j}{(x+j-\gamma)^2}.$$

On the other hand, we can factor $f = \alpha \prod_{j=1}^r (x - \alpha_j)^{n_j}$ for suitable $\alpha \in \mathbb{C}$, pairwise distinct elements $\alpha_j \in \mathbb{C}$ and $n_j \in \mathbb{Z}$. Thus

$$\frac{\delta f}{f} = \alpha \sum_{j=1}^r \frac{n_j}{(x - \alpha_j)^2},$$

a contradiction. $\square$

We next treat the case $\mathcal{H} = \mathbb{G}_a$.

Lemma 4.2. *For $\gamma \in \mathbb{C}$, the (infinite) set of functions*

$$\log\left(\frac{1}{x-\gamma} + 1\right), \log\left(\frac{1}{x+1-\gamma} + 1\right), \log\left(\frac{1}{x+2-\gamma} + 1\right), \dots$$

is algebraically independent over $\mathbb{C}(x)$.

Proof. Assume to the contrary that the set is algebraically dependent. Then there exists an $m \in \mathbb{N}$ such that $\log\left(\frac{1}{x-\gamma} + 1\right), \log\left(\frac{1}{x+1-\gamma} + 1\right), \dots, \log\left(\frac{1}{x+m-\gamma} + 1\right)$ are algebraically dependent. By the theorem of Kolchin-Ostrowski (see e.g. Section 2 in [Kol68]), this can only happen if there exist a non-zero vector $(c_0, \dots, c_m) \in \mathbb{C}^{m+1}$ with

$$f := \sum_{j=0}^m c_j \log\left(\frac{1}{x+j-\gamma} + 1\right) \in \mathbb{C}(x).$$

We compute

$$\delta(f) = \sum_{j=0}^m \frac{-c_j}{(x+j-\gamma)(x+j+1-\gamma)} = \sum_{j=0}^m \left(\frac{-c_j}{(x+j-\gamma)} + \frac{c_j}{(x+j+1-\gamma)} \right).$$

As f is contained in $\mathbb{C}(x)$, all terms need to cancel. As the terms with denominator $(x-\gamma)$ and $(x+m+1-\gamma)$ cannot cancel with any other term, we conclude $c_0 = c_m = 0$. Inductively, we obtain that $c_j = 0$ for all j , a contradiction. $\square$

Finally, we treat the cyclic case.

Lemma 4.3. *For $\gamma \in \mathbb{C}$ and $d \geq 1$ a natural number, consider the (infinite) set of functions*

$$f_0 = \sqrt[d]{\frac{1}{x-\gamma} + 1}, f_1 = \sqrt[d]{\frac{1}{x+1-\gamma} + 1}, f_2 = \sqrt[d]{\frac{1}{x+2-\gamma} + 1}, \dots$$

Then for every $j \in \mathbb{N}$, f_j has degree d over $\mathbb{C}(x)(f_0, f_1, \dots, f_{j-1})$.

Proof. Define $E = \mathbb{C}(x)(f_0, \dots, f_{j-1})$ and $L = \mathbb{C}(x)(f_j)$. We claim that E and L are linearly disjoint over $\mathbb{C}(x)$. Recall that $L/\mathbb{C}(x)$ is cyclic and in particular Galois of degree d . Hence it suffices to prove $E \cap L = \mathbb{C}(x)$.

Since $f_j^d = (x+j+1-\gamma)/(x+j-\gamma)$, $L/\mathbb{C}(x)$ is totally ramified at $(x+j+1-\gamma)$ as f_j has valuation $1/d$ at this place. Therefore, the subfield $E \cap L$ is also totally ramified at $(x+j+1-\gamma)$ of ramification index

$$e = [E \cap L : \mathbb{C}(x)].$$

On the other hand, $E/\mathbb{C}(x)$ is unramified at $(x+j+1-\gamma)$. To see this, consider the chain of fields $E_0 = \mathbb{C}(x)(f_0)$, $E_1 = E_0(f_1)$, $\dots$, $E_{j-1} = E_{j-2}(f_{j-1}) = E$. Then it suffices to show that E_m/E_{m-1} is unramified at all places $\mathfrak{p}$ lying over $(x+j+1-\gamma)$ for $m = 1, \dots, j-1$. The minimal polynomial of E_m/E_{m-1} divides $T^d - (x+m+1-\gamma)/(x+m-\gamma)$ which has coefficients in $\mathcal{O}_{\mathfrak{p}}$ and reduces to a separable polynomial modulo $\mathfrak{p}$. Hence $\mathfrak{p}$ splits completely in E_m/E_{m-1} ([Sti09, Thm. 3.3.7]) and the claim follows.

Therefore, the subfield $E \cap L$ of E is also unramified at $(x+j+1-\gamma)$, so $e = 1$ and we conclude $E \cap L = \mathbb{C}(x)$. $\square$

4.2. Constructing explicit σ -Picard-Vessiot rings. In this section, we work over the base field $F = \mathbb{C}(x)$ considered as a $\delta\sigma$ -field via $\delta = \frac{d}{dx}$ and $\sigma(f(x)) = f(x+1)$. For $m \in \mathbb{N}$, let $L(m)$ be as defined in Example 3.8.(i).

Proposition 4.4. *Let $m \in \mathbb{N}$ and let $\mathcal{H}$ be either $\mathbb{G}_m$, $\mathbb{G}_a$ or a finite cyclic group. Then there exists a σ -Picard-Vessiot ring R over $F = \mathbb{C}(x)$ with σ -Galois group isomorphic to $[\sigma]_{\mathbb{C}}\mathcal{H}$ such that $R \subseteq L(m)$ as an F - $\delta\sigma$ -subalgebra.*

Proof. We begin with $\mathcal{H} = \mathbb{G}_m$. Define $a = -(x - mi)^2 \in F$. Then $y = \exp(\frac{1}{x-mi})$ solves the differential equation $\delta(y) = ay$ and y is holomorphic on $\mathbb{P}_{\mathbb{C}}^1 \setminus \{mi\} \supseteq V_{1,m}$. Hence $y \in F_{V_{1,m}} \subseteq L(m)$. Define $R = F\{y, y^{-1}\} \subseteq L(m)$. Then R/F is a σ -Picard-Vessiot ring for $\delta(y) = ay$ by Lemma 1.4, since $L(m)^{\delta} = \mathbb{C} = F^{\delta}$. Using Proposition 2.3 together with Lemma 4.1, we conclude that its σ -Galois group is isomorphic to $[\sigma]_{\mathbb{C}}\mathbb{G}_m$.

