J. reine angew. Math. **768** (2020), 93–147
DOI 10.1515/crelle-2019-0039

Journal für die reine und angewandte Mathematik

The cohomology of semi-infinite Deligne–Lusztig varieties

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Abstract. We prove a 1979 conjecture of Lusztig on the cohomology of semi-infinite Deligne–Lusztig varieties attached to division algebras over local fields. We also prove the two conjectures of Boyarchenko on these varieties. It is known that in this setting, the semi-infinite Deligne–Lusztig varieties are ind-schemes comprised of limits of certain finite-type schemes X_h. Boyarchenko's two conjectures are on the maximality of X_h and on the behavior of the torus-eigenspaces of their cohomology. Both of these conjectures were known in full generality only for division algebras with Hasse invariant $1/n$ in the case $h = 2$ (the "lowest level") by the work of Boyarchenko–Weinstein on the cohomology of a special affinoid in the Lubin–Tate tower. We prove that the number of rational points of X_h attains its Weil–Deligne bound, so that the cohomology of X_h is pure in a very strong sense. We prove that the torus-eigenspaces of the cohomology group $H_c^i(X_h)$ are irreducible representations and are supported in exactly one cohomological degree. Finally, we give a complete description of the homology groups of the semi-infinite Deligne–Lusztig varieties attached to any division algebra, thus giving a geometric realization of a large class of supercuspidal representations of these groups. Moreover, the correspondence $\theta \mapsto H_c^i(X_h)[\theta]$ agrees with local Langlands and Jacquet–Langlands correspondences. The techniques developed in this paper should be useful in studying these constructions for p-adic groups in general.

Contents

This work was partially supported by NSF grants DMS-0943832 and DMS-1160720.

1. Introduction

The seminal work of Deligne and Lusztig on the representations of finite reductive groups [9] has influenced an industry studying parallel constructions in the same theme. Classically, one begins with a reductive group $\mathbb{G}$ over $\mathbb{F}_q$ together with a maximal torus $\mathbb{T} \subset \mathbb{G}$, and writing $G := \mathbb{G}(\mathbb{F}_q)$, $T := \mathbb{T}(\mathbb{F}_q)$, one considers the G-representations arising from the torus-eigenspaces $H_c^i(\widetilde{X}_{T\subset G}, \overline{\mathbb{Q}}_\ell)[\theta]$ of the cohomology of the Deligne–Lusztig variety $\widetilde{X}_{T\subset G}$. The Deligne–Lusztig variety is a T-torsor over a subvariety of the flag variety, and can be defined to be

$$\widetilde{X}_{T\subset G} := (\widetilde{U} \cap F^{-1}(\widetilde{U}))\backslash\{g \in \widetilde{G} : F(g)g^{-1} \in \widetilde{U}\},$$

where $\widetilde{T} = \mathbb{T}(\overline{\mathbb{F}}_q)$, $\widetilde{G} = \mathbb{G}(\overline{\mathbb{F}}_q)$, $\widetilde{U} \subset \widetilde{G}$ is the unipotent radical of a Borel $\widetilde{B} \subset \widetilde{G}$ containing $\widetilde{T}$, and $F\colon \widetilde{G} \to \widetilde{G}$ is the group automorphism induced by the Frobenius element in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Following this philosophy, it is natural to ask whether a similar construction can be used to study representations of reductive groups over finite rings or over local fields. Both of these set-ups have been studied by Lusztig – for reductive groups over finite rings, see for example [20], and for reductive groups over local fields, see [19]. In the present paper, we study the latter situation using the construction proposed by Lusztig in [19]. We recall Lusztig's construction now. The analogy with the classical Deligne–Lusztig varieties $\widetilde{X}_{T\subset G}$ will be apparent.

For a reductive group $\mathbb{G}$ over a non-Archimedean local field K with residue field $\mathbb{F}_q$, let $\mathbb{T} \subset \mathbb{G}$ be an elliptic unramified maximal torus over K, and write $T := \mathbb{T}(K)$, $G := \mathbb{G}(K)$, and $\widetilde{T} := \mathbb{T}(\widehat{K}^{\mathrm{nr}})$, $\widetilde{G} := \mathbb{G}(\widehat{K}^{\mathrm{nr}})$. Assume that there exists a Borel subgroup $\widetilde{B} \subset \widetilde{G}$ defined over $\widehat{K}^{\mathrm{nr}}$ and containing $\widetilde{T}$, and let $\widetilde{U}$ be its unipotent radical. Let $F\colon \widetilde{G} \to \widetilde{G}$ denote the group automorphism induced by the Frobenius element in $\mathrm{Gal}(\widehat{K}^{\mathrm{nr}}/K)$. Lusztig's construction, which we call the *semi-infinite Deligne–Lusztig set*, is the quotient

$$\widetilde{X} := (\widetilde{U} \cap F^{-1}(\widetilde{U}))\backslash\{g \in \widetilde{G} : F(g)g^{-1} \in \widetilde{U}\}.$$

In [19], Lusztig suggests that $\widetilde{X}$ should have the structure of an infinite-dimensional variety over $\overline{\mathbb{F}}_q$ and that for a fixed character $\theta\colon T \to \overline{\mathbb{Q}}_\ell^\times$, the subspace $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ wherein T acts by θ should be zero for large i and be concentrated in a single cohomological degree if θ is in general position. The long-term goal is to give a uniform construction of supercuspidal representations of p-adic groups by realizing them in the cohomology of $\widetilde{X}$.

We make this precise and realize this goal in the case when $G = D^\times$ for an n^2-dimensional division algebra $D = D_{k/n}$ over K and $T = L^\times$, where L is the degree-n unramified extension of K. Following Boyarchenko [2], we define an ind-scheme structure on $\widetilde{X}$ together with homology groups $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ that yield smooth representations of $T \times G$. We now state the main theorem, which gives a complete description of the subspaces $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$.

Main Theorem 1. *Let $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be any smooth character.*

(a) *There exists an $r_\theta \in \mathbb{Z}_{\geq 0}$ such that*

$$H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \neq 0 \iff i = r_\theta.$$

Furthermore, r_θ can be determined in terms of the Howe factorization of θ, which measures the extent to which θ arises from characters that factor through norm maps.

(b) *For any very regular element* $x \in \mathcal{O}_L^\times \subset D^\times$,

$$\mathrm{Tr}(x; H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]) = (-1)^{r_\theta} \cdot \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x).$$

(c) *If* θ *has trivial* $\mathrm{Gal}(L/K)$*-stabilizer, then the* $D^\times$*-representation* $H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ *is irreducible.*

When θ has trivial $\mathrm{Gal}(L/K)$-stabilizer, there is a canonical way of assigning a corresponding irreducible representation of the Weil group $\mathcal{W}_K$. In this setting, by work of Henniart [13, 14] (see [7] for a discussion), Main Theorem 1 is enough to characterize the correspondence $\theta \mapsto H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ in terms of the local Langlands and Jacquet–Langlands correspondences:

Corollary. *Let* ϵ *be any character of* $K^\times$ *with* $\ker(\epsilon) = \mathrm{Nm}_{L/K}(L^\times)$. *Let* $\mathfrak{X}$ *denote the set of smooth characters* $L^\times \to \overline{\mathbb{Q}}_\ell^\times$ *with trivial* $\mathrm{Gal}(L/K)$*-stabilizer and let* $\mathcal{A}_K(D^\times)$ *denote the set of isomorphism classes of irreducible* $D^\times$*-representations* ρ *such that* $\rho \cong \rho \otimes (\epsilon \circ \mathrm{Nrd})$, *where* Nrd *is the reduced norm of* $D^\times$. *The correspondence* $\theta \mapsto H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ *induces a bijection*

$$\mathfrak{X}/\mathrm{Gal}(L/K) \xrightarrow{\sim} \mathcal{A}_K(D^\times)$$

which agrees with the local Langlands and Jacquet–Langlands correspondences. In particular, the representations in $\mathcal{A}_K(D^\times)$ *are exactly the irreducible representations of* $D^\times$ *appearing in the cohomology of* $\widetilde{X}$.

The proof of Main Theorem 1 splits into two parts:

(1) Prove that if $\theta|_{U_L^h} = 1$, where $U_L^h = 1 + \pi^h \mathcal{O}_L$, then

$$H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] = \mathrm{Ind}_{K^\times \cdot \mathcal{O}_D^\times}^{D^\times}\big(\text{“}H_c^{2d-i}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]\text{”}\big),$$

where X_h is a finite-type variety of pure dimension d and $\chi := \theta|_{U_L^1}$. Here, $K^\times$ is the center of $D^\times$, the ring $\mathcal{O}_D$ is the unique maximal order of D, and the cohomology group $H_c^{2d-i}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ – which *a priori* is a representation of a subquotient of $\mathcal{O}_D^\times$ – is extended to a representation of $K^\times \cdot \mathcal{O}_D^\times$ in a way uniquely determined by the character θ.

(2) For any $\chi: U_L^1/U_L^h \to \overline{\mathbb{Q}}_\ell^\times$, characterize $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ as a representation of a certain finite unipotent group $\mathbb{U}_{h,k}(\mathbb{F}_{q^n})$.

Part (1) was proved by Boyarchenko in [2], where he made two conjectures about Part (2) (see [2, Conjectures 5.16 and 5.18]). In light of this, the obstruction to understanding $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ reduces to the study of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$. For the reader's convenience, we recall the two conjectures on these cohomology groups here.

Conjecture A (Boyarchenko). The finite-type $\mathbb{F}_{q^n}$-scheme X_h is a maximal variety in the sense of Boyarchenko–Weinstein [4]. Equivalently, we have $H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = 0$ unless i or n is even, and the geometric Frobenius Fr_{q^n} acts on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ by the scalar $(-1)^i q^{\frac{ni}{2}}$.

Conjecture B (Boyarchenko). Given a character $\chi: U_L^1/U_L^h \to \overline{\mathbb{Q}}_\ell^\times$, there exists an $r_\chi \geq 0$ such that $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi] = 0$ for all $i \neq r_\chi$. Moreover, $H_c^{r_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is an irreducible representation of $\mathbb{U}_{h,k}(\mathbb{F}_{q^n})$.

Remark 1.0.1. A finite-type $\mathbb{F}_Q$-scheme S is a *maximal variety* if $S(\mathbb{F}_Q)$ attains its Weil–Deligne bound. More explicitly, by the Grothendieck–Lefschetz trace formula and Deligne's work on the Weil conjectures [8, Theorem 3.3.1],

$$\begin{aligned}\#S(\mathbb{F}_Q) &= \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{Tr}(\mathrm{Fr}_Q, H_c^i(S, \overline{\mathbb{Q}}_\ell)) \\ &\leq \sum_{i\in\mathbb{Z}} Q^{\frac{i}{2}} \dim H_c^i(S, \overline{\mathbb{Q}}_\ell).\end{aligned}$$

Observe that this bound is realized exactly when Fr_Q acts by the scalar $(-1)^i Q^{\frac{i}{2}}$.

We describe the progress on these two conjectures prior to the present work. Under the assumption $k = 1$, Conjectures A and B were proved in the $h = 2$ case by Boyarchenko and Weinstein in [4], where they show that the perfection of X_2 is the special fiber of a particular open affinoid $\mathcal{V}$ in the Lubin–Tate tower, and then prove that the cohomology of $\mathcal{V}$ realizes certain cases of the local Langlands and Jacquet–Langlands correspondences by calculating $H_c^i(X_2, \overline{\mathbb{Q}}_\ell)[\chi]$. For $h > 2$, the results are more sparse. Nearly nothing was known about Conjecture A, and the only work on Conjecture B required the assumption that χ is a *primitive* character (i.e., the restriction of χ to $U_L^{h-1}/U_L^h \cong \mathbb{F}_{q^n}$ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer). For primitive χ, Conjecture B was proved: in [2] for $k = 1$, $h = 3$, $n = 2$, and $\operatorname{char} K > 0$; in [5] for k, h arbitrary, $n = 2$, and $\operatorname{char} K > 0$; and in [6] for k, h, n arbitrary, and $\operatorname{char} K > 0$. In these works, it was also shown that Fr_{q^n} acts on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ in the predicted way for χ primitive, and this is the extent to which Conjecture A was known for $h > 2$.

We remark that the restriction to the class of primitive characters also appears in various other works, for example [20, 21], where they are called regular characters. The present paper is the first paper to study in detail cohomology groups of this type for the class of *all* characters.

The approach of this paper is of a somewhat different nature to the earlier work towards Boyarchenko's conjectures. Before this paper, the idea was to define a unipotent group scheme $U_{h,k}^{n,q}$ over $\mathbb{F}_{q^n}$ such that $U_{h,k}^{n,q}(\mathbb{F}_{q^n})$ is a subquotient of $\mathcal{O}_D^\times$ and stabilizes an $\mathbb{F}_{q^n}$-subscheme $X_h \subset U_{h,k}^{n,q}$ via the natural multiplication action. This approach has the disadvantage that $U_{h,k}^{n,q}$ does not arise naturally from the set-up and also does not have a mixed characteristic analogue. (Only when $h = 2$ does the unipotent group scheme work in mixed characteristic, and [4] makes use of this.) In this paper we work with a unipotent group scheme $\mathbb{U}_{h,k}$ over $\mathbb{F}_q$ that arises naturally from the set-up and has the feature that $\mathbb{U}_{h,k}(\mathbb{F}_q)$ is a subquotient of $\mathcal{O}_D^\times$ and stabilizes an $\mathbb{F}_{q^n}$-subscheme $X_h \subset \mathbb{U}_{h,k}$ via the natural multiplication action. There is a subtle issue that X_h is not defined over $\mathbb{F}_q$, but this can be dealt with by relating the cohomology of X_h to the cohomology of the union of q-Frobenius translates of X_h in $\mathbb{U}_{h,k}$.

Main Theorem 2. *The two conjectures of Boyarchenko are true.*

The proof of the two conjectures of Boyarchenko comprises most of the present paper. We expect that the techniques we develop and use to prove these conjectures should be applicable to studying semi-infinite Deligne–Lusztig varieties for more general reductive groups. Indeed, since $\mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$ is a subquotient of the standard Iwahori subgroup of $\mathrm{GL}_n(\widehat{K}^{\mathrm{nr}})$, we expect that these ideas should apply to Deligne–Lusztig varieties associated to more general parahoric subgroups of p-adic groups. In forthcoming work with A. Ivanov, we investigate this program for any inner form G of $\mathrm{GL}_n(K)$.

Remark 1.0.2. The term "semi-infinite Deligne–Lusztig variety" is intended to be reminiscent of the following analogies:

$$\begin{aligned} \text{Deligne–Lusztig variety} &: \text{flag variety},\\ \text{affine Deligne–Lusztig variety} &: \text{affine flag variety},\\ \text{semi-infinite Deligne–Lusztig variety} &: \text{semi-infinite flag variety}. \end{aligned}$$

Here, we use the term "semi-infinite" in the sense of Feigin–Frenkel [12]. We also remark that by recent work joint with A. Ivanov [7], there is a close relationship between semi-infinite Deligne–Lusztig varieties and affine Deligne–Lusztig varieties of higher level in the sense of [16].

1.1. Outline of the paper. Fix coprime integers $k, n \geq 1$. In Section 2, we define semi-infinite Deligne–Lusztig sets and recall the (ind-pro-)scheme structure on the set $\widetilde{X}$ attached to a division algebra. This naturally leads us to study a family of finite type $\mathbb{F}_{q^n}$-schemes X_h that arise as subschemes of a unipotent group scheme $\mathbb{U}_{h,k}$ over $\mathbb{F}_q$. There is a natural subgroup scheme $\mathbb{T}_{h,k} \subset \mathbb{U}_{h,k}$ analogous to $\mathbb{T} \subset \mathbb{G}$ as in the introduction, and the natural left- and right-multiplication actions of $\mathbb{T}_{h,k}(\mathbb{F}_q)$ and $\mathbb{U}_{h,k}(\mathbb{F}_q)$ on $\mathbb{U}_{h,k}$ stabilize X_h. In Remark 2.2.5, we discuss how $\mathbb{U}_{h,k}$ differs from the unipotent group schemes $U_{h,k}^{n,q}$ appearing in [2, 4–6].

We also define the notion of the Howe factorization of a character of $\mathbb{T}_{h,k}(\mathbb{F}_q)$ in the sense of [15]. The Howe factorization gives rise to a pair of sequences $(\{m_i\}, \{h_i\})$ from which one can define a stratification of an indexing set $\mathcal{A}^+$ (see Section 3). One of the most important features of $\mathcal{A}^+$ is that it is normed and satisfies Lemma 3.2.6. As a quick application, we prove in Section 3.3 that X_h is smooth, affine, and has dimension $(n-1)(h-1)$.

In Section 4, we prove several general results on the cohomology of constructible $\overline{\mathbb{Q}}_\ell$-sheaves coming from pullbacks of local systems. Proposition 4.1.1 is a generalization of [2, Proposition 2.3] that allows one to calculate spaces of homomorphisms between representations of $G(\mathbb{F}_q)$ and cohomology groups of $X \subset G$ even if X is not defined over $\mathbb{F}_q$. Proposition 4.2.1 relates the cohomology of a scheme S to the cohomology of a subscheme of smaller dimension, and Proposition 4.2.2 is a particular specialization of this proposition which will be one of the main structural techniques used in the proof of Theorem 5.1.1. We remark that one can view [2, Proposition 2.10], [6, Propositions 3.4, 3.5], and Proposition 4.2.2 as variations of the same theme under the umbrella of Proposition 4.2.1.

The content of Section 5 is Theorem 5.1.1, and this is really the heart of the proof of Conjectures A and B. In this theorem, we prove

$$\operatorname{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_{q^n})}\left(\operatorname{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_{q^n})}^{\mathbb{U}_{h,k}(\mathbb{F}_{q^n})}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) \neq 0 \iff i = r_\chi,$$

where r_χ can be given explicitly in terms of the Howe factorization of χ. The driving idea of the proof is to calculate certain cohomology groups by inducting along linear fibrations (using Proposition 4.2.2) determined by the stratification of $\mathcal{A}^+$ associated to the Howe factorization of χ.

We begin Section 6 by proving (Theorem 6.1.1) that the alternating sum

$$\sum (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$$

is an irreducible representation of $\mathbb{U}_{h,k}(\mathbb{F}_q)$. This is very close to results of Lusztig [19,20] and Stasinski [21]. This result together with Theorem 5.1.1 immediately implies Main Theorem 2,

and in particular Conjecture B. There is a subtlety that occurs here: while it follows from Theorem 5.1.1 and Theorem 6.1.1 that the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ are concentrated in a single degree, it requires a nontrivial argument to show that this nonvanishing cohomological degree is the r_χ from Theorem 5.1.1. This is the content of Theorem 6.2.4. In Section 6.4, we show that r_χ indeed specializes to the formulas obtained in [4] and [6].

We also prove related results in the same theme. We prove a multiplicity-one statement (Theorem 6.2.2): the association $\chi \mapsto H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ defines an injection into the set of irreducible representations of $\mathbb{U}_{h,k}(\mathbb{F}_q)$. In addition, we compute the character of $H_c^*(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ on the set of very regular elements of $L^\times$ (Theorem 6.2.3), which we utilize in the proof of Theorem 6.2.4. Finally, we give an explicit formula for the zeta function of X_h (Theorem 6.3.1).

In the concluding section, Section 7, we use the discussion of semi-infinite Deligne–Lusztig varieties in Section 2.1 and the theorems of Sections 5 and 6 to prove Main Theorem 1. We then finish the paper with a discussion of the local Langlands and Jacquet–Langlands correspondences and prove the above stated Corollary to Main Theorem 1 (see Theorem 7.2.1).

The techniques of this paper should be directly applicable to studying the homology groups of semi-infinite Deligne–Lusztig varieties attached to an arbitrary pair $T \subset G$, where T is a maximal unramified torus of a reductive group G over a non-Archimedean local field. For example, this can be done when G is any pure inner form of GL_n, and this is part of ongoing work with A. Ivanov. See Remark 5.1.4 for a more technical discussion.

Acknowledgement. I would like to thank Bhargav Bhatt for several helpful conversations and Alex Ivanov for helpful comments on an earlier draft. I would also like to thank the anonymous referee for numerous observations and suggestions that have improved this paper.

2. Definitions

We fix, once and for all, an integer $n \geq 1$, a non-Archimedean local field K with finite residue field $\mathbb{F}_q$ of characteristic p, and a uniformizer π of K. Let L be the unique degree-n unramified extension of K and let $\mathcal{O}_L$ be its ring of integers with unique maximal ideal P_L. Write $U_L^h := 1 + P_L^h$. For a division algebra D over K, we denote by $\mathcal{O}_D$ its ring of integers (i.e., its unique maximal order) and denote by P_D the unique maximal ideal of $\mathcal{O}_D$. Write $U_D^h := 1 + P_D^h$. If D has Hasse invariant $k/n \in \mathbb{Q}/\mathbb{Z}$, where $(k, n) = 1$, we sometimes write $D = D_{k/n}$.

If K has characteristic p, we let $\mathbb{W}(A) = A[\![\pi]\!]$ for any $\mathbb{F}_q$-algebra A and if $a_i \in A$ for $i \geq 0$, we write $[a_i]_{i\geq 0}$ to denote the element $\sum_{i\geq 0} a_i \pi^i \in \mathbb{W}(A)$. If K has characteristic 0, we let $\mathbb{W} = W_{\mathcal{O}_K} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathbb{F}_q$, where $W_{\mathcal{O}_K}$ is the $\mathcal{O}_K$-ring scheme of $\mathcal{O}_K$-Witt vectors [11, Section 1.2]. Following the notation of *op. cit.,* for an $\mathbb{F}_q$-algebra A, we write the elements of $\mathbb{W}(A)$ as $[a_i]_{i\geq 0}$ where each $a_i \in A$. As usual, we have the Frobenius and Verschiebung morphisms

$$\varphi\colon \mathbb{W} \to \mathbb{W}, \quad [a_i]_{i\geq 0} \mapsto [a_i^q]_{i\geq 0},$$
$$V\colon \mathbb{W} \to \mathbb{W}, \quad [a_i]_{i\geq 0} \mapsto [0, a_0, a_1, \ldots].$$

One also has a morphism

$$r\colon A \to \mathbb{W}(A), \quad a \mapsto [a, 0, \ldots]$$

for any $\mathbb{F}_q$-algebra A. Note that

$$\pi \cdot [a_i]_{i\geq 0} = \begin{cases} [0, a_0, a_1, \ldots] & \text{if char } K > 0, \\ [0, a_0^q, a_1^q, \ldots] & \text{if char } K = 0. \end{cases}$$

For any $h \in \mathbb{N}$, let $\mathbb{W}_h := \mathbb{W}/V^h\mathbb{W}$ be the corresponding truncated ring scheme. For any $r \in \mathbb{N}$, consider the group schemes

$$\mathbb{W}^{(r)} := 1 + V^r\mathbb{W} \subset \mathbb{W}^\times,$$
$$\mathbb{W}_h^{(r)} := 1 + V^r\mathbb{W}_{h-r} \subset \mathbb{W}_h^\times.$$

These are all defined over $\mathbb{F}_q$.

For any integer m, define $[m]$ to be the unique integer with $1 \leq [m] \leq n$ such that $m \equiv [m]$ modulo n.

2.1. Semi-infinite Deligne–Lusztig varieties for division algebras. Let $\mathbb{G}$ be a connected reductive group over K and write $G = \mathbb{G}(K)$ and $\widetilde{G} = \mathbb{G}(\widehat{K}^{\mathrm{nr}})$ so that $G = \widetilde{G}^F$, where $F\colon \widetilde{G} \to \widetilde{G}$ is the Frobenius map induced by the arithmetic Frobenius in $\mathrm{Gal}(\widehat{K}^{\mathrm{nr}}/K)$. Let $\mathbb{T} \subset \mathbb{G}$ be an elliptic maximal torus over K and assume that there exists a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$ defined over $\widehat{K}^{\mathrm{nr}}$ such that $\mathbb{T}(\widehat{K}^{\mathrm{nr}}) \subset \mathbb{B}(\widehat{K}^{\mathrm{nr}}) =: \widetilde{B}$. Let $\widetilde{U}$ denote the unipotent radical of $\widetilde{B}$.

Definition 2.1.1. The *semi-infinite Deligne–Lusztig set* associated to $G := \mathbb{G}(K) = \widetilde{G}^F$ and $T := \mathbb{T}(K) = \widetilde{T}^F$ is the quotient

$$X := (\widetilde{U} \cap F^{-1}(\widetilde{U}))\backslash\{g \in \widetilde{G} : F(g)g^{-1} \in \widetilde{U}\}.$$

It is clear that X is endowed with a left-multiplication action of T and a right-multiplication action of G.

Remark 2.1.2. Definition 2.1.1 is due to Lusztig [19], where he defined X to be

$$\{g \in \widetilde{G} : g^{-1}F(g) \in \widetilde{U}\}/(\widetilde{U} \cap F^{-1}(\widetilde{U})),$$

which has a right-multiplication action of T and a left-multiplication action of G. In this paper, we follow the convention set in [2], where the quotient is taken on the other side.

The Brauer group of the local field K is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Hence for any integer $1 \leq k \leq n$ with $(k, n) = 1$, there is a corresponding division algebra $D_{k/n}$ of dimension n^2 over K. The group $L^\times$ is an unramified anisotropic torus in $D_{k/n}^\times$, and we can realize

$$L^\times \hookrightarrow D_{k/n}^\times$$

in this framework in two ways. Set $\widetilde{G} := \mathrm{GL}_n(\widehat{K}^{\mathrm{nr}})$ and consider the automorphisms

$$F_1\colon \widetilde{G} \to \widetilde{G}, \quad g \mapsto \varpi_k^{-1}\varphi(g)\varpi_k, \quad \varpi_k := \begin{pmatrix} 0 & 1_{n-1} \\ \pi^k & 0 \end{pmatrix}, \tag{2.1}$$

$$F_2\colon \widetilde{G} \to \widetilde{G}, \quad g \mapsto \varpi^{-k}\varphi(g)\varpi^k, \quad \varpi := \begin{pmatrix} 0 & 1_{n-1} \\ \pi & 0 \end{pmatrix}, \tag{2.2}$$

where 1_{n-1} denotes the identity matrix of size $(n-1)\times(n-1)$ and $\varphi(g)$ is the matrix obtained by applying the arithmetic Frobenius $\varphi \in \mathrm{Gal}(\widehat{K}^{\mathrm{nr}}/K)$ to each entry of g. Then for $i = 1, 2$, the morphism F_i is a Frobenius relative to a K-rational structure and we denote the corresponding algebraic group by $\mathbb{G}_i$. Consider the diagonal torus $\widetilde{T} \subset \widetilde{G}$ and let $\widetilde{B} \subset \widetilde{G}$ be the standard Borel. Since F_i stabilizes $\widetilde{T}$, it defines a K-rational structures on $\widetilde{T}$, and we can denote the corresponding algebraic group by $\mathbb{T}_i$. Note that neither F_1 nor F_2 stabilizes $\widetilde{B}$ and we have

$$\mathbb{G}_1(K) \xrightarrow{\cong} \mathbb{G}_2(K), \quad \mathbb{T}_1(K) \xrightarrow{\cong} \mathbb{T}_2(K),$$

where the isomorphism is given by $f : g \mapsto \gamma^{-1} \cdot g \cdot \gamma$, where $\gamma = \gamma_0 \cdot \mathrm{diag}(\pi^{\lambda_1}, \ldots, \pi^{\lambda_n})$ for a permutation matrix γ_0 and for some $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. Since the image of ϖ in the Weyl group has order n, we may assume that $e_1 \cdot \gamma_0 = e_1$, where e_1 is the first elementary row vector. For $i = 1, 2$, set $G_i := \mathbb{G}_i(K) \supset \mathbb{T}_i(K) =: T_i$. We have $G_1 \cong G_2 \cong D^{\times}_{k/n}$ and $T_1 \cong T_2 \cong L^{\times}$.

