

# HONDA–TATE THEORY FOR SHIMURA VARIETIES

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## Abstract

*A Shimura variety of Hodge type is a moduli space for abelian varieties equipped with a certain collection of Hodge cycles. We show that the Newton strata on such varieties are nonempty provided that the corresponding group  $G$  is quasisplit at  $p$ , confirming a conjecture of Fargues and Rapoport in this case. Under the same condition, we conjecture that every mod  $p$  isogeny class on such a variety contains the reduction of a special point. This is a refinement of Honda–Tate theory. We prove a large part of this conjecture for Shimura varieties of PEL type. Our results make no assumption on the availability of a good integral model for the Shimura variety. In particular, the group  $G$  may be ramified at  $p$ .*

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## Introduction

A Shimura variety  $\mathrm{Sh}(G, X)$  of Hodge type may be thought of as a moduli space for abelian varieties equipped with a particular family of Hodge cycles. This interpretation gives rise to a natural integral model  $\mathcal{S} = \mathcal{S}(G, X)$ . For a mod  $p$  point,  $x \in \mathcal{S}(\bar{\mathbb{F}}_p)$ , one has the attached abelian variety  $\mathcal{A}_x$  and its  $p$ -divisible group  $\mathcal{G}_x = \mathcal{A}_x[p^\infty]$ . In this article, we study the two related questions of classifying the isogeny classes of  $\mathcal{G}_x$  and  $\mathcal{A}_x$ . We are able to do this for quite general groups  $G$ , as our methods do not require any particular information about  $\mathcal{S}$ ; for example, we do not assume that  $\mathcal{S}$  has good reduction.

The isogeny class of  $\mathcal{G}_x$  is determined by its rational Dieudonné module  $\mathbb{D}_x$ , which is an  $(L = W(\bar{\mathbb{F}}_p)[1/p])$ -vector space equipped with a Frobenius semilin-

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ear operator  $b_x \sigma$ , where  $b_x \in G(L)$  is an element which is well defined up to  $\sigma$ -conjugacy,  $b_x \mapsto g^{-1} b_x \sigma(g)$ , and  $\sigma$  denotes the Frobenius automorphism of  $L$ . The element  $b_x$  is subject to a group-theoretic analogue of Mazur's inequality (see [36, Theorem 4.2]), and the set consisting of  $\sigma$ -conjugacy classes which satisfy this condition is denoted  $B(G, \mu)$ , where  $\mu : \mathbb{G}_m \rightarrow G$  is the inverse of the cocharacter  $\mu_X$  (up to conjugacy) attached to  $X$ . (See Sections 1.1.5 and 1.2.3 below for precise definitions.) Let  $\mathbb{D}$  denote the protorus whose character group is  $\mathbb{Q}$ . Each  $b \in G(L)$  gives rise to the so-called *Newton cocharacter*  $v_b : \mathbb{D} \rightarrow G$ , defined over  $L$ , whose conjugacy class is defined over  $\mathbb{Q}_p$  and depends only on the  $\sigma$ -conjugacy class  $[b]$ . The slope decomposition of  $\mathbb{D}_x$  is given by  $v_{b_x}$ . For  $[b] \in B(G, \mu)$ , the corresponding subset  $S_{[b]} \subset \mathcal{S}(\overline{\mathbb{F}}_p)$  is called the *Newton stratum* corresponding to  $[b]$  so that a point  $x \in \mathcal{S}(\overline{\mathbb{F}}_p)$  belongs to  $S_{[b]}$  if and only if  $[b_x] = [b]$ . Our first result is on the nonemptiness of Newton strata. (The converse is known; i.e., if  $S_{[b]}$  is nonempty, then  $b \in B(G, \mu)$ . See Lemma 1.3.9.)

#### THEOREM 1

*Suppose that  $b \in B(G, \mu)$  and that the  $G(L)$ -conjugacy class of  $v_b$  has a representative which is defined over  $\mathbb{Q}_p$ . Then  $S_{[b]}$  is nonempty. In particular,  $S_{[b]}$  is always nonempty either when  $G_{\mathbb{Q}_p}$  is quasisplit or when  $[b]$  is basic.*

Fargues [11, Conjecture 3.1.1] and Rapoport [35, Conjecture 7.1] have conjectured that  $S_{[b]}$  is nonempty for every  $b \in B(G, \mu)$  (see also He and Rapoport [14]). Previous results on the nonemptiness of  $S_{[b]}$  have been obtained by a number of authors (see, e.g., Wedhorn [41] and Wortmann [44] for the  $\mu$ -ordinary case (of hyperspecial level), Viehmann and Wedhorn [40] for the PEL case of type A or C (of hyperspecial level), and the recent work of Zhou [46] for many cases of parahoric level). These all rely on an understanding of the fine structure of a suitable integral model of  $\mathrm{Sh}(G, X)$ .

Our method involves constructing a special point whose reduction lies in  $S_{[b]}$ . This is essentially a group-theoretic problem, as the Newton stratum of a special point can be computed in terms of the torus and cocharacter attached to that point. When  $G_{\mathbb{Q}_p}$  is unramified, this problem was already solved by Langlands and Rapoport [28, Lemma 5.2]. This was independently observed by Lee [29], who also used it to show the nonemptiness of Newton strata in this case. If  $S_{[b]}$  contains the reduction of a special point, then it is easy to see that the  $G(L)$ -conjugacy class of  $v_b$  has a representative which is defined over  $\mathbb{Q}_p$ . Thus, the result of Theorem 1 is the best possible using this method.

Along the way, we confirm an expectation of Rapoport and Viehmann [37, Remark 8.3] on cocharacters and isocrystals. (See Remark 1.1.14 below.) We also show that the Newton stratification has some of the expected properties.

#### THEOREM 2

For every  $b \in B(G, \mu)$ ,  $S_{[b]} \subset \mathcal{S}(\bar{\mathbb{F}}_p)$  is locally closed for the Zariski topology. One has the following closure relations, where  $\preceq$  is the partial order on the set of conjugacy classes of Newton cocharacters (see Section 1.1.1):

$$\overline{S}_{[b]} \subset \bigcup_{\nu_{b'} \preceq \nu_b} S_{[b']}.$$

This theorem is proved by showing the existence of isocrystals with  $G$ -structure on  $\mathcal{S}$ . This may be of independent interest, but the proof is rather technical, so we leave it to the appendix. (Recently, Hamacher and Kim [12] proved similar results for the case of Kisin–Pappas models by a different argument.) We remark that inclusion in the theorem is expected to be an equality for hyperspecial level, but not in general. As a corollary, we obtain generalizations of the theorems of Wedhorn and Wortmann on the density of the  $\mu$ -ordinary locus.

#### THEOREM 3

If the special fiber of  $\mathcal{S}$  is locally integral, then the  $\mu$ -ordinary locus is dense in the special fiber.

We now discuss the problem of classifying  $\mathcal{A}_x$  up to isogeny. For the moduli space of polarized abelian varieties, this is closely related to Honda–Tate theory, which asserts that the isogeny class of an abelian variety  $A$  over  $\mathbb{F}_q$  is determined by the characteristic polynomial of the  $q$ -Frobenius on the  $\ell$ -adic cohomology  $H^1(A, \mathbb{Q}_\ell)$ , with  $\ell \nmid q$ , and that the isogeny class of  $A$  contains the reduction of a special point. Using this fact, one can describe precisely which characteristic polynomials can occur. For  $x \in \mathcal{S}(G, X)(\mathbb{F}_q)$ , one expects that the  $q$ -Frobenius arises from a  $\gamma \in G(\mathbb{Q})$  whose  $G(\bar{\mathbb{Q}})$ -conjugacy class is independent of  $\ell$ , although it is in general not a complete invariant for the isogeny class of  $A$ . We make the following conjecture.

#### CONJECTURE 1

If  $G_{\mathbb{Q}_p}$  is quasisplit, then the isogeny class of any  $x \in \mathcal{S}(\bar{\mathbb{F}}_p)$  contains the reduction of a special point.

Here if  $x, x' \in \mathcal{S}(\bar{\mathbb{F}}_p)$ , then  $\mathcal{A}_x, \mathcal{A}_{x'}$  are defined to be in the same isogeny class if there is an isogeny  $i : \mathcal{A}_x \rightarrow \mathcal{A}_{x'}$  such that for each of the Hodge cycles  $s_{\alpha,x}$  carried by  $\mathcal{A}_x$ ,  $i$  takes  $s_{\alpha,x}$  to  $s_{\alpha,x'}$ . More precisely, the Hodge cycles  $s_{\alpha,x}$  can be viewed via either  $\ell$ -adic cohomology for  $\ell \neq p$ , or crystalline cohomology. We require that  $i$  takes  $s_{\alpha,x}$  to  $s_{\alpha,x'}$  in each of these cohomology theories.

When  $G$  is unramified, this conjecture was proved by the first author [17] (see also [46] for some cases of parahoric Shimura varieties). The methods of [46] require rather fine information about the special fiber of  $\mathcal{S}$ , and are rather different from the ones employed in this paper which require almost no information about integral models.

Even for the moduli space of polarized abelian varieties, the conjecture is a more refined statement than Honda–Tate theory, since the definition of isogeny class involves isogenies which respect polarizations. As we shall explain, it can nevertheless be deduced from Honda–Tate theory with some extra arguments, but remarkably these do not seem to be in the literature; the closest is perhaps [22, Section 17]. (See Section 2.3.6 below.)

To state our main result in the direction of the conjecture, we recall that the group of automorphisms of  $\mathcal{A}_x$  in the isogeny category is naturally the  $\mathbb{Q}$ -points of an algebraic group  $I'_x = \underline{\text{Aut}}_{\mathbb{Q}} \mathcal{A}_x$  over  $\mathbb{Q}$ . Similarly, one can define the subgroup  $I = I_x \subset I'_x$  consisting of isogenies which respect Hodge cycles in  $\ell$ -adic and crystalline cohomology. The set of isogenies (respecting Hodge cycles) between  $\mathcal{A}_x$  and  $\mathcal{A}_{x'}$  is likewise the  $\mathbb{Q}$ -points of a scheme  $\mathcal{P}(x, x')$  which is either empty or a torsor under  $I_x$ . We say that  $\mathcal{A}_x$  and  $\mathcal{A}_{x'}$  are  $\mathbb{Q}$ -isogenous if  $\mathcal{P}(x, x')$  is nonempty. This is equivalent to asking that there be a finite extension  $F/\mathbb{Q}$  and an isomorphism  $\mathcal{A}_x \otimes F \rightarrow \mathcal{A}_{x'} \otimes F$  (e.g., as fppf sheaves) respecting Hodge cycles. We say that  $\mathcal{A}_x$  and  $\mathcal{A}_{x'}$  are  $\mathbb{Q}$ -isogenous if  $\mathcal{P}(x, x')$  is a trivial torsor.

#### THEOREM 4

*Suppose that  $G$  is quasisplit at  $p$  and that  $(G, X)$  is a PEL Shimura datum of type A or C. Then for any  $x \in \mathcal{S}(\bar{\mathbb{F}}_p)$ , the abelian variety  $\mathcal{A}_x$  is  $\mathbb{Q}$ -isogenous to  $\mathcal{A}_{x'}$ , with  $x'$  the reduction of a special point.*

Our main result is actually more precise, as we show that one can construct special points associated to any maximal torus  $T \subset I$ . There is also a slightly weaker version of the theorem in the case of PEL type D (see Remark 2.3.16). In fact, we prove an analogous theorem for  $(G, X)$  of Hodge type conditional on a version of Tate’s theorem for abelian varieties equipped with Hodge cycles (see below).

When  $G$  is unramified, a result closely related to the above theorem was proved by Zink [47]. Note that in [47], Zink’s theorem says that  $\mathcal{A}_x$  is isogenous (not just  $\mathbb{Q}$ -

isogenous) to the reduction of a special point; however, his definition does not require that isogenies respect polarizations, and it is not hard to see that one can then produce a  $\mathbb{Q}$ -isogeny from a  $\bar{\mathbb{Q}}$ -isogeny (the corresponding torsor turns out to be trivial).

When  $G^{\text{ab}}$  satisfies the Hasse principle, one can replace  $\bar{\mathbb{Q}}$ -isogenies by  $\mathbb{Q}$ -isogenies in Theorem 4. For example, one has the following.

#### THEOREM 5

*Suppose that  $G$  is quasisplit at  $p$  and that  $(G, X)$  is a PEL Shimura datum of type  $C$  or of type  $A_n$  with  $n$  odd. Then for any  $x \in \mathcal{S}(\bar{\mathbb{F}}_p)$ ,  $\mathcal{A}_x$  is  $\mathbb{Q}$ -isogenous to  $\mathcal{A}_{x'}$ , with  $x'$  the reduction of a special point, so that Conjecture 1 holds in this case.*

One of the key ingredients in Honda–Tate theory is Tate’s theorem on the Tate conjecture for morphisms between abelian varieties over finite fields (see [39]). We prove an analogue of this result for  $(G, X)$  of Hodge type, and for *automorphisms* of abelian varieties equipped with the corresponding collection of Hodge cycles. To explain this, for each  $\ell \neq p$ , let  $I_\ell \subset \text{Aut}(H^1(\mathcal{A}_x, \mathbb{Q}_\ell))$  be the subgroup which fixes the Hodge cycles  $s_{\alpha, x}$  and commutes with the  $q$ -Frobenius for  $q = p^r$  and  $r$  sufficiently divisible. We define a similar group  $I_p$  using crystalline cohomology.

#### THEOREM 6

*For every  $\ell$  (including  $\ell = p$ ), the natural map*

$$I \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \rightarrow I_\ell$$

*is an isomorphism. In particular, the (absolute) rank of  $I$  is equal to the rank of  $G$ .*

The proof uses the finiteness of  $\mathcal{S}(\mathbb{F}_q)$  (when level is fixed) as in [17], as well as a result of Noot on the independence of  $\ell$  of the conjugacy class of Frobenius as an element of  $G(\mathbb{Q}_\ell)$ . Note that a similar finiteness condition plays a crucial role in [39].

Using this result, one knows that any maximal torus  $T \subset I$  has the same rank as  $G$ . We show that, when  $G_{\mathbb{Q}_p}$  is quasisplit, any such  $T$  can be viewed as (transferred to) a subgroup of  $G$ . Our results on nonemptiness of Newton strata then imply that there is a special point  $\tilde{x}' \in \text{Sh}(G, X)$  with associated torus  $T$ . If  $x'$  is the reduction of  $\tilde{x}'$ , then  $\mathcal{A}_x$  and  $\mathcal{A}_{x'}$  should be  $\bar{\mathbb{Q}}$ -isogenous. Indeed, this follows from a version of Tate’s theorem with Hodge cycles. When  $x = x'$ , this is Theorem 6 above, but we do not know how to prove such a theorem when  $x \neq x'$ , except in the PEL case, when one can use Tate’s original result to deduce the first part of Theorem 4. Finally, the second part is proved via an analysis of the local behavior of the torsor  $\mathcal{P}(x, x')$ .

### Notational conventions

Given a connected reductive group  $G$  over a field  $F$ , we write  $G^{\text{der}} \subset G$  for its derived subgroup and  $G^{\text{sc}} \rightarrow G^{\text{der}}$  for the simply connected cover of its derived group.

Fix an algebraic closure  $\bar{F}$  for  $F$ . For any torus  $T$  over  $F$ , we set

$$X_*(T) = \text{Hom}(\mathbb{G}_{m, \bar{F}}, T_{\bar{F}}), \quad X^*(T) = \text{Hom}(T_{\bar{F}}, \mathbb{G}_{m, \bar{F}})$$

for the cocharacter and character groups of  $T$ , respectively. Write  $\mathbb{D}$  for the multiplicative progroup scheme over  $\mathbb{Q}_p$  with character group  $\mathbb{Q}$ . A homomorphism  $\mathbb{D}_{\bar{F}} \rightarrow T_{\bar{F}}$  gives an element of  $X_*(T)_{\mathbb{Q}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ , and vice versa. We often refer to a homomorphism  $\mathbb{D} \rightarrow G$  (defined over an extension of  $F$ ) as a *cocharacter* of  $G$  by standard abuse of terminology.

For a maximal torus  $T$  in the reductive group  $G$ , we write  $W(G, T)$  for the absolute Weyl group of  $G$  relative to  $T$ , and we denote by  $\pi_1(G)$  the algebraic fundamental group of  $G$  (see [3]). It is a  $\text{Gal}(\bar{F}/F)$ -module, functorial in  $G$ , and canonically isomorphic to  $X_*(T)/X_*(T^{\text{sc}})$ , where  $T^{\text{sc}}$  is the preimage of  $T$  in  $G^{\text{sc}}$ .

For  $v$  a place of  $\mathbb{Q}$ , we fix an algebraic closure  $\bar{\mathbb{Q}}_v$  for  $\mathbb{Q}_v$  (here,  $\mathbb{Q}_{\infty} = \mathbb{R}$  and  $\bar{\mathbb{Q}}_{\infty} = \mathbb{C}$ ). We also fix an algebraic closure  $\bar{\mathbb{Q}}$ , along with embeddings  $\iota_v : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_v$ , for every place  $v$ . Set  $\Gamma_v = \text{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)$  and  $\Gamma = \Gamma_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . We will use our chosen embeddings to view  $\Gamma_v$  as a subgroup of  $\Gamma$ .

When  $E$  is a number field, the ring of integers of  $E$  is denoted by  $\mathcal{O}_E$ .

## 1. Nonemptiness of Newton strata

### 1.1. Local results

Fix a rational prime  $p$ . Let  $G$  be a connected reductive group over  $\mathbb{Q}_p$ . Fix a maximal torus  $T \subset G$  defined over  $\mathbb{Q}_p$  and a Borel subgroup  $B \subset G_{\bar{\mathbb{Q}}_p}$  containing  $T_{\bar{\mathbb{Q}}_p}$ . Positive roots and coroots of  $T$  in  $G$  will be determined by  $B$ .

#### 1.1.1

Set

$$\mathcal{N}(G) = (X_*(T)_{\mathbb{Q}}/W(G, T))^{\Gamma_p}.$$

This space has a more canonical description that  $\mathcal{N}(G)$  is the space of  $G(\bar{\mathbb{Q}}_p)$ -conjugacy classes of homomorphisms  $\mathbb{D}_{\bar{\mathbb{Q}}_p} \rightarrow G_{\bar{\mathbb{Q}}_p}$  which are defined over  $\mathbb{Q}_p$ .

Let  $\bar{\mathcal{C}} \subset X_*(T)_{\mathbb{R}}$  be the closed dominant Weyl chamber determined by  $B$ . Each class  $\bar{v} \in \mathcal{N}(G)$  has a unique representative  $v \in X_*(T)_{\mathbb{Q}} \cap \bar{\mathcal{C}}$ . There is a natural partial order  $\leq_G$  on  $X_*(T)_{\mathbb{R}}$  and  $\mathcal{N}(G)$ , also denoted by  $\leq$  if there is no danger of confusion, determined as follows (cf. [36, Lemma 2.2, Section 2.3]). Given  $\bar{v}_1, \bar{v}_2 \in \mathcal{N}(G)$  with representatives  $v_1, v_2 \in X_*(T)_{\mathbb{Q}} \cap \bar{\mathcal{C}}$ , we have  $\bar{v}_1 \leq \bar{v}_2$  if and only if

$\nu_2 - \nu_1$  is a nonnegative linear combination of positive coroots. Similarly,  $\preceq$  is defined on  $X_*(T)_{\mathbb{R}}$  using dominant representatives.

There is a unique map  $\mathcal{N}(G) \rightarrow \pi_1(G)^{\Gamma_p} \otimes \mathbb{Q}$  which is functorial in  $G$  and induces the identity map when  $G$  is a torus (see [36, Theorem 1.15]).

### 1.1.2

Let  $W = W(\bar{\mathbb{F}}_p)$  be the ring of Witt vectors for an algebraic closure  $\bar{\mathbb{F}}_p$  of  $\mathbb{F}_p$ , and write  $L$  for its fraction field. We fix an algebraic closure  $\bar{L}$  for  $L$  along with an embedding  $\bar{\mathbb{Q}}_p \hookrightarrow \bar{L}$ . Let  $\sigma : W \rightarrow W$  be the unique automorphism lifting the  $p$ -power Frobenius on  $\bar{\mathbb{F}}_p$ . As in [20], we will denote by  $B(G)$  the set of  $\sigma$ -conjugacy classes in  $G(L)$ , so that two elements  $b_1, b_2 \in G(L)$  are in the same class in  $B(G)$  if and only if there exists  $c \in G(L)$  with  $b_1 = cb_2\sigma(c)^{-1}$ .

Recall the following maps from [36, Theorem 1.15], which are functorial in  $G$ :

$$\kappa_G : B(G) \rightarrow \pi_1(G)_{\Gamma_p}, \quad \bar{\nu}_G : B(G) \rightarrow \mathcal{N}(G).$$

A class  $[b] \in B(G)$  is *basic* if  $\bar{\nu}_G([b])$  is the class of a central cocharacter of  $G$ . We write  $B(G)_b \subset B(G)$  for the subset of basic classes.

The maps  $\kappa_G, \bar{\nu}_G$  have the following properties:

- (1) The diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{\kappa_G} & \pi_1(G)_{\Gamma_p} \\ \bar{\nu}_G \downarrow & & \downarrow \\ \mathcal{N}(G) & \longrightarrow & (\pi_1(G) \otimes \mathbb{Q})^{\Gamma_p} \end{array}$$

commutes. Here, the vertical map on the right-hand side is induced by the usual isomorphism averaging over each  $\Gamma_p$ -orbit (cf. [36, p. 162]):

$$(\pi_1(G) \otimes \mathbb{Q})_{\Gamma_p} \xrightarrow{\simeq} (\pi_1(G) \otimes \mathbb{Q})^{\Gamma_p}.$$

The bottom horizontal map is uniquely characterized as a functorial map in  $G$  that is the natural identification when  $G$  is a torus. (See [36, Theorem 1.15] for details.)

- (2) Given  $b \in G(L)$  representing a class  $[b] \in B(G)$ , the conjugacy class  $\bar{\nu}_G([b])$  is represented by a cocharacter  $\nu_b : \mathbb{D}_L \rightarrow G_L$  that is characterized uniquely by the following property. There exist  $c \in G(L)$  and an integer  $r \in \mathbb{Z}_{>0}$  such that  $r\nu_b$  factors through a cocharacter  $\mathbb{G}_{m,L} \rightarrow G_L$ ,  $c(r\nu_b)c^{-1}$  is defined over the fixed field of  $\sigma^r$  on  $L$ , and

$$cb\sigma(b)\sigma^2(b)\cdots\sigma^r(b)\sigma^r(c)^{-1} = c(r\nu_b)(p)c^{-1}.$$

This implies that  $\nu_{\sigma(b)} = \sigma(\nu_b)$  and that, for every  $g \in G(L)$ ,

$$\nu_g b \sigma(g)^{-1} = g \nu_b g^{-1}.$$

(See [20, Sections 4.3, 4.4].)

(3) The map

$$(\kappa_G, \bar{\nu}_G) : B(G) \rightarrow \pi_1(G)_{\Gamma_p} \times \mathcal{N}(G)$$

is injective. Furthermore, the restriction of  $\kappa_G$  to  $B(G)_b$  induces a bijection:

$$B(G)_b \xrightarrow{\simeq} \pi_1(G)_{\Gamma_p}.$$

(See [23, Section 4.13].)

(4) When  $G = T$  is a torus,  $\kappa_T$  is an isomorphism which can be described explicitly. Let  $E/L$  be a finite extension over which  $T$  is split, and let  $N_{E/L} : T(E) \rightarrow T(L)$  be the associated norm map. Fix a uniformizer  $\pi \in E$ . Then we have a commutative diagram:

$$\begin{array}{ccc} X_*(T) & \xrightarrow{v \mapsto [N_{E/L}(v(\pi))]} & B(T) \\ & \searrow & \swarrow \kappa_T \\ & & X_*(T)_{\Gamma_p} \end{array} \quad \begin{array}{c} \\ \\ \simeq \end{array}$$

(See [20, Section 2.5].)

### 1.1.3

Later we will often make the following hypothesis on  $G$  and  $[b]$ :

(1) The class  $[b]$  contains a representative  $b \in G(L)$  such that the cocharacter  $\nu_b$  is defined over  $\mathbb{Q}_p$ .

Given  $[b]$  satisfying the above condition, we fix such a representative and denote the corresponding cocharacter by  $\nu_G([b])$ . Let  $M_{[b]} \subset G$  be the centralizer of  $\nu_G([b])$ . This is a  $\mathbb{Q}_p$ -rational Levi subgroup of  $G$ .

Note that hypothesis 1.1.3(1) is always satisfied if  $G$  is quasisplit over  $\mathbb{Q}_p$  as one can see from 1.1.2(2) (cf. [20, p. 219]). If  $[b]$  is basic (but  $G$  is possibly not quasisplit), then hypothesis 1.1.3(1) is still satisfied as 1.1.2(2) shows that  $\nu_b$  is a  $\sigma$ -invariant central cocharacter of  $G$  for any representative  $b$ .

### 1.1.4

Suppose that  $b \in G(L)$ . Consider the group scheme  $J_b$  over  $\mathbb{Q}_p$  that attaches to every  $\mathbb{Q}_p$ -algebra  $R$  the group



$$J_b(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} L) : gb = b\sigma(g)\}.$$

By construction, there is a natural map of group schemes over  $L$ :  $J_{b,L} \rightarrow G_L$ .

