



Affine Deligne–Lusztig varieties at infinite level

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Abstract

We initiate the study of affine Deligne–Lusztig varieties with arbitrarily deep level structure for general reductive groups over local fields. We prove that for GL_n and its inner forms, Lusztig’s semi-infinite Deligne–Lusztig construction is isomorphic to an affine Deligne–Lusztig variety at infinite level. We prove that their homology groups give geometric realizations of the local Langlands and Jacquet–Langlands correspondences in the setting that the Weil parameter is induced from a character of an unramified field extension. In particular, we resolve Lusztig’s 1979 conjecture in this setting for minimal admissible characters.

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1 Introduction

In their fundamental paper [13], Deligne and Lusztig gave a powerful geometric approach to the construction of representations of finite reductive groups. To a reductive group G over a finite field $\mathbb{F}_q$ and a maximal $\mathbb{F}_q$ -torus $T \subseteq G$, they attach a variety given by the set of Borel subgroups of G lying in a fixed relative position (depending on T) to their Frobenius translate. This variety has a T -torsor called the *Deligne–Lusztig variety*. The Deligne–Lusztig variety has commuting actions of G and T , and its ℓ -adic étale cohomology realizes a natural correspondence between characters of $T(\mathbb{F}_q)$ and representations of $G(\mathbb{F}_q)$.

Two possible ways of generalizing this construction to reductive groups over local fields are to consider subsets cut out by Deligne–Lusztig conditions in the semi-infinite flag manifold (in the sense of Feigin–Frenkel [15]) or in affine flag manifolds of increasing level. The first approach is driven by an outstanding conjecture of Lusztig [25] that the semi-infinite Deligne–Lusztig set has an algebro-geometric structure, one can define its ℓ -adic homology groups, and the resulting representations should be irreducible supercuspidal. This conjecture was studied in detail in the case of division algebras by Boyarchenko and the first named author in [5, 10, 11], and ultimately resolved in this setting in [12]. Prior to the present paper, Lusztig’s conjecture was completely open outside the setting of division algebras.

The second approach is based on Rapoport’s affine Deligne–Lusztig varieties [29], which are closely related to the reduction of (integral models of) Shimura varieties. Affine Deligne–Lusztig varieties for arbitrarily deep level structure were introduced and then studied in detail for GL_2 by the second named author in [21–23], where it was shown that their ℓ -adic cohomology realizes many irreducible supercuspidal representations for this group.

The goals of the present paper are to show that these constructions

- (A) are isomorphic for all inner forms of GL_n and their maximal unramified elliptic torus
- (B) realize the local Langlands and Jacquet–Langlands correspondences for supercuspidal representations coming from unramified field extensions

The first goal is achieved by computing both sides and defining an explicit isomorphism between Lusztig’s semi-infinite construction and an inverse limit of coverings of affine Deligne–Lusztig varieties. In particular, this allows us to use the known scheme structure of affine Deligne–Lusztig varieties to define a natural scheme structure on the semi-infinite side, which was previously only known in the case of division algebras. This resolves the algebro-geometric conjectures of [25] for all inner forms of GL_n .

To attain the second goal, we study the cohomology of this infinite-dimensional variety using a wide range of techniques. To show irreducibility of certain eigenspaces under the torus action, we generalize a method of Lusztig [26, 31] to quotients of parahoric subgroups which do not come from reductive groups over finite rings. We study the geometry and its behavior under certain group actions to prove an analogue of cuspidality for representations of such quotients. To obtain a comparison to the local Langlands correspondence, we use the Deligne–Lusztig fixed-point formula to determine the character on the maximal unramified elliptic torus and use characterizations of automorphic induction due to Henniart [19, 20]. In particular, for minimal admissible characters, we resolve the remaining part of Lusztig’s conjecture (supercuspidality) for all inner forms of GL_n .

We now give a more detailed overview. Let K be a non-archimedean local field with finite residue field $\mathbb{F}_q$, let $\tilde{K}$ be the completion of the maximal unramified extension of K and let σ denote the Frobenius automorphism of $\tilde{K}/K$. For any algebro-geometric object X over K , we write $\tilde{X} := X(\tilde{K})$ for the set of its $\tilde{K}$ -points. Let $\mathcal{G}$ be a connected reductive group over K . For simplicity assume that $\mathcal{G}$ is split. For $b \in \tilde{\mathcal{G}}$, let J_b be the

σ -stabilizer of b

$$J_b(R) := \{g \in \mathcal{G}(R \otimes_K \check{K}) : g^{-1}b\sigma(g) = b\}$$

for any K -algebra R . Then J_b is an inner form of a Levi subgroup of $\mathcal{G}$, and if b is basic, J_b is an inner form of $\mathcal{G}$. Let $\mathcal{T}$ be a maximal split torus in $\mathcal{G}$. For an element w in the Weyl group of $(\mathcal{G}, \mathcal{T})$, let

$$T_w(R) := \{t \in \mathcal{T}(R \otimes_K \check{K}) : t^{-1}\dot{w}\sigma(t) = \dot{w}\}$$

for any K -algebra R , where $\dot{w}$ is a lift of w to $\check{\mathcal{G}}$.

The semi-infinite Deligne–Lusztig set $X_w^{DL}(b)$ is the set of all Borel subgroups of $\check{\mathcal{G}}$ in relative position w to their $b\sigma$ -translate. It has a cover

$$\dot{X}_w^{DL}(b) := \{g\check{U} \in \check{\mathcal{G}}/\check{U} : g^{-1}b\sigma(g) \in \check{U}\dot{w}\check{U}\} \subseteq \check{\mathcal{G}}/\check{U}$$

with a natural action by $J_b(K) \times T_w(K)$, and this set coincides with Lusztig's construction [25]. On the other hand, for arbitrarily deep congruence subgroups $J \subseteq \check{\mathcal{G}}$, one can define affine Deligne–Lusztig sets of higher level J ,

$$X_x^J(b) := \{gJ \in \check{\mathcal{G}}/J : g^{-1}b\sigma(g) \in JxJ\} \subseteq \check{\mathcal{G}}/J,$$

where x is a J -double coset in $\check{\mathcal{G}}$. Under some technical conditions on x , we prove that these sets can be endowed with a structure of an $\mathbb{F}_q$ -scheme (Theorem 4.9). We remark that when K has mixed characteristic, $\check{\mathcal{G}}/J$ is a ind-(perfect scheme), so $X_x^J(b)$ will also carry the structure of a perfect scheme.

We now specialize to the following setting. Consider $\check{\mathcal{G}} = \mathrm{GL}_n(\check{K})$ and $G = J_b(K)$ for some basic $b \in \mathrm{GL}_n(\check{K})$ so that G is an inner form of $\mathrm{GL}_n(K)$. Let w be a Coxeter element so that $T := T_w(K) \cong L^\times$ for the degree- n unramified extension L of K . Let $G_{\mathcal{O}}$ be a maximal compact subgroup of G and let $T_{\mathcal{O}} = T \cap G_{\mathcal{O}} \cong \mathcal{O}_L^\times$. We consider a particular tower of affine Deligne–Lusztig varieties $\dot{X}_{\dot{w}_r}^m(b)$ for congruence subgroups of $\check{\mathcal{G}}$ indexed by m , where the image of each $\dot{w}_r$ in the Weyl group is w . We form the inverse limit $\dot{X}_w^\infty(b) = \varprojlim_{r > m \geq 0} \dot{X}_{\dot{w}_r}^m(b)$, which carries a natural action of $G \times T$.

Theorem (6.9) *There is a $(G \times T)$ -equivariant map of sets*

$$\dot{X}_w^{DL}(b) \xrightarrow{\sim} \dot{X}_w^\infty(b).$$

In particular, this gives $\dot{X}_w^{DL}(b)$ the structure of a scheme over $\overline{\mathbb{F}}_q$.

We completely determine the higher level affine Deligne–Lusztig varieties $\dot{X}_{\dot{w}_r}^m(b)$. They are $(\mathcal{O}_L/\mathfrak{p}_L^{m+1})^\times$ -torsors over the schemes $X_{\dot{w}_r}^m(b)$, which are interesting in their own right. In particular, $X_{\dot{w}_r}^0(b)$ provide examples of explicitly described Iwahori-level affine Deligne–Lusztig varieties. We prove the following.

Theorem (6.17) *The scheme $X_{\dot{w}_r}^m(b)$ is a disjoint union, indexed by $G/G_{\mathcal{O}}$, of classical Deligne–Lusztig varieties for the reductive quotient of $G_{\mathcal{O}} \times T_{\mathcal{O}}$ times finite-dimensional affine space.*

The disjoint union decomposition is deduced from Viehmann [32]. We point out the similarity between the Iwahori level varieties $X_{\dot{w}_r}^0(b)$ and those considered by Görtz and He [16, e.g. Proposition 2.2.1], though in our setting, the elements $\dot{w}_r$ can have arbitrarily large length in the extended affine Weyl group.

One of the key insights throughout our paper is the flexibility of working with different representatives b of a σ -conjugacy class. For example, when $G = \mathrm{GL}_n(K)$, switching between $b = 1$ and b being a Coxeter element allows us to use techniques that are otherwise inaccessible.

Having established the isomorphism $\dot{X}_w^{DL}(b) \xrightarrow{\sim} \dot{X}_w^{\infty}(b)$, the main objective in the rest of the paper is to study the virtual G -representation

$$R_T^G(\theta) := \sum_i (-1)^i H_i(\dot{X}_w^{\infty}(b), \overline{\mathbb{Q}}_{\ell})[\theta]$$

for smooth characters $\theta: T \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$, where $[\theta]$ denotes the subspace where T acts by θ . We write $|R_T^G(\theta)|$ to denote the genuine representation when one of $\pm R_T^G(\theta)$ is genuine.

One could try to calculate $R_T^G(\theta)$ by calculating the cohomology of the affine Deligne–Lusztig varieties $\dot{X}_{\dot{w}_r}^m(b)$. These finite-level varieties have somewhat strange descriptions (see the equivalence relation $\sim_{b,m,r}$ in Sect. 6.2), though it is conceivable that one could use the results of Part 4 to study the cohomology of these higher-level affine Deligne–Lusztig varieties.

Instead of passing through affine Deligne–Lusztig varieties, we approximate our infinite-level object $\dot{X}_w^{DL}(b)$ by using an analogue of Deligne–Lusztig varieties for parahoric subgroups, which are easier to explicitly describe than affine Deligne–Lusztig varieties. Using the decomposition of $\dot{X}_w^{\infty}(b)$ into G -translates of $G_{\mathcal{O}}$ -stable components (as in Theorem 6.17), the computation of the cohomology of $\dot{X}_b^{\infty}(b)$ reduces to the computation for one such component, which can in turn be written as an inverse limit $\varprojlim_h X_h$ of finite-dimensional varieties X_h , each endowed with an action of level- h quotients $G_h \times T_h$ of $G_{\mathcal{O}} \times T_{\mathcal{O}}$. We write $R_{T_h}^{G_h}(\theta)$ for the virtual G_h -representation corresponding to $\theta: T_h \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$. We note that X_1 is a classical Deligne–Lusztig variety for the reductive subquotient of $T_{\mathcal{O}}$ in the reductive quotient of $G_{\mathcal{O}}$.

However, the infinite-level object $\dot{X}_w^{\infty}(b)$ has a very natural description, so we proceed by defining another tower of finite-dimensional objects X_h , which are analogues of Deligne–Lusztig varieties for parahoric subgroups. Using the Deligne–Lusztig fixed-point formula, we compute (part of) the character of $R_{T_h}^{G_h}(\theta)$ on T_h , which when combined with Henniart’s characterizations [19, 20] of automorphic induction yields:

Theorem (11.3) *Let $\theta: T \rightarrow \overline{\mathbb{Q}}_{\ell}^{\times}$ be a smooth character. If $|R_T^G(\theta)|$ is irreducible supercuspidal, then the assignment $\theta \mapsto |R_T^G(\theta)|$ is a geometric realization of automorphic induction and the Jacquet–Langlands correspondence.*

Proving that $|R_T^G(\theta)|$ is irreducible supercuspidal involves two main steps: proving that $|R_{T_h}^{G_h}(\theta)|$ is irreducible and proving its induction to G (after extending by the center) is irreducible. In [26], Lusztig studies the irreducibility of $R_{T_h}^{G_h}(\theta)$ for reductive groups over finite rings under a regularity assumption on θ . In our setting, this regularity assumption corresponds to θ being *minimal admissible*. We extend Lusztig's arguments to the non-reductive setting to handle the non-quasi-split inner forms of $\mathrm{GL}_n(K)$ and prove that $R_{T_h}^{G_h}(\theta)$ is irreducible under the same regularity assumption on θ (Theorem 8.1). In this context, we prove a cuspidality result (Theorem 9.1) for $|R_{T_h}^{G_h}(\theta)|$, which allows us to emulate the arguments from [27, Proposition 6.6] that inducing classical Deligne–Lusztig representations gives (depth zero) irreducible supercuspidal representations of p -adic groups. This approach was carried out in the GL_2 case for arbitrary depth in [21, Propositions 4.10, 4.22]. Note that the $|R_T^G(\theta)|$ can have arbitrarily large depth, depending on the level of the smooth character θ .

Theorem (12.5) *If $\theta: T \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is minimal admissible, then $|R_T^G(\theta)|$ is irreducible supercuspidal.*

1.1 Outline

This paper is divided into four parts. The first part of the article is devoted to purely geometric properties of the Deligne–Lusztig constructions for arbitrary reductive groups over local fields. In Sects. 3 and 4, we define and recall the two types of Deligne–Lusztig constructions. The main result of this part is Theorem 4.9, where we prove that, under a technical hypothesis, affine Deligne–Lusztig sets of arbitrarily deep level can be endowed with a scheme structure. After Part 1, we work only in the context of the inner forms of $\mathrm{GL}_n(K)$.

We begin Part 2 with a discussion of the group-theoretic constructions we will use at length throughout the rest of the paper (Sect. 5). We emphasize the importance of the seemingly innocuous Sect. 5.2, where we define two representatives b for each basic σ -conjugacy class of $\mathrm{GL}_n(\bar{K})$. In Sect. 6, we define the affine Deligne–Lusztig varieties $\dot{X}_{w_r}^m(b)$, construct an isomorphism between $\dot{X}_w^\infty(b)$ and $\dot{X}_w^{DL}(b)$ using the isocrystal $(\bar{K}^n, b\sigma)$, and explicate the scheme structure of $\dot{X}_w^\infty(b)$. In Sect. 7, we introduce a family of smooth finite-type schemes X_h whose limit is a component of $\dot{X}_w^\infty(b)$ corresponding to G_O and study its geometry. This plays the role of a Deligne–Lusztig variety for subquotients of G (see Proposition 7.12).

In Part 3, we calculate the cohomology $R_{T_h}^{G_h}(\theta)$ under a certain regularity assumption on θ . We prove irreducibility (Theorem 8.1) using a generalization of [26, 31] discussed in Sect. 8.4. We prove a result about the restriction of $R_{T_h}^{G_h}(\theta)$ to the “deepest part” of unipotent subgroups (Theorem 9.1) which can be viewed as an analogue of cuspidality for G_h -representations. This is a long calculation using fixed-point formulas.

Finally, in Part 4, we combine the results of the preceding two parts to deduce our main theorems about $R_T^G(\theta)$, the homology of the affine Deligne–Lusztig variety at infinite level $\dot{X}_w^\infty(b)$. We review the methods of Henniart [19, 20] in Sect. 10, define

and discuss some first properties of the homology of $\dot{X}_w^\infty(b)$ in Sect. 11, and prove the irreducible supercuspidality of $R_T^G(\theta)$ for minimal admissible θ in Sect. 12.

2 Notation

Throughout the paper we will use the following notation. Let K be a non-archimedean local field with residue field $\mathbb{F}_q$ of prime characteristic p , and let $\check{K}$ denote the completion of a maximal unramified extension of K . We denote by $\mathcal{O}_K$, $\mathfrak{p}_K$ (resp. $\mathcal{O}$, $\mathfrak{p}$) the integers and the maximal ideal of K (resp. of $\check{K}$). The residue field of $\check{K}$ is an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. We write σ for the Frobenius automorphism of $\check{K}$, which is the unique K -automorphism of $\check{K}$, lifting the $\mathbb{F}_q$ -automorphism $x \mapsto x^q$ of $\overline{\mathbb{F}}_q$. Finally, we denote by ϖ a uniformizer of K (and hence of $\check{K}$) and by $\text{ord} = \text{ord}_{\check{K}}$ the valuation of $\check{K}$, normalized such that $\text{ord}(\varpi) = 1$.

If K has positive characteristic, we let $\mathbb{W}$ denote the ring scheme over $\mathbb{F}_q$ where for any $\mathbb{F}_q$ -algebra A , $\mathbb{W}(A) = A[[\pi]]$. If K has mixed characteristic, we let $\mathbb{W}$ denote the K -ramified Witt ring scheme over $\mathbb{F}_q$ so that $\mathbb{W}(\mathbb{F}_q) = \mathcal{O}_K$ and $\mathbb{W}(\overline{\mathbb{F}}_q) = \mathcal{O}$. Let $\mathbb{W}_h = \mathbb{W}/V^h\mathbb{W}$ be the truncated ring scheme, where $V: \mathbb{W} \rightarrow \mathbb{W}$ is the Verschiebung morphism. For any $1 \leq r \leq h$, we write $\mathbb{W}_h^r$ to denote the kernel of the natural projection $\mathbb{W}_h \rightarrow \mathbb{W}_r$. As the Witt vectors are only well behaved on perfect $\mathbb{F}_q$ -algebras, algebro-geometric considerations when K has mixed characteristic are taken up to perfection. We fix the following convention.

Convention If K has mixed characteristic, whenever we speak of a scheme (resp. ind-scheme) over its residue field $\mathbb{F}_q$, we mean a *perfect scheme* (resp. *ind-(perfect scheme)*), that is a set-valued functor on perfect $\mathbb{F}_q$ -algebras, representable by the perfection of a scheme (resp. ind-scheme).

For results on perfect schemes we refer to [2, 34]. Note that passing to perfection does not affect the ℓ -adic étale cohomology; thus for purposes of this paper, we could in principle pass to perfection in all cases. However, in the equal characteristic case working on non-perfect rings does not introduce complications, and we prefer to work in this slightly greater generality.

Fix a prime $\ell \neq p$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. The field of coefficients of all representations is assumed to be $\overline{\mathbb{Q}}_\ell$ and all cohomology groups throughout are compactly supported ℓ -adic étale cohomology groups.

2.1 List of terminology

Our paper introduces some notions for a general group G (Part 1) and then studies these notions for G an inner form of GL_n (Parts 2 through 4). The investigations for G an inner form of GL_n involve many different methods. For the reader's reference, we give a brief summary of the most important notation introduced and used in Parts 2 through 4.