Let X denote the semi-infinite Deligne–Lusztig set associated to G_1 and T_1. We now recollect how to realize X as the $\overline{\mathbb{F}}_q$-points of an infinite-dimensional scheme using a method suggested by Lusztig in [19] and formalized by Boyarchenko in [2]. By [2, Corollary 4.3], X can be identified with the set

$$\widetilde{X} := \{g \in \widetilde{G} : F_1(g)g^{-1} \in \widetilde{U} \cap F_1(\widetilde{U}^-)\}, \tag{2.3}$$

where $\widetilde{U}^-$ is the unipotent radical of the opposite Borel to $\widetilde{B}$. By [2, Lemma 4.4], a matrix $A \in \widetilde{G}$ belongs to $\widetilde{X}$ if and only if it has the form

$$A = x(a_1, \ldots, a_n) \tag{2.4}$$
$$:= \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ \pi^k\varphi(a_n) & \varphi(a_1) & \varphi(a_2) & \cdots & \varphi(a_{n-1}) \\ \pi^k\varphi^2(a_{n-1}) & \pi^k\varphi^2(a_n) & \varphi^2(a_1) & \cdots & \varphi^2(a_{n-2}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \pi^k\varphi^{n-1}(a_2) & \pi^k\varphi^{n-1}(a_3) & \pi^k\varphi^{n-1}(a_4) & \cdots & \varphi^{n-1}(a_1) \end{pmatrix},$$

where $a_i \in \widehat{K}^{\mathrm{nr}}$ and $\det(A) \in K^{\times}$. (Note the indexing difference between equation (2.4) and [2, equation (4.5)].) We may therefore write

$$\widetilde{X} = \bigsqcup_{m\in\mathbb{Z}} \widetilde{X}^{(m)},$$

where $\widetilde{X}^{(m)}$ consists of all $A \in \widetilde{X}$ with $\det(A) \in \pi^m \mathcal{O}_K^{\times}$. Note that the action of ϖ_k takes each $\widetilde{X}^{(m)}$ isomorphically onto $\widetilde{X}^{(m+k)}$, and the action of π takes each $\widetilde{X}^{(m)}$ isomorphically onto $\widetilde{X}^{(m+n)}$. Since $(k, n) = 1$ by assumption, the $\widetilde{X}^{(m)}$ are all isomorphic. It is therefore sufficient to show that $\widetilde{X}^{(0)}$ can be realized as the $\overline{\mathbb{F}}_q$-points of a scheme. To do this, we use Lemma 2.1.3, whose proof (see [6, Lemma 7.1]) is nearly exactly the same as that of [2, Lemma 4.5].

Lemma 2.1.3 (Boyarchenko, [2, Lemma 4.5]). *Assume* $(k, n) = 1$. *If a matrix* A *of the form* (2.4) *satisfies* $\det(A) \in \mathcal{O}_K^{\times}$, *then* $a_j \in \pi^{-\lfloor (j-1)k/n \rfloor}\widehat{\mathcal{O}}_K^{nr}$ *for* $1 \le j \le n$ *and* $a_1 \in (\widehat{\mathcal{O}}_K^{nr})^{\times}$.

We have now shown that $\widetilde{X}^{(0)}$ consists of matrices of the form

$$A(a_1, a_2, \ldots, a_n) := x(a'_1, a'_2, \ldots, a'_n)$$

for some $a_1 \in (\widehat{\mathcal{O}}_K^{\mathrm{nr}})^\times$ and $a_j \in \widehat{\mathcal{O}}_K^{\mathrm{nr}}$ for $2 \le j \le n$, where we write

$$a'_j := \pi^{-\lfloor (j-1)k/n \rfloor} a_j \quad \text{for } 1 \le j \le n.$$

Note that the $(L^\times \times D^\times)$-action on $\widetilde{X}$ induces an $(\mathcal{O}_L^\times \times \mathcal{O}_D^\times)$-action on $\widetilde{X}^{(0)}$. (The stabilizer of $\widetilde{X}^{(0)}$ in $L^\times \times D^\times$ is actually slightly bigger, but we save this discussion for Section 7.)

For each integer $h \ge 1$, define $\widetilde{X}_h^{(0)}$ to be the set of matrices

$$\begin{aligned}\widetilde{X}_h^{(0)} := \big\{ A(a_1, a_2, \dots, a_n) : a_1 &\in (\widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^h \widehat{\mathcal{O}}_K^{\mathrm{nr}})^\times, \\ a_j &\in \widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^{h-1}\widehat{\mathcal{O}}_K^{\mathrm{nr}} \text{ for } 2 \le j \le n, \\ \det(A(a_1, \dots, a_n)) &\in (\mathcal{O}_K/\pi^h \mathcal{O}_K)^\times \big\}.\end{aligned}$$

This can be naturally viewed as the set of $\overline{\mathbb{F}}_q$-points of a scheme of finite type over $\mathbb{F}_q$ (see [2, Section 4.5] for $k = 1$ and [6, Section 7.1] for the completely analogous general case).

Note that $\widetilde{X}_h^{(0)}$ has a left-multiplication action of $\mathcal{O}_L^\times / U_L^h$ and a right-multiplication action of $\mathcal{O}_D^\times / U_D^{n(h-1)+1}$, and these actions are defined over $\mathbb{F}_{q^n}$. Because of this, *from now on*, we will regard $\widetilde{X}_h^{(0)}$ as an $\mathbb{F}_{q^n}$-scheme. By Lemma 2.1.3, we have

$$\widetilde{X}^{(0)} = \varprojlim_h \widetilde{X}_h^{(0)},$$

so that $\widetilde{X}^{(0)}$ is the set of $\overline{\mathbb{F}}_q$-points of a (pro-)scheme over $\mathbb{F}_{q^n}$. Similarly, set

$$\widetilde{X}_h^{(m)} := \widetilde{X}_h^{(0)} \cdot (\varpi_k^a \pi^b),$$

where $a, b \in \mathbb{Z}$ are (any!) such that $ak + bn = m$. Note that $\widetilde{X}_h^{(m)}$ does not depend on the choice of a, b and that like $X_h^{(0)}$ it has a left-multiplication action of $\mathcal{O}_L^\times / U_L^h$ and a right-multiplication action of $\mathcal{O}_D^\times / U_D^{n(h-1)+1}$. Having this, we now define ℓ-adic homology groups of $\widetilde{X}^{(m)}$.

Lemma 2.1.4 (Boyarchenko, [2, Lemma 4.7]). *The left-multiplication action of the group $W_h := U_L^{h-1}/U_L^h$ on $\widetilde{X}_h^{(m)}$ preserves every fiber of the natural map $\widetilde{X}_h^{(m)} \to \widetilde{X}_{h-1}^{(m)}$, the induced morphism $W_h \backslash \widetilde{X}_h^{(m)} \to \widetilde{X}_{h-1}^{(m)}$ is smooth, and each of its fibers is isomorphic to the affine space $\mathbb{A}^{n-1}$ over $\overline{\mathbb{F}}_q$.*

For a scheme S of pure dimension d, set $H_i(S, \overline{\mathbb{Q}}_\ell) := H_c^{2d-i}(S, \overline{\mathbb{Q}}_\ell)$. By Lemma 2.1.4,

$$H_i(\widetilde{X}_{h-1}^{(m)}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\cong} H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell)^{W_h}$$

and in particular, we have a natural embedding $H_i(\widetilde{X}_{h-1}^{(m)}, \overline{\mathbb{Q}}_\ell) \hookrightarrow H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell)$. We set

$$\begin{aligned} H_i(\widetilde{X}^{(m)}, \overline{\mathbb{Q}}_\ell) &:= \varinjlim_h H_i(\widetilde{X}_h^{(m)}, \overline{\mathbb{Q}}_\ell), \\ H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell) &:= \bigoplus_m H_i(\widetilde{X}^{(m)}, \overline{\mathbb{Q}}_\ell). \end{aligned}$$

For each $i \ge 0$, the vector space $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ inherits commuting smooth actions of $L^\times$ and $D_{k/n}^\times$.

We will need a slightly different incarnation of $\widetilde{X}_h^{(0)}$. The morphism

$$f\colon \mathbb{G}_1 \to \mathbb{G}_2. \quad g \mapsto \gamma^{-1} \cdot g \cdot \gamma$$

induces an $(\mathcal{O}_L^\times/U_L^h) \times (\mathcal{O}_D^\times/U_D^{n(h-1)+1})$-equivariant isomorphism of $\mathbb{F}_{q^n}$-schemes

$$\widetilde{X}_h^{(0)} \xrightarrow{\cong} \widetilde{X}_h'^{(0)},$$

where if we write $A'(a_0, \dots, a_{n-1}) := \gamma^{-1} \cdot A(a_0, \dots, a_{n-1}) \cdot \gamma$, then

$$\begin{aligned}\widetilde{X}_h'^{(0)} = \big\{A'(a_0, \dots, a_{n-1}) : a_0 &\in (\widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^h \widehat{\mathcal{O}}_K^{\mathrm{nr}})^\times, \\ a_j &\in \widehat{\mathcal{O}}_K^{\mathrm{nr}}/\pi^{h-1}\widehat{\mathcal{O}}_K^{\mathrm{nr}} \text{ for } 1 \le j \le n-1, \\ &\det(A'(a_0, \dots, a_{n-1})) \in (\mathcal{O}_K/\pi^h\mathcal{O}_K)^\times\big\}.\end{aligned}$$

Observe that the determinant condition holds by multiplicativity. This proves:

Lemma 2.1.5. *For all $i \ge 0$, as representations of $\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1}$,*

$$H_c^i(\widetilde{X}_h^{(0)}, \overline{\mathbb{Q}}_\ell) \cong H_c^i(\widetilde{X}_h'^{(0)}, \overline{\mathbb{Q}}_\ell)$$

In the next subsections, we will define a subvariety $X_h \subset \widetilde{X}_h'^{(0)}$ satisfying the following: the stabilizer of X_h in $\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1}$ is equal to the subgroup

$$\Gamma_h := \{(x^{-1}, x) : x \in \mathcal{O}_L^\times/U_L^h\} \cdot (U_L^1/U_L^h \times U_D^1/U_D^{n(h-1)+1}),$$

and $\widetilde{X}_h'^{(0)}$ is equal to the $(\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1})$-translates of X_h. This implies that

$$H_c^i(\widetilde{X}_h'^{(0)}, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Ind}_{\Gamma_h}^{\mathcal{O}_L^\times/U_L^h \times \mathcal{O}_D^\times/U_D^{n(h-1)+1}} \big(H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big).$$

The bulk of this paper is devoted to studying the cohomology of X_h, and we only return to the setting of the semi-infinite Deligne–Lusztig variety $\widetilde{X}$ in Section 7, the final section of this paper.

2.2. The unipotent group scheme $\mathbb{U}_{h,k}$ and the subscheme X_h. For $r \in \mathbb{N}$, let I_r denote the rth subgroup in the standard Iwahori filtration for $\mathrm{GL}_n(\mathbb{W})$ over $\overline{\mathbb{F}}_q$. Explicitly, for any $\overline{\mathbb{F}}_q$-algebra A, let $\mathfrak{b}_0(A)$ be the preimage of the standard upper-triangular Borel subalgebra of $M_n(A)$ under the reduction $M_n(\mathbb{W}(A)) \to M_n(A)$. Consider the morphism

$$\mathcal{V}\colon M_n(\mathbb{W}(A)) \to M_n(\mathbb{W}(A)), \quad g \mapsto \begin{pmatrix} 0 & 1_{n-1} \\ V & 0 \end{pmatrix} g,$$

where if $g = (a_{ij})_{i,j=1}^n$, its image has (i,j)th coordinate

$$(\mathcal{V}g)_{ij} = \begin{cases} a_{i+1,j} & \text{if } i = 1, \dots, n-1, \\ Va_{1,j} & \text{if } i = n. \end{cases}$$

For integers $r \ge 1$, define $\mathfrak{b}_{r/n} := \mathcal{V}^r\mathfrak{b}_0$. Then the Iwahori subgroup $I_0 := \mathfrak{b}_0^\times$ has a filtration given by $I_r := 1 + \mathfrak{b}_r$, and for any $0 \le r \le s$, we may consider the quotient $I_{r,s} := I_r/I_s$. We write $I_{r+} := \bigcup_{s>r} I_s$.

Let $F = F_2$ from equation (2.2). Note that I_r is stable under F since $\varpi^{-1}\mathfrak{b}_0\varpi \subset \mathfrak{b}_0$.

Definition 2.2.1. The quotient $I_{0,(h-1)^+}$ admits an $\mathbb{F}_q$-rational structure with associated Frobenius F. We denote by $\mathbb{G}_{h,k}$ the resulting group scheme defined over $\mathbb{F}_q$ with associated Frobenius Fr_q. Define $\mathbb{T}'_{h,k} \subset \mathbb{G}_{h,k}$ to be the subgroup consisting of diagonal matrices and define

$$\begin{aligned}\mathbb{U}_{h,k} &:= I_{0^+,(h-1)^+} \subset \mathbb{G}_{h,k},\\ \mathbb{T}_{h,k} &:= \{\text{diagonal matrices in } \mathbb{U}_{h,k}\},\end{aligned}$$

so that we have group schemes

$$\begin{array}{ccc}\mathbb{T}'_{h,k} & \hookrightarrow & \mathbb{G}_{h,k}\\ \uparrow & & \uparrow\\ \mathbb{T}_{h,k} & \hookrightarrow & \mathbb{U}_{h,k}\end{array}$$

all defined over $\mathbb{F}_q$. We view the determinant as a morphism

$$\det\colon \mathbb{G}_{h,k} \to \mathbb{W}_h^\times.$$

Remark 2.2.2. Over $\overline{\mathbb{F}}_q$, the above groups have the following explicit description:

(i) $\mathbb{G}_{h,k}$ can be identified with the group of $n \times n$ matrices (M_{ij}) with $M_{ii} \in \mathbb{W}_h^\times$ and $M_{ij} \in \mathbb{W}_{h-1}$ for $i < j$, $M_{ij} \in V\mathbb{W}_{h-1}$ for $i > j$,

(ii) $\mathbb{T}'_{h,k}$ is the subgroup consisting of matrices $(M_{ij}) \in \mathbb{G}_{h,k}$ with $M_{ij} = 0$ if $i \neq j$,

(iii) $\mathbb{U}_{h,k}$ is the subgroup consisting of matrices $(M_{ij}) \in \mathbb{G}_{h,k}$ with $M_{ii} \in \mathbb{W}_h^{(1)}$,

(iv) $\mathbb{T}_{h,k}$ is the subgroup consisting of matrices $(M_{ij}) \in \mathbb{U}_{h,k}$ with $M_{ij} = 0$ if $i \neq j$.

In addition,

$$\begin{aligned}\mathbb{G}_{h,k}(\mathbb{F}_q) &\cong \mathcal{O}_D^\times / U_D^{n(h-1)+1}, \quad \mathbb{T}'_{h,k}(\mathbb{F}_q) \cong \mathcal{O}_L^\times / U_L^h,\\ \mathbb{U}_{h,k}(\mathbb{F}_q) &\cong U_D^1 / U_D^{n(h-1)+1}, \quad \mathbb{T}_{h,k}(\mathbb{F}_q) \cong U_L^1 / U_L^h.\end{aligned}$$

Observe also that $\mathbb{G}_{h,k}(\mathbb{F}_q) \cong \mathbb{F}_{q^n}^\times \ltimes \mathbb{U}_{h,k}(\mathbb{F}_q)$.

By construction, $\widetilde{X}'^{(0)}_h(\overline{\mathbb{F}}_q) \subset \mathbb{G}_{h,k}(\overline{\mathbb{F}}_q)$. Moreover, it is stabilized by Fr_q^n (but not by Fr_q!) and the resulting $\mathbb{F}_{q^n}$-rational structure agrees with the standard $\mathbb{F}_{q^n}$-rational structure on $\widetilde{X}'^{(0)}_h$. Hence $\widetilde{X}'^{(0)}_h$ is a subscheme of $\mathbb{G}_{h,k}$ defined over $\mathbb{F}_{q^n}$.

Definition 2.2.3. For any $\mathbb{F}_{q^n}$-algebra A, define

$$X_h(A) := \widetilde{X}'^{(0)}_h(A) \cap \mathbb{U}_{h,k}(A).$$

The finite group $\mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{G}_{h,k}(\mathbb{F}_q)$ acts on $\mathbb{U}_{h,k}$ by

$$(\zeta, t, g) * x := \zeta \cdot t \cdot x \cdot g \cdot \zeta^{-1}, \quad t \in \mathbb{T}_{h,k}(\mathbb{F}_q),\ (\zeta, g) \in \mathbb{F}_{q^n}^\times \ltimes \mathbb{U}_{h,k}(\mathbb{F}_q) \cong \mathbb{G}_{h,k}(\mathbb{F}_q).$$

This action stabilizes the $\mathbb{F}_{q^n}$-subscheme $X_h \subset \mathbb{U}_{h,k}$.

Lemma 2.2.4. *Let $\widetilde{U}_h \subset \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$ denote the subgroup of upper-triangular unipotent matrices. Then*

$$X_h(\overline{\mathbb{F}}_q) = \{x \in \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q) : \mathrm{Fr}_q(x)x^{-1} \in \widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)\}.$$

Proof. We use (2.3). If $g \in \widetilde{G}$ is such that $F'(g)g^{-1} \in \widetilde{U} \cap F'(\widetilde{U}^-)$, then

$$F(\gamma^{-1}g\gamma)\gamma^{-1}g^{-1}\gamma = \gamma^{-1}F'(g)g^{-1}\gamma \in \gamma^{-1}(\widetilde{U} \cap F'(\widetilde{U}^-))\gamma = \widetilde{U} \cap F(\widetilde{U}^-),$$

where the last equality holds since $e_1 \cdot \gamma_0 = e_1$ by construction and $\widetilde{U} \cap F'(\widetilde{U}^-)$ exactly consists of upper-triangular unipotent matrices with nonzero nondiagonal entries only in the first row. The conclusion follows. □

Remark 2.2.5. Note that the unipotent group scheme $\mathbb{U}_{h,k}$ is a rather different object to the unipotent group schemes appearing in previous work.

(i) In [4, Section 4.4.1], the unipotent group $U^{n,q}$ over $\mathbb{F}_{q^n}$ is defined to be the group consisting of formal expressions $1 + a_1 \cdot e_1 + \cdots + a_n \cdot e_n$ which are multiplied according to the rule $e_i \cdot a = a^{q^i} \cdot e_i$ for all $1 \leq i \leq n$ and

$$e_i \cdot e_j = \begin{cases} e_{i+j} & \text{if } i + j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

This can be viewed as the unipotent group associated to the parameters: $h = 2$, $k = 1$, arbitrary n, and $\mathrm{char}(K)$ arbitrary.

(ii) In [2, Definition 5.5], for any $\mathbb{F}_p$-algebra A, the unipotent group $U_h^{n,q}(A)$ is defined to be the elements of $A\langle\tau\rangle/(\tau^{n(h-1)+1})$ with constant term 1. Here, $A\langle\tau\rangle$ is the twisted polynomial ring with the commutation relation $\tau \cdot a = a^q \cdot \tau$ for $a \in A$. This can be viewed as the unipotent group associated to the parameters: arbitrary h, $k = 1$, arbitrary n, and $\mathrm{char}(K) = p$. Note that $U_2^{n,q} = U^{n,q}$ and $h = 2$ is the only h such that $U_2^{n,q}$ can be used when $\mathrm{char}(K) = 0$.

(iii) The unipotent groups in [6] and [5] are specializations of those in (ii). In [6], we define a unipotent group $U_{h,k}^{n,q}$ together with a subscheme $X_h \subset U_{h,k}^{n,q}$. The unipotent group is isomorphic to U_h^{n,q^l} described in (ii) where l is an integer with $lk \equiv 1$ modulo n, but the variety X_h depends on k (and hence l).

When $\mathrm{char}(K) = p$, one can prove the theorems of this paper using $U_{h,k}^{n,q}$ instead of $\mathbb{U}_{h,k}$, and the proofs are very similar. However, it does not seem possible to formulate a characteristic zero analogue of $U_{h,k}^{n,q}$. The upshot of the unipotent group scheme $\mathbb{U}_{h,k}$ over $\mathbb{F}_q$ defined in Definition 2.2.1 is that it removes all hypotheses on the parameters, and in particular, we are able to work with arbitrary h over K of arbitrary characteristic. Furthermore, the definition of $\mathbb{U}_{h,k}$ seems to lend itself to generalizations to other reductive groups over local fields.

2.3. An explicit description of X_h.

Definition 2.3.1. We define three bijections associated to $F := F_2$ and $F' := F_1$ (see equations (2.1) and (2.2)).

(1) Let $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation such that

$$F(\mathrm{diag}(y_1, \ldots, y_n)) = \mathrm{diag}(\varphi(y_{\sigma(1)}), \ldots, \varphi(y_{\sigma(n)})).$$

(2) Let $\sigma'\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation such that

$$F'(\mathrm{diag}(y_1, \ldots, y_n)) = \mathrm{diag}(\varphi(y_{\sigma'(1)}), \ldots, \varphi(y_{\sigma'(n)})).$$

(3) Let $\tau\colon\{1,\dots,n\}\to\{0,\dots,n-1\}$ be the bijection such that

$$\gamma_0\cdot\operatorname{diag}(y_1,\dots,y_n)\cdot\gamma_0^{-1}=\operatorname{diag}(y_{\tau(1)+1},\dots,y_{\tau(n)+1}).$$

By the assumption that $e_1\cdot\gamma_0=e_1$, we have $\tau(1)=0$.

The following lemma is an easy calculation.

Lemma 2.3.2. *For each* $1\le i\le n$,

$$\tau(\sigma(i))+1=\sigma'(\tau(i)+1)=[\tau(i)+1-1]=[\tau(i)].$$

Proof. By construction,

$$\gamma_0^{-1}\cdot F'(\operatorname{diag}(y_1,\dots,y_n))\cdot\gamma_0=F(\gamma_0^{-1}\cdot\operatorname{diag}(y_1,\dots,y_n)\cdot\gamma_0).$$

Thus we have

$$\begin{aligned}\gamma_0^{-1}\cdot F'(\operatorname{diag}(y_1,\dots,y_n))\cdot\gamma_0&=\gamma_0^{-1}\cdot\operatorname{diag}(\varphi(y_{\sigma'(1)},\dots,y_{\sigma'(n)}))\cdot\gamma_0\\&=\operatorname{diag}(\varphi(y_{\tau^{-1}(\sigma'(1)-1)},\dots,y_{\tau^{-1}(\sigma'(n)-1)})),\\F(\gamma_0^{-1}\cdot\operatorname{diag}(y_1,\dots,y_n)\cdot\gamma_0)&=F(\operatorname{diag}(y_{\tau^{-1}(0)},\dots,y_{\tau^{-1}(n-1)}))\\&=\operatorname{diag}(\varphi(y_{\sigma(\tau^{-1}(0))},\dots,y_{\sigma(\tau^{-1}(n-1))})).\end{aligned}$$

Hence for each $1\le i\le n$, we have

$$\tau^{-1}(\sigma'(i)-1)=\sigma(\tau^{-1}(i-1))$$

and therefore $\sigma'(\tau(j)+1)=\tau(\sigma(j))+1$. The remaining equalities follow from the explicit computation that $F'(\operatorname{diag}(y_1,\dots,y_n))=\operatorname{diag}(\varphi(y_n),\varphi(y_1),\dots,\varphi(y_{n-1}))$. □

Definition 2.3.3. A useful description of X_h is the following. Let $M=(M_{i,j})_{i,j=1,\dots,n}$ be an element of $\mathbb{U}_{h,k}$. Then we may write

$$M=M_1'+M_2'\varpi^k+M_3'\varpi^{[2k]}+\cdots+M_n'\varpi^{[k(n-1)]},\tag{2.5}$$

where

$$\begin{aligned}M_i'&=\operatorname{diag}(M_{1,[(i-1)k+1]},M_{2,[(i-1)k+2]},\dots,M_{n,[(i-1)k+n]})\\&=\operatorname{diag}(M_{1,\sigma^{i-1}(1)},M_{2,\sigma^{i-1}(2)},\dots,M_{n,\sigma^{i-1}(n)})&&\text{for } 1\le i\le n,\\M_{i,j}&=\begin{cases}[1,M_{(i,j,1)},\dots,M_{(i,j,h-1)}]\in\mathbb{W}_h^{(1)} & \text{if } i=j,\\ [M_{(i,j,1)},\dots,M_{(i,j,h-1)}]\in\mathbb{W}_{h-1} & \text{if } i\ne j,\end{cases}&&\text{for } 1\le i,j\le n.\end{aligned}$$

We know that $M\in X_h$ if and only if $\gamma^{-1}M\gamma\in\widetilde{X}_h^{(0)}$. Therefore if $M\in X_h$, then $\gamma^{-1}M_i'\gamma$ is of the form $\operatorname{diag}(A,\varphi(A),\varphi^2(A),\dots,\varphi^{n-1}(A))$. We have

$$\gamma^{-1}M_i'\gamma=\operatorname{diag}(M_{\tau^{-1}(0),\sigma^{i-1}(\tau^{-1}(0))},M_{\tau^{-1}(1),\sigma^{i-1}(\tau^{-1}(1))},\dots,M_{\tau^{-1}(n-1),\sigma^{i-1}(n-1)})$$

and therefore

$$M_{j,\sigma^{i-1}(j)}=\varphi^{\tau(j)}(M_{1,\sigma^{i-1}(1)})\quad\text{for } 1\le i,j\le n.\tag{2.6}$$

In particular, $M_{i,i}=\varphi^{\tau(i)}(M_{1,1})$ for $1\le i\le n$. For any $\mathbb{F}_{q^n}$-algebra A, we have $M\in X_h(A)$ if and only if M satisfies (2.6) and $\varphi(\det(M))=\det(M)$, where $\det(M)\in\mathbb{W}_h^{(1)}(A)$. We call M the *standard form* of an A-point of X_h.

2.4. The Howe factorization. Let $\mathscr{T}_{n,h}$ denote the set of all $\overline{\mathbb{Q}}_\ell$-valued characters of $\mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})$. Recall that we have natural surjections $\mathrm{pr}\colon \mathbb{W}_h^{(1)} \to \mathbb{W}_{h-1}^{(1)}$ and injections $\mathbb{G}_a \to \mathbb{W}_h^{(1)}$ given by $x \mapsto [1, 0, \dots, 0, x]$. Furthermore, for any subfield $F \subset L$, the norm map $L^\times \to F^\times$ induces a map $\mathrm{Nm}\colon \mathbb{W}_h^{(1)}(k_L) \to \mathbb{W}_h^{(1)}(k_F)$. These maps induce

$$\begin{aligned} \mathrm{pr}^*\colon \mathscr{T}_{n,h'} &\to \mathscr{T}_{n,h} \quad \text{for } h' < h, \\ \mathrm{Nm}^*\colon \mathscr{T}_{m,h} &\to \mathscr{T}_{n,h} \quad \text{for } m \mid n. \end{aligned}$$

By pulling back along the map $\mathbb{G}_a \to \mathbb{W}_h^{(1)}$, $x \mapsto [1, 0, \dots, 0, x]$, we may restrict characters of $\mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})$ to characters of $\mathbb{F}_{q^n}$. We say that $\chi \in \mathscr{T}_{n,h}$ has *conductor* m if the stabilizer of $\chi|_{\mathbb{F}_{q^n}}$ in $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_{q^m})$. If $\chi \in \mathscr{T}_{n,h}$ has conductor n, we say that χ is *primitive*. We write $\mathscr{T}^0_{n,h} \subset \mathscr{T}_{n,h}$ to denote the subset of primitive characters.

We can decompose $\chi \in \mathscr{T}_{n,h}$ into primitive components in the sense of Howe [15, Corollary after Lemma 11]. (Howe's formulation involves a non-degeneracy condition on the partial products $\chi_1\chi_2\cdots\chi_i$ of the factorization below, but this condition is equivalent to the primitivity condition on the individual factors χ_i.)

Definition 2.4.1. A *Howe factorization* of a character $\chi \in \mathscr{T}_{n,h}$ is a decomposition

$$\chi = \prod_{i=1}^{r} \chi_i, \quad \text{where } \chi_i = \mathrm{pr}^* \mathrm{Nm}^* \chi_i^0 \text{ and } \chi_i^0 \in \mathscr{T}^0_{m_i,h_i},$$

such that $m_i < m_{i+1}$, $m_i \mid m_{i+1}$, and $h_i > h_{i+1}$. It is automatic that $m_i \le n$ and $h \ge h_i$. For any integer $0 \le t \le r$, set χ_0 to be the trivial character and define

$$\chi_{\ge t} := \prod_{i=t}^{r} \chi_i \in \mathscr{T}_{n,h_t}.$$

Observe that the choice of χ_i in a Howe factorization $\chi = \prod_{i=1}^r \chi_i$ is not unique, but the m_i and h_i only depend on χ. Hence the Howe factorization attaches to each character $\chi \in \mathscr{T}_{n,h}$ a pair of well-defined sequences

$$\begin{aligned} 1 =: m_0 \le m_1 < m_2 < \cdots < m_r \le m_{r+1} := n, \\ h =: h_0 \ge h_1 > h_2 > \cdots > h_r \ge h_{r+1} := 1 \end{aligned}$$

satisfying the divisibility $m_i \mid m_{i+1}$ for $0 \le i \le r$.