If  $b' = gb\sigma(g)^{-1}$  is another representative of  $[b] \in B(G)$ , then conjugation by  $g$  induces an isomorphism of  $\mathbb{Q}_p$ -groups:

$$\text{int}(g) : J_b \xrightarrow{\cong} J_{b'}.$$

As shown in [36, Section 1.11],  $J_b$  is a reductive group over  $\mathbb{Q}_p$ . A more precise statement holds. Let  $M_{v_b} \subset G_L$  be the centralizer of  $v_b$ . By replacing  $b$  by a  $\sigma$ -conjugate if necessary, we can arrange that property 1.1.2(2) holds:

$$b\sigma(b)\sigma^2(b)\cdots\sigma^{r-1}(b) = (rv_b)(p), \quad (1.1.4.1)$$

with  $v_b$  defined over  $\mathbb{Q}_{p^r}$  and  $r \in \mathbb{Z}_{\geq 1}$ . Then  $M_{v_b}$  is also defined over  $\mathbb{Q}_{p^r}$ , and  $b$  belongs to  $G(\mathbb{Q}_{p^r})$ . Moreover, the natural map  $J_{b,L} \rightarrow G_L$  is defined over  $\mathbb{Q}_{p^r}$  and identifies  $J_{b,\mathbb{Q}_{p^r}}$  with  $M_{v_b}$ .

Under hypothesis 1.1.3(1), the discussion in 1.1.2(2) and 1.1.3 tells us that  $M_{v_b}$  is a pure inner twist of  $M_{[b]}$  by the  $M_{[b]}$ -torsor (trivial by Steinberg's theorem) of elements of  $G_{\mathbb{Q}_{p^r}}$  conjugating  $v_b$  to  $v_G([b])$ .

Combining the previous two paragraphs, we find that  $J_b$  is equipped with an inner twisting  $J_b \xrightarrow{\cong} M_{[b]}$  over  $\mathbb{Q}_p$  (cf. also [20, Section 5.2]).

### 1.1.5

We return to the general setup, disregarding hypothesis 1.1.3(1) until Proposition 1.1.13 below. Let  $G^*$  be the quasisplit inner form of  $G$  over  $\mathbb{Q}_p$ , and let  $\xi : G \xrightarrow{\cong} G^*$  be an inner twisting. Let  $B^* \subset G^*$  be a Borel subgroup over  $\mathbb{Q}_p$ , and let  $T^* \subset B^*$  be a maximal torus over  $\mathbb{Q}_p$ . Write  $\overline{\mathcal{C}^*} \subset X_*(T^*)_{\mathbb{R}}$  for the  $B^*$ -dominant chamber.

If the  $G(\bar{\mathbb{Q}}_p)$ -conjugacy class of a cocharacter  $\nu : \mathbb{D}_{\bar{\mathbb{Q}}_p} \rightarrow G_{\bar{\mathbb{Q}}_p}$  is defined over  $\mathbb{Q}_p$ , then so is the  $G^*(\bar{\mathbb{Q}}_p)$ -conjugacy class of  $\xi \circ \nu$ . Thus,  $\xi$  induces a map  $\mathcal{N}_{\xi} : \mathcal{N}(G) \rightarrow \mathcal{N}(G^*)$  depending only on the  $G^*(\bar{\mathbb{Q}}_p)$ -conjugacy class of  $\xi$ .

Let  $\{\mu\}$  be a conjugacy class of cocharacters  $\mathbb{G}_{m,\bar{\mathbb{Q}}_p} \rightarrow G_{\bar{\mathbb{Q}}_p}$ , and let  $\mu^* \in X_*(T^*) \cap \overline{\mathcal{C}^*}$  be the dominant representative for  $\xi \circ \{\mu\}$ . Let  $\Gamma_{\mu^*} \subset \Gamma_p$  be the stabilizer of  $\mu^*$ , and set

$$N\mu^* = \frac{1}{[\Gamma_p : \Gamma_{\mu^*}]} \sum_{\sigma \in \Gamma_p / \Gamma_{\mu^*}} \sigma\mu^* \in X_*(T^*)_{\mathbb{Q}}^{\Gamma_p}.$$

We will write  $\bar{\mu}^*$  for the image of  $N\mu^*$  in  $\mathcal{N}(G^*)$ .

Let  $\mu^\#$  be the image of  $\{\mu\}$  in  $\pi_1(G)_{\Gamma_p}$ . (Note that the image of  $\mu^*$  in  $\pi_1(G^*)_{\Gamma_p}$  is equal to  $\mu^\#$  via the canonical isomorphism  $\pi_1(G)_{\Gamma_p} = \pi_1(G^*)_{\Gamma_p}$ .) Given  $[b] \in B(G)$ , we will say that the pair  $([b], \{\mu\})$  is *G-admissible*, or simply *admissible*, if two conditions hold:

- (1)  $\kappa_G([b]) = \mu^\#$ , and
- (2)  $\mathcal{N}_\xi(\bar{\nu}_G([b])) \leq \bar{\mu}^*$ .

If  $G$  is quasisplit, then we may and will take  $G = G^*$  and  $\xi$  to be the identity map so that  $\mathcal{N}_\xi$  is also the identity map.

#### LEMMA 1.1.6

Given a conjugacy class  $\{\mu\}$  as above, let  $[b_{\text{bas}}(\mu)] \in B(G)_b$  denote the unique basic class such that  $\kappa_G([b_{\text{bas}}(\mu)]) = \mu^\#$ . Then  $([b_{\text{bas}}(\mu)], \{\mu\})$  is admissible.

#### Proof

Condition 1.1.5(1) is tautological, and 1.1.5(2) follows from [36, Proposition 2.4(ii)] and the commutativity of 1.1.2(1).  $\square$

#### Definition 1.1.7

Let  $T' \subset G$  be a maximal torus over  $\mathbb{Q}_p$ . We will call an admissible pair  $([b], \{\mu\})$  *T'-special* if there exists a representative  $b' \in T'(L)$  (resp.,  $\mu' \in X_*(T')$ ) of  $[b]$  (resp.,  $\{\mu\}$ ) such that the pair  $([b']_{T'}, \mu')$  is an admissible pair for  $T'$ . Here we write  $[b']_{T'}$  for the  $\sigma$ -conjugacy class of  $b'$  in  $T'(L)$ . We say that  $([b], \{\mu\})$  is *special* if it is *T'-special* for some maximal torus  $T' \subset G$ .

#### LEMMA 1.1.8

Suppose that  $([b], \{\mu\})$  is an admissible pair for  $G$  with  $[b]$  basic. Then  $([b], \{\mu\})$  is *T'-special* for any elliptic maximal torus  $T' \subset G$ . More precisely, for any  $\mu' \in X_*(T')$  in  $\{\mu\}$ ,  $[b_{\text{bas}}(\mu')] \in B(T')$  maps to  $[b] \in B(G)$ .

#### Proof

Let  $T' \subset G$  be an elliptic maximal torus, and let  $\mu' \in X_*(T')$  be a representative for  $\{\mu\}$ . As  $T'$  is elliptic,  $[b_{\text{bas}}(\mu')] \in B(T')$  maps to a basic class  $[b'] \in B(G)$  (see [20, Proposition 5.3]). Moreover,  $\kappa_G([b'])$  is the image in  $\pi_1(G)_{\Gamma_p}$  of  $\mu'^\# = \kappa_{T'}([b_{\text{bas}}(\mu')])$ , and so must be equal to  $\mu^\#$ . Hence,  $[b'] = [b_{\text{bas}}(\mu)] = [b]$ .  $\square$

#### 1.1.9

From here until Proposition 1.1.13, we are concerned with quasisplit groups. Let  $H_0$  be an absolutely simple quasisplit adjoint group over a finite extension  $F/\mathbb{Q}_p$ . Fix a Borel subgroup  $B_0 \subset H_0$  and a maximal torus  $T_0 \subset B_0$  over  $F$ .

Set  $H = \text{Res}_{F/\mathbb{Q}_p} H_0$ ,  $B = \text{Res}_{F/\mathbb{Q}_p} B_0$ ,  $T = \text{Res}_{F/\mathbb{Q}_p} T_0$ , and  $X = X_*(T)$ . The last is a free  $\mathbb{Z}$ -module with an action of  $\Gamma_p$ , and the choice of  $B_0$  equips it with a  $\Gamma_p$ -invariant positive chamber  $\mathcal{C} \subset X_{\mathbb{Q}}$ . As above, we have a Galois averaging map  $N : \mathcal{C} \rightarrow \mathcal{C}$  with image in  $\mathcal{C}^{\Gamma_p}$ .

LEMMA 1.1.10

Let  $F'/\mathbb{Q}_p$  be the unramified extension with  $[F' : \mathbb{Q}_p] = [F : \mathbb{Q}_p]$ . Then there is an absolutely simple quasisplit adjoint group  $H'_0$  over  $F'$  equipped with a Borel subgroup  $B'_0$  and a maximal torus  $T'_0 \subset B'_0$  with the following properties.

(1) Let  $(H', B', T') = \text{Res}_{F'/\mathbb{Q}_p}(H'_0, B'_0, T'_0)$ . Then there is an isomorphism of triples:

$$(H, B, T) \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p \xrightarrow{\sim} (H', B', T') \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p.$$

(2) Let  $\mathcal{C}' \subset X'_{\mathbb{Q}}$  be the positive chamber of  $X' = X_*(T')$  determined by  $B'$ , and let  $N' : \mathcal{C}' \rightarrow \mathcal{C}'$  be the Galois averaging map. Then the isomorphism in (1) can be chosen such that the induced isomorphism  $\mathcal{C}' \xrightarrow{\sim} \mathcal{C}$  carries the endomorphism  $N'$  to  $N$ .

*Proof*

We begin by explicating the averaging map  $N$ . Let  $D$  be the Dynkin diagram of  $H$ . It is a disjoint union

$$\bigsqcup_{\sigma: F \rightarrow \bar{\mathbb{Q}}_p} D_0,$$

where  $D_0$  is the Dynkin diagram of  $H_0$ . The action of  $\Gamma_p$  permutes the connected components of this diagram in the usual way, and for each  $\sigma : F \rightarrow \bar{\mathbb{Q}}_p$ , the stabilizer  $\Gamma_{\sigma} \subset \Gamma_p$  of  $\sigma$  (i.e., the pointwise stabilizer of  $\sigma(F)$ ) acts on  $D_0$  via a homomorphism

$$\rho_{\sigma} : \Gamma_{\sigma} \rightarrow \text{Aut}(D_0).$$

Fix an embedding  $\sigma_0 : F \rightarrow \bar{\mathbb{Q}}_p$ , and let  $\tau \in \Gamma_p$  be such that  $\tau \circ \sigma_0 = \sigma$ . Then  $\rho_{\sigma}$  is equal to the composition

$$\Gamma_{\sigma} \xrightarrow{\gamma \mapsto \tau^{-1} \gamma \tau} \Gamma_{\sigma_0} \xrightarrow{\rho_{\sigma_0}} \text{Aut}(D_0).$$

The simple coroots in  $X$  are in canonical bijection with pairs  $(\sigma, d_0)$ , where  $\sigma : F \rightarrow \bar{\mathbb{Q}}_p$  and  $d_0 \in D_0$  is a vertex. Write  $\alpha^{\vee}(\sigma, d_0)$  for the simple coroot associated with such a pair.

The  $\Gamma_p$ -orbit of  $\alpha^{\vee}(\sigma, d_0)$  consists of simple coroots  $\alpha^{\vee}(\sigma', d'_0)$ , where  $d'_0 \in D_0$  is in the  $\Gamma_{\sigma}$ -orbit of  $\alpha^{\vee}$ , and  $\sigma' : F \rightarrow \bar{\mathbb{Q}}_p$  is arbitrary. Therefore, if  $d_{0,1}, \dots, d_{0,r} \in D_0$  comprise the  $\Gamma_{\sigma}$ -orbit of  $d_0$ , then we have

$$N\alpha^\vee(\sigma, d_0) = \frac{1}{r[F:\mathbb{Q}_p]} \sum_{\substack{\sigma': F \rightarrow \bar{\mathbb{Q}}_p \\ 1 \leq i \leq r}} \alpha^\vee(\sigma', d_{0,i}).$$

Fix an embedding  $\sigma'_0 : F' \rightarrow \bar{\mathbb{Q}}_p$ . We now claim that we can find a quasisplit group  $H'_0$  over  $F'$  with a Borel subgroup  $B'_0 \subset H'_0$  and a maximal torus  $T'_0 \subset B'_0$  with the following properties.

- There is an isomorphism

$$(H'_0, B'_0, T'_0) \otimes_{F', \sigma'_0} \bar{\mathbb{Q}}_p \xrightarrow{\cong} (H_0, B_0, T_0) \otimes_{F, \sigma_0} \bar{\mathbb{Q}}_p.$$

- If  $D'_0$  is the Dynkin diagram of  $H'_0$ , identified with  $D_0$  via the above isomorphism, then the induced action of  $\Gamma_{\sigma'_0}$  on  $D_0$  has the same orbits as those of the action of  $\Gamma_{\sigma_0}$ .

The claim implies the lemma by choosing a bijection between  $\text{Hom}(F, \bar{\mathbb{Q}}_p)$  and  $\text{Hom}(F', \bar{\mathbb{Q}}_p)$  carrying  $\sigma_0$  to  $\sigma'_0$ . Indeed, (1) follows from the first part of the claim, and (2) from the second; since  $N'$  and  $N$  are linear, it suffices to compare them on the set of simple coroots.

Let us prove the claim. Suppose first that the image of  $\Gamma_{\sigma_0}$  in  $\text{Aut}(D_0)$  is cyclic. Consider a map  $\Gamma_{\sigma'_0} \rightarrow \text{Aut}(D_0)$  which has the same image as  $\Gamma_{\sigma_0}$  and factors through the Galois group of an unramified extension of  $F'$ . Then we can take  $H'_0$  to be the quasisplit outer form of  $H_0$  over  $F'$  associated to this map.

The only remaining case is when  $D_0$  is of type  $D_4$ , and  $\Gamma_{\sigma_0}$  surjects onto  $\text{Aut}(D_0)$ . In this case, the subgroup of index 2 still acts transitively on each orbit of  $\text{Aut}(D_0)$  in  $D_0$ , and we choose  $\Gamma_{\sigma'_0} \rightarrow \text{Aut}(D_0)$  with image this index-2 subgroup, which factors through the Galois group of an unramified extension of  $F'$ , and  $H'_0$  the corresponding quasisplit outer form of  $H_0$ . The proof of the claim is complete.  $\square$

### 1.1.11

Assume that  $G$  is quasisplit over  $\mathbb{Q}_p$ . Let  $B$  be a Borel subgroup of  $G$  over  $\mathbb{Q}_p$ , and let  $T \subset B$  be a maximal torus over  $\mathbb{Q}_p$ . Let  $M \subset G$  be a standard Levi subgroup. Recall that this means that  $M$  is the centralizer of a split torus  $T_1 \subset T$ . Note that we may regard  $X_*(Z_M)_{\mathbb{Q}}^{\Gamma_p}$  as a subset of  $\mathcal{N}(M)$ .

#### LEMMA 1.1.12

Let  $\mu, \mu_M \in X_*(T)$  be cocharacters having the same image in  $\pi_1(G)$ , and let  $[b_M] \in B(M)_b$  be the unique basic class with  $\kappa_M([b_M]) = \mu_M^\#$ . Then

- (1)  $\bar{v}_M([b_M])$  is equal to the image of  $\mu_M^\#$  in

$$(\pi_1(M) \otimes \mathbb{Q})^{\Gamma_p} \simeq (X_*(Z_M) \otimes \mathbb{Q})^{\Gamma_p},$$

(2)  $([b_M], \{\mu\})$  is  $G$ -admissible if and only if  $\bar{v}_M([b_M]) \leq_G \bar{\mu}$ .

*Proof*

The first claim follows from the commutativity of 1.1.2(1). By definition, the  $G$ -admissibility of  $([b_M], \{\mu\})$  is equivalent to asking that  $\bar{v}_M([b_M]) \leq_G \bar{\mu}$ , and that  $\mu_M^\#$  maps to  $\mu^\#$  in  $\pi_1(G)_{\Gamma_p}$ . However, since  $\mu_M$  and  $\mu$  have the same image in  $\pi_1(G)$ , the second condition is automatic.  $\square$

PROPOSITION 1.1.13

Suppose that  $G$  is quasisplit over  $\mathbb{Q}_p$ . Let  $\mu \in X_*(T)$  be minuscule, and let  $[b_M] \in B(M)_b$  be such that  $([b_M], \{\mu\})$  is  $G$ -admissible. Then there exists  $w \in W(G, T)$  such that  $([b_M], \{w \cdot \mu\})$  is  $M$ -admissible.

*Proof*

First, suppose that  $G$  is unramified. We fix a reductive model of  $G$  over  $\mathbb{Z}_p$ , again denoted by  $G$ , such that  $T$  extends to a maximal torus  $T \subset G$  over  $W$ . Then  $M$  extends to a Levi subgroup  $M \subset G$  over  $W$ .

By a theorem of Wintenberger [43], the admissibility of  $([b_M], \{\mu\})$  implies that there exists  $g \in G(L)$  such that  $g^{-1}b_M\sigma(g)$  belongs to  $G(W)\mu(p)G(W)$ . By the Iwasawa decomposition, after modifying  $g$  by an element of  $G(W)$ , we can assume that  $g = nm$ , where  $m \in M(L)$  and  $n \in N(L)$ , where  $N \subset G$  is the unipotent radical of the (positive) parabolic subgroup of  $G$  with Levi subgroup  $M$ . Then an argument with the Satake transform (see [28, Lemma 5.2]) shows that  $m^{-1}b_M\sigma(m)$  belongs to  $M(W)\mu'(p)M(W)$ , where  $\mu' \in X_*(T)$  is a cocharacter of  $M$  which is  $G(L)$ -conjugate to  $\mu$ . More precisely, the Satake transform is used to show that  $\mu' \leq_G \mu$  (in the notation of Section 1.1.1), and the minuscule nature of  $\mu$  allows us to conclude that  $\mu'$  is conjugate to  $\mu$ . (See the proofs of [24, Theorems 1.1, 4.1] and the proof of [17, Proposition 2.2.2] for alternative arguments to show the conjugacy.) Write  $\mu' = w \cdot \mu$  with  $w \in W(G, T)$ . By a result of Rapoport and Richartz [36, Theorem 4.2],  $([b_M], \{w \cdot \mu\})$  is  $M$ -admissible.

Now, let  $G$  be an arbitrary quasisplit group. We can assume that  $G$  is adjoint. Indeed, let  $\tilde{M} \subset G^{\text{ad}}$  denote the image of  $M$ , and let  $[b_M^{\text{ad}}] \in B(\tilde{M})_b$  denote the image of  $[b_M]$ . If  $w \in W(G, T)$  is such that  $([b_M^{\text{ad}}], \{w \cdot \mu^{\text{ad}}\})$  is  $\tilde{M}$ -admissible, then we claim that  $([b_M], \{w \cdot \mu\})$  is  $M$ -admissible. To see this, note that the difference  $\kappa_M([b_M]) - (w \cdot \mu)^\#$  is contained in the intersection of the kernels of the maps

$$\pi_1(M)_{\Gamma_p} \rightarrow \pi_1(\tilde{M})_{\Gamma_p} \quad \text{and} \quad \pi_1(M)_{\Gamma_p} \rightarrow \pi_1(G)_{\Gamma_p}.$$

The kernel of the first map is the image of  $X_*(Z_G)_{\Gamma_p} \rightarrow \pi_1(M)_{\Gamma_p}$ . The composite  $X_*(Z_G)_{\Gamma_p} \rightarrow \pi_1(G)_{\Gamma_p} \rightarrow X_*(G^{\text{ab}})_{\Gamma_p}$  has torsion kernel, so the intersection must be

a torsion group. However, by [4, Corollary 2.5.12(2)], the kernel of the second map is torsion-free. Hence the intersection is trivial.

Next, by considering the simple factors of  $G$  separately, we can assume that  $G$  is also simple. Therefore,  $G = \text{Res}_{F/\mathbb{Q}_p} G_0$ , where  $F/\mathbb{Q}_p$  is a finite extension, and  $G_0$  is an absolutely simple quasisplit adjoint group over  $F$ . We may also assume that

$$T = \text{Res}_{F/\mathbb{Q}_p} T_0, \quad B = \text{Res}_{F/\mathbb{Q}_p} B_0,$$

where  $T_0 \subset G_0$  (resp.,  $B_0 \subset G_0$ ) is a maximal torus (resp., Borel subgroup).

By Lemma 1.1.10, we can find an *unramified* group  $G'$ , a Borel subgroup  $B' \subset G'$ , and a maximal torus  $T' \subset B'$ , as well as an isomorphism

$$\xi : (G, B, T) \otimes \bar{\mathbb{Q}}_p \xrightarrow{\sim} (G', B', T') \otimes \bar{\mathbb{Q}}_p$$

such that the induced isomorphism of positive chambers  $\eta : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$  commutes with Galois averaging maps.

Recall that  $M$  is the centralizer of  $T_1$ , which is a split torus in  $T$ . Set  $\mu' = \eta(\mu)$  and  $T'_1 = \xi(T_1)$ . Since  $\eta$  commutes with Galois averaging maps, the elements in  $X_*(T'_1)$  are equal to their own Galois averages, and hence are  $\Gamma_p$ -invariant. Hence the subtorus  $T'_1 \subset G'$  is defined over  $\mathbb{Q}_p$  and is again split. Let  $M' \subset G'$  be the centralizer of  $T'_1$ . Then  $\xi$  carries  $M$  onto  $M'$ .

Let  $\mu_M \in X_*(T)$  be a cocharacter such that  $\mu_M^\# = \kappa_M([b_M])$ , and such that  $\mu_M$  and  $\mu$  have the same image in  $\pi_1(G)$ , and set  $\mu'_M = \eta(\mu_M)$ . Let  $[b'_M] \in B(M')_b$  be the unique basic class with  $\mu_M^\# = \kappa_{M'}([b'_M])$ . Then, using Lemma 1.1.12, one sees that  $([b'_M], \{\mu'\})$  is  $G'$ -admissible. Hence, by what we saw in the unramified case, there exists  $w \in W(G', T') = W(G, T)$  such that  $([b'_M], \{w \cdot \mu'\})$  is  $M'$ -admissible. By Lemma 1.1.6, this is equivalent to  $\mu_M^\# = (w \cdot \mu')^\#$  in  $\pi_1(M')_{\Gamma_p}$ . This implies that  $\mu_M^\# - (w \cdot \mu)^\#$  in  $\pi_1(M)_{\Gamma_p}$  is torsion, since its image under the averaging map in 1.1.2(1) is zero. Since this difference maps to zero in  $\pi_1(G)_{\Gamma_p}$ , it follows, as above, that  $\mu_M^\# = (w \cdot \mu)^\#$ , and hence, applying Lemma 1.1.6 again, that  $([b_M], \{w \cdot \mu\})$  is  $M$ -admissible.  $\square$

*Remark 1.1.14*

The previous proposition confirms that [37, Lemma 8.2(ii)] holds generally for quasisplit groups as expected. (See their Remark 8.3. In fact they do not assume that  $[b_M]$  is basic in  $B(M)$ , but one can reduce to the basic case by [20, Proposition 6.2].) We further extend the proposition to non-quasisplit groups below.

**COROLLARY 1.1.15**

*Let  $G$  be an arbitrary connected reductive group over  $\mathbb{Q}_p$  with a  $\mathbb{Q}_p$ -rational Levi*

subgroup  $M$ . Let  $\mu : \mathbb{G}_m \rightarrow M$  be a minuscule cocharacter, and let  $[b_M] \in B(M)_b$  be such that  $([b_M], \{\mu\})$  is  $G$ -admissible. Then there exists  $w \in W(G, M) := N_G(M)/M$  such that  $([b_M], \{w \cdot \mu\})$  is  $M$ -admissible.

The assumptions of the corollary imply hypothesis 1.1.3(1) for  $[b_M]$  (as an element of  $B(M)$  or  $B(G)$ ). In other words, the corollary is vacuous unless hypothesis 1.1.3(1) is satisfied.

*Proof*

We reduce the proof to the quasisplit case. We will freely use the notation from 1.1.5. So let  $\xi : G \xrightarrow{\sim} G^*$  denote an inner twisting. Let  $P$  be a  $\mathbb{Q}_p$ -rational parabolic subgroup with  $M$  as a Levi factor. Then the  $G^*(\bar{\mathbb{Q}}_p)$ -conjugacy class of  $\xi(P)$  is defined over  $\mathbb{Q}_p$ . Since  $G^*$  is quasisplit, there exists  $g \in G^*(\bar{\mathbb{Q}}_p)$  such that  $P^* := g\xi(P)g^{-1}$  is  $\mathbb{Q}_p$ -rational. We replace  $\xi$  by  $g\xi g^{-1}$  so that  $\xi(P) = P^*$ . Put  $M^* := \xi(M)$  so that  $\xi|_M : M \xrightarrow{\sim} M^*$  is an inner twisting. We use  $\xi$  to identify  $W(G, M) \simeq W(G^*, M^*) := N_{G^*}(M^*)/M^*$ . We may assume that  $B^* \subset P^*$  and  $T^* \subset M^*$ .