- L The degree- n unramified extension of K . Its ring of integers $\mathcal{O}_L$ has a unique maximal ideal $\mathfrak{p}_L$ and its residue field is $\mathcal{O}_L/\mathfrak{p}_L \cong \mathbb{F}_{q^n}$. For any $h \geq 1$, we write $U_L^h = 1 + \mathfrak{p}_L^h$
- $[b]$ Fixed basic σ -conjugacy class of $\mathrm{GL}_n(\check{K})$. Typically we take representatives b of $[b]$ to be either the Coxeter-type or special representative (Sect. 5.2)
- κ $\kappa_{\mathrm{GL}_n}([b])$, where κ_{GL_n} is the Kottwitz map. We assume that $0 \leq \kappa \leq n-1$ and set $n' = \gcd(n, \kappa)$, $n_0 = n/n'$, $k_0 = \kappa/n'$
- F Twisted Frobenius morphism $F: \mathrm{GL}_n(\check{K}) \rightarrow \mathrm{GL}_n(\check{K})$ given by $F(g) = b\sigma(g)b^{-1}$
- G $= J_b(K) = \mathrm{GL}_n(\check{K})^F \cong \mathrm{GL}_{n'}(D_{k_0/n_0})$, where D_{k_0/n_0} is the division algebra with Hasse invariant k_0/n_0
- T $= L^\times$, an unramified elliptic torus in G
- $g_b^{\mathrm{red}}(x)$ $(n \times n)$ -matrix whose i th column is $\varpi^{-\lfloor (i-1)k_0/n_0 \rfloor} (b\sigma)^{i-1}(x)$ with $x \in V$ (Eq. (6.5))
- $\dot{X}_w^{DL}(b)$ a semi-infinite Deligne–Lusztig variety, with a natural action of $G \times T$ (Sect. 3)
- $\dot{X}_{w_r}^m(b)$ an affine Deligne–Lusztig variety with a natural action of $G \times T$ (Sect. 6.2)
- $\dot{X}_w^\infty(b)$ $= \varprojlim_{r>m} \dot{X}_{w_r}^m(b) = V_b^{\mathrm{adm}, \mathrm{rat}, w_0} \cong V_b^{\mathrm{adm}, \mathrm{rat}} = \{x \in V_b^{\mathrm{adm}} : \det g_b(x) \in K^\times\}$ an affine Deligne–Lusztig variety at the infinite level, with a natural $G \times \mathcal{O}_L^\times$ -action (Corollary 6.18)
- $\dot{X}_w^\infty(b)_{\mathcal{L}_0} = \mathcal{L}_{0,b}^{\mathrm{adm}, \mathrm{rat}, w_0} \cong \mathcal{L}_{0,b}^{\mathrm{adm}, \mathrm{rat}} = \{x \in \mathcal{L}_0 : \det g_b^{\mathrm{red}}(x) \in \mathcal{O}_K^\times\}$ is the union of connected components of $\dot{X}_w^\infty(b)$ associated to the lattice $\mathcal{L}_0$ (Definition 6.10)
- G_h $= \mathbb{G}_h(\mathbb{F}_q) = (\check{G}_{\mathbf{x},0}/\check{G}_{\mathbf{x},(h-1)+})^F$ where $F(g) = b\sigma(g)b^{-1}$ for b the Coxeter-type or special representative. G_h is a subquotient of G (Sect. 5.3)
- T_h $= \mathbb{T}_h(\mathbb{F}_q) \cong \mathcal{O}_L^\times/U_L^h$
- X_h A quotient of $\dot{X}_{w_r}^m(b)_{\mathcal{L}_0}$ for any $r > m \geq 0$ (Sect. 7.6). It has a $(G_h \times T_h)$ -action and is a finite-ring analogue of a Deligne–Lusztig variety (Proposition 7.12)
- $R_{T_h}^{G_h}(\theta)$ $= \sum_i (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta]$, where $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta] \subset H_c^i(X_h, \overline{\mathbb{Q}}_\ell)$ is the subspace where T_h acts by $\theta: T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$
- $R_T^G(\theta)$ $= \sum_i (-1)^i H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta] = \sum_i (-1)^i H_i(\dot{X}_w^{DL}(b), \overline{\mathbb{Q}}_\ell)[\theta]$, where the homology groups of the scheme $\dot{X}_w^\infty(b)$ are defined in Sect. 11 and where $[\theta]$ denotes the subspace where T acts by $\theta: T \rightarrow \overline{\mathbb{Q}}_\ell^\times$
- $\mathcal{X}$ The set of all smooth characters of $L^\times$ that are in general position; i.e., they have trivial stabilizer in $\mathrm{Gal}(L/K)$ (Part 4)
- $\mathcal{X}^{\min}$ The set of all characters of $L^\times$ that are minimal admissible (Sect. 12)

The action of $G \times T$ on each of the schemes $\dot{X}_{w_r}^m(b)$, $\dot{X}_w^\infty(b)$, $\dot{X}_w^{DL}(b)$ is given by $x \mapsto gxt$. These actions descend to an action of $G_h \times T_h$ on X_h .

Part 1. Deligne–Lusztig constructions for p -adic groups

In this part we discuss two analogues of Deligne–Lusztig constructions attached to a reductive group over K : semi-infinite Deligne–Lusztig sets and affine Deligne–Lusztig varieties at higher level. We begin by fixing some notation.

Let G be a connected reductive group over K . Let S be a maximal $\check{K}$ -split torus in G . By [8, 5.1.12] it can be chosen to be defined over K . Let $T = \mathcal{Z}_G(S)$ and $\mathcal{N}_G(S)$ be the centralizer and normalizer of S , respectively. By Steinberg’s theorem, $G_{\check{K}}$ is quasi-split, hence T is a maximal torus. The Weyl group W of S in G is the quotient $W = \mathcal{N}_G(S)/T$ of the normalizer of S by its centralizer. By [3, Theorem 21.2], every connected component of $\mathcal{N}_G(S)$ meets $\check{G}$, so $W = \mathcal{N}_G(S)(\check{K})/\check{T}$. In particular, the action of the absolute Galois group of K on W factors through a $\text{Gal}(\check{K}/K)$ -action.

For a scheme X over K , the loop space LX of X is the functor on $\mathbb{F}_q$ -algebras given by $LX(R) = X(\mathbb{W}(R)[\varpi^{-1}])$. For a scheme $\mathfrak{X}$ over $\mathcal{O}$, the space of positive loops $L^+\mathfrak{X}$ of $\mathfrak{X}$ is the functor on $\mathbb{F}_q$ -algebras given by $L^+\mathfrak{X}(R) = \mathfrak{X}(\mathbb{W}(R))$, and the functor L_h^+ of truncated positive loops is given by $L_h^+\mathfrak{X}(R) = \mathfrak{X}(\mathbb{W}_h(R))$.

For any algebro-geometric object X over K , we write $\check{X}$ for the set of its $\check{K}$ -rational points.

3 Semi-infinite Deligne–Lusztig sets in G/B

Assume that G is quasi-split. Pick a K -rational Borel $B \subseteq G$ containing T and let U be the unipotent radical of B . We have the following direct analogue of classical Deligne–Lusztig varieties [13].

Definition 3.1 Let $w \in W$, $\dot{w} \in \mathcal{N}_G(S)(\check{K})$ a lift of w , and $b \in \check{G}$. The *semi-infinite Deligne–Lusztig sets* $X_w^{DL}(b)$, $\dot{X}_w^{DL}(b)$ are

$$\begin{aligned} X_w^{DL}(b) &= \{g \in \check{G}/\check{B} : g^{-1}b\sigma(g) \in \check{B}w\check{B}\}, \\ \dot{X}_w^{DL}(b) &= \{g \in \check{G}/\check{U} : g^{-1}b\sigma(g) \in \check{U}\dot{w}\check{U}\}. \end{aligned}$$

There is a natural map $\dot{X}_w^{DL}(b) \rightarrow X_w^{DL}(b)$, $g\check{U} \mapsto g\check{B}$.

For $b \in \check{G}$, we denote by J_b the σ -stabilizer of b , which is the K -group defined by

$$J_b(R) := \{g \in G(R \otimes_K \check{K}) : g^{-1}b\sigma(g) = b\}$$

for any K -algebra R (cf. [30, 1.12]). Then J_b is an inner form of the centralizer of the Newton point b (which is a Levi subgroup of G). In particular, if b is *basic*, i.e., the Newton point of b is central, then J_b is an inner form of G . Let $w \in W$ and let $\dot{w} \in \mathcal{N}_G(S)(\check{K})$ be a lift. We denote by T_w the σ -stabilizer of $\dot{w}$ in T , which is the K -group defined by

$$T_w(R) := \{t \in T(R \otimes_K \check{K}) : t^{-1}\dot{w}\sigma(t) = \dot{w}\}.$$

for any K -algebra R . As T is commutative, this only depends on w , not on $\dot{w}$.

Lemma 3.2 *Let $b \in \check{G}$ and let $w \in W$ with lift $\dot{w} \in \mathcal{N}_G(S)(\check{K})$.*

- (i) *Let $g \in \check{G}$. The map $x\check{B} \mapsto gx\check{B}$ defines a bijection $X_w^{DL}(b) \xrightarrow{\sim} X_w^{DL}(g^{-1}b\sigma(g))$.*
- (ii) *Let $g \in \check{G}$ and $t \in \check{T}$. The map $x\check{U} \mapsto gxt\check{U}$ defines a bijection $\dot{X}_{\dot{w}\sigma(t)}^{DL}(b) \xrightarrow{\sim} \dot{X}_w^{DL}(g^{-1}b\sigma(g))$.*
- (iii) *There are actions of $J_b(K)$ on $X_w^{DL}(b)$ given by $(g, x\check{B}) \mapsto gx\check{B}$ and of $J_b(K) \times T_w(K)$ on $\dot{X}_w^{DL}(b)$ given by $(g, t, x\check{U}) \mapsto gxt\check{U}$. They are compatible with $\dot{X}_w^{DL}(b) \rightarrow X_w^{DL}(b)$, and if this map is surjective, then $\dot{X}_w^{DL}(b)$ is a right $T_w(K)$ -torsor over $X_w^{DL}(b)$.*

Proof (i) and (ii) follow from the definitions by immediate computations. (iii) follows from (i) and (ii). $\square$

Remark 3.3 (i) Whereas the classical Deligne–Lusztig varieties are always non-empty, $X_w^{DL}(b)$ is non-empty if and only if the σ -conjugacy class $[b]$ of b in $G(\check{K})$ intersects the double coset $\check{B}w\check{B}$. For example, if $G = \mathrm{GL}_n$ ($n \geq 2$) and b is superbasic, then $X_1^{DL}(b) = \emptyset$, as was observed by E. Viehmann.

- (ii) L. Fargues pointed out the following way to endow the semi-infinite Deligne–Lusztig set $X_w^{DL}(1)$ (and $\dot{X}_w^{DL}(b)$ if T_w is elliptic) with a scheme structure: assume that G (and B) come from a reductive group over $\mathcal{O}_K$ (again denoted G), such that G/B is a projective $\mathcal{O}_K$ -scheme. Then

$$(G/B)(\check{K}) = (G/B)(\mathcal{O}) = \varprojlim_r (G/B)(\mathcal{O}/\mathfrak{p}^r).$$

Now $(G/B)(\mathcal{O}/\mathfrak{p}^r) = L_r^+(G/B)(\overline{\mathbb{F}}_q)$ is a finite dimensional $\mathbb{F}_q$ -scheme via L_r^+ . For a given element w in the finite Weyl group, the corresponding Deligne–Lusztig condition is given by a finite set of open and closed conditions in G/B which involve σ . The closed conditions cut a closed, hence projective, subscheme of G/B , and replacing G/B by this closed subscheme Z , we may assume that there are only open conditions. These define an open subscheme Y_r in each L_r^+Z . Set $X_w^{DL}(1)_r := \mathrm{pr}_r^{-1}(Y_r)$, where $\mathrm{pr}_r: L^+Z \rightarrow L_r^+Z$ is the projection. This gives $X_w^{DL}(1)_r$ the structure of an open subscheme of L^+Z and $X_w^{DL}(1) = \bigcup_{r=1}^{\infty} X_w^{DL}(1)_r$ is now an (ascending) union of open subschemes of L^+Z . Note that since the transition morphisms are not closed immersions, this union does not define an ind-scheme. Now if w is such that T_w is elliptic, then $T_w(K)$ is compact modulo $Z(K)$, where Z is the center of G , and $\dot{X}_w^{DL}(1)$ —being a $T_w(K)$ -torsor over $X_w^{DL}(1)$ —is a scheme.

However, this scheme structure appears to be the “correct” one only on the subscheme $X_w^{DL}(1)_1$, as the action of $G(K) = J_1(K)$ on $X_w^{DL}(1)$ cannot in general be an action by algebraic morphisms (whereas the action of $G(\mathcal{O}_K)$ on $X_w^{DL}(1)_1$ is). This will become clear from the SL_2 -example discussed in Sect. 6.5 below. $\diamond$

Finally we investigate the relation of $\dot{X}_w^{DL}(b)$ with Lusztig's constructions from [25, 26]. In fact, consider the map $F: \check{G} \rightarrow \check{G}, g \mapsto b\sigma(g)b^{-1}$. Assuming that (w, b) satisfies $w\check{B} = b\sigma(\check{B})$, so that $w\check{B}b^{-1} = F(\check{B})$,

$$\begin{aligned} X_w^{DL}(b) &= \{g\check{B} \in \check{G}/\check{B} : g^{-1}b\sigma(g) \in \check{B}w\check{B}\} \\ &= \{g\check{B} \in \check{G}/\check{B} : g^{-1}F(g) \in \check{B}F(\check{B})\} \\ &= \{g \in \check{G} : g^{-1}F(g) \in F(\check{B})\}/(\check{B} \cap F(\check{B})) \end{aligned}$$

Similarly, assuming that $(\dot{w}, b)$ satisfies $\dot{w}\check{U} = b\sigma(\check{U})$, so that $\dot{w}\check{U}b^{-1} = F(\check{U})$,

$$\dot{X}_w^{DL}(b) = \{g \in \check{G} : g^{-1}F(g) \in F(\check{U})\}/(\check{U} \cap F(\check{U})).$$

This is precisely the definition of the semi-infinite Deligne–Lusztig set in [25]. It was studied by Boyarchenko [5] and the first named author [10–12] in the case when $G = \mathrm{GL}_n$ and b superbasic, i.e., $J_b(K)$ are the units of a division algebra over K , where it admits an ad hoc scheme structure.

In the setting of Part 2 of this paper (see Theorem 6.5), it will turn out that $X_w^{DL}(b) = \{g \in \check{G} : g^{-1}F(g) \in F(\check{U})\}/(\check{T}^F(\check{U} \cap F(\check{U}))) = \dot{X}_w^{DL}(b)/\check{T}^F$. This is quite nontrivial. In the finite field setting [13, Definition 1.17(i)], this is true because the Lang map $g \mapsto g^{-1}F(g)$ is surjective. In the setting of p -adic groups (even in our GL_n setting), the Lang map is no longer surjective. However, a corollary of Theorem 6.5 is that for any $x \in X_w^{DL}(b)$, there exists a representative $g \in \check{G}$ such that $g^{-1}F(g) = tu \in F(\check{B})$ with t in the image of the Lang map on $\check{T}$.

4 Affine Deligne–Lusztig varieties and covers

Let the notation be as in the beginning of part 1. In this section we recall from [21] the definition of affine Deligne–Lusztig varieties of higher level, and prove that they are locally closed in the affine Grassmannian (Theorem 4.9 and Corollary 4.10).

4.1 Affine Grassmannian

We will use representability results on affine Grassmannians attached to G , which were proven by Pappas–Rapoport [28] in the equal characteristic case, and by Zhu [34] and Bhatt–Scholze [2] in the mixed characteristic case. Let $\mathcal{G}$ be a smooth affine $\mathcal{O}_K$ -scheme with generic fiber G and with connected special fiber. The functor $L^+\mathcal{G}$ is represented by an (infinite-dimensional) affine group scheme over $\mathbb{F}_q$. The functor LG is represented by a strict ind-scheme of ind-finite type; that is, LG can be written as a direct limit of schemes of finite type, with transition morphisms being closed immersions.

The affine Grassmannian associated with $\mathcal{G}$ is the fpqc-sheaf $LG/L^+\mathcal{G}$, which is the sheafification for the fpqc-topology of the functor on $\mathbb{F}_q$ -algebras given by

$$R \mapsto LG(R)/L^+\mathcal{G}(R).$$

It possesses the following representability properties.

Theorem 4.1 (cf. [28, Theorem 1.4], [34, Theorem 1.4] and [2, Corollary 9.6]) *The fpqc-sheaf $LG/L^+\mathcal{G}$ on $\mathbb{F}_q$ -algebras is represented by a strict ind-scheme. The quotient morphism $LG \rightarrow LG/L^+\mathcal{G}$ has sections locally for the étale topology (i.e., $\mathrm{Spec}(R) \times_{LG/L^+\mathcal{G}} LG \cong \mathrm{Spec}(R) \times_{\mathrm{Spec}(\mathbb{F}_q)} L^+\mathcal{G}$ for each point of $LG/L^+\mathcal{G}$ with values in a strictly henselian ring R).*

Moreover, if $\mathcal{G}$ is parahoric, then $LG/L^+\mathcal{G}$ is ind-proper, but we will not use this in the following. In general, the affine Grassmannian is not reduced. We have $LG/L^+\mathcal{G}(\mathbb{F}_q) = \check{G}/\mathcal{G}(\mathcal{O})$.

4.2 Level subgroups

Let $\Phi = \Phi(G_{\check{K}}, S)$ denote the set of roots of S in $G_{\check{K}}$ and let U_α denote the root subgroup for $\alpha \in \Phi$. Put $U_0 := T$.

Let $\mathbf{x}$ be a point in the apartment of S inside the Bruhat–Tits building of the adjoint group of G over $\check{K}$. Attached to it, there is a valuation of the root datum of G in the sense of Bruhat–Tits [8]. In particular, for each $\alpha \in \Phi$, it induces a descending filtration $\check{U}_{\alpha,r}$ on $\check{U}_\alpha$ with $r \in \check{\mathbb{R}}$, where $\check{\mathbb{R}} := \mathbb{R} \cup \{r+ : r \in \mathbb{R}\} \cup \{\infty\}$ is the ordered monoid as in [8, 6.4.1]. Further, a choice of an admissible schematic filtration on tori (in the sense of [33, §4]) also defines a descending filtration $\check{U}_{0,r} := \check{U}_{0,r}$ on $\check{U}_0$. (If G is either simply connected or adjoint, or split over a tamely ramified extension, this filtration coincides with the Moy–Prasad filtration, and hence is independent on the choice.) For any concave function $f : \Phi \cup \{0\} \rightarrow \check{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$, let $\check{G}_f$ denote the subgroup of $\check{G}$ (depending on $\mathbf{x}$) generated by $U_{\alpha, f(\alpha)}$ ($\alpha \in \Phi \cup \{0\}$). In [33, Theorem 8.3] it is shown that there exists a smooth affine group scheme $\mathcal{G}_f$ over $\mathcal{O}$ with generic fiber G , satisfying $\mathcal{G}_f(\mathcal{O}) = \check{G}_f$. Moreover, assume that $\mathbf{x}$ is stable under the action of σ on the adjoint building. Then $\mathcal{G}_f$ descends to a smooth affine group scheme over $\mathcal{O}_K$, again denoted $\mathcal{G}_f$ [33, §9.1].

Proposition 4.2 *Let $f, g : \Phi \cup \{0\} \rightarrow \check{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$ be two concave functions with $g \geq f$.*

(i) *$L^+\mathcal{G}_g$ is a closed subgroup scheme of $L^+\mathcal{G}_f$.*

Assume that $\mathcal{G}_g$ is normal in $\mathcal{G}_f$, and that $L^+\mathcal{G}_g$ is pro-unipotent.