We next treat the case $\mathcal{H} = \mathbb{G}_a$. Recall that the complex logarithm is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{-}$, where $\mathbb{R}_{-}$ denotes the interval $(-\infty, 0]$. As $x \mapsto \frac{1}{x-mi} + 1$ defines a holomorphic function $V_{1,m} \rightarrow \mathbb{C} \setminus \mathbb{R}_{-}$, we conclude that $y = \log(\frac{1}{x-mi} + 1)$ is holomorphic on $V_{1,m}$ and in particular, $y \in F_{V_{1,m}} \subseteq L(m)$. Also note that $\delta(y)$ is contained in $\mathbb{C}(x) = F$. Hence $R = F\{y\} \subseteq L(m)$ is a σ -Picard-Vessiot ring over F with σ -Galois group isomorphic to $[\sigma]_{\mathbb{C}}\mathbb{G}_a$ by Proposition 2.4 together with Lemma 4.2.

Finally we treat the case that $\mathcal{H}$ is a finite cyclic group. Let $d \in \mathbb{N}$ be the order of $\mathcal{H}$. Recall that the complex d -th root $\sqrt[d]{x} = \exp(\log(x)/d)$ is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}_{-}$, where $\mathbb{R}_{-}$ denotes the interval $(-\infty, 0]$. As $x \mapsto \frac{1}{x-mi} + 1$ defines a holomorphic function $V_{1,m} \rightarrow \mathbb{C} \setminus \mathbb{R}_{-}$, we conclude that $y = \sqrt[d]{\frac{1}{x-mi} + 1}$ is holomorphic on $V_{1,m}$ and in particular, $y \in F_{V_{1,m}} \subseteq L(m)$. Hence $R = F\{y, y^{-1}\} \subseteq L(m)$ is a σ -Picard-Vessiot ring over F with σ -Galois group $[\sigma]_{\mathbb{C}}\mathcal{H}$ by Proposition 2.5 together with Lemma 4.3. $\square$

5. MAIN RESULT

The following lemma allows to base change a σ -Picard-Vessiot R/F from F to a $\delta\sigma$ -overfield $F_2 \supseteq F$ without shrinking the σ -Galois group under certain assumptions. An analogous statement for Picard-Vessiot rings has been proved in [BHHW18, Lemma 2.9].

Lemma 5.1. *Let (F, F_1, F_2, F_0) be a quadruple of $\delta\sigma$ -fields with $F \subseteq F_1, F_2 \subseteq F_0$ such that $F_1 \cap F_2 = F$ and $F_0^{\delta} = F^{\delta}$. Let R/F be a σ -Picard-Vessiot ring such that $R \subseteq F_1$ as an F - $\delta\sigma$ -subalgebra. Then the compositum $F_2R \subseteq F_0$ is a σ -Picard-Vessiot ring over F_2 with the same σ -Galois group as R/F .*

Proof. Let G be the σ -Galois group of R/F . By Lemma 1.9 (ii), F_2R/F_2 is a σ -Picard-Vessiot ring with σ -Galois group H a σ -closed subgroup of G . We consider the functorial invariants $\text{Frac}(R)^H \subseteq \text{Frac}(R) \subseteq F_1$. As $\text{Frac}(R)^H \subseteq \text{Frac}(F_2R)^H = F_2$ by Theorem 1.6, we conclude $\text{Frac}(R)^H \subseteq F_1 \cap F_2 = F$ and thus $H = G$ by Theorem 1.6. $\square$

We are now in a position to prove our main theorem:

Theorem 5.2. *Consider $F = \mathbb{C}(x)$ as $\delta\sigma$ -field with $\delta = \frac{d}{dx}$ and $\sigma(f(x)) = f(x+1)$. Then, for every linear algebraic group $\mathcal{G}$ over $\mathbb{C}$, there exists a σ -Picard-Vessiot ring over F with σ -Galois group isomorphic to $[\sigma]_{\mathbb{C}}\mathcal{G}$.*

Proof. Set $G = [\sigma]_{\mathbb{C}}\mathcal{G}$. By Proposition 2.2, there exist σ -closed subgroups $H_1, \dots, H_m$ of G with $G = \langle H_1, \dots, H_m \rangle$ and such that for each i , $H_i \simeq [\sigma]_{\mathbb{C}}\mathcal{H}_i$, where $\mathcal{H}_i$ is $\mathbb{G}_m$, $\mathbb{G}_a$, or a finite cyclig group (depending on i).

For $r \in \mathbb{N}$, let

$$U_{r,m} = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid x \notin \bigcup_{l=1}^m ([-r, 0] + li)\}$$

as in Example 3.8.(ii) and let $F_{U_{r,m}}$ denote the field of meromorphic functions on $U_{r,m}$. As explained in Example 3.8, the direct limit $F(m) = \varinjlim_{r \in \mathbb{N}} F_{U_{r,m}}$ is a $\delta\sigma$ -field extension of F with

$F(m)^{\delta} = \mathbb{C}$. We claim that there exists a σ -Picard-Vessiot ring $R(m)$ over F with σ -Galois group G and $R(m) \subseteq F(m)$ as a F - $\delta\sigma$ -subalgebra. We prove the claim by induction on m .

If $m = 1$, then $F(m) = F(1) = L(1)$ with $L(1)$ as defined in Example 3.8.(ii) and the claim follows from Proposition 4.4 applied to $m = 1$.

Now assume that we constructed a σ -Picard-Vessiot ring $R(m-1) \subseteq F(m-1)$ with σ -Galois group $\langle H_1, \dots, H_{m-1} \rangle$. We define connected metric open subsets

$$V_{r,m} = \{x \in \mathbb{P}_{\mathbb{C}}^1 \mid x \notin [-r, 0] + mi\}$$

as in Example 3.8.(i). Hence $U_{r,m-1} \cap V_{r,m} = U_{r,m}$ is connected and $U_{r,m-1} \cup V_{r,m} = \mathcal{X}$ for all $r \in \mathbb{N}$. We define

$$\begin{aligned} F_1 &= \varinjlim_{r \in \mathbb{N}} F_{U_{r,m-1}} = F(m-1) \\ F_2 &= \varinjlim_{r \in \mathbb{N}} F_{V_{r,m}} = L(m) \\ F_0 &= \varinjlim_{r \in \mathbb{N}} F_{U_{r,m-1} \cap V_{r,m}} = F(m). \end{aligned}$$

As explained in Example 3.11, (F, F_1, F_2, F_0) is a $\delta\sigma$ -diamond with the factorization property and moreover, $F_0^{\delta} = \mathbb{C} = F^{\delta}$ holds. By Lemma 5.1, we can thus lift $R(m-1) \subseteq F_1$ to a σ -Picard-Vessiot ring $R_2 = F_2 R(m-1)$ over F_2 with σ -Galois group $\langle H_1, \dots, H_{m-1} \rangle$ and $R_2 \subseteq F_0$.