We have a chain of isomorphisms

$$B(M)_b \xrightarrow{\kappa_M^M} \pi_1(M)_{\Gamma_p} = \pi_1(M^*)_{\Gamma_p} \xrightarrow{\kappa_{M^*}^{-1}} B(M^*)_b,$$

where the second map is a canonical isomorphism (cf. [36, Section 1.13]). Write  $[b_{M^*}] \in B(M^*)_b$  for the image of  $[b_M]$ . Let  $\mu^*$  be the  $(B^* \cap M^*)$ -dominant representative in  $X_*(T^*)$  of the  $M^*(\bar{\mathbb{Q}}_p)$ -conjugacy class of  $\xi|_M \circ \mu$ . We claim that  $([b_{M^*}], \{\mu^*\})$  is  $G^*$ -admissible. Once this is shown, Proposition 1.1.13 implies that there exists  $w^* \in W(G^*, M^*)$  such that  $([b_{M^*}], \{w^* \cdot \mu^*\})$  is  $M^*$ -admissible. Writing  $w \in W(G, M)$  for the image of  $w^*$ , the  $M$ -admissibility of  $([b_M], \{w \cdot \mu\})$  follows from this.

It remains to prove the claim, that is, to verify that  $\kappa_{G^*}([b_{M^*}]) = (\mu^*)^\#$  and that  $\bar{v}_{G^*}([b_{M^*}]) \leq_{G^*} \bar{\mu}^*$ . We will deduce this from the assumption that  $([b_M], \{\mu\})$  is  $G$ -admissible via compatibility of various maps. The former condition follows from the construction of  $[b_{M^*}]$  and  $\mu^*$ , using the functoriality of the Kottwitz map and the fact that the canonical isomorphisms  $\pi_1(M) = \pi_1(M^*)$  and  $\pi_1(G) = \pi_1(G^*)$  are compatible with the Levi embeddings  $M \subset G$  and  $M^* \subset G^*$ . For the latter condition, since we know that  $\mathcal{N}_\xi(\bar{v}_G([b_M])) \leq_{G^*} \bar{\mu}^*$ , it suffices to check that

$$\mathcal{N}_\xi(\bar{v}_G([b_M])) = \bar{v}_{G^*}([b_{M^*}]).$$

By [23, Section 4.4], the Newton maps  $\mathcal{N}_{\xi|_M} \circ \bar{v}_M : B(M)_b \rightarrow \mathcal{N}(M^*)$  and  $\bar{v}_{M^*} : B(M^*)_b \rightarrow \mathcal{N}(M^*)$  factor through the natural inclusion  $X_*(A_{M^*})_{\mathbb{Q}} \subset \mathcal{N}(M^*)$ , where  $A_{M^*}$  is the maximal split torus in the center of  $M^*$ . Also, the images

$\mathcal{N}_{\xi|M}(\bar{v}_M([b_M]))$  and  $\bar{v}_{M^*}([b_{M^*}])$  in  $X_*(A_{M^*})_{\mathbb{Q}}$  are determined by  $\kappa_M([b_M])$  and  $\kappa_{M^*}([b_{M^*}])$  as elements of  $\pi_1(M)_{\Gamma_p} = \pi_1(M^*)_{\Gamma_p}$  (via the canonical isomorphism  $X_*(A_{M^*})_{\mathbb{Q}} \simeq \pi_1(M^*)_{\Gamma_p} \otimes \mathbb{Q}$ ). Since  $\kappa_M([b_M]) = \kappa_{M^*}([b_{M^*}])$  by construction, we obtain that  $\mathcal{N}_{\xi|M}(\bar{v}_M([b_M])) = \bar{v}_{M^*}([b_{M^*}])$ . This implies that  $\mathcal{N}_{\xi}(\bar{v}_G([b_M])) = \bar{v}_{G^*}([b_{M^*}])$  since the maps  $\mathcal{N}(M) \rightarrow \mathcal{N}(G)$  and  $\mathcal{N}(M^*) \rightarrow \mathcal{N}(G^*)$  induced by Levi embeddings are compatible with  $\mathcal{N}_{\xi|M}$ ,  $\mathcal{N}_{\xi}$ , and likewise for the maps  $B(M) \rightarrow B(G)$  and  $B(M^*) \rightarrow B(G^*)$ . The proof is complete.  $\square$

### 1.1.16

Let  $b \in G(L)$ . We continue to allow  $G$  to be non-quasisplit but assume hypothesis 1.1.3(1) on  $G$  and  $[b]$ . Recall that the group  $J_b$  defined in 1.1.4 is equipped with an inner twisting  $J_b \xrightarrow{\sim} M_{[b]}$ . In particular,  $\nu_G([b])$  induces a central cocharacter  $\nu_{b,J} : \mathbb{D} \rightarrow J_b$  defined over  $\mathbb{Q}_p$ .

If  $T' \subset J_b$  is a maximal torus over  $\mathbb{Q}_p$ , then a *transfer* of  $T'$  to  $M_{[b]}$  is an embedding  $T' \hookrightarrow M_{[b]}$  over  $\mathbb{Q}_p$  which is  $M_{[b]}(\bar{\mathbb{Q}}_p)$ -conjugate to the composite

$$T' \hookrightarrow J_b \xrightarrow{\sim} M_{[b]}.$$

A transfer of  $T'$  to  $M_{[b]}$  always exists either if  $G$  is quasisplit (see [27, Lemma 2.1]) or if  $T'$  is elliptic (see [21, Section 10]).

### COROLLARY 1.1.17

Assume hypothesis 1.1.3(1). Let  $([b], \{\mu\})$  be an admissible pair for  $G$  with  $\{\mu\}$  minuscule. Let  $T' \subset J_b$  be a maximal torus. Assume that its transfer  $j : T' \hookrightarrow M_{[b]}$  exists. Then  $([b], \{\mu\})$  is  $j(T')$ -special.

In particular, there exists  $\mu_{T'} \in X_*(T')$  such that  $j \circ \mu_{T'}$  lies in the  $G$ -conjugacy class  $\{\mu\}$ , and such that we have

$$\nu_{b,J} = N\mu_{T'} \in X_*(T')_{\mathbb{Q}}^{\Gamma_p}.$$

### Proof

Note that  $J_b$  and  $M_{[b]}$  are both subgroups of  $G$  over  $L$ . After replacing  $b$  by a  $\sigma$ -conjugate satisfying (1.1.4.1), we may assume that  $J_{b,L}$  is identified with  $M_{\nu_b}$  and that the inner twisting  $J_b \xrightarrow{\sim} M_{[b]}$  is given by composing this identification with conjugation by an element  $h \in G(L)$  that carries  $\nu_b$  to  $\nu_G([b])$ . Then in particular we have  $\nu_G([b]) = \text{int}(h)(\nu_{b,J})$  as  $G$ -valued cocharacters.

Now, view  $T'$  as a subtorus of  $G$ , via  $j$ , let  $T_1 \subset T'$  be the maximal split subtorus, and let  $M \subset G$  be the centralizer of  $T_1$ , so that  $T'$  is an elliptic maximal torus of  $M$ . Let  $T_2 \supset T_1$  be a maximal split torus in  $G$  containing  $T_1$ . After conjugating our fixed torus  $T \subset G$ , we may assume that  $T$  contains  $T_2$ , so that  $M \supset T$  is a standard Levi subgroup.



The scheme of elements of  $M_{[b],L}$  which conjugate the inclusion  $j_0 : T' \hookrightarrow J_b \xrightarrow{\sim} M_{[b]}$  into  $j$  is a  $T'$ -torsor over  $L$ . By Steinberg's theorem, this torsor is trivial. Hence, there exists  $m \in M_{[b]}(L)$  such that  $m j_0 m^{-1} = j$ . Now a simple computation, using the definition of  $J_b$ , shows that  $b_M = mh \cdot b \cdot \sigma(mh)^{-1}$  commutes with  $j(T'(\mathbb{Q}_p))$ . Since  $T_1(\mathbb{Q}_p)$  is Zariski-dense in  $T_1$ , this shows that  $b_M$  belongs to  $M(L)$ . Moreover, since  $v_G([b])$  is defined over  $\mathbb{Q}_p$ , by definition, it factors through  $T_1$ , so  $v_{b_M} = v_G([b])$  is central in  $M$ , and  $b_M$  is in fact basic in  $M$ .

By Lemma 1.1.13, there exists  $w \in W(G, T)$  such that  $([b_M], \{w \cdot \mu\})$  is  $M$ -admissible. (Here we may take  $\mu \in X_*(T)$  to be the dominant representative of  $\{\mu\}$ .) It follows by Lemma 1.1.8 that  $([b_M], \{w \cdot \mu\})$  is  $T'$ -special. In particular, there exists  $\mu_{T'} \in X_*(T')$  in  $\{\mu\}$  such that  $v_{b_M} = N\mu_{T'}$ . Hence, if we think of  $N\mu_{T'}$  as a  $J_b$ -valued cocharacter via the natural inclusion  $T' \subset J_b$ , then  $v_{b,J} = N\mu_{T'}$ .  $\square$

## 1.2. Global results

### LEMMA 1.2.1

Let  $T$  be a torus over  $\mathbb{Q}$ . For any prime  $p$ , the restriction map

$$\ker(H^1(\mathbb{Q}, T) \rightarrow H^1(\mathbb{Q}_p, T)) \rightarrow H^1(\mathbb{R}, T)$$

is surjective.

### Proof

For each place  $v$  of  $\mathbb{Q}$ , there is a canonical isomorphism (see [21, (1.1.1)])

$$j_v : H^1(\mathbb{Q}_v, T) \xrightarrow{\sim} X_*(T)_{\Gamma_v}^{\text{tors}}.$$

Write  $\bar{j}_v$  for the composition of this map with the natural projection  $X_*(T)_{\Gamma_v}^{\text{tors}} \rightarrow X_*(T)_{\Gamma}^{\text{tors}}$ . We then have an exact sequence (see [21, Proposition 2.6])

$$H^1(\mathbb{Q}, T) \rightarrow \oplus_v H^1(\mathbb{Q}_v, T) \xrightarrow{\oplus \bar{j}_v} X_*(T)_{\Gamma}^{\text{tors}}.$$

So, given a class  $\alpha_{\infty} \in H^1(\mathbb{R}, T)$ , it suffices to find  $\ell \neq p$  and a class  $\alpha_{\ell} \in H^1(\mathbb{Q}_{\ell}, T)$  such that  $\bar{j}_{\ell}(\alpha_{\ell}) = -\bar{j}_{\infty}(\alpha_{\infty})$ . Indeed, once we have done this, we can take the element  $(\alpha_v) \in \oplus_v H^1(\mathbb{Q}_v, T)$ , with  $\alpha_v = 0$  for  $v \neq \infty, \ell$ . This will be the image of an element  $\alpha \in H^1(\mathbb{Q}, T)$  mapping to  $\alpha_{\infty} \in H^1(\mathbb{R}, T)$  and to the trivial element in  $H^1(\mathbb{Q}_p, T)$ .

The remainder of the proof now proceeds as in [26, Lemme 7.16]. We choose a finite Galois extension  $E \subset \bar{\mathbb{Q}}$  over which  $T$  splits. Then complex conjugation on  $\mathbb{C}$  induces an automorphism  $\sigma_{\infty}$  of  $E$ . We now choose  $\ell \neq p$  such that  $E$  is unramified over  $\ell$  and such that, for some place  $v|\ell$  of  $E$ , the Frobenius  $\sigma_v$  at  $v$  is conjugate

to  $\sigma_\infty$ . We can further assume that  $v$  is induced from the embedding  $E \hookrightarrow \bar{\mathbb{Q}}_\ell$ . If  $g \in \Gamma$  conjugates  $\sigma_v$  into  $\sigma_\infty$ , then the automorphism of  $X_*(T)$ , given by  $g$ , induces an isomorphism  $X_*(T)_{\Gamma_\infty} \xrightarrow{\cong} X_*(T)_{\Gamma_\ell}$ , which is compatible with projections onto  $X_*(T)_\Gamma$ . We use this isomorphism to identify  $X_*(T)_{\Gamma_\infty}^{\text{tors}}$  with  $X_*(T)_{\Gamma_\ell}^{\text{tors}}$ . Now we may take  $\alpha_\ell = -j_\ell^{-1}(j_\infty(\alpha_\infty))$ .  $\square$

### LEMMA 1.2.2

Let  $G$  be a connected reductive group over  $\mathbb{Q}$ . Suppose that we are given a finite set of places  $S$  of  $\mathbb{Q}$  and, for each  $v \in S$ , a maximal torus  $T_v \subset G_{\mathbb{Q}_v}$ . Then there exists a maximal torus  $T \subset G$  such that, for all  $v \in S$ , the inclusion  $T_{\mathbb{Q}_v} \subset G_{\mathbb{Q}_v}$  is  $G(\mathbb{Q}_v)$ -conjugate to  $T_v \subset G_{\mathbb{Q}_v}$ .

#### Proof

This is [13, Lemma 5.5.3] (cf. [3, Lemma 5.6.3]).  $\square$

### 1.2.3

Let  $(G, X)$  be a Shimura datum. Given  $x \in X$ , we have the associated homomorphism of  $\mathbb{R}$ -groups:

$$h_x : \mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}.$$

We also have the associated (minuscule) cocharacter

$$\mu_x : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{\cong} \mathbb{S}_{\mathbb{C}} \xrightarrow{h_x} G_{\mathbb{C}}.$$

The  $G(\mathbb{R})$ -conjugacy class of  $h_x$ , and hence the  $G(\mathbb{C})$ -conjugacy class  $\{\mu_x\}_\infty$  of  $\mu_x$ , is independent of the choice of  $x$ . Let  $E \subset \mathbb{C}$  be the reflex field for  $(G, X)$ . This is the field of definition of  $\{\mu_x\}_\infty$ , and is a finite extension of  $\mathbb{Q}$ .

The embedding  $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$  allows us to view  $E \subset \mathbb{C}$  as a subfield of  $\bar{\mathbb{Q}}$ , so that we may regard  $\{\mu_x\}_\infty$  as a conjugacy class  $\{\mu_x\}$  of cocharacters of  $G_{\bar{\mathbb{Q}}}$ .

### 1.2.4

We will use the embedding  $\iota_p$  to view  $\{\mu_x\}$  as a conjugacy class  $\{\mu_x\}_p$  of cocharacters of  $G_{\bar{\mathbb{Q}}_p}$ .

### PROPOSITION 1.2.5

Let  $[b] \in B(G_{\mathbb{Q}_p})$  be a class such that  $([b], \{\mu_x^{-1}\}_p)$  is admissible. Assume that hypothesis 1.1.3(1) holds for  $[b]$ . Then there exist a maximal torus  $T \subset G$  and an element  $x \in X$  with  $h_x$  factoring through  $T_{\mathbb{R}}$  (in which case  $\mu_x^{-1} \in X_*(T)$ ) such that  $[b_{\text{bas}}(\mu_x)] \in B(T_{\mathbb{Q}_p})$  maps to  $[b] \in B(G_{\mathbb{Q}_p})$ .

*Proof*

This proof is directly inspired by that of [28, Lemma 5.12].

By Corollary 1.1.17, there exist a maximal torus  $T_p \subset G_{\mathbb{Q}_p}$  (chosen to be elliptic if  $G_{\mathbb{Q}_p}$  is not quasisplit so that the transfer to  $M_{[b]}$  exists) and a representative  $\mu_p \in X_*(T_p)$  of  $\{\mu_X\}_p$  such that  $[b_{\text{bas}}(\mu_p^{-1})] \in B(T_p)$  maps to  $[b] \in B(G_{\mathbb{Q}_p})$ .

Choose  $y \in X$ , and let  $T_\infty \subset G_{\mathbb{R}}$  be a maximal torus such that  $h_y$  factors through  $T_\infty$ . By Lemma 1.2.2, we can find a maximal torus  $T \subset G$  such that  $T_{\mathbb{Q}_p}$  (resp.,  $T_{\mathbb{R}}$ ) is  $G(\mathbb{Q}_p)$ -conjugate to  $T_p$  (resp.,  $G(\mathbb{R})$ -conjugate to  $T_\infty$ ).

Choose  $g_p \in G(\mathbb{Q}_p)$  such that  $g_p T_p g_p^{-1} = T_{\mathbb{Q}_p}$ , and let  $\mu_T : \mathbb{G}_{m, \bar{\mathbb{Q}}} \rightarrow T_{\bar{\mathbb{Q}}}$  be the unique cocharacter, which, after base change along  $\iota_p$ , is identified with  $\text{int}(g_p)(\mu_p)$ . Then  $[b_{\text{bas}}(\mu_T^{-1})]$  maps to  $[b]$ .

Choose  $g_\infty \in G(\mathbb{R})$  such that  $g_\infty T_\infty g_\infty^{-1} = T_{\mathbb{R}}$ . After base change along  $\iota_\infty$ , the cocharacter  $\mu_T$  is  $G(\mathbb{C})$ -conjugate to  $\mu_\infty = \text{int}(g_\infty)(\mu_y)$ . Therefore, there exists an element  $\omega \in W(G, T)(\mathbb{C})$  such that  $\omega(\mu_\infty) = \mu_T$ .

We can identify  $W(G, T)$  with  $N_{G^{\text{sc}}}(T^{\text{sc}})/T^{\text{sc}}$ . Let  $n \in N_{G^{\text{sc}}}(T^{\text{sc}})(\mathbb{C})$  be any element mapping to  $\omega$ . Since  $T^{\text{sc}}$  is anisotropic over  $\mathbb{R}$ , the element  $\omega$  acts on  $T^{\text{sc}}$  by an  $\mathbb{R}$ -automorphism. Hence  $n\bar{n}^{-1} \in T^{\text{sc}}(\mathbb{C})$ . The cocycle carrying complex conjugation to  $n\bar{n}^{-1}$  determines a class  $\alpha_\infty \in H^1(\mathbb{R}, T^{\text{sc}})$  depending only on  $\omega$  (not on the choice of  $n$ ). By Lemma 1.2.1, we can find a class  $\alpha \in H^1(\mathbb{Q}, T^{\text{sc}})$  mapping to  $\alpha_\infty \in H^1(\mathbb{R}, T^{\text{sc}})$ , as well as to the trivial class in  $H^1(\mathbb{Q}_p, T^{\text{sc}})$ .

By construction, the image of  $\alpha_\infty$  in  $H^1(\mathbb{R}, G^{\text{sc}})$  is trivial. Therefore, by the Hasse principle and Kneser's vanishing theorem for simply connected groups, the image of  $\alpha$  in  $H^1(\mathbb{Q}, G^{\text{sc}})$  is trivial. This means that we can find  $g \in G^{\text{sc}}(\bar{\mathbb{Q}})$  such that, for any  $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ ,  $g\sigma(g)^{-1} \in T^{\text{sc}}(\bar{\mathbb{Q}})$ , and such that  $\alpha$  is represented by the  $T^{\text{sc}}(\bar{\mathbb{Q}})$ -valued cocycle  $\sigma \mapsto g\sigma(g)^{-1}$ .

In particular, if we view  $g$  as an element of  $G^{\text{sc}}(\mathbb{C})$  via  $\iota_\infty$ , then there exists  $t \in T^{\text{sc}}(\mathbb{C})$  such that  $g\bar{g}^{-1} = t n \bar{n}^{-1} \bar{t}^{-1}$ .

Now,  $\mu_\infty$  and  $\text{int}(g^{-1})(\mu_T)$  are conjugate under  $h = g^{-1}tn \in G(\mathbb{R})$ , and the maximal torus  $\text{int}(g^{-1})(T_{\bar{\mathbb{Q}}}) \subset G_{\bar{\mathbb{Q}}}$  is defined over  $\mathbb{Q}$ . Replacing  $T$  with this torus, and  $\mu_T$  with  $\text{int}(g^{-1})(\mu_T)$ , we see that  $\mu_T$  is of the form  $\mu_x$  for  $x \in X$ , and that the pair  $(T, \mu_x)$  satisfies the conclusions of the proposition.  $\square$

### 1.3. Shimura varieties of Hodge type

One may view Proposition 1.2.5 as showing the nonemptiness of Newton strata in the special fiber of the Shimura variety associated with  $(G, X)$ . We will now make this assertion precise in the case where  $(G, X)$  is of Hodge type, where the moduli spaces of abelian varieties give us a natural way to construct integral models.

### 1.3.1

Recall that, given a symplectic space  $(V, \psi)$  over  $\mathbb{Q}$ , we can attach to it the Siegel Shimura datum  $(\mathcal{G}_V, \mathcal{H}_V)$ , where  $\mathcal{G}_V = \mathrm{GSp}(V, \psi)$  is the group of symplectic similitudes and  $\mathcal{H}_V$  is the union of the Siegel half-spaces associated with  $(V, \psi)$ .

Let  $(G, X)$  be a Shimura datum of *Hodge type*. This means that there exists a faithful symplectic representation  $(V, \psi)$  of  $G$  over  $\mathbb{Q}$  such that the associated map of  $\mathbb{Q}$ -groups  $G \hookrightarrow \mathcal{G}_V$  extends to an embedding of Shimura data  $(G, X) \hookrightarrow (\mathcal{G}_V, \mathcal{H}_V)$ . We denote by  $E = E(G, X)$  the reflex field of  $(G, X)$ .

### 1.3.2

Fix a  $\mathbb{Z}_{(p)}$ -lattice  $V_{(p)} \subset V$  on which  $\psi$  is  $\mathbb{Z}_{(p)}$ -valued. Set  $V_p = \mathbb{Z}_p \otimes V_{(p)}$ , and let  $\mathcal{K}_p \subset \mathcal{G}_V(\mathbb{Q}_p)$  (resp.,  $K_p \subset G(\mathbb{Q}_p)$ ) be the stabilizer of  $V_p \subset V_{\mathbb{Q}_p}$ .

Given a sufficiently small compact open subgroup  $K^p \subset G(\mathbb{A}_f^p)$ , we can find a neat compact open subgroup  $\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)$  such that, with  $K = K_p K^p$  and  $\mathcal{K} = \mathcal{K}_p \mathcal{K}^p$ , the map of Shimura varieties

$$\mathrm{Sh}_K := \mathrm{Sh}_K(G, X) \rightarrow \mathrm{Sh}_{\mathcal{K}} := \mathrm{Sh}_{\mathcal{K}}(\mathcal{G}_V, \mathcal{H}_V) \otimes E$$

is a closed immersion (see [16, Lemma 2.1.2]).

The variety  $\mathrm{Sh}_{\mathcal{K}}$  admits an integral model  $\mathcal{S}_{\mathcal{K}}$  over  $\mathbb{Z}_{(p)}$ , which is an open and closed subscheme of the moduli scheme parameterizing polarized abelian schemes  $(A, \lambda)$  up to prime-to- $p$  isogeny, and equipped with additional level structures away from  $p$ . Let  $\mathcal{A}$  denote the universal abelian scheme over  $\mathcal{S}_{\mathcal{K}}$  up to prime-to- $p$  isogeny.

The set of compact open subgroups  $K_p \subset G(\mathbb{Q}_p)$  for which one can choose  $V$  and  $V_{(p)}$  so that this construction applies, includes the stabilizers of points  $x$  in the building  $B(G, \mathbb{Q}_p)$ , and is closed under finite intersections. For the first point, note that a result of Landvogt [25] implies that for any faithful representation  $V$  of  $G$ , there is an injective map of buildings  $i : B(G, \mathbb{Q}_p) \rightarrow B(\mathrm{GL}(V), \mathbb{Q}_p)$ . If  $(V, \psi)$  is a symplectic representation of  $G$ , and  $L_1, \dots, L_m \subset V$  are the lattices corresponding to the vertices in the facet which is the closure of  $i(x)$ , then  $K_p$  is the stabilizer of  $L_1 \oplus \dots \oplus L_m$  in  $(V^m, \psi^m)$ . The closure under intersections follows in the same way, by taking direct sums of lattices.

### 1.3.3

We will now use the notation from 1.1.2. Given a point  $s_0 \in \mathcal{S}_{\mathcal{K}}(\bar{\mathbb{F}}_p)$ , we obtain the associated Dieudonné  $F$ -crystal  $\mathbb{D}(\mathcal{A}_{s_0})$  over  $W$ . Set  $D_{s_0} = \mathbb{D}(\mathcal{A}_{s_0})_{\mathbb{Q}}$ . This is an  $F$ -isocrystal over  $L = W[p^{-1}]$ , so that it is equipped with a  $\sigma$ -semilinear bijection  $\varphi : D_{s_0} \rightarrow D_{s_0}$ .

Given a finite extension  $L' \subset \bar{L}$  of  $L$  and a point  $s \in \mathcal{S}_{\mathcal{K}}(L')$  specializing to  $s_0$ , we obtain two canonical comparison isomorphisms:

- (1) The Berthelot–Ogus isomorphism:

$$H_{\mathrm{dR}}^1(\mathcal{A}_s/L') \xrightarrow{\sim} L' \otimes_L D_{s_0}.$$

- (2) The
- $p$
- adic comparison isomorphism:

$$B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} H_{\mathrm{\acute{e}t}}^1(\mathcal{A}_{s,\bar{L}}, \mathbb{Q}_p) \xrightarrow{\sim} B_{\mathrm{cris}} \otimes_L D_{s_0}.$$

The two isomorphisms are compatible with the de Rham comparison isomorphism:

$$B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H_{\mathrm{\acute{e}t}}^1(\mathcal{A}_{s,\bar{L}}, \mathbb{Q}_p) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{L'} H_{\mathrm{dR}}^1(\mathcal{A}_s/L'). \quad (1.3.3.3)$$

### 1.3.4

Let  $V_{\mathrm{dR}}$  be the (cohomological) de Rham realization of  $\mathcal{A}$ . It is a vector bundle over  $\mathrm{Sh}_{\mathcal{K}}$  with integrable connection, and its fiber at each point  $s \in \mathrm{Sh}_{\mathcal{K}}(\kappa)$  ( $\kappa$  a field of characteristic 0) is the de Rham cohomology  $H_{\mathrm{dR}}^1(\mathcal{A}_s/\kappa)$ .