- (ii) *The fpqc quotient sheaf $L^+\mathcal{G}_f/L^+\mathcal{G}_g$ is representable by a smooth affine $\mathbb{F}_q$ -group scheme. The morphism $L^+\mathcal{G}_f \rightarrow L^+\mathcal{G}_f/L^+\mathcal{G}_g$ splits Zariski-locally on the target.*
- (iii) *The fpqc sheaf morphism $LG/L^+\mathcal{G}_g \rightarrow LG/L^+\mathcal{G}_f$ is represented in the category of ind-schemes. It is thus an $L^+\mathcal{G}_f/L^+\mathcal{G}_g$ -torsor in the category of ind-schemes. It admits sections locally for the étale topology on $LG/L^+\mathcal{G}_f$.*

Proof When $\mathcal{G}_g, \mathcal{G}_f$ are parahoric models of G , part (i) is shown in [28, Proposition 8.7(a)]. In the general case, (i) follows by the same argument. To see (ii), first observe that $L^+\mathcal{G}_g \hookrightarrow L^+\mathcal{G}_f$ is a monomorphism of sheaves (although $\mathcal{G}_g \rightarrow \mathcal{G}_f$ is not an immersion if $f \neq g$), as $\mathcal{G}_g$ is obtained from $\mathcal{G}_f$ by a series of dilatations (see [4, §3]) of closed subschemes in the special fiber. Put $\mathcal{G}_f^{(0)} = \mathcal{G}_f$ and for $h \geq 1$, let $\mathcal{G}_f^{(h)}$ be the dilatation of $\mathcal{G}_f^{(h-1)}$ along the unit section of the special fiber. Then $L^+\mathcal{G}_f^{(h)} = \ker(L^+\mathcal{G}_f \rightarrow L_h^+\mathcal{G}_f)$ (cf. [34, p. 414]). We can find an $h \geq 1$, such that the natural morphism $\mathcal{G}_f^{(h)} \rightarrow \mathcal{G}_f$ factors through $\mathcal{G}_g \rightarrow \mathcal{G}_f$. This gives closed immersions $L^+\mathcal{G}_f^{(h)} \hookrightarrow L^+\mathcal{G}_g \hookrightarrow L^+\mathcal{G}_f$. Applying the natural morphism of functors $L^+ \rightarrow L_h^+$ to the arrow $\mathcal{G}_g \rightarrow \mathcal{G}_f$, and using that $L^+ \rightarrow L_h^+$ is surjective when evaluated at a flat $\mathcal{O}_K$ -scheme, we thus obtain the following commutative diagram of fpqc-sheaves on $\mathbb{F}_q$ -algebras, with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & L^+\mathcal{G}_f^{(h)} & & & \\
 & & \swarrow & \downarrow & & & \\
 0 & \longrightarrow & L^+\mathcal{G}_g & \longrightarrow & L^+\mathcal{G}_f & \longrightarrow & L^+\mathcal{G}_f/L^+\mathcal{G}_g \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & L_h^+\mathcal{G}_g & \xrightarrow{\alpha} & L_h^+\mathcal{G}_f & \longrightarrow & L_h^+\mathcal{G}_f/\mathrm{im}(\alpha) \longrightarrow 0
 \end{array}$$

Now a diagram chase shows that the right vertical map is a monomorphism. Hence it is an isomorphism. We have presented $L^+\mathcal{G}_f/L^+\mathcal{G}_g$ as a quotient of two finite dimensional smooth affine group schemes. The last claim of (ii) follows as in the proof of [28, Proposition 8.7(b)].

Finally, we prove (iii). It is clear that the morphism of fpqc sheaves $p: LG/L^+\mathcal{G}_g \rightarrow LG/L^+\mathcal{G}_f$ is an $L^+\mathcal{G}_f/L^+\mathcal{G}_g$ -torsor. A (sheaf-)torsor under an affine group scheme is always relatively representable, so we deduce from (ii) that for any scheme T and any morphism $t: T \rightarrow LG/L^+\mathcal{G}_f$, the pullback $p_t: T \times_{LG/L^+\mathcal{G}_f} LG/L^+\mathcal{G}_g \rightarrow T$ is a morphism of schemes. This implies that $LG/L^+\mathcal{G}_g \rightarrow LG/L^+\mathcal{G}_f$ is a morphism of ind-schemes. The last claim follows from Theorem 4.1. $\square$

4.3 Affine Deligne–Lusztig varieties of higher level

Until the end of Sect. 4, we fix a σ -stable $\mathbf{x}$ as above, and a σ -stable Iwahori subgroup $I \subseteq \check{G}$, whose corresponding alcove in the building contains $\mathbf{x}$. There is a function $f_I: \Phi \cup \{0\} \rightarrow \mathbb{R}_{\geq 0} \setminus \{\infty\}$ satisfying $\check{G}_{\mathbf{x}, f_I} = I$, and we have the corresponding integral model $\mathcal{I} := \mathcal{G}_{\mathbf{x}, f_I}$. The extended affine Weyl group of S in G is $\tilde{W} = \mathcal{N}_G(S)(\check{F})/\mathcal{N}_G(S)(\check{F}) \cap I$.

In [29] Rapoport introduced an *affine Deligne–Lusztig variety* attached to elements $w \in \tilde{W}$ and $b \in \check{G}$,

$$X_w(b) = \{gI \in \check{G}/I : g^{-1}b\sigma(g) \in IwI\} \subseteq \check{G}/I = (LG/L^+\mathcal{I})(\overline{\mathbb{F}}_q).$$

It is a locally closed subset of $LG/L^+\mathcal{I}$, hence it inherits the reduced induced sub-ind-scheme structure (see also Theorem 4.9 below). It is even a scheme locally of finite type over $\mathbb{F}_q$. Covers of $X_w(b)$ were introduced by the second named author in [21]. We recall the definition (cf. [22, Sections 2.1–2.2] for a discussion in a more general setup).

Definition 4.3 Let $b \in \check{G}$. Let $f: \Phi \cup \{0\} \rightarrow \check{R}_{\geq 0} \setminus \{\infty\}$ be a concave function such that $\check{G}_f$ is σ -stable. Let $x \in \check{G}_f \backslash \check{G} / \check{G}_f$ be a double coset. Then we define the corresponding *affine Deligne–Lusztig set of level f* ,

$$X_x^f(b) := \{g\check{G}_f \in \check{G} / \check{G}_f : g^{-1}b\sigma(g) \in \check{G}_f x \check{G}_f\} \subseteq \check{G} / \check{G}_f = (LG/L^+\mathcal{G}_f)(\overline{\mathbb{F}}_q).$$

If $J = \check{G}_f$ satisfies the assumptions in the definition, we sometimes also write $X_x^J(b)$ for $X_x^f(b)$. We will prove that $X_x^f(b)$ is in certain cases a locally closed subset of $LG/L^+\mathcal{G}_f$ (Theorem 4.9). There is a natural $J_b(K)$ -action by left multiplication on $X_x^f(b)$ for all f and all x . If $f' \geq f$ and $x' \in \check{G}_{f'} \backslash \check{G} / \check{G}_{f'}$ lies over $x \in \check{G}_f \backslash \check{G} / \check{G}_f$, then the natural projection $\check{G} / \check{G}_{f'} \rightarrow \check{G} / \check{G}_f$ restricts to a map $X_{x'}^{f'}(b) \rightarrow X_x^f(b)$. Concerning the right action, we have the following lemma.

Lemma 4.4 Let $J' = \check{G}_{f'}$ and $J = \check{G}_f$ be two subgroups as in Definition 4.3, such that J' is a normal subgroup of J . Let $x' \in J' \backslash \check{G} / J'$ lie over $x \in J \backslash \check{G} / J$ and let $b \in \check{G}$.

- (i) Any $i \in J$ defines an $X_x^J(b)$ -isomorphism $X_{x'}^{J'}(b) \rightarrow X_{i^{-1}x'\sigma(i)}^{J'}(b)$ given by $gJ' \mapsto giJ'$.
- (ii) If $X_{x'}^{J'}(b) \rightarrow X_x^J(b)$ is surjective, then $X_{x'}^{J'}(b)$ is set-theoretically a $(J/J')_{x'}$ -torsor over $X_x^J(b)$, where

$$(J/J')_{x'} := \{i \in J : i^{-1}x'\sigma(i) = x'\} / J'.$$

Proof Since J' is normal in J , we see that $iJ'x'J'\sigma(i)^{-1} = J'ix'\sigma(i)^{-1}J'$. This implies (i). For (ii) we need to show that $(J/J')_{x'}$ acts faithfully and transitively on the fibers of $\varphi: X_{x'}^{J'}(b) \rightarrow X_x^J(b)$. By definition, $\varphi^{-1}(gJ) = \{ghJ' : h \in J \text{ and } (gh)^{-1}b\sigma(gh) \in J'x'J'\}$. The claim follows from normality of J' in J and the definition of $(J/J')_{x'}$. $\square$

4.4 Scheme structure

We need some notation. Write $\widehat{\Phi} := \Phi \cup \{0\}$. Let Φ_{aff} denote the set of affine roots of S in G and let $\widehat{\Phi}_{\text{aff}}$ be the disjoint union of Φ_{aff} with the set of all pairs $(0, r)$ with $r \in \check{\mathbb{R}}_{<\infty}$, for which the filtration step $\check{U}_{0,r} / \check{U}_{0,r+}$ is non-trivial. There is a natural projection $p: \widehat{\Phi}_{\text{aff}} \rightarrow \widehat{\Phi}$, mapping an affine root to its vector part and $(0, r)$ to 0. We extend the action of $\check{W}$ on Φ , Φ_{aff} to an action on $\widehat{\Phi}$, $\widehat{\Phi}_{\text{aff}}$ by letting it act trivially on 0 and all $(0, r)$.

By [33], for any $\alpha \in \widehat{\Phi}$ and $r \in \check{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$, there is an $\mathcal{O}$ -scheme $\mathcal{U}_{\alpha,r}$ satisfying $\mathcal{U}_{\alpha,r}(\mathcal{O}) = \check{U}_{\alpha,r}$ whose generic fibre is $U_{\alpha,\check{K}}$. If $f: \widehat{\Phi} \rightarrow \check{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$ is concave, the

schematic closure of U_α in $\mathcal{G}_f$ is $\mathcal{U}_{\alpha, f(\alpha)}$. If $r < s$ in $\widetilde{\mathbb{R}}_{<\infty}$, there is a unique morphism of group schemes $\mathcal{U}_{\alpha, s} \rightarrow \mathcal{U}_{\alpha, r}$ which induces the natural inclusion $\check{U}_{\alpha, s} \hookrightarrow \check{U}_{\alpha, r}$ on $\mathcal{O}$ -points. Let $L_{[r, s)}\mathcal{U}_\alpha$ be the fpqc quotient sheaf

$$L_{[r, s)}\mathcal{U}_\alpha = L^+\mathcal{U}_{\alpha, r}/L^+\mathcal{U}_{\alpha, s}.$$

It is represented by a finite-dimensional group scheme over $\overline{\mathbb{F}}_q$.

Lemma 4.5 *Let $f: \widehat{\Phi} \rightarrow \widetilde{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$ be a concave function such that $\check{G}_f \subseteq I$ is a normal subgroup. Then there is an isomorphism of $\overline{\mathbb{F}}_q$ -schemes*

$$\prod_{\alpha \in \widehat{\Phi}} L_{[f_I(\alpha), f(\alpha))}\mathcal{U}_\alpha \rightarrow L^+\mathcal{I}/L^+\mathcal{G}_f,$$

which on geometric points is given by $(a_\alpha)_{\alpha \in \widehat{\Phi}} \rightarrow \prod_\alpha \tilde{a}_\alpha$, where $\tilde{a}_\alpha$ is any lift of a_α to $\check{U}_{\alpha, f_I(\alpha)}$ and the product can be taken in any order.

Proof The conclusion of [8, 6.4.48] also holds for the Iwahori subgroup, i.e., for the function f_I (this follows from the Iwahori decomposition). Thus there is a bijection

$$\prod_{\alpha \in \widehat{\Phi}} L^+\mathcal{U}_{\alpha, f_I(\alpha)}(\overline{\mathbb{F}}_q) \rightarrow I, \quad (a_\alpha)_{\alpha \in \Phi \cup \{0\}} \rightarrow \prod_\alpha a_\alpha,$$

given by multiplication in any order, and a similar statement for I, f_I replaced by $\check{G}_f, f$. The statement of the lemma on geometric points follows from these bijections by normality of $\check{G}_f$ in I . Now, the map $(a_\alpha)_{\alpha \in \widehat{\Phi}} \rightarrow \prod_\alpha \tilde{a}_\alpha$ in the lemma is an algebraic morphism between smooth varieties that is bijective on geometric points and hence an isomorphism. $\square$

Let $x \in \widetilde{W}$. We give an explicit parametrization of the set of double cosets $\check{G}_f \backslash IxI/\check{G}_f$ in certain cases. For simplicity, we abuse the notation in the following few lemmas and write x again for any lift of x to $\check{G}$. We say also that $(\alpha, m) \in \widehat{\Phi}_{\text{aff}}$ occurs in a subgroup J of $\check{G}$, if $\check{U}_{\alpha, m}$ is contained in J . Then (α, m) occurs in $\check{G}_f$ if and only if $m \geq f(\alpha)$. Let $\widehat{\Phi}_{\text{aff}}(J) \subseteq \widehat{\Phi}_{\text{aff}}$ denote the set of all pairs (α, m) occurring in J . If $J' \subseteq J$ is a normal subgroup, let $\widehat{\Phi}_{\text{aff}}(J/J') := \widehat{\Phi}_{\text{aff}}(J) \setminus \widehat{\Phi}_{\text{aff}}(J')$.

Let $f: \widehat{\Phi} \rightarrow \widetilde{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$ be a concave function such that $\check{G}_f \subseteq I$ is a normal subgroup. For $x \in \widetilde{W}$, we can divide the set of all affine roots $\Phi_{\text{aff}}(I/\check{G}_f)$ into three disjoint parts A_x, B_x, C_x :

$$\begin{aligned} A_x &= \{(\alpha, m) \in \widehat{\Phi}_{\text{aff}}(I/\check{G}_f) : x.(\alpha, m) \notin \widehat{\Phi}_{\text{aff}}(I)\} \\ B_x &= \{(\alpha, m) \in \widehat{\Phi}_{\text{aff}}(I/\check{G}_f) : x.(\alpha, m) \in \widehat{\Phi}_{\text{aff}}(I/\check{G}_f)\} \\ C_x &= \{(\alpha, m) \in \widehat{\Phi}_{\text{aff}}(I/\check{G}_f) : x.(\alpha, m) \in \widehat{\Phi}_{\text{aff}}(\check{G}_f)\} \end{aligned} \quad (4.1)$$

Lemma 4.6 *Let $f: \widehat{\Phi} \rightarrow \widetilde{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$ be a concave, such that $\check{G}_f \subseteq I$ is a normal subgroup. Let $x \in \widetilde{W}$. Assume that $p(A_x), p(B_x)$ and $p(C_x)$ are mutually disjoint,*

and that the same is true for $p(A_{x^{-1}})$, $p(B_{x^{-1}})$, $p(C_{x^{-1}})$. Then there is a well-defined bijective map

$$\begin{aligned} & \prod_{\alpha \in p(A_{x^{-1}})} L_{[f_I(\alpha), f(\alpha)]} \mathcal{U}_\alpha(\overline{\mathbb{F}}_q) \\ & \times \prod_{\alpha \in p(B_x)} L_{[f_I(\alpha), f(\alpha)]} \mathcal{U}_\alpha(\overline{\mathbb{F}}_q) \\ & \times \prod_{\alpha \in p(A_x)} L_{[f_I(\alpha), f(\alpha)]} \mathcal{U}_\alpha(\overline{\mathbb{F}}_q) \rightarrow \check{G}_f \backslash IxI / \check{G}_f \end{aligned}$$

given by $((a_\alpha)_{\alpha \in p(A_{x^{-1}})}, (b_\alpha)_{\alpha \in p(B_x)}, (a_\alpha)_{\alpha \in p(A_x)}) \mapsto \prod_{\alpha \in p(A_{x^{-1}})} \tilde{a}_\alpha \cdot x \cdot \prod_{\alpha \in p(B_x)} \tilde{b}_\alpha \cdot \prod_{\alpha \in p(A_x)} \tilde{a}_\alpha$, where $\tilde{a}_\alpha$ is any lift of a_α to an element of $\check{U}_{\alpha, f_I(\alpha)}$, and similarly for $\tilde{b}_\alpha, b_\alpha$. This endows the set $\check{G}_f \backslash IxI / \check{G}_f$ with the structure of a reduced $\overline{\mathbb{F}}_q$ -scheme of finite type.

Proof That the claimed map is well-defined follows from Lemma 4.5. We have an obvious surjective map $I / \check{G}_f \times I / \check{G}_f \rightarrow \check{G}_f \backslash IxI / \check{G}_f$, given by $(i\check{G}_f, j\check{G}_f) \mapsto \check{G}_f i x j \check{G}_f$. By Lemma 4.5, we may write any element of the left $I / \check{G}_f$ as product $a_{x^{-1}} b_{x^{-1}} c_{x^{-1}}$, where $a_{x^{-1}} = \prod_{\alpha \in p(A_{x^{-1}})} a_\alpha$, etc. Thus any element of $\check{G}_f \backslash IxI / \check{G}_f$ may be written in the form

$$\check{G}_f \tilde{a}_{x^{-1}} \tilde{b}_{x^{-1}} \tilde{c}_{x^{-1}} \cdot x \cdot j \check{G}_f, \quad (4.2)$$

for some $j \in I$, where $\tilde{(\cdot)}$ denotes an arbitrary lift of an element to the root subgroup. Bringing $\tilde{b}_{x^{-1}} \tilde{c}_{x^{-1}}$ to the right side of x changes it to $x^{-1} \tilde{b}_{x^{-1}} \tilde{c}_{x^{-1}} x$, which is a product of elements of certain filtration steps of root subgroups, all of which lie in I by definition of $B_{x^{-1}}, C_{x^{-1}}$. Thus we may eliminate $\tilde{b}_{x^{-1}} \tilde{c}_{x^{-1}}$ from (4.2). Now, by Lemma 4.5, we may write any element of the right $I / \check{G}_f$ as the product $c_x b_x a_x$, with $c_x = \prod_{\alpha \in p(C_x)} c_\alpha$, etc. That is, any element of $\check{G}_f \backslash IxI / \check{G}_f$ may be written as

$$\check{G}_f \tilde{a}_{x^{-1}} \cdot x \cdot \tilde{c}_x \tilde{b}_x \tilde{a}_x \check{G}_f, \quad (4.3)$$

for some lifts $\tilde{c}_x, \tilde{b}_x, \tilde{a}_x$ of c_x, b_x, a_x . Bringing $\tilde{c}_x$ to the left side of x in (4.3), makes it to $x^{-1} \tilde{c}_x x$, which is a product of elements of certain filtration steps of root subgroups, all of which lie in $\check{G}_f$ by definition of C_x . By normality of $\check{G}_f$, we may eliminate $\tilde{c}_x$ from the (4.3). It finally follows that we may write any element of $\check{G}_f \backslash IxI / \check{G}_f$ as a product

$$\check{G}_f \tilde{a}_{x^{-1}} \cdot x \cdot \tilde{b}_x \tilde{a}_x \check{G}_f, \quad (4.4)$$

with $\tilde{a}_{x^{-1}}, \tilde{b}_x, \tilde{a}_x$ as above. This shows the surjectivity of the map in the lemma. It remains to show injectivity.