On the other hand, Proposition 4.4 provides us with a σ -Picard-Vessiot ring $\tilde{R}/F$ with σ -Galois group H_m and $\tilde{R} \subseteq L(m) = F_2$ as a F - $\delta\sigma$ -subalgebra. Again by Lemma 5.1, we may lift $\tilde{R}$ to a σ -Picard-Vessiot ring $R_1 = F_1 \tilde{R}$ over F_1 with σ -Galois group H_m and $R_1 \subseteq F_0$.

Using the patching result Theorem 3.12, we obtain a σ -Picard-Vessiot ring $R(m)$ over F with σ -Galois group $\langle H_1, \dots, H_m \rangle$ and with $R(m) \subseteq F_0 = F(m)$ as claimed. $\square$

6. DIFFERENCE ALGEBRAIC GROUPS THAT ARE NOT σ -GALOIS GROUPS

Since, as shown above, all linear algebraic groups over $\mathbb{C}$, considered as σ -algebraic groups, occur as σ -Galois groups over $\mathbb{C}(x)$, one may feel tempted to conjecture that in fact all σ -algebraic groups over $\mathbb{C}$ occur as σ -Galois groups over $\mathbb{C}(x)$. However, this is not true. For example, as we will see below, the σ -closed subgroup G of GL_1 defined by $G(S) = \{g \in \mathrm{GL}_1(S) \mid g^2 = 1, \sigma(g) = g\}$ for any $\mathbb{C}$ - σ -algebra S is not a σ -Galois group over $\mathbb{C}(x)$. Moreover, we show that no proper non-trivial subgroup of the additive group $\mathbb{G}_a$ is a σ -Galois group over $\mathbb{C}(x)$.

6.1. A necessary criterion. In this section, we will isolate two properties that any σ -Galois group over $\mathbb{C}(x)$ must have. The main result of this section is the following.

Theorem 6.1. *If G is a σ -Galois group over the $\delta\sigma$ -field $\mathbb{C}(x)$ with derivation $\delta = \frac{d}{dx}$ and endomorphism σ given by $\sigma(f(x)) = f(x+1)$, then G is σ -reduced and σ -connected.*

The definition of σ -reduced and σ -connected is given below. Theorem 6.1 is proved at the end of this section. The fact that every σ -Galois group over $\mathbb{C}(x)$ is σ -reduced follows rather directly from [DVHW14] and the fact that $\sigma: \mathbb{C}(x) \rightarrow \mathbb{C}(x)$ is surjective. On the other hand, the fact that every σ -Galois group over $\mathbb{C}(x)$ is σ -connected, essentially goes back to the fact that $\mathbb{C}(x)$ does not have any non-trivial finite difference field extensions.

We first discuss σ -connected σ -algebraic groups and the related notion of the group $\pi_0^\sigma(G)$ of σ -connected components of a σ -algebraic group G . We do not strive for a comprehensive study of these notions. The interested reader is referred to Section 4.2 of [Wib15]. We will only introduce the definitions and results necessary for proving Theorem 6.1. Most of the required difference algebraic results appeared in [TW18].

Let k be a σ -field. To motivate the definition of the group of σ -connected components of a σ -algebraic group, let us first recall the definition of the group of connected components of an algebraic group. See e.g., [Wat79, Chapter 6] or [Mil17, Section 2, g].

Recall that a group scheme $\mathcal{G}$ of finite type over k is *étale* if $k[\mathcal{G}]$ is an étale k -algebra, i.e., $k[\mathcal{G}] \otimes_k \bar{k}$ is a finite direct product of copies of $\bar{k}$. The group $\pi_0(\mathcal{G})$ of connected components of $\mathcal{G}$ can be defined through the following universal property: There exists a morphism $\mathcal{G} \rightarrow \pi_0(\mathcal{G})$ of affine group schemes over k such that $\pi_0(\mathcal{G})$ is étale and for every étale group scheme $\mathcal{H}$ with a morphism $\mathcal{G} \rightarrow \mathcal{H}$, there exists a unique morphism $\pi_0(\mathcal{G}) \rightarrow \mathcal{H}$ such that

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad} & \pi_0(\mathcal{G}) \\ & \searrow & \swarrow \text{dotted} \\ & \mathcal{H} & \end{array}$$

commutes. The identity component $\mathcal{G}^\circ$ of $\mathcal{G}$ can be defined as the kernel of $\mathcal{G} \rightarrow \pi_0(\mathcal{G})$. The existence of $\pi_0(\mathcal{G})$ can be established as follows: Let $\pi_0(k[\mathcal{G}])$ denote the union of the étale k -subalgebras of $k[\mathcal{G}]$. Then one can show that $\pi_0(k[\mathcal{G}])$ is an étale algebra and a Hopf subalgebra of $k[\mathcal{G}]$. So $\pi_0(\mathcal{G}) = \text{Spec}(\pi_0(k[\mathcal{G}]))$ is an étale algebraic group and the morphism $\mathcal{G} \rightarrow \pi_0(\mathcal{G})$ corresponding to the inclusion $\pi_0(k[\mathcal{G}]) \rightarrow k[\mathcal{G}]$ has the desired universal property.

To follow a similar path for difference algebraic groups we first need to define an appropriate difference analog of étale algebras. Following [TW18] we make the following definition.

Definition 6.2. *A k - σ -algebra R is σ -separable if $\sigma: R \otimes_k k' \rightarrow R \otimes_k k'$ is injective for every σ -field extension k' of k . A σ -separable k - σ -algebra is strongly σ -étale if it is an étale k -algebra.*

For a k - σ -algebra R , we denote the union of all strongly σ -étale k - σ -subalgebras of R with $\pi_0^\sigma(R/k)$. If the base σ -field k is clear from the context we will usually write $\pi_0^\sigma(R)$ instead of $\pi_0^\sigma(R/k)$. We know from [TW18, Rem. 1.18] that $\pi_0^\sigma(R)$ is a k - σ -subalgebra of R .