Let  $\widehat{V}^p(\mathcal{A})$  be the prime-to- $p$  Tate module of  $\mathcal{A}$ . This is a smooth  $\mathbb{A}_f^p$ -sheaf over  $\mathrm{Sh}_{\mathcal{K}}$ . Write  $V^p$  for its dual; then the fiber of  $V^p$  at any point  $s \in \mathrm{Sh}_{\mathcal{K}}(\kappa)$ , with  $\kappa$  algebraically closed, is identified with the étale cohomology group  $H_{\mathrm{\acute{e}t}}^1(\mathcal{A}_s, \mathbb{A}_f^p)$ .

Finally, write  $T_p(\mathcal{A})$  for the  $p$ -adic Tate module of  $\mathcal{A}$ , and set  $V_p(\mathcal{A}) = \mathbb{Q}_p \otimes T_p(\mathcal{A})$ . Write  $V_p$  for the dual  $(V_p(\mathcal{A}))^\vee$ . We will set

$$\widehat{V}(\mathcal{A}) = \widehat{V}^p(\mathcal{A}) \times V_p(\mathcal{A}) \quad \text{and} \quad V_{\mathrm{\acute{e}t}} = V^p \times V_p.$$

Fix tensors  $\{s_\alpha\} \subset V^\otimes$  such that  $G$  is their pointwise stabilizer in  $\mathrm{GL}(V)$ . Here and below, the superscript  $\otimes$  means the direct sum of  $V^{\otimes n} \otimes V^{*\otimes m}$  for all  $m, n \geq 0$ . Then there exist global sections

$$\{s_{\alpha,\mathrm{dR}}\} \subset H^0(\mathrm{Sh}_K, V_{\mathrm{dR}}^\otimes), \quad \{s_{\alpha,\mathrm{\acute{e}t}}\} \subset H^0(\mathrm{Sh}_K, V_{\mathrm{\acute{e}t}}^\otimes)$$

with the following properties:

- (1) Given an algebraically closed field  $\kappa$  of characteristic 0 and a point  $s \in \mathrm{Sh}_K(\kappa)$ , there exists an isomorphism

$$V_{\mathbb{A}_f} \xrightarrow{\sim} H_{\mathrm{\acute{e}t}}^1(\mathcal{A}_s, \mathbb{A}_f) = V_{\mathrm{\acute{e}t},s},$$

determined up to translation by  $G(\mathbb{A}_f)$ , carrying  $\{s_\alpha\}$  to  $\{s_{\alpha,\mathrm{\acute{e}t},s}\}$ .

- (2) For each  $\alpha$ , let  $s_{\alpha,p}$  be the projection of  $s_{\alpha,\mathrm{\acute{e}t}}$  onto  $V_p$ . Then, given a finite extension  $L'/L$  and a point  $s \in \mathrm{Sh}_K(L')$ , the isomorphism (1.3.3.3) carries  $\{1 \otimes s_{\alpha,p,s}\}$  to  $\{1 \otimes s_{\alpha,\mathrm{dR},s}\}$ .

The construction of these tensors is described in [16, Section 2.2]. The key point is a theorem of Deligne showing that all Hodge cycles on abelian varieties over  $\mathbb{C}$  are absolutely Hodge. Property (1) now holds by construction. Property (2) is a theorem of Blasius and Wintenberger [2].

## 1.3.5

Fix a place  $v|p$  of  $E$  and an embedding  $k(v) \hookrightarrow \bar{\mathbb{F}}_p$ . We denote by

$$\mathcal{S}_K = \mathcal{S}_K(G, X) \hookrightarrow \mathcal{O}_{E,(v)} \otimes_{\mathbb{Z}_{(p)}} \mathcal{S}_{\mathcal{K}}$$

the normalization of the Zariski closure of  $\mathrm{Sh}_K$  in  $\mathcal{O}_{E,(v)} \otimes \mathcal{S}_{\mathcal{K}}$ .

We shall use that  $\mathcal{S}_K$  has the following extension property.

## LEMMA 1.3.6

Let  $S$  be the spectrum of a discrete valuation ring  $R$  of mixed characteristic  $(0, p)$ , with generic point  $\eta$ , and a map  $s : \eta \rightarrow \mathcal{S}_K$ . Then the following are equivalent:

- (i)  $s$  extends to  $S \rightarrow \mathcal{S}_K$ .
- (ii)  $\mathcal{A}_\eta$  has good reduction.
- (iii)  $\mathcal{A}_\eta$  has potentially good reduction.

*Proof*

By construction, (i) is equivalent to  $s$  extending to a map  $S \rightarrow \mathcal{S}_{\mathcal{K}}$ . Thus (i) and (ii) are equivalent and imply (iii). If  $\mathcal{A}_\eta$  has potentially good reduction, then there is a finite flat  $R'/R$  such that  $s$  induces a map  $\mathrm{Spec} R' \rightarrow \mathcal{S}_{\mathcal{K}}$ , and this necessarily factors through  $S$  as this is true on the generic fiber.  $\square$

## PROPOSITION 1.3.7

For every point  $s_0 \in \mathcal{S}_{K,k(v)}(\bar{\mathbb{F}}_p)$ , there exists a canonical collection of  $\varphi$ -invariant tensors  $\{s_{\alpha,\mathrm{cris},s_0}\} \subset D_{s_0}^{\otimes}$  characterized by the following property: for any lift  $s \in \mathcal{S}_K(\bar{L})$  of  $s_0$ , the isomorphism 1.3.3(2) carries  $\{s_{\alpha,p,s}\}$  to  $\{s_{\alpha,\mathrm{cris},s_0}\}$ .

*Proof*

The proof of this can essentially be found in [16, Proposition 2.3.5]; however, since it is not given there in the generality we require, we review the key steps here. Write  $L' = E_v L \subset \bar{L}$ ; here, we are embedding  $E_v \hookrightarrow \bar{L}$  via the fixed embedding  $\bar{\mathbb{Q}}_p \hookrightarrow \bar{L}$ . Let  $\widehat{\mathcal{U}}$  be the formal scheme over  $W$  prorepresenting the deformation functor for the  $p$ -divisible group  $\mathcal{A}_{s_0}[p^\infty]$ : this is formally smooth over  $W$ . Let  $\widehat{\mathcal{U}}$  be the formal scheme obtained by completing  $\mathcal{S}_K \otimes_{\mathcal{O}_{E,(v)}} \mathcal{O}_{L'}$  along  $s_0$ .

We have a finite map of normal formal schemes over  $\mathcal{O}_{L'}$ ,  $\widehat{\mathcal{U}} \rightarrow \widehat{\mathcal{U}}_{L'}$ . Taking their rigid analytic fibers (in the sense of Berthelot; cf. [5, Section 7.3]), we obtain a map  $\widehat{\mathcal{U}}^{\mathrm{an}} \rightarrow \widehat{\mathcal{U}}_{L'}^{\mathrm{an}}$  of smooth, irreducible rigid analytic spaces over  $L'$ . This map is a closed immersion, since the map  $\mathrm{Sh}_K \rightarrow \mathrm{Sh}_{\mathcal{K}}$  is.

Since  $\widehat{\mathcal{U}}_{L'}$  is formally smooth,  $\widehat{\mathcal{U}}_{L'}^{\mathrm{an}}$  is a rigid analytic open ball over  $L'$ , and, for any two points  $s, s' \in \widehat{\mathcal{U}}^{\mathrm{an}}(\bar{L})$ ,  $p$ -adic parallel transport using the Gauss–Manin connection on  $V_{\mathrm{dR}}$  gives us a canonical isomorphism

$$H_{\mathrm{dR}}^1(\mathcal{A}_s/\bar{L}) \xrightarrow{\sim} H_{\mathrm{dR}}^1(\mathcal{A}_{s'}/\bar{L}). \quad (1.3.7.1)$$

Suppose now that  $s, s'$  lie in  $\widehat{U}^{\mathrm{an}}(\bar{L})$ . Since the sections  $s_{\alpha, \mathrm{dR}}$  over  $\mathrm{Sh}_K$  are horizontal for the connection, and since  $\widehat{U}^{\mathrm{an}}$  is smooth and irreducible over  $L'$ , for each  $\alpha$ , this isomorphism carries  $s_{\alpha, \mathrm{dR}, s}$  to  $s_{\alpha, \mathrm{dR}, s'}$ .

Any  $s \in \widehat{U}^{\mathrm{an}}(\bar{L})$  is defined over a finite extension  $L''/L'$ . Since the tensors  $\{s_{\alpha, p, s}\}$  are  $\mathrm{Gal}(\bar{L}/L'')$ -invariant, by construction, the isomorphism 1.3.3(2) carries  $\{s_{\alpha, p, s}\}$  to  $\varphi$ -invariant tensors  $\{s_{\alpha, \mathrm{cris}, s}\} \subset D_{s_0}^{\otimes}$ . To prove the proposition, it is now enough to show that if  $s'$  is a different lift, giving rise to  $\varphi$ -invariant tensors  $\{s_{\alpha, \mathrm{cris}, s'}\} \subset D_{s_0}^{\otimes}$ , then, for each  $\alpha$ , we have  $s_{\alpha, \mathrm{cris}, s'} = s_{\alpha, \mathrm{cris}, s}$ .

By the compatibility of the isomorphisms 1.3.3(2) and 1.3.3(1), and by 1.3.4(2), the preimage of  $1 \otimes s_{\alpha, \mathrm{cris}, s}$  (resp.,  $1 \otimes s_{\alpha, \mathrm{cris}, s'}$ ) in  $H_{\mathrm{dR}}^1(\mathcal{A}_s/\bar{L})^{\otimes}$  (resp., in  $H_{\mathrm{dR}}^1(\mathcal{A}_{s'}/\bar{L})^{\otimes}$ ) under isomorphism 1.3.3(1) is exactly  $s_{\alpha, \mathrm{dR}, s}$  (resp.,  $s_{\alpha, \mathrm{dR}, s'}$ ). Therefore, we only need to show that the composition

$$H_{\mathrm{dR}}^1(\mathcal{A}_s/\bar{L}) \xrightarrow{\sim} \bar{L} \otimes D_{s_0} \xrightarrow{\sim} H_{\mathrm{dR}}^1(\mathcal{A}'_s/\bar{L})$$

is the parallel transport isomorphism (1.3.7.1). This follows from [1, Section 2.9].  $\square$

### 1.3.8

It follows from Proposition 1.3.7 and property 1.3.4(1) that there exists an isomorphism  $L \otimes_{\mathbb{Q}} V \xrightarrow{\sim} D_{s_0}$  carrying  $\{1 \otimes s_{\alpha}\}$  to  $\{s_{\alpha, \mathrm{cris}, s_0}\}$ . Indeed, the scheme of such isomorphisms is a  $G$ -torsor by 1.3.4(1), and a  $G$ -torsor over  $L$  is trivial by Steinberg's theorem. Under this isomorphism, the map  $\varphi : D_{s_0} \rightarrow D_{s_0}$  pulls back to an automorphism of  $L \otimes V$  of the form  $\sigma \otimes b_{s_0}$ , with  $b_{s_0} \in G(L)$  well-determined up to  $\sigma$ -conjugacy. Therefore,  $s_0$  determines a canonical class  $[b_{s_0}] \in B(G_{\mathbb{Q}_p})$ .

Assume that  $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$  has been chosen such that the associated embedding  $E \hookrightarrow \bar{\mathbb{Q}}_p$  induces the place  $v$ .

### LEMMA 1.3.9

*The pair  $([b_{s_0}], \{\mu_X^{-1}\}_p)$  is admissible.*

#### *Proof*

This is a consequence of a result of Wintenberger (cf. corollary to [42, Proposition 4.5.3]).  $\square$

### PROPOSITION 1.3.10

*Assume hypothesis 1.1.3(1) for  $G_{\mathbb{Q}_p}$  and  $[b]$ . Then the pair  $([b], \{\mu_X^{-1}\}_p)$  is admissible if and only if there exists  $s_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p)$  such that  $[b] = [b_{s_0}]$ .*

*Proof*

The “if” part is Lemma 1.3.9

Suppose that  $[b] \in B(G_{\mathbb{Q}_p})$  with  $([b], \{\mu_X^{-1}\}_p)$  admissible. Then Proposition 1.2.5 gives us a maximal torus  $T \subset G$  and an  $x \in X$  such that  $h_x$  factors through  $T_{\mathbb{R}}$ , and such that  $[b_{\text{bas}}(\mu_x^{-1})] \in B(T_{\mathbb{Q}_p})$  maps to  $[b] \in B(G_{\mathbb{Q}_p})$ .

Now, consider the 0-dimensional Shimura variety  $\text{Sh}_0 = \text{Sh}_{K \cap T(\mathbb{A}_f)}(T, h_x)$ . This is a finite étale scheme over the reflex field  $E_T = E(T, h_x)$ . Fix a place  $v' | p$  of  $E_T$  lying above  $v$ . The normalization of  $\text{Spec } \mathcal{O}_{E_T, (v')}$  in  $\text{Sh}_0$  gives us a canonical normal integral model  $\mathcal{S}_0$  for  $\text{Sh}_0$  over  $\mathcal{O}_{E_T, (v')}$ . Since all CM abelian varieties over number fields have everywhere potentially good reduction, the map  $\text{Sh}_0 \rightarrow E_T \otimes_E \text{Sh}_K$  extends to a map of  $\mathcal{O}_{E_T, (v')}$ -schemes  $\mathcal{S}_0 \rightarrow \mathcal{O}_{E_T, (v')} \otimes_{\mathcal{O}_{E, (v)}} \mathcal{S}_K$ , by Lemma 1.3.6.

Therefore, to prove the theorem, we may replace  $(G, [b], \{\mu_X^{-1}\})$  with the triple  $(T, [b_{\text{bas}}(\mu_x^{-1})], \mu_x^{-1})$ , and reduce to the case where  $G = T$  is a torus. Choose any point  $s_0 \in \mathcal{S}_0(\bar{\mathbb{F}}_p)$ . By Lemma 1.3.9, the pair  $([b_{s_0}], \mu_x^{-1})$  is admissible for  $T_{\mathbb{Q}_p}$ . But then we must have  $[b_{s_0}] = [b]$ .  $\square$

### 1.3.11

Given a scheme  $S$  in characteristic  $p$ , let  $\mathbf{F} - \text{Isoc}(S)$  be the category of  $F$ -isocrystals over  $S$  (cf. [36, Section 3]). This is the isogeny category obtained by localizing the category of  $F$ -crystals over  $S$ . It is a  $\mathbb{Q}_p$ -linear (nonneutral) Tannakian category, whose identity object  $\mathbf{1}$  corresponds to the structure sheaf on the crystalline site of  $S$  over  $\mathbb{Z}_p$ .

Recall that for  $G$  a reductive group over  $\mathbb{Q}_p$ , an  $F$ -isocrystal with  $G$ -structure over  $S$  (see [36, Definition 3.3]) is an exact faithful tensor functor

$$\text{Rep}_{\mathbb{Q}_p} G \rightarrow \mathbf{F} - \text{Isoc}(S).$$

Here  $\text{Rep}_{\mathbb{Q}_p} G$  denotes the category of finite-dimensional  $\mathbb{Q}_p$ -representations of  $G$ .

The crystalline realization of the universal abelian scheme  $\mathcal{A}$  over  $\mathcal{S}_K$  gives us a canonical object  $\mathcal{D}$  in  $\mathbf{F} - \text{Isoc}(\mathcal{S}_K \otimes_{\mathcal{O}_{E, (v)}} \bar{\mathbb{F}}_p)$ . For each point  $s_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p)$ , the restriction of  $\mathcal{D}$  over  $s_0$  is realized by the  $F$ -isocrystal  $D_{s_0}$ .

The proof of the following proposition is rather technical. Since it is used only in Theorem 1.3.14 and Corollary 1.3.16 below, and the rest of the paper does not depend on it, we relegate it to an appendix, where we prove a stronger statement (see Corollary A.7 below).

### PROPOSITION 1.3.12

*For each  $\alpha$ , there exists a morphism*

$$s_\alpha : \mathbf{1} \rightarrow \mathcal{D}^{\otimes}$$



whose restriction to any point  $s_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p)$  is  $s_{\alpha, \text{cris}, s_0}$ .

### COROLLARY 1.3.13

The association  $V \mapsto \mathcal{D}$  extends to an  $F$ -isocrystal with  $G$ -structure over  $\mathcal{S}_K \otimes \bar{\mathbb{F}}_p$ .

#### Proof

Let  $S$  be a connected component of  $\mathcal{S}_K$ . We shall again write  $\mathcal{D}$  for  $\mathcal{D}|_S$ . Let  $C_{\mathcal{D}}$  be the smallest full Tannakian subcategory of  $F\text{-Isoc}(S)$  containing  $\mathcal{D}$ . It suffices to construct, for each  $S$ , an exact faithful tensor functor  $\omega : \text{Rep}_{\mathbb{Q}_p} G \rightarrow C_{\mathcal{D}}$  which sends  $V$  to  $\mathcal{D}$ .

First consider the associated  $L$ -linear category  $C_{\mathcal{D}, L} = C_{\mathcal{D}} \otimes L$ , which is obtained from  $C_{\mathcal{D}}$  by tensoring the Hom sets by  $L$ , and adjoining the direct summands corresponding to idempotents in the endomorphism algebra of each object (see [7, Section 2.1]). Choose  $s_0 \in S(\bar{\mathbb{F}}_p)$ . Pulling isocrystals back to  $s_0$  induces an  $L$ -fiber functor  $\omega_{s_0} : C_{\mathcal{D}, L} \rightarrow F\text{-Isoc}(s_0)$  which takes  $\mathcal{D}$  to  $D_{s_0}$ , and  $C_{\mathcal{D}, L}$  is equivalent to the category  $\text{Rep}_L G_{s_0}$ , where  $G_{s_0} = \text{Aut}_{\{s_{\alpha, \text{cris}, s_0}\}} D_{s_0}$ , the group of automorphisms of  $D_{s_0}$  respecting the tensors  $s_{\alpha, \text{cris}, s_0}$ .

Let  $P(s_0) = \text{Isom}_{s_{\alpha}}(V_L, D_{s_0})$ , the scheme of  $L$ -linear maps from  $V_L$  to  $D_{s_0}$  taking  $s_{\alpha}$  to  $s_{\alpha, \text{cris}, 0}$ . Then  $P(s_0)$  is a  $G$ -torsor. (It is necessarily a trivial  $G$ -torsor by Steinberg's theorem.) If  $W$  is in  $\text{Rep}_{\mathbb{Q}_p} G$ , then  $W^{\mathcal{D}} = G \backslash (W \times P(s_0))$  is an  $L$ -representation of  $G_{s_0}$ . We consider the composite functor

$$\omega_L : \text{Rep}_{\mathbb{Q}_p} G \xrightarrow{W \mapsto W^{\mathcal{D}}} \text{Rep}_L G_{s_0} \simeq C_{\mathcal{D}, L}.$$

It remains to show that the above functor factors through  $C_{\mathcal{D}}$ . For this, note that any object of  $\text{Rep}_{\mathbb{Q}_p} G$  is the kernel of a map  $e : W \rightarrow W$ , where  $W$  is a direct sum of objects of the form  $V_{m,n} := V^{\otimes n} \otimes V^{*\otimes m}$ . Now  $\omega_L(V_{m,n}) = \mathcal{D}^{\otimes m} \otimes \mathcal{D}^{*\otimes n}$  lies in  $C_{\mathcal{D}}$ . Since  $e$  can be considered as a morphism  $1 \rightarrow W^* \otimes W$ , we see that by Proposition 1.3.12,  $\omega_L(e)$  lies in  $C_{\mathcal{D}}$ , and so does its kernel. Similarly, if  $e : W_1 \rightarrow W_2$  is any map in  $\text{Rep}_{\mathbb{Q}_p} G$ , then  $e$  may be regarded as a map  $1 \rightarrow W_1^* \otimes W_2$ , so  $\omega_L(e)$  is in  $C_{\mathcal{D}}$  by Proposition 1.3.12.  $\square$

### THEOREM 1.3.14

(1) If  $s_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p)$ , then

$$\{s'_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p) : \bar{v}_G([b_{s'_0}]) \leq \bar{v}_G([b_{s_0}])\} \subset \mathcal{S}_K(\bar{\mathbb{F}}_p)$$

is a Zariski-closed subset.

(2) Let  $B(G_{\mathbb{Q}_p}, \{\mu_X^{-1}\}_p) \subset B(G_{\mathbb{Q}_p})$  be the subset consisting of those classes  $[b]$  such that  $([b], \{\mu_X^{-1}\}_p)$  is admissible. Then, for every  $[b] \in B(G_{\mathbb{Q}_p}, \{\mu_X^{-1}\}_p)$

satisfying hypothesis (1), the subset

$$S_{[b]} = \{s_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p) : [b_{s_0}] = [b]\}$$

is nonempty and locally closed in  $\mathcal{S}_K(\bar{\mathbb{F}}_p)$  for the Zariski topology.

- (3) Let  $\bar{S}_{[b]}$  be the closure of  $S_{[b]}$  in  $\mathcal{S}_K(\bar{\mathbb{F}}_p)$ ; then we have an inclusion of Zariski-closed subsets:

$$\bar{S}_{[b]} \subset \bigsqcup_{\bar{v}_G([b']) \leq \bar{v}_G([b])} S_{[b']}.$$

*Proof*

Assertions (1) and (3) follow from Corollary 1.3.13 and the argument of [36, Theorem 3.6]. One reduces to the case  $G = \mathrm{GL}_n$  using [36, Lemma 2.2(iv)], and applies Grothendieck's semicontinuity theorem for Newton polygons of  $F$ -isocrystals (see [15, Theorem 2.3.1]). Assertion (2) follows from (1) and Proposition 1.3.10.  $\square$

As we noted in 1.1.3, the second part of the theorem implies that the stratum  $S_{[b]}$  is nonempty if either  $[b]$  is basic or  $G_{\mathbb{Q}_p}$  is quasisplit.

### 1.3.15

As in 1.1.5, we fix an inner twisting  $\xi : G_{\mathbb{Q}_p} \rightarrow G^*$  over  $\bar{\mathbb{Q}}_p$ , a Borel  $B^* \subset G^*$ , and a maximal torus  $T^* \subset B^*$  over  $\mathbb{Q}_p$ . Let  $\mu$  be the  $B^*$ -dominant representative of  $\xi \circ \{\mu_X\}_p$ . There is a unique  $[b_\mu] \in B(G_{\mathbb{Q}_p}, \{\mu^{-1}\})$  with  $\mathcal{N}_\xi(\bar{v}_G([b_\mu])) = \bar{\mu}^{-1}$  (which, of course, does not depend on the choice of  $B^*$  or  $T^*$ ). The corresponding subset  $S_{[b_\mu]} \subset \mathcal{S}_K(\bar{\mathbb{F}}_p)$  is the  $\mu$ -ordinary stratum. By Proposition 1.3.10 and Theorem 1.3.14, this stratum is a nonempty Zariski-open subspace.

### COROLLARY 1.3.16

Suppose that the special fiber  $\mathcal{S}_{K,k(v)}$  is locally integral. Then  $S_{[b_\mu]}$  is dense in  $\mathcal{S}_{K,k(v)}$ .

*Proof*

If the special fiber is locally integral, it follows from [30, Corollary 4.1.11] that every connected component of  $\mathcal{S}_K$  has irreducible special fiber. This implies that  $S_{[b_\mu]}$  is dense in any connected component of  $\mathcal{S}_{K,k(v)}$  it intersects, and since  $S_{[b_\mu]}$  is nonempty, it is dense in *some* connected component.

To see that it is dense in all connected components, suppose that  $s, s' \in \mathcal{S}_K(\bar{\mathbb{Q}})$  with reductions  $s_0, s'_0 \in \mathcal{S}_K(\bar{\mathbb{F}}_p)$ . If there is an isogeny  $\mathcal{A}_s \rightarrow \mathcal{A}_{s'}$  taking  $s_{\alpha, \text{ét}, s}$  to  $s_{\alpha, \text{ét}, s'}$ , then there is an induced isogeny  $\mathcal{A}_{s_0} \rightarrow \mathcal{A}_{s'_0}$  taking  $s_{\alpha, \text{cris}, s_0}$  to  $s_{\alpha, \text{cris}, s'_0}$ , so

that if  $s_0 \in S_{[b_\mu]}$ , then  $s'_0 \in S_{[b_\mu]}$ . Since the group  $G(\mathbb{A}_f)$  acts transitively on the set of connected components of  $\mathcal{S}_{K,k(v)}$ , this implies that  $S_{[b_\mu]}$  is dense in  $\mathcal{S}_{K,k(v)}$ .  $\square$

*Remark 1.3.17*

In the situation where  $p > 2$  and  $K_p$  is hyperspecial, so that it is of the form  $G_{(p)}(\mathbb{Z}_p)$  for a reductive model  $G_{(p)}$  of  $G$  over  $\mathbb{Z}_{(p)}$ , the main theorem of [16] shows that  $\mathcal{S}_{K,k(v)}$  is smooth. So the corollary applies to give the density of the  $\mu$ -ordinary locus in this situation. This special case is already known due to Wortmann [44].

Using the results of the first author and Pappas, we can prove the following.