Suppose there are tuples $(a_{x^{-1}}, b_x, a_x)$ and $(a'_{x^{-1}}, b'_x, a'_x)$ giving the same double coset, i.e., $\tilde{a}_{x^{-1}}x\tilde{b}_x\tilde{a}_x = i\tilde{a}'_{x^{-1}}x\tilde{b}'_x\tilde{a}'_xj$ for some $i, j \in \check{G}_f$. This equation is equivalent to

$$x^{-1}(\tilde{a}'_{x^{-1}})^{-1}i\tilde{a}_{x^{-1}}x = \tilde{b}'_x\tilde{a}'_xj\tilde{a}_x^{-1}\tilde{b}_x^{-1}.$$

Here, the right hand side lies in I , hence it follows that $(\tilde{a}'_{x^{-1}})^{-1}i\tilde{a}_{x^{-1}} \in I \cap xIx^{-1}$. We now apply Lemma 4.5: any element of $I/\check{G}_f$ can be written uniquely as a product $s_{x^{-1}}r_{x^{-1}}$ with $s_{x^{-1}} = \prod_{\alpha \in p(A_{x^{-1}})} s_\alpha$ and $r_{x^{-1}} = \prod_{\alpha \in p(B_{x^{-1}} \cup C_{x^{-1}})} r_\alpha$ with $s_\alpha, r_\alpha \in L_{[f_I(\alpha), f(\alpha)]} U_\alpha(\overline{\mathbb{F}_q})$. By definition, the affine roots in $A_{x^{-1}}$ are precisely those affine roots in $\widehat{\Phi}_{\text{aff}}(I/\check{G}_f)$ which do not occur in $I \cap xIx^{-1}$. Hence we see that the image of the composed map $I \cap xIx^{-1} \hookrightarrow I \rightarrow I/\check{G}_f$ is equal to the set of all elements of $I/\check{G}_f$ with $s_{x^{-1}} = 1$ in the above decomposition. Now we have inside $I/\check{G}_f$ (so in particular, the element $i \in \check{G}_f$ can be ignored)

$$a_{x^{-1}} = a_{x^{-1}} \cdot 1 = a'_{x^{-1}} \cdot (a'_{x^{-1}})^{-1}i a_{x^{-1}},$$

which gives two decompositions of the element $a_{x^{-1}} \in I/\check{G}_f$. By uniqueness of such a decomposition, we must have $a'_{x^{-1}} = a_{x^{-1}}$. Now analogous computations (first done for a'_x, a_x and then for b'_x, b_x) show that we also must have $a'_x = a_x$ and $b'_x = b_x$. This finishes the proof of injectivity. $\square$

The Schubert cell attached to $x \in \tilde{W}$ is the reduced subscheme of $LG/L^+\mathcal{I}$ whose underlying set of $\overline{\mathbb{F}_q}$ -points is $IxI/I \subseteq \check{G}/I$. We denote it by $C(x)$. As $\overline{\mathbb{F}_q}$ -schemes $C(x) \cong \mathbb{A}^{\ell(x)}$, where $\ell(x)$ is the length of the element x in $\tilde{W}$. We now consider the reduced subscheme of $LG/L^+\mathcal{G}_f$, whose underlying set of $\overline{\mathbb{F}_q}$ -points is $IxI/\check{G}_f \subseteq \check{G}/\check{G}_f$ and we denote it by $C_f(x)$. The étale $L^+\mathcal{I}/L^+\mathcal{G}_f$ -torsor $LG/L^+\mathcal{G}_f \rightarrow LG/L^+\mathcal{I}$ pulls back to the étale $L^+\mathcal{I}/L^+\mathcal{G}_f$ -torsor $C_f(x) \rightarrow C(x)$.

Lemma 4.7 *Let $f: \widehat{\Phi} \rightarrow \mathbb{R}_{\geq 0} \setminus \{\infty\}$ be a concave function such that $\check{G}_f \subseteq I$ is a normal subgroup. Let $x \in \tilde{W}$. The étale $L^+\mathcal{I}/L^+\mathcal{G}_f$ -torsor $C_f(x) \rightarrow C(x)$ is trivial. If $p^{-1}(p(A_{x^{-1}})) \cap \widehat{\Phi}_{\text{aff}}(I/\check{G}_f) = A_{x^{-1}}$, then there is an isomorphism of $\overline{\mathbb{F}_q}$ -schemes*

$$\prod_{\alpha \in p(A_{x^{-1}})} L_{[f_I(\alpha), f(\alpha)]} \mathcal{U}_\alpha \times L^+\mathcal{I}/L^+\mathcal{G}_f \rightarrow C_f(x)$$

given by $((a_\alpha)_{\alpha \in p(A_{x^{-1}})}, i) \mapsto \prod_{\alpha \in p(A_{x^{-1}})} \tilde{a}_\alpha \cdot x \cdot i \check{G}_f$, where $\tilde{a}_\alpha \in \check{U}_{\alpha, f_I(\alpha)}$ is any lift of a_α .

Proof The group $L^+\mathcal{I}/L^+\mathcal{G}_f$ has a composition series with all subquotients either $\mathbb{G}_a$ or $\mathbb{G}_m$. The cohomology of both vanishes on an affine space. This proves that $C_f(x) \rightarrow C(x)$ is a trivial torsor. The explicit isomorphism is proven in the same way as in Lemma 4.6. $\square$

Lemma 4.8 *Under the assumptions of Lemma 4.6, the projection $p: C_f(x) \rightarrow \check{G}_f \backslash IxI / \check{G}_f$ is a geometric quotient in the sense of Mumford for the left multiplication action of $L^+ \mathcal{G}_f$ on $C_f(x)$. Here $\check{G}_f \backslash IxI / \check{G}_f$ is endowed with the structure of an $\mathbb{F}_q$ -scheme using the parametrization from Lemma 4.6.*

Proof First note that from the assumptions of Lemma 4.6 it follows that $p^{-1}(p(A_{x^{-1}})) \cap \widehat{\Phi}_{\text{aff}}(I / \check{G}_f) = A_{x^{-1}}$ (as $A_{x^{-1}} \dot{\cup} B_{x^{-1}} \dot{\cup} C_{x^{-1}} = \widehat{\Phi}_{\text{aff}}(I / \check{G}_f)$), so we may use Lemma 4.7. The action of $L^+ \mathcal{G}_f$ on $C_f(x)$ factors through a finite-dimensional quotient (any subgroup $J \subseteq \check{G}_f \cap x \check{G}_f x^{-1}$ which is normal in $\check{G}_f$ acts trivially on $C_f(x)$). Now, since p is a surjective orbit map, $\check{G}_f \backslash IxI / \check{G}_f$ is normal and the irreducible components of $C_x(f)$ are open. Thus by [3, Proposition 6.6], it remains to show that p is a separable morphism of varieties. But this is true since, in terms of the parameterizations given in Lemmas 4.6 and 4.7, it is given by $(a_{x^{-1}}, i = c_x b_x a_x) \mapsto (a_{x^{-1}}, b_x, a_x)$. $\square$

Theorem 4.9 *Assume G is split. Let $f: \widehat{\Phi} \rightarrow \widetilde{\mathbb{R}}_{\geq 0} \setminus \{\infty\}$ be a concave function such that $\check{G}_f \subseteq I$ is a normal subgroup. Let $\dot{x}$ be an $\check{G}_f$ -double coset in $\check{G}$ with image x in $\widetilde{W}$. Assume that $p(A_x)$, $p(B_x)$ and $p(C_x)$ are mutually disjoint, and that the same is true for $p(A_{x^{-1}})$, $p(B_{x^{-1}})$, $p(C_{x^{-1}})$, where A, B, C are as in (4.1). For $b \in \check{G}$ arbitrary, $X_x^f(b)$ is locally closed in $\check{G} / \check{G}_f$.*

Proof By Lemma 4.8, the theorem is now a special case of [22, Proposition 2.4]. We recall the proof. It is well known that the Iwahori-level sets $X_x^I(b) = X_x(b)$ are locally closed in $LG / L^+ \mathcal{I}$. Let $\widetilde{X}$ be the pullback of $X_x^I(b)$ along $LG / L^+ \mathcal{G}_f \rightarrow LG / L^+ \mathcal{I}$. As $X_w(b)$ are schemes locally of finite type over $\overline{\mathbb{F}}_q$, the same is true for $\widetilde{X}$. By Theorem 4.1, the map $\beta: LG \rightarrow LG / L^+ \mathcal{G}_f$ admits sections étale locally. Let $U \rightarrow \widetilde{X}$ be étale such that there is a section $s: U \rightarrow \beta^{-1}(U)$ of β . Consider the composition

$$\psi: U \rightarrow \beta^{-1}(U) \times U \rightarrow LG / L^+ \mathcal{G}_f,$$

where the first map is $g \mapsto (s(g^{-1}), b\sigma(g))$ and the second map is the restriction of the left multiplication action of LG on $LG / L^+ \mathcal{G}_f$. As U lies over $\widetilde{X}$, this composition factors through $C_f(x) \rightarrow LG / L^+ \mathcal{G}_f$. Denote the resulting morphism by $\psi_0: U \rightarrow C_f(x)$. Let $p: C_f(x) \rightarrow \check{G}_f \backslash IxI / \check{G}_f$ denote the geometric quotient from Lemma 4.8. The composition $p \circ \psi_0$ is independent of the choice of the section s . It sends an $\overline{\mathbb{F}}_q$ -point $g \check{G}_f$ to the double coset $\check{G}_f g^{-1} b \sigma(g) \check{G}_f$. Thus étale locally, $X_x^f(b)$ is just the preimage of the point $\dot{x}$ under $p \circ \psi_0$. The theorem now follows by using étale descent for closed subschemes. $\square$

Corollary 4.10 *Under the assumptions as in Theorem 4.9, $X_x^f(b)$ endowed with the induced reduced sub-ind-scheme structure is a scheme locally of finite type over $\overline{\mathbb{F}}_q$.*

Proof $X_w^I(b)$ is a scheme locally of finite type over $\overline{\mathbb{F}}_q$. Since $p: LG / L^+ \mathcal{G}_f \rightarrow LG / L^+ \mathcal{I}$ is a morphism of ind-schemes which is a torsor under the finite-dimensional affine group scheme $L^+ \mathcal{I} / L^+ \mathcal{G}_f$ (by Proposition 4.2), it follows that $\widetilde{X} := p^{-1}(X_w^I(b))$ is also a scheme locally of finite type over $\overline{\mathbb{F}}_q$. Now the proof of Theorem 4.9 shows that $X_x^f(b)$ has the same property. $\square$

Part 2. Geometry of Deligne–Lusztig varieties for inner forms of GL_n

From now and until the end of the paper, we fix an integer $n \geq 1$ and study in detail the constructions in Part 1 for $\mathrm{GL}_n(K)$ and its inner forms. Inner forms of GL_n over K can be naturally parametrized by $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$. Fix an integer $0 \leq \kappa < n$, put $n' = \gcd(\kappa, n)$, and let n_0, k_0 be the non-negative integers such that

$$n = n'n_0, \quad \kappa = n'k_0.$$

The group of K -points of the inner form corresponding to κ/n is isomorphic to $G := \mathrm{GL}_{n'}(D_{k_0/n_0})$, where D_{k_0/n_0} denotes the central division algebra over K with invariant k_0/n_0 . Let $\mathcal{O}_{D_{k_0/n_0}}$ denote the ring of integers of D_{k_0/n_0} and set $G_{\mathcal{O}} := \mathrm{GL}_{n'}(\mathcal{O}_{D_{k_0/n_0}})$. Note that $G_{\mathcal{O}}$ is a maximal compact subgroup of G .

We let L denote the unramified extension of K of degree n , and write $\mathcal{O}_L$ for its integers, $\mathfrak{p}_L$ for the maximal ideal in $\mathcal{O}_L$. For $h \geq 1$, we write $U_L^h = 1 + \mathfrak{p}_L^h$ for the h -units of L .

Up to conjugacy there is only one maximal unramified elliptic torus $T \subseteq G$. We have $T \cong L^\times$. Moreover, we say a smooth character $\theta: L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ has level $h \geq 0$, if θ is trivial on U_L^{h+1} and non-trivial on U_L^h .

We let V be an n -dimensional vector space over $\check{K}$ with a fixed K -rational structure V_K . Fix a basis $\{e_1, \dots, e_n\}$ of V_K . This gives an identification of $\mathrm{GL}(V_K)$ with GL_n over K . Set $\mathcal{L}_0$ to be the $\mathcal{O}$ -lattice generated by $\{e_1, \dots, e_n\}$.

5 Inner forms of GL_n

5.1 Presentation as σ -stabilizers of basic elements

For $b \in \mathrm{GL}_n(\check{K})$, recall from Sect. 3 the σ -stabilizer J_b of b . Then J_b is an inner form of the centralizer of the Newton point b (which is a Levi subgroup of GL_n). In particular, if b is *basic*, i.e. the Newton point of b is central, then J_b is an inner form of GL_n , and every inner form of GL_n arises in this way. If

$$\kappa = \kappa_{\mathrm{GL}_n}(b) := \mathrm{ord} \circ \det(b),$$

then J_b is the inner form corresponding to κ/n modulo $\mathbb{Z}$. Note that κ_{GL_n} is the Kottwitz map

$$\kappa_{\mathrm{GL}_n}: B(\mathrm{GL}_n(\check{K})) := \{\sigma\text{-conj classes in } \mathrm{GL}_n(\check{K})\} \rightarrow \mathbb{Z}$$

and induces a bijection between the set of basic σ -conjugacy classes and $\mathbb{Z}$. Consider

$$F: \mathrm{GL}_n(\check{K}) \rightarrow \mathrm{GL}_n(\check{K}), \quad g \mapsto b\sigma(g)b^{-1}.$$

This is a twisted Frobenius on $\mathrm{GL}_n(\check{K})$ and J_b is the K -group corresponding to this Frobenius on $\mathrm{GL}_n(\check{K})$. In particular, if b is in the basic σ -conjugacy class with $\kappa_{\mathrm{GL}_n}(b) = \kappa$, then

$$G = \mathrm{GL}_{n'}(D_{k_0/n_0}) \cong \mathrm{GL}_n(\check{K})^F = J_b(K).$$

5.2 Two different choices for b

We will need to choose representatives b of the basic σ -conjugacy class $[b]$ with $\kappa_{\mathrm{GL}_n}(b) = \kappa$. Depending on the context, we will work with either a *Coxeter-type representative* or a *special representative*.

5.2.1 Coxeter-type representatives

Set

$$b_0 := \begin{pmatrix} 0 & 1 \\ 1_{n-1} & 0 \end{pmatrix}, \quad \text{and} \quad t_{\kappa,n} := \begin{cases} \underbrace{\mathrm{diag}(1, \dots, 1)}_{n-\kappa}, \underbrace{(\varpi, \dots, \varpi)}_{\kappa} & \text{if } (\kappa, n) = 1, \\ \underbrace{\mathrm{diag}(t_{k_0, n_0}, \dots, t_{k_0, n_0})}_{n'} & \text{otherwise.} \end{cases}$$

Fix an integer $e_{\kappa,n}$ such that $(e_{\kappa,n}, n) = 1$ and $e_{\kappa,n} \equiv k_0 \pmod{n_0}$. (It is clear that $e_{\kappa,n}$ exists.) If κ divides n , (i.e. $k_0 = 1$), always take $e_{\kappa,n} = 1$.

Definition 5.1 The *Coxeter-type representative* attached to κ is $b_0^{e_{\kappa,n}} \cdot t_{\kappa,n}$.

The main advantage of this choice is that the maximal torus of $\mathrm{GL}_n(\check{K})$ consisting of diagonal matrices gives an unramified elliptic torus of J_b (as the image of b in the Weyl group of the diagonal torus is a cycle of length n). Thus when we use the explicit presentation $G = J_b(K)$ for the Coxeter-type b , then our unramified elliptic torus $T \subseteq G$ is the diagonal torus.

5.2.2 Special representatives

Definition 5.2 The *special representative* attached to κ is the block-diagonal matrix of size $n \times n$ with $(n_0 \times n_0)$ -blocks of the form $\begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}^{k_0}$.

Note that the special representative and the Coxeter-type representative agree if $(\kappa, n) = 1$ (see the proof of Lemma 5.6), though in general they may differ (for example, when $\kappa = 0$, the special representative is the identity and the Coxeter representative is b_0).

Remark 5.3 If b is the special representative, $b\sigma$ acts on the standard basis $\{\mathbf{e}_i\}_{i=1}^n$ of V in the same way as in [32, Section 4.1] the operator F considered there acts on the basis $\{e_{j,i,l}\}_{j,i,l}$. To be more precise, in our situation, there is only one j (that is $j = 1$) as the isocrystal $(V, b\sigma)$ is isoclinic. Then our basis element $\mathbf{e}_i$ for $1 \leq i \leq n$ corresponds to Viehmann's basis element $e_{1,i'+1,l}$, where $i = i'n_0 + l$ for $0 \leq i' < n'$, $0 \leq l < n_0$. $\diamond$

Remark 5.4 If $(\kappa, n) = 1$, the special representative b is a length-0 element of the extended affine Weyl group of GL_n and therefore is a standard representative in the sense of [17, Section 7.2]. In general, b is block-diagonal with blocks consisting of the standard representative of size $n_0 \times n_0$ and determinant k_0 . $\diamond$

5.2.3 Properties of the representatives

Lemma 5.5 Let $\check{T}_{\mathrm{diag}}$ denote the maximal torus of $\mathrm{GL}_n(\check{K})$ given by the subgroup of diagonal matrices. Then the Coxeter-type and special representatives lie in the normalizer $N_{\mathrm{GL}_n(\check{K})}(\check{T}_{\mathrm{diag}})$. Moreover, both representatives are basic elements whose Newton polygon has slope κ/n .

Proof The first statement is clear. For $b \in N_{\mathrm{GL}_n(\check{K})}(\check{T}_{\mathrm{diag}})$, the Newton point can be computed as $\frac{1}{a} v_{b^a}$, where $a \in \mathbb{Z}_{>0}$ is appropriate such that $b^a \in \check{T}_{\mathrm{diag}}$. Thus the second statement follows from an easy calculation (for the Coxeter type, it uses the condition on $e_{\kappa, n}$). $\square$

Let $b, b' \in \mathrm{GL}_n(\check{K})$. We say b, b' are integrally σ -conjugate if there is $g \in \mathrm{GL}_n(\mathcal{O})$ such that $g^{-1}b\sigma(g) = b'$.

Lemma 5.6 The Coxeter-type and special representatives attached to κ/n are integrally σ -conjugate.