Example 2.4.2. We give some examples of the sequences associated to characters $\chi \in \mathscr{T}_{n,h}$.

(a) If χ is the trivial character, the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, 1, n\}, \quad \{h_0, h_1, h_2\} = \{h, 1, 1\}.$$

(b) We say that χ is a primitive character of level h' if $\chi|_{U_L^{h'}} = 1$ and $\chi|_{U_L^{h'-1}/U_L^{h'}}$ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer. In this setting, $\chi \in \mathscr{T}_{n,h}$ if we have $h \ge h'$, and then the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, n, n\}, \quad \{h_0, h_1, h_2\} = \{h, h', 1\}.$$

The case $h = h'$ is studied in [5, 6].

(c) If $\chi|_{U_L^2} = 1$ and $\chi|_{U_L^1/U_L^2}$ has conductor m, then the associated sequences are

$$\{m_0, m_1, m_2\} = \{1, m, n\}, \quad \{h_0, h_1, h_2\} = \{h, 2, 1\}.$$

The case $h = 2$ is studied in [2,4].

3. Indexing sets

We first recall some basic facts about the ramified Witt vectors, which we will need to work with to handle the char $K = 0$ setting. In Section 3.2, we define indexing sets and prove some fundamental properties of these indexing sets that will be heavily used in Section 5. As a quick application of Section 3.2, we calculate the dimension of X_h in Section 3.3.

3.1. Ramified Witt vectors. In this subsection, we assume K has characteristic 0. We first define a "simplified version" of the ramified Witt ring $\mathbb{W}$.

Definition 3.1.1. For any $\mathbb{F}_q$-algebra A, let $W(A)$ be the set $A^{\mathbb{N}}$ endowed with the following coordinatewise addition and multiplication rule:

$$[a_i]_{i\geq 0} +_W [b_i]_{i\geq 0} = [a_i + b_i]_{i\geq 0},$$
$$[a_i]_{i\geq 0} *_W [b_i]_{i\geq 0} = \left[\sum_{j=0}^{i} a_j^{q^{i-j}} b_{i-j}^{q^i}\right]_{i\geq 0}.$$

It is a straightforward check that W is a commutative ring scheme over $\mathbb{F}_q$. It comes with Frobenius and Verschiebung morphisms φ and V.

Lemma 3.1.2. *Let A be an $\mathbb{F}_q$-algebra.*

(a) *For any $[a] = [a_i]_{i\geq 0}, [b] = [b_i]_{i\geq 0} \in A^{\mathbb{N}}$,*

$$[a] *_{\mathbb{W}} [b] = [a] *_W [b] +_W [c],$$

where $[c] = [c_i]_{i\geq 0}$ for some $c_i \in A[a_{i_1}^{e_1} b_{i_2}^{e_2} : i_1 + i_2 < i,\ e_1, e_2 \in \mathbb{Z}_{\geq 0}]$.

(b) *For any $[a] = [a_i]_{i\geq 0}, [b] = [b_i]_{i\geq 0} \in A^{\mathbb{N}}$,*

$$[a] +_{\mathbb{W}} [b] = [a] +_W [b] +_W [c],$$

where $[c] = [c_i]_{i\geq 0}$ for some $c_i \in A[a_j, b_j : j < i]$.

(c) *For any $[a] = [a_i]_{i\geq 0} \in A^{\mathbb{N}}$,*

$$\pi *_{\mathbb{W}} [a] = [0, 1, 0, \ldots] *_W [a] = [0, a_0^q, a_1^q, \ldots].$$

Lemma 3.1.3. *Let A be an $\mathbb{F}_q$-algebra.*

(a) *For any $[a_1], \ldots, [a_n] \in A^{\mathbb{N}}$ where $[a_j] = [a_{j,i}]_{i\geq 0}$,*

$$\prod_{\substack{1\leq j\leq n \\ \text{w.r.t. } \mathbb{W}}} [a_j] = \left(\prod_{\substack{1\leq j\leq n \\ \text{w.r.t. } W}} [a_j]\right) +_W [c],$$

where $[c] = [c_i]_{i\geq 0}$ for some $c_i \in A[a_{1,i_1}^{e_1} \cdots a_{n,i_n}^{e_n} : i_1 + \cdots + i_n < i,\ e_1, \ldots, e_n \in \mathbb{Z}_{\geq 0}]$.

(b) *For any* $[a_1], \dots, [a_n] \in A^{\mathbb{N}}$ *where* $[a_j] = [a_{j,i}]_{i \geq 0}$,

$$\sum_{\substack{1 \leq j \leq n \\ \text{w.r.t. } \mathbb{W}}} [a_j] = \left(\sum_{\substack{1 \leq j \leq n \\ \text{w.r.t. } W}} [a_j] \right) +_W [c],$$

where $[c] = [c_i]_{i \geq 0}$ *for some* $c_i \in A[a_{1,j}, \dots, a_{n,j} : j < i]$.

We call the portion coming from W *the "major contribution" and* $[c]$ *the "minor contribution."*

3.2. Normed indexing sets. We define indexing sets associated to the unipotent group $\mathbb{U}_{h,k}$ that will be crucial to the proof of Theorem 5.1.1.

Define

$$\begin{aligned} \mathcal{A}^+ &:= \{(i,j,l) : 1 \leq i, j \leq n,\ 1 \leq l \leq h-1\}, \\ \mathcal{A} &:= \{(i,j,l) \in \mathcal{A}^+ : i \neq j\}, \\ \mathcal{A}^- &:= \{(i,j,l) \in \mathcal{A} : i = 1\}. \end{aligned}$$

Write $\mathbb{A}[\mathcal{A}^+]$ to denote the affine space of dimension $\#\mathcal{A}^+$ over $\overline{\mathbb{F}}_q$ with coordinates indexed by $\mathcal{A}^+$. We fix an identification $\mathbb{U}_{h,k} = \mathbb{A}[\mathcal{A}^+]$ (as $\overline{\mathbb{F}}_q$-schemes) as follows: Every point of $\mathbb{U}_{h,k}$ is of the form $(A_{i,j})_{1 \leq i,j \leq n}$, where

$$A_{i,j} = \sum_{l \geq 0} V^l r(A_{(i,j,l^*)}), \quad \text{where } l^* := \begin{cases} l & \text{if } i \geq j, \\ l+1 & \text{if } i < j. \end{cases} \tag{3.1}$$

Thus the element $(i,j,l^*) \in \mathcal{A}^+$ corresponds to the coefficient of π^l in the (i,j)th entry of an element of $\mathbb{U}_{h,k}$. Continuing this dictionary, $\mathcal{A}$ corresponds to the elements of $\mathbb{U}_{h,k}$ with 1's along the diagonal, and $\mathcal{A}^-$ corresponds to the elements of $\mathbb{U}_{h,k}$ with 1's along the diagonal and zeros everywhere else but the top row.

Definition 3.2.1. Define a norm on $\mathcal{A}^+$:

$$\begin{aligned} \mathcal{A}^+ &\to \{1, 2, \dots, n(h-1)\}, \\ (i,j,l) &\mapsto |(i,j,l)| := [j-i] + n(l-1). \end{aligned}$$

Given two sequences of integers

$$1 =: m_0 \leq m_1 < m_2 < \cdots < m_r \leq m_{r+1} := n, \quad m_i \mid m_{i+1}, \tag{3.2}$$

$$h =: h_0 \geq h_1 > h_2 > \cdots > h_r \geq h_{r+1} := 1, \tag{3.3}$$

we can define the following subsets of $\mathcal{A}$ for $0 \leq s, t \leq r$:

$$\begin{aligned} \mathcal{A}_{s,t} &:= \{(i,j,l) \in \mathcal{A} : j \equiv i \pmod{m_s},\ j \not\equiv i \pmod{m_{s+1}},\ 1 \leq l \leq h_t - 1\}, \\ \mathcal{A}^-_{s,t} &:= \mathcal{A}_{s,t} \cap \mathcal{A}^-. \end{aligned}$$

For the proof of Theorem 5.1.1, we will need to consider the following decomposition of $\mathcal{A}^-_{s,t}$:

$$\begin{aligned} \mathcal{I}_{s,t} &:= \{(1,j,l) \in \mathcal{A}^-_{s,t} : |(1,j,l)| > \tfrac{1}{2} n(h_t - 1)\}, \\ \mathcal{J}_{s,t} &:= \{(1,j,l) \in \mathcal{A}^-_{s,t} : |(1,j,l)| \leq \tfrac{1}{2} n(h_t - 1)\}. \end{aligned}$$

For any real number $\nu \geq 0$, define

$$\mathcal{A}_{\geq \nu,t} := \bigsqcup_{s=\lceil \nu \rceil}^{r} \mathcal{A}_{s,t}, \quad \mathcal{I}_{\geq \nu,t} := \bigsqcup_{s=\lceil \nu \rceil}^{r} \mathcal{I}_{s,t},$$
$$\mathcal{J}_{\geq \nu,t} := \bigsqcup_{s=\lceil \nu \rceil}^{r} \mathcal{J}_{s,t}, \quad \mathcal{A}^{-}_{\geq \nu,t} := \mathcal{A}_{\geq \nu,t} \cap \mathcal{A}^{-},$$

and observe that

$$\mathcal{A}_{\geq s,t} = \{(i,j,l) \in \mathcal{A} : j \equiv i \pmod{m_s},\ 1 \leq l \leq h_t - 1\}.$$

Remark 3.2.2. Recall that the Howe factorization of $\chi \in \mathcal{T}_{n,h}$ gives rise to two sequences of integers exactly of the form (3.2) and (3.3), where (3.2) corresponds to the conductors appearing in the Howe factorization and (3.3) corresponds to the levels in which the conductor jumps.

Lemma 3.2.3. *There is an order-reversing injection $\mathcal{I}_{s,t} \hookrightarrow \mathcal{J}_{s,t}$ that is a bijection if and only if $\#\mathcal{A}_{s,t}$ is even. Explicitly, it is given by*

$$\mathcal{I}_{s,t} \hookrightarrow \mathcal{J}_{s,t}, \quad (1, \sigma^j(1), l) \mapsto (1, \sigma^{n-j}(1), h_t - l),$$

where σ is the permutation defined in Definition 2.3.1. *Note that $\#\mathcal{A}_{s,t}$ is even unless n and h_t are both even.*

Proof. It is clear that if $m_s \mid j$ and $m_{s+1} \nmid j$, then $m_s \mid (n-j)$ and $m_{s+1} \nmid (n-j)$. By assumption, $|(1, \sigma^j(1), l)| = \sigma^j(1) - 1 + n(l-1) > \frac{1}{2}n(h_t - 1)$. Then

$$\begin{aligned} |(1, \sigma^{n-j}(1), h_t - l)| &= (n - \sigma^j(1) + 1) + n(h_t - l - 1) \\ &= n(h_t - 1) - |(1, \sigma^j(1), l)| < \tfrac{1}{2}n(h_t - 1). \end{aligned}$$

It is clear that the map is a bijection if and only if the indexing set $\mathcal{J}_{s,t}$ contains an element of norm $\frac{1}{2}n(h_t - 1)$. □

Remark 3.2.4. Note that the divisibility condition on $\mathcal{J}_{s,t}$ implies that there is at most one s such that $(1, \sigma^{n/2}(1), h_t/2) \in \mathcal{J}_{s,t}$. Hence outside this s, the sets $\mathcal{I}_{s,t}$ and $\mathcal{J}_{s,t}$ are in bijection.

Definition 3.2.5. For $\lambda = (i,j,l) \in \mathcal{A}^+$, define

$$\lambda^\vee := \begin{cases} (j, i, h_t - 1 - l) & \text{if } i = j. \\ (j, i, h_t - l) & \text{if } i \neq j. \end{cases}$$

Lemma 3.2.6. *Write $A = (A_{i,j})_{1 \leq i,j \leq n} \in \mathbb{U}_{h,k}$, where $A_{i,j}$ is as in equation* (3.1). *Assume that for $\lambda_1, \lambda_2 \in \mathcal{A}^+$, the variables A_{λ_1} and A_{λ_2} appear in the same monomial in $\det(A) \in \mathbb{W}^{(1)}_{h'}$ for some $h' \leq h$.*

(a) *Then $|\lambda_1| + |\lambda_2| \leq n(h'-1)$.*

(b) *If $|\lambda_1| + |\lambda_2| = n(h'-1)$, then $\lambda_2 = \lambda_1^\vee$.*

Proof. By definition,

$$\det(A) = \sum_{\gamma \in S_n} \prod_{1 \le i \le n} A_{i,\gamma(i)} \in \mathbb{W}_{h'}^{(1)}(\overline{\mathbb{F}}_q).$$

Let $l \le h' - 1$. If K has characteristic p, then the contributions to the π^l-coefficient coming from $\gamma \in S_n$ are of the form

$$\prod_{i=1}^{n} A_{(i,\gamma(i),l_i^*)},$$

where $(l_1, \dots, l_n)$ is a partition of l. Then

$$|(i, \gamma(i), l_i^*)| = [\gamma(i) - i] + n(l_i^* - 1) = \gamma(i) - i + nl_i,$$

and therefore

$$\sum_{i=1}^{n} |(i, \gamma(i), l_i^*)| = \sum_{i=1}^{n} \gamma(i) - i + nl_i = \sum_{i=1}^{n} nl_i = nl \le n(h' - 1). \tag{3.4}$$

If K has characteristic 0, then by Lemma 3.1.3, the major contributions to the π^l-coefficient coming from γ are of the form

$$\prod_{i=1}^{n} A_{(i,\gamma(i),l_i^*)}^{e_i},$$

where the e_i are some nonnegative integers and where $(l_1, \dots, l_n)$ is a partition of l. Hence

$$\sum_{i=1}^{n} |(i, \gamma(i), l_i^*)| = nl \le n(h' - 1). \tag{3.5}$$

The minor contributions to the π^l-coefficient coming from γ are polynomials in

$$\prod_{i=1}^{n} A_{i,\gamma(i),l_i^*}^{e_i'},$$

where $l_1 + \cdots + l_n < l$ and the e_i' are some nonnegative integers. Hence

$$\sum_{i=1}^{n} |(i, \gamma(i), l_i^*)| < n(h' - 1).$$

Suppose now that $\lambda_1 = (i_1, j_1, l_1), \lambda_2 = (i_2, j_2, l_2) \in \mathcal{A}^+$ are such that A_{λ_1} and A_{λ_2} contribute to the same monomial in $\det(M) \in \mathbb{W}_{h'}^{(1)}$. Then there exists some $\gamma \in S_n$ such that $\gamma(i_1) = j_1$ and $\gamma(i_2) = j_2$, and by equations (3.4) and (3.5),

$$|\lambda_1| + |\lambda_2| \le n(h' - 1).$$

Observe that if K has characteristic 0 and λ_1 and λ_2 occur in a minor contribution, then $|\lambda_1| + |\lambda_2| < n(h' - 1)$. This proves (a) and furthermore, we see that if $|\lambda_1| + |\lambda_2| = n(h' - 1)$, then the simultaneous contribution of A_{λ_1} and A_{λ_2} comes from a major contribution. But now (b) follows: since $|\lambda| > 0$ for any $\lambda \in \mathcal{A}$, it follows from equations (3.4) and (3.5) that if $|\lambda_1| + |\lambda_2| = n(h' - 1)$, then necessarily the associated $\gamma \in S_n$ must be a transposition. □

Example 3.2.7. We illustrate a way to visualize the indexing sets $\mathcal{A}_{\geq s,t}$ in a small example. Consider the sequences

$$\{m_0, m_1, m_2, m_3, m_4\} = \{1, 2, 4, 8, 8\}, \quad \{h_0, h_1, h_2, h_3, h_4\}.$$

This corresponds to a character of the form

$$\chi = \chi_1^0(\mathrm{Nm}_{\mathbb{F}_{q^8}/\mathbb{F}_{q^2}}) \cdot \chi_2^0(\mathrm{Nm}_{\mathbb{F}_{q^8}/\mathbb{F}_{q^4}}) \cdot \chi_3^0,$$

where $\chi_i^0 \colon \mathbb{W}_{h_t}^{(1)}(\mathbb{F}_{q^{m_t}}) \to \overline{\mathbb{Q}}_\ell^\times$ is primitive. Then we have

$$\mathcal{A}_{\geq 0,t} = \left(\left(\begin{smallmatrix} 1 & * & * & * & * & * & * & * \\ * & 1 & * & * & * & * & * & * \\ * & * & 1 & * & * & * & * & * \\ * & * & * & 1 & * & * & * & * \\ * & * & * & * & 1 & * & * & * \\ * & * & * & * & * & 1 & * & * \\ * & * & * & * & * & * & 1 & * \\ * & * & * & * & * & * & * & 1 \end{smallmatrix} \right), \operatorname{depth} h_t \right),$$

$$\mathcal{A}_{\geq 1,t} = \left(\left(\begin{smallmatrix} 1 & * & * & * & * \\ * & 1 & * & * & * \\ * & * & 1 & * & * \\ * & * & * & 1 & * \\ * & * & * & * & 1 \end{smallmatrix} \right), \operatorname{depth} h_t \right),$$

$$\mathcal{A}_{\geq 2,t} = \left(\left(\begin{smallmatrix} 1 & & & * & \\ & 1 & & & * \\ & & 1 & & \\ * & & & 1 & \\ & * & & & 1 \end{smallmatrix} \right), \operatorname{depth} h_t \right),$$

$$\mathcal{A}_{\geq 3,t} = \left(\left(\begin{smallmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{smallmatrix} \right), \operatorname{depth} h_t \right).$$

In Section 5, we will calculate certain cohomology groups $H_c^i(\mathcal{X}, \mathcal{F})$ inductively:

$$\begin{aligned} \mathcal{X} \longleftrightarrow \mathcal{A} = \mathcal{A}_{\geq 0,1} &\rightsquigarrow \mathcal{A}_{\geq 1,1} && \text{(Proposition 5.3.2)} \\ &\rightsquigarrow \mathcal{A}_{\geq 1,2} && \text{(Proposition 5.3.1)} \\ &\rightsquigarrow \mathcal{A}_{\geq 2,2} && \text{(Proposition 5.3.2)} \\ &\rightsquigarrow \mathcal{A}_{\geq 2,3} = \varnothing && \text{(Proposition 5.3.1).} \end{aligned}$$

This reduces the calculation of $H_c^i(\mathcal{X}, \mathcal{F})$ to the calculation of the cohomology of a point. (Warning: There may be a step in the inductive process where the reduction is to the cohomology of a curve. This happens exactly when $\#\mathcal{A}_{s,t}$ is odd, which can occur for at most one s.)

3.3. Dimension of X_h.

Proposition 3.3.1. *The $\mathbb{F}_{q^n}$-scheme X_h is smooth, affine, and of dimension $(n-1)(h-1)$.*

Proof. By Definition 2.3.3, $X_h \subset \mathbb{A}[\mathcal{A}^+]$ is defined by the following polynomials:

$$\begin{aligned} f_{(i,\sigma^j(i),l)} &:= M_{(i,\sigma^j(i),l)} - M_{(1,\sigma^j(1),l)}^{q^{\tau(i)}} && \text{for } 2 \leq i \leq n,\ 1 \leq j \leq n,\ 1 \leq l \leq h-1, \\ g_l &:= \mathrm{pr}_l(\det(M)) - \mathrm{pr}_l(\det(M))^q && \text{for } 1 \leq l \leq h-1, \end{aligned}$$

where

$$\mathrm{pr}_l \colon \mathbb{W}_h^{(1)} \to \mathbb{A}^1, \quad [1, a_1, \ldots, a_{h-1}] \mapsto a_l.$$

This gives us $n(n-1)(h-1)+(h-1)$ equations and $\#\mathcal{A}^+ = n^2(h-1)$, and so to prove the proposition, it suffices to find a submatrix of size $(n^2-n+1)(h-1)\times(n^2-n+1)(h-1)$ of the Jacobian matrix that is nonsingular for every point of X_h.

Consider the submatrix J_0 of the Jacobian corresponding to the partial derivatives with respect to the following subset of $\mathcal{A}^+$:

$$A_0 := \{(i,\sigma^j(i),l)\in\mathcal{A}^+ : i\neq 1\}\cup\{(1,1,l)\in\mathcal{A}^+\}.$$

Since we are working in characteristic p, we have

$$\frac{\partial f_\lambda}{\partial M_{\lambda'}} = \begin{cases} 1 & \text{if } \lambda=\lambda', \\ 0 & \text{otherwise,}\end{cases} \tag{3.6}$$

$$\frac{\partial g_l}{\partial M_{(1,1,l')}} = \begin{cases} 1 & \text{if } l'=l, \\ 0 & \text{if } l'>l, \\ ? & \text{if } l'<l.\end{cases} \tag{3.7}$$

Reorder the rows of J_0 so that the first $n(n-1)(h-1)$ rows correspond to the f_λ for $\lambda\in A_0$ and the remaining $h-1$ rows correspond to $g_1,\dots,g_{h-1}$. Reorder the columns of J_0 so that the first $n(n-1)(h-1)$ columns correspond to the M_λ for $\lambda\in A_0$ and the remaining $h-1$ columns correspond to $M_{(1,1,1)},\dots,M_{(1,1,h-1)}$. Then

$$J_0 = \begin{pmatrix} A & B \\ C & D\end{pmatrix},$$

where A is a permutation matrix of size $n(n-1)(h-1)\times n(n-1)(h-1)$ (by equation (3.6)), B is the zero matrix of size $n(n-1)(h-1)\times(h-1)$ (by equation (3.6)), and D is a unipotent lower-triangular matrix of size $(h-1)\times(h-1)$ (by equation (3.7)). Hence for any point in X_h, the matrices A and D are nonsingular, and so J_0 is as well. This shows that X_h is a smooth complete intersection of dimension $n^2(h-1)-(n^2-n+1)(h-1)=(n-1)(h-1)$. □

4. On the cohomology of certain $\overline{\mathbb{Q}}_\ell$-local systems

In this section, we prove some general results on the cohomology of constructible $\overline{\mathbb{Q}}_\ell$-sheaves. Proposition 4.1.1 is a generalization of [2, Proposition 2.3] that allows one to study the $G(\mathbb{F}_q)$-representations arising from the cohomology of a variety $X\subset G$ defined over $\mathbb{F}_{q^N}$. From this perspective, [2, Proposition 2.3] is the $N=1$ setting of Proposition 4.1.1.

The other main result of this section is Proposition 4.2.1, which is a general result on the cohomology of constructible $\overline{\mathbb{Q}}_\ell$-sheaves coming from pullbacks of local systems. Its proof is similar to that of [2, Proposition 2.10] and [6, Proposition 3.4]. We then apply Proposition 4.2.1 to the special case when the local system is an Artin–Schreier-like sheaf (see Proposition 4.2.2).

Throughout this paper, we choose the convention that Fr_q is the geometric Frobenius so that Fr_q acts on the Tate twist $\overline{\mathbb{Q}}_\ell(-1)$ by multiplication by q.

4.1. Cohomology and induced representations. If X is a locally closed subvariety of G that is stable under the action of $G(\mathbb{F}_q)$, it is natural to ask what $G(\mathbb{F}_q)$-representations arise in the cohomology $H_c^i(X,\overline{\mathbb{Q}}_\ell)$. To this end, it is useful to understand spaces of $G(\mathbb{F}_q)$-equi-

variant homomorphisms between $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ and certain nice representations. The following proposition relates such spaces to the cohomology of an explicitly constructed variety with coefficients in a (often nonconstant) $\overline{\mathbb{Q}}_\ell$-sheaf. The upshot is that in the cases we deal with, these new cohomology groups can be explicitly computed (see Section 5 where we use Proposition 4.1.1 to rephrase the calculation so that we may inductively apply Proposition 4.2.2).

Proposition 4.1.1. *Assume that we are given the following data:*

- *an algebraic group G together with a closed connected subgroup $H \subset G$ over $\mathbb{F}_q$ with Frobenius Fr_q,*
- *a section $s\colon G/H \to G$ of the quotient morphism $G \to G/H$,*
- *a character $\chi\colon H(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$,*
- *a closed subscheme Y of G defined over $\mathbb{F}_{q^N}$ such that for any $0 \le i \ne j \le N-1$, the intersection $\mathrm{Fr}_q^i(Y) \cap \mathrm{Fr}_q^j(Y)$ is independent of i, j and is equal to a finite set of points S.*

Set $X := L_q^{-1}(Y)$, where L_q is the Lang map $g \mapsto \mathrm{Fr}_q(g)g^{-1}$ on G. The right multiplication action of $G(\mathbb{F}_q)$ on X induces linear representations of $G(\mathbb{F}_q)$ on the cohomology $H_c^i(X, \overline{\mathbb{Q}}_\ell)$, and for each $i \ge 0$, we have a vector space isomorphism

$$\mathrm{Hom}_{G(\mathbb{F}_q)}\big(\mathrm{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\chi), H_c^i(X, \overline{\mathbb{Q}}_\ell)\big) \cong H_c^i(\beta^{-1}(Y), P^*\mathcal{L}_\chi)$$

compatible with the action of Fr_{q^N} on both sides, where $\mathcal{L}_\chi$ is the rank 1 local system on H corresponding to χ, the morphism $\beta\colon (G/H) \times H \to G$ is given by

$$\beta(x, h) = s(\mathrm{Fr}_q(x)) \cdot h \cdot s(x)^{-1},$$

and the morphism $P\colon \beta^{-1}(Y) \to H$ is the composition $\beta^{-1}(Y) \hookrightarrow (G/H) \times H \xrightarrow{\mathrm{pr}_2} H$.

First observe the following easy fact about the morphism β:

Lemma 4.1.2. *Let G be an algebraic group with a closed connected subgroup $H \subset G$ over $\mathbb{F}_q$ with Frobenius Fr_q. Assume that we have a section s of the quotient morphism $G \to G/H$. Define $\beta\colon (G/H) \times H \to G$ to be the morphism $\beta(x, h) = s(\mathrm{Fr}_q(x)) \cdot h \cdot s(x)^{-1}$. For any $(x, y) \in (G/H) \times H$ and any closed subscheme $Y \subset G$,*

$$(x, L_q(y)) \in \beta^{-1}(Y) \iff s(x) \cdot y \in L_q^{-1}(Y). \tag{4.1}$$

Moreover, $\beta^{-1}(Y)$ is affine.

Proof. We have $\beta(x, L_q(y)) = s(\mathrm{Fr}_q(x)) \cdot \mathrm{Fr}_q(y) \cdot y^{-1} \cdot s(x)^{-1} = L_q(s(x) \cdot y)$, which establishes (4.1). For the second assertion, consider the composition

$$G/H \times H \xrightarrow{(\mathrm{id}, L_q)} G/H \times H \xrightarrow{\beta} G.$$

Observe that this composition maps $(x, y) \mapsto s(\mathrm{Fr}_q(x)) \cdot L_q(y) \cdot s(x)^{-1} = L_q(s(x) \cdot y)$, so this composition is finite étale and surjective. The first map (id, L_q) is finite étale and surjective, and therefore β must also be finite étale and surjective. Hence $\beta^{-1}(Y)$ is affine. □

We will need a couple of lemmas in the proof of Proposition 4.1.1.