Definition 6.3. A σ -algebraic group G is strongly σ -étale if $k\{G\}$ is a strongly σ -étale k - σ -algebra.

Example 6.4. Let G be the σ -closed subgroup of GL_1 defined by $G(S) = \{g \in \mathrm{GL}_1(S) \mid g^2 = 1, \sigma(g) = g\}$ for any k - σ -algebra S . Then $k\{G\} = k \times k$, with σ given by $\sigma(a, b) = (\sigma(a), \sigma(b))$. Thus G is strongly σ -étale.

On the other hand, the group G defined by $G(S) = \{g \in \mathrm{GL}_1(S) \mid g^2 = 1, \sigma(g) = 1\}$ for any k - σ -algebra S is not strongly σ -étale.

The following proposition was essentially proved in [TW18], but there the results were formulated in an algebraic manner. Here we give a more geometric interpretation.

Proposition 6.5. Let G be a σ -algebraic group. Then there exists a strongly σ -étale σ -algebraic group $\pi_0^\sigma(G)$ together with a morphism $G \rightarrow \pi_0^\sigma(G)$ of σ -algebraic groups satisfying the following universal property: If H is a strongly σ -étale σ -algebraic group with a morphism $G \rightarrow H$, then there exists a unique morphism $\pi_0^\sigma(G) \rightarrow H$ such that

$$\begin{array}{ccc} G & \xrightarrow{\quad} & \pi_0^\sigma(G) \\ & \searrow & \swarrow \text{dotted} \\ & H & \end{array}$$

commutes.

Proof. A k - σ -Hopf algebra ([Wib, Def. 2.2]) is a Hopf algebra over k that has the structure of a k - σ -algebra such that the Hopf algebra structure maps are morphisms of k - σ -algebras. It follows directly from the definitions that the category of σ -algebraic groups is equivalent to the category of k - σ -Hopf algebras that are finitely σ -generated as k - σ -algebra ([Wib, Rem. 2.3]).

By Theorem 3.2 of [TW18] the k - σ -subalgebra $\pi_0^\sigma(k\{G\})$ of $k\{G\}$ is a k - σ -Hopf-subalgebra and by Theorem 4.5 of [Wib] every k - σ -Hopf-subalgebra of a finitely σ -generated k - σ -Hopf algebra is finitely σ -generated. It follows that the σ -algebraic group $\pi_0^\sigma(G)$ represented by $\pi_0^\sigma(k\{G\})$ is strongly σ -étale. Moreover, since a quotient of a strongly σ -étale k - σ -algebra is strongly σ -étale ([TW18, Lemma 1.15]) the inclusion $\pi_0^\sigma(k\{G\}) \subseteq k\{G\}$ has the following property: If $k\{H\}$ is a strongly σ -étale k - σ -Hopf-algebra with a morphism $\psi: k\{H\} \rightarrow k\{G\}$ then $\psi(k\{H\}) \subseteq \pi_0^\sigma(k\{G\})$. Geometrically, this translates to the required universal property. $\square$

Definition 6.6. Let G be a σ -algebraic group. The strongly σ -étale σ -algebraic group $\pi_0^\sigma(G)$ from Proposition 6.5 is called the group of σ -connected components of G . If $\pi_0^\sigma(G)$ is the trivial group, then G is σ -connected.

Remark 6.7. The terminology “ σ -connected” is justified by the following fact ([Wib15, Lemma 4.2.35]): A σ -algebraic group G is σ -connected if and only if $\mathrm{Spec}(k\{G\})$ is connected with respect to the σ -topology. The closed sets of the σ -topology on $\mathrm{Spec}(k\{G\})$ are the invariant (Zariski) closed sets.

Example 6.4 gives an example of σ -algebraic group that is not σ -connected. Any σ -algebraic group G such that $k\{G\}$ is an integral domain is σ -connected because in that case

$\text{Spec}(k\{G\})$ is connected and so a fortiori σ -connected (Remark 6.7). One can also show that any (even non-connected) algebraic group is σ -connected when considered as a σ -algebraic group ([Wib15, Prop. 4.2.43]).

Example 6.8. Let G be the σ -closed subgroup of GL_2 such that for a k - σ -algebra S and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(S)$ we have $g \in G(S)$ if and only if

$$\begin{aligned} c\sigma(a) + d\sigma(b) &= 0, \\ a\sigma(c) + b\sigma(d) &= 0, \\ a\sigma(a) + b\sigma(b) &= c\sigma(c) + d\sigma(d), \\ (a\sigma(a) + b\sigma(b))^2 &= 1, \\ \sigma(a)\sigma^2(a) + \sigma(b)\sigma^2(b) &= a\sigma(a) + b\sigma(b). \end{aligned}$$

To see that these equations indeed define a subgroup, note the following alternative description of G . Let D be the σ -closed subgroup of GL_2 defined by

$$D(S) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \text{GL}_2(S) \mid a^2 = 1, \sigma(a) = a \right\}$$

for any k - σ -algebra S . Then $G(S) = \{g \in \text{GL}_2(S) \mid \sigma(g)g^T \in D(S)\}$. This is a subgroup of $\text{GL}_2(S)$ because $D(S)$ lies in the center of $\text{GL}_2(S)$.

We claim that $\phi: G \rightarrow D, g \mapsto \sigma(g)g^T$ satisfies the universal property of Proposition 6.5. In particular, $\pi_0^\sigma(G)$ is isomorphic to the σ -algebraic group defined in Example 6.4. The easiest way to see this is to use some results from [Wib15]. In [Wib15, Prop. 4.2.41] it is shown that if $\phi: G \rightarrow D$ is a morphism of σ -algebraic groups such that the dual map $\phi^*: k\{D\} \rightarrow k\{G\}$ is injective, D is strongly σ -étale and $\ker(\phi)$ is σ -connected, then ϕ satisfies the universal property.