**COROLLARY 1.3.18**

*Suppose that  $p > 2$ , that  $G$  splits over a tamely ramified extension, and that  $K_p$  is a special parahoric. Then the embedding  $G \hookrightarrow \mathcal{G}_V$  can be chosen such that  $S_{[b_\mu]}$  is dense in  $\mathcal{S}_{K,k(v)}$ .*

*Proof*

This follows from Corollary 1.3.16 and [18, Corollary 0.3].  $\square$

## 2. CM lifts and independence of $\ell$

### 2.1. Tate's theorem with additional structures

#### 2.1.1

We keep the notation introduced in Section 1.3, so that  $(G, X)$  is a Shimura datum of Hodge type, equipped with an embedding of Shimura data  $\iota : (G, X) \hookrightarrow (\mathcal{G}_V, \mathcal{H}_V)$ , and we have a finite map  $\mathcal{S}_K \rightarrow \mathcal{O}_{E,(v)} \otimes \mathcal{S}_{\mathcal{K}}$ , which is an embedding on generic fibers.

We set

$$\mathcal{S}_{\mathcal{K}_p} = \varprojlim_{\mathcal{K}^p \subset \mathcal{G}_V(\mathbb{A}_f^p)} \mathcal{S}_{\mathcal{K}^p \mathcal{K}_p}, \quad \mathcal{S}_{K_p} = \varprojlim_{K^p \subset G(\mathbb{A}_f^p)} \mathcal{S}_{K^p K_p}.$$

The transition maps in the inverse systems are finite étale, and so the limits are schemes over  $\mathbb{Z}_{(p)}$  (resp.,  $\mathcal{O}_{E,(v)}$ ). By construction, we have a map

$$\iota_p : \mathcal{S}_{K_p} \rightarrow \mathcal{O}_{E,(v)} \otimes \mathcal{S}_{\mathcal{K}_p}.$$

Since  $G(\mathbb{A}_f^p)$  acts naturally on the right on  $\mathcal{S}_{\mathcal{K}_p}$  and the generic fiber  $\mathrm{Sh}_{K_p} = E \otimes_{\mathcal{O}_{E,(v)}} \mathcal{S}_{K_p}$ , compatibly with the map  $\iota_p$ , this action extends to  $\mathcal{S}_{K_p}$ .

The scheme  $\mathcal{S}_{\mathcal{K}_p}$  is open and closed in the moduli space of triples  $(A, \lambda, \varepsilon)$ , where  $(A, \lambda)$  is a polarized abelian scheme up to prime-to- $p$  isogeny, and

$$\varepsilon : \underline{\mathbb{A}}_f^p \otimes V \xrightarrow{\sim} \widehat{V}^p(A)$$

is an isomorphism of smooth  $\mathbb{A}_f^p$ -sheaves carrying the symplectic form  $\psi$  to an  $\underline{\mathbb{A}}_f^{p,\times}$ -multiple of the Weil pairing  $\widehat{T}^p(\lambda)$  on the prime-to- $p$  Tate module

$$\widehat{V}^p(A) = \left( \varprojlim_{p \nmid n} A[n] \right) \otimes \mathbb{Q}.$$

### 2.1.2

For each  $\alpha$ , let  $s_{\alpha, \mathbb{A}_f^p}$  be the projection of  $s_{\alpha, \text{ét}}$  onto  $H^0(\text{Sh}_{K_p}, (\widehat{V}^p(\mathcal{A}))^\otimes)$ . Since  $\mathcal{S}_{K_p}$  is normal,  $s_{\alpha, \mathbb{A}_f^p}$  extends to a section over  $\mathcal{S}_{K_p}$ .

Over  $\mathcal{S}_{K_p}$ , the map  $\iota_p$  induces an isomorphism

$$\eta : \mathbb{A}_f^p \otimes V \xrightarrow{\sim} \widehat{V}^p(\mathcal{A}) \quad (2.1.2.1)$$

carrying  $s_\alpha$  to  $s_{\alpha, \mathbb{A}_f^p}$  for each  $\alpha$ . In particular, for any  $s_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$ , the stabilizer of the collection  $\{s_{\alpha, \mathbb{A}_f^p, s_0}\}$  in  $\text{GL}(\widehat{V}^p(\mathcal{A}_{s_0}))$  is canonically identified with  $G(\mathbb{A}_f^p)$ .

### 2.1.3

Let  $\underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  be the algebraic group over  $\mathbb{Q}$  attached to the group of units in the endomorphism algebra  $\text{End}_{\mathbb{Q}}(\mathcal{A}_{s_0}) := \mathbb{Q} \otimes \text{End}(\mathcal{A}_{s_0})$ . We have the subgroup  $\mathbb{G}_m \subset \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  which acts on  $\mathcal{A}_{s_0}$  by scalar multiplication. Let  $\underline{\text{Aut}}_{\mathbb{Q}, \psi}(\mathcal{A}_{s_0}) \subset \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  denote the subgroup which preserves the polarization on  $\mathcal{A}_{s_0}$ , up to a scalar. There is a map  $c : \underline{\text{Aut}}_{\mathbb{Q}, \psi}(\mathcal{A}_{s_0}) \rightarrow \mathbb{G}_m$  which takes an automorphism to its action on the polarization. The kernel of  $c$  and  $\underline{\text{Aut}}_{\mathbb{Q}, \psi}(\mathcal{A}_{s_0})/\mathbb{G}_m$  are compact over  $\mathbb{R}$ . In particular, any closed subgroup of  $\underline{\text{Aut}}_{\mathbb{Q}, \psi}(\mathcal{A}_{s_0})$  is a reductive group over  $\mathbb{Q}$ .

Now,  $\underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  acts naturally on  $\widehat{V}^p(\mathcal{A}_{s_0})$  and  $D_{s_0}$ . Let  $I_{s_0}^p \subset \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  be the closed subgroup that fixes the tensors  $\{s_{\alpha, \mathbb{A}_f^p, s_0}\} \subset \widehat{V}^p(\mathcal{A}_{s_0})^\otimes$ , and let  $I_{s_0} \subset I_{s_0}^p$  be the largest closed subgroup that also fixes the tensors  $\{s_{\alpha, \text{cris}, s_0}\} \subset D_{s_0}^\otimes$ . Since  $I_{s_0} \subset I_{s_0}^p \subset G(\mathbb{A}_f^p)$ , we have  $I_{s_0} \subset I_{s_0}^p \subset \underline{\text{Aut}}_{\mathbb{Q}, \psi}(\mathcal{A}_{s_0})$ . In particular,  $I_{s_0}$  and  $I_{s_0}^p$  are reductive groups, and their quotients by the subgroup of scalars  $\mathbb{G}_m$  are compact over  $\mathbb{R}$ .

Recall that  $\mathcal{A}_{s_0}$  is an abelian variety up to prime-to- $p$  isogeny (so the notion of automorphism is understood accordingly). Set

$$I_{s_0}(\mathbb{Z}_{(p)}) = I_{s_0}(\mathbb{Q}_p) \cap \text{Aut}(\mathcal{A}_{s_0}).$$

We can view this as a subgroup of  $G(\mathbb{A}_f^p)$  via the embeddings

$$I_{s_0}(\mathbb{Z}_{(p)}) \subset I_{s_0}(\mathbb{A}_f^p) \subset I_{s_0}^p(\mathbb{A}_f^p) \subset G(\mathbb{A}_f^p).$$

LEMMA 2.1.4

Suppose that  $g_1, g_2 \in G(\mathbb{A}_f^p)$  are such that

$$s_0 \cdot g_1 = s_0 \cdot g_2 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p).$$

Then  $g_1$  and  $g_2$  have the same image in  $I_{s_0}(\mathbb{Z}_{(p)}) \backslash G(\mathbb{A}_f^p)$ .

*Proof*

The proof is essentially contained in [17, Proposition 2.1.3].

The image of  $s_0$  in  $\mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  corresponds to the triple  $(\mathcal{A}_{s_0}, \lambda_{s_0}, \varepsilon_{s_0})$  over  $\bar{\mathbb{F}}_p$  under the moduli interpretation of  $\mathcal{S}_{K_p}$ . For  $g \in G(\mathbb{A}_f^p)$ , the image of  $s_0 \cdot g$  in  $\mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  corresponds to the triple  $(A_{s_0}, \lambda_{s_0}, \varepsilon_{s_0} \circ g)$ . Therefore, if  $s_0 \cdot g_1 = s_0 \cdot g_2$ , then in particular, we have an isomorphism of triples

$$(A_{s_0}, \lambda_{s_0}, \varepsilon_{s_0} \circ g_1) \xrightarrow{\sim} (A_{s_0}, \lambda_{s_0}, \varepsilon_{s_0} \circ g_2).$$

This corresponds to an automorphism  $\phi \in \text{Aut}(\mathcal{A}_{s_0})$  (necessarily unique) such that

$$\widehat{V}^p(\phi) \circ \varepsilon_{s_0} \circ g_1 = \varepsilon_{s_0} \circ g_2$$

and  $\phi$  carries  $\{s_{\alpha, \text{cris}, s_0 \cdot g_1}\} \subset D_{s_0 \cdot g_1}^{\otimes}$  to  $\{s_{\alpha, \text{cris}, s_0 \cdot g_2}\} \subset D_{s_0 \cdot g_2}^{\otimes}$ . Note that here we are using Proposition 1.3.7.

The first condition implies that  $\widehat{V}^p(\phi)$  fixes  $\{s_{\alpha, \mathbb{A}_f^p, s_0}\}$ . Since under the natural identifications  $D_{s_0} = D_{s_0 \cdot g_i}$  induced by the identifications  $\mathcal{A}_{s_0} = \mathcal{A}_{s_0 \cdot g_i}$ , for  $i = 1, 2$ , the tensors  $\{s_{\alpha, \text{cris}, s_0}\}$  are carried to  $\{s_{\alpha, \text{cris}, s_0 \cdot g_i}\}$ , and  $\phi$  preserves the  $\{s_{\alpha, \text{cris}, s_0}\}$ . Hence,  $\phi$  must belong to

$$I_{s_0}(\mathbb{Z}_{(p)}) = I_{s_0}(\mathbb{Q}) \cap \text{Aut}(\mathcal{A}_{s_0}). \quad \square$$

### 2.1.5

Choose a neat compact open  $K^p \subset G(\mathbb{A}_f^p)$ . Set  $K = K_p K^p$ , and suppose that the image of  $s_0$  in  $\mathcal{S}_K(\bar{\mathbb{F}}_p)$  is defined over  $\mathbb{F}_q$ .

Then, for any  $m \in \mathbb{Z}_{\geq 1}$ , let  $\gamma_{m, s_0}$  denote the geometric,  $q^m$ -power Frobenius of  $\mathcal{A}_{s_0}$ . Then  $\gamma_{m, s_0}$  fixes the absolute Hodge cycle components  $\{s_{\alpha, \mathbb{A}_f^p, s_0}\}$ , and it fixes the crystalline components  $\{s_{\alpha, \text{cris}, s_0}\}$  as these are  $\varphi$ -invariant. Hence,  $\gamma_{m, s_0} \in I_{s_0}(\mathbb{Q})$ . In particular,  $\gamma_{m, s_0}$  induces a semisimple automorphism  $\gamma_{m, s_0}^p$  of  $\widehat{V}^p(\mathcal{A}_{s_0})$  which preserves  $\{s_{\alpha, \mathbb{A}_f^p, s_0}\}$ , and thus lies in  $G(\mathbb{A}_f^p)$ . Set

$$I_{\mathbb{A}_f^p, m, s_0} = \text{Cent}_{G_{\mathbb{A}_f^p}}(\gamma_{m, s_0}^p).$$

If  $m \mid m'$ , then  $\gamma_{m', s_0}^p = (\gamma_{m, s_0}^p)^{m'/m}$ , and so we have a natural inclusion

$$I_{\mathbb{A}_f^p, m, s_0} \subset I_{\mathbb{A}_f^p, m', s_0}.$$

Set

$$I_{\mathbb{A}_f^p, s_0} = \varinjlim_m I_{\mathbb{A}_f^p, m, s_0}.$$

Then, for  $m$  sufficiently divisible, the Zariski closure of the subgroup of  $I_{s_0}$  generated by  $\gamma_{m, s_0}$  is a torus, and we have  $I_{\mathbb{A}_f^p, s_0} = I_{\mathbb{A}_f^p, m, s_0}$ , which is independent of the choice of  $q$ .

For each  $\ell \neq p$ , write  $I_{\ell, s_0}$  for the projection of  $I_{\mathbb{A}_f^p, s_0}$  onto  $G_{\mathbb{Q}_\ell}$ . For  $m$  sufficiently divisible, this is the centralizer in  $G_{\mathbb{Q}_\ell}$  of the projection  $\gamma_{m, \ell, s_0}$  of  $\gamma_{m, s_0}^p$ .

For  $m$  sufficiently divisible, the Zariski closure in  $G_{\mathbb{A}_f^p}$  of the subgroup generated by  $\gamma_{m, s_0}^p$  is a torus. Therefore,  $I_{\ell, s_0}$  is a Levi subgroup of  $G_{\mathbb{Q}_\ell}$  (over  $\bar{\mathbb{Q}}_\ell$ ) and, in particular, connected reductive.

The action of  $I_{s_0}$  on  $\widehat{V}^p(\mathcal{A}_{s_0})$  gives us a canonical map of  $\mathbb{A}_f^p$ -groups

$$\mathbb{A}_f^p \otimes I_{s_0} \rightarrow I_{\mathbb{A}_f^p, s_0}.$$

For each  $\ell \neq p$ , this gives us a map  $i_\ell : \mathbb{Q}_\ell \otimes I_{s_0} \rightarrow I_{\ell, s_0}$ , which is injective.

#### PROPOSITION 2.1.6

Let  $\ell \neq p$  be a prime such that  $G_{\mathbb{Q}_\ell}$  is split and such that the characteristic polynomial of  $\gamma_{m, \ell, s_0}$  is split over  $\mathbb{Q}_\ell$ . Then  $i_\ell$  is an isomorphism.

*Proof*

By Lemma 2.1.4, we have surjective maps

$$G(\mathbb{A}_f^p) \rightarrow s_0 \cdot G(\mathbb{A}_f^p) \rightarrow I_{s_0}(\mathbb{Z}_{(p)}) \backslash G(\mathbb{A}_f^p),$$

where the first map is the orbit map  $g \mapsto s_0 \cdot g$ , and the composite is the natural projection.

For any neat compact open  $K^p \subset G(\mathbb{A}_f^p)$  with  $\ell$ -primary factor  $K_\ell \subset G(\mathbb{Q}_\ell)$ , this implies that the image in  $\mathcal{S}_K(\bar{\mathbb{F}}_p)$  of  $s_0 \cdot I_{\ell, s_0}(\mathbb{Q}_\ell)$  surjects onto the quotient  $I_{s_0}(\mathbb{Q}_\ell) \backslash I_{\ell, s_0}(\mathbb{Q}_\ell) / (K_\ell \cap I_{\ell, s_0}(\mathbb{Q}_\ell))$ . Since  $I_{\ell, s_0}(\mathbb{Q}_\ell)$  commutes with  $\gamma_{m, \ell, s_0}$  for  $m$  sufficiently divisible, this image is in fact contained in  $\mathcal{S}_K(\mathbb{F}_{q^m})$ . In particular,  $I_{s_0}(\mathbb{Q}_\ell) \backslash I_{\ell, s_0}(\mathbb{Q}_\ell) / (K_\ell \cap I_{\ell, s_0}(\mathbb{Q}_\ell))$  is finite.

The proposition is now deduced just as in [17, Corollary 2.1.7].  $\square$

#### 2.1.7

We will prove that  $i_\ell$  is an isomorphism for every  $\ell$ , including  $\ell = p$ . This will be done using a result of Noot. We first explain the definition of the  $I_{\ell, s_0}$  and  $i_\ell$  when  $\ell = p$ .

For any  $m \in \mathbb{Z}_{>0}$ , the crystalline realization of  $\mathcal{A}_{s_0}$  is defined over  $\mathbb{Q}_{q^m} = W(\mathbb{F}_{q^m})[p^{-1}]$ ; therefore, the isocrystal  $D_{s_0}$  has a natural descent to an  $F$ -isocrystal  $D_{m,s_0}$  over  $\mathbb{Q}_{q^m}$ , and the  $\varphi$ -invariant tensors  $\{s_{\alpha,\text{cris},s_0}\}$  belong to  $D_{m,s_0}^{\otimes}$ . Write  $q = p^r$ , and let  $\gamma_{m,\text{cris},s_0} = \varphi^{rm} : D_{m,s_0} \rightarrow D_{m,s_0}$  be the crystalline realization of  $\gamma_{m,s_0}$ . It is a  $\varphi$ -equivariant isomorphism fixing the tensors  $\{s_{\alpha,\text{cris},s_0}\}$ .

As in Section 1.3.8, for  $m$  sufficiently divisible (which we now assume), we can find an isomorphism

$$\mathbb{Q}_{q^m} \otimes V \xrightarrow{\sim} D_{m,s_0} \quad (2.1.7.1)$$

carrying, for each  $\alpha$ ,  $1 \otimes s_{\alpha}$  to  $s_{\alpha,\text{cris},s_0}$ . Let  $\delta_{s_0} \in G(\mathbb{Q}_{q^m})$  be such that  $\varphi : D_{m,s_0} \rightarrow D_{m,s_0}$  pulls back to the automorphism  $\delta_{s_0}(\sigma \otimes 1)$  of  $\mathbb{Q}_{q^m} \otimes V$  under this isomorphism. Then, by construction, the class  $[b_{s_0}] \in B(G_{\mathbb{Q}_p})$  associated with  $s_0$  is exactly the  $\sigma$ -conjugacy class of  $\delta_{s_0}$ .

Similarly, the automorphism  $\gamma_{m,\text{cris},s_0}$  of  $D_{m,s_0}$  pulls back to an element  $\gamma_{m,p,s_0} \in G(\mathbb{Q}_{q^m})$ , whose conjugacy class under  $\varinjlim_m G(\mathbb{Q}_{q^m})$  is independent of all choices.

We have the relation

$$\gamma_{m,p,s_0} = \delta_{s_0} \sigma(\delta_{s_0}) \cdots \sigma^{rm-2}(\delta_{s_0}) \sigma^{rm-1}(\delta_{s_0}) \in G(\mathbb{Q}_{q^m}). \quad (2.1.7.2)$$

Define an algebraic group  $I_{m,\delta_{s_0}}$  over  $\mathbb{Q}_p$  as follows. For any  $\mathbb{Q}_p$ -algebra  $R$ , we have

$$I_{m,\delta_{s_0}}(R) = \{g \in G(\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} R) : g\delta_{s_0} = \delta_{s_0}\sigma(g)\}.$$

Then  $\mathbb{Q}_{q^m} \otimes_{\mathbb{Q}_p} I_{m,\delta_{s_0}}$  is naturally identified with the centralizer in  $G_{\mathbb{Q}_{q^m}}$  of  $\gamma_{m,p,s_0}$ . Since  $\gamma_{m,p,s_0}$  is semisimple (which follows from semisimplicity of  $\gamma_{m,s_0}$ ),  $I_{m,\delta_{s_0}}$  is a reductive group over  $\mathbb{Q}_p$ , and is connected for  $m$  sufficiently divisible. Set

$$I_{p,s_0} = \varinjlim_m I_{m,\delta_{s_0}},$$

which is equal to  $I_{m,\delta_{s_0}}$  for  $m$  sufficiently divisible. We have a canonical inclusion

$$i_p : I_{s_0} \otimes_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow I_{p,s_0},$$

and an inclusion  $I_{p,s_0} \hookrightarrow G$  defined over  $\mathbb{Q}_{q^m}$  for  $m$  sufficiently divisible.

Let  $J_{\delta_{s_0}}$  be the  $\mathbb{Q}_p$ -group defined in 1.1.4. For any  $m$ , we have the obvious inclusion  $I_{m,\delta_{s_0}} \subset J_{\delta_{s_0}}$ , and in particular  $I_{p,s_0} \subset J_{\delta_{s_0}}$ .

### 2.1.8

Given a connected reductive group  $H$  over a field  $F$  of characteristic 0, write  $\text{Conj}(H)$  for the scheme over  $F$  parameterizing semisimple conjugacy classes in  $H$ .

More precisely, the conjugation action of  $H$  on itself induces an action on the Hopf algebra  $\mathcal{O}_H$ , and  $\text{Conj}(H) = \text{Spec}(\mathcal{O}_H)^H$ .

Following Noot [32, Section 1.5], we will also define a certain quotient  $\text{Conj}'(H)$  of  $\text{Conj}(H)$  as follows. Let  $\bar{F}$  be an algebraic closure of  $F$ . Then  $H_{\bar{F}}^{\text{der}}$  is an almost direct product of simple reductive factors  $H_i$  with  $i$  in some indexing set  $I$ .

Write  $I_D \subset I$  for the subset of indices  $i$  such that  $H_i \simeq \text{SO}(2n_i)$  for some  $n_i \geq 4$ . For each  $i \in I_D$ , set  $H'_i = \text{O}(2n_i)$ . Since  $I_D \subset I$  is  $\text{Gal}(\bar{F}/F)$ -stable, the finite  $\bar{F}$ -group scheme

$$\text{Out}'(H)_{\bar{F}} = \prod_{i \in I_D} H'_i / H_i$$

descends to a finite group scheme  $\text{Out}'(H)$  over  $F$ , which acts canonically on  $\text{Conj}(H)$ . We will write  $\text{Conj}'(H)$  for the quotient of  $\text{Conj}(H)$  for this action.

We call an element  $\gamma \in H(F)$  *neat* if  $\gamma$  is semisimple and the Zariski closure of  $\langle x \rangle$ , the group of points generated by  $x$ , is connected (i.e., a torus).

COROLLARY 2.1.9

For every  $\ell$ , the map

$$i_\ell : \mathbb{Q}_\ell \otimes I_{s_0} \rightarrow I_{\ell, s_0}$$

is an isomorphism.

*Proof*

Choose  $\ell_0 \neq p$  a prime satisfying the conditions of Proposition 2.1.6 so that  $i_{\ell_0}$  is an isomorphism. Let  $m$  be sufficiently divisible such that  $\gamma_{m, s_0} \in \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  is neat,  $I_{\ell, s_0}$  is the centralizer of  $\gamma_{m, \ell, s_0}$  in  $G_{\mathbb{Q}_\ell}$  if  $\ell \neq p$  (resp., in  $G_{\mathbb{Q}_{q^m}}$  if  $\ell = p$ ), and  $I_{\ell_0, s_0}$  is the centralizer of  $\gamma_{m, \ell_0, s_0}$  in  $G_{\mathbb{Q}_{\ell_0}}$ .

By [32, Théorèmes 1.8, 4.2], the images of the elements  $\gamma_{m, \ell, s_0}$  and  $\gamma_{m, \ell_0, s_0}$  in  $\text{Conj}'(G)$  lie in  $\text{Conj}'(G)(\mathbb{Q})$ , and are equal. In particular,  $I_{\ell, s_0}$  and  $I_{\ell_0, s_0}$  have the same dimension. Thus  $\mathbb{Q}_\ell \otimes I_{s_0}$  and  $I_{\ell, s_0}$  have the same dimension by Proposition 2.1.6, and since  $I_{\ell, s_0}$  is connected  $i_\ell$  is an isomorphism.  $\square$

## 2.2. Independence of $\ell$ and conjugacy classes

### 2.2.1

Let  $l$  be a prime (possibly equal to  $p$ ). An element  $\alpha \in \bar{\mathbb{Q}}$  is called an  *$l$ -Weil number* of weight  $w \in \mathbb{Z}$  if  $\alpha$  is an  $l$ -unit and all its complex embeddings have absolute value  $l^{w/2}$ . (This  $l$  should not be conflated with the prime  $\ell$  as in independence of  $\ell$ .)

Let  $H$  be an algebraic group over  $\mathbb{Q}$ . We call an element  $\gamma \in H(\mathbb{Q})$  an  *$l$ -Weil point* if for some faithful representation  $W$  of  $H$  (defined over any field of charac-



teristic 0), the eigenvalues of  $\gamma$  on  $W$  are  $l$ -Weil numbers. If  $W'$  is any other representation of  $H$ , then  $W'$  is isomorphic to a representation in the Tannakian category generated by  $W$ . Hence the eigenvalues of  $\gamma$  acting on  $W'$  are also  $l$ -Weil numbers, and the definition does not depend on  $W$ .

We call  $\gamma \in H(\mathbb{Q})$  a Weil point if it is an  $l$ -Weil point for some  $l$ .

Keeping the notation introduced in Section 2.1, our first goal in this subsection is to prove the following analogue of the result of Noot on independence of Frobenius elements, used above.

### PROPOSITION 2.2.2

*Let  $\gamma \in I_{s_0}(\mathbb{Q})$  be a neat Weil point. For each  $\ell$ , the image of  $i_\ell(\gamma)$  in  $\text{Conj}'(G)$  lies in  $\text{Conj}'(G)(\mathbb{Q})$  and does not depend on  $\ell$ .*

### 2.2.3

To prepare for the proof of Proposition 2.2.2, we first show two lemmas. Recall that a  $\mathbb{Q}$ -torus  $T$  satisfies the *Serre condition* if its maximal  $\mathbb{R}$ -split subtorus  $T_1 \subset T$  is  $\mathbb{Q}$ -split. For an algebraic  $\mathbb{Q}$ -group  $H$ , and  $F$  a number field, an element  $\gamma \in H(F)$  is called an  $l$ -unit if for every place  $v \nmid l$  of  $F$  the group  $\gamma$  generated is bounded in  $H(F_v)$ .