Lemma 4.1.3. *Let $\mathcal{F}$ be a pure rank-1 lisse $\overline{\mathbb{Q}}_\ell$-sheaf on a smooth affine curve X over $\mathbb{F}_q$. Let $\overline{X}$ be a compactification of X and let $j: X \to \overline{X}$. Then*

$$H_c^1(X, \mathcal{F}) \cong H^0(\overline{X} \smallsetminus X, j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F}) \oplus H^1(\overline{X}, j_*\mathcal{F})$$

as Fr_q*-vector spaces.*

Proof. The open/closed decomposition

$$j: X \hookrightarrow \overline{X} \hookleftarrow \overline{X} \smallsetminus X : i.$$

implies that we have the following short exact sequence of sheaves on $\overline{X}$:

$$0 \to j_!j^*(j_*\mathcal{F}) \to j_*\mathcal{F} \to i_*i^*(j_*\mathcal{F}) \to 0.$$

Since X is an affine curve, $H_c^0(X, \mathcal{F}) = 0$, so the first part of the induced long exact sequence is

$$0 \to H^0(\overline{X}, j_*\mathcal{F}) \to H^0(\overline{X} \smallsetminus X, j_*\mathcal{F}) \to H_c^1(X, \mathcal{F}) \to H^1(\overline{X}, j_*\mathcal{F}) \to 0,$$

which implies

$$0 \to H^0(\overline{X} \smallsetminus X, j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F}) \to H_c^1(X, \mathcal{F}) \to H^1(\overline{X}, j_*\mathcal{F}) \to 0. \tag{4.2}$$

We now show that this sequence splits. By Deligne's work on the Weil conjectures [8, Corollary 3.3.9 and Theorem 3.2.3], we know that $H^0(\overline{X} \smallsetminus X, j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F})$ has weight $\mathrm{wt}(j_*\mathcal{F})$ and $H^1(\overline{X}, j_*\mathcal{F})$ has weight $\mathrm{wt}(j_*\mathcal{F}) + 1$, and so the extension $H_c^1(X, j_*\mathcal{F})$ is classified by an element of $H^1(\widehat{\mathbb{Z}}, V)$, where

$$V := H^0(\overline{X} \smallsetminus X, j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F}) \otimes H^1(\overline{X}, j_*\mathcal{F})^*$$

has weight -1. Now,

$$H^1(\widehat{\mathbb{Z}}, V) = Z^1(\widehat{\mathbb{Z}}, V)/B^1(\widehat{\mathbb{Z}}, V),$$

where

$$Z^1(\widehat{\mathbb{Z}}, V) = \{f: \widehat{\mathbb{Z}} \to V : f(gh) = gf(h) + f(g) \text{ for all } g, h \in \widehat{\mathbb{Z}}\},$$
$$B^1(\widehat{\mathbb{Z}}, V) = \{f \in Z^1(\widehat{\mathbb{Z}}, V) : \text{there exists } v \in V \text{ such that } f(g) = ga - a \text{ for all } g \in \widehat{\mathbb{Z}}\}.$$

Observe that

$$Z^1(\widehat{\mathbb{Z}}, V) \cong V, \quad B^1(\widehat{\mathbb{Z}}, V) \cong (\mathrm{Fr}_q - 1)V.$$

Since V has weight -1, this implies that $(\mathrm{Fr}_q - 1)V = V$ and so

$$H^1(\widehat{\mathbb{Z}}, V) = 0.$$

Thus the short exact sequence in (4.2) splits, and the desired conclusion follows. □

Lemma 4.1.4. *Let $\mathcal{F}$ be a pure rank-1 lisse $\overline{\mathbb{Q}}_\ell$-sheaf on an affine curve X over $\mathbb{F}_q$. If S is a finite set of points in X, then the short exact sequence*

$$0 \to H_c^0(S, \mathcal{F}) \to H_c^1(X \smallsetminus S, \mathcal{F}) \to H_c^1(X, \mathcal{F}) \to 0$$

of Fr_q*-vector spaces splits.*

Proof. First assume that X is smooth. Let $\overline{X}$ be a compactification of X. Then it must also be a compactification of $X \smallsetminus S$. Therefore by Lemma 4.1.3,

$$\begin{aligned} H_c^1(X, \mathcal{F}) &\cong H^0(\overline{X} \smallsetminus X, j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F}) \oplus H^1(\overline{X}, j_*\mathcal{F}), \\ H_c^1(X \smallsetminus S, \mathcal{F}) &\cong H^0(S \cup (\overline{X} \smallsetminus X), j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F}) \oplus H^1(\overline{X}, j_*\mathcal{F}) \\ &\cong H^0(S, \mathcal{F}) \oplus H^0(\overline{X} \smallsetminus X, j_*\mathcal{F})/H^0(\overline{X}, j_*\mathcal{F}) \oplus H^1(\overline{X}, j_*\mathcal{F}) \\ &\cong H_c^0(S, \mathcal{F}) \oplus H_c^1(X, \mathcal{F}). \end{aligned}$$

Thus the desired short exact sequence splits.

Now assume that X is not smooth and let $\widetilde{X}$ be the normalization of X. Then $\widetilde{X}$ is a smooth affine curve and by the previous paragraph, the upper short exact sequence of Fr_q-vector spaces of the following commutative diagram splits:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_c^0(\widetilde{S}, \mathcal{F}) & \longrightarrow & H_c^1(\widetilde{X} \smallsetminus \widetilde{S}, \mathcal{F}) & \longrightarrow & H_c^1(\widetilde{X}, \mathcal{F}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H_c^0(S, \mathcal{F}) & \longrightarrow & H_c^1(X \smallsetminus S, \mathcal{F}) & \longrightarrow & H_c^1(X, \mathcal{F}) & \longrightarrow & 0, \end{array}$$

where we view $\mathcal{F}$ as a sheaf on $\widetilde{X}$ by pulling back along the morphism $\widetilde{X} \to X$, and this morphism induces the upward vertical maps. Restriction gives the dotted downward arrow. The composition

$$H_c^1(X \smallsetminus S, \mathcal{F}) \to H_c^1(\widetilde{X} \smallsetminus \widetilde{S}, \mathcal{F}) \to H_c^0(\widetilde{S}, \mathcal{F}) \to H_c^0(S, \mathcal{F})$$

gives a splitting of Fr_q-vector spaces of the lower short exact sequence. □

We are now ready to prove Proposition 4.1.1.

Proof of Proposition 4.1.1. For convenience, set $V_\chi := \mathrm{Ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\chi)$. Define Y^+ to be the closed reduced subscheme of G such that

$$Y^+(\overline{\mathbb{F}}_q) = Y(\overline{\mathbb{F}}_q) \cup \mathrm{Fr}_q(Y(\overline{\mathbb{F}}_q)) \cup \cdots \cup \mathrm{Fr}_q^{N-1}(Y(\overline{\mathbb{F}}_q)) \subset G(\overline{\mathbb{F}}_q).$$

Note Y^+ is defined over $\mathbb{F}_q$. By [2, Proposition 2.3], we have Fr_q-compatible vector-space isomorphisms

$$\mathrm{Hom}_{G(\mathbb{F}_q)}\big(V_\chi, H_c^i(X^+, \overline{\mathbb{Q}}_\ell)\big) \cong H_c^i(\beta^{-1}(Y^+), (P^+)^*\mathcal{L}_\chi) \quad \text{for all } i \geq 0, \tag{4.3}$$

where $X^+ = L_q^{-1}(Y^+)$ and P^+ is the composition $\beta^{-1}(Y^+) \hookrightarrow (G/H) \times H \to H$. We will use open/closed decompositions and the associated long exact sequences to relate the cohomology of X^+ to the cohomology of X, and the cohomology of $\beta^{-1}(Y^+)$ to the cohomology of $\beta^{-1}(Y)$.

Let $Y'^+ := Y^+ \smallsetminus S$. The open/closed decomposition

$$\beta^{-1}(Y'^+) \overset{j}{\hookrightarrow} \beta^{-1}(Y^+) \overset{i}{\hookleftarrow} \beta^{-1}(S)$$

induces a short exact sequence of sheaves

$$0 \to j_! j^*(P^+)^*\mathcal{L}_\chi \to (P^+)^*\mathcal{L}_\chi \to i_* i^*(P^+)^*\mathcal{L}_\chi \to 0$$

whose associated long exact sequence in cohomology gives rise to Fr_q-compatible isomorphisms

$$H_c^i(\beta^{-1}(Y'^+), \mathcal{F}^+) \cong H_c^i(\beta^{-1}(Y^+), \mathcal{F}^+) \quad \text{for } i \geq 2, \tag{4.4}$$

and an exact sequence

$$\begin{aligned} 0 \to H_c^0(\beta^{-1}(Y'^+), \mathcal{F}^+) \to H_c^0(\beta^{-1}(Y^+), \mathcal{F}^+) \to H_c^0(\beta^{-1}(S), \mathcal{F}^+) \\ \to H_c^1(\beta^{-1}(Y'^+), \mathcal{F}^+) \to H_c^1(\beta^{-1}(Y^+), \mathcal{F}^+) \to H_c^1(\beta^{-1}(S), \mathcal{F}^+) = 0. \end{aligned} \tag{4.5}$$

Here, we write for convenience

$$\mathcal{F}^+ := (P^+)^* \mathcal{L}_\chi$$

and abuse notation by writing $\mathcal{F}^+$ for $(P^+|_{\beta^{-1}(Y'^+)})^* \mathcal{L}_\chi$ and $(P^+|_{\beta^{-1}(S)})^* \mathcal{L}_\chi$ as well. The same argument applied to the $\mathbb{F}_{q^N}$-scheme $Y' := Y \smallsetminus S$ shows that we have Fr_{q^N}-compatible isomorphisms

$$H_c^i(\beta^{-1}(Y'), P^* \mathcal{L}_\chi) \cong H_c^i(\beta^{-1}(Y), P^* \mathcal{L}_\chi) \quad \text{for } i \geq 2, \tag{4.6}$$

and an exact sequence

$$\begin{aligned} 0 \to H_c^0(\beta^{-1}(Y'), \mathcal{F}) \to H_c^0(\beta^{-1}(Y), \mathcal{F}) \to H_c^0(\beta^{-1}(S), \mathcal{F}) \\ \to H_c^1(\beta^{-1}(Y'), \mathcal{F}) \to H_c^1(\beta^{-1}(Y), \mathcal{F}) \to H_c^1(\beta^{-1}(S), \mathcal{F}) = 0, \end{aligned} \tag{4.7}$$

where $P = P^+|_{\beta^{-1}(Y)}$ and we write $\mathcal{F} := P^* \mathcal{L}_\chi$ and abuse notation as before.

Applying the same argument to $X'^+ := X^+ \smallsetminus L_q^{-1}(S)$ and $X' := X \smallsetminus L_q^{-1}(S)$ together with the constant sheaf $\overline{\mathbb{Q}}_\ell$ gives Fr_q-compatible isomorphisms

$$H_c^i(X'^+, \overline{\mathbb{Q}}_\ell) \cong H_c^i(X^+, \overline{\mathbb{Q}}_\ell) \quad \text{for } i \geq 2, \tag{4.8}$$

exact sequences

$$\begin{aligned} 0 \to H_c^0(X'^+, \overline{\mathbb{Q}}_\ell) \to H_c^0(X^+, \overline{\mathbb{Q}}_\ell) \to H_c^0(L_q^{-1}(S), \overline{\mathbb{Q}}_\ell) \\ \to H_c^1(X'^+, \overline{\mathbb{Q}}_\ell) \to H_c^1(X^+, \overline{\mathbb{Q}}_\ell) \to H_c^1(L_q^{-1}(S), \overline{\mathbb{Q}}_\ell) = 0, \end{aligned} \tag{4.9}$$

Fr_{q^N}-compatible isomorphisms

$$H_c^i(X', \overline{\mathbb{Q}}_\ell) \cong H_c^i(X, \overline{\mathbb{Q}}_\ell) \quad \text{for } i \geq 2, \tag{4.10}$$

and exact sequences

$$\begin{aligned} 0 \to H_c^0(X', \overline{\mathbb{Q}}_\ell) \to H_c^0(X, \overline{\mathbb{Q}}_\ell) \to H_c^0(L_q^{-1}(S), \overline{\mathbb{Q}}_\ell) \\ \to H_c^1(X', \overline{\mathbb{Q}}_\ell) \to H_c^1(X, \overline{\mathbb{Q}}_\ell) \to H_c^1(L_q^{-1}(S), \overline{\mathbb{Q}}_\ell) = 0. \end{aligned} \tag{4.11}$$

By construction, $F^i(\beta^{-1}(Y')) \cap F^j(\beta^{-1}(Y')) = \varnothing$ for all $0 \leq i \neq j \leq N-1$, and hence we have Fr_{q^N}-compatible isomorphisms

$$H_c^i(\beta^{-1}(Y'^+), \mathcal{F}^+) \cong \bigoplus_{j=0}^{N-1} H_c^i\big(\mathrm{Fr}_q^j(\beta^{-1}(Y')), \mathcal{F}^+|_{\mathrm{Fr}_q^j(\beta^{-1}(Y'))}\big)$$

for all $i \geq 0$. For any $0 \leq j \leq N-1$, we have a commutative diagram

$$\begin{array}{ccccc}
 & & \overset{P}{\frown} & & \\
\beta^{-1}(Y') & \hookrightarrow & \beta^{-1}(Y'^{+}) & \xrightarrow{P^{+}} & H \\
\downarrow{\scriptstyle \mathrm{Fr}_q^j} & & \downarrow{\scriptstyle \mathrm{Fr}_q^j} & & \downarrow{\scriptstyle \mathrm{Fr}_q^j} \\
\mathrm{Fr}_q^j(\beta^{-1}(Y')) & \hookrightarrow & \beta^{-1}(Y'^{+}) & \xrightarrow{P^{+}} & H
\end{array}$$

and so it follows that $\mathcal{F}^{+}|_{\mathrm{Fr}_q^j(\beta^{-1}(Y'))} = (\mathrm{Fr}_q^j)_*\mathcal{F}$ and

$$H_c^i(\beta^{-1}(Y'), \mathcal{F}) = H_c^i\big(\mathrm{Fr}_q^j(\beta^{-1}(Y')), \mathcal{F}^{+}|_{\mathrm{Fr}_q^j(\beta^{-1}(Y'))}\big).$$

Thus

$$H_c^i(\beta^{-1}(Y'^{+}), \mathcal{F}^{+}) \cong H_c^i(\beta^{-1}(Y'), \mathcal{F})^{\oplus N} \quad \text{for all } i \geq 0.$$

We first prove the proposition for $i \geq 2$. By equations (4.4) and (4.6), for all $i \geq 2$, we have

$$\begin{aligned} H_c^i(\beta^{-1}(Y^{+}), \mathcal{F}^{+}) &\cong H_c^i(\beta^{-1}(Y'^{+}), \mathcal{F}^{+}) \cong H_c^i(\beta^{-1}(Y'), \mathcal{F})^{\oplus N} \\ &\cong H_c^i(\beta^{-1}(Y), \mathcal{F})^{\oplus N}. \end{aligned}$$

Similarly, by equations (4.8) and (4.10), for all $i \geq 2$, we have

$$H_c^i(X^{+}, \overline{\mathbb{Q}}_\ell) \cong H_c^i(X'^{+}, \overline{\mathbb{Q}}_\ell) \cong H_c^i(X', \overline{\mathbb{Q}}_\ell)^{\oplus N} \cong H_c^i(X, \overline{\mathbb{Q}}_\ell)^{\oplus N}.$$

Combining this with equation (4.3), we have Fr_{q^N}-compatible vector-space isomorphisms

$$\begin{aligned} \mathrm{Hom}_{G(\mathbb{F}_q)}\big(V_\chi, H_c^i(X, \overline{\mathbb{Q}}_\ell)\big)^{\oplus N} &\cong \mathrm{Hom}_{G(\mathbb{F}_q)}\big(V_\chi, H_c^i(X^{+}, \overline{\mathbb{Q}}_\ell)\big) \\ &\cong H_c^i(\beta^{-1}(Y^{+}), P^{*}\mathcal{L}_\chi) \\ &\cong H_c^i(\beta^{-1}(Y), P^{*}\mathcal{L}_\chi)^{\oplus N} \qquad \text{for all } i \geq 2. \end{aligned}$$

It remains to prove the proposition for $i = 0, 1$. Since X is an affine scheme, we see that

$$H_c^0(X, \overline{\mathbb{Q}}_\ell) \neq 0 \implies \dim X = 0 \quad \text{and} \quad H_c^1(X, \overline{\mathbb{Q}}_\ell) \neq 0 \implies \dim X = 1.$$

First observe that S is defined over $\mathbb{F}_q$ and by [2, Proposition 2.3], we have

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(V_\chi, H_c^0(L_q^{-1}(S), \overline{\mathbb{Q}}_\ell)) \cong H_c^0(\beta^{-1}(S), \mathcal{F}) \tag{4.12}$$

as Fr_q-vector spaces. We will use this to complete the proof.

Assume $\dim X = 0$. Then $\dim Y = 0$, and by Lemma 4.1.2, we have $\dim \beta^{-1}(Y) = 0$ and $\dim \beta^{-1}(S) = 0$. By equations (4.5), (4.7), (4.9), and (4.11), we have

$$H_c^0(\beta^{-1}(Y^{+}), \mathcal{F}^{+}) \cong H_c^0(\beta^{-1}(Y'^{+}), \mathcal{F}^{+}) \oplus H_c^0(\beta^{-1}(S), \mathcal{F}^{+}),$$

and similarly for $\beta^{-1}(Y)$, X^{+}, and X. Thus we have

$$\begin{aligned} &\mathrm{Hom}_{G(\mathbb{F}_q)}\big(V_\chi, H_c^0(X', \overline{\mathbb{Q}}_\ell)\big)^{\oplus N} \oplus \mathrm{Hom}_{G(\mathbb{F}_q)}\big(V_\chi, H_c^0(L_q^{-1}(S), \overline{\mathbb{Q}}_\ell)\big) \\ &\quad\cong \mathrm{Hom}_{G(\mathbb{F}_q)}\big(V_\chi, H_c^0(X^{+}, \overline{\mathbb{Q}}_\ell)\big) \\ &\quad\cong H_c^0(\beta^{-1}(Y^{+}), \mathcal{F}^{+}) \\ &\quad\cong H_c^0(\beta^{-1}(Y'), \mathcal{F})^{\oplus N} \oplus H_c^0(\beta^{-1}(S), \mathcal{F}). \end{aligned}$$

The last summand of the first and last lines are isomorphic by equation (4.12). Therefore as Fr_{q^N}-vector spaces,

$$\mathrm{Hom}_{G(\mathbb{F}_q)}(V_\chi, H_c^0(X', \overline{\mathbb{Q}}_\ell)) \cong H_c^0(\beta^{-1}(Y'), \mathcal{F}),$$

and the desired conclusion now follows.

Finally, assume that $\dim X = 1$. Since X is an affine variety, equations (4.5), (4.7), (4.9), and (4.11) reduce to short exact sequences, and by Lemma 4.1.4,

$$H_c^1(\beta^{-1}(Y'^+), \mathcal{F}^+) \cong H_c^1(\beta^{-1}(Y^+), \mathcal{F}^+) \oplus H_c^0(\beta^{-1}(S), \mathcal{F}),$$

and similarly for $\beta^{-1}(Y)$, X^+, and X. Using equation (4.12), we have Fr_{q^N}-vector space isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{G(\mathbb{F}_q)}\left(V_\chi, H_c^1(X, \overline{\mathbb{Q}}_\ell)\right)^{\oplus N} \oplus H_c^0(\beta^{-1}(S), \mathcal{F})^{\oplus N} \\
&\quad\cong \mathrm{Hom}_{G(\mathbb{F}_q)}\left(V_\chi, H_c^1(X', \overline{\mathbb{Q}}_\ell)\right)^{\oplus N} \\
&\quad\cong \mathrm{Hom}_{G(\mathbb{F}_q)}\left(V_\chi, H_c^1(X'^+, \overline{\mathbb{Q}}_\ell)\right) \\
&\quad\cong H_c^1(\beta^{-1}(Y'^+), \mathcal{F}^+) \\
&\quad\cong H_c^1(\beta^{-1}(Y'), \mathcal{F})^{\oplus N} \\
&\quad\cong H_c^1(\beta^{-1}(Y), \mathcal{F})^{\oplus N} \oplus H_c^0(\beta^{-1}(S), \mathcal{F})^{\oplus N},
\end{aligned}$$

and this completes the proof of the proposition. □

4.2. An inductive strategy for calculating cohomology. In [2], Boyarchenko formulated an inductive strategy for calculating the cohomology of (pullbacks of) Artin–Schreier local systems $P^*\mathscr{F}$ arising from (a simplification of) Proposition 4.1.1. The morphisms P in Boyarchenko's setting (see Theorem 5.20 and more specifically Lemma 6.18 of [2]) are such that at each inductive step, they are of the form

$$P(x, y) = \eta(x, y) + P_2(x)$$

for $\eta(x, y) = f(x)^{q^j} y - f(x)^{q^n} y^{q^{n-j}}$ so that [2, Proposition 2.10] applies. This then allows one to write $H_c^i(-, P^*\mathscr{F})$ in terms of the cohomology of either $\mathbb{G}_a$ or a point (depending on a parity issue).

We will execute this strategy in Section 5 to calculate $H_c^i(-, P^*\mathscr{F})$ except that our situation is more complicated in two ways: $\mathscr{F}$ is a multiplicative local system on $\mathbb{W}_h^{(1)}$ (rather than just on $\mathbb{G}_a$, as in the Artin–Schreier setting) and P is more complicated than in Boyarchenko's setting. We therefore need a generalization of [2, Proposition 2.10].

We remark that in previous works [5, 6], the relevant sheaves $P^*\mathscr{F}$ were pullbacks of Artin–Schreier sheaves, but the η were slightly more complicated than Boyarchenko's. These previous generalizations of [2, Proposition 2.10] are [5, Proposition 4.4] and [6, Propositions 3.4, 3.5]. The proofs of these all use essentially the same idea, so we have written down a general statement (Proposition 4.2.1) that applies to $P^*\mathscr{F}$, where $\mathscr{F}$ is a general multiplicative local system and η can be arbitrarily complicated. We then specialize this (Proposition 4.2.2) to the particular class of $\mathscr{F}$ and η we will need in Section 5.

Proposition 4.2.1. *Let G, G' be algebraic groups over $\mathbb{F}_q$, let $\mathcal{F}$ be a multiplicative local system on G, and consider a filtration of finite type $\mathbb{F}_q$-schemes $S_3 \subset S_2 \subset S$. Assume*

that $H_c^i(G', \overline{\mathbb{Q}}_\ell)$ is concentrated in a single degree $i = r$ and assume that $S = S_2 \times G'$. Let $\eta\colon S_2 \times G' \to G$ and $P_2\colon S_2 \to G$ be any morphisms, set $P_3 := P_2|_{S_3}$, and define

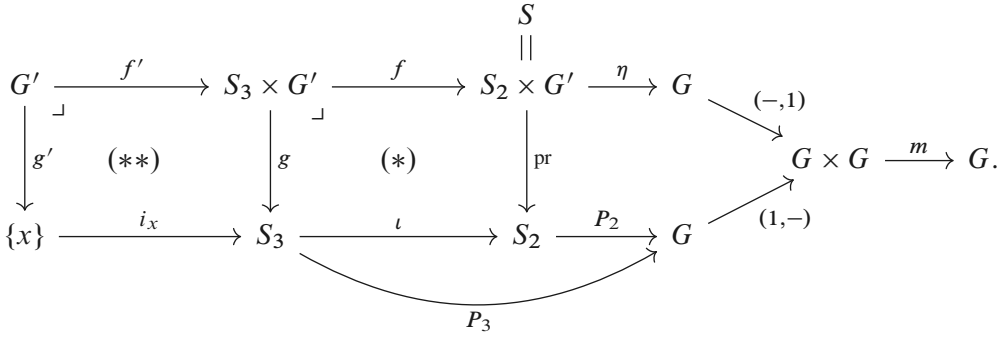
$$P\colon S = S_2 \times G' \to G, \quad (x, g) \mapsto \eta(x, g) \cdot P_2(x),$$
$$\eta_x\colon G' \to G, \qquad g \mapsto \eta(x, g) \qquad \text{for } x \in S_2.$$

If η has the property that $\eta^*\mathcal{F}|_{S_3 \times G'}$ is the constant local system and $\eta_x^*\mathcal{F}$ is a nontrivial multiplicative local system on G' for every $x \in S_2(\overline{\mathbb{F}}_q) \smallsetminus S_3(\overline{\mathbb{F}}_q)$, then for all $i \in \mathbb{Z}$,

$$H_c^i(S, P^*\mathcal{F}) \cong H_c^{i-r}(S_3, P_3^*\mathcal{F} \otimes \underline{H_c^r(G', \overline{\mathbb{Q}}_\ell)})$$

as Fr_q*-vector spaces, where $\underline{H_c^r(G', \overline{\mathbb{Q}}_\ell)}$ is a constant sheaf.*

Proof. Consider the following commutative diagram, where $(*)$ and $(**)$ are Cartesian squares and x is any point in $S_3(\overline{\mathbb{F}}_q)$:



By construction,

$$P^*\mathcal{F} \cong (\eta^*\mathcal{F}) \otimes \mathrm{pr}^*(P_2^*\mathcal{F}),$$

hence by the projection formula,

$$R\,\mathrm{pr}_!(P^*\mathcal{F}) \cong P_2^*\mathcal{F} \otimes R\,\mathrm{pr}_!(\eta^*\mathcal{F}) \quad \text{in } D_c^b(S_2, \overline{\mathbb{Q}}_\ell).$$

Since $\eta^*\mathcal{F}|_{S_3 \times G'} = \overline{\mathbb{Q}}_\ell$ by assumption, the proper base change theorem applied to $(*)$ implies that

$$\iota^* R\,\mathrm{pr}_!(\eta^*\mathcal{F}) \cong Rg_!f^*(\eta^*\mathcal{F}) = Rg_!(\overline{\mathbb{Q}}_\ell).$$

For any $x \in S_3(\overline{\mathbb{F}}_q)$, the proper base change theorem applied to $(**)$ implies that

$$(R^i g_!\overline{\mathbb{Q}}_\ell)_x = i_x^*(R^i g_!\overline{\mathbb{Q}}_\ell) = R^i g'_!f'^*\overline{\mathbb{Q}}_\ell = R^i g'_!\overline{\mathbb{Q}}_\ell = H_c^i(G', \overline{\mathbb{Q}}_\ell).$$

It therefore follows that

$$\iota^* R^i\,\mathrm{pr}_!(\eta^*\mathcal{F}) \cong \underline{H_c^i(G', \overline{\mathbb{Q}}_\ell)},$$

where the right-hand side is a constant sheaf on S_3.

We now show that $R^i\,\mathrm{pr}_!(\eta^*\mathcal{F})$ is supported on S_3. Now let $x \in S_2(\overline{\mathbb{F}}_q) \smallsetminus S_3(\overline{\mathbb{F}}_q)$. Then by the proper base change theorem applied to the Cartesian square

$$\begin{array}{ccc} G' & \xrightarrow{f''} & S_2 \times G' \\ {\scriptstyle g''}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{pr}} \\ \{x\} & \xrightarrow[i_x]{} & S_2 \end{array}$$

we have

$$i_x^* Rg_!(\eta^*\mathcal{F}) \cong Rg''_!f''^*(\eta^*\mathcal{F}) \cong Rg''_!(\eta_x^*\mathcal{F}).$$

On the other hand, by [1, Lemma 9.4], we have

$$H_c^i(G', \eta_x^* \mathcal{F}) = 0 \quad \text{for all } i \geq 0.$$

Therefore

$$(R^i g_!(\eta^* \mathcal{F}))_x \cong R^i g_!''(\eta_x^* \mathcal{F}) = H_c^i(G', \eta_x^* \mathcal{F}) = 0 \quad \text{for all } i \geq 0.$$

Combining all statements above, we obtain

$$R^i \operatorname{pr}_!(P^* \mathcal{F}) \cong P_3^* \mathcal{F} \otimes \underline{H_c^i(G', \overline{\mathbb{Q}}_\ell)}. \qquad \square$$

In this paper, the most important application of Proposition 4.2.1 is Proposition 4.2.2, which we will use repeatedly to prove Theorem 5.1.1.

Proposition 4.2.2. *Let $\chi\colon \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell^\times$ be a character of conductor m and let $\mathcal{L}_\chi$ be the corresponding local system on $\mathbb{W}_h^{(1)}$. Let S_2 be a finite-type scheme over $\mathbb{F}_{q^n}$, put $S = S_2 \times \mathbb{A}^1$ and suppose that a morphism $P\colon S \to \mathbb{W}_h^{(1)}$ has the form*

$$P(x, y) = g(y^{q^{j+\tau}}(f(x)^{q^n} - f(x))^{q^\sigma} + f(x)^{q^{n-j+\sigma}}(y^{q^n} - y)^{q^\tau}) \cdot P_2(x),$$

where

- *$j, \sigma, \tau \geq 0$ are integers and m does not divide j,*
- *$f\colon S_2 \to \mathbb{A}^1$ and $P_2\colon S_2 \to \mathbb{W}_h^{(1)}$ are morphisms defined over $\mathbb{F}_{q^n}$, and*
- *$g\colon \mathbb{A}^1 \to \mathbb{W}_h^{(1)}$ is the morphism $z \mapsto (1, 0, \ldots, 0, z)$.*

Let $S_3 \subset S_2$ be the subscheme defined by $f(x)^{q^n} - f(x) = 0$ and let $P_3 = P_2|_{S_3}\colon S_3 \to \mathbb{W}_h^{(1)}$. Then for all $i \in \mathbb{Z}$, we have

$$H_c^i(S, P^* \mathcal{L}_\chi) \cong H_c^{i-2}(S_3, P_3^* \mathcal{L}_\chi)(-1)$$

as vector spaces equipped with an action of Fr_{q^n}.