Proof If κ is coprime to n , then necessarily $e_{\kappa, n} = \kappa$. We have $b_0^{-1} \mathrm{diag}(1, \dots, 1, \varpi) b_0 = \mathrm{diag}(1, \dots, 1, \varpi, 1)$ and it follows that $(b_0 \cdot t_{1, n})^\kappa = b_0^\kappa \cdot t_{\kappa, n}$, so the special representative and the Coxeter representative agree. Now assume that $(\kappa, n) = n' > 1$. For convenience of notation, let b denote the Coxeter-type representative and let b' denote the special representative. Recall that $b = b_0^{e_{\kappa, n}} \cdot t_{\kappa, n} = \begin{pmatrix} 0 & 1 \\ 1_{n-1} & 0 \end{pmatrix}^{e_{\kappa, n}} \cdot \mathrm{diag}(t_{k_0, n_0}, \dots, t_{k_0, n_0})$ and that b' is the block-diagonal matrix with $\begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}^{k_0}$ in each block. We would like to produce a $g \in \mathrm{GL}_n(\mathcal{O})$ such that $b\sigma(g) = gb'$. Write

$$g = (g_1 \mid g_2 \mid \cdots \mid g_n), \text{ where } g_i = \begin{pmatrix} g_{i,1} \\ g_{i,2} \\ \vdots \\ g_{i,n} \end{pmatrix}.$$

Then the first n_0 columns of the equation $b\sigma(g) = gb'$ is the equality of $n \times n_0$ matrices

$$(b\sigma(g_1) \mid b\sigma(g_2) \mid \cdots \mid b\sigma(g_{n_0})) = (g_{k_0+1} \mid g_{k_0+2} \mid \cdots \mid g_{n_0} \mid \varpi g_1 \mid \varpi g_2 \mid \cdots \mid \varpi g_{k_0}).$$

In other words, we have

$$g_i = \begin{cases} \varpi^{-1} \cdot b\sigma(g_{[i-k_0]_{n_0}}) & \text{if } 1 \leq i \leq k_0, \\ b\sigma(g_{[i-k_0]_{n_0}}) & \text{if } k_0 + 1 \leq i \leq n_0, \end{cases} \quad (5.1)$$

and hence

$$\varpi^{k_0} g_{k_0} = \varpi^{k_0-1} b \sigma(g_{n_0}) = \varpi^{k_0-1} b^2 \sigma^2(g_{n_0-k_0}) = \dots = b^{n_0} \sigma^{n_0}(g_{k_0}), \quad (5.2)$$

where the exponent of ϖ changes periodically according to (5.1). Observe that since $b_0^{-1} t_{\kappa,n} b_0$ is the block-diagonal matrix with blocks equal to $\begin{pmatrix} 0 & 1 \\ 1_{n_0-1} & 0 \end{pmatrix}^{-1} t_{k_0,n_0} \begin{pmatrix} 0 & 1 \\ 1_{n_0-1} & 0 \end{pmatrix}$, we see that $\varpi^{-k_0} b^{n_0}$ is a permutation matrix of order n' ; writing $e' := [e_{\kappa,n} n_0]_n$, we have $\varpi^{-k_0} b^{n_0} = \begin{pmatrix} 0 & 1_{e'} \\ 1_{n-e'} & 0 \end{pmatrix}$ and $(e', n) = n_0$. Write

$$g_{k_0} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then from (5.2) it follows that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{e'} \\ x_{e'+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sigma^{n_0}(x_{n-e'+1}) \\ \sigma^{n_0}(x_{n-e'+2}) \\ \vdots \\ \sigma^{n_0}(x_n) \\ \sigma^{n_0}(x_1) \\ \vdots \\ \sigma^{n_0}(x_{n-e'}) \end{pmatrix}.$$

In particular, the vector g_{k_0} is determined by $x_1, x_2, \dots, x_{n_0}$.

Let $\bar{\alpha} \in \mathbb{F}_q^\times$ be an element such that $\bar{\alpha}, \sigma(\bar{\alpha}), \dots, \sigma^{n-1}(\bar{\alpha})$ are linearly independent over $\mathbb{F}_q$. Let $\alpha \in \mathcal{O}^\times$ be any lift of $\bar{\alpha}$. Let g_{k_0} be the vector associated to the choice $x_{n_0-k_0+1} = \alpha$ and $x_i = 0$ otherwise. We next show that for this choice of g_{k_0} , each of the columns $g_1, \dots, g_{n_0}$ (all of which are determined by g_{k_0} by (5.1)) have coefficients in $\mathcal{O}$ and that moreover the entries are either zero or in $\mathcal{O}^\times$. For any positive integer j , we know that

$$b \sigma(\mathbf{e}_{[i_0-e_{\kappa,n}j]_n}) = \begin{cases} \varpi \cdot \mathbf{e}_{[i_0-e_{\kappa,n}(j+1)]_n} & \text{if } [i_0 - e_{\kappa,n}j]_{n_0} \geq n_0 - k_0 + 1 \\ \mathbf{e}_{[i_0-e_{\kappa,n}(j+1)]_n} & \text{if } [i_0 - e_{\kappa,n}j]_{n_0} \leq n_0 - k_0 \end{cases}$$

where $\mathbf{e}_i$ denotes the i th elementary column vector. Comparing this to (5.1), we see that the condition for when the scalar ϖ^{-1} appears in the equation for $g_{[(j+1)k_0]_{n_0}}$ coincides with the condition for when the scalar ϖ appears in the equation for $b \sigma(\mathbf{e}_{[i_0-e_{\kappa,n}j]_n})$ when $[e_{\kappa,n}(j+1)]_{n_0} = n_0 + 1 - [i_0 - e_{\kappa,n}j]_{n_0}$. Since $e_{\kappa,n} \equiv k_0$ modulo n_0 by definition, this implies $i_0 = n_0 - k_0 + 1$, and it follows now that the entries of $g_1, \dots, g_{n_0}$ are either zero or in $\mathcal{O}^\times$.

We may repeat the above argument for the next n_0 columns of the equation $b \sigma(g) = gb'$. We then obtain that for $n_0 + 1 \leq i \leq 2n_0$, the columns g_i of g are uniquely determined by $g_{n_0+k_0}$, and that the vector $g_{n_0+k_0}$ is uniquely determined by its first n_0 entries $x_1, \dots, x_{n_0}$. Let $g_{n_0+k_0}$ be the vector associated to the choice $x_{n_0-k_0+1} = \sigma(\alpha)$ and $x_i = 0$ otherwise. Repeating this for the remaining columns of g , with $g_{in_0+k_0}$

being determined by setting $x_{n_0-k_0+1} = \sigma^i(\alpha)$ for $1 \leq i \leq n' - 1$, we have now obtained an $n \times n$ matrix g satisfying $b\sigma(g) = gb'$ whose entries are either 0 or in $\mathcal{O}^\times$.

To complete the proof that b, b' are integrally σ -conjugate, it remains to show that the $\mathcal{O}$ -valued $n \times n$ matrix g lies in $\mathrm{GL}_n(\mathcal{O})$; that is, it remains to show that $\det(g) \in \mathcal{O}^\times$. To see this, observe that by construction, the rows and columns of g can be permuted so that it is block-diagonal with i th ($1 \leq i \leq n_0$) block equal to the matrix

$$\sigma^{(i-1)n'} \begin{pmatrix} \alpha & \sigma(\alpha) & \cdots & \sigma^{n'-1}(\alpha) \\ \sigma(\alpha) & \sigma^2(\alpha) & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{n'-1}(\alpha) & \alpha & \cdots & \sigma^{n'-2}(\alpha) \end{pmatrix}.$$

The reduction-modulo- ϖ of this matrix is a Vandermonde matrix, and since $\bar{\alpha}$ is such that $\bar{\alpha}, \sigma(\bar{\alpha}), \dots, \sigma^{n-1}(\bar{\alpha}) \in \mathbb{F}_{q^n}$ are linearly dependent over $\mathbb{F}_q$, it has nonzero determinant. Hence $\det(g) \in \mathcal{O}^\times$.

5.3 Integral models

Let $\mathcal{B}^{\mathrm{red}} := \mathcal{B}^{\mathrm{red}}(\mathrm{GL}_n, \check{K})$ be the reduced building of GL_n over $\check{K}$. For any point $x \in \mathcal{B}^{\mathrm{red}}$, the Moy–Prasad filtration is a collection of subgroups $\check{G}_{x,r} \subset \mathrm{GL}_n(\check{K})$ indexed by real numbers $r \geq 0$ [27, Section 3.2]. We write $\check{G}_{x,r+} = \bigcup_{s>r} \check{G}_{x,s} \subset \mathrm{GL}_n(\check{K})$.

Let $\mathcal{A}^{\mathrm{red}}$ denote the apartment of $\mathcal{B}^{\mathrm{red}}$ associated to the maximal split torus given by the subgroup of diagonal matrices in $\mathrm{GL}_n(\check{K})$ and let b be the Coxeter-type representative so that b acts on $\mathcal{A}^{\mathrm{red}}$ with a unique fixed point $\mathbf{x} \in \mathcal{A}^{\mathrm{red}}$. By construction, each $\check{G}_{\mathbf{x},r}$ is stable under the Frobenius $F(g) = b\sigma(g)b^{-1}$ and $\check{G}_{\mathbf{x},0}^F \cong G_{\mathcal{O}}$.

We now define $\mathbb{G}$ to be the smooth affine group scheme over $\mathbb{F}_q$ such that

$$\mathbb{G}(\bar{\mathbb{F}}_q) = \check{G}_{\mathbf{x},0}, \quad \mathbb{G}(\mathbb{F}_q) = \check{G}_{\mathbf{x},0}^F.$$

For $h \in \mathbb{Z}_{\geq 1}$, we define $\mathbb{G}_h$ to be the smooth affine group scheme over $\mathbb{F}_q$ such that

$$\mathbb{G}_h(\bar{\mathbb{F}}_q) = \check{G}_{\mathbf{x},0}/\check{G}_{\mathbf{x},(h-1)+}, \quad \mathbb{G}_h(\mathbb{F}_q) = \check{G}_{\mathbf{x},0}^F/\check{G}_{\mathbf{x},(h-1)+}^F.$$

We have a well-defined determinant morphism

$$\det: \mathbb{G}_h \rightarrow \mathbb{W}_h^\times.$$

Define $\mathbb{T}_h$ to be the subgroup scheme of $\mathbb{G}_h$ defined over $\mathbb{F}_q$ given by the diagonal matrices. Set:

$$G_h := \mathbb{G}_h(\mathbb{F}_q), \quad T_h := \mathbb{T}_h(\mathbb{F}_q).$$

Note that $\mathbb{G}_h(\mathbb{F}_q)$ is a subquotient of G and $\mathbb{T}_h(\mathbb{F}_q) \cong (\mathcal{O}_L/\varpi^h)^\times \cong \mathbb{W}_h^\times(\mathbb{F}_{q^n})$ is a subquotient of the unramified elliptic torus T of G .

We remark that each $\check{G}_{\mathbf{x},r}$ is also stable under the Frobenius $F(g) = b\sigma(g)b^{-1}$ for the special representative b and that $\check{G}_{\mathbf{x},0}^F \cong G_{\mathcal{O}}$. Thus we also can regard $\mathbb{G}_h$ as a group scheme over $\mathbb{F}_q$ as above with $\mathbb{G}_h(\mathbb{F}_q)$ a subquotient of $J_b(K)$ with b being the special representative. However, the induced $\mathbb{F}_q$ -rational structure on $\mathbb{T}_h$ gives that $\mathbb{T}_h(\mathbb{F}_q) \cong (\mathbb{W}_h^\times(\mathbb{F}_{q^{n_0}}))^{\times n'}$, which is *not* a subquotient of any elliptic torus in G .

Explicitly, $\mathbb{G}_h(\mathbb{F}_q)$ is the group of invertible $n \times n$ -matrices, whose $n_0 \times n_0$ -blocks are matrices $(a_{ij})_{1 \leq i, j \leq n_0}$ with $a_{ii} \in \mathcal{O}/\mathfrak{p}^h$, $a_{ij} \in \mathcal{O}/\mathfrak{p}^{h-1}$ ($\forall i > j$), $a_{ij} \in \mathfrak{p}/\mathfrak{p}^h$ ($\forall i < j$). For example, for $n_0 = 3$, the $n_0 \times n_0$ -blocks are

$$\begin{pmatrix} \mathcal{O}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h \\ \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h \\ \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^h \end{pmatrix}.$$

The following lemma describes the F -fixed part of the Weyl group of $\mathbb{T}_1$ in $\mathbb{G}_1$ explicitly. Note that $b^{n_0}\varpi^{-k_0}$ is a permutation matrix in $\mathrm{GL}_n(K)$.

Lemma 5.7 *Let b be the Coxeter-type representative. We have*

- (i) *We have $N_{\mathbb{G}_h}(\mathbb{T}_h)/\mathbb{T}_h = N_{\mathbb{G}_1}(\mathbb{T}_1)/\mathbb{T}_1 = S_{n'} \times \cdots \times S_{n'} (n_0 \text{ copies})$.*
- (ii) *$N_{G_h}(T_h)/T_h = (N_{\mathbb{G}_h}(\mathbb{T}_h)/\mathbb{T}_h)^F = \langle b^{n_0}\varpi^{-k_0} \rangle \cong \mathrm{Gal}(L/K)[n']$, the n' -torsion subgroup of $\mathrm{Gal}(L/K)$.*

Proof Part (i) is clear by the explicit description of $\mathbb{G}_h$. To prove (ii), we need to make the action of F on $N_{\mathbb{G}_h}(\mathbb{T}_h)/\mathbb{T}_h$ explicit. Indeed, F is an automorphism of order n , it permutes the copies of $S_{n'}$ cyclically, and each of the copies is stabilized by F^{n_0} . We can think of the first $S_{n'}$ as permutation matrices with entries 0 and 1 in $\mathrm{GL}(\langle \mathbf{e}_i : i \equiv 1 \pmod{n_0} \rangle) \cong \mathrm{GL}_{n'}$. Then the F^{n_0} -action $S_{n'}$ comes from the conjugation by b^{n_0} on $\mathrm{GL}(\langle \mathbf{e}_i : i \equiv 1 \pmod{n_0} \rangle)$. But b^{n_0} is the order- n' cycle $e_1 \mapsto e_{1+n_0} \mapsto \cdots \mapsto e_{1+n_0(n'-1)} \mapsto e_1$, and the subgroup of $S_{n'}$ stable by it is $\langle b^{n_0}\varpi^{-k_0} \rangle$. We can identify it with $\mathrm{Gal}(L/K)[n']$ by sending $b^{n_0}\varpi^{-k_0}$ to σ^{n_0} (see also Lemma 5.8). $\square$

5.4 Twisted polynomial rings

Let L_0 be the degree- n_0 unramified extension of K and consider the twisted polynomial ring $L_0\langle \Pi_0 \rangle$ determined by the commutation relation $a \cdot \Pi_0 = \Pi_0 \cdot \sigma^{l_0}(a)$, where l_0 is the integer in the range $1 \leq l_0 \leq n_0$ with $l_0 k_0 = 1$ modulo n_0 .

On the other hand, consider the Frobenius map $F_0: M_{n_0}(\check{K}) \rightarrow M_{n_0}(\check{K})$ defined by $F_0(g) = \begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}^{k_0} \sigma(g) \begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}^{-k_0}$. The diagonal matrices in $M_{n_0}(\check{K})^{F_0}$ are exactly of the form

$$D_0(a) := \mathrm{diag}(a, \sigma^{l_0}(a), \dots, \sigma^{(n_0-1)l_0}(a)), \quad \text{for } a \in L_0.$$

By a direct calculation, we see that we can define an isomorphism

$$L_0\langle \Pi_0 \rangle / (\Pi_0^{n_0} - \varpi^{k_0}) \rightarrow M_{n_0}(\check{K})^{F_0} = D_{k_0/n_0}$$

by sending $a \in L_0$ to $D_0(a)$, and sending Π_0 to $\begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}$. Note that under this identification, the ring of integers $\mathcal{O}_{D_{k_0/n_0}}$ of D_{k_0/n_0} is $\mathcal{O}_{L_0} \langle \Pi_0 \rangle / (\Pi_0^{n_0} - \varpi^{k_0})$.

5.5 Cartan decomposition

Let b be the special representative and let $\Pi_0 = \begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}$. We use the description of D_{k_0/n_0} in Sect. 5.4. Let $\check{T}_{\text{diag}}$ be the subgroup of diagonal matrices in $\text{GL}_n(\check{K})$. Then the set of F -fixed points of the cocharacters $X_*(\check{T}_{\text{diag}})^F$ is given by

$$X_*(\check{T}_{\text{diag}})^F = \{v = (v_1, \dots, v_1, v_2, \dots, v_2, \dots, v_{n'}, \dots, v_{n'}) : v_i \in \mathbb{Z}\},$$

where each v_i repeated n_0 times. Let $X_*(\check{T}_{\text{diag}})_{\text{dom}}^F \subset X_*(\check{T}_{\text{diag}})^F$ be the subset consisting of v with $v_1 \leq v_2 \leq \dots \leq v_{n'}$. For $v \in X_*(\check{T}_{\text{diag}})^F$, we write Π_0^v for the $n \times n$ block-diagonal matrix whose i th $n_0 \times n_0$ -block is $\Pi_0^{v_i}$. The Cartan decomposition of $G = \text{GL}_{n'}(D_{k_0/n_0})$ with respect to the maximal compact subgroup $G_{\mathcal{O}} = \text{GL}_{n'}(\mathcal{O}_{D_{k_0/n_0}})$ is given by

$$G = \bigsqcup_{v \in X_*(\check{T}_{\text{diag}})^F, \text{dom}} G_{\mathcal{O}} \Pi_0^v G_{\mathcal{O}}$$

Note that Π_0^v normalizes $G_{\mathcal{O}}$ if and only if all v_i are equal, and Π_0^v centralizes $G_{\mathcal{O}}$ if and only if all v_i are equal and divisible by n_0 so that

$$N_G(G_{\mathcal{O}})/Z_G G_{\mathcal{O}} \cong \mathbb{Z}/n_0\mathbb{Z}.$$

5.6 Reductive quotient $\mathbb{G}_1$

Let b be either the Coxeter-type or the special representative. The group $\mathbb{G}_1$ is equal to the reductive quotient of $\mathbb{G}$. Recall the $\mathcal{O}$ -lattice $\mathcal{L}_0$ and its basis $\{\mathbf{e}_i\}_{i=1}^n$ from the beginning of Part 2. The following lemma describes the reductive quotient in terms of $\mathcal{L}_0$. Its proof reduces to some elementary explicit calculations, so we omit it.

Lemma 5.8 *Let $c, d \in \mathbb{Z}$ with $k_0c + n_0d = 1$.*

- (i) *We have $(b\sigma)^c \varpi^d(\mathcal{L}_0) \subseteq \mathcal{L}_0$, and $(b\sigma)^c \varpi^d(\mathcal{L}_0)$ is independent of the choice of c, d .¹ The quotient space*

$$\overline{V} := \mathcal{L}_0 / (b\sigma)^c \varpi^d(\mathcal{L}_0)$$

is n' -dimensional $\overline{\mathbb{F}}_q$ -vector space. The images of $\{\mathbf{e}_i\}_{i=1 \pmod{n_0}}$ form a basis of $\overline{V}$.

¹ $(b\sigma)^c \varpi^d(\mathcal{L}_0)$ coincides with the operator defined in [32, Equation (4.3)].