### LEMMA 2.2.4

*Let  $T$  be a  $\mathbb{Q}$ -torus which satisfies the Serre condition. An element  $\gamma \in T(\mathbb{Q})$  is an  $l$ -Weil point if and only if it is an  $l$ -unit. In particular, an element  $\gamma \in I_{s_0}(\mathbb{Q})$  is an  $l$ -Weil point if and only if it is an  $l$ -unit.*

### *Proof*

An element  $\gamma \in T(\mathbb{Q})$  is an  $l$ -Weil point if and only if  $\chi(\gamma)$  is an  $l$ -Weil number for any  $\chi \in X^*(T)$ , as the direct sum of a basis of  $X^*(T)$  is a faithful representation of  $T$ . In particular, if  $\gamma$  is an  $l$ -Weil point, then, for every  $\chi \in X^*(T)$ , the subgroup of  $\mathbb{Q}(\chi(\gamma))^\times$  generated by  $\chi(\gamma)$  is  $v$ -adically bounded for every place  $v \nmid l$  of  $\mathbb{Q}(\chi(\gamma))$ . Hence the subgroup generated by  $\gamma$  is bounded in  $T(\mathbb{Q}_v)$  for every place  $v \nmid l$  of  $\mathbb{Q}$ , and  $\gamma$  is an  $l$ -unit in  $T(\mathbb{Q})$ .

Conversely, if  $\gamma$  is an  $l$ -unit, let  $T_2 \subset T$  be the maximal subtorus such that  $T_2(\mathbb{R})$  is compact. Then  $T_2$  is defined over  $\mathbb{Q}$ . If we think of  $\chi$  as defined over  $\mathbb{C}$ , then  $\chi\bar{\chi}$  is trivial on  $T_2$ , and factors through  $T/T_2$ . Hence,  $\chi\bar{\chi}(\gamma) \in \mathbb{Q}^\times$  is an  $l$ -unit and equal to  $l^w$  for some integer  $w$ . This shows that  $\chi(\gamma)$  has absolute value  $l^{w/2}$  under all complex embeddings.

The final statement follows from the fact that every  $\gamma \in I_{s_0}(\mathbb{Q})$  is semisimple, and so is contained in some maximal torus  $T \subset I_{s_0}$ . Any such maximal torus satisfies

the Serre condition. In fact, the maximal  $\mathbb{R}$ -split torus of  $T$  is either trivial, or the subtorus  $\mathbb{G}_m \subset I_{s_0}$ , consisting of scalars, as in Section 2.1.3.  $\square$

#### LEMMA 2.2.5

Let  $\gamma \in I_{s_0}(\mathbb{Q})$  be an  $l$ -Weil point. Then for  $\ell \neq p$  the set of eigenvalues of  $i_\ell(\gamma)$  acting on  $V_{\tilde{\mathbb{Q}}_\ell}$  does not depend on  $\ell$ , and for some  $w \in \mathbb{Z}$ , these eigenvalues are all  $l$ -Weil numbers of weight  $w$ .

#### Proof

The independence of  $\ell$  is standard and follows from the Lefschetz trace formula. Now, recall from Section 2.1.3 that we have the homomorphism  $c : I_{s_0} \rightarrow \mathbb{G}_m$  whose kernel  $I_{s_0}^1$  is compact over  $\mathbb{R}$ . For the second claim, it suffices to replace  $\gamma$  by some power, when we can write  $\gamma = l^i \cdot \gamma^1$ , where  $\gamma^1 \in I_{s_0}^1(\mathbb{Q})$  is an  $l$ -Weil point, and  $l^i$  denotes scalar multiplication by  $l^i$  on  $\mathcal{A}_{s_0}$ . It suffices to show that for any  $i_\ell$ , the eigenvalues of  $i_\ell(\gamma^1)$  acting on  $V_{\tilde{\mathbb{Q}}_\ell}$  have all their complex absolute values equal to 1.

Let  $T \subset I_{s_0}^1$  be a maximal torus containing  $\gamma^1$ . Fix an isomorphism  $\mathbb{C} \simeq \tilde{\mathbb{Q}}_\ell$ . For each eigenspace of  $T$  acting on  $V_{\mathbb{C}}$ , the corresponding  $\chi \in X^*(T)$  satisfies  $\chi \bar{\chi} = 1$ , as  $T$  is compact over  $\mathbb{R}$ . Thus  $\chi(\gamma^1) \bar{\chi}(\gamma^1) = 1$ , as  $\gamma^1 \in T(\mathbb{Q})$ .  $\square$

#### 2.2.6

The proof of Proposition 2.2.2 will follow Noot's arguments with a modification at one point where we will need to use Corollary 2.1.9. We begin by recalling some definitions from [32, 2.3].

Let  $H$  be an absolutely almost simple group of classical type over a field of characteristic 0, and let  $W$  be a finite-dimensional  $H$ -representation. We say that  $W$  is *admissible* if it is a multiple of one of the following:

- The direct sum of the standard representation and its dual if  $H$  is of type  $A$ .
- The spin representation if  $H$  is of type  $B$ .
- The standard representation if  $H$  is of type  $C$ .
- The standard representation if  $H$  is of type  $D$ .
- The direct sum of the two half-spin representations if  $H$  is of type  $D$ .

In the case of type  $D$ , in the fourth (resp., fifth) case we say that  $(H, W)$  is of type  $D^{\mathbb{H}}$  (resp.,  $D^{\mathbb{R}}$ ).

Now recall our embedding of Shimura data  $\iota : (G, X) \hookrightarrow (\mathcal{G}_V, \mathcal{H}_V)$ . We say that  $\iota$  is *strictly accommodating* if:

- For some totally real field  $K$ ,  $G^{\text{der}} = \text{Res}_{K/\mathbb{Q}} G^s$  with  $G^s$  absolutely almost simple, and the  $G^{\text{der}}$ -representation  $V$  has the form  $\text{Res}_{K/\mathbb{Q}} V^s$  for an admissible  $G^s$ -representation  $V^s$ .

- If  $(G^s, V^s)$  is of type  $D^{\mathbb{R}}$ , then for any character  $\chi : Z_G \rightarrow \mathbb{G}_m$ , over  $\bar{\mathbb{Q}}$ , the  $\chi$ -part of  $V$  is an admissible representation of a factor of  $G_{\bar{\mathbb{Q}}}^{\text{der}}$ .
- For any proper, nonzero,  $G$ -stable subspace  $V' \subset V$ , if  $G'$  denotes the image of  $G$  in  $\text{Aut } V'$ , then we require that  $(G', V')$  *not* satisfy the first two conditions above.

Finally, we say that  $\iota$  is *accommodating* if there is a finite collection of accommodating embeddings of Shimura data  $\iota_j : (G_j, X_j) \hookrightarrow (\mathcal{G}_{V_j}, \mathcal{H}_{V_j})$ ,  $j = 1, \dots, s$ , and an isomorphism of symplectic spaces  $\prod_{j=1}^s V_j \simeq V$  which induces a commutative diagram

$$\begin{array}{ccc} (G, X) & \longrightarrow & (\mathcal{G}_V, \mathcal{H}_V) \\ \downarrow & & \downarrow \\ \prod_{j=1}^s (G_j, X_j) & \longrightarrow & (\prod_{j=1}^s \mathcal{G}_{V_j}, \prod_{j=1}^s \mathcal{H}_{V_j}) \end{array}$$

such that the map on the left induces an isomorphism  $G^{\text{der}} \simeq \prod_{j=1}^s G_j^{\text{der}}$ .

Note that Noot's definitions are formulated for the Mumford–Tate group of an abelian variety, rather than for Shimura data. The embedding  $\iota$  is accommodating in our sense if and only if for some (or equivalently any)  $y \in \text{Sh}_K(G, X)(\mathbb{C})$  such that the corresponding abelian variety  $\mathcal{A}_y$  has Mumford–Tate group  $G$ ,  $\mathcal{A}_y$  is accommodating in the sense of Noot.

### 2.2.7. Proof of Proposition 2.2.2

Suppose first that  $\iota : (G, X) \hookrightarrow (\mathcal{G}_V, \mathcal{H}_V)$  is accommodating. In this case, the proof is the same as [32, Théorème 2.4]. For the reader's convenience, we indicate the argument.

Let  $V \otimes \bar{\mathbb{Q}} = \bigoplus_{i=1}^n W_i$  be a decomposition of the  $G$ -representation  $V$  into its isotypic components over  $\bar{\mathbb{Q}}$ . The subalgebra  $\bar{\mathbb{Q}}^n \subset \text{End}_{\bar{\mathbb{Q}}} V_{\bar{\mathbb{Q}}}$ , which acts by scalars on each factor  $W_i$ , descends to a product of fields  $L = \prod_{i=1}^k L_i \subset \text{End}_{\mathbb{Q}} V$ , which corresponds to a decomposition  $V = \prod_{i=1}^k V_i$ .

Let  $P_{i,\gamma,\ell}$  denote the characteristic polynomial of  $\gamma$  acting on  $V_{i,\bar{\mathbb{Q}}_\ell}$ . One first shows that  $P_{i,\gamma,\ell}$  does not depend on  $\ell$  (see the proof of [31, Lemma 6.13]). Note that since  $\gamma$  is an  $l$ -Weil point for some  $l$ , the eigenvalues of  $P_{i,\gamma,\ell}$  are  $l$ -Weil numbers. Since  $\gamma$  is neat, no two of these roots differ by a non-trivial root of 1; this is the condition Noot calls *faiblement net*. Then applying [32, Lemmes 2.5, 2.6], one finds that since  $P_{i,\gamma,\ell}$  does not depend on  $\ell$ , the element  $i_\ell(\gamma) \in \text{Conj}'(G)(\bar{\mathbb{Q}}_\ell)$  is also independent of  $\ell$ , and lies in  $\text{Conj}'(G)(\bar{\mathbb{Q}})$ .

To reduce to the accommodating case, we again follow Noot's argument in [32, Section 3], though we formulate it in terms of Shimura data rather than Mumford–

Tate groups. Lift  $s_0$  to a point  $s \in \mathrm{Sh}_K(G, X)(\bar{\mathbb{Q}}_p)$ . The statement of the proposition depends only on the abelian variety  $\mathcal{A}_s$  equipped with the Hodge cycles corresponding to  $\{s_\alpha\}$ , and not on level structures. Thus, fixing an isomorphism  $\bar{\mathbb{Q}}_p \simeq \mathbb{C}$ , we may assume that  $s$  is the image of a point of the form  $(h_0, 1) \in X \times G(\mathbb{A}_f)$ .

The results of Deligne [8, Proposition 2.3.10] (see also [31, Theorem 2.12]) imply that there exists an accommodating embedding  $\iota' : (G', X') \hookrightarrow (\mathcal{G}_{V'}, \mathcal{H}_{V'})$  together with a map  $G'^{\mathrm{der}} \rightarrow G^{\mathrm{der}}$  which induces an isomorphism  $(G'^{\mathrm{ad}}, X'^{\mathrm{ad}}) \simeq (G^{\mathrm{ad}}, X^{\mathrm{ad}})$  of adjoint Shimura data. Here  $V'$  denotes a  $\mathbb{Q}$ -vector space equipped with a symplectic form  $\psi'$ . By the real approximation theorem applied to  $G^{\mathrm{ad}}$ , after conjugating the map  $G'^{\mathrm{der}} \rightarrow G^{\mathrm{der}}$  by an element of  $G^{\mathrm{ad}}(\mathbb{Q})$ , we may assume that the image of  $X'$  in  $X^{\mathrm{ad}}$  contains  $h_0$ . Identifying  $G'^{\mathrm{ad}}$  and  $G^{\mathrm{ad}}$ , let  $G''$  be the connected component of the identity of  $G' \times_{G^{\mathrm{ad}}} G$ , and let  $X''$  be a  $G''(\mathbb{R})$ -orbit of  $(h_0, h_0) \in X \times_{X^{\mathrm{ad}}} X'$ . Finally, we set  $V'' = V \oplus V'$ , where  $V''$  is equipped with the symplectic form  $\psi'' = \psi \oplus \psi'$ , and consider the embedding

$$\iota'' : (G'', X'') \rightarrow (\mathcal{G}_{V''}, \mathcal{H}_{V''})$$

induced by  $\iota$  and  $\iota'$ .

Applying our previous constructions to each of  $\iota'$  and  $\iota''$ , we obtain maps between integral models

$$\mathcal{S}_K(G, X) \leftarrow \mathcal{S}_{K''}(G'', X'') \rightarrow \mathcal{S}_{K'}(G', X'),$$

where  $K''$  and  $K'$  are suitable level structures. Since  $h_0 \in X''$ ,  $s$  lifts to a point  $s'' \in \mathcal{S}_{K''}(G'', X'')(\bar{\mathbb{Q}}_p)$ . As in [32, p. 68], using the Néron–Ogg–Shafarevich criterion one sees that  $\mathcal{A}_{s''}$  has good reduction. So by Lemma 1.3.6,  $s''$  specializes to  $s''_0 \in \mathcal{S}_{K''}(G'', X'')(\bar{\mathbb{F}}_p)$  lifting  $s_0$ . Let  $s'_0 \in \mathcal{S}_{K'}(G', X')(\bar{\mathbb{F}}_p)$  be the image of  $s''_0$ . By the construction of  $\iota'$  and  $\iota''$ , there are maps of abelian varieties

$$\mathcal{A}_{s_0} \leftarrow \mathcal{A}_{s''_0} \rightarrow \mathcal{A}_{s'_0}$$

corresponding to the projections of  $V''$  onto  $V$  and  $V'$ .

Note that the action of  $G''$  on  $V''$  respects the decomposition  $V \oplus V'$ . Thus, the projections  $V'' \rightarrow V$  and  $V'' \rightarrow V'$  are  $G''$ -invariant elements of  $\mathrm{End}(V'')$ , and we may include them in the set of Hodge cycles used to define  $I_{s''_0}$ . This shows that the surjections of  $G''$  onto  $G$  and  $G'$  induce the maps

$$I_{s_0} \leftarrow I_{s''_0} \rightarrow I_{s'_0}.$$

By Corollary 2.1.9, these maps are surjective and induce the isomorphisms

$$I_{s_0}/Z_G \simeq I_{s''_0}/Z_{G''} \simeq I_{s'_0}/Z_{G'}.$$

Let  $T \subset I_{s_0}$  be a maximal torus containing  $\gamma$ , and let  $T'' \subset I_{s'_0}$  be the preimage of  $T$ . For some positive integer  $n$ , there exists a map  $T \rightarrow T''$  whose composite with the projection  $T'' \rightarrow T$  is multiplication by  $n$ . Let  $\gamma''$  be the image of  $\gamma$  in  $T''(\mathbb{Q})$  via the map  $T \rightarrow T''$ . This is an  $l$ -unit in  $T''(\mathbb{Q})$ , and hence a neat Weil point in  $I_{s'_0}(\mathbb{Q})$  by Lemma 2.2.4. It suffices to show Proposition 2.2.2 for  $\gamma''$ , as the result then follows for  $\gamma''$ , and  $\gamma$  by [32, Proposition 3.2].

By Lemma 2.2.8 below, there is a map over  $\mathbb{Q}$ -groups  $I_{s'_0} \rightarrow G''^{\text{ab}}$  which agrees with the map induced by  $i_\ell$  for any  $\ell$ . Replacing  $\gamma''$  by a power as above, we may assume that the image of  $\gamma''$ ,  $\gamma''^{\text{ab}} \in G''^{\text{ab}}(\mathbb{Q})$ , lifts to  $z \in Z_{G''}(\mathbb{Q})$ . Write  $\gamma'' = \gamma'_1 z$ . Note that  $\gamma''^{\text{ab}}$  is an  $l$ -Weil point as is  $z$ , for example, using Lemma 2.2.4. Since  $z$  and  $\gamma$  commute,  $\gamma'_1$  is again an  $l$ -Weil point. It suffices to show that the image of  $i_\ell(\gamma'_1)$  in  $\text{Conj}'(G''^{\text{der}}) \subset \text{Conj}'(G'')$  is a  $\mathbb{Q}$ -point which is independent of  $\ell$ . Since  $G''^{\text{der}} \simeq G'^{\text{der}}$ , this is a consequence of the corresponding statement for the image of  $\gamma'_1$  in  $I_{s'_0}$ , which is the accommodating case considered above.  $\square$

#### LEMMA 2.2.8

*There is a map of  $\mathbb{Q}$ -groups  $I_{s_0} \rightarrow G^{\text{ab}}$  which agrees with the map induced by  $i_\ell$  for every  $\ell$ .*

#### *Proof*

Recall that for  $\ell \neq p$ , we have the composite

$$I_{s_0} \otimes \mathbb{Q}_\ell \simeq I_{\ell, s_0} \rightarrow G_{\mathbb{Q}_\ell} \rightarrow G_{\mathbb{Q}_\ell}^{\text{ab}}.$$

Similarly, we have a map  $I_{s_0, \mathbb{Q}_{q^m}} \rightarrow G_{\mathbb{Q}_{q^m}}^{\text{ab}}$  defined for  $m$  sufficiently divisible. We have to show that all these maps are induced by a map of  $\mathbb{Q}$ -groups  $I_{s_0} \rightarrow G^{\text{ab}}$ .

Consider a special point on  $\text{Sh}_K(G, X)$ , corresponding to a pair  $(T, h_T)$ , where  $T \subset G$  is a maximal torus and  $h_T : \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow T$  is a cocharacter. Let  $G' = G \times T$ , equipped with the symplectic representation  $V' = V \oplus V$ . Let  $X' = X \times \{h_T\}$ . Then we have  $(G', X') \hookrightarrow (\mathcal{G}_{V'}, \mathcal{H}_{V'})$ . Applying our constructions, we obtain a map of integral models  $\mathcal{S}_{K'}(G', X') \rightarrow \mathcal{S}_K(G, X)$ . As in the proof of Proposition 2.2.2, after possibly conjugating the map  $T \rightarrow G$  by a point of  $G^{\text{ad}}(\mathbb{Q})$ , we may assume that  $s_0$  lifts to  $s'_0 \in \mathcal{S}_{K'}(G', X')(\bar{\mathbb{F}}_p)$ .

By construction,  $\mathcal{A}_{s'_0}$  is isogenous to  $\mathcal{A}_{s_0} \times \mathcal{A}_T$ , where  $\mathcal{A}_T$  is the reduction of a CM abelian variety with  $T$ -action. The action of  $I_{s'_0}$  on  $\mathcal{A}_{s_0} \times \mathcal{A}_T$  preserves this decomposition. This follows, for example, from the fact that the action of  $G(\mathbb{Q}_\ell)$  preserves the corresponding decomposition on  $\ell$ -adic Tate modules for any  $\ell \neq p$ . Restricting the action of  $I_{s'_0}$  to  $\mathcal{A}_T$  induces a map of  $\mathbb{Q}$ -groups  $I_{s'_0} \rightarrow T$ , and we consider the composite  $I_{s'_0} \rightarrow T \rightarrow G^{\text{ab}}$ . By Corollary 2.1.9,  $I_{s'_0} \rightarrow I_{s_0}$  is surjective, so the map  $I_{s'_0} \rightarrow G^{\text{ab}}$  factors through  $I_{s_0}$ , as this is true over  $\mathbb{Q}_\ell$  for every  $\ell \neq p$ .

This gives us the map  $I_{s_0} \rightarrow G^{\text{ab}}$ . One checks easily, using the construction, that it has the required property.  $\square$

### 2.2.9

In the remainder of this subsection, we will apply Proposition 2.2.2 to show a kind of prerequisite for the existence of special points which reduce into a given isogeny class. This asserts that maximal tori in  $I_{s_0}$  transfer to  $G$  when  $G$  is quasisplit at  $p$ . We begin with two lemmas.

#### LEMMA 2.2.10

*Let  $T$  be a torus over  $\mathbb{Q}$  satisfying the Serre condition. If  $l$  is a prime such that  $T_{\mathbb{Q}_l}$  is a split torus, then the set of  $l$ -Weil points in  $T(\mathbb{Q})$  forms a Zariski-dense subgroup of  $T$ . Moreover, the set of neat  $l$ -Weil points contains a Zariski-dense subgroup of  $T$ .*

#### *Proof*

It is clear that the  $l$ -Weil points form a subgroup, and we denote by  $T' \subset T$  its Zariski closure. Then  $T/T'$  is again a torus which is split at  $l$ . Suppose that  $T/T'$  is nontrivial. Then there is a nontrivial  $\chi \in X^*(T/T') \subset X^*(T)$ .

Let  $y_l \in T(\mathbb{Q}_l)$  be a point such that  $\chi(y_l) \in \mathbb{Q}_l^\times$  has positive valuation, and let  $y \in T(\mathbb{A}_f)$  be the point with component  $y_l$  at  $l$  and trivial components away from  $l$ . For any compact open subgroup  $K_T \subset T(\mathbb{A}_f)$  the quotient  $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T$  is finite. Hence there exist  $x \in T(\mathbb{Q})$  and a positive integer  $m$  with  $x = y^m \bmod K_T$ . Then  $x$  is an  $l$ -Weil point by Lemma 2.2.4, and  $\chi(x) \in \mathbb{Q}_l^\times$  has positive valuation, so  $x \notin T'(\mathbb{Q})$ , which is a contradiction. It follows that  $T' = T$ , and the subgroup of  $l$ -Weil points is dense in  $T$ .

For the second claim, let  $k$  be a number field which splits  $T$ , and let  $n$  denote the number of roots of unity in  $k$ . Suppose that  $x \in T(\mathbb{Q})$  is an  $l$ -Weil point, let  $\tilde{S} \subset T$  be the Zariski closure of  $\langle x \rangle$ , and let  $S \subset \tilde{S}$  be the connected component of 1. Then  $n \cdot \tilde{S}/S = \{0\}$ , so  $x^n$  is a neat  $l$ -Weil point. As multiplication by  $n$  induces an isogeny on  $T$ , this implies that the set of neat  $l$ -Weil points contains a Zariski-dense subgroup of  $T$ .  $\square$

#### LEMMA 2.2.11

*Let  $S$  be an irreducible scheme of finite type over a field  $k$ , and let  $\Gamma \subset S(k)$  be a Zariski-dense subset. Let  $W \subset \text{Aut}_k S$  be a finite subgroup, and let  $\sigma \in \text{Aut}_k S$ . Suppose that for every  $\gamma \in \Gamma$ , there exists  $w \in W$  such that  $w(\gamma) = \sigma(\gamma)$ . Then  $\sigma = w$  for some  $w \in W$ .*

*Proof*

We are grateful to the referee for supplying the following proof, which is simpler and more general than our original one. For  $w \in W$ , let  $\Gamma_w = \{\gamma \in \Gamma : w(\gamma) = \sigma(\gamma)\}$ . Then  $\Gamma = \bigcup_{w \in W} \Gamma_w$  and  $\bigcup_{w \in W} \overline{\Gamma_w} = \overline{\Gamma} = S$ , where  $\overline{\Gamma_w}$  and  $\overline{\Gamma}$  denote the closures of  $\Gamma_w$  and  $\Gamma$  in  $S$ , respectively. Since  $S$  is irreducible, this implies that  $\Gamma_{w_0}$  is dense in  $S$  for some  $w_0 \in W$ , and it follows that  $\sigma = w_0$ .  $\square$

## 2.2.12

Suppose that  $C$  and  $H$  are reductive algebraic groups over a field  $F$  of characteristic 0. We denote by  $\text{Aut}'(H)$  the preimage of  $\text{Out}'(H)$  in the group scheme of automorphisms  $\text{Aut } H$ . (Recall  $\text{Out}'(H)$  from 2.1.8.) Consider two maps  $i_1, i_2 : C \rightarrow H$  defined over some extensions  $F_1, F_2$  respectively, of  $F$ . We say that  $i_1$  and  $i_2$  are conjugate (resp., conjugate by an element of  $\text{Aut}'(H)$ ) if there exists an extension  $F_3/F$  containing  $F_1$  and  $F_2$  as well as  $g \in H(F_3)$  (resp.,  $g \in \text{Aut}'(H)(F_3)$ ) such that  $i_2 = gi_1g^{-1}$  (resp.,  $i_2 = g(i_1) := g \circ i_1$ ).

## PROPOSITION 2.2.13

*The maps  $i_\ell : I_{s_0} \rightarrow G$ , defined over  $\mathbb{Q}_\ell$  if  $\ell \neq p$  and over  $\mathbb{Q}_{q^m}$  for  $m$  sufficiently divisible if  $\ell = p$ , are all conjugate by elements of  $\text{Aut}'(G)$ . In particular, if  $G^{\text{ad}}$  has no factors of type  $D$ , then the  $i_\ell$  are all conjugate.*

*Proof*

We consider all maps of groups over an algebraically closed field  $k$  containing all  $\mathbb{Q}_\ell$  for  $\ell \neq p$  and  $\mathbb{Q}_{q^m}$  for all  $m$ . Let us write  $I = I_{s_0}$  for simplicity.

Suppose that  $T_1, T_2 \subset G$  are maximal tori over  $k$ , and suppose that  $\gamma \in T_1(k) \cap T_2(k)$ . Then there exists  $g \in G(k)$  conjugating  $T_1$  into  $T_2$  and fixing  $\gamma$ . Indeed, let  $M$  be the connected component of the identity in the centralizer of  $\gamma$  in  $G$ . Then  $M$  is a Levi subgroup of  $G$ , and  $T_1, T_2 \subset M$  are maximal tori, so conjugate in  $M$ . Now if  $\gamma_1 \in T_1(k)$ ,  $\gamma_2 \in T_2(k)$ , and if  $\sigma(\gamma_1) = \gamma_2$  for some  $\sigma \in \text{Aut}'(G)(k)$ , then there exists  $\sigma' \in \text{Aut}'(G)(k)$  taking  $\gamma_1$  to  $\gamma_2$  and  $T_1$  to  $T_2$ . To see this, apply the previous remark to  $\sigma(\gamma_1) = \gamma_2 \in \sigma(T_1) \cap T_2$ . We will use this observation below.