Proof. It is clear that the conclusion holds if we can apply Proposition 4.2.1 to the situation when:

- $G = \mathbb{W}_h^{(1)}$ and $G' = \mathbb{A}^1 = \mathbb{G}_a$,
- $\mathcal{F} = \mathcal{L}_\chi$,
- $\eta\colon S_2 \times \mathbb{A}^1 \to H$ is $(x, y) \mapsto g(y^{q^{j+\tau}}(f(x)^{q^n} - f(x))^{q^\sigma} + f(x)^{q^{n-j+\sigma}}(y^{q^n} - y)^{q^\tau})$.

To this end, we must verify that:

(a) $\eta^* \mathcal{L}_\chi|_{S_3 \times \mathbb{A}^1}$ is the constant local system.

(b) For any $x \in S_2(\overline{\mathbb{F}}_q) \smallsetminus S_3(\overline{\mathbb{F}}_q)$, the sheaf $\eta_x^* \mathcal{L}_\chi$ is a nontrivial local system on $\mathbb{W}_h^{(1)}$.

Part (a) is clear since $S_3 = \mathbb{V}(f^{q^n} - f)$ by definition and χ is a character of $\mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})$. To see (b), we use the approach of [2, Section 6.4]. First, note that $\eta = g \circ \eta_0$, where

$$\eta_0\colon S_2 \times \mathbb{A}^1 \to \mathbb{A}^1 = \mathbb{G}_a, \quad (x, y) \mapsto y^{q^{j+\tau}}(f(x)^{q^n} - f(x))^{q^\sigma} + f(x)^{q^{n-j+\sigma}}(y^{q^n} - y)^{q^\tau},$$

and hence

$$\eta^* \mathcal{L}_\chi \cong \eta_0^* \mathcal{L}_\psi, \quad \text{where } \psi = \chi|_{\{(1,0,\ldots,0,*) \in \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})\}}.$$

Note that $\mathcal{L}_\psi$ is a multiplicative local system on $\mathbb{G}_a$. Fix an auxiliary nontrivial additive character $\psi_0\colon \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$, and for any $z \in \overline{\mathbb{F}}_p$, define

$$\mathcal{L}_z := m_z^* \mathcal{L}_{\psi_0}, \quad \text{where } m_z\colon \mathbb{G}_a \to \mathbb{G}_a \text{ is the map } x \mapsto xz.$$

Then there exists a unique $z \in \overline{\mathbb{F}}_q$ such that $\mathcal{L}_\psi = \mathcal{L}_z$. Since ψ has conductor q^m, the stabilizer of z in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is exactly $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_{q^m})$, and hence $z \in \mathbb{F}_{q^m}$. By [2, Corollary 6.5], we have $\eta_x^* \mathcal{L}_\psi \cong \mathcal{L}_{\eta_x^*(z)}$, where

$$\begin{aligned}
\eta_x^*(z) &= f(x)^{q^{n+\sigma-j-\tau}} z^{q^{-j-\tau}} - f(x)^{q^{\sigma-j-\tau}} z^{q^{-j-\tau}} \\
&\quad + f(x)^{q^{n-j+\sigma-n-\tau}} z^{q^{-n-\tau}} - f(x)^{q^{n-j+\sigma-\tau}} z^{q^{-\tau}} \\
&= f(x)^{q^{n-j+\sigma-\tau}} (z^{q^{-j-\tau}} - z^{q^{-\tau}}) - f(x)^{q^{-j+\sigma-\tau}} (z^{q^{-j-\tau}} - z^{q^{-n-\tau}}) \\
&= (f(x)^{q^{n-j+\sigma-\tau}} - f(x)^{q^{-j+\sigma-\tau}})(z^{q^{-j-\tau}} - z^{q^{-\tau}}).
\end{aligned}$$

But this is nonzero by assumption, so $\eta_x^* \mathcal{L}_\psi$ is a nontrivial local system on $\mathbb{G}_a$. □

5. Morphisms to the cohomology

In this section we prove a theorem calculating the space of homomorphisms

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big).$$

This result is crucial to the proofs of many of the theorems in Section 6. The finale of the proof of Theorem 5.1.1 is in Section 5.4. Throughout this section, for $x \in \mathbb{A}[\mathcal{A}^+]$, we write $x_{(i,j,k)}$ to mean the coordinate of x corresponding to $(i, j, k) \in \mathcal{A}^+$.

5.1. Nonvanishing in a single degree: Statements of results. Recall from Section 2.4 that the Howe factorization attaches to any character of $\mathbb{T}_{h,k}(\mathbb{F}_q) \cong \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})$ two sequences of integers

$$\begin{aligned}
1 =: m_0 \le m_1 < m_2 < \cdots < m_r \le m_{r+1} := n \\
h =: h_0 = h_1 > h_2 > \cdots > h_r \ge h_{r+1} := 1
\end{aligned}$$

satisfying the divisibility $m_i \mid m_{i+1}$ for $0 \le i \le r$.

Theorem 5.1.1. *For any character $\chi\colon \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big) = \begin{cases} \overline{\mathbb{Q}}_\ell^{\oplus q^{n d_\chi/2}} \otimes ((-q^{n/2})^{r_\chi})^{\deg} & \text{if } i = r_\chi, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned}
d_\chi &= \sum_{t=1}^{r+1} \left(\frac{n}{m_{t-1}} - \frac{n}{m_t}\right)(h_t - 1), \\
r_\chi &= \sum_{t=1}^{r+1} \left(\left(\frac{n}{m_{t-1}} - \frac{n}{m_t}\right)(h_t - 1) + 2\left(\frac{n}{m_{t-1}} - 1\right)(h_{t-1} - h_t)\right).
\end{aligned}$$

Moreover, Fr_{q^n} *acts on* $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ *by multiplication by the scalar* $(-1)^i q^{ni/2}$.

Notation 5.1.2. Following Katz [17], we write $\alpha^{\deg}$ to mean the rank-1 $\overline{\mathbb{Q}}_\ell$-sheaf with the action of Fr_{q^n} given by multiplication by α. Given an algebraic group $\mathbb{G}$ with $\mathbb{F}_q$-rational structure given by some Frobenius Fr_q, we let $L_q\colon \mathbb{G} \to \mathbb{G}$ denote the Lang morphism $x \mapsto \mathrm{Fr}_q(x)x^{-1}$. We will abuse notation by using L_q to denote the Lang morphism for different $\mathbb{G}$; we hope the meaning is still clear from the context.

Before we prove Theorem 5.1.1, we first note an easy but crucial consequence.

Corollary 5.1.3. *Let π be an irreducible constituent of $H_c^r(X_h, \overline{\mathbb{Q}}_\ell)$ for some r. Then*

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\pi, H_c^i(X_h, \overline{\mathbb{Q}}_\ell)) = 0 \quad \textit{for all } i \neq r.$$

Proof. The representation π of $\mathbb{U}_{h,k}(\mathbb{F}_q)$ is a constituent of $\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi)$ for some χ. Hence

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^r(X_h, \overline{\mathbb{Q}}_\ell)\big) \neq 0.$$

By Theorem 5.1.1, it follows that $r = r_\chi$ and

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big) = 0 \quad \text{for all } i \neq r_\chi.$$

The desired conclusion now holds. □

Remark 5.1.4. The arguments presented in this section can be applied to more general contexts. For example, observe that once an analogue of the Section 5.2 is established, the general result Proposition 4.1.1 reduces Theorem 5.1.1 to a statement about the cohomology groups of a pullback $P^*\mathcal{L}_\chi$ of an Artin–Schreier sheaf to an affine space S which can then be calculated using an inductive method on certain linear fibrations of S. To achieve this in the proof of Theorem 5.1.1, we use the carefully defined indexing sets $\mathcal{A}_{s,t}$ determined by a Howe decomposition of the character χ, where the main features of these indexing sets are Lemmas 3.2.3 and 3.2.6.

It is possible to axiomatize the main steps of the proof of Theorem 5.1.1 so that we specify exactly to what context one can apply the proof of Propositions 5.3.1 and 5.3.2, which would allow us to reduce the problem of computing the cohomology of certain subschemas of unipotent groups to a combinatorial problem about defining indexing sets analogous to $\mathcal{A}_{s,t}$. However, we choose to forgo this approach as we feel the present exposition is clearer and more explicit.

We will prove Theorem 5.1.1 by combining Proposition 5.2.3 with a calculation of the cohomology groups of the Artin–Schreier sheaves $P^*\mathcal{L}_\chi$. This calculation is driven by two ideas: the first (Proposition 5.3.2) is an inductive argument that can be viewed as an instance of the techniques established in Section 4, and the second (Proposition 5.3.1) comes from factoring the morphism P through appropriate Lang maps (see Section 5.2).

5.2. Compatibility of the morphism P with the Howe factorization.

5.2.1. Induced representations and cohomology. We make the first reduction in the calculation of $\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell))$ by using Proposition 4.1.1 to relate this space to the cohomology of the pullback of Artin–Schreier sheaves along certain mor-

phisms P. To make this precise, first recall that over $\mathbb{F}_q$, the quotient $\mathbb{U}_{h,k}/\mathbb{T}_{h,k}$ of affine schemes can be identified with the affine space $\mathbb{A}[\mathcal{A}]$. There is a section of the quotient map $\mathbb{U}_{h,k} \to \mathbb{U}_{h,k}/\mathbb{T}_{h,k}$ given by

$$s\colon (x_{(i,j,l)})_{(i,j,l)\in\mathcal{A}} \mapsto x_1 + x_2\varpi^{[k]} + \cdots + x_n\varpi^{[k(n-1)]},$$

where x_1 is the $n \times n$ identity matrix and for $j = 2, \ldots, n$,

$$x_j = \operatorname{diag}(x_{1,j}, x_{2,[j+1]}, \ldots, x_{n,[j+n-1]}), \quad x_{i,j} = [x_{(i,j,1)}, \ldots, x_{(i,j,h-1)}].$$

Since the $\mathbb{F}_{q^n}$-subscheme $X_h \subset \mathbb{U}_{h,k}$ is stable under the action of $\mathbb{U}_{h,k}(\mathbb{F}_q)$, there exists a closed $\mathbb{F}_{q^n}$-subscheme $Y_h \subset \mathbb{U}_{h,k}$ such that $X_h = L_q^{-1}(Y_h)$. Since Y_h is a closed subscheme of an affine scheme, it must be affine. Moreover, by Lemma 2.2.4, we may take Y_h to be the reduced subscheme such that $Y_h(\overline{\mathbb{F}}_q) = \widetilde{U}_h \cap \operatorname{Fr}_q(\widetilde{U}_h^-)$. Hence we see that

$$\operatorname{Fr}_q^i(Y_h) \cap \operatorname{Fr}_q^j(Y_h) = \{1\}$$

for all $i \neq j$.

Define

$$\beta\colon (\mathbb{U}_{h,k}/\mathbb{T}_{h,k}) \times \mathbb{T}_{h,k} \to \mathbb{U}_{h,k}, \quad (x, g) \mapsto s(F(x)) \cdot g \cdot s(x)^{-1}.$$

The $\mathbb{F}_{q^n}$-scheme $\beta^{-1}(Y_h) \subset (\mathbb{U}_{h,k}/\mathbb{T}_{h,k}) \times \mathbb{T}_{h,k}$ comes with two maps:

$$\mathrm{pr}_1\colon \beta^{-1}(Y_h) \to \mathbb{U}_{h,k}/\mathbb{T}_{h,k} = \mathbb{A}[\mathcal{A}], \quad \mathrm{pr}_2\colon \beta^{-1}(Y_h) \to \mathbb{T}_{h,k}.$$

The Lang morphism L_q is surjective. By Lemma 4.1.2, for any $y \in \mathbb{T}_{h,k}$ such that $L_q(y) = g$,

$$(x, g) \in \beta^{-1}(Y_h) \iff s(x) \cdot y \in X_h.$$

Lemma 5.2.1. *The scheme $\beta^{-1}(Y_h)$ is the graph of $P\colon \mathrm{pr}_1(\beta^{-1}(Y_h)) \to \mathbb{W}_h^{(1)}$, where P is the restriction of the map $\mathbb{A}[\mathcal{A}] \to \mathbb{W}_h^{(1)}$ given by*

$$x \mapsto L_q(\det(s(x)))^{-1} = \varphi(\det(s(x))) \cdot \det(s(x))^{-1}.$$

Proof. Let $x \in \mathbb{A}[\mathcal{A}]$ and $y \in \mathbb{T}_{h,k}$ be such that $s(x) \cdot y \in X_h$. This implies that we have

$$L_q(\det(s(x))) \cdot L_q(\det(y)) = L_q(\det(s(x) \cdot y)) = (1, 0, \ldots, 0) \in \mathbb{W}_h^{(1)},$$

and hence we see that $L_q(\det(s(x)))^{-1} = L_q(\det(y))$. In the remainder of the proof, we show that $L_q(\det(y)) = L_{q^n}(y_1)$ for some $y_1 \in \mathbb{W}_h^{(1)}$ and that $L_q(y)$ is determined by y_1.

By construction, the entries along the diagonal of $s(x) \cdot y$ are $y_1, \ldots, y_n \in \mathbb{W}_h^{(1)}$ and by Definition 2.3.3,

$$y_i = \varphi^{\tau(i)}(y_1) \quad \text{for } 1 \leq i \leq n.$$

Thus the ith coordinate of

$$L_q(y) = \operatorname{diag}(\varphi(y_{\sigma(1)}), \varphi(y_{\sigma(2)}), \ldots, \varphi(y_{\sigma(n)})) \cdot \operatorname{diag}(y_1, y_2, \ldots, y_n)^{-1}$$

is the expression

$$\varphi(y_{\sigma(i)}) \cdot y_i^{-1} = \varphi(\varphi^{\tau(\sigma(i))}(y_1)) \cdot \varphi^{\tau(i)}(y_1^{-1}),$$

where σ is the bijection defined in Definition 2.3.1. By Lemma 2.3.2, we have

$$\tau(\sigma(i)) + 1 = [\tau(i)]$$

for all $1 \le i \le n$, and therefore

$$\varphi(y_{\sigma(i)}) \cdot y_i^{-1} = \begin{cases} \varphi^{[\tau(i)]}(y_1) \cdot \varphi^{\tau(i)}(y_1^{-1}) = 1 & \text{if } i \neq 1, \\ \varphi^{[\tau(i)]}(y_1) \cdot \varphi^{\tau(i)}(y_1^{-1}) = \varphi^n(y_1) \cdot y_1^{-1} & \text{if } i = 1. \end{cases}$$

Therefore

$$L_q(y) = \operatorname{diag}(\varphi^n(y_1) y_1^{-1}, 1, \dots, 1). \qquad \square$$

Lemma 5.2.2. *The following diagram commutes:*

$$\begin{array}{ccc} \beta^{-1}(Y_h) & \xrightarrow{\quad \mathrm{pr}_2 \quad} & \mathbb{T}_{h,k} \\ & {\scriptstyle P \circ \mathrm{pr}_1} \searrow \quad \nearrow {\scriptstyle x \mapsto \mathrm{diag}(x,1,\dots,1)} & \\ & \mathbb{W}_h^{(1)}. & \end{array}$$

Proof. This follows from the proof of Lemma 5.2.1. $\square$

Proposition 5.2.3. *For any character $\chi\colon \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n}) \cong \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, let $\mathcal{L}_\chi$ denote the corresponding $\overline{\mathbb{Q}}_\ell$-local system on $\mathbb{W}_h^{(1)}$. Then for $i \ge 0$, we have Fr_{q^n}-compatible isomorphisms*

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big) \cong H_c^i(\mathrm{pr}_1(\beta^{-1}(Y_h)), P^* \mathcal{L}_\chi),$$

where $P\colon \mathbb{A}[\mathcal{A}] \to \mathbb{W}_h^{(1)}$ is the morphism $x \mapsto L_q(\det(s(x)))^{-1} = (\varphi(\det(s(x))) \cdot s(x))^{-1}$.

Proof. By Proposition 4.1.1, we have Fr_{q^n}-compatible isomorphisms

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big) \cong H_c^i(\beta^{-1}(Y_h), \mathrm{pr}_2^* \mathcal{L}'_\chi),$$

where $\mathcal{L}'_\chi$ is the rank-1 local system on $\mathbb{T}_{h,k}$ corresponding to χ and pr_2 is the composition $\beta^{-1}(Y_h) \hookrightarrow (\mathbb{U}_{h,k}/\mathbb{T}_{h,k}) \times \mathbb{T}_{h,k} \xrightarrow{\mathrm{pr}} \mathbb{T}_{h,k}$. By Lemma 5.2.1, $\beta^{-1}(Y_h)$ is the graph of $P\colon \mathrm{pr}_1(\beta^{-1}(Y_h)) \to \mathbb{W}_h^{(1)}$, and by Lemma 5.2.2, $\mathrm{pr}_2 = i \circ (P \circ \mathrm{pr}_1)$, where $i\colon \mathbb{W}_h^{(1)} \to \mathbb{T}_{h,k}$ is the morphism of $\mathbb{F}_{q^n}$-schemes given by $x \mapsto \mathrm{diag}(x, 1, \dots, 1)$. Moreover, the pullback of $\mathcal{L}'_\chi$ is $i^* \mathcal{L}'_\chi = \mathcal{L}_\chi$, where $\mathcal{L}_\chi$ is the rank-1 local system on $\mathbb{W}_h^{(1)}$ corresponding to χ. Therefore

$$H_c^i(\beta^{-1}(Y_h), \mathrm{pr}_2^* \mathcal{L}'_\chi) = H_c^i(\mathrm{pr}_1(\beta^{-1}(Y_h)), P^* \mathcal{L}_\chi). \qquad \square$$

5.2.2. Subschemes of $\mathrm{pr}_1(\beta^{-1}(Y_h))$. We will calculate $H_c^i(\mathrm{pr}_1(\beta^{-1}(Y_h)), P^* \mathcal{L}_\chi)$ by relating it to the cohomology of certain subschemes associated to the indexing sets $\mathcal{A}_{\ge s,t}$ defined in Section 3.2. We now define these subschemes and prove some first lemmas about them.

Definition 5.2.4. Define

$$P_{\ge s,t}\colon \mathcal{X}_{\ge s,t} := \mathrm{pr}_1(\beta^{-1}(Y_h)) \cap \mathbb{A}[\mathcal{A}_{\ge s,t}] \to \mathbb{W}_{h_t}^{(1)}$$

by

$$x \mapsto L_q(\det(s(x)))^{-1} = (\varphi(\det(s(x))) \det(s(x)))^{-1}.$$

Lemma 5.2.5. *The morphism $P_{\geq s,t}$ factors through L_q via the morphism*

$$Q_{\geq s,t}\colon \mathfrak{X}_{\geq s,t} \to \mathbb{W}_{h_t}^{(1)}, \quad x \mapsto \det(s(x))^{-1}.$$

Equivalently, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_{\geq s,t} & \xrightarrow{P_{\geq s,t}} & \mathbb{W}_h^{(1)} \\ & {\scriptstyle Q_{\geq s,t}} \searrow \quad \nearrow {\scriptstyle L_q} & \\ & \mathbb{W}_h^{(1)}. & \end{array}$$

Proof. This follows by definition of $P_{\geq s,t}$. □

Lemma 5.2.6. *The projection map*

$$\mathfrak{X}_{\geq s,t} \to \mathbb{A}[\mathcal{A}^-_{\geq s,t}]$$

is an isomorphism. Moreover, we have an isomorphism

$$\mathfrak{X}_{\geq s,t} \cong \mathbb{A}[\mathcal{A}^-_{\geq s,t} \smallsetminus \mathcal{A}^-_{\geq s,t+1}] \times \mathfrak{X}_{\geq s,t+1}.$$

Proof. We will show that this map is surjective and then use a counting argument to show injectivity.

Let $x \in \mathbb{A}[\mathcal{A}_{\geq s,t}]$ and $y \in \mathbb{T}_{h_t,k}$ be such that $z := s(x) \cdot y \in X_{h_t}$. Recall that from the proof of Lemma 5.2.1, y is determined by its first coordinate $y_1 = [1, a_1, \ldots, a_{h_t-1}] \in \mathbb{W}_{h_t}^{(1)}$. Then for any $\lambda = (i, j, l) \in \mathcal{A}_{\geq s,t}$,

$$z_\lambda = \begin{cases} x_\lambda + (\text{terms with } x_{(i,j,l')} \text{ for } l' < l \text{ and } a_{l''} \text{ for } l'' < l) & \text{if } i < j, \\ x_\lambda + (\text{terms with } x_{(i,j,l')} \text{ for } l' < l \text{ and } a_{l''} \text{ for } l'' \leq l) & \text{if } i > j. \end{cases} \tag{5.1}$$

This implies that the condition $\det(s(x) \cdot y) \in \mathbb{W}_{h_t}^{(1)}(\mathbb{F}_q)$ is equivalent to the vanishing of $h_t - 1$ polynomials in y and x_ν for $\nu \in \mathcal{A}^-_{\geq s,t}$. Explicitly, if we write

$$\det(s(x) \cdot y) = [1, d_1, \ldots, d_{h_t-1}] \in \mathbb{W}_{h_t}^{(1)},$$

then the $h_t - 1$ polynomials are $d_i^q - d_i$ for $1 \leq i \leq h_t - 1$.

Now fix $x^- \in \mathbb{A}[\mathcal{A}^-_{\geq s,t}]$. Observe that d_1 is a polynomial of degree q^{n-1} in a_1, so there are at most q^n roots of the polynomial $d_1^q - d_1$. Let a_1 be a root, and observe that d_2 is a polynomial of degree q^{n-1} in a_2, so that there are at most q^n roots of the polynomial $d_2^q - d_2$. In this way, we see that there exists a $y \in \mathbb{T}_{h_t,k}$ such that $\det(s(x) \cdot y) \in \mathbb{W}_{h_t}^{(1)}(\mathbb{F}_q)$ and that there are at most $q^{n(h_t-1)}$ such y.

The existence of such a $y \in \mathbb{T}_{h_t,k}$ now allows us to extend any $x^- \in \mathbb{A}[\mathcal{A}^-_{\geq s,t}]$ to an element $x \in \mathfrak{X}_{\geq s,t} \subset \mathbb{A}[\mathcal{A}_{\geq s,t}]$ satisfying $s(x) \cdot y \in X_{h_t}$. This shows the surjectivity of the projection.

Recall that $x \in \mathfrak{X}_{\geq s,t}$ determines y up to $\mathbb{T}_{h_t,k}(\mathbb{F}_q)$-translates. There are exactly $q^{n(h_t-1)}$ such translates, and since there are at most $q^{n(h_t-1)}$ points y satisfying $d_i^q - d_i$ for all i with $1 \leq i \leq h_t - 1$, we see that each of these choices of $y \in \mathbb{T}_{h_t,k}$ determine the same extension of $x^- \in \mathbb{A}[\mathcal{A}^-_{\geq s,t}]$ to $x \in \mathfrak{X}_{\geq s,t} \subset \mathbb{A}[\mathcal{A}_{\geq s,t}]$. This shows injectivity. □

Lemma 5.2.7. *Let $x \in \mathfrak{X}_{\geq s,t}$. If $x_\nu = 0$ for some $\nu \in \mathcal{A}^-_{\geq s,t}$, then we have $x_\lambda = 0$ for all $\lambda \in \mathcal{A}_{\geq s,t}$ with $|\lambda| = |\nu|$.*

Proof. This follows easily from the proof of Lemma 5.2.6 (see equation (5.1)). □

For a moment, we write $\mathfrak{X}^{n,q}_{\geq s,t}$ to emphasize the dependence on n, q. For each t with $0 \leq t \leq r+1$, the sequences

$$\begin{aligned} &1 =: m_0 \leq m_1 < m_2 < \cdots < m_r \leq m_{r+1} := n, \\ &h =: h_0 = h_1 > h_2 > \cdots > h_r \geq h_{r+1} := 1 \end{aligned}$$

induce corresponding sequences

$$\begin{aligned} &1 = m'_0 \leq m'_1 < m'_2 < \cdots < m'_{r-t} \leq m'_{r+1-t} = n/m_t, \quad m'_{i-t} := m_i/m_t, \\ &h' =: h'_0 \geq h'_1 > h'_2 > \cdots > h'_{r-t} \geq h'_{r+1-t} = 1, \qquad\qquad h'_{i-t} := h_i \end{aligned}$$

where we define

$$m'_{i-t} := m_i/m_t \quad \text{and} \quad h'_{i-t} := h_i \quad \text{for } t \leq i \leq r+1.$$

Note that $h'_0 = h'_1$ if and only if $t = 0$. For $0 \leq s_0, t_0 \leq r-t$, let $\mathcal{A}^{n/m_t, q^{m_t}}_{s_0,t_0}$ denote the associated indexing sets as in Section 3.2. Under this correspondence, the scheme $\mathfrak{X}^{n,q}_{\geq t+s_0, t+t_0}$ can be matched with $\mathfrak{X}^{n/m_t,q^{m_t}}_{\geq s_0,t_0}$. We make this precise in the next lemma.

Lemma 5.2.8. *The natural projection*

$$\mathcal{A}^{n,q}_{\geq t,t} \to \mathcal{A}^{n/m_t,q^{m_t}}_{\geq 0,0}, \qquad (i,j,l) \mapsto \left(i, \tfrac{j-i}{m_t} + 1, l\right) \quad \text{if } i, j \equiv 1 \pmod{m_t} \tag{5.2}$$

induces an isomorphism $\mathfrak{X}^{n,q}_{\geq t,t} \cong \mathfrak{X}^{n/m_t,q^{m_t}}_{\geq 0,0}$ under which $P^{n,q}_{\geq t,t} = P^{n/m_t,q^{m_t}}_{\geq 0,0}$.

Proof. Recall that

$$\begin{aligned} \mathcal{A}^{n,q}_{\geq t,t} &= \big\{(i,j,l) : 1 \leq i, j \leq n,\ 1 \leq l \leq h_t - 1,\ j \equiv i \pmod{m_t}\big\}, \\ \mathcal{A}^{n/m_t,q^{m_t}}_{\geq 0,0} &= \big\{(i,j,l) : 1 \leq i, j \leq n/m_t,\ 1 \leq l \leq h'_0 - 1 = h_t - 1\big\}. \end{aligned}$$

It is clear that the map (5.2) defines a bijection

$$\{(i,j,l) : 1 \leq i, j \leq n,\ 1 \leq l \leq h_t - 1,\ j \equiv i \equiv 1 \pmod{m_t}\} \to \mathcal{A}^{n/m_t,q^{m_t}}_{\geq 0,0}.$$

In particular, (5.2) induces a bijection $(\mathcal{A}^{n,q}_{\geq t,t})^- = (\mathcal{A}^{n/m_t,q^{m_t}}_{\geq 0,0})^-$ and therefore an isomorphism between the associated affine spaces. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X}^{n,q}_{\geq t,t} & \longrightarrow & \mathfrak{X}^{n/m_t,q^{m_t}}_{\geq 0,0} \\ \downarrow & & \downarrow \\ \mathbb{A}[(\mathcal{A}^{n,q}_{\geq t,t})^-] & \longrightarrow & \mathbb{A}[(\mathcal{A}^{n/m_t,q^{m_t}}_{\geq 0,0})^-], \end{array}$$

where the vertical maps are isomorphisms by Lemma 5.2.6. It therefore follows that the top map must be an isomorphism. □

5.3. Calculation of Artin–Schreier sheaves. In this section, we prove the two key propositions required to calculate the cohomology of the (pullbacks of) Artin–Schreier sheaves at hand. Fix a character $\chi\colon \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n}) \cong \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ together with a Howe factorization $\chi = \prod_{i=1}^r \chi_i$ and recall from Section 2 that this gives rise to associated indexing sets $\mathcal{T}_{h_t}, \mathcal{I}_{s,t}, \mathcal{J}_{s,t}, \mathcal{A}_{s,t}$.