- (ii) The map $(b\sigma)^{n_0}\varpi^{-k_0}$ induces a σ^{n_0} -linear automorphism $\overline{\sigma}_b$ of $\overline{V}$, equipping it with a $\mathbb{F}_{q^{n_0}}$ -linear structure. If b is the special representative, the σ^{n_0} -linear operator $\overline{\sigma}_b$ of $\overline{V}$ is given by $\mathbf{e}_i \mapsto \mathbf{e}_i$ for $1 \leq i \leq n$ with $i \equiv 1 \pmod{n_0}$. If b is Coxeter-type, then it is given by $e_{1+n_0i} \mapsto e_{1+n_0(i+e_{\kappa,n})}$.
- (iii) We have a canonical identification

$$\mathbb{G}_1 = \text{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q} \text{GL}_{n'} \overline{V}.$$

5.7 Isocrystals

We recall that an $\overline{\mathbb{F}}_q$ -isocrystal is an $\check{K}$ -vector space together with an σ -linear isomorphism. For $b \in \text{GL}_n(\check{K})$, we have the isocrystal $(V, b\sigma)$. Assume now that b is basic with $\kappa_G(b) = \kappa$. Then $(V, b\sigma)$ is isomorphic to the direct sum of n' copies of the simple isocrystal with slope k_0/n_0 . We observe that $(V, b\sigma)$ up to isomorphism only depends on the σ -conjugacy class $[b]$, and that its group of automorphisms is $G = J_b(K)$.

6 Comparison in the case GL_n , b basic, w Coxeter

We will compare the two Deligne–Lusztig type constructions from Part 1 in this special situation and describe both explicitly using the isocrystal $(V, b\sigma)$. In Sects. 6.1 and 6.2, we let $b \in \text{GL}_n(\check{K})$ be any basic element with $\kappa_{\text{GL}_n}(b) = \kappa$. From Sect. 6.3 onwards, we take b to be the special representative defined in Sect. 5.2.2.

6.1 The admissible subset of $(V, b\sigma)$

We will describe the various Deligne–Lusztig varieties using certain subsets of V , which we now define. Let $x \in V$. Put

$$\begin{aligned} g_b(x) &= \text{matrix in } M_n(\check{K}) \text{ with columns } x, b\sigma(x), \dots, (b\sigma)^{n-1}(x) \\ V_b^{\text{adm}} &= \{x \in V : \det g_b(x) \in \check{K}^\times\} \\ V_b^{\text{adm, rat}} &= \{x \in V : \det g_b(x) \in K^\times\} \end{aligned}$$

If $g^{-1}b'\sigma(g) = b$, then the isomorphism of isocrystals $(V, b\sigma) \rightarrow (V, b'\sigma)$, $x \mapsto gx$, maps V_b^{adm} to $V_{b'}^{\text{adm}}$. In particular, $J_b(K)$ acts on V_b^{adm} by left multiplication. Moreover, $L^\times$ acts on $V_b^{\text{adm, rat}}$ by scaling. Note also that $x \in V$ lies in V_b^{adm} if and only if the $\mathcal{O}$ -submodule of V generated by $x, (b\sigma)(x), \dots, (b\sigma)^{n-1}(x)$ is an $\mathcal{O}$ -lattice.

We have the following useful lemma, which essentially follows from basic properties of Newton polygons. Its simple proof was explained to the authors by E. Viehmann.

Lemma 6.1 *Let $x \in V_b^{\text{adm}}$. The $\mathcal{O}$ -lattice generated by $\{(b\sigma)^i(x)\}_{i=0}^{n-1}$ is $b\sigma$ -stable, i.e., there exist unique elements $\lambda_i \in \mathcal{O}$ such that $(b\sigma)^n(x) = \sum_{i=0}^{n-1} \lambda_i (b\sigma)^i(x)$. Moreover, $\text{ord}(\lambda_0) = \kappa_G(b)$.*

Proof The Newton polygon of $(V, b\sigma)$ is the straight line segment connecting the points $(0, 0)$ and (n, κ) in the plane. Now, let $K[\Sigma]$ be the non-commutative ring defined by the relation $a\Sigma = \Sigma\sigma(a)$, and let Σ act on V by $b\sigma$. Then the Newton polygon of the characteristic polynomial of x (which is an element of $K[\Sigma]$) is equal to the Newton polygon of $(V, b\sigma)$ (see e.g. [1]). Observe that any $x \in V_b^{\text{adm}}$ generates V as a $K[\Sigma]$ -module. Then the point $(i, \text{ord}(a_i))$ in the plane, where a_i is the coefficient of Σ^{n-i} in the characteristic polynomial, lies over that Newton polygon. This simply means $\text{ord}(a_i) \geq \frac{i\kappa}{n} \geq 0$, as $\kappa \geq 0$. Hence $\Sigma^n(v) = \sum_{i=1}^n a_i \Sigma^{n-i}(x)$ lies in the $\mathcal{O}$ -lattice generated by $x, \Sigma(x), \dots, \Sigma^{n-1}(x)$. This proves the first assertion. The second statement follows as $(n, \text{ord}(a_n))$ has necessarily to be the rightmost vertex of the Newton polygon, which is (n, κ) . $\square$

Example 6.2 For $b = 1$, the set V_1^{adm} is just the Drinfeld upper halfspace. If $(\kappa, n) = 1$, then $V_b^{\text{adm}} = V \setminus \{0\}$ as $(V, b\sigma)$ has no proper non-trivial sub-isocrystals.

6.2 Set-theoretic description

We need the following notation:

- Let T_{diag} denote the diagonal torus of GL_n and W its Weyl group.
- Let w be the image in W of the element b_0 from Sect. 5.2.1. Then the form $T_w := T_{\text{diag}, w}$ of T_{diag} (as in Sect. 3) is elliptic with $T_w(K) \cong L^\times$ and has a natural model over $\mathcal{O}_K$, again denoted T_w , with $T_w(\mathcal{O}_K) \cong \mathcal{O}_L^\times$.
- I^m (with $m \geq 0$) denotes the preimage under the projection $\text{GL}_n(\mathcal{O}) \rightarrow \text{GL}_n(\mathcal{O}/\varpi^{m+1}\mathcal{O})$, of all lower triangular matrices in $\text{GL}_n(\mathcal{O}/\varpi^{m+1}\mathcal{O})$ whose entries under the main diagonal lie in $\varpi^m\mathcal{O}/\varpi^{m+1}\mathcal{O}$.
- $\dot{I}^m$ (with $m \geq 0$) denotes the subgroup of I^m consisting of all elements whose diagonal entries are congruent 1 modulo ϖ^{m+1} .
- $X_*^m(b), \dot{X}_*^m(b)$ denote affine Deligne–Lusztig varieties of level $I^m, \dot{I}^m$ respectively (for appropriate $*$).
- For $r \geq 0$ and $x \in V_b^{\text{adm}}$, let $g_{b,r}(x) \in \text{GL}_n(\check{K})$ denote the matrix whose i th column is $\varpi^{r(i-1)}(b\sigma)^{i-1}(x)$. We have $g_b(x) = g_{b,0}(x)$.
- For $r, m \geq 0$, define $\sim_{b,m,r}$ and $\dot{\sim}_{b,m,r}$ on V_b^{adm} by

$$\begin{aligned} x \sim_{b,m,r} y \in V_b^{\text{adm}} &\Leftrightarrow y \in g_{b,r}(x) \cdot (\mathcal{O}^\times \mathfrak{p}^m \dots \mathfrak{p}^m)^\top, \\ x \dot{\sim}_{b,m,r} y \in V_b^{\text{adm}} &\Leftrightarrow y \in g_{b,r}(x) \cdot (1 + \mathfrak{p}^{m+1} \mathfrak{p}^m \dots \mathfrak{p}^m)^\top. \end{aligned}$$

It follows from Lemma 6.6 that $\sim_{b,m,r}$ and $\dot{\sim}_{b,m,r}$ are equivalence relations.

- For $r \geq 0$, set $\dot{w}_r = b_0\varpi^{(-r, \dots, -r, \kappa + (n-1)r)} \in \text{GL}(\check{K})$ and denote again by $\dot{w}_r$ the image of $\dot{w}_r$ in all the sets $I^m \backslash \text{GL}_n(\check{K})/I^m$ and $\dot{I}^m \backslash \text{GL}_n(\check{K})/\dot{I}^m$ for $m \geq 0$. The image of $\dot{w}_r$ in W is the Coxeter element w .

Furthermore, we define

$$V_b^{\text{adm}, \text{rat}, \dot{w}_0} := \{x \in V : \sigma(\det g_b(x)) = \det(\dot{w}_0) \det(b)^{-1} \det(g_b(x))\}. \quad (6.1)$$

Observe that $\det(\dot{w}_0) \det(b)^{-1} \in \mathcal{O}^\times$, so picking any $\alpha \in \mathcal{O}^\times$ such that $\sigma(\alpha \sigma(\alpha) \sigma^2(\alpha) \cdots \sigma^{n-1}(\alpha)) = \det(\dot{w}_0) \det(b)^{-1}$ induces a $(J_b(K) \times L^\times)$ -equivariant isomorphism $V_b^{\text{adm, rat}} \rightarrow V_b^{\text{adm, rat}, \dot{w}_0}$ given by $x \mapsto \alpha x$.

Remark 6.3 We will study the scheme structure on $X_{\dot{w}_r}^m(b)$, $\dot{X}_{\dot{w}_r}^m(b)$ in detail below in Sect. 6.4. But we want to point out already here that both are locally closed in $\text{GL}_n(\check{K})/I^m$, $\text{GL}_n(\check{K})/\dot{I}^m$, hence are reduced $\mathbb{F}_q$ -schemes locally of finite type. Indeed, $\dot{I}^m$ is normal in I and the image of $\dot{w}_r$ in $\tilde{W}$ satisfies the assumptions of Theorem 4.9 (see Lemma 6.4 below), hence it follows that $\dot{X}_{\dot{w}_r}^m(b) \subseteq \text{GL}_n(\check{K})/\dot{I}^m$ is locally closed. The same argument does not apply to $X_{\dot{w}_r}^m(b)$ as $I^m \subseteq I$ is not normal. Still $X_{\dot{w}_r}^m(b) \subseteq \text{GL}_n(\check{K})/I^m$ is locally closed. Indeed, let $p: \text{GL}_n(\check{K})/I^m \rightarrow \text{GL}_n(\check{K})/I$ denote the natural projection. As we will see below in Proposition 6.12, the Iwahori level variety $X_{\dot{w}_r}^0(b) = \bigsqcup_{G/G \circ} g \cdot X_{\dot{w}_r}^0(b)_{\mathcal{L}_0} \subseteq \text{GL}_n(\check{K})/I$ is the scheme-theoretic disjoint union of translates of a certain locally closed subset $X_{\dot{w}_r}^0(b)_{\mathcal{L}_0}$. It thus suffices to show that $X_{\dot{w}_r}^m(b)_{\mathcal{L}_0} = X_{\dot{w}_r}^m(b) \cap p^{-1}(X_{\dot{w}_r}^0(b)_{\mathcal{L}_0}) \subseteq p^{-1}(X_{\dot{w}_r}^0(b)_{\mathcal{L}_0})$ is locally closed. But this follows from the explicit coordinates on $X_{\dot{w}_r}^m(b)_{\mathcal{L}_0}$ given in the proof of Theorem 6.17. $\diamond$

Lemma 6.4 *Let w_r denote the image of $\dot{w}_r$ in $\tilde{W}$. Then w_r and $\dot{I}_m$ satisfy the assumptions of Theorem 4.9 (with respect to the Iwahori subgroup I^0).*

Proof We use the same notation as in Theorem 4.9. We have to show that the subsets $p(A_{w_r})$, $p(B_{w_r})$, $p(C_{w_r})$ of $\hat{\Phi}$ are disjoint and that the same holds for $p(A_{w_r^{-1}})$, $p(B_{w_r^{-1}})$, $p(C_{w_r^{-1}})$. Write $\hat{\Phi} = \{\alpha_{i,j}: 1 \leq i \neq j \leq n\} \cup \{0\}$ where $\alpha_{i,j}$ corresponds to the i, j -th entry of a matrix. Let w be the image of w_r in the finite Weyl group W . Then w acts on $\hat{\Phi}$ by $w \cdot \alpha_{i,j} = \alpha_{i+1,j+1}$, $w \cdot 0 = 0$, where we consider i, j as elements of $\mathbb{Z}/n\mathbb{Z}$. Then $\hat{\Phi}_{\text{aff}}(I/I^m)$ is equal to

$$\{(\alpha_{i,j}, \lambda): \alpha_{i,j} \in \Phi \text{ and } 0 \leq \lambda \leq m-1 \text{ (if } i > j) \text{ resp. } 1 \leq \lambda \leq m \text{ (if } i < j)\} \cup \{(0, \lambda): 0 \leq \lambda \leq m\}.$$

Now one computes that $w_r \cdot (0, \lambda) = (0, \lambda)$ and $w_r \cdot (\alpha_{i,j}, \lambda) = (\alpha_{i+1,j+1}, \lambda)$ if either $(n > i > j)$ or $(n > j > i)$. Further, $w_r \cdot (\alpha_{n,j}, \lambda) = (\alpha_{1,j+1}, \lambda + nr + \kappa)$ and $w_r \cdot (\alpha_{i,n}, \lambda) = (\alpha_{i+1,1}, \lambda - nr - \kappa)$. Thus we deduce that $p(B_{w_r}) = \{\alpha_{i,j} \in \Phi: n > i > j \text{ or } n > j > i\} \cup \{0\}$, $p(A_{w_r}) = \{\alpha_{i,n} \in \Phi: 1 \leq i \leq n-1\}$ and $p(C_{w_r}) = \{\alpha_{n,j} \in \Phi: 1 \leq j \leq n-1\}$. Similarly, we compute that $p(B_{w_r^{-1}}) = \{\alpha_{i,j} \in \Phi: i > j > 1\} \cup \{0\}$, $p(A_{w_r^{-1}}) = \{\alpha_{1,j} \in \Phi: 1 \leq j \leq n-1\}$ and $p(C_{w_r^{-1}}) = \{\alpha_{i,1} \in \Phi: 1 \leq i \leq n-1\}$. $\square$

Recall from Sect. 3 that $G = J_b(K)$ acts on $X_w^{DL}(b)$ and $\dot{X}_{\dot{w}_0}^{DL}(b)$ by left multiplication. By Theorem 6.5 below, the maps $\dot{X}_{\dot{w}_0}^{DL}(b) \rightarrow X_w^{DL}(b)$ and $\dot{X}_{\dot{w}_r}^m(b) \rightarrow X_{\dot{w}_r}^m(b)$ are both surjective. Hence the former is a $T_w(K)$ -torsor via the right-multiplication action of $T_w(K)$ on $\dot{X}_{\dot{w}_0}^{DL}(b)$ (by Lemma 3.2(iii)) and the latter is a $(I^m/\dot{I}^m)_{\dot{w}_r} \cong T_w(\mathcal{O}_K/\varpi^{m+1})$ -torsor via the right-multiplication action of $I^m/\dot{I}^m$ on $\dot{X}_{\dot{w}_r}^m(b)$.

Theorem 6.5 (i) *There is a commutative diagram of sets*

$$\begin{array}{ccc} V_b^{\text{adm}, \text{rat}} & \xrightarrow{\sim} & X_{\dot{w}_0}^{DL}(b) \\ \downarrow & & \downarrow T_w(K) \\ V_b^{\text{adm}} / \check{K}^\times & \xrightarrow{\sim} & X_w^{DL}(b) \end{array}$$

in which horizontal arrows are $G \times T_w(K)$ -equivariant isomorphisms. Moreover, the vertical arrows in the diagram are surjective.

(ii) *Assume that $r \geq m \geq 0$. There is a commutative diagram of sets*

$$\begin{array}{ccc} V_b^{\text{adm}, \text{rat}} / \sim_{b,m,r} & \xrightarrow{\sim} & \dot{X}_{\dot{w}_r}^m(b)(\overline{\mathbb{F}}_q) \\ \downarrow & & \downarrow T_w(\mathcal{O}_K / \varpi^{m+1} \mathcal{O}_K) \\ V_b^{\text{adm}} / \sim_{b,m,r} & \xrightarrow{\sim} & X_{\dot{w}_r}^m(b)(\overline{\mathbb{F}}_q) \end{array}$$

in which horizontal arrows are $G \times T_w(\mathcal{O}_K / \varpi^{m+1})$ -equivariant isomorphisms. Moreover, the vertical arrows in the diagram are surjective.

Before proving the theorem, we need some preparations. Observe that by Lemmas 3.2 and 4.4 in the proof of Theorem 6.5, we may replace b by an σ -conjugate element of $\check{G}$.

Lemma 6.6 *Let $r > 0$. Let $x, y \in V_b^{\text{adm}}$. Then*

$$x \sim_{b,m,r} y \Leftrightarrow g_{b,r}(x)I^m = g_{b,r}(y)I^m, \quad (6.2)$$

$$x \dot{\sim}_{b,m,r} y \Leftrightarrow g_{b,r}(x)\dot{I}^m = g_{b,r}(y)\dot{I}^m. \quad (6.3)$$

Proof Indeed, $g_{b,r}(y) \in g_{b,r}(x)I^m$ is equivalent to

$$\begin{aligned} y &\in x\mathcal{O}^\times + \varpi^{m+r}b\sigma(x)\mathcal{O} + \dots + \varpi^{m+r(n-1)}(b\sigma)^{n-1}(x)\mathcal{O} \\ \varpi^r(b\sigma)(y) &\in \varpi^{m+1}x\mathcal{O} + \varpi^rb\sigma(x)\mathcal{O}^\times + \varpi^{m+2r}(b\sigma)^2(x)\mathcal{O} \dots + \varpi^{m+(n-1)r}(b\sigma)^{n-1}(x)\mathcal{O} \\ &\vdots \\ \varpi^{r(n-1)}(b\sigma)^{n-1}(y) &\in \varpi^{m+1}x\mathcal{O} + \dots + \varpi^{m+1+r(n-2)}(b\sigma)^{n-2}(x)\mathcal{O} + \varpi^{r(n-1)}(b\sigma)^{n-1}(x)\mathcal{O}^\times. \end{aligned}$$

By definition, the first equation is equivalent to $x \sim_{b,m,r} y$. But once the first equation holds, then the $(i+1)$ th equation must also hold by applying $\varpi^{ri}(b\sigma)^i$ to the first equation and using Lemma 6.1. Hence (6.2) follows, and a similar proof gives (6.3). $\square$

Lemma 6.7 *Let $r \geq 0$ and $x \in V_b^{\text{adm}}$. Then*

$$b\sigma(g_{b,r}(x)) = g_{b,r}(x)\dot{w}_rA,$$

where $A \in \mathrm{GL}_n(\check{K})$ is a matrix, which can differ from the identity matrix only in the last column. Moreover, the lower right entry of A lies in $\mathcal{O}^\times$, and if $r > m \geq 0$, then $A \in I^m$.