Choose  $m$  sufficiently divisible such that  $\gamma_{m,s_0}$  is neat. By the Weil conjecture for abelian varieties,  $\gamma_{m,s_0} \in I(\mathbb{Q})$  is a Weil point. Hence, by Proposition 2.2.2 (or Noot's original result), there is a  $\gamma_0 \in G(k)$  such that for each  $\ell$ ,  $i_\ell(\gamma_{m,s_0})$  differs from  $\gamma_0$  by an element of  $\text{Aut}'(G)(k)$ . Let  $I_0 \subset G$  denote the centralizer of  $\gamma_0$ . After modifying  $i_\ell$  by an element of  $\text{Aut}'(G)$ , we obtain maps  $j_\ell : I \rightarrow I_0$  taking  $\gamma_{m,s_0}$  to  $\gamma_0$ . Choose  $T \subset I$  and  $T_0 \subset I_0$  maximal tori. By the observation above, applied with  $I$  in place of  $G$ , after conjugating each  $j_\ell$  by an element of  $I_0(k)$  we may also assume that  $j_\ell$  maps  $T$  to  $T_0$ .

Now fix primes  $\ell, \ell'$ , and set  $\sigma = j_{\ell'} \circ j_{\ell}^{-1}$ . Let  $\gamma \in T(\mathbb{Q})$  be a Weil point. By Proposition 2.2.2, there exists an element  $g \in \text{Aut}'(G)(k)$  which conjugates  $j_{\ell}(\gamma)$  to  $j_{\ell'}(\gamma)$ . By the observation above (applied with  $T_1 = T_2 = T_0$ ), we may assume that  $g$  induces an automorphism of  $T_0$ . Note that the group of automorphisms of  $T_0$  induced by an element of  $\text{Aut}'(G)$  is finite. By Lemma 2.2.10, the set of neat Weil points in  $T(\mathbb{Q})$  is Zariski-dense. It follows by Lemma 2.2.11 that  $\sigma|_T$  is induced by a point  $g \in \text{Aut}'(G)(k)$ .

By construction,  $\sigma$  fixes  $\gamma_0$ , so  $g$  does also, and thus  $g$  induces an automorphism of  $I_0$ . As  $\sigma$  and  $g$  are automorphisms of  $I_0$  which agree on  $T$ , they differ by conjugation by an element of  $t \in T(k)$ . Replacing  $g$  by  $gt$ , we may assume that  $g$  induces  $\sigma$  on  $I_0$ . This implies that  $i_{\ell}$  and  $i_{\ell'}$  are conjugate by an element of  $\text{Aut}'(G)(k)$ .  $\square$

#### COROLLARY 2.2.14

Let  $T \subset I_{s_0}$  be a maximal torus, and suppose that  $G$  is quasisplit at  $p$  and has no factors of type  $D$ . Then there is an embedding of  $\mathbb{Q}$ -groups  $i^T : T \hookrightarrow G$  which is conjugate to each of the embeddings  $i_{\ell}|_T$ . In particular, for each  $m > 0$ , there is an element  $\gamma_{m,0,s_0} \in G(\mathbb{Q})$  conjugate to  $\gamma_{m,\ell,s_0}$  in  $G(\bar{\mathbb{Q}}_{\ell})$  for each  $\ell$ .

#### Proof

Let  $G^*$  be the quasisplit inner form of  $G$ , and choose an inner twisting  $G \xrightarrow{\sim} G^*$  over  $\mathbb{Q}$ . Let  $i_{\ell}^{T*} : T \hookrightarrow G^*$  be the embedding over  $\bar{\mathbb{Q}}_{\ell}$  induced by  $i_{\ell}|_T$  and the chosen inner twisting. By Proposition 2.2.13, there exists an embedding  $\bar{i}^T : T \hookrightarrow G^*$  defined over  $\bar{\mathbb{Q}}$  and conjugate to each of the  $i_{\ell}^{T*}$ . For  $\ell \neq p$ ,  $i_{\ell}$  is defined over  $\mathbb{Q}_{\ell}$  so the conjugacy class of  $i_{\ell}^{T*}$  is invariant by  $\text{Gal}(\bar{\mathbb{Q}}_{\ell}/\mathbb{Q}_{\ell})$ . Hence, by Chebotarev's density theorem, the stabilizer of the conjugacy class of  $\bar{i}^T$  in  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  is an open subgroup which meets every conjugacy class in  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . This implies that the conjugacy class of  $\bar{i}^T$  is invariant by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . It follows by [19, Corollary 2.2] that  $\bar{i}^T$  is conjugate to an embedding  $i^{T*} : T \hookrightarrow G^*$  defined over  $\mathbb{Q}$ . We view  $T$  as a subgroup of  $G^*$  via  $i^{T*}$ .

Now  $T$  transfers to  $G$  at every prime  $\ell \neq p, \infty$  as  $i_{\ell}$  is defined over  $\mathbb{Q}_{\ell}$ . It transfers to  $G$  at  $p$ , since  $G$  is quasisplit at  $p$ , and it transfers to  $G$  at infinity as the image of  $T$  in  $G^{\text{ad}}$  is anisotropic at infinity. Hence  $T$  transfers to  $G$  by [28, Lemma 5.6].

For the final statement, writing  $i^T : T \hookrightarrow G$  for the transfer, we take  $\gamma_{m,0,s_0} = i^T(\gamma_{m,s_0})$ .  $\square$

### 2.3. CM lifts and the conjugacy class of Frobenius

#### 2.3.1

We again return to the notation and assumptions of Section 2.1. Let  $s_0, s'_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$ . Then  $s_0, s'_0$  are defined over  $\mathbb{F}_q$  for some  $q$ , and we use the notation of Section 2.1.7.



Write  $\underline{\mathrm{Hom}}_{\mathbb{Q}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  for the scheme over  $\mathbb{Q}$  that assigns to any  $\mathbb{Q}$ -algebra  $R$  the group  $R \otimes \mathrm{Hom}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$ . (Here the  $\mathrm{Hom}$ -spaces are taken in the prime-to- $p$  isogeny categories.) For any  $\mathbb{Q}$ -algebra  $R$ , an  $R$ -isogeny from  $\mathcal{A}_{s_0}$  to  $\mathcal{A}_{s'_0}$  is an element

$$f \in \underline{\mathrm{Hom}}_{\mathbb{Q}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})(R)$$

such that there exists

$$f' \in \underline{\mathrm{Hom}}_{\mathbb{Q}}(\mathcal{A}_{s'_0}, \mathcal{A}_{s_0})(R)$$

with  $f' \circ f \in \underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})(R)$ .

Let  $\underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  be the functor on  $\mathbb{Q}$ -algebras that assigns to any  $\mathbb{Q}$ -algebra  $R$  the set of  $R$ -isogenies from  $\mathcal{A}_{s_0}$  to  $\mathcal{A}_{s'_0}$ . Note that this functor is either empty or representable by a torsor over  $\mathbb{Q}$  under  $\underline{\mathrm{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$ .

### 2.3.2

For any prime  $\ell \neq p$ , denote by  $V_{\ell}(\mathcal{A}_{s_0})$  be the  $\ell$ -adic Tate module of  $\mathcal{A}_{s_0}$ , and let  $\underline{\mathrm{Isog}}_{\ell}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  be the  $\mathbb{Q}_{\ell}$ -scheme that assigns to any  $\mathbb{Q}_{\ell}$ -algebra  $R$  the set of  $R$ -linear isomorphisms

$$R \otimes_{\mathbb{Q}_{\ell}} V_{\ell}(\mathcal{A}_{s_0}) \xrightarrow{\sim} R \otimes_{\mathbb{Q}_{\ell}} V_{\ell}(\mathcal{A}_{s'_0})$$

that carry  $1 \otimes \gamma_{m,\ell,s_0}$  to  $1 \otimes \gamma_{m,\ell,s'_0}$  for all  $m$  sufficiently divisible.

For any  $\ell \neq p$ , cohomological realization gives us a natural map of  $\mathbb{Q}_{\ell}$ -schemes:

$$i_{\ell}(s_0, s'_0) : \mathbb{Q}_{\ell} \otimes \underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0}) \rightarrow \underline{\mathrm{Isog}}_{\ell}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0}).$$

Similarly, let  $\underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0})$  be the  $\mathbb{Q}_p$ -scheme that assigns to every  $\mathbb{Q}_p$ -algebra  $R$  the set of  $1 \otimes \varphi$ -equivariant,  $R \otimes L$ -linear isomorphisms  $R \otimes_{\mathbb{Q}_p} D_{s'_0} \xrightarrow{\sim} R \otimes_{\mathbb{Q}_p} D_{s_0}$  that carry  $D_{m,s'_0}$  to  $D_{m,s_0}$  for  $m$  sufficiently large. We have a natural map of  $\mathbb{Q}_p$ -schemes:

$$i_p(s_0, s'_0) : \mathbb{Q}_p \otimes \underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0}) \rightarrow \underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0}).$$

By Tate's theorem on endomorphisms of abelian varieties and its crystalline analogue,  $i_{\ell}(s_0, s'_0)$  is an isomorphism for all  $\ell$ .

### 2.3.3

For  $\ell \neq p$ , let  $P_{\ell}(s_0, s'_0) \subset \underline{\mathrm{Isog}}_{\ell}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  (resp.,  $P_p(s_0, s'_0) \subset \underline{\mathrm{Isog}}(D_{s_0}, D_{s'_0})$ ) be the closed subscheme parameterizing isomorphisms that carry, for each  $\alpha$ ,  $1 \otimes s_{\alpha,\ell,s_0}$  to  $1 \otimes s_{\alpha,\ell,s'_0}$  (resp.,  $1 \otimes s_{\alpha,\mathrm{cris},s'_0}$  to  $1 \otimes s_{\alpha,\mathrm{cris},s_0}$ ). Let  $P(s_0, s'_0) \subset \underline{\mathrm{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  be

the largest closed subscheme (defined over  $\mathbb{Q}$ ) that maps into  $P_\ell(s_0, s'_0)$  for every  $\ell$ , including  $\ell = p$ . Note that  $P(s_0, s'_0)$  is either empty or an  $I_{s_0}$ -torsor.

We make the following conjecture.

#### CONJECTURE 2.3.4

*For every  $\ell$ , the map*

$$P(s_0, s'_0) \otimes \mathbb{Q}_\ell \rightarrow P_\ell(s_0, s'_0)$$

*induced by  $i_\ell$  is an isomorphism.*

When  $s'_0 = s_0$  this is simply Corollary 2.1.9.

#### LEMMA 2.3.5

*The schemes  $P(s_0, s'_0)$  and  $P_\ell(s_0, s'_0)$  depend only on  $s_0$  and  $s'_0$  and not on the choice of the collection of Hodge cycles  $\{s_\alpha\}$ . In particular, the truth of Conjecture 2.3.4 depends only on  $s_0, s'_0$  and not on  $\{s_\alpha\}$ .*

*If  $(G, X)$  is of PEL type A or C, then Conjecture 2.3.4 holds.*

#### *Proof*

From the definitions, it suffices to prove the first statement for  $P_\ell(s_0, s'_0)$  for each  $\ell$ . If  $\{t_\beta\}$  is another collection of Hodge cycles defining  $G$ , then it suffices to consider the case  $\{s_\alpha\} \subset \{t_\beta\}$ . If  $P_{\ell,1}(s_0, s'_0)$  is the analogue of  $P_\ell(s_0, s'_0)$  defined using  $\{t_\beta\}$ , then  $P_{\ell,1}(s_0, s'_0) \subset P_\ell(s_0, s'_0)$  and it suffices to show that if one scheme is nonempty, then so is the other, as then each is an  $I_\ell$ -torsor. However, each scheme is nonempty if and only if  $\gamma_{m,s_0,\ell}$  and  $\gamma_{m,s'_0,\ell}$  are conjugate in  $G(\bar{\mathbb{Q}}_\ell)$  (even for  $\ell = p$ ).

Now suppose that  $(G, X)$  is of PEL type A or C. In this case,  $G$  is the group preserving a collection of endomorphisms  $\{t_\beta\}$  together with the polarization  $\psi$  up to a scalar. (Note that  $\psi$  does not have weight 0, and so does not quite fit into our formalism involving  $\{s_\alpha\}$ .) Then  $\psi$  induces a pairing

$$V_\ell(\mathcal{A}_{s_0}) \times V_\ell(\mathcal{A}_{s'_0}) \rightarrow \mathbb{Q}_\ell(1),$$

well defined up to a nonzero scalar, and similarly for  $D_{m,s_0}$ . We refer to these pairings as *polarizations*.

Define  $P_1(s_0, s'_0)$  to be the subscheme of  $\text{Isog}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  which preserves the  $\{t_\beta\}$  and polarizations up to a scalar. For  $\ell \neq p$ , let  $P_{\ell,1}(s_0, s'_0) \subset \text{Isog}_\ell(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  (resp.,  $P_{p,1}(s_0, s'_0) \subset \text{Isog}(D_{s_0}, D_{s'_0})$ ) be the closed subscheme parameterizing isomorphisms that carry, for each  $\beta$ ,  $1 \otimes t_{\beta,\ell,s_0}$  to  $1 \otimes t_{\beta,\ell,s'_0}$  (resp.,  $1 \otimes t_{\beta,\text{cris},s_0}$  to  $1 \otimes t_{\beta,\text{cris},s'_0}$ ) and which preserve polarizations up to a scalar.

By Tate's theorem, for each  $\ell$  the map

$$P_1(s_0, s'_0) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} P_{\ell,1}(s_0, s'_0)$$

is an isomorphism. An argument as in the proof of the first part of the lemma shows that this map can be identified with the map of Conjecture 2.3.4.  $\square$

### 2.3.6

In the PEL case, when  $G$  is unramified at  $p$ , the above result is due to Kottwitz (see [22, Lemmas 17.1, 17.2] and their proofs).

The restriction that  $(G, X)$  be of type  $A$  or  $C$  in the lemma above is in some sense a question of definitions. When  $(G, X)$  is of PEL type  $D$ , one cannot actually define  $G \subset \mathrm{GSp}(V)$  using endomorphisms and polarizations. Instead, there is a collection  $\{t_\beta\} \subset V^\otimes$  of a polarization and endomorphisms which define a group  $G' \subset \mathrm{GSp}(V)$  whose connected component is  $G$  (see [22, p. 393]). An analogue of the last statement of the lemma then holds for  $G'$ .

We will say that  $s_0$  and  $s'_0$  are  $\bar{\mathbb{Q}}$ -isogenous if the space  $P(s_0, s'_0)$  of 2.3.3 is nonempty. We will say that they are isogenous if  $P(s_0, s'_0)(\mathbb{Q})$  is nonempty. If  $s_0, s'_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$ , then we will say that  $s'_0$  and  $s_0$  are  $\bar{\mathbb{Q}}$ -isogenous (resp., isogenous) if this condition holds when  $s_0, s'_0$  are viewed as  $\mathbb{F}_q$  points for some  $q = p^r$ .

### 2.3.7

Let  $s_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$ . Suppose that  $T \subset I_{s_0}$  is a maximal torus. Let  $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow G_{\mathbb{R}}$  be an  $\mathbb{R}$ -morphism. Let  $\mathrm{Sh}_{K_{T,p}}(h)$  be the pro-Shimura variety associated with  $(T, \{h\})$  and  $K_{T,p} = K_p \cap T(\mathbb{Q}_p)$ . An isogeny CM lift (resp., a  $\bar{\mathbb{Q}}$ -isogeny CM lift) of  $s_0$  with respect to  $T$  will consist of a triple  $(j, x, s'_0)$ , where

- $j : T \hookrightarrow G$  is an embedding defined over  $\mathbb{Q}$  such that for each  $\ell$ ,  $j$  is conjugate over  $\bar{\mathbb{Q}}_\ell$  to the embedding

$$i_\ell : T_{\mathbb{Q}_\ell} \hookrightarrow I_{\ell, s_0} \hookrightarrow G_{\mathbb{Q}_\ell},$$

- $x \in X$  is a point with  $h_x$  factoring through  $j(T_{\mathbb{R}})$ , and
- $s'_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  is a point admitting a lift to  $\mathrm{Sh}_{K_{T,p}}(h_x)$

such that  $s'_0$  is isogenous (resp.,  $\bar{\mathbb{Q}}$ -isogenous) to  $s_0$ .

Of course isogeny CM lifts can exist only when the  $i_\ell$  are conjugate for all  $\ell$ . We make the following conjecture.

### CONJECTURE 2.3.8

If  $G$  is quasisplit at  $p$ , then for any  $s_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  and any maximal torus  $T \subset I_{s_0}$ ,  $s_0$  admits an isogeny CM lift with respect to  $T$ .

When  $K_p$  is hyperspecial this conjecture is proved in [16]. The main point of this section is to show that Conjecture 2.3.4 implies a version of Conjecture 2.3.8 with

$\bar{\mathbb{Q}}$ -isogenies, when  $G^{\text{ad}}$  has no factors of type  $D$ . In particular, we will show that a  $\bar{\mathbb{Q}}$ -version of this conjecture holds for  $(G, X)$  of PEL type  $A$  or  $C$ .

### 2.3.9

Let  $T \subset G$  be a maximal torus, and let  $x \in X$  with  $h_x$  factoring through  $T$ . Let  $s_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  be defined over  $\mathbb{F}_q$  for some  $q = p^r$ . Suppose that  $s_0$  is a reduction of a  $\bar{\mathbb{Q}}_p$ -valued point  $s$  of  $\text{Sh}_{K_{T,p}}(h_x)$ .

For any  $m \in \mathbb{Z}_{>0}$ , the  $q^m$ -Frobenius acts on  $\mathcal{A}_{s_0}$ , and the corresponding automorphism  $\gamma_{m,s_0} \in \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0})$  lies in  $I_{s_0}(\mathbb{Q})$ . Since  $T$  contains the Mumford–Tate group of  $\mathcal{A}_s$  (defined via some embedding  $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ ), there are natural embeddings:

$$T \hookrightarrow \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_s) \hookrightarrow \underline{\text{Aut}}_{\mathbb{Q}}(\mathcal{A}_{s_0}).$$

It follows from the definitions that this embedding exhibits  $T$  as a subtorus of  $I_{s_0}$ .

Recall from Section 2.2.3 that an element  $\gamma \in T(\mathbb{Q})$  is called a  $p$ -unit if the subgroup it generates is contained in a compact subset of  $T(\mathbb{Q}_\ell)$  for all  $\ell \neq p$ .

### LEMMA 2.3.10

*The element  $\gamma_{m,s_0}$  lies in  $T(\mathbb{Q}) \subset I_{s_0}(\mathbb{Q})$ . It has the following properties:*

- (1)  $\gamma_{m,s_0}$  is a  $p$ -unit.
- (2) Set  $\mu = \mu_x^{-1} \in X_*(T)$ . Under the composition

$$T(\mathbb{Q}) \rightarrow T(\mathbb{Q}_p) \rightarrow B(T) \xrightarrow[\simeq]{\kappa_T} X_*(T)_{\Gamma_p},$$

$\gamma_{m,s_0}$  is mapped to  $m \log_p q \cdot \mu^\sharp$ .

Given any other element  $\gamma \in T(\mathbb{Q})$  satisfying the two conditions above, there exists  $r \in \mathbb{Z}_{>0}$  such that  $\gamma_{m,s_0}^r = \gamma^r$ .

### Proof

It was already remarked in the proof of Proposition 2.2.13 that  $\gamma_{m,s_0} \in I_{s_0}(\mathbb{Q})$  is a Weil point, and hence a  $p$ -unit by Lemma 2.2.4.

Let us show (2). First, we note that for  $m$  sufficiently large, the embedding

$$T_{\mathbb{Q}_p} \hookrightarrow \mathbb{Q}_p \otimes I_{s_0} \hookrightarrow \underline{\text{Aut}}(D_{m,s_0})$$

arises from an isomorphism  $\mathbb{Q}_{q^m} \otimes V \xrightarrow{\simeq} D_{m,s_0}$ . We can choose this isomorphism so that the semilinear map  $\varphi : D_{m,s_0} \rightarrow D_{m,s_0}$  is identified with the automorphism  $\delta_{s_0}(\sigma \otimes 1)$  of  $\mathbb{Q}_{q^m} \otimes V$  for some element  $\delta_{s_0} \in T(\mathbb{Q}_{q^m})$ . By Lemma 1.3.9, the image of  $\delta_{s_0}$  in  $X_*(T)_{\Gamma_p}$  is  $\mu^\sharp$ . The assertion now follows from (2.1.7.2).

For the final assertion, note that since  $(T/\mathbb{G}_m)_{\mathbb{R}}$  is compact,  $T(\mathbb{Q})$  is a discrete subgroup of  $T(\mathbb{A}_f)$ . Given  $\gamma$  satisfying (1) and (2), set  $\beta = \gamma^{-1}\gamma_{m,s_0}$ . We have to show that  $\beta^r = 1$  for some  $r \in \mathbb{Z}_{>0}$ .

For  $\ell \neq p$ , the eigenvalues of  $\beta$  acting on  $V_\ell(\mathcal{A}_{s_0})$  all belong to  $\mathbb{Z}_\ell^\times$ ; therefore,  $\beta$  lies in a compact subgroup of  $T(\mathbb{A}_f^p)$ . Moreover,  $\beta$  is in the kernel of  $T(\mathbb{Q}_p) \rightarrow B(T)$ , and so it lies in the compact subgroup in  $T(\mathbb{Q}_p)$  consisting of elements  $\sigma$ -conjugate to 1 over  $L$ . In sum, we find that  $\beta$  lies in both the discrete subgroup  $T(\mathbb{Q})$  and a compact subgroup of  $T(\mathbb{A}_f)$ , and must therefore be of finite order.  $\square$

### PROPOSITION 2.3.11

*Suppose that  $G$  is quasisplit at  $p$ , that  $G^{\text{ad}}$  has no factors of type  $D$ , and that Conjecture 2.3.4 holds for  $(G, X)$ . Then for any maximal torus  $T \subset I_{s_0}$ ,  $s_0$  admits a  $\mathbb{Q}$ -isogeny CM lift with respect to  $T$ .*

#### *Proof*

We can view  $T_{\mathbb{Q}_p}$  as a maximal torus in  $I_{p,s_0}$ . By Corollary 1.1.17, there exists a cocharacter  $\mu_T \in X_*(T)$  defined over  $\mathbb{Q}$  whose image in  $G_{\bar{\mathbb{Q}}}$  lies in the conjugacy class  $\{\mu_X\}_p$ , and such that  $v_{\delta_{s_0}} = N\mu_T \in X_*(T)_{\mathbb{Q}}^{\Gamma_p}$ .

By Corollary 2.2.14, there is an embedding  $i : T \hookrightarrow G$  such that for all  $\ell$ ,  $i$  is  $G(\bar{\mathbb{Q}}_\ell)$ -conjugate to the embeddings

$$i_\ell : T_{\mathbb{Q}_\ell} \hookrightarrow I_{\ell,s_0} \hookrightarrow G_{\mathbb{Q}_\ell}.$$

The cocharacter

$$\mu_{T,\infty} : \mathbb{G}_{m,\mathbb{C}} \rightarrow T_{\mathbb{C}}$$

obtained from  $\mu_T$  via the embedding  $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$  is  $G(\mathbb{C})$ -conjugate to  $\mu_y$ , for  $y \in X$ . By modifying  $i$  within its  $G(\bar{\mathbb{Q}})$ -conjugacy class, as in Proposition 1.2.5, we can assume that  $\mu_{T,\infty}$  is  $G(\mathbb{R})$ -conjugate to  $\mu_y$ , and so arises from a homomorphism  $h_x : \mathbb{S} \rightarrow T_{\mathbb{R}}$ , for  $x \in X$ .

Let  $s'_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  be the reduction of a point of  $\text{Sh}_{K_{T,p}}(h_x)$ . Recall from the preceding lemma that the  $q^m$ -Frobenius  $\gamma_{m,s_0} \in I_{s_0}(\mathbb{Q})$  is contained in  $T(\mathbb{Q})$ . We claim that for  $m$  sufficiently divisible,

$$\gamma_{m,s_0} = \gamma_{m,s'_0} \in T(\mathbb{Q}).$$

Here we view  $s_0, s'_0 \in \mathcal{S}_{K_p}(\mathbb{F}_{q^m})$ . Assuming this, we see that since  $i$  and  $i_\ell$  are conjugate for any  $\ell$ ,  $\gamma_{m,\ell,s_0}$  and  $\gamma_{m,\ell,s'_0}$  are conjugate in  $G(\bar{\mathbb{Q}}_\ell)$ . This implies that  $P_\ell(s_0, s'_0)$  is nonempty, and hence  $P(s_0, s'_0)$  is nonempty by Conjecture 2.3.4, which implies that  $s'_0$  is a  $\mathbb{Q}$ -isogeny CM lift of  $s_0$  with respect to  $T$ .