The main idea behind Proposition 5.3.1 is to use the conductor-lowering compatibility lemmas of Section 5.2 to induct according to the Howe factorization. This proposition allows us to reduce a cohomological calculation about $\chi_{\geq t} = \prod_{i=t}^r \chi_i$ to a calculation about $\chi_{\geq t+1}$, *as long as everything is "divisible" by the conductor of* χ_t. Proposition 5.3.1 is proved in Section 5.3.1.

Proposition 5.3.1. *For* $0 \leq t \leq r$*, we have* Fr_{q^n}*-compatible isomorphisms*

$$H_c^i(\mathcal{X}_{\geq t,t}, P^*\mathcal{L}_{\chi_{\geq t}}) \cong H_c^i(\mathcal{X}_{\geq t,t+1}, P^*\mathcal{L}_{\chi_{\geq t+1}})[2e_t] \otimes ((-q^{n/2})^{2e_t})^{\deg},$$

where $e_t = \#(\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1})$.

Proposition 5.3.2 is essentially proved by applying Proposition 4.2.2 inductively. The main calculation is to show that the hypotheses of Proposition 4.2.2 hold. Proposition 5.3.2 allows us to *'get rid of' all the parts that are not 'divisible' by the conductor of* χ_t, which returns us to the setting of Proposition 5.3.1. Proposition 5.3.2 is proved in Section 5.3.2.

Proposition 5.3.2. *For* $1 \leq t \leq r$*, we have* Fr_{q^n}*-compatible isomorphisms*

$$H_c^i(\mathcal{X}_{\geq t-1,t}, P^*\mathcal{L}_{\chi_{\geq t}}) \cong H_c^i(\mathcal{X}_{\geq t,t}, P^*\mathcal{L}_{\chi_{\geq t}})^{\oplus q^{nd_t/2}}[d_t] \otimes ((-q^{n/2})^{d_t})^{\deg},$$

where $d_t = \#\mathcal{A}^-_{t-1,t}$.

5.3.1. Proof of Proposition 5.3.1. By definition, as characters of $\mathbb{W}_{h_t}^{(1)}(\mathbb{F}_{q^n})$, we have

$$\chi_{\geq t} = \chi_t \cdot \chi_{\geq t+1},$$

where χ_t has level h_t and conductor m_t, and $\chi_{\geq t+1}$ has level $h_{t+1} < h_t$ and conductor $m_{t+1} > m_t$, where $m_t \mid m_{t+1}$. This implies that

$$\mathcal{L}_{\chi_{\geq t}} = \mathcal{L}_{\chi_t} \otimes \mathrm{pr}^*\, \mathcal{L}_{\chi_{\geq t+1}}$$

as sheaves on $\mathbb{W}_{h_t}^{(1)}$, where $\mathrm{pr}\colon \mathbb{W}_{h_t}^{(1)} \to \mathbb{W}_{h_{t+1}}^{(1)}$.

By Lemma 5.2.8, we have an isomorphism $\mathcal{X}^{n,q}_{\geq t,t} \cong \mathcal{X}^{n/m_t,q^{m_t}}_{\geq 0,0}$ and $P^{n,q}_{\geq t,t} = P^{n/m_t,q^{m_t}}_{\geq 0,0}$. Therefore, by Lemma 5.2.5, we have the following equality of $\mathbb{Q}_\ell$-sheaves on $\mathcal{X}^{n,q}_{\geq t,t}$:

$$P^*\mathcal{L}_{\chi_{\geq t}} = P^*\mathcal{L}_{\chi_t} \otimes P^*\,\mathrm{pr}^*\,\mathcal{L}_{\chi_{\geq t+1}} = Q^*(L^*_{q^{m_t}}\mathcal{L}_{\chi_t}) \otimes P^*\,\mathrm{pr}^*\,\mathcal{L}_{\chi_{\geq t+1}},$$

where $Q = Q^{n/m_t,q^{m_t}}_{\geq 0,0}$. By construction, χ_t factors through $\mathbb{W}_{h_t}^{(1)}(\mathbb{F}_{q^{m_t}})$ and hence $L^*_{q^{m_t}}\mathcal{L}_{\chi_t}$ is the trivial local system. By Lemma 5.2.6,

$$\mathcal{X}_{\geq t,t} = \mathbb{A}[\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1}] \times \mathcal{X}_{\geq t,t+1}.$$

Since the push-forward of $P^*\,\mathrm{pr}^*\,\mathcal{L}_{\chi_{\geq t+1}}$ to $\mathbb{A}[\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1}]$ is the trivial local system, we have

$$P^*\mathcal{L}_{\chi_{\geq t}} = \overline{\mathbb{Q}}_\ell \boxtimes P^*\mathcal{L}_{\chi_{\geq t+1}},$$

where $\overline{\mathbb{Q}}_\ell$ is the constant sheaf on $\mathbb{A}[\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1}]$ and $P^*\mathcal{L}_{\chi_{\geq t+1}}$ is the pullback along $P^*\colon \mathcal{X}_{\geq t,t+1} \to \mathbb{W}^{(1)}_{h_{t+1}}$. Thus by the Künneth formula,

$$\begin{aligned} H^i_c(\mathcal{X}_{\geq t,t}, P^*\mathcal{L}_{\chi_{\geq t}}) &= H^i_c\big(\mathbb{A}[\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1}] \times \mathcal{X}_{\geq t,t+1}, \overline{\mathbb{Q}}_\ell \boxtimes P^*\mathcal{L}_{\chi_{\geq t+1}}\big) \\ &\cong \bigoplus_{r+s=i} H^r_c\big(\mathbb{A}[\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1}], \overline{\mathbb{Q}}_\ell\big) \otimes H^s_c(\mathcal{X}_{\geq t,t+1}, P^*\mathcal{L}_{\chi_{\geq t+1}}) \\ &= H^i_c(\mathcal{X}_{\geq t,t+1}, P^*\mathcal{L}_{\chi_{\geq t+1}})[2\#(\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1})] \\ &\qquad \otimes \big((-q^{n/2})^{2\#(\mathcal{A}^-_{\geq t,t} \smallsetminus \mathcal{A}^-_{\geq t,t+1})}\big)^{\deg}. \end{aligned}$$

5.3.2. Proof of Proposition 5.3.2. The main ideas in the proof of Proposition 5.3.2 are Lemmas 5.3.3 and 5.3.5. We will need some notation that will be used in these two lemmas and also in the proof of the proposition at hand. Recall that

$$\mathcal{A}^-_{t-1,t} := \{(1,j,l) \in \mathcal{A} : j \equiv 1 \pmod{m_{t-1}},\ j \not\equiv 1 \pmod{m_t},\ 1 \leq l \leq h_t - 1\},$$

and define

$$\begin{aligned} I_0 := \mathcal{I}_{t-1,t} &= \{(1,j,l) \in \mathcal{A}^-_{t-1,t} : |(1,j,l)| > \tfrac{1}{2}n(h_t-1)\}, \\ J_0 := \mathcal{J}_{t-1,t} &= \{(1,j,l) \in \mathcal{A}^-_{t-1,t} : |(1,j,l)| \leq \tfrac{1}{2}n(h_t-1)\}. \end{aligned}$$

Recall that, as in Lemma 3.2.3, these two sets are totally ordered and come with an order-reversing injection

$$I_0 \hookrightarrow J_0, \quad (1,\sigma^j(1),l) \mapsto (1,\sigma^j(1),l)' := (1,\sigma^{n-j}(1),h_t-l).$$

This map is a bijection if and only if n, h_t are even and $(1,\sigma^{n/2}(1),h_t/2) \notin J_0$. Otherwise, $\#J_0 - \#I_0 = 1$. Given this, there is a natural filtration on I_0 by iteratively removing a highest-norm element, and analogously a natural filtration on J_0 by iteratively removing a lowest-norm element. These filtrations can be specified so that they are compatible under the map $I_0 \to J_0$ above. Precisely, fix a labeling $\nu_0, \nu_1, \ldots, \nu_{\#I_0-1}$ of the elements of I_0 where $|\nu_i| \geq |\nu_{i+1}|$ for $0 \leq i < \#I_0 - 1$. We may define

$$I_{\kappa+1} := I_\kappa \smallsetminus \{\nu_\kappa\}, \quad J_{\kappa+1} := J_\kappa \smallsetminus \{\nu'_\kappa\}.$$

Define

$$\begin{aligned} \widetilde{I_\kappa} &:= \{(i,[j-i+1],l) \in \mathcal{A}_{t-1,t} : (1,\sigma^j(1),l) \in I_\kappa\}, \\ \widetilde{J_\kappa} &:= \{(i,[j-i+1],l) \in \mathcal{A}_{t-1,t} : (1,\sigma^j(1),l) \in J_\kappa\}. \end{aligned}$$

Lemma 5.3.3. *For each $\kappa \leq \#I_0 - 1$, we have Fr_{q^n}-compatible isomorphisms*

$$\begin{aligned} &H^i_c(\mathcal{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (\widetilde{I_\kappa} \cup \widetilde{J_\kappa})], P^*\mathcal{L}_{\chi_{\geq t}}) \\ &\quad \cong H^i_c(\mathcal{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (\widetilde{I_{\kappa+1}} \cup \widetilde{J_{\kappa+1}})], P^*\mathcal{L}_{\chi_{\geq t}})^{\oplus q^n}[2] \otimes (q^n)^{\deg}. \end{aligned}$$

Remark 5.3.4. A variation of Lemma 5.3.3 for the equal characteristic case with $(\{1,n,n\},\{h,h,1\})$ was proved in [6, Lemma 5.11]. We give a different proof here that does not refer to juggling sequences or the explicit equations cutting out X_h.

Proof. Recall that by definition I_κ and J_κ depend on t. This proof is driven by the following simple goal: to apply Proposition 4.2.2. To this end, the content of this lemma is the calculation that the polynomial

$$P\colon \mathfrak{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (\widetilde{I_\kappa} \cup \widetilde{J_\kappa})] \to \mathbb{W}_h^{(1)}, \quad x \mapsto L_q(\det(s(x)))^{-1}$$

has the required form for $f(x) = x_{\nu_\kappa}$ and $y = x_{\nu'_\kappa}$.

Let $x \in \mathfrak{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (\widetilde{I_\kappa} \cup \widetilde{J_\kappa})]$ and let $y \in \mathbb{T}_{h,k}$ be such that

$$z := s(x) \cdot y \in X_h.$$

Set $\nu := \nu_\kappa := (1, \sigma^j(1), l) \in I_\kappa$ and $\nu' := \nu'_\kappa := (1, \sigma^{n-j}(1), h_t - l) \in J_\kappa$, and note that $|\nu| + |\nu'| = n(h_t - 1)$.

Take $\lambda_1 := (i, \sigma^j(i), l) \in \widetilde{I_\kappa}$ so that $|\lambda_1| = |\nu|$. Let $\lambda_2 \in \mathcal{A}_{\geq t,t} \cup \widetilde{I_\kappa} \cup \widetilde{J_\kappa}$ be such that both z_{λ_1} and z_{λ_2} contribute to a monomial in $\det(s(x))$. I first claim that necessarily $\lambda_2 \in \widetilde{J_\kappa}$. Indeed, if $\gamma \in S_n$ is such that $\gamma(i) = \sigma^j(i)$, then $m_t \nmid \gamma(i) - i$, and hence there must be another i' such that $m_t \nmid \gamma(i') - i'$. If γ corresponds to a nontrivial summand of $\det(z)$, then $|(i', \gamma(i'), l')| \leq n(h_t - 1) - |(i, \gamma(i), l)| \leq n(h_t - 1)/2$ and so $(i', \gamma(i'), l') \in \widetilde{J_\kappa}$. But now $|\lambda_1| + |(i', \gamma(i'), l')| \geq |\nu| + |\nu'| = n(h_t - 1)$ and by Lemma 3.2.6, this inequality must be an equality. Thus γ must be a transposition, and we see that $\lambda_2 \in \widetilde{J_\kappa}$ and in fact

$$\lambda_2 = (\sigma^j(i), i, n - l) = (\sigma^j(i), \sigma^{n-j}(\sigma^j(i)), n - l).$$

By equation (2.6), z_{λ_1} and z_{λ_2} are powers of z_ν and $z_{\nu'}$, respectively:

$$z_{\lambda_1} = z_\nu^{q^{\tau(i)}} \quad \text{and} \quad z_{\lambda_2} = z_{\nu'}^{q^{\tau(\sigma^j(i))}} = z_{\nu'}^{q^{[\tau(i)-j+1]-1}}.$$

Therefore the contribution of λ_1 and λ_2 is

$$(-1)^{\operatorname{sgn}\gamma} z_{\lambda_1} z_{\lambda_2} = -z_{\lambda_1} z_{\lambda_2} = \begin{cases} z_\nu^{q^{\tau(i)}} \cdot z_{\nu'}^{q^{[\tau(i)-j+1]-1}} & \text{if char } K > 0, \\ z_\nu^{q^{\tau(i)+h_t-l}} \cdot z_{\nu'}^{q^{[\tau(i)-j+1]-1+l}} & \text{if char } K = 0. \end{cases} \tag{5.3}$$

The contribution of z_ν to $L_q(\det(z))^{-1}$ is given by

$$\sum_{\lambda_1} (z_{\lambda_1} z_{\lambda_2})^q - z_{\lambda_1} z_{\lambda_2}, \tag{5.4}$$

and after writing this sum in terms of z_ν and $z_{\nu'}$ as in equation (5.3), all terms cancel except for those corresponding to when

$$\tau(i) = 0, \quad \tau(i) = n - 1, \quad [\tau(i) - j + 1] - 1 = 0, \quad \text{or} \quad [\tau(i) - j + 1] - 1 = n - 1,$$

which exactly corresponds to when

$$\tau(i) = 0, \quad \tau(i) = n - 1, \quad \tau(i) = j, \quad \text{or} \quad \tau(i) = j - 1.$$

Hence the sum in (5.4) simplifies to

$$\begin{cases} (z_\nu^{q^n} - z_\nu) \cdot z_{\nu'}^{q^j} + (z_{\nu'}^{q^n} - z_{\nu'}) \cdot z_\nu^{q^{n-j}} & \text{if char } K > 0, \\ (z_\nu^{q^n} - z_\nu)^{q^{h_t-l}} \cdot z_{\nu'}^{q^{j+l}} + (z_{\nu'}^{q^n} - z_{\nu'})^{q^l} \cdot z_\nu^{q^{n-j+h_t-l}} & \text{if char } K = 0. \end{cases} \tag{5.5}$$

We now investigate how equation (5.5) allows us to understand the contribution of x_ν and $x_{\nu'}$ to $L_q(\det(z)) = L_q(\det(s(x)\cdot y)) \in \mathbb{W}_{h_t}$. If x_λ appears in z_{λ_0}, then $|\lambda| \le |\lambda_0|$. Necessarily λ_0 does not sit on the diagonal since λ does not, and hence if z_{λ_0} contributes to $L_q(\det(z))$, there must exist a λ_0' such that $z_{\lambda_0'}$ also contributes to the same monomial. But by Lemma 3.2.6, this implies that $|\lambda_0'| \le n(h_t-1) - |\lambda_0| \le |\lambda'|$. This shows that the contribution of x_λ and $x_{\lambda'}$ is contained in the contribution of z_λ and $z_{\lambda'}$. Now,

$$\begin{aligned} z_{\lambda'} &= x_{\lambda'} + (\text{terms that each include a factor of } x_{\lambda_0'} \text{ for } \lambda_0' \in \widetilde{J}_0 \text{ and } |\lambda_0'| < |\nu'|) \\ &= z_{\lambda'}, \\ z_\lambda &= x_\lambda + (\text{terms that each include a factor of } x_{\lambda_0} \text{ for } \lambda_0 \in \widetilde{J}_0 \text{ and } |\lambda_0| < |\nu|) \end{aligned}$$

and hence we see that the contribution of x_λ is captured by the contribution of z_λ which is captured by the contribution of z_ν. That is, the contribution of x_λ is equal to

$$\begin{cases} (x_\nu^{q^n} - x_\nu)\cdot x_{\nu'}^{q^j} + (x_{\nu'}^{q^n} - x_{\nu'})\cdot x_\nu^{q^{n-j}} & \text{if } \operatorname{char} K > 0, \\ (x_\nu^{q^n} - x_\nu)^{q^{h_t-l}}\cdot x_{\nu'}^{q^{j+l}} + (x_{\nu'}^{q^n} - x_{\nu'})^{q^l}\cdot x_\nu^{q^{n-j+h_t-l}} & \text{if } \operatorname{char} K = 0. \end{cases} \tag{5.6}$$

By assumption, m_t does not divide j since $\nu \notin \mathcal{A}_{\ge t,t}$. Since $\chi_{\ge t}$ has conductor m_t, equation (5.6) shows that P has the form required to apply Proposition 4.2.2 in the case $f(x) = x_{\nu'}$ and $y = x_\nu$. Therefore

$$H_c^i(\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup (\widetilde{I}_\kappa \cup \widetilde{J}_\kappa)], P^*\mathcal{L}_{\chi_{\ge t}}) \cong H_c^i(S_3, P^*\mathcal{L}_{\chi_{\ge t}})[2] \otimes (q^n)^{\deg},$$

where $S_3 \subset \mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup (\widetilde{I}_\kappa \cup \widetilde{J}_\kappa) \smallsetminus \{\nu\}]$ is the subscheme defined by

$$f(x)^{q^n} - f(x) = 0.$$

Since S_3 is the disjoint union of q^n copies of $\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup (\widetilde{I}_\kappa \cup \widetilde{J}_\kappa) \smallsetminus \{\nu, \nu'\}]$,

$$\begin{aligned} &H_c^i(S_3, P^*\mathcal{L}_{\chi_{\ge t}})[2] \otimes (q^n)^{\deg} \\ &\quad\cong H_c^i(\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup (\widetilde{I}_\kappa \cup \widetilde{J}_\kappa) \smallsetminus \{\nu, \nu'\}], P^*\mathcal{L}_{\chi_{\ge t}})^{\oplus q^n}[2] \otimes (q^n)^{\deg} \\ &\quad= H_c^i(\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup (\widetilde{I}_{\kappa+1} \cup \widetilde{J}_{\kappa+1})], P^*\mathcal{L}_{\chi_{\ge t}})^{\oplus q^n}[2] \otimes (q^n)^{\deg}, \end{aligned}$$

where the last equality holds by Lemma 5.2.7. □

Lemma 5.3.5. *Suppose* $\#J_0 - \#I_0 = 1$. *Then* $J_{\#I_0} = \{v := (1, n/2+1, h_t/2) \in J_0\}$ *and we have* Fr_{q^n}*-compatible isomorphisms*

$$\begin{aligned} &H_c^i(\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup \widetilde{J}_{\#I_0}], P^*\mathcal{L}_{\chi_{\ge t}}) \\ &\quad= H_c^i(\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t}], P^*\mathcal{L}_{\chi_{\ge t}})^{\oplus q^{n/2}}[1] \otimes (-q^{n/2})^{\deg}. \end{aligned}$$

Proof. For ease of notation, set $i_0 := \#I_0$ in this proof. By the divisibility assumption on $\mathcal{A}_{\ge t,t}$, we see that

$$\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t} \cup \widetilde{J}_{i_0}] = (\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t}]) \times (\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\widetilde{J}_{i_0}]).$$

Moreover, by Lemma 5.2.6, the projection

$$f\colon (\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t}]) \times (\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\widetilde{J}_{i_0}]) \to (\mathfrak{X}_{\ge t-1,t} \cap \mathbb{A}[\mathcal{A}_{\ge t,t}]) \times \mathbb{A}[\{v\}]$$

is an isomorphism of varieties. By Lemma 3.2.6, it follows that any $\lambda \in \widetilde{J}_{i_0}$ can only contribute nontrivially to the last coordinate of $\mathbb{W}_{h_t}^{(1)}$ in $\det(s(x))$. Therefore, for $P_0 = P|_{\mathfrak{X}_{\geq t,t}}$ and some morphism $\eta\colon \mathbb{A}[\{v\}] \to \mathbb{W}_{h_t}^{(1)}$, we have

$$f_* P^* \mathcal{L}_{\chi_{\geq t}} = P_0^* \mathcal{L}_{\chi_{\geq t}} \boxtimes \eta^* \mathcal{L}_{\chi_{\geq t}},$$

and therefore

$$\begin{aligned} &H_c^i(\mathfrak{X}_{\geq t,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup \widetilde{J}_{i_0}], P^* \mathcal{L}_{\chi_{\geq t}}) \\ &\quad = H_c^i(\mathfrak{X}_{\geq t,t} \times \mathbb{A}[\{v\}], P_0^* \mathcal{L}_{\chi_{\geq t}} \boxtimes \eta^* \mathcal{L}_{\chi_{\geq t}}) \\ &\quad \cong \bigoplus_{r+s=i} H_c^r(\mathfrak{X}_{\geq t,t}, P_0^* \mathcal{L}_{\chi_{\geq t}}) \otimes H_c^s(\mathbb{A}[\{v\}], \eta^* \mathcal{L}_{\chi_{\geq t}}). \end{aligned}$$

It now remains to calculate η, which we do by a similar calculation to the one in Lemma 5.3.3.

By Lemma 3.2.3, $v = (1, n/2+1, h_t/2)$. The contribution of x_v to the last coordinate of $L_q(\det(s(x)))^{-1}$ is exactly given by

$$\begin{cases} \sum_{i=1}^n x_v^{q^{\tau(i)+1}} \cdot x_v^{q^{[\tau(i)+n/2]+1}} - x_v^{q^{\tau(i)}} \cdot x_v^{q^{[\tau(i)+n/2]}} & \text{if char } K > 0, \\ \sum_{i=1}^n x_v^{q^{\tau(i)+(h_t-1)/2+1}} \cdot x_v^{q^{[\tau(i)+n/2]+(h_t-1)/2+1}} & \\ \quad -x_v^{q^{\tau(i)+(h_t-1)/2}} \cdot x_v^{q^{[\tau(i)+n/2]+(h_t-1)/2}} & \text{if char } K = 0. \end{cases}$$

This simplifies to

$$\begin{cases} (x_v^{q^n} - x_v) \cdot x_v^{q^{n/2}} & \text{if char } K > 0, \\ (x_v^{q^n} - x_v)^{q^{h_t/2}} \cdot x_v^{q^{(n+h_t)/2}} & \text{if char } K = 0, \end{cases}$$

and it follows that

$$\eta\colon \mathbb{A}[\{v\}] \to \mathbb{W}_{h_t}^{(1)}, \quad x \mapsto \begin{cases} (0, \ldots, 0, x^{q^{n/2}}(x^{q^n} - x)) & \text{if char } K > 0, \\ (0, \ldots, 0, x^{q^{(n+h_t)/2}}(x^{q^n} - x)^{q^{h_t/2}}) & \text{if char } K = 0. \end{cases}$$

We can now make the final conclusion. By [4, equation (6.5.7)] (or [6, Step 4: Case 2 of Proposition 6.1]), we have

$$\dim H_c^i(\mathbb{G}_a, \eta^* \mathcal{L}_{\chi_{\geq t}}) = \begin{cases} q^{n/2} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Fr_{q^n} acts on $H_c^1(\mathbb{G}_a, \eta^* \mathcal{L}_{\chi_{\geq t}})$ by multiplication by $-q^{n/2}$. □

The desired conclusion of Proposition 5.3.2 now follows by combining Lemmas 5.3.3 and 5.3.5. Explicitly, by Lemma 5.3.3,

$$\begin{aligned} &H_c^i(\mathfrak{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t-1,t}], P^* \mathcal{L}_{\chi_{\geq t}}) \\ &\quad = H_c^i(\mathfrak{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (\widetilde{I}_0 \cup \widetilde{J}_0)], P^* \mathcal{L}_{\chi_{\geq t}}) \\ &\quad \cong H_c^i(\mathfrak{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (\widetilde{I}_\kappa \cup \widetilde{J}_\kappa)], P^* \mathcal{L}_{\chi_{\geq t}})^{\oplus q^{\kappa n}}[2\kappa] \otimes ((-q^{n/2})^{2\kappa})^{\deg} \end{aligned}$$

for any $\kappa \leq \#I_0$. Clearly $I_{\#I_0} = \varnothing$. If $J_{\#I_0} = \varnothing$, then we are done. Otherwise, $J_{\#I_0}$ is the singleton

$$J_{\#I_0} = \{(1, 1 + n/2, (h_t - 1)/2)\}$$

and by Lemma 5.3.5,

$$H_c^i(\mathfrak{X}_{\geq t-1,t} \cap \mathbb{A}[\mathcal{A}_{\geq t,t} \cup (I_{\#I_0} \cup J_{\#I_0})], P^*\mathcal{L}_{\chi_{\geq t}})[2\#I_0] \otimes ((-q^{n/2})^{2\#I_0})^{\deg}$$
$$\cong H_c^i(\mathfrak{X}_{\geq t,t}, P^*\mathcal{L}_{\chi_{\geq t}})^{\oplus q^{n/2}}[2\#I_0+1] \otimes ((-q^{n/2})^{2\#I_0+1})^{\deg}.$$

It remains to observe that if $J_{\#I_0} = \varnothing$, then $2\#I_0 = \#(I_0 \cup J_0) = \#(\mathcal{I}_{t-1,t} \cup \mathcal{J}_{t-1,t})$, and otherwise, $2\#I_0 + 1 = \#(I_0 \cup J_0) = \#(\mathcal{I}_{t-1,t} \cup \mathcal{J}_{t-1,t})$. This completes the proof. □

5.4. Morphisms: Proof of Theorem 5.1.1. Let $\chi\colon \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n}) \cong \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ be any character and consider the sequences of integers

$$1 =: m_0 \leq m_1 < m_2 < \cdots < m_r \leq m_{r+1} := n,$$
$$h =: h_0 = h_1 > h_2 > \cdots > h_r \geq h_{r+1} := 1$$

associated to a(ny) Howe factorization of χ. (Recall from Section 2.4 that these sequences do not depend on the choice of Howe factorization.) Let $\mathcal{A}_{s,t}$ be indexing sets associated to a Howe factorization of χ. By combining Propositions 5.2.3, 5.3.2, and 5.3.1, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right)\\
&\quad\cong H_c^i(\mathrm{pr}_1(\beta^{-1}(Y_h)), P^*\mathcal{L}_\chi) && \text{(Prop. 5.2.3)}\\
&\quad= H_c^i(\mathfrak{X}_{\geq 0,0}, P^*\mathcal{L}_{\chi_{\geq 0}}) && \text{(Lem. 5.2.1)}\\
&\quad\cong H_c^i(\mathfrak{X}_{\geq 0,1}, P^*\mathcal{L}_{\chi_{\geq 1}})[2e_0]\left((-q^{n/2})^{2e_0}\right)^{\deg} && \text{(Prop. 5.3.1)}\\
&\quad\cong H_c^i(\mathfrak{X}_{\geq 1,1}, P^*\mathcal{L}_{\chi_{\geq 1}})^{\oplus q^{nd_1/2}}[d_1+2e_0]\left((-q^{n/2})^{d_1+2e_0}\right)^{\deg} && \text{(Prop. 5.3.2)}\\
&\quad\cong H_c^i(\mathfrak{X}_{\geq 1,2}, P^*\mathcal{L}_{\chi_{\geq 2}})^{\oplus q^{nd_1/2}}[d_1+2(e_0+e_1)]\left((-q^{n/2})^{d_1+2(e_0+e_1)}\right)^{\deg} && \text{(Prop. 5.3.1)}
\end{aligned}$$

and so forth by iteratively applying Propositions 5.3.1 and 5.3.2. Recall that

$$\begin{aligned}
d_t &= \#\mathcal{A}_{t-1,t}^-\\
&= \{(1,j,l) : m_{t-1} \mid j-1,\ m_t \nmid j-1,\ 1 \leq l \leq h_t - 1\}\\
&= \left(\tfrac{n}{m_{t-1}} - \tfrac{n}{m_t}\right)(h_t - 1) && \text{for } 1 \leq t \leq r+1,\\
e_t &= \#(\mathcal{A}_{\geq t,t}^- \smallsetminus \mathcal{A}_{\geq t,t+1}^-)\\
&= \#\{(1,j,l) : m_t \mid j-1,\ j \neq 1,\ h_{t+1} - 1 < l \leq h_t - 1\}\\
&= \left(\tfrac{n}{m_t} - 1\right)(h_t - h_{t+1}) && \text{for } 0 \leq t \leq r.
\end{aligned}$$

Thus setting

$$\begin{aligned}
d_\chi &:= d_1 + \cdots + d_{r+1}\\
&= \sum_{t=1}^{r+1}\left(\frac{n}{m_{t-1}} - \frac{n}{m_t}\right)(h_t - 1),\\
r_\chi &:= (d_1 + \cdots + d_{r+1}) + 2(e_0 + \cdots + e_r)\\
&= \sum_{t=1}^{r+1}\left(\left(\frac{n}{m_{t-1}} - \frac{n}{m_t}\right)(h_t - 1) + 2\left(\frac{n}{m_{t-1}} - 1\right)(h_{t-1} - h_t)\right),
\end{aligned}$$

the cohomology groups above are isomorphic to

$$H_c^i(\mathfrak{X}_{\geq r,r+1}, P^*\mathcal{L}_{\chi_{\geq r}})^{q^{nd_\chi/2}}[r_\chi]\left((-q^{n/2})^{r_\chi}\right)^{\deg}.$$

Note that $\mathcal{X}_{\geq r,r+1}$ is a single point and hence we obtain that as vector spaces,

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) = \begin{cases} \overline{\mathbb{Q}}_\ell^{\oplus q^{nd_\chi/2}} & \text{if } i = r_\chi, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, since Fr_{q^n} acts trivially on $H_c^0(*, \overline{\mathbb{Q}}_\ell)$, it follows that Fr_{q^n} acts by multiplication by $(-1)^{r_\chi} q^{nr_\chi/2}$ on the above space of $\mathbb{U}_{h,k}(\mathbb{F}_q)$-homomorphisms. Since every representation of $\mathbb{U}_{h,k}(\mathbb{F}_q)$ occurs in $\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi)$ for at least one character χ, the above gives a complete description of the Fr_{q^n}-action on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$, and this action is always by multiplication by $(-1)^i q^{ni/2}$.