Proof By definition, we have

$$\begin{aligned} b\sigma(g_{b,r}(x)) &= (b\sigma(x) \varpi^r(b\sigma)^2(x) \cdots \varpi^{r(n-2)}(b\sigma)^{n-1}(x) \varpi^{r(n-1)}(b\sigma)^n(x)), \\ g_{b,r}(x)\dot{w}_r &= (b\sigma(x) \varpi^r(b\sigma)^2(x) \cdots \varpi^{r(n-2)}(b\sigma)^{n-1}(x) \varpi^{r(n-1)+\kappa_G(b)}x), \end{aligned}$$

As the first $n-1$ columns of these matrices coincide, it follows that A can at most differ from the identity matrix in the last column. By Lemma 6.1, we may write

$$\begin{aligned} (b\sigma)^n(x) &= \sum_{i=0}^{n-1} \alpha_i \cdot (b\sigma)^i(x) \\ &= \frac{\alpha_0}{\varpi^{r(n-1)+\kappa_G(b)}} \cdot \varpi^{r(n-1)+\kappa_G(b)}x + \sum_{i=1}^{n-1} \frac{\alpha_i}{\varpi^{r(i-1)}} \cdot \varpi^{r(i-1)}(b\sigma)^i(x), \end{aligned}$$

where $\alpha_0, \dots, \alpha_{n-1} \in \mathcal{O}$ and $\mathrm{ord}(\alpha_0) = \kappa$. By construction, the last column of A is

$$\left(\varpi^{r(n-1)}\alpha_1, \varpi^{r(n-2)}\alpha_2, \varpi^{r(n-3)}\alpha_3, \dots, \varpi^r\alpha_{n-1}, \frac{\alpha_0}{\varpi^{\kappa_G(b)}} \right)^\top.$$

We then see that the lower right entry of A is $\frac{\alpha_0}{\varpi^\kappa} \in \mathcal{O}^\times$ and that if $r \geq m+1$, then all the entries above $\frac{\alpha_0}{\varpi^\kappa}$ lie in $\varpi^{m+1}\mathcal{O}$ and $A \in I^m$. $\square$

Proof of Theorem 6.5 (i) As in [13, §1], the sets $X_w^{DL}(b)$ do not depend on the choice of the Borel subgroup, so we may choose $B \subseteq \mathrm{GL}_n$ to be the Borel subgroup of the upper triangular matrices and U its unipotent radical. Lemma 6.7 for $r = 0$ implies the existence of the map

$$V_b^{\mathrm{adm}} \rightarrow X_w^{DL}(b), \quad x \mapsto g_b(x)\check{B}.$$

We claim this map is surjective. Let $g\check{B} \in X_w^{DL}(b)$, i.e., $g^{-1}b\sigma(g) \in \check{B}\dot{w}_0\check{B}$. Replacing g by another representative in $g\check{B}$ if necessary, we may assume that $b\sigma(g) \in g\dot{w}_0\check{B}$. Moreover, this assumption does not change, whenever we replace g by another representative $g' = gc$ with $c \in \check{B} \cap \dot{w}_0\check{B}$ (here $\dot{w}_0\check{B} = \dot{w}_0\check{B}\dot{w}_0^{-1}$). A direct computation shows that replacing g by gc for an appropriate $c \in B \cap \dot{w}_0\check{B}$, we find a representative g of $g\check{B}$ with columns $g_1, g_2, \dots, g_n$ satisfying $g_{i+1} = b\sigma(g_i)$ for $i = 1, \dots, n-1$. This means precisely $g = g_b(x)$. All this shows the surjectivity claim. For $x, y \in V_b^{\mathrm{adm}}$, one has $g_b(x)\check{B} = g_b(y)\check{B}$ if and only if x, y differ by a constant in $\check{K}^\times$. This shows the lower horizontal isomorphism in part (i) of the theorem.

We now construct the upper isomorphism. We may write an element of $\check{g}\check{U} \in \check{G}/\check{U}$ lying over $g_b(x)\check{B} \in X_w^{DL}(b)$ as $\check{g}\check{U} = g_b(x)t\check{U}$ for some $t \in \check{T}$. Using Lemma 6.7

(and the notation from there) we see that

$$\dot{g}^{-1}b\sigma(\dot{g}) = t^{-1}g_b(x)^{-1}b\sigma(g_b(x))\sigma(t) = t^{-1}\dot{w}_0A\sigma(t)$$

Now write $A = \lambda A_0$ with $A_0 \in \check{U}$ and $\lambda = \text{diag}(1, \dots, 1, \lambda_0)$ a diagonal matrix with $\lambda_0 \in \mathcal{O}^\times$. We then see that $\dot{g}^{-1}b\sigma(\dot{g}) = t^{-1}\dot{w}_0\lambda\sigma(t)^{\cdot\sigma(t)}A_0$. Hence $g_b(x)t \in X_{\dot{w}_0}^{DL}(b)$ if and only if $\dot{w}_0^{-1}t^{-1}\dot{w}_0\sigma(t) = \lambda^{-1}$. Thus, if we write $t_0, t_1, \dots, t_{n-1} \in \check{K}^\times$ for the diagonal entries of t , then we have $t_{i+1} = \sigma(t_i)$ for $0 \leq i \leq n-2$. We may assume this. In particular, it implies that $g_b(x)t = g_b(x t_0)$. In other words, replacing x by $x t_0$, we may assume that $\dot{g} = g_b(x)$. It remains to determine all $x \in V_b^{\text{adm}}$, for which $g_b(x)\check{U} \in X_{\dot{w}_0}^{DL}(b)$, i.e., $g_b(x)^{-1}b\sigma(g_b(x)) \in \check{U}\dot{w}_0\check{U}$. Comparing the determinants on both sides we deduce that $\sigma(\det g_b(x)) = \det(\dot{w}_0)\det(b)^{-1}\det g_b(x)$ (i.e. that $x \in V_b^{\text{adm, rat}, \dot{w}_0}$) as a necessary condition. Assume this holds. With notations as in Lemma 6.7, we deduce $\det(A) = 1$. Moreover, since A can differ from the identity matrix in at most the last column by Lemma 6.7, $\det(A) = 1$ is equivalent to $A \in \check{U}$. All this, together with the fact that $V_b^{\text{adm, rat}, \dot{w}_0} \cong V_b^{\text{adm, rat}}$ (see (6.1)) shows the upper isomorphism in part (i). The commutativity of the diagram and $J_b(K)$ -equivariance of the involved maps are clear from the construction. The surjectivity claim for the vertical maps is shown in exactly the same way as the analogous claim in part (ii) below.

(ii) Lemma 6.7 for $r > m \geq 0$ implies the existence of the map

$$V_b^{\text{adm}} \rightarrow X_{\dot{w}_r}^m(b), \quad x \mapsto g_{b,r}(x)I^m.$$

We claim it is surjective. Let $gI^m \in X_{\dot{w}_r}^m(b)$, i.e., $g^{-1}b\sigma(g) \in I^m\dot{w}_rI^m$. Replacing g by another representative of gI^m if necessary, we may assume that $b\sigma(g) \in g\dot{w}_rI^m$. Moreover, this assumption does not change, whenever we replace g by another representative $g' = gj$ with $j \in I^m \cap \dot{w}_rI^m$. In the rest of the proof, we call such transformations *allowed*. We compute

$$I^m \cap \dot{w}_rI^m = \begin{pmatrix} \mathcal{O}^\times & \mathfrak{p}^{rn+m} & \dots & \dots & \mathfrak{p}^{rn+m} \\ \mathfrak{p}^m & \mathcal{O}^\times & \mathfrak{p}^{m+1} & \dots & \mathfrak{p}^{m+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathfrak{p}^m & \dots & \mathfrak{p}^m & \mathcal{O}^\times & \mathfrak{p}^{m+1} \\ \mathfrak{p}^m & \dots & \dots & \mathfrak{p}^m & \mathcal{O}^\times \end{pmatrix}$$

(on the main diagonal entries can lie in $\mathcal{O}^\times$, under the main diagonal in $\mathfrak{p}^m$, in the first row, beginning from the second entry, in $\mathfrak{p}^{rn+m}$, and above the main diagonal, except

for the first row, in $\mathfrak{p}^{m+1}$). Let $g_1, \dots, g_n$ denote the columns of g , seen as elements of V . Then $g\dot{w}_r \in b\sigma(g)I^m$ is equivalent to the following n equations:

$$\begin{aligned} g_2 &\in \varpi^r b\sigma(g_1)\mathcal{O}^\times + \varpi^{r+m} b\sigma(g_2)\mathcal{O} + \dots + \varpi^{r+m} b\sigma(g_n)\mathcal{O} \\ g_3 &\in \varpi^{r+m+1} b\sigma(g_1)\mathcal{O} + \varpi^r b\sigma(g_2)\mathcal{O}^\times + \varpi^{r+m} b\sigma(g_3)\mathcal{O} + \dots + \varpi^{r+m} b\sigma(g_n)\mathcal{O} \\ &\vdots \\ g_n &\in \varpi^{r+m+1} b\sigma(g_1)\mathcal{O} + \dots + \varpi^{r+m+1} b\sigma(g_{n-2})\mathcal{O} + \varpi^r b\sigma(g_{n-1})\mathcal{O}^\times + \varpi^{r+m} b\sigma(g_n)\mathcal{O} \\ \varpi^{r+n+m+\kappa} g_1 &\in \varpi^{r+2m+1} b\sigma(g_1)\mathcal{O} + \dots + \varpi^{r+2m+1} b\sigma(g_{n-1})\mathcal{O} + \varpi^{r+m} b\sigma(g_n)\mathcal{O}^\times. \end{aligned}$$

A linear algebra exercise shows that after some allowed transformations these equations can be rewritten as

$$\begin{aligned} g_2 &\in \varpi^r b\sigma(g_1)\mathcal{O}^\times \\ g_3 &\in \varpi^r b\sigma(g_2)\mathcal{O}^\times \\ &\vdots \\ g_n &\in \varpi^r b\sigma(g_{n-1})\mathcal{O}^\times \\ \varpi^{r(n-1)+\kappa} g_1 &\in \varpi^{m+1} b\sigma(g_1)\mathcal{O} + \dots + \varpi^{m+1} b\sigma(g_{n-1})\mathcal{O} + b\sigma(g_n)\mathcal{O}^\times. \end{aligned}$$

This shows that $g = g_{b,r}(g_1)$, and hence the claimed surjectivity. Lemma 6.6 shows that the lower map in part (ii) is an isomorphism. Exactly as in the proof of (i), one shows that the claim of (ii) is true if one replaces the upper left entry by $\left\{x \in V_b^{\text{adm}} : \begin{array}{l} \det(g_{b,r}(x)) \bmod \varpi^{m+1} \\ \text{is fixed by } \sigma \end{array} \right\}$. As $x \sim_{b,m,r} xu$ for any $u \in 1 + \mathfrak{p}^{m+1}$, the original claim of (ii) follows from this modified claim along with the surjectivity of the map $1 + \mathfrak{p}^{m+1} \rightarrow 1 + \mathfrak{p}^{m+1}$, $u \mapsto \prod_{i=0}^{n-1} \sigma^i(u)$, and the fact that $\det g_b(x) \in K^\times \Leftrightarrow \det g_{b,r}(x) \in K^\times$.

It remains to show that the vertical arrows in the diagram in (ii) are surjective. It suffices to handle the left arrow. Let $x \in V_b^{\text{adm}}$. By definition of $\sim_{b,m,r}$, for any $\lambda \in \mathcal{O}^\times$ we have $\lambda x \sim_{b,m,r} x$. Now we have $\det g_b(\lambda x) = \varpi^a \cdot \prod_{i=0}^{n-1} \sigma^i(\lambda) \cdot \det g_b(x)$ for an appropriate $a \in \mathbb{Z}$. Now the map $\lambda \mapsto \prod_{i=0}^{n-1} \sigma^i(\lambda) : \mathcal{O}^\times \rightarrow \mathcal{O}^\times$ is surjective, hence by rescaling x with an appropriate $\lambda \in \mathcal{O}^\times$ (thus not changing its class in $V_b^{\text{adm}} / \sim_{b,m,r}$) we can arrange that $\det g_b(x) \in K^\times$, i.e., that $x \in V_b^{\text{adm, rat}}$. $\square$

The natural projection maps $X_{\dot{w}_r}^{m+1}(b) \rightarrow X_{\dot{w}_r}^m(b)$ and $\dot{X}_{\dot{w}_r}^{m+1}(b) \rightarrow \dot{X}_{\dot{w}_r}^m(b)$ are obviously morphisms of schemes. However, Theorem 6.5 implies that there are G - and $G \times T_w(\mathcal{O}_K/\varpi^{m+1})$ -equivariant maps of sets (on $\overline{\mathbb{F}}_q$ -points)

$$X_{\dot{w}_{r+1}}^m(b) \rightarrow X_{\dot{w}_r}^m(b), \quad \text{and} \quad \dot{X}_{\dot{w}_{r+1}}^m(b) \rightarrow \dot{X}_{\dot{w}_r}^m(b) \quad (6.4)$$

induced by $g_{b,r+1}(x) \mapsto g_{b,r}(x)$. In Sect. 6.4, we explicate the scheme structure on $X_{\dot{w}_r}^m(b)$, $\dot{X}_{\dot{w}_r}^m(b)$ and prove that these maps of sets are actually morphisms of schemes (Theorem 6.17). Taking Theorem 6.17 for granted at the moment, we have a notion of an *affine Deligne–Lusztig variety at infinite level*.

Definition 6.8 Define the (infinite-dimensional) $\overline{\mathbb{F}}_q$ -scheme

$$X_w^\infty(b) := \varprojlim_{r,m: r>m} X_{\dot{w}_r}^m(b) \quad \text{and} \quad \dot{X}_w^\infty(b) := \varprojlim_{r,m: r>m} \dot{X}_{\dot{w}_r}^m(b).$$

Both have actions by G and the natural G -equivariant map $\dot{X}_w^\infty(b) \rightarrow X_w^\infty(b)$ is a $T_w(\mathcal{O}_K)$ -torsor.

Passing to the infinite level in Theorem 6.5 gives the following result.

Theorem 6.9 *There is a commutative diagram of sets with G -equivariant maps:*

$$\begin{array}{ccccc} \dot{X}_w^{DL}(b) & \xleftarrow{\sim} & V_b^{\text{adm, rat}} & \xrightarrow{\sim} & \dot{X}_w^\infty(b) \\ \downarrow T_w(K) & & \downarrow & & \downarrow T_w(\mathcal{O}_K) \\ X_w^{DL}(b) & \xleftarrow{\sim} & V_b^{\text{adm}}/\check{K}^\times \xleftarrow{\mathbb{Z}} V_b^{\text{adm}}/\mathcal{O}^\times & \xrightarrow{\sim} & X_w^\infty(b) \end{array}$$

The upper horizontal maps are $T_w(\mathcal{O}_K)$ -equivariant. This extends the natural $T_w(\mathcal{O}_K)$ -action on $\dot{X}_w^\infty(b)$ to a $T_w(K)$ -action.

Using the set-theoretic isomorphism in Theorem 6.9, we will see in Sect. 6.4 that by endowing V_b^{adm} with the natural scheme structure over $\overline{\mathbb{F}}_q$ coming from the lattice $\mathcal{L}_0$, we can view the semi-infinite Deligne–Lusztig sets $X_w^{DL}(b)$, $\dot{X}_w^{DL}(b)$ as (infinite-dimensional) $\overline{\mathbb{F}}_q$ -schemes. Moreover, every isomorphism in Theorem 6.9 is an isomorphism of $\overline{\mathbb{F}}_q$ -schemes (Corollary 6.19).

6.3 Connected components

To “minimize” powers of the uniformizer, we define

$$g_b^{\text{red}}(v) := \left(v \mid \frac{1}{\varpi^{\lfloor k_0/n_0 \rfloor}} b\sigma(v) \mid \frac{1}{\varpi^{\lfloor 2k_0/n_0 \rfloor}} (b\sigma)^2(v) \mid \cdots \mid \frac{1}{\varpi^{\lfloor (n-1)k_0/n_0 \rfloor}} (b\sigma)^{n-1}(v) \right) \quad (6.5)$$

to be the $n \times n$ matrix whose i th column is $\frac{1}{\varpi^{\lfloor (i-1)k_0/n_0 \rfloor}} \cdot (b\sigma)^{i-1}(v)$ for $v \in V$. Observe that

$$g_b(v) = g_b^{\text{red}}(v) \cdot D_{k,n},$$

where $D_{k,n}$ is the diagonal matrix whose (i, i) th entry is $\varpi^{\lfloor (i-1)k_0/n_0 \rfloor}$.

Definition 6.10 For any basic b (with $\kappa_{\text{GL}_n}(b) = \kappa$) which is integrally σ -conjugate to the special representative as in Sect. 5.2.2, we define

$$\begin{aligned} \mathcal{L}_{0,b}^{\text{adm}} &:= \left\{ v \in \mathcal{L}_0 : \det g_b^{\text{red}}(v) \in \mathcal{O}^\times \right\}, \\ \mathcal{L}_{0,b}^{\text{adm, rat}} &:= \left\{ v \in \mathcal{L}_0 : \det g_b^{\text{red}}(v) \in \mathcal{O}_K^\times \right\}, \\ \mathcal{L}_{0,b}^{\text{adm, rat}, \dot{w}_0} &:= \left\{ v \in \mathcal{L}_0 : \sigma(\det g_b^{\text{red}}(v)) = \det(\dot{w}_0) \det(b)^{-1} \det g_b^{\text{red}}(v) \in \mathcal{O}^\times \right\}. \end{aligned}$$

As in (6.1), note that $\mathcal{L}_{0,b}^{\text{adm, rat}, w_0} \cong \mathcal{L}_{0,b}^{\text{adm, rat}}$ by rescaling appropriately by an element of $\mathcal{O}^\times$.

Let now b be the special representative with $\kappa_G(b) = \kappa$. As $G_{\mathcal{O}} \subseteq \text{GL}_n(\mathcal{O}) = \text{Stab}(\mathcal{L}_0)$ inside $\text{GL}_n(\check{K})$, we see that $\mathcal{L}_{0,b}^{\text{adm, rat}}$, $\mathcal{L}_{0,b}^{\text{adm}}$ are stable under $G_{\mathcal{O}} \times T_w(\mathcal{O}_K)$. We have

$$\mathcal{L}_{b,0}^{\text{adm}} = \left\{ v = \sum_{i=1}^n a_i \mathbf{e}_i \in \mathcal{L}_0 : \begin{array}{l} a_i \in \mathcal{O} \text{ for } 1 \leq i \leq n; \{a_i \mathbf{e}_i \pmod{\varpi}\}_{i \equiv 1 \pmod{n_0}} \\ \text{generate the } \overline{\mathbb{F}}_q\text{-vector space } \overline{V} \end{array} \right\}, \quad (6.6)$$

where $\overline{V}$ is as in Sect. 5.6 (compare [32, Lemma 4.8]). For $x \in \mathcal{L}_{b,0}^{\text{adm}}$ with reduction-modulo- ϖ $\bar{x}$, define $\overline{g_b}(\bar{x}) \in \text{GL}_{n'}(\overline{\mathbb{F}}_q)$ to be the matrix from reducing every entry of $g_b^{\text{red}}(x)$ modulo ϖ and deleting the i th row and j th column for all $(i, j) \not\equiv (1, 1)$ modulo n_0 .