We first note that ϕ is a morphism of σ -algebraic groups because D lies in the center of GL_2 and that D is strongly σ -étale because it is isomorphic to the group in Example 6.4. To see that ϕ^* is injective, note that ϕ^* sends the image of x in $k[x]/(x^2 - 1) = k \times k = k\{D\}$ to the image of $a\sigma(a) + b\sigma(b)$ in $k\{G\}$. Thus if ϕ^* was not injective, the latter image would need to equal 1 or -1 . However, neither is the case. For example, for the identity matrix $a\sigma(a) + b\sigma(b)$ evaluates to 1, but for the matrix $\begin{pmatrix} (1, -1) & 0 \\ 0 & (1, -1) \end{pmatrix} \in G(S)$, where $S = k \times k$ with $\sigma(\lambda, \mu) = (\sigma(\mu), \sigma(\lambda))$, the expression $a\sigma(a) + b\sigma(b)$ evaluates to -1 . Finally, the kernel N of ϕ is the σ -closed subgroup of GL_2 defined by the equation $\sigma(g)g^T = I_2$. Since this equation can be rewritten as $\sigma(g) = (g^T)^{-1}$, we see that $k\{N\}$ is isomorphic to $k[\text{GL}_2]$. In particular, $k\{N\}$ is an integral domain. Thus N is σ -connected.

We will need two lemmas from [TW18].

Lemma 6.9 ([TW18, Lemma 1.25]). *Let R and S be k - σ -algebras. Then $\pi_0^\sigma(R \otimes_k S) = \pi_0^\sigma(R) \otimes_k \pi_0^\sigma(S)$.*

Lemma 6.10 ([TW18, Lemma 1.24]). *Let R be a k - σ -algebra and k'/k an extension of σ -fields. Then $\pi_0^\sigma(R \otimes_k k'/k') = \pi_0^\sigma(R/k) \otimes_k k'$.*

We now return to the σ -Picard-Vessiot theory. As before, F is a $\delta\sigma$ -field of characteristic zero and k is the σ -field of δ -constants of F .

Proposition 6.11. *Let R/F be a σ -Picard-Vessiot ring with σ -Galois group G . Then $\pi_0^\sigma(R/F)$ is a σ -Picard-Vessiot ring over F with σ -Galois group isomorphic to $\pi_0^\sigma(G)$.*

Proof. We will first show that $\pi_0^\sigma(R/F)$ is a finite field extension of F . Note that if an étale algebra (over a field) is an integral domain, then it is a field. Since R is an integral domain and $\pi_0^\sigma(R/F)$ is a union of étale algebras contained in R it follows that $\pi_0^\sigma(R/F)$ is a field. So $\pi_0^\sigma(R/F)$ is an algebraic field extension of F and a σ -subfield of the field of fractions E of R . Since E is finitely σ -generated as a σ -field extension of F and any intermediate σ -field of a finitely σ -generated σ -field extension is itself finitely σ -generated ([Lev08, Theorem 4.4.1]), it follows that $\pi_0^\sigma(R/F)$ is finitely σ -generated over F . Since $\pi_0^\sigma(R/F)$ is a union of finite σ -field extensions of F , we see that indeed $\pi_0^\sigma(R/F)$ is a finite σ -field extension of F .

To see that $\pi_0^\sigma(R/F)$ is stable under the derivation $\delta: R \rightarrow R$, let $a \in \pi_0^\sigma(R/F)$ have minimal polynomial f over F . Then $0 = \delta(f(a)) = f^\delta(a) + f'(a)\delta(a)$, where f^δ is the polynomial obtained from f by applying δ to the coefficients of f . Since $f'(a) \neq 0$, it follows that $\delta(a) \in \pi_0^\sigma(R/F)$. So $\pi_0^\sigma(R/F)$ is an F - $\delta\sigma$ -algebra.

We next apply $\pi_0^\sigma(-/F)$ to the identity $R \otimes_F R = R \otimes_k k\{G\} = R \otimes_F (F \otimes_k k\{G\})$ from equation (1). Using Lemmas 6.9 and 6.10 we find

$$\begin{aligned} \pi_0^\sigma(R/F) \otimes_F \pi_0^\sigma(R/F) &= \pi_0^\sigma(R \otimes_F R/F) = \pi_0^\sigma(R/F) \otimes_F \pi_0^\sigma(F \otimes_k k\{G\}/F) = \\ &= \pi_0^\sigma(R/F) \otimes_F (F \otimes_k (\pi_0^\sigma(k\{G\}/k))) = \pi_0^\sigma(R/F) \otimes_k \pi_0^\sigma(k\{G\}/k) = \\ &= \pi_0^\sigma(R/F) \otimes_k k\{\pi_0^\sigma(G)\}. \end{aligned}$$

Since $\pi_0^\sigma(k\{G\}/k) \subseteq k\{G\} = (R \otimes_F R)^\delta$, we see that $\pi_0^\sigma(R/F) \otimes_F \pi_0^\sigma(R/F)$ is generated by $(\pi_0^\sigma(R/F) \otimes_F \pi_0^\sigma(R/F))^\delta$ as a left $\pi_0^\sigma(R/F)$ -module. Moreover, $\pi_0^\sigma(R/F)^\delta = F^\delta$. Thus the extension $\pi_0^\sigma(R/F)/F$ of differential fields is a Picard-Vessiot extension in the sense of [AMT09, Def. 1.8]. Since this definition is equivalent to the standard one ([AMT09, Theorem 3.11]) it follows that $\pi_0^\sigma(R/F)$ is a Picard-Vessiot extension for some linear differential equation $\delta(y) = Ay$, with $A \in F^{n \times n}$. Since $\pi_0^\sigma(R/F)$ is an algebraic extension, the Picard-Vessiot ring and the Picard-Vessiot extension coincide. So the F - δ -algebra $\pi_0^\sigma(R/F)$ is a Picard-Vessiot ring for $\delta(y) = Ay$. It is then clear that the F - $\delta\sigma$ -algebra $\pi_0^\sigma(R/F)$ is a σ -Picard-Vessiot ring for $\delta(y) = Ay$.

Since $(\pi_0^\sigma(R/F) \otimes_F \pi_0^\sigma(R/F))^\delta = \pi_0^\sigma(k\{G\}/k)$, we see that the σ -Galois group $\pi_0^\sigma(R/F)$ is isomorphic to $\pi_0^\sigma(G)$. $\square$

Corollary 6.12. *Let F be a $\delta\sigma$ -field such that F does not have any non-trivial finite difference field extensions. Then every σ -Galois group over F is σ -connected.*

Proof. Let R be a σ -Picard-Vessiot ring over F with σ -Galois group G . As shown in the first paragraph of the proof of Proposition 6.11, the F - σ -algebra $\pi_0^\sigma(R/F)$ is a finite σ -field extension of F . Thus, by assumption, it is trivial and consequently also its σ -Galois group $\pi_0^\sigma(G)$ is trivial. $\square$

Following [DVHW14], a σ -algebraic group G is σ -reduced if $\sigma: k\{G\} \rightarrow k\{G\}$ is injective. The group in Example 6.4 is σ -reduced. An example of a σ -algebraic group G that is not σ -reduced would be $G = \{g \in \mathrm{GL}_n \mid \sigma(g) = I_n\}$.