6. Deligne–Lusztig theory for finite unipotent groups

In this section, we prove the main theorems of this paper. We first calculate the alternating sum of the cohomology groups (Theorem 6.1.1) using a technique of [19]. This is very similar to the results of Lusztig [20] and Stasinski [21], which study closely related groups in a reductive setting. Combining Theorem 5.1.1 and Theorem 6.1.1 gives Theorems 6.2.1 and 6.2.2, which prove Boyarchenko's conjectures [2, Conjectures 5.16 and 5.18] in full generality. Note that strictly speaking, as stated in [2] these two conjectures assume that char $K > 0$ and $k = 1$; however, they can be easily extended and formulated without these assumptions (see [6, Conjectures 7.4, 7.5]). Theorems 6.2.1 and 6.2.2 can be viewed naturally as the higher-dimensional analogues of the results of Boyarchenko and Weinstein on the cohomology of X_2 in [4, Sections 4-6].

We remark that in all previous work (i.e., the $h = 2$ work of Boyarchenko–Weinstein [4] and the primitive-χ, equal-characteristic work of the author in [6], [5]), pinning down the nonvanishing cohomological degree $i = s_\chi$ of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ was a trivial consequence of (the analogues of) Theorem 5.1.1. This is because the central character of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ determines s_χ in these cases and so s_χ agrees with the r_χ appearing in Theorem 5.1.1. However, in the general setting, this no longer holds, and it is a nontrivial theorem that the equality $s_\chi = r_\chi$ is still true (Theorem 6.2.4). The proof is an application of the Deligne–Lusztig trace formula [9, Theorem 3.2].

The trio of Theorems 6.2.1, 6.2.2, and 6.2.4 gives us a complete description of the $\mathbb{T}_{h,k}(\mathbb{F}_q)$-eigenspaces $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ together with the Frobenius action on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$. Combining these theorems with Theorem 5.1.1 and the fact that the multiplicity of an irreducible ρ in the regular representation is equal to the dimension of ρ, we may write down an explicit formula for the zeta function of X_h. This is done in Theorem 6.3.1.

In Section 6.4, we demonstrate how to realize the main theorems of [2], [4], [6], and [5] as corollaries of the theorems in this paper.

6.1. Alternating sums of eigenspaces.

We study the virtual $\mathbb{U}_{h,k}(\mathbb{F}_q)$-representation

$$R_\chi := \sum_i (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi].$$

Theorem 6.1.1. *For each $\chi: \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, the $\mathbb{U}_{h,k}(\mathbb{F}_{q^n})$-representation $\pm R_\chi$ is irreducible. If $\chi \neq \chi'$, then $\pm R_\chi, \pm R_{\chi'}$ are nonisomorphic.*

Proof. Since the dual of the virtual $\mathbb{U}_{h,k}(\mathbb{F}_q)$-representation $\sum_i (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is isomorphic to $\sum_i (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi^{-1}]$,

$$\begin{aligned}
&\sum_{i,j} (-1)^{i+j} \dim \mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)} \left(H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi], H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi']\right) \\
&\quad = \sum_{i,j} (-1)^{i+j} \dim \left(H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi^{-1}] \otimes H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi']\right)^{\mathbb{U}_{h,k}(\mathbb{F}_q)} \\
&\quad = \sum_i (-1)^i \dim H_c^i(\Sigma, \overline{\mathbb{Q}}_\ell)_{\chi^{-1},\chi'},
\end{aligned}$$

where the subscript χ^{-1}, χ' denotes the $(\mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{T}_{h,k}(\mathbb{F}_q))$-eigenspace corresponding to $\chi^{-1} \otimes \chi'$, and

$$\Sigma := (X_h \times X_h)/\mathbb{U}_{h,k}(\mathbb{F}_q).$$

Hence the statement of the theorem is equivalent to showing

$$\sum (-1)^i \dim H_c^i(\Sigma, \overline{\mathbb{Q}}_\ell)_{\chi^{-1},\chi'} = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases}$$

We follow a technique of Lusztig demonstrated in [19] wherein we construct an action of a connected torus $\mathcal{T}$ over $\overline{\mathbb{F}}_q$ on Σ and then use the fact (see, for example, [10, Proposition 10.15]) that

$$\sum (-1)^i H_c^i(\Sigma, \overline{\mathbb{Q}}_\ell) = \sum (-1)^i H_c^i(\Sigma^{\mathcal{T}}, \overline{\mathbb{Q}}_\ell).$$

For the remainder of the proof, we extend scalars to $\overline{\mathbb{F}}_q$. Recall from Lemma 2.2.4 that

$$X_h = \{g \in \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q) : \mathrm{Fr}_q(g) g^{-1} \in \widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)\}.$$

It is clear that the map

$$\begin{aligned}
X_h \times X_h \to \{(x, x', y, y') \in (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q) \times \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q) : \\
xy = \mathrm{Fr}_q(y),\ y' = \mathrm{Fr}_q(y')x'\}, \\
(g, g') \mapsto (\mathrm{Fr}_q(g) g^{-1}, \mathrm{Fr}_q(g') g'^{-1}, g, g'^{-1})
\end{aligned}$$

is an isomorphism. Since $(\mathbb{U}_{h,k}(\overline{\mathbb{F}}_q))^{\mathrm{Fr}_q} = \mathbb{U}_{h,k}(\mathbb{F}_q)$, for any $g \in X_h$ and $h \in \mathbb{U}_{h,k}(\mathbb{F}_q)$, we have $gh \in X_h$, and the image of $(gh, g'h)$ is $(\mathrm{Fr}_q(g) g^{-1}, \mathrm{Fr}_q(g') g'^{-1}, gh, h^{-1} g'^{-1})$. Moreover, $\mathrm{Fr}_q(gh)(gh)^{-1} = \mathrm{Fr}_q(g) g^{-1}$ if and only if $h \in \mathbb{U}_{h,k}(\mathbb{F}_q)$. It follows from this that the map

$$\begin{aligned}
\Sigma \to \{(x, x', y) \in (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q) : xy = \mathrm{Fr}_q(y)x'\}, \\
(g, g') \mapsto (\mathrm{Fr}_q(g) g^{-1}, \mathrm{Fr}_q(g') g'^{-1}, g g'^{-1})
\end{aligned}$$

is a bijection.

Since $\mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$ has an Iwahori factorization, any $y \in \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$ can be written uniquely in the form

$$\begin{aligned}
y = y_1' y_2' y_1'' y_2'', \quad & y_1' \in \widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h), && y_2' \in \widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-), \\
& y_1'' \in \widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-)), && y_2'' \in \widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h),
\end{aligned}$$

where $\widetilde{D}_h \subset \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$ is the subgroup of diagonal matrices. Hence Σ is in bijection with the set of tuples

$$\begin{aligned}(x, x', y_1', y_2', y_1'', y_2'') \in (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times (\widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h)) \\ \times (\widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-)) \times (\widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-))) \\ \times (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h))\end{aligned}$$

satisfying

$$x y_1' y_2' y_1'' y_2'' = \mathrm{Fr}_q(y_1' y_2' y_1'' y_2'') x'.$$

As in [19, p. 73], note that any $z \in \widetilde{U}_h$ can be written uniquely in the form $F(y_1')^{-1} x y_1'$ for $y_1' \in \widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h)$ and $x \in \widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)$ (this can be checked by the same calculation as [2, Section 6.8]). Using this together with the substitution $\widetilde{x}' = \mathrm{Fr}_q(y_2'') x' \in \widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)$, we have that Σ is in bijection with tuples

$$\begin{aligned}(\widetilde{x}', y_2', y_1'', y_2'') \in (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times (\widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-)) \\ \times (\widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-))) \times (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h))\end{aligned}$$

satisfying

$$\mathrm{Fr}_q(y_2' y_1'') \widetilde{x}' \in \widetilde{U}_h y_2' y_1'' y_2'' = \widetilde{U}_h y_1'' y_2''.$$

Consider the subgroup of $\mathbb{U}_{h,k}(\overline{\mathbb{F}}_q) \times \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$ given by

$$\begin{aligned}H := \{(t, t') : t, t' \text{ diag}, t \in \widetilde{D}_h t', \\ t^{-1} \mathrm{Fr}_q(t) = t'^{-1} \mathrm{Fr}_q(t') \text{ centralizes } \widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1} \widetilde{U}_h^-)\}.\end{aligned}$$

For $(t, t') \in H$ consider the map

$$a_{(t,t')}\colon (\widetilde{x}', y_2', y_1'', y_2'') \mapsto (\mathrm{Fr}_q(t) \widetilde{x}' \mathrm{Fr}_q(t)^{-1}, t' y_2' t'^{-1}, t' y_1'' t^{-1}, \mathrm{Fr}_q(t) y_2'' \mathrm{Fr}_q(t)^{-1}).$$

We first show that $a_{(t,t')}$ defines a map $\Sigma \to \Sigma$ for $(t, t') \in H$. For

$$\begin{aligned}(\widetilde{x}', y_2', y_1'', y_2'') \in (\widetilde{U}_h \cap \mathrm{Fr}_q(\widetilde{U}_h^-)) \times (\widetilde{U}_h \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-)) \\ \times (\widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h^-))) \times (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}(\widetilde{U}_h)),\end{aligned}$$

we have

$$\mathrm{Fr}_q(t' y_2' t'^{-1} t' y_1'' t^{-1}) \mathrm{Fr}_q(t) \widetilde{x}' \mathrm{Fr}_q(t)^{-1} \in \widetilde{U}_h t' y_1'' t^{-1} \mathrm{Fr}_q(t) y_2'' \mathrm{Fr}_q(t)^{-1}$$

if and only if

$$\mathrm{Fr}_q(t' y_2' y_1'') \widetilde{x}' \mathrm{Fr}_q(t)^{-1} \in \widetilde{U}_h t' y_1'' t^{-1} \mathrm{Fr}_q(t) y_2'' \mathrm{Fr}_q(t)^{-1} = \widetilde{U}_h \mathrm{Fr}_q(t') y_1'' y_2'' \mathrm{Fr}_q(t)^{-1},$$

where we used that $t^{-1} \mathrm{Fr}_q(t) = t'^{-1} \mathrm{Fr}_q(t')$ commutes with y_1''. Since $\mathrm{Fr}_q(t')$ normalizes $\widetilde{U}_h$, the above holds if and only if

$$\mathrm{Fr}_q(y_2' y_1'') \widetilde{x}' \in \widetilde{U}_h y_1'' y_2'',$$

which exactly means $(\widetilde{x}, y_2', y_1'', y_2'') \in \Sigma$. Since H is abelian, it is now clear that $a_{(t,t')}$ for $(t, t') \in H$ defines an action on Σ.

Observe that H contains $\mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{T}_{h,k}(\mathbb{F}_q)$ as a subgroup. Moreover, the action of H restricts to the left action of $\mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{T}_{h,k}(\mathbb{F}_q)$ inherited from the left-multiplication action of $\mathbb{T}_{h,k}(\mathbb{F}_q)$ on X_h. We now pinpoint an algebraic torus $\mathcal{T} \subset H$. By an explicit calculation,

$\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}\,\widetilde{U}_h^-$ consists of matrices $\left(\begin{smallmatrix} u_{n-k} & 0 \\ 0 & u_k \end{smallmatrix}\right) \in \mathbb{U}_{h,k}(\overline{\mathbb{F}}_q)$, where u_{n-k} is a lower-triangular matrix of rank $n-k$ and u_k is a lower-triangular matrix of rank k. Thus for any $a_0 \in \mathbb{W}_h(\overline{\mathbb{F}}_q)^\times$, the element $a := \mathrm{diag}(1,\dots,1,a_0,\dots,a_0) \in \mathbb{G}_{h,k}(\overline{\mathbb{F}}_q)$ (where there are k coordinates of a_0) centralizes $\widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}\,\widetilde{U}_h^-)$. If $t = \mathrm{diag}(t_1,\dots,t_n)$, then

$$\mathrm{Fr}_q(t) = \mathrm{diag}(\varphi(t_{\sigma(1)}), \varphi(t_{\sigma(2)}), \dots, \varphi(t_{\sigma(n)})),$$

and $t = \mathrm{Fr}_q(t)a$ is equivalent to the condition

$$t_i = \begin{cases} \varphi(t_{\sigma(i)}) & \text{if } 1 \le i \le n-k, \\ \varphi(t_{\sigma(i)}) \cdot a_0 & \text{if } n-k+1 \le i \le n. \end{cases} \tag{6.1}$$

Since the permutation σ has order n, it is clear from the above that t is determined by a single entry, say t_n. Let $t(\xi)$ denote the unique t satisfying (6.1) with $t_n = \xi$. Then $\mathrm{Fr}_q(t(\xi))^{-1}t(\xi)$ centralizes $\widetilde{D}_h \cdot (\widetilde{U}_h^- \cap \mathrm{Fr}_q^{-1}\,\widetilde{U}_h^-)$ and

$$\mathcal{T} := \{(t(\xi), t(\xi)) : \xi \in \overline{\mathbb{F}}_q^\times\} \subset H$$

is a connected algebraic torus over $\overline{\mathbb{F}}_q$ whose action on $\Sigma = (X_h \times X_h)/\mathbb{U}_{h,k}(\mathbb{F}_q)$ commutes with the $(\mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{T}_{h,k}(\mathbb{F}_q))$-action. One can easily see that the $\mathcal{T}$-fixed point set of Σ is

$$\begin{aligned} \Sigma^{\mathcal{T}} &= \{(1,1,y_2',1) : y_2' \in \widetilde{D}_h,\ y_2' \in \mathrm{Fr}_q(y_2') \cdot \widetilde{U}_h\} \\ &= \{(1,1,y_2',1) : y_2' \in \widetilde{D}_h^{\mathrm{Fr}_q}\} \cong \mathbb{T}_{h,k}(\mathbb{F}_q), \end{aligned}$$

where under the final identification, $\mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{T}_{h,k}(\mathbb{F}_q)$ acts by $(t,t') * y_2' = ty_2't'^{-1}$. Thus

$$\begin{aligned} \sum (-1)^i H_c^i(\Sigma^{\mathcal{T}}, \overline{\mathbb{Q}}_\ell)_{\chi^{-1},\chi'} &= \sum (-1)^i H_c^i(\mathbb{T}_{h,k}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)_{\chi^{-1},\chi'} \\ &= H_c^0(\mathbb{T}_{h,k}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)_{\chi^{-1},\chi'} \end{aligned}$$

and the theorem follows from

$$\dim H_c^0(\mathbb{T}_{h,k}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)_{\chi^{-1},\chi'} = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases}$$

□

6.2. Boyarchenko's conjectures. In this subsection, we combine the results of Sections 5.4 and 6.1 to prove Conjectures A and B (Theorems 6.2.1 and 6.2.2). Theorem 6.2.4 can be viewed as a refinement of Conjecture B in the sense that it proves that the nonvanishing cohomological degree of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is equal to the explicit integer r_χ defined in Theorem 5.1.1.

Theorem 6.2.1. *The finite-type $\mathbb{F}_{q^n}$-scheme X_h is maximal in the sense of Boyarchenko–Weinstein* [4]. *That is, for each $i \ge 0$, we have $H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = 0$ unless i or n is even, and the Frobenius morphism Fr_{q^n} acts on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ by the scalar $(-1)^i q^{ni/2}$.*

Proof. By Theorem 5.1.1, Fr_{q^n} acts on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ by multiplication by $(-1)^i q^{ni/2}$. To finish, we show that $H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = 0$ if i and n are both odd. Assume that n is odd. By definition of r_χ, it is enough to show that the sum $d_1 + \cdots + d_{r+1}$ is always even. We have

$$d_t = \left(\frac{n}{m_{t-1}} - \frac{n}{m_t}\right)(h_t - 1),$$

and since n is odd by assumption, then n/m_{t-1} and n/m_t must also be odd, and hence d_t is even. This completes the proof. □

Theorem 6.2.2. *For any $\chi \in \mathscr{T}_{n,h}$, the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ are nonzero in a single degree $i = s_\chi$, and $H_c^{s_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is an irreducible representation of $\mathbb{U}_{h,k}(\mathbb{F}_q)$. Moreover, $H_c^{s_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi] \cong H_c^{s_{\chi'}}(X_h, \overline{\mathbb{Q}}_\ell)[\chi']$ if and only if $\chi = \chi'$.*

Proof. Let π be an irreducible constituent in $H_c^{s_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ for some s_χ. Then by Corollary 5.1.3,

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\pi, H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]) = 0 \quad \text{for all } i \neq s_\chi.$$

But this implies that the alternating sum $\pm R_\chi = \sum_i (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ can have no cancellation, and $\pi \cong \pm R_\chi$. By Theorem 6.1.1,

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi] = \begin{cases} \text{irreducible} & \text{if } i = s_\chi, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if $\chi \neq \chi'$, then $\pm R_\chi$ and $\pm R_{\chi'}$ are nonisomorphic by Theorem 6.1.1 and it follows easily that $H_c^{s_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ and $H_c^{s_{\chi'}}(X_h, \overline{\mathbb{Q}}_\ell)[\chi']$ must also be nonisomorphic. □

As in Definition 2.2.3, for any $\zeta \in \mathbb{F}_{q^n}$, $t \in \mathbb{T}_{h,k}(\mathbb{F}_q)$, and $g \in \mathbb{U}_{h,k}(\mathbb{F}_q)$, let (ζ, t, g) denote the map $X_h \to X_h$ given by $x \mapsto \zeta \cdot t \cdot x \cdot g \cdot \zeta^{-1}$.

Theorem 6.2.3. *If $\zeta \in \mathbb{F}_{q^n}^\times$ has trivial stabilizer in* $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$*, then*

$$\mathrm{Tr}\big((\zeta, 1, g)^*; H_c^{s_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]\big) = (-1)^{s_\chi}\chi(g) \quad \textit{for any } g \in \mathbb{T}_{h,k}(\mathbb{F}_q).$$

Proof. This is identical to [6, proof of Proposition 6.2]. The argument is very similar to the proof of Lemma 6.2.5 in the present paper. □

Up to now, we have only shown that $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ is concentrated in a single degree s_χ. It is natural to expect, based on Theorem 5.1.1, that $s_\chi = r_\chi$. We resolve this question in Theorem 6.2.4. The proof uses purely cohomological techniques and essentially is a combination of Theorem 6.2.3 together with the Deligne–Lusztig fixed point formula.

Theorem 6.2.4. *For any $\chi\colon \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$,*

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^{s_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]\right) \neq 0.$$

In particular, $s_\chi = r_\chi$.

Proof. By Frobenius reciprocity, it is enough to show

$$\dim \mathrm{Hom}_{\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}(\chi, H_c^{s_\chi^{n,q}}(X_h^{n,q}, \overline{\mathbb{Q}}_\ell)[\chi]) \neq 0, \tag{6.2}$$

where we write $\mathbb{T}_{h,k} = \mathbb{T}_{h,k}^{n,q}$, $s_\chi = s_\chi^{n,q}$, and $X_h = X_h^{n,q}$ to emphasize the dependence on n, q. It is clear that once this is established, then by Theorem 5.1.1, it follows immediately that $s_\chi = r_\chi$. For notational simplicity, we write $H_c^i(X)$ to mean $H_c^i(X, \overline{\mathbb{Q}}_\ell)$. Recall from Definition 2.2.3 that for $\zeta \in \mathbb{F}_{q^n}^\times$, $t \in \mathbb{T}_{h,k}(\mathbb{F}_q)$, $g \in \mathbb{U}_{h,k}(\mathbb{F}_q)$, we write (ζ, t, g) to denote the map $X_h \to X_h$ given by $x \mapsto \zeta \cdot t \cdot x \cdot g \cdot \zeta^{-1}$. We first prove two lemmas.

Lemma 6.2.5. *Let p_0 be a prime dividing n. Then, for any $x \in \mathbb{F}_{q^{p_0}}^{\times} \smallsetminus \mathbb{F}_q^{\times}$ and any $g \in \mathbb{T}_{h,k}(\mathbb{F}_q)$,*

$$\begin{aligned}&(-1)^{s_\chi^{n,q}} \operatorname{Tr}\big((x,1,g); H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big)\\ &\quad = (-1)^{s_\chi^{n/p_0,q^{p_0}}} \operatorname{Tr}\big((1,1,g); H_c^{s_\chi^{n/p_0,q^{p_0}}}(X_h^{n/p_0,q^{p_0}})[\chi]\big).\end{aligned}$$

Proof. Fix $x \in \mathbb{F}_{q^{p_0}}^{\times} \smallsetminus \mathbb{F}_q^{\times}$. Recall that x acts by conjugation on

$$M = M_1 + M_2\varpi^k + \cdots + M_n\varpi^{[k(n-1)]} \in X_h^{n,q}$$

(recall the standard form of a point of $X_h^{n,q}$ in Definition 2.3.3):

$$x * M = x^{-1} \cdot M \cdot x.$$

If l is the unique integer $0 \le l \le n-1$ such that $lk = 1$ modulo n, then

$$x^{-1} \cdot \varpi^k \cdot x = x^{-1} \cdot \varphi^l(x) \cdot \varpi^k.$$

We have $\varphi^{l(j-1)}(x) = x$ if and only if $p_0 \mid (j-1)$ and therefore we see that if $x * M = M$, then necessarily $M_j = 0$ for $j \not\equiv 1$ modulo p_0 and

$$M = M_1 + M_{p_0+1}\varpi^{[kp_0]} + \cdots + M_{n-p_0+1}\varpi^{[k(n-p_0)]}.$$

For any integer m, let $[m]'$ be the unique integer $1 \le [m]' \le \frac{n}{p_0}$ such that $m \equiv [m]'$ modulo $\frac{n}{p_0}$. I now claim that there is a $\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)$-equivariant morphism

$$f\colon (X_h^{n,q})^x \to X_h^{n/p_0,q^{p_0}}$$

given by

$$\begin{aligned}&M_1 + M_{p_0+1}\varpi^{k[p_0]} + \cdots + M_{n-p_0+1}\varpi^{[k(n-p_0)]}\\ &\quad \mapsto M_1' + M_{p_0+1}'\varpi_{n/p_0}^{[k]'} + \cdots + M_{n-p_0+1}'\varpi_{n/p_0}^{[k(\frac{n}{p_0}-1)]'},\end{aligned}$$

where M_j' denotes the top-left-justified $\frac{n}{p_0} \times \frac{n}{p_0}$ matrix in M_j and

$$\varpi_{n/p_0} = \begin{pmatrix} 0 & 1_{\frac{n}{p_0}-1} \\ \pi & 0 \end{pmatrix}.$$

Using Definition 2.3.3, it is a straightforward check to see that this morphism is well-defined since $f(M)$ is of the form (2.5), satisfies (2.6), and the determinant condition

$$\varphi(\det(M)) = \det(M)$$

implies that $\varphi^{p_0}(\det(f(M))) = \det(f(M))$. (This last claim can be seen by observing that the rows and columns of M can be swapped so that the matrix becomes block diagonal of the form $\operatorname{diag}(f(M), \varphi^{\tau(2)}(f(M)), \ldots, \varphi^{\tau(p_0-1)}(f(M)))$.) The equivariance under

$$\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \cong \mathbb{T}_{h,k}^{n/p_0,q^{p_0}}(\mathbb{F}_q)$$

is clear.

By Proposition 3.3.1, $X_h^{n,q}$ is a separated, finite-type scheme over $\mathbb{F}_{q^n}$, and the action of $(x,t,g) \in \mathbb{F}_{q^n}^\times \times \mathbb{T}_{h,k}(\mathbb{F}_q) \times \mathbb{T}_{h,k}(\mathbb{F}_q)$ defines a finite-order automorphism. Moreover,

$$(x,t,g) = (1,t,g)\cdot(x,1,1),$$

where $(1,t,g)$ is a p-power-order automorphism and $(x,1,1)$ has prime-to-p order. Hence by the Deligne–Lusztig fixed point formula [9, Theorem 3.2], we have

$$\sum_i (-1)^i \operatorname{Tr}\big((x,t,g)^*; H_c^i(X_h^{n,q})\big) = \sum_i (-1)^i \operatorname{Tr}\big((1,t,g)^*; H_c^i((X_h^{n,q})^x)\big).$$

Therefore

$$\begin{aligned}
&\#\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \sum_i (-1)^i \operatorname{Tr}\big((x,1,g)^*; H_c^i(X_h^{n,q})[\chi]\big) \\
&\quad = \sum_{t\in\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)} \chi(t)^{-1} \sum_i (-1)^i \operatorname{Tr}\big((x,t,g)^*; H_c^i(X_h^{n,q})\big) \\
&\quad = \sum_{t\in\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)} \chi(t)^{-1} \sum_i (-1)^i \operatorname{Tr}\big((1,t,g)^*; H_c^i((X_h^{n,q})^x)\big) \\
&\quad = \#\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \sum_i (-1)^i \operatorname{Tr}\big((1,1,g)^*; H_c^i((X_h^{n,q})^x)[\chi]\big) \\
&\quad = \#\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \sum_i (-1)^i \operatorname{Tr}\big((1,1,g)^*; H_c^i(X_h^{n/p_0,q^{p_0}})[\chi]\big).
\end{aligned}$$

The desired equality now follows since by Theorem 6.2.2, $H_c^i(X_h^{n,q})[\chi]$ and $H_c^i(X_h^{n/p_0,q^{p_0}})[\chi]$ are nonzero only when $i = s_\chi^{n,q}$ and $i = s_\chi^{n/p_0,q^{p_0}}$, respectively. □

Lemma 6.2.6. *Let $\chi\colon \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$. Assume that we are in one of the following cases:*

(1) *$n > 1$ is odd and p_0 is a prime divisor of n.*

(2) *$n > 1$ is even and $p_0 = 2$.*

Fix a $\zeta \in \mathbb{F}_{q^{p_0}}^\times$ such that $\langle\zeta\rangle = \mathbb{F}_{q^{p_0}}^\times$ and consider the corresponding extension of χ defined as

$$\widetilde{\chi}\colon \mathbb{F}_{q^{p_0}}^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times, \quad (\zeta^i, g) \mapsto \begin{cases} \chi(g) & \text{if } q \text{ is even,} \\ ((-1)^{s_\chi^{n,q}+s_\chi^{n/p_0,q^{p_0}}})^i \cdot \chi(g) & \text{if } q \text{ is odd.} \end{cases}$$

Then

$$\sum_{x\in\mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q^\times} \widetilde{\chi}(x,1)^{-1} \neq 0.$$

Proof. If q is even, then

$$\widetilde{\chi}|_{\mathbb{F}_{q^{p_0}}^\times \times \{1\}} = 1,$$

so the conclusion holds. For the remainder of the proof, assume that q is odd. If we are in Case (1), then by Theorem 6.2.1, we know that $s_\chi^{n,q}$ and $s_\chi^{n/p_0,q^{p_0}}$ are both even. Hence

$$\widetilde{\chi}|_{\mathbb{F}_{q^{p_0}}^\times \times \{1\}} = 1$$

and we are done. The same conclusion holds if we are in Case (2) and $s_\chi^{n,q} + s_\chi^{n/p_0,q^{p_0}}$ is even, so assume $s_\chi^{n,q} + s_\chi^{n/p_0,q^{p_0}}$ is odd. Then $\widetilde{\chi}|_{\mathbb{F}_{q^{p_0}}^\times \times \{1\}}$ is nontrivial and

$$\sum_{x \in \mathbb{F}_{q^{p_0}}^\times} \widetilde{\chi}(x,1)^{-1} = 0.$$

Thus to show the conclusion of the lemma, it suffices to show that $\sum_{x\in\mathbb{F}_q^\times} \widetilde{\chi}(x,1)^{-1} \neq 0$. If $\zeta^m \in \mathbb{F}_q^\times$, then m must be a multiple of $q+1$, which is even. This implies that $\widetilde{\chi}(x,1) = 1$ if $x \in \mathbb{F}_q^\times$, and we are done. □

We are now ready to prove the theorem. First observe that $X_h^{1,q} = \mathbb{T}_{h,k}^{1,q}(\mathbb{F}_q)$ and hence for any $\chi\colon \mathbb{T}_{h,k}^{1,q}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, we have

$$H_c^{s_\chi^{1,q}}(X_h^{1,q})[\chi] = H_c^0(\mathbb{T}_{h,k}^{1,q}(\mathbb{F}_q))[\chi] = \chi,$$

so equation (6.2) holds for $n = 1$ and q arbitrary. We now induct on the number of prime divisors l of n. Assume that for a fixed integer $l \geq 0$, equation (6.2) holds for any $n = \prod_{i=1}^l p_i$ and arbitrary q, where the p_i are (possibly nondistinct) primes. We will show that equation (6.2) holds for any $n = \prod_{i=0}^l p_i$ and arbitrary q.