To see the claim, note that the eigenvalues of  $\gamma_{m,s_0}$  acting on  $V_\ell(\mathcal{A}_{s_0})$  for  $\ell \neq p$  are  $q^m$ -Weil numbers. So  $\gamma_{m,s_0} \in T(\mathbb{Q})$  is a  $p$ -unit as in Lemma 2.3.10. We have

$$\gamma_{m,p,s_0} = \delta_{s_0} \sigma(\delta_{s_0}) \cdots \sigma^{rm-2}(\delta_{s_0}) \sigma^{rm-1}(\delta_{s_0})$$

so using 1.1.2(4) we see that the image of  $\gamma_{m,p,s_0}$  under the composite

$$T(\mathbb{Q}_p) \xrightarrow{\kappa} X_*(T)_{\Gamma_p} \rightarrow X_*(T)_{\Gamma_p} \otimes \mathbb{Q} \xrightarrow{\sim} X_*(T)_{\mathbb{Q}}^{\Gamma_p}.$$

is equal to the image of  $rmv_{\delta_{s_0}}(p) = rmN\mu_T(p)$ , which is just the image of  $rm\mu_T = m \log_p q \cdot \mu_T$  in  $X_*(T)_{\Gamma_p} \otimes \mathbb{Q}$  by 1.1.2(4). Hence for  $m$  divisible enough, the image of  $\gamma_{m,s_0}$  in  $X_*(T)_{\Gamma_p}$  is  $m \log_p q \cdot \mu^\#$ . It follows by Lemma 2.3.10 that  $\gamma_{m,s_0} = \gamma_{m,s'_0}$  for  $m$  sufficiently divisible.  $\square$

### 2.3.12

We will show that in some cases, the result of Proposition 2.3.11 can be improved to produce  $\mathbb{Q}$ -isogeny lifts of  $s_0$ . To do that, we need the following.

#### LEMMA 2.3.13

*Suppose that  $s_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$ , that  $T \subset I_{s_0}$  is a maximal torus, and that  $s_0$  admits a  $\bar{\mathbb{Q}}$ -isogeny CM lift  $(j, x, s'_0)$  with respect to  $T$ . Let  $P^T = P^T(s_0, s'_0)$  be the subscheme of  $P(s_0, s'_0)$  consisting of isomorphisms which respect the action of  $T$ . Then  $P^T$  is a  $T$ -torsor whose class in  $H^1(\mathbb{Q}_v, G)$  is trivial for every place  $v$  of  $G$ .*

#### *Proof*

By construction,  $\mathcal{A}_{s_0}$  and  $\mathcal{A}_{s'_0}$  are equipped with an action of  $T$ , so the subscheme  $P^T$  is well defined. For each  $\ell$ , we denote by  $P_\ell^T(s_0, s'_0)$  the subscheme of  $P_\ell(s_0, s'_0)$  consisting of isomorphisms which respect the action of  $T$ . Since  $j$  is conjugate to  $i_\ell$  by an element of  $G(\bar{\mathbb{Q}}_\ell)$ ,  $P_\ell^T(s_0, s'_0)$  is nonempty. Hence, by Tate's theorem,  $P^T := P^T(s_0, s'_0)$  is nonempty, and thus is a  $T$ -torsor, which is a reduction of the  $I_{s_0}$ -torsor  $P(s_0, s'_0)$ .

Let  $I'_{s_0}$  denote the group of automorphisms of  $\mathcal{A}_{s_0}$  respecting polarizations up to a  $\mathbb{Q}^\times$ -scalar. Consider the subscheme  $P \subset \underline{\text{Isog}}(\mathcal{A}_{s_0}, \mathcal{A}_{s'_0})$  parameterizing isogenies respecting polarizations up to a  $\mathbb{Q}^\times$ -scalar. Then  $P$  is an  $I'_{s_0}$ -torsor. By [22, Lemma 17.1], the class of  $P$  in  $H^1(\mathbb{R}, I'_{s_0})$  is trivial, so the class of  $P^T$  in  $H^1(\mathbb{R}, T)$  is trivial by [17, Lemma 4.4.5]. In particular, the class of  $P^T$  in  $H^1(\mathbb{R}, G)$  is trivial.

Next, for  $\ell \neq p$  a finite prime, consider  $\underline{\text{Isom}}_{\{s_\alpha\}}(V_\ell(\mathcal{A}_{s_0}), V_\ell(\mathcal{A}_{s'_0}))$ , the scheme of isomorphisms which take  $s_{\alpha,\ell,s_0}$  to  $s_{\alpha,\ell,s'_0}$ . (Note that we do not require that the isomorphisms respect Frobenius.) This scheme is a  $G$ -torsor over  $\mathbb{Q}_\ell$ , obtained from  $P^T$  via the natural map  $T \rightarrow G$  over  $\mathbb{Q}_\ell$ . If  $\tilde{s}_0, \tilde{s}'_0 \in \mathcal{S}_{K_p}$  are lifts of  $\tilde{s}_0, \tilde{s}'_0$ , then this  $G$ -torsor may also be identified with  $\underline{\text{Isom}}_{\{s_\alpha\}}(V_\ell(\mathcal{A}_{\tilde{s}_0}), V_\ell(\mathcal{A}_{\tilde{s}'_0}))$ . However, from the definition of the universal abelian scheme over  $\mathcal{S}_{K_p}$ , one sees that this last torsor is trivial.

It remains to check that the image of  $P^T$  in  $H^1(\mathbb{Q}_p, G)$  is trivial. Fix  $q$  such that  $s_0, s'_0$  are defined over  $\mathbb{F}_q$ . As above, by Steinberg's theorem, for  $m$  sufficiently large,

we may fix isomorphisms

$$D_{m,s_0} \simeq \mathbb{Q}_{q^m} \otimes V \simeq D_{m,s'_0}$$

which take  $s_\alpha$  to  $s_{\alpha,\text{cris},s_0}$  and  $s_{\alpha,\text{cris},s'_0}$ , respectively, and respect the action of  $T$ . Then  $\varphi$  on  $D_{m,s_0}$  and  $D_{m,s'_0}$  are given by  $\delta_{s_0}(\sigma \otimes 1)$ ,  $\delta_{s'_0}(\sigma \otimes 1)$ , respectively, for  $\delta_{s_0}, \delta_{s'_0} \in T(\mathbb{Q}_{q^m})$ .

Recall that for any reductive group  $H$  over  $\mathbb{Q}_p$ , we have isomorphisms (see [20])

$$H^1(\mathbb{Q}_p, H) \simeq H^1(\text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p), H) \simeq (\pi_1(H)_{\Gamma_p})_{\text{tors}}.$$

Here the first isomorphism is given by Steinberg's theorem, and the second isomorphism takes a cocycle  $c$  to  $\kappa_H(c_\sigma)$ ,  $\sigma$  the Frobenius.

The class of  $P^T$  in  $H^1(\mathbb{Q}_p, T)$  corresponds to the cocycle sending  $\sigma$  to  $\delta_{s'_0} \delta_{s_0}^{-1}$ . By Lemma 1.3.9,  $\delta_{s_0}$  and  $\delta_{s'_0}$  have the same image in  $\pi_1(G)_{\Gamma_p}$ , so that the class of this torsor in  $H^1(\mathbb{Q}_p, G)$  is trivial, as required.  $\square$

#### COROLLARY 2.3.14

*With the assumptions of Proposition 2.3.11, suppose that  $G^{\text{der}}$  is simply connected and that  $G^{\text{ab}}$  satisfies the Hasse principle*

$$\ker^1(\mathbb{Q}, G^{\text{ab}}) := \ker\left(H^1(\mathbb{Q}, G^{\text{ab}}) \rightarrow \prod_v H^1(\mathbb{Q}_v, G^{\text{ab}})\right) = 0.$$

*Then for any maximal torus  $T \subset I_{s_0}$ ,  $s_0$  admits a  $\mathbb{Q}$ -isogeny CM lift with respect to  $T$ .*

*Proof*

By Corollary 2.3.11,  $s_0$  admits a  $\bar{\mathbb{Q}}$ -isogeny CM lift with respect to  $T$ , say,  $(j, x, s'_0)$ . Let  $P^T$  be as in Lemma 2.3.13. For every place  $v$  of  $\mathbb{Q}$ , the class of  $P^T$  is trivial in  $H^1(\mathbb{Q}_v, G)$  and hence in  $H^1(\mathbb{Q}_v, G^{\text{ab}})$ . Since  $G^{\text{ab}}$  satisfies the Hasse principle, the class of  $P^T$  in  $H^1(\mathbb{Q}, G^{\text{ab}})$  is trivial.

As  $P^T$  has trivial image in  $H^1(\mathbb{R}, G)$  and  $H^1(\mathbb{Q}, G^{\text{ab}})$ , and  $G^{\text{der}}$  is simply connected,  $P^T$  has trivial image in  $H^1(\mathbb{Q}, G)$  by [3, Theorem 5.12], so  $P^T$  arises from a point  $\omega \in (G/T)(\mathbb{Q})$ . Now let  $j' = \omega^{-1}j\omega$ . Then  $j' : T \rightarrow G$  is defined over  $\mathbb{Q}$ . Since the image of  $\omega$  in  $H^1(\mathbb{R}, T)$  is trivial,  $\omega^{-1}h_x\omega$  corresponds to a point  $x' \in X$  and factors through  $j'(T_{\mathbb{R}})$  (cf. [17, Section 4.2.2]). If  $s''_0 \in \mathcal{S}_{K_p}(\bar{\mathbb{F}}_p)$  is a point admitting a lift to  $\text{Sh}_{K_{T,p}}(h_x)$ , then  $P(s_0, s''_0)$  is a trivial  $I_{s_0}$ -torsor by [17, Proposition 4.2.6], so  $(j', x', s''_0)$  is an isogeny CM lift with respect to  $T$ .  $\square$

#### COROLLARY 2.3.15

*Suppose that  $G$  is quasisplit at  $p$ , and that  $(G, X)$  is of PEL type A or C. Then for any*

maximal torus  $T \subset I_{s_0}$ ,  $s_0$  admits a  $\bar{\mathbb{Q}}$ -isogeny CM lift with respect to  $T$ . Moreover,  $s_0$  admits a  $\mathbb{Q}$ -isogeny CM lift with respect to  $T$  unless  $G$  is of type  $A_n$  with  $n$  even.

*Proof*

The first statement follows from Proposition 2.3.11 and Lemma 2.3.5. For the second statement, note that if  $(G, X)$  is of PEL type  $A$  or  $C$ , then  $G^{\text{der}}$  is simply connected, and  $G^{\text{ab}}$  satisfies the Hasse principle unless  $G$  is of type  $A_n$  with  $n$  even (see [22, Section 7]). Hence the second statement follows from Corollary 2.3.14.  $\square$

*Remarks 2.3.16*

(1) In fact, the corollary can be shown for certain groups  $G$  of type  $A_n$  with  $n$  even. Namely, if it is a unitary similitude group (in  $n + 1$  variables) arising from a CM quadratic extension  $F$  of a totally real field  $F^+$  with  $[F^+ : \mathbb{Q}]$  odd, then the Hasse principle holds for  $G$  by the proof of Lemma 3.1.1 of [38], so the above proof goes through.

(2) As in Section 2.3.6, one can extend the proof of the first statement of the last corollary to the case of type  $D$  if one works with the disconnected group  $G'$ . For an algebraically closed field  $k$ , two points of  $G(k)$  give rise to the same point of  $\text{Conj}'(G)$  if and only if they are conjugate in  $G'(k)$ . Using this, one can deduce a version of Corollary 2.2.14 from Proposition 2.2.13 and use it to deduce an analogue of the first part of Corollary 2.3.15, but where  $\bar{\mathbb{Q}}$ -isogeny is defined using the tensors  $\{t_\beta\}$ . We leave this as an exercise for the reader.

(3) In [47], Zink proves that for PEL Shimura varieties, and primes of good reduction, every point has an isogeny CM lift with respect to  $T$ . However, his definition of isogeny is required to respect only endomorphisms and not polarizations. In that case, the analogue of  $P(s_0, s'_0)$  is a torsor under the group of units in a product of (possibly skew) fields. Any such torsor is trivial, for example, because a  $\mathbb{Q}$ -vector space has a Zariski-dense set of rational points, or alternatively because in this case the group is a product of inner forms of  $\text{GL}_n$ .

Thus, the first part of Corollary 2.3.15 recovers Zink's result in this case. However, the second part is really stronger. Even for the moduli space of principally polarized abelian varieties the deduction of this statement using Honda–Tate theory does not quite seem to be in the literature. Although it is a special case of a result of [17], the techniques used there are quite different.

(4) The condition on  $G^{\text{ab}}$  in Corollary 2.3.14 and the second part of Corollary 2.3.15 is used to show that the class of  $P^T$  in  $H^1(\mathbb{Q}, G^{\text{ab}})$  is trivial. In fact, this should follow from the fact that  $s_0, s'_0$  lie on the same Shimura variety, since the motive obtained from  $\mathcal{A}_{s_0}$  and any representation of  $G$  which factors through  $G^{\text{ab}}$  should be constant; for example, this holds in characteristic 0 at the level of variations



of Hodge structure. Even when  $G^{\text{der}}$  is not simply connected, there is a corresponding cohomology group  $H^1(\mathbb{Q}, G/\tilde{G})$  in which the image of  $P^T$  should be trivial (here  $\tilde{G}$  is the simply connected cover of  $G^{\text{der}}$ ), which would be enough for the argument of Corollary 2.3.14. Unfortunately, we do not know how to make these motivic heuristics rigorous.

(5) We have not thought seriously about which of these results can be generalized to the case of abelian-type Shimura varieties. Integral models for these are usually defined using those for an auxiliary Shimura variety of Hodge type. Thus, it is quite plausible that one can directly deduce analogues of our results on nonemptiness of Newton strata and special point liftings. Of course, in this case the *construction* of the Newton strata would usually also involve the auxiliary Shimura variety. A more interesting problem is the definition and nonemptiness of the torsors  $P(s_0, s'_0)$ , given the lack of a good general definition of an isogeny of motives (see the recent paper of Yang [45] for the case of K3 surfaces).

### Appendix. Construction of isocrystals with $G$ -structure

The purpose of this appendix is to prove Proposition 1.3.12. The main tool is Faltings's comparison theorem (see [10, p. 62]), as well as de Jong's theorem on alterations in [6] and a result of Ogus on proper descent for convergent isocrystals in [33].

#### A.1

Let  $k$  be a perfect field of characteristic  $p$ , and let  $W = W(k)$ . We equip  $k$  and  $W$  with the trivial log structure.

Let  $X$  be a scheme over  $W$  equipped with a fine saturated log structure. A  $p$ -adic formal log scheme  $T$  over  $W$  is a  $p$ -adic formal scheme  $T/W$  together with the data of a compatible system of log structures on  $T_n = T \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$  for  $n \geq 1$  such that the inclusions  $T_n \hookrightarrow T_{n+1}$  are exact.

An *enlargement* of  $X$  is a triple  $(T, I, i_T)$  consisting of a  $p$ -adic formal log scheme  $T$  over  $W$ , an ideal of definition  $I$  of  $T$ , and a map of log schemes  $i_T : T_0 \rightarrow X$ , where  $T_0$  is the subscheme of  $T$  defined by  $I$ . We say that  $(T, I, i_T)$  is reduced if  $T_0$  is reduced. We say that  $(T, I, i_T)$  is a PD-enlargement if  $I$  is equipped with divided powers extending the divided powers on  $pW$ .

As in [33, Section 2.7], we can use the definition of an enlargement to define the category of convergent log isocrystals (cf. [34, Section 3]). This category does not change if we allow  $I$  to be any  $p$ -adically closed ideal as in [10, p. 258]. Indeed, the value of a convergent log isocrystal on such an enlargement can be defined as the inverse limit of its values on  $(T, (I, p^n), i_{T,n})$  for  $n \geq 1$ , where  $i_{T,n}$  is the composite  $T_n \hookrightarrow T_0 \xrightarrow{i_T} X$  and  $T_n$  is defined by  $(I, p^n)$ .

The category of convergent log isocrystals also does not change if we define it using only reduced enlargements. In particular, it depends only on  $X \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ , and not on  $X$ , and is equipped with a Frobenius pullback functor  $F^*$ . Thus, we have the notion of a convergent log  $F$ -isocrystal (again cf. [34, Section 3]). When the log structure on  $X$  is trivial, this agrees with the definitions of convergent isocrystal and  $F$ -isocrystal in [33].

The log crystalline site of  $X$  is the site whose objects consist of PD-enlargements. As in [30, Section 1.3.3], a log Dieudonné crystal over  $X$  is a crystal  $M$  in the log crystalline site of  $X$  together with maps  $F^*M \rightarrow M$  and  $M \rightarrow F^*M$  whose composite in either order is multiplication by  $p$ . As in [33, Proposition 2.18] or [34, Remark 16], a log Dieudonné crystal over  $X$  gives rise to a convergent  $F$ -isocrystal on  $X$ .

## A.2

Let  $S$  be a flat, normal, finite-type  $W$ -scheme, let  $D \subset S$  be a relative Cartier divisor, and let  $j : U = D - S \hookrightarrow S$ , the inclusion. We consider  $S$  as a log scheme equipped with the log structure  $j_*\mathcal{O}_U$ , and for  $n \geq 1$ , we give  $S \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$  the induced log structure.

Let  $\pi : \mathcal{A} \rightarrow U$  be an abelian scheme which extends to a semi-abelian scheme over  $S$ . We denote by  $\mathbb{L}$  the étale local system  $R^1\pi_*\mathbb{Q}_p$  on  $U_{K,\text{ét}}$ . We denote by  $\mathcal{E}$  the convergent  $F$ -isocrystal on  $U$  attached to the  $p$ -divisible group  $\mathcal{A}[p^\infty]$ .

By [30, Proposition 1.3.5], there is a log Dieudonné crystal on  $S$  attached to  $\mathcal{A}$ , and hence a convergent log  $F$ -isocrystal  $\mathcal{E}^{\log}$  on  $S$ , whose restriction to  $U$  is  $\mathcal{E}$ , and whose formation is compatible with Cartier duality.

## A.3

Let  $K_0 = W[1/p]$ , and let  $K/K_0$  be a finite extension. Fix an algebraic closure  $\bar{K} \supset K$ , and let  $G_K = \text{Gal}(\bar{K}/K)$ . We keep the above notation, but we now assume that  $S$  is semistable over  $\mathcal{O}_K$  and that  $S_0 \cup D \subset S$  is a normal crossing divisor. Here  $S_0 = S \otimes_{\mathcal{O}_K} k$ .

Above we considered  $S$  with the log structure given by  $D$ . We denote by  $S^{\text{vlog}}$  the scheme  $S$  considered with the log structure given by  $S_0 \cup D$ . There is a map of log schemes  $i : S^{\text{vlog}} \rightarrow S$ . We set  $\mathcal{E}^{\text{vlog}} = i^*(\mathcal{E}^{\log})$ , a convergent  $F$ -isocrystal on  $S^{\text{vlog}}$ .

## LEMMA A.4

With the above notation,  $\mathbb{L}$  and  $\mathcal{E}^{\text{vlog}}$  are associated in the sense of [10, p. 258].

## Proof

As already remarked in [10],  $\mathcal{E}^{\text{vlog}}$  gives rise to a convergent isocrystal in the sense of

[10, p. 258]. The proof of the lemma is entirely analogous to the argument given in [9, Section 6] (cf. also [30, Proposition A.22]) for the case of log schemes.  $\square$

### A.5

We now return to the assumptions of A.2, so we no longer assume that  $S$  is semistable.

Let  $s : \mathbf{1} \rightarrow \mathbb{L}^\otimes$  be a map of étale local systems over  $U$ . That is,  $s$  is a global section of  $\mathbb{L}^\otimes$ . For any finite  $K'/K$  in  $\bar{K}$ , with residue field  $k'$ , and any  $\xi \in U(\mathcal{O}_{K'})$ ,  $\xi^*(s)$  corresponds to a section

$$s_{0,\xi} = D_{\text{cris}}(\xi^*(s)) : \mathbf{1} \rightarrow \xi^*(\mathcal{E})^\otimes.$$

### PROPOSITION A.6

*If  $S$  is proper and semistable over  $\mathcal{O}_K$ , and  $S_0 \cup D \subset S$  is a normal crossing divisor, then there is a morphism of convergent log  $F$ -isocrystals  $s_0 : \mathbf{1} \rightarrow \mathcal{E}^{\log \otimes}$  over  $S$  such that  $\xi^*(s_0)(W(k')) = s_{0,\xi}$  for all  $K'/K, k'$ , and  $\xi$  as above*

#### *Proof*

Let  $\pi \in \mathcal{O}_K$  be a uniformizer, and let  $E(T)$  be an Eisenstein polynomial for  $\pi$ . Let  $R = W[[T]]$ , and for  $n \geq 1$ , let  $R_n$  be the  $p$ -adic completion of  $W[T, E(T)^{n!}/i!]$ . We view  $\mathcal{O}_K$  as an  $R_n$ -algebra, and so an  $R$ -algebra via  $T \mapsto \pi$ . It suffices to construct  $s_0$  étale locally on  $S$ .

Let  $\text{Spec } A$  be an étale neighborhood of  $S$  which admits an étale map

$$\varpi : \text{Spec } A \rightarrow \mathcal{O}_K[t_1, \dots, t_d]/(t_1 \cdots t_e - \pi)$$

for some  $e \leq d$ , and such that the log structure on  $\text{Spec } A$  is given by the preimage of the Cartier divisor defined by  $t_1 \cdots t_r$  for some  $e \leq r \leq d$ . Let  $\hat{A}$  be the  $p$ -adic completion of  $A$ . Thus,  $\text{Spf } \hat{A}$  is a  $p$ -adic formal log scheme over  $\mathcal{O}_K$ , which is formally smooth when  $\mathcal{O}_K$  is equipped with the log structure  $\mathcal{O}_K - \{0\}$ . Lift  $\text{Spf } \hat{A}$  to a formally smooth  $(p, T)$ -adic formal log scheme  $Y_R = \text{Spf } \hat{A}_R$  over  $R$  (defined as in the  $p$ -adic case). Thus,  $\hat{A}_R$  is formally étale over the  $(p, T)$ -adic completion of  $R[t_1, \dots, t_d]/(t_1 \cdots t_e - T)$ , with the log structure given by the preimage of the Cartier divisor defined by  $t_1 \cdots t_r$ .

We consider the Frobenius lift  $F$  on  $\hat{A}_R$  induced by  $t_i \mapsto t_i^p$ , and  $T \mapsto T^p$ . Let  $Y_n$  be the base change of  $Y_R$  to  $R_n$ . Then  $F$  induces a lift of Frobenius on  $Y_n$ . Note that  $Y_n$  is an enlargement of  $S^{\text{vlog}}$ , and so we may evaluate  $\mathcal{E}^{\text{vlog}}$  on it.

By Lemma A.4 and [10, Section 5, Corollary 4, Remark 1)],  $s$  gives rise to a Frobenius-invariant, parallel section  $s_0$  of  $\mathcal{E}^{\text{vlog}}(Y_1)^\otimes$ . Note that the result there applies because  $\mathcal{E}^{\text{vlog}}$  arises from a log Dieudonné crystal on  $S^{\text{vlog}}$ . Hence for any  $m$ ,  $n$  we can apply that result to the log  $F$ -crystal obtained by multiplying the Frobenius on  $\mathcal{E}^{\text{vlog} \otimes n} \otimes \mathcal{E}^{\text{vlog} * \otimes m}$  by a high enough power of  $p$ , and replacing  $\mathbb{L}^{\otimes n} \otimes \mathbb{L}^{* \otimes m}$

by a suitable Tate twist. Since  $s_0$  is Frobenius-invariant, it gives rise to a section of  $\mathcal{E}^{\text{vlog}}(Y_n)^{\otimes}$  for any  $n \geq 1$ .

Now let  $Y_n^h$  be the  $p$ -adic formal log scheme with the same underlying formal scheme as  $Y_n$ , but with the log structure defined by  $t_{e+1} \cdots t_r$ . Then  $Y_n^h$  is an enlargement of  $S$ , and from the definitions we have  $\mathcal{E}^{\text{log}}(Y_n^h) = \mathcal{E}^{\text{vlog}}(Y_n)$ . Since  $Y_R$  is formally smooth over  $W$  (as in [33, Theorem 2.11]), the sections  $s_0 \in \mathcal{E}^{\text{log}}(Y_n^h)^{\otimes}$  give rise to a morphism of convergent  $F$ -isocrystals  $s_0 : \mathbf{1} \rightarrow \mathcal{E}^{\text{log} \otimes}$  over  $\text{Spec } A$ . The relation  $\xi^*(s_0)(W(k')) = s_{0,\xi}$  follows from the functoriality of the map constructed in [10].  $\square$

#### COROLLARY A.7

For any  $S$  (not assumed proper or semistable), and  $s : \mathbf{1} \rightarrow \mathbb{L}^{\otimes}$  as above, there exists a unique morphism of convergent  $F$ -isocrystals over  $U$ ,

$$s_0 : \mathbf{1} \rightarrow \mathcal{E}^{\otimes}$$

such that for every  $K'/K$  finite, and  $\xi$  as above,  $\xi^*(s_0)(W(k')) = s_{0,\xi}$ .

#### Proof

By [6, Theorem 6.5], after replacing  $K$  by a finite extension, there exists a proper truncated hypercovering

$$U_1 \rightrightarrows U_0 \rightarrow U$$

such that for  $i = 0, 1$  there is a dense open immersion  $U_i \hookrightarrow S_i$ , with  $S_i$  proper and semistable, and  $(S_i \setminus U_i) \cup S_i \otimes_{\mathcal{O}_K} k$  is a normal crossing divisor in  $S_i$ . By proper descent for convergent isocrystals (see [33, Theorem 4.6]), it suffices to prove the proposition with  $U_i$  in place of  $U$ . Thus we may replace  $U$  by  $U_i$ , and  $S$  by  $S_i$ , and assume that  $S$  is proper and semistable, and  $S_0 \cup D \subset S$  is a normal crossing divisor. Then the required map is obtained by restricting the map  $s_0 : \mathbf{1} \rightarrow \mathcal{E}^{\text{log} \otimes}$  of Proposition A.6 to  $U$ . The uniqueness is easily deduced from [33, Theorem 4.1].  $\square$

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