Proof of Theorem 6.1. Let R be a σ -Picard-Vessiot ring over $\mathbb{C}(x)$ with σ -Galois group G . In [DVHW14, Cor. 4.4] it is shown that a σ -Galois group over a $\delta\sigma$ -field F such that $\sigma: F \rightarrow F$ is an automorphism is σ -reduced. Since $\sigma: \mathbb{C}(x) \rightarrow \mathbb{C}(x)$, $f(x) \mapsto f(x+1)$ is an automorphism, it follows that G is σ -reduced.

The difference field $\mathbb{C}(x)$ (with $\sigma(f(x)) = f(x+1)$) does not have any non-trivial finite difference field extensions (see the proof of Theorem XIX in Chapter 9 of [Coh65]). It thus follows from Corollary 6.12 that G is σ -connected. $\square$

6.2. Unipotent groups. In this section, we show that being σ -reduced and σ -connected is far from sufficient for being a σ -Galois group over $\mathbb{C}(x)$. In fact, we show that no proper non-trivial σ -closed subgroup of $\mathbb{G}_a$ is a σ -Galois group over $\mathbb{C}(x)$. As a consequence of this result we also deduce that the constant subgroups of unipotent linear algebraic groups do not occur as σ -Galois groups over $\mathbb{C}(x)$.

Lemma 6.13. *Let F be a $\delta\sigma$ -field of characteristic zero with field of constants $k = F^\delta$. Let G be a σ -closed subgroup of the additive group $\mathbb{G}_a$ over k and let R/F be a σ -Picard-Vessiot ring with σ -Galois group G . Then there exists an element $y \in R$ with $R = F\{y\}$ and $\delta(y) \in F$.*

Proof. Let X be the σ -variety over the σ -field F represented by the F - σ -algebra R . As explained in the first paragraph of the proof of Prop 3.1, there is a canonical structure on X as a right G_F -torsor and we may thus also consider X as a left G_F -torsor and use the results on left G_F -torsors in [BW]. Either $G = \mathbb{G}_a$ and thus X is the trivial G -torsor ([BW, Cor. 3.6]) or there exists an expression $\mathcal{L}(y) = \sigma^n(y) + \lambda_{n-1}\sigma^{n-1}(y) + \cdots + \lambda_1\sigma(y) + \lambda_0y$ with $\lambda_i \in k$ such that $G(S) = \{g \in S \mid \mathcal{L}(g) = 0\}$ for all k - σ -algebras S ([DVHW17, Cor. A.3]). In the latter case, it was shown in Example 5.4 of [BW] that there exists an $a \in F$ such that X is isomorphic as G_F -torsor to the torsor X_a defined as the σ -closed σ -subvariety of $\mathbb{A}_k^1$ given by the equation $\mathcal{L}(y) = a$ with G -action given by addition. We conclude that in both cases, X is a σ -closed σ -subvariety of $\mathbb{A}_k^1$, thus there exists an $y \in R$ with $R = F\{X\} = F\{y\}$. Moreover, the G_F -action is given by addition, so

$$g(\delta(y)) = \delta(g(y)) = \delta(y + g) = \delta(y) + \delta(g) = \delta(y)$$

for every $g \in G(S)$. It thus follows from the Galois correspondence that $\delta(y) \in F$. $\square$

Recall that every element of $\mathbb{C}(x)$ has a unique partial fraction decomposition

$$g(x) + \sum_{j=1}^r \frac{\alpha_j}{x + \beta_j} + \sum_{j=1}^{r_2} \frac{\alpha_{2j}}{(x + \beta_{2j})^2} + \cdots + \sum_{j=1}^{r_m} \frac{\alpha_{mj}}{(x + \beta_{mj})^m}$$

with $g \in \mathbb{C}[x]$, $m, r, r_2, \dots, r_m \in \mathbb{N}$ and $\alpha_j, \alpha_{ij}, \beta_j, \beta_{ij} \in \mathbb{C}$. The term $\sum_{j=1}^r \frac{\alpha_j}{x + \beta_j}$ is called the *logarithmic part*. An element in $\mathbb{C}(x)$ has an antiderivative inside $\mathbb{C}(x)$ if and only if its logarithmic part vanishes.

Proposition 6.14. *Let G be a non-trivial proper σ -closed subgroup of the additive group $\mathbb{G}_a$ over $\mathbb{C}$. Then G is not a σ -Galois group over the $\delta\sigma$ -field $\mathbb{C}(x)$ with derivation $\delta = \frac{d}{dx}$ and endomorphism σ given by $\sigma(f(x)) = f(x+1)$.*

Proof. Suppose, for a contradiction, that $R/\mathbb{C}(x)$ is a σ -Picard-Vessiot ring with σ -Galois group G . By Lemma 6.13, there exists an element $y \in R$ with $R = \mathbb{C}(x)\{y\}$ and $\delta(y) \in \mathbb{C}(x)$. Set $a = \delta(y) \in \mathbb{C}(x)$. As δ and σ commute, $\delta(\sigma^l(y)) = \sigma^l(a)$ for all $l \in \mathbb{N}$. By Proposition 2.4, there exists an $n \in \mathbb{N}$ such that $y, \sigma(y), \sigma^2(y), \dots, \sigma^n(y)$ are algebraically dependent over $\mathbb{C}(x)$. By the theorem of Kolchin-Ostrowski (see e.g. Section 2 in [Kol68]), this can only happen if there exists a non-zero vector $(c_0, \dots, c_n) \in \mathbb{C}^{n+1}$ with $\sum_{l=0}^n c_l \sigma^l(y) \in \mathbb{C}(x)$. We may assume that $c_n \neq 0$. We differentiate and obtain that $\sum_{l=0}^n c_l \sigma^l(a)$ has no logarithmic part. Let $\sum_{j=1}^r \frac{\alpha_j}{x+\beta_j}$ be the logarithmic part of a with pairwise distinct $\beta_1, \dots, \beta_r$ and non-zero elements α_j . Then the logarithmic part of $\sum_{l=0}^n c_l \sigma^l(a)$ equals