If n is odd, let p_0 be any prime divisor of n, and if n is even, let $p_0 = 2$. Define the character $\widetilde{\chi}\colon \mathbb{F}_{q^{p_0}}^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ as in Lemma 6.2.6. Then

$$\begin{aligned}
(6.3)\quad & \#(\mathbb{F}_{q^{p_0}}^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)) \cdot \dim \operatorname{Hom}_{\mathbb{F}_{q^{p_0}}^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}\big(\widetilde{\chi}, H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big) \\
&= \sum_{(x,g)\in \mathbb{F}_{q^{p_0}}^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)} \widetilde{\chi}(x,g)^{-1} \operatorname{Tr}\big((x,1,g); H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big) \\
&= \#(\mathbb{F}_q^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)) \cdot \dim \operatorname{Hom}_{\mathbb{F}_q^\times \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}\big(\widetilde{\chi}, H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big) \\
&\qquad + \sum_{(x,g)\in (\mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q^\times) \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)} \widetilde{\chi}(x,g)^{-1} \operatorname{Tr}\big((x,1,g); H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big).
\end{aligned}$$

By Lemma 6.2.5,

$$\begin{aligned}
(6.4)\quad & \sum_{(x,g)\in (\mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q^\times) \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)} \widetilde{\chi}(x,g)^{-1} \operatorname{Tr}\big((x,1,g); H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big) \\
&= (-1)^{s_\chi^{n,q} + s_\chi^{n/p_0,q^{p_0}}} \cdot \sum_{\substack{(x,g) \\ x\in \mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q^\times}} \widetilde{\chi}(x,g)^{-1} \\
&\qquad\qquad \cdot \operatorname{Tr}\big((1,1,g); H_c^{s_\chi^{n/p_0,q^{p_0}}}(X_h^{n/p_0,q^{p_0}})[\chi]\big) \\
&= (-1)^{s_\chi^{n,q} + s_\chi^{n/p_0,q^{p_0}}} \cdot \sum_{x\in \mathbb{F}_{q^{p_0}}^\times \smallsetminus \mathbb{F}_q^\times} \widetilde{\chi}(x,1)^{-1} \cdot N,
\end{aligned}$$

where

$$N := \#\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) \cdot \dim \operatorname{Hom}_{\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}\big(\chi, H_c^{s_\chi^{n/p_0,q^{p_0}}}(X_h^{n/p_0,q^{p_0}})[\chi]\big).$$

By the inductive hypothesis together with the fact that

$$\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q) = \mathbb{T}_{h,k}^{n/p_0,q^{p_0}}(\mathbb{F}_{q^{p_0}}) \cong \mathbb{W}_h^{(1)}(\mathbb{F}_{q^n}),$$

we have $N \neq 0$. Since p_0 satisfies the hypothesis of Lemma 6.2.6, we now see that (6.4) is nonzero. By (6.3), this implies that at least one of the following numbers is nonzero:

$$\dim \operatorname{Hom}_{\mathbb{F}_{q^{p_0}}^{\times} \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}\big(\widetilde{\chi}, H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big)$$

or

$$\dim \operatorname{Hom}_{\mathbb{F}_q^{\times} \times \mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}\big(\widetilde{\chi}, H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big).$$

In either case, we must have

$$\dim \operatorname{Hom}_{\mathbb{T}_{h,k}^{n,q}(\mathbb{F}_q)}\big(\chi, H_c^{s_\chi^{n,q}}(X_h^{n,q})[\chi]\big) \neq 0. \qquad \square$$

6.3. The zeta function of X_h. Thanks to Theorems 5.1.1 and 6.2.4, we now have an explicit description of the nonvanishing cohomological degree r_χ. In the next theorem, we give a formula for the zeta function of X_h.

Theorem 6.3.1. *The Hasse–Weil zeta function of X_h is*

$$Z(X_h, t) = \prod_{i=0}^{2\dim X_h} \big(1 - (-q^{n/2})^i \cdot t\big)^{(-1)^{i+1} \dim H_c^i(X_h, \overline{\mathbb{Q}}_\ell)},$$

where

$$\dim H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = \sum_{\substack{\chi: \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^{\times}, \\ r_\chi = i}} q^{n d_\chi/2}.$$

Moreover, if n is odd, then $Z(X_h, t)^{-1}$ is a polynomial.

Proof. By the Grothendieck–Lefschetz trace formula,

$$Z(X_h, t) = \prod_{i=0}^{2\dim X_h} \big(\det((1 - t \operatorname{Fr}_{q^n}); H_c^i(X_h, \overline{\mathbb{Q}}_\ell))\big)^{(-1)^{i+1}}.$$

By Theorem 6.2.1, we know that Fr_{q^n} acts on $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ by multiplication by $(-q^{n/2})^i$, so

$$\det((1 - t \operatorname{Fr}_{q^n}); H_c^i(X_h, \overline{\mathbb{Q}}_\ell)) = \big(1 - (-q^{n/2})^i \cdot t\big)^{\dim H_c^i(X_h, \overline{\mathbb{Q}}_\ell)}.$$

It remains to prove the dimension formula.

Let $H_c^\bullet(X_h, \overline{\mathbb{Q}}_\ell) = \bigoplus_i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$. By Theorem 6.2.2, $H_c^\bullet(X_h, \overline{\mathbb{Q}}_\ell)$ is a direct sum of distinct irreducible representations of $\mathbb{U}_{h,k}(\mathbb{F}_q)$ (parametrized by $\chi: \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$). Write

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = \pi_{i,1} \oplus \cdots \oplus \pi_{i,k_i},$$

where the π_{ij} are nonisomorphic irreducible representations of $\mathbb{U}_{h,k}(\mathbb{F}_q)$. Recall that the regular representation $\operatorname{Reg} := \operatorname{Ind}_{\{1\}}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(1)$ has the property that an irreducible $\mathbb{U}_{h,k}(\mathbb{F}_q)$-representation π has multiplicity $\dim \pi$ in Reg. Then

$$\begin{aligned}\dim \operatorname{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\big(\operatorname{Reg}, H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\big) &= \dim \operatorname{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\operatorname{Reg}, \bigoplus_{j=1}^{k_i} \pi_{ij}\right) \\ &= \sum_{j=1}^{k_i} \dim \pi_{ij}.\end{aligned}$$

On the other hand, by Theorem 5.1.1,

$$\begin{aligned}
&\dim \mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\mathrm{Reg}, H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) \\
&= \dim \mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\bigoplus_{\chi \text{ s.t. } r_\chi = i} \mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) \\
&= \dim \bigoplus_{\chi \text{ s.t. } r_\chi = i} \mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) \\
&= \sum_{\chi \text{ s.t. } r_\chi = i} q^{n d_\chi/2},
\end{aligned}$$

where each sum ranges over $\chi\colon \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ such that $r_\chi = i$. This proves the dimension formula. The final assertion now follows from Theorem 6.2.1 since if n is odd and

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell) \neq 0,$$

then i must be even, and hence $Z(X_h, t)$ has no nontrivial factors in the numerator. □

Example 6.3.2. We demonstrate how to calculate the Hasse–Weil zeta function in the case that n is prime. First observe that the Howe decomposition of a character of $\mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})$ must be of the form

$$\chi = \chi_1^0(\mathrm{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}) \cdot \chi_2^0,$$

where χ_1^0 is a character of $\mathbb{W}_h^{(1)}(\mathbb{F}_q)$ and χ_2^0 is a primitive character of $\mathbb{W}_{h'}^{(1)}(\mathbb{F}_{q^n})$. After fixing a level h' with $1 \leq h' \leq h$, the number of such χ is equal to

$$N_{h'} := (q^{h-1} - q^{h'-1} + 1) \cdot (q^{p(h'-1)} - q^{p(h'-2)} - q + 1),$$

and

$$d_\chi = (n-1)(h'-1), \quad r_\chi = (n-1)(h-1) + (n-1)(h-h').$$

Thus by Theorem 6.3.1,

$$\dim H_c^i(X_h, \overline{\mathbb{Q}}_\ell) = \begin{cases} q^{h-1} & \text{if } i = 2(n-1)(h-1), \\ N_{h'} \cdot q^{n(n-1)(h'-1)/2} & \text{if } i = (n-1)(h-1) + (n-1)(h-h'). \end{cases}$$

We can now write down explicit formulas for the zeta function of X_h. For example:

Corollary 6.3.3. *If $n = 2$, then*

$$Z(X_3, t) = \frac{(1 + q^3 \cdot t)^{(q^2-q+1)(q^2-q)q}}{(1 - q^2 \cdot t)^{q^2} \cdot (1 - q^4 \cdot t)^{(q^4-q^2-q+1)q^2}}.$$

6.4. Examples. Prior to this work, the only cases in which the $\mathbb{U}_{h,k}(\mathbb{F}_q)$-representations $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ had been studied were in the following cases:

(1) For $h = 2$ and $k = 1$, this was done by Boyarchenko–Weinstein in [4, Theorem 4.5.1].

(2) For h, k arbitrary, χ primitive, and char $K > 0$, this was done by the author in [6]. Before this some smaller cases were done:
   (a) The $n = 2$, $h = 3$, $k = 1$, char $K > 0$ case was done by Boyarchenko in [2, Theorem 5.20].
   (b) The $n = 2$, h arbitrary, $k = 1$, char $K > 0$ case was done by the author in [5].

We explain how to specialize Theorems 5.1.1, 6.2.1, and 6.2.2 to recover these results.

Note that in previous work, the unipotent group schemes are called $U_{h,k}^{n,q}$ and are defined over $\mathbb{F}_{q^n}$, whereas the unipotent group schemes in this paper are called $\mathbb{U}_{h,k}$ and are defined over $\mathbb{F}_q$ (see Remark 2.2.5 for a more detailed discussion). However, $U_{h,k}^{n,q}(\mathbb{F}_{q^n}) \cong \mathbb{U}_{h,k}(\mathbb{F}_q)$, and there is a natural way to realize $\mathbb{W}_h^{(1)}(\mathbb{F}_{q^n})$ as a subgroup of each, so the distinction between working in the ambient $U_{h,k}^{n,q}$ and $\mathbb{U}_{h,k}$ only appears in the proofs and not in the theorem statements.

6.4.1. The case $h = 2, k = 1$. In this setting, the subquotients of the multiplicative group of the division algebra in equal and mixed characteristic are isomorphic, and so one does not run into the mixed characteristic difficulties that arise when $h > 2$.

Note that $\mathbb{T}_{2,1}(\mathbb{F}_q) \cong \mathbb{W}_2^{(1)}(\mathbb{F}_{q^n}) \cong \mathbb{F}_{q^n}$. Let $\chi\colon \mathbb{F}_{q^n} \to \overline{\mathbb{Q}}_\ell^\times$ be a character. If χ is trivial, then it corresponds to $(\{1,1,n\},\{2,1,1\})$, and if χ is nontrivial of conductor m, then it corresponds to $(\{1,m,n\},\{2,2,1\})$. Then by Theorem 5.1.1,

$$r_\chi = (n - \tfrac{n}{m}) + 2(\tfrac{n}{m} - 1) = n + \tfrac{n}{m} - 2$$

and

$$\mathrm{Hom}_{\mathbb{U}_{2,1}(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{F}_{q^n}}^{\mathbb{U}_{2,1}(\mathbb{F}_q)}(\chi), H_c^i(X_2, \overline{\mathbb{Q}}_\ell)\right) \neq 0 \iff i = n + \tfrac{n}{m} - 2.$$

The center of $\mathbb{U}_{2,q}(\mathbb{F}_q)$ is $\mathbb{T}_{2,1}(\mathbb{F}_q) \cong \mathbb{F}_{q^n}$. Since the actions of $\mathbb{T}_{2,1}(\mathbb{F}_q)$ and $\mathbb{U}_{2,1}(\mathbb{F}_q)$ on X_2 agree on the center of $\mathbb{U}_{2,1}(\mathbb{F}_q)$, the above equation implies that

$$H_c^i(X_2, \overline{\mathbb{Q}}_\ell)[\chi] \neq 0 \iff i = n + \tfrac{n}{m} - 2.$$

The centrality of $\mathbb{T}_{2,1}(\mathbb{F}_q)$ in $\mathbb{U}_{2,1}(\mathbb{F}_q)$ (which is not true for $h > 2$) allowed us to obtain Theorem 6.2.4 from Theorem 5.1.1 automatically. We now see that maximality of X_2 holds by Theorem 6.2.1 (this is [4, Theorem 4.5.1 (b)]), and the irreducibility and multiplicity-one properties of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ follow from Theorem 6.2.2 (this is [4, Theorem 4.5.1 (a)]).

6.4.2. The case h, k arbitrary, χ primitive, char $K > 0$. Let $\chi\colon \mathbb{T}_{h,k}(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ be primitive. Then the sequences $\{m_i\}, \{h_i\}$ associated to the Howe factorization of χ are $(\{1,n,n\},\{h,h,1\})$. By Theorem 5.1.1,

$$r_\chi = (n - \tfrac{n}{n})(h-1) + 2(\tfrac{n}{n} - \tfrac{n}{n})(h-1) = (n-1)(h-1)$$

and

$$\mathrm{Hom}_{\mathbb{U}_{h,k}(\mathbb{F}_q)}\left(\mathrm{Ind}_{\mathbb{T}_{h,k}(\mathbb{F}_q)}^{\mathbb{U}_{h,k}(\mathbb{F}_q)}(\chi), H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) \neq 0 \iff i = (n-1)(h-1).$$

The subgroup $\mathbb{T}_{h,k}(\mathbb{F}_q)$ is not central in $\mathbb{U}_{h,k}(\mathbb{F}_q)$, but the center of $\mathbb{U}_{h,k}(\mathbb{F}_q)$ contains $H_0(\mathbb{F}_q)$, where H_0 is the subgroup of $\mathbb{U}_{h,k}$ consisting of diagonal matrices with entries in

$$\{(1, 0, \ldots, 0, *)\} \subset \mathbb{W}_h^{(1)}.$$

Moreover, χ is primitive if and only if its restriction to $H_0(\mathbb{F}_q)$ is primitive, and it therefore follows that r_χ only depends on $\chi|_{H_0(\mathbb{F}_q)}$. Hence

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi] \neq 0 \iff i = (n-1)(h-1).$$

Note that again, it was the fact that the restriction of χ to the center of $\mathbb{U}_{h,k}(\mathbb{F}_q)$ determines r_χ (which is not true for nonprimitive χ), which allowed us to immediately pinpoint the nonvanishing cohomological degree of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ from Theorem 5.1.1. The irreducibility

and multiplicity-one properties of $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ follow from Theorem 6.2.2. In this generality, this is [6, Theorem 6.3], in the case (2a), this is [2, Theorem 5.20], and in the case (2b), irreducibility is [5, Theorem 5.1] and multiplicity-one follows from the trace formula given in [5, Theorem 5.2].

7. Torus eigenspaces of the homology of semi-infinite Deligne–Lusztig varieties

We return our discussion to the semi-infinite Deligne–Lusztig variety $\widetilde{X}$ associated to $L^\times \hookrightarrow D^\times$. Our goal is to understand the representations of $D^\times$ arising from the θ-eigenspaces $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$, where $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$. Once we understand the relationship between the (ind-pro-)scheme structure we defined on $\widetilde{X}$ in Section 2.1 and the action of $L^\times \times D^\times$, the problem of computing the representations $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ can be reduced to the problem of computing the representations $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$. This reduction was already known by Boyarchenko [2], and therefore, after we summarize [2, Lemma 6.11, Corollary 6.12], Theorems 7.1.1, 7.1.2, and 7.1.3 follow from the theorems in Section 6. We then use these results together with a method of Henniart [13, 14] to characterize the $D^\times$-representations $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ in terms of the local Langlands and Jacquet–Langlands correspondences (Theorem 7.2.1).

7.1. Semi-infinite Deligne–Lusztig varieties. We first define some terminology. If $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ is a smooth character, then there exists an h such that the restriction $\theta|_{U_L^h}$ is trivial. We call the smallest such h the *level* of θ. We say that $x \in L^\times$ is *very regular* if $x \in \mathcal{O}_L^\times$ and its image in the residue field $\mathbb{F}_{q^n}^\times$ has trivial $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-stabilizer. Later in this section, we will give a character formula for $H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ on the subgroup of $D^\times$ consisting of very regular elements of $L^\times$. By work of Henniart [13, 14], such character formulas are often enough to pinpoint these irreducible representations. We learned of this strategy from [3] where these ideas were applied to $\mathrm{GL}_n(K)$ and $D_{1/n}^\times$, combined this with the theory of χ-datum of Langlands–Shelstad [18] in [6, Section 7.2], and in joint work with A. Ivanov, carried it out for any inner form of $\mathrm{GL}_n(K)$ in [7, Section 10].

The action of $L^\times \times D^\times$ on $\widetilde{X}$ induces an $(L^\times/U_L^h) \times (D^\times/U_D^{n(h-1)+1})$-action on

$$\widetilde{X}_h := \bigsqcup_{m\in\mathbb{Z}} \widetilde{X}_h{}^{(m)}.$$

Recall that X_h is a subvariety of $\widetilde{X}_h'^{(0)} \cong \widetilde{X}_h^{(0)} \hookrightarrow \widetilde{X}_h$ whose stabilizer is

$$\begin{aligned}\widetilde{\Gamma}_h &:= \langle(\pi, \pi^{-1})\rangle \cdot \langle(\zeta, \zeta^{-1})\rangle \cdot (U_L^1/U_L^h \times U_D^1/U_D^{n(h-1)+1}) \\ &\subset L^\times/U_L^h \times D^\times/U_D^{n(h-1)+1},\end{aligned}$$

and $\widetilde{X}_h$ is equal to the union of $(L^\times/U_L^h \times D^\times/U_D^{n(h-1)+1})$-translates of (the image of) X_h. It therefore follows that there is a natural isomorphism

$$H_i(\widetilde{X}_h, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Ind}_{\widetilde{\Gamma}_h}^{(L^\times/U_L^h)\times(D^\times/U_D^{n(h-1)+1})}(H_i(X_h, \overline{\mathbb{Q}}_\ell)). \tag{7.1}$$

By Proposition 3.3.1,

$$H_i(X_h, \overline{\mathbb{Q}}_\ell) \cong H_c^{2(n-1)(h-1)-i}(X_h, \overline{\mathbb{Q}}_\ell) \otimes (q^{n(n-1)(h-1)})^{\deg}.$$

Moreover, we have the following result.

Theorem 7.1.1. *Let $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ be a smooth character whose restriction to U_L^h is trivial and set $\chi := \theta|_{U_L^1}$. Then for $r_\theta := 2(n-1)(h-1) - r_\chi$,*

$$H_i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \neq 0 \iff i = r_\theta.$$

Moreover,

$$\eta_\theta := H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \mathrm{Ind}_{\pi^{\mathbb{Z}}\cdot\mathcal{O}_D^\times}^{D^\times}(\eta'_\theta),$$

where η'_θ is a representation naturally obtained from θ and $H_c^{r_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$. Explicitly:

(i) *$H_c^{r_\chi}(X_h, \overline{\mathbb{Q}}_\ell)[\chi]$ extends to a representation η°_θ of*

$$\mathbb{G}_{h,k}(\mathbb{F}_q) \cong \mathbb{F}_{q^n}^\times \ltimes \mathbb{U}_{h,k}(\mathbb{F}_q) \cong \mathcal{O}_D^\times / U_D^{n(h-1)+1}$$

with $\mathrm{Tr}(\eta^\circ_\theta(\zeta)) = (-1)^{r_\theta}\theta(\zeta)$, where $\zeta \in \mathcal{O}_L^\times$ is very regular. We may view η°_θ as a representation of $\mathcal{O}_D^\times$.

(ii) *We can extend η°_θ to a representation η'_θ of $\pi^{\mathbb{Z}}\cdot\mathcal{O}_D^\times$ by demanding $\pi \mapsto \theta(\pi)$.*

Proof. The statement is exactly parts (a) and (b) of [2, Proposition 5.19]: Part (i) holds by Theorem 6.2.3 (see [2, Step 1 of Section 6.15]) and part (ii) essentially follows by (7.1) (see [2, Section 6.14.5 and Step 3 of Section 6.15]). □

Theorem 7.1.2. *If $\theta\colon L^\times \to \overline{\mathbb{Q}}_\ell^\times$ has trivial $\mathrm{Gal}(L/K)$-stabilizer, then the $D^\times$-representation $H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ is irreducible.*

Proof. To prove this, we need to show that the normalizer of the $(\mathbb{Z}\cdot\mathcal{O}_D^\times)$-representation η'_θ in $D^\times$ is exactly $\mathbb{Z}\cdot\mathcal{O}_D^\times$. To see this, it is sufficient to show that η'_θ is not invariant under the conjugation action of Π. Let $x \in \mathcal{O}_L^\times \subset \mathcal{O}_D^\times$ be very regular. Recall that by Theorem 6.2.3 and the definition of η'_θ given in Theorem 7.1.1,

$$\mathrm{Tr}\,\eta'_\theta(x) = (-1)^{r_\theta}\cdot\theta(x).$$

For any generator Π of the unique maximal ideal of $\mathcal{O}_D$, we have $\Pi\cdot x\cdot\Pi^{-1} = \varphi^k(x)$. Thus conjugation by Π normalizes the set of very regular elements and

$$\mathrm{Tr}\,\eta'_\theta(\Pi\cdot x\cdot\Pi^{-1}) = (-1)^{r_\chi}\cdot\theta(\varphi^k(x)).$$

Therefore, if θ has trivial $\mathrm{Gal}(L/K)$-stabilizer, then $\pi^{\mathbb{Z}}\cdot\mathcal{O}_D^\times$ is the normalizer of η'_θ in $D^\times$. □

Theorem 7.1.3. *Let $x \in \mathcal{O}_L^\times$ be very regular. Then*

$$\mathrm{Tr}\,\eta_\theta(x) = (-1)^{r_\theta}\cdot\sum_{\gamma\in\mathrm{Gal}(L/K)}\theta^\gamma(x).$$

Proof. We have

$$\mathrm{Tr}\,\eta_\theta(x) = \sum_{\substack{g\in D^\times/(\mathbb{Z}\cdot\mathcal{O}_D^\times)\\ gxg^{-1}\in\mathbb{Z}\cdot\mathcal{O}_D^\times}}\mathrm{Tr}\,\eta'_\theta(gxg^{-1}) = \sum_{\gamma\in\mathrm{Gal}(L/K)}(-1)^{r_\theta}\cdot\theta^\gamma(x),$$

where the second equality holds by [3, Lemma 5.1 (b)] together with Theorem 6.2.3. □

7.2. The local Langlands and Jacquet–Langlands correspondences. To conclude the paper, we will use the above results to describe the supercuspidal $D^\times$-representation $H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$ in terms of the local Langlands and Jacquet–Langlands correspondences. We use a method of Henniart [13, 14] in the form discussed in [7, Section 10].

Fix a character $\epsilon\colon K^\times \to \overline{\mathbb{Q}}_\ell^\times$ with $\ker(\epsilon) = \mathrm{Nm}_{L/K}(L^\times)$ and let $\mathcal{G}_K(n)$ denote the set of irreducible n-dimensional representations σ of the Weil group $\mathcal{W}_K$ such that

$$\sigma \cong \sigma \otimes (\epsilon \circ \mathrm{rec}_K^{-1}),$$

where $\mathrm{rec}_K\colon K^\times \to \mathcal{W}_K^{\mathrm{ab}}$ is the reciprocity isomorphism from local class field theory. Let $\mathcal{X}$ denote the set of characters $L^\times \to \overline{\mathbb{Q}}_\ell^\times$ with trivial $\mathrm{Gal}(L/K)$-stabilizer. Then there is a canonical bijection

$$\mathcal{X}/\operatorname{Gal}(L/K) \to \mathcal{G}_K(n), \quad \theta \mapsto \sigma_\theta$$

obtained by invoking the theory of χ-datum of Langlands–Shelstad [18, Section 2.5] (also see [6, Section 7.2] for an exposition). Now let $\mathcal{A}_K(D^\times)$ denote the set of isomorphism classes of irreducible supercuspidal representations ρ of $D^\times$ such that

$$\rho \cong \rho \otimes (\epsilon \circ \mathrm{Nrd}),$$

where Nrd is the reduced norm of $D^\times$. Then by local Langlands and Jacquet–Langlands correspondences, there is a canonical bijection

$$\mathcal{G}_K(n) \to \mathcal{A}_K(D^\times), \quad \sigma_\theta \mapsto \rho_\theta$$

satisfying certain natural properties.

Theorem 7.2.1. *The bijection*

$$\mathcal{X}/\operatorname{Gal}(L/K) \xrightarrow{\sim} \mathcal{A}_K(D^\times), \quad [\theta] \longmapsto H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta]$$

agrees with the composition of the local Langlands and Jacquet–Langlands correspondences.

Proof. Since $\mathrm{Nrd}(D^\times) \subset \mathrm{Nm}_{L/K}(L^\times)$, it follows that $H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \in \mathcal{A}_K(D^\times)$. By this together with Theorems 7.1.2 and 7.1.3, we may apply [7, Proposition 10.5] to obtain $H_{r_\theta}(\widetilde{X}, \overline{\mathbb{Q}}_\ell)[\theta] \cong \rho_\theta$. □

In particular, by Theorem 7.2.1, the cohomology of semi-infinite Deligne–Lusztig varieties for division algebras gives a geometric realization of the Jacquet–Langlands correspondence on $\mathcal{A}_K(-)$: if D_1, D_2 are any two division algebras over K of dimension n^2 and $\widetilde{X}_1$, $\widetilde{X}_2$ are the corresponding varieties, then

$$\mathcal{A}_K(D_1^\times) \xrightarrow{\mathrm{JLC}} \mathcal{A}_K(D_2^\times), \quad H_{r_\theta}(\widetilde{X}_1, \overline{\mathbb{Q}}_\ell)[\theta] \longmapsto H_{r_\theta}(\widetilde{X}_2, \overline{\mathbb{Q}}_\ell)[\theta].$$

(Recall from Theorem 5.1.1 that r_θ does not depend on the Hasse invariant of D.)

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Eingegangen 5. September 2018, in revidierter Fassung 10. September 2019