$$0 = \sum_{l=0}^n \sum_{j=1}^r \frac{c_l \alpha_j}{x+l+\beta_j}.$$

We claim that the logarithmic part of a is zero, i.e., $\sum_{j=1}^r \frac{\alpha_j}{x+\beta_j}$ is an empty sum. Otherwise, we can choose j_0 with $1 \leq j_0 \leq r$ such that $\operatorname{Re}(\beta_{j_0})$ is maximal among all elements $\operatorname{Re}(\beta_j)$. Then the term with denominator $x+n+\beta_{j_0}$ cannot cancel with any other term (here we use that $\beta_1, \dots, \beta_r$ are pairwise distinct) and hence $c_n \alpha_{j_0} = 0$, a contradiction. Therefore, the logarithmic part of a vanishes and hence $y \in \mathbb{C}(x)$, $R = \mathbb{C}(x)$ and $G = \{1\}$, contradicting that G is non-trivial. $\square$

We consider expressions of the form $\mathcal{L}(y) = \sigma^n(y) + \lambda_{n-1}\sigma^{n-1}(y) + \dots + \lambda_1\sigma(y) + \lambda_0 y$ with $\lambda_i \in \mathbb{C}$ and the corresponding σ -closed subgroups $G_{\mathcal{L}}$ of $\mathbb{G}_a$ with $G_{\mathcal{L}}(S) = \{g \in S \mid \mathcal{L}(g) = 0\}$ for all $\mathbb{C}$ - σ -algebras S . Every proper σ -closed subgroup of the additive group $\mathbb{G}_a$ is isomorphic to such a $G_{\mathcal{L}}$ ([DVHW17, Cor. A.3]).

Remark 6.15. *Note that $G_{\mathcal{L}}$ is σ -reduced if and only if $\lambda_0 \neq 0$ and it is always σ -connected. Indeed, $\sigma: \mathbb{C}\{G_{\mathcal{L}}\} \rightarrow \mathbb{C}\{G_{\mathcal{L}}\}$ is injective if and only if $\lambda_0 \neq 0$ and $\mathbb{C}\{G_{\mathcal{L}}\}$ is an integral domain, thus $G_{\mathcal{L}}$ is connected and in particular σ -connected. Hence most of the groups in Proposition 6.14 satisfy the necessary conditions given in Theorem 6.1 but yet do not occur as σ -Galois groups.*

We remark that the multiplicative case differs from the additive case. Indeed, the constant points of the multiplicative group $\mathbb{G}_m$ do occur as σ -Galois group over $\mathbb{C}(x)$ (see Example 1.5), whereas Proposition 6.14 implies that the constant subgroup of the additive group does not occur as σ -Galois group over $\mathbb{C}(x)$.

To generalize this result from $\mathbb{G}_a$ to all unipotent groups, we need some basics about quotients of σ -algebraic groups ([DVHW14, A.9] or [Wib15, Chapter 3]): Let G be a σ -algebraic group and N a normal σ -closed subgroup. The quotient G/N can be defined through the usual universal property. A morphism $\phi: G \rightarrow H$ of σ -algebraic groups is a quotient map (i.e., can be identified with the canonical map $G \rightarrow G/N$ for some normal σ -closed subgroup N of G) if and only if the dual map $\phi^*: k\{H\} \rightarrow k\{G\}$ is injective.

Corollary 6.16. *Let $\mathcal{G} \leq \mathrm{GL}_n$ be a non-trivial unipotent linear algebraic group over $\mathbb{C}$ and let G be the constant subgroup of $\mathcal{G}$, i.e., $G(S) = \{g \in \mathcal{G}(S) \mid \sigma(g) = g\}$ for all $\mathbb{C}$ - σ -algebras S . Then G is not a σ -Galois group over the $\delta\sigma$ -field $\mathbb{C}(x)$ with derivation $\delta = \frac{d}{dx}$ and endomorphism σ given by $\sigma(f(x)) = f(x+1)$.*

Proof. Over a field of characteristic zero, every non-trivial unipotent linear algebraic group has a quotient isomorphic to $\mathbb{G}_a$ ([Mil17, Prop. 14.21 and Rem. 14.24 (a)]). Let $\pi: \mathcal{G} \rightarrow \mathbb{G}_a$ be such a quotient map. We claim that π induces a quotient map $\phi: G \rightarrow H$, where H is the constant subgroup of $\mathbb{G}_a$, i.e., $H(S) = \{g \in S \mid \sigma(g) = g\}$ for all $\mathbb{C}$ - σ -algebras S .

The morphism $\pi: \mathcal{G} \rightarrow \mathbb{G}_a$ is given by a polynomial $p \in \mathbb{C}[X_{ij}, 1/\det(X)] = \mathbb{C}[\mathrm{GL}_n]$. So $\pi(g) = p(g)$ for $g \in \mathcal{G}(T) \leq \mathrm{GL}_n(T)$ and T a $\mathbb{C}$ -algebra. The induced morphism $G \rightarrow [\sigma]_k \mathbb{G}_a$ of σ -algebraic groups is also given by $g \mapsto p(g)$ for $g \in G(S)$ and S a $\mathbb{C}$ - σ -algebra and therefore maps into H . To see that the dual of $\phi: G \rightarrow H$ is injective, note that the dual $\pi^*: \mathbb{C}[t] = \mathbb{C}[\mathbb{G}_a] \rightarrow \mathbb{C}[\mathcal{G}]$, $t \mapsto \bar{p}$ of π is injective because π is dominant. The coordinate ring of G is $\mathbb{C}\{G\} = \mathbb{C}[\mathcal{G}]$ with σ the identity and similarly for H . Moreover, ϕ^* agrees with π^* as a morphism of $\mathbb{C}$ -algebras. In particular, ϕ^* is injective and so ϕ is a quotient map.

Now suppose, for a contradiction, that G is a σ -Galois group over $\mathbb{C}(x)$. Then, by the second fundamental theorem of σ -Galois theory ([DVHW14, Thm. 3.3]), also H would be a σ -Galois group $\mathbb{C}(x)$. This contradicts Proposition 6.14. $\square$

REFERENCES

- [AMT09] Katsutoshi Amano, Akira Masuoka, and Mitsuhiro Takeuchi. Hopf algebraic approach to Picard-Vessiot theory. In *Handbook of algebra*, volume 6, pages 127–171. Elsevier/North-Holland, Amsterdam, 2009.
- [Arr13] Carlos E. Arreche. A Galois-theoretic proof of the differential transcendence of the incomplete Gamma function. *J. Algebra*, 389:119–127, 2013.
- [AS17] Carlos E. Arreche and Michael F. Singer. Galois groups for integrable and projectively integrable linear difference equations. *J. Algebra*, 480:423–449, 2017.
- [Bac18] Annette Bachmayr. New classes of parameterized differential Galois groups. *Math. Z.*, 288(1-2):361–381, 2018.
- [BHH16] Annette Bachmayr, David Harbater, and Julia Hartmann. Differential Galois groups over Laurent series fields. *Proc. Lond. Math. Soc. (3)*, 112(3):455–476, 2016.
- [BHH18] Annette Bachmayr, David Harbater, and Julia Hartmann. Differential embedding problems over Laurent series fields. *J. Algebra*, 513:99–112, 2018.
- [BHHP20] Annette Bachmayr, David Harbater, Julia Hartmann, and Florian Pop. Large fields in differential Galois theory. *J. Inst. Math. Jussieu*, 2020. <https://doi.org/10.1017/S1474748020000018>.
- [BHHW] Annette Bachmayr, David Harbater, Julia Hartmann, and Michael Wibmer. Free differential Galois groups. arXiv:1904.07806.
- [BHHW18] Annette Bachmayr, David Harbater, Julia Hartmann, and Michael Wibmer. Differential embedding problems over complex function fields. *Doc. Math.*, 23:241–291, 2018.
- [BS64] A. Borel and J.-P. Serre. Théorèmes de finitude en cohomologie galoisienne. *Comment. Math. Helv.*, 39:111–164, 1964.
- [BW] Annette Bachmayr and Michael Wibmer. Torsors for difference algebraic groups. arXiv:1607.07035v1.
- [Coh65] Richard M. Cohn. *Difference algebra*. Interscience Publishers John Wiley & Sons, New York-London-Sydney, 1965.

- [CS07] Phyllis J. Cassidy and Michael F. Singer. Galois theory of parameterized differential equations and linear differential algebraic groups. In *Differential equations and quantum groups*, volume 9 of *IRMA Lect. Math. Theor. Phys.*, pages 113–155. Eur. Math. Soc., Zürich, 2007.
- [DHR18] Thomas Dreyfus, Charlotte Hardouin, and Julien Roques. Hypertranscendence of solutions of Mahler equations. *J. Eur. Math. Soc. (JEMS)*, 20(9):2209–2238, 2018.
- [Dre14] Thomas Dreyfus. A density theorem in parametrized differential Galois theory. *Pacific J. Math.*, 271(1):87–141, 2014.
- [DV12] Lucia Di Vizio. Approche galoisienne de la transcendance différentielle. In *Transcendance et irrationalité*, SMF Journée Annuelle [SMF Annual Conference], pages 1–20. Société Mathématique de France, 2012. arXiv:1404.3611.
- [DVHW14] Lucia Di Vizio, Charlotte Hardouin, and Michael Wibmer. Difference Galois theory of linear differential equations. *Adv. Math.*, 260:1–58, 2014.
- [DVHW17] Lucia Di Vizio, Charlotte Hardouin, and Michael Wibmer. Difference algebraic relations among solutions of linear differential equations. *J. Inst. Math. Jussieu*, 16(1):59–119, 2017.
- [Har05] Julia Hartmann. On the inverse problem in differential Galois theory. *J. Reine Angew. Math.*, 586:21–44, 2005.
- [HH10] David Harbater and Julia Hartmann. Patching over fields. *Israel J. Math.*, 176:61–107, 2010.
- [HMO17] Charlotte Hardouin, Andrei Minchenko, and Alexey Ovchinnikov. Calculating differential Galois groups of parametrized differential equations, with applications to hypertranscendence. *Math. Ann.*, 368(1-2):587–632, 2017.
- [HS08] Charlotte Hardouin and Michael F. Singer. Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.
- [Kol68] E. R. Kolchin. Algebraic groups and algebraic dependence. *Amer. J. Math.*, 90:1151–1164, 1968.
- [Lan08] Peter Landesman. Generalized differential Galois theory. *Trans. Amer. Math. Soc.*, 360(8):4441–4495, 2008.
- [Lev08] Alexander Levin. *Difference algebra*, volume 8 of *Algebra and Applications*. Springer, New York, 2008.
- [Mai15] Annette Maier. On the parameterized differential inverse Galois problem over $k((t))(x)$. *J. Algebra*, 428:43–53, 2015.
- [Mil17] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.
- [MOS14] Andrey Minchenko, Alexey Ovchinnikov, and Michael F. Singer. Unipotent differential algebraic groups as parameterized differential Galois groups. *J. Inst. Math. Jussieu*, 13(4):671–700, 2014.
- [MS12] Claude Mitschi and Michael F. Singer. Monodromy groups of parameterized linear differential equations with regular singularities. *Bull. Lond. Math. Soc.*, 44(5):913–930, 2012.
- [Sin13] Michael F. Singer. Linear algebraic groups as parameterized Picard-Vessiot Galois groups. *J. Algebra*, 373:153–161, 2013.
- [Spr09] T. A. Springer. *Linear algebraic groups*. Birkhäuser Boston Inc., second edition, 2009.
- [Sti09] Henning Stichtenoth. *Algebraic function fields and codes*, volume 254 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, second edition, 2009.
- [Sza09] Tamás Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2009.
- [TT79] Carol Tretkoff and Marvin Tretkoff. Solution of the inverse problem of differential Galois theory in the classical case. *Amer. J. Math.*, 101(6):1327–1332, 1979.
- [TW18] Ivan Tomašić and Michael Wibmer. Strongly étale difference algebras and Babbitt’s decomposition. *J. Algebra*, 504:10–38, 2018.
- [vdPS03] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations*, volume 328 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2003.

- [Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979.
- [Wib] Michael Wibmer. Finiteness properties of affine difference algebraic groups. arXiv:1405.6603.
- [Wib15] Michael Wibmer. Affine difference algebraic groups, 2015. Habilitation thesis, RWTH Aachen University, 146 pages, <http://www.math.tugraz.at/~wibmer/habil.pdf>.
- [Wib20] Michael Wibmer. Free proalgebraic groups. *Épjournal Geom. Algébrique*, 4(1):1–36, 2020.

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