Lemma 6.11 *We have a disjoint decomposition*

$$V_b^{\text{adm}} = \coprod_{g \in G/G_{\mathcal{O}}} g(\mathcal{L}_{0,b}^{\text{adm}})$$

Proof Let c, d be as in Lemma 5.8. On V we have the operators considered in [32, 4.1]: $b\sigma$, $\varpi(b\sigma)^{-1}$, $\varpi^d(b\sigma)^c$, $\varpi^{-\kappa_0}(b\sigma)^{n_0} = \sigma^{n_0}$ (in [32], these operators are denoted F, V, π_1, σ_1 respectively). For $v \in V_b^{\text{adm}}$ we may consider the smallest $\mathcal{O}$ -submodule $P(v)$ of V , containing v and stable under these four operators (this is a indeed lattice: for $\kappa > 0$ see [32, p. 354, paragraph before Lemma 4.10]; for $\kappa = 0$, Lemma 6.1 shows that this is the lattice generated by $\{\sigma^i(v)\}_{i=0}^{n-1}$). Further we have

$$\mathcal{L}_{0,b}^{\text{adm}} = \{v \in \mathcal{L}_0 : P(v) = \mathcal{L}_0\}.$$

This follows from the explicit description of both sides: (6.6) above and [32, Lemma 4.8 and beginning of §4.4]. Next, the set of lattices in V stable under the four operators above is in bijection with $G/G_{\mathcal{O}}$ via $gG_{\mathcal{O}} \mapsto g\mathcal{L}_0$. Indeed, for $\kappa > 0$ this follows from Lemma 4.8 and 4.10 of [32] (note that there Viehmann only considers bi-infinitesimal p -divisible groups, so that the slope(s) of V must be strictly between 0 and 1); and for $\kappa = 0$, this is clear from the description of the affine Grassmannian of GL_n in terms of lattices.

Any $g \in G$ commutes with the four operators above, hence $P(gv) = gP(v)$ and hence $g(\mathcal{L}_{0,b}^{\text{adm}}) = \{v \in \mathcal{M} : P(v) = \mathcal{M}\}$ if $\mathcal{M} = g(\mathcal{L}_0)$. Finally, $\bigcup_{\mathcal{M}} \{v \in \mathcal{M} : P(v) = \mathcal{M}\}$ (where $\mathcal{M}$ runs through all lattices stable under the four operators) is disjoint and equal to V_b^{adm} . $\square$

For each $h \in G/G_{\mathcal{O}}$, let $X_{\tilde{w}_r}^m(b)_{h(\mathcal{L}_0)}$ be the subset of $X_{\tilde{w}_r}^m(b)$ consisting of all points which under the isomorphism $X_{\tilde{w}_r}^m(b)(\overline{\mathbb{F}}_q) \cong V_b^{\text{adm}} / \sim_{b,m,r}$ of Theorem 6.5 correspond to $h(\mathcal{L}_{0,b}^{\text{adm}})$.

The Iwahori level variety $X_{\check{w}_r}^0(b)$ is known to be locally closed in the ind-scheme $\mathrm{GL}_n(\check{K})/I$ and locally of finite type over $\overline{\mathbb{F}}_q$. The following result (as well as the preceding lemma) is based on ideas from [32], which were explained to the authors by E. Viehmann.

Proposition 6.12 *Let $r > 0$. For any $h \in G$, $X_{\check{w}_r}^0(b)_{h(\mathcal{L}_0)}$ is a closed and open subset of $X_{\check{w}_r}^0(b)$. In other words, $X_{\check{w}_r}^0(b) = \coprod_{h \in G/G_O} X_{\check{w}_r}^0(b)_{h(\mathcal{L}_0)}$ is a scheme-theoretic disjoint union decomposition.*

Proof The proof follows along the same lines as [32, Lemma 4.16]. First we show that $X_{\check{w}_r}^0(b)_{\mathcal{L}_0} \subseteq X_{\check{w}_r}^0(b)$ is closed. To this end, recall that points of $\mathrm{GL}_n(\check{K})/I$ can be interpreted as complete chains of $\mathcal{O}$ -lattices in $\check{K}^n$. Let $\mathcal{L}_\bullet = \{\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \cdots \supseteq \mathcal{L}_{n-1} \supseteq \varpi \mathcal{L}_0\}$ be the standard chain (stabilized by I). Then gI corresponds to the lattice chain $g\mathcal{L}_\bullet = \{g\mathcal{L}_i\}_{i=0}^{n-1}$. We claim that there is an integer $C > 0$ only depending on n, κ, r such that

$$X_{\check{w}_r}^0(b)_{\mathcal{L}_0} = \{\mathcal{M}_\bullet = (\mathcal{M}_i)_{i=0}^{n-1} \in X_{\check{w}_r}^0(b) : \mathcal{M}_0 \subseteq \mathcal{L}_0 \text{ and } \mathrm{vol}(\mathcal{M}_0) = C\}. \quad (6.7)$$

Let $\mathcal{M}_\bullet \in X_{\check{w}_r}^0(b)_{\mathcal{L}_0}$. By construction, there is some $v \in \mathcal{L}_{0,b}^{\mathrm{adm}}$ such that $\mathcal{M}_\bullet = g_{b,r}(v)\mathcal{L}_\bullet$. Then $\mathcal{M}_0 = g_{b,r}(v)(\mathcal{L}_0)$ is the lattice generated by $\{\varpi^{ri}(b\sigma)^i(v)\}_{i=0}^{n-1}$, which is contained in $\mathcal{L}_0$. Moreover,

$$\mathrm{vol}(\mathcal{M}_0) = \mathrm{vol}(g_{b,r}(v)(\mathcal{L}_0)) = \mathrm{ord} \det g_{b,r}(v) = C + \mathrm{ord} \det g_b^{\mathrm{red}}(v) = C,$$

with $C > 0$ some explicit constant depending on n, κ, r . Conversely, let $\mathcal{M}_\bullet \in X_{\check{w}_r}^0(b)$ be such that $\mathcal{M}_0 \subseteq \mathcal{L}_0$ and $\mathrm{vol}(\mathcal{M}_0) = C$ with the same C as above. By Theorem 6.5, there is some $v \in V_b^{\mathrm{adm}}$ with $\mathcal{M}_\bullet = g_{b,r}(v)\mathcal{L}_\bullet$. Then $\mathcal{M}_0 = g_{b,r}(v)\mathcal{L}_0$ is the lattice generated by $\{\varpi^{ri}(b\sigma)^i(v)\}_{i=0}^{n-1}$. In particular, we must have $v \in \mathcal{M}_0 \subseteq \mathcal{L}_0$ and

$$\mathrm{ord} \det(g_b^{\mathrm{red}}(v)) = \mathrm{ord} \det(g_{b,r}(v)) - C = \mathrm{vol}(\mathcal{M}_0) - C = 0.$$

This proves claim (6.7). From this claim it follows that $X_{\check{w}_r}^0(b)_{\mathcal{L}_0}$ is the intersection in $\mathrm{GL}_n(\check{K})/I$ of $X_{\check{w}_r}^0(b)$ with the preimage under

$$\mathrm{GL}_n(\check{K})/I \rightarrow \mathrm{GL}_n(\check{K})/\mathrm{Stab}(\mathcal{L}_0) \cong \{\mathcal{O}\text{-lattices in } \check{K}^n\}, \quad \mathcal{M}_\bullet = \{\mathcal{M}_i\}_{i=0}^{n-1} \mapsto \mathcal{M}_0$$

of the closed (see [32, Remark 4.15]) subset of lattices with fixed volume and contained in $\mathcal{L}_0$. Hence $X_{\check{w}_r}^0(b)_{\mathcal{L}_0}$ is closed in $X_{\check{w}_r}^0(b)$.

Applying the G -action on $X_{\check{w}_r}(b)$ we deduce that also $X_{\check{w}_r}^0(b)_{h(\mathcal{L}_0)}$ is closed in $X_{\check{w}_r}^0(b)$ for each $h \in G$. Now the closed subvarieties $X_{\check{w}_r}^0(b)_{h(\mathcal{L}_0)}$ form a disjoint cover of $X_{\check{w}_r}^0(b)$, as this holds on geometric points by Lemma 6.11 and Theorem 6.5. As $X_{\check{w}_r}^0(b)$ is locally of finite type, this disjoint union is locally finite. Hence $X_{\check{w}_r}^0(b)_{h(\mathcal{L}_0)}$ is also open. $\square$

Remark 6.13 We explain the differences between our setting and that of [32, §4]. Viehmann proved a similar decomposition for an open subset $\mathcal{S}_1$ of some minuscule affine Deligne–Lusztig varieties at the hyperspecial level (in particular, in the bi-infinitesimal case, which in our notation corresponds to $\kappa > 0$). A point of $\mathcal{S}_1$ corresponds to a Dieudonné lattice $\mathcal{M}$ in V , that is a lattice stable by the operators $b\sigma$ and $p(b\sigma)^{-1}$. Such a lattice also possesses a single generator $v \in V_b^{\text{adm}}$, but the difference between $\text{vol}(\mathcal{M})$ and $\text{vol}(P(\mathcal{M}))$ (where $P(\mathcal{M})$ is the smallest lattice containing $\mathcal{M}$ and stable under the four operators as in the proof of Lemma 6.11), is quite inexplicit, and it is a complicated task [32, Theorem 4.11(ii)] to show that this difference is constant. In our situation the lattice (chain) corresponding to a point of $X_{\check{w}_r}^0(b)$ is completely explicit, and the difference of volumes is immediate to compute. Note also that we work with the cocharacter $(-r, \dots, -r, (n-1)r)$ for $r \geq 1$, and since this is never minuscule, there is no direct comparison between our setting and the varieties from [32]. $\diamond$

Corollary 6.14 *Let $b \in \check{G}$ be integrally σ -conjugate to the special representative attached to κ . Then the conclusion of Proposition 6.12 holds for this b .*

Proof If $h \in \text{GL}_n(\mathcal{O})$ is such that $b = h^{-1}b_{sp}\sigma(h)$, where b_{sp} is the special representative, then $g \mapsto h^{-1}g$ defines an isomorphism $X_{\check{w}_r}^m(b_{sp}) \xrightarrow{\sim} X_{\check{w}_r}^m(b)$. Further, $g_b^{\text{red}}(v) = h^{-1}g_{b_{sp}}^{\text{red}}(hv)$ and the corollary follows from the commutativity of the obvious diagram. $\square$

By Lemma 5.6, Corollary 6.14 applies to the Coxeter-type representatives from Sect. 5.2.1.

6.4 The structure of $X_{\check{w}}^m(b)$

Let b be the special representative with $\kappa_{\text{GL}_n}(b) = \kappa$. Let $X_{\check{w}_r}^m(b)_{\mathcal{L}_0}$ be as defined before Proposition 6.12. The following auxiliary elements of $\text{GL}_n(\check{K})$ will be used in this subsection only. For $r \geq 1$, put $\mu_r = (1, r, 2r, \dots, (n-1)r) \in X_*(T_{\text{diag}})$. For an integer a , let $0 \leq [a]_{n_0} < n_0$ denote its residue modulo n_0 . Let $v_1 \in \text{GL}_{n_0}(\check{K})$ be the permutation matrix whose only non-zero entries are concentrated in the entries $(1 + [(i-1)k_0]_{n_0}, i)$ and are all equal to 1. Let $v \in \text{GL}_n(\check{K})$ denote the block-diagonal matrix, whose diagonal $n_0 \times n_0$ -blocks are each equal to v_1 . We begin with the following key proposition.

Proposition 6.15 *For $r \geq 1$, the Iwahori level variety $X_{\check{w}_r}^0(b)_{\mathcal{L}_0}$ is contained in the Schubert cell $IvD_{\kappa,n}\mu_r I/I \subseteq \text{GL}_n(\check{K})/I$. In particular, $X_{\check{w}_r}^m(b)_{\mathcal{L}_0} \subseteq IvD_{\kappa,n}\mu_r I/I^m \subseteq \text{GL}_n(\check{K})/I^m$.*

Proof We have to show that for $x \in \mathcal{L}_0^{\text{adm}}$ one has $Ig_{b,r}(x)I = IvD_{\kappa,n}\mu_r I$, i.e., that by successively multiplying by elements from I on the left and right side we can bring $g_{b,r}(x) = g_b^{\text{red}}(x)D_{\kappa,n}\mu_r$ to the form $vD_{\kappa,n}\mu_r$. For $1 \leq i \leq n'$, we call a matrix in $\text{GL}_n(\check{K})$ *i-nice*, if the following two conditions hold:

- (i) each of its n'^2 blocks of size $n_0 \times n_0$ has the following shape: in its ℓ th column ($1 \leq \ell \leq n_0$), the entries above the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry lie in $\mathfrak{p}$ and the other entries lie in $\mathcal{O}$;
- (ii) for $1 \leq \ell \leq n_0$, the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry of the (i, i) th $(n_0 \times n_0)$ -block lies in $\mathcal{O}^\times$.

The inductive algorithm to prove the lemma is as follows: put $A_1 := g_b^{\text{red}}(x)$ and let $1 \leq i \leq n'$. Assume that by modifying $g_b^{\text{red}}(x)D_{\kappa,n}\mu_r$ (by multiplication from left and right with I) we have constructed the i -nice matrix A_i , such that $Ig_b^{\text{red}}(x)D_{\kappa,n}\mu_r I = IA_i D_{\kappa,n}\mu_r I$ and such that the first $i - 1$ rows and $i - 1$ columns of $n_0 \times n_0$ -blocks of $A_i D_{\kappa,n}\mu_r$ coincide with $vD_{\kappa,n}\mu_r$ up to $\mathcal{O}^\times$ -multiplies of the non-zero entries. Now we do the following steps:

- (1) *Annihilate the entries of the (i, i) th $n_0 \times n_0$ -block of A_i lying over $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry (for each $1 \leq \ell \leq n_0$).*

By assumption, the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry lies in $\mathcal{O}^\times$. By multiplying upper triangular unipotent elements from I (with non-diagonal entries in $\mathfrak{p}$) from the left to $A_i D_{\kappa,n}\mu_r$ (i.e., performing elementary row operations on matrices), we obtain a nice matrix A'_i (uniquely determined by A_i) whose entries have the same images in $\mathcal{O}/\mathfrak{p}$ as those of A_i . Moreover, $IA_i D_{\kappa,n}\mu_r I = IA'_i D_{\kappa,n}\mu_r I$.

Put $A'_{i,0} := A'_i$. For $\ell = 1, 2, \dots, n_0$ do successively the following step:

- (2) $_\ell$ *Annihilate the $(n_0(i - 1) + \ell)$ th column and $(n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0})$ th row of $A'_{i,\ell-1}$.*

By assumption, the $(n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}, n_0(i - 1) + \ell)$ th entry of the i -nice matrix $A'_{i,\ell-1}$ lies in $\mathcal{O}^\times$. By multiplying $A'_{i,\ell-1} D_{\kappa,n}\mu_r$ successively from the left by lower triangular matrices from I which have 1's on the main diagonal and only further non-zero entries in the $n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}$ th column, we can kill all entries of the $n_0(i - 1) + \ell$ th column of $A'_{i,\ell-1}$ except for the $(n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}, n_0(i - 1) + \ell)$ th entry itself, which remains unchanged. After this we can, using the $(n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}, n_0(i - 1) + \ell)$ th entry, easily eliminate all entries $n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}$ th row except for $(n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}, n_0(i - 1) + \ell)$ th entry itself, which remains unchanged (by multiplying $A'_{i,\ell-1} D_{\kappa,n}\mu_r$ from the right with unipotent upper triangular matrices in I). This does not change the rest of the matrix, because $n_0(i - 1) + \ell$ th column contains precisely one non-zero entry.

As an output we obtain the matrix $A_{i+1} := A'_{i,n_0}$ which we claim is $(i + 1)$ -nice. Assume for now that this is true. Proceeding the described algorithm for all $1 \leq i \leq n'$, we obtain the matrix $A_{n'+1}$, which differs from v only by some diagonal matrix with entries in $\mathcal{O}^\times$, so that $IA_{n'+1} D_{\kappa,n}\mu_r I = IvD_{\kappa,n}\mu_r I$ is now clear.

Observe that when looking modulo $\mathfrak{p}$, the step (2) $_\ell$ in the algorithm for a single ℓ affects the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry of the $(i + 1, i + 1)$ th $n_0 \times n_0$ -block, but does not affect the entries $(1 + [(\ell' - 1)k_0]_{n_0}, \ell')$ th ($\forall \ell' \neq \ell$) of the same block. In particular, the steps (2) $_\ell$ can be applied in any order of the ℓ 's, and when the (2) $_{\ell_0}$ is applied first to A'_i (to kill its $(n_0(i - 1) + \ell_0)$ th column and $(n_0(i - 1) + 1 + [(\ell_0 - 1)k_0]_{n_0})$ th row) giving the matrix A''_{i,ℓ_0} , then the $(1 + [(\ell_0 - 1)k_0]_{n_0}, \ell_0)$ th entry of $(i + 1, i + 1)$ th $n_0 \times n_0$ -block of A''_{i,ℓ_0} already coincides modulo $\mathfrak{p}$ with the same entry of A_{i+1} .

We now show that for $1 \leq i \leq n$, the matrix A_i appearing in the algorithm is i -nice. (By induction we may assume that $A_{i'}$ is i' -nice for $1 \leq i' < i$, which is sufficient to run the first $i - 1$ steps of the algorithm to obtain A_i). For $1 \leq j \leq i' \leq n$, $1 \leq \ell \leq n'$, let $\alpha_{i',j,\ell} \in \mathcal{O}/\mathfrak{p}$ denote the residue modulo $\mathfrak{p}$ of the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry of the (j, j) th $n_0 \times n_0$ -block of $A_{i'}$. Note that $\alpha_{i',j,\ell} = \alpha_{i'',j,\ell}$ for all $1 \leq j \leq i' \leq i''$. Indeed, if $j < i'$, this is obvious as the first $i' - 1$ diagonal blocks of $A_{i'}$ and $A_{i''}$ coincide. If $j = i'$ observe that the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entries (for all $1 \leq \ell \leq n_0$) of the (i', i') th $n_0 \times n_0$ -block of $A_{i'}$ can only be affected by step (1) of the algorithm, which does not change the residue modulo $\mathfrak{p}$.

Recall the image $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n'})^T$ of x in $\bar{V}$ and the corresponding matrix $\overline{g_b}(\bar{x}) \in \mathrm{GL}_{n'}(\overline{\mathbb{F}}_q)$ defined in Sect. 6.3. For $1 \leq i \leq n'$, let $m_i \in \overline{\mathbb{F}}_q$ denote the determinant of the upper left $i \times i$ -minor of $\overline{g_b}(\bar{x})$. By Lemma 6.16, $m_i \in \overline{\mathbb{F}}_q^\times$ for all i . We claim that for $1 \leq \ell \leq n_0$,

$$\alpha_{i,j,\ell} = \begin{cases} \sigma^{\ell-1}(m_1) & \text{if } j = 1 \\ \sigma^{\ell-1}\left(\frac{m_j}{m_{j-1}}\right) & \text{if } 2 \leq j \leq i \end{cases} \quad (6.8)$$

By induction we may assume that this holds for all $1 \leq i' < i$, from which (6.8) follows for all $j < i$. It thus remains to compute $\alpha_{i,i,\ell}$. Note that for $1 \leq \ell \leq n_0$, the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ -entry of $A_1 = g_b^{\mathrm{red}}(x)$ is equal to $\sigma^{\ell-1}(x_{1,0}) = \sigma^{\ell-1}(\bar{x}_1)$. This finishes the case $i = 1$. Assume $i \geq 2$ and fix some $1 \leq \ell \leq n_0$. By the observation above, $\alpha_{i,i,\ell}$ is equal to the residue modulo $\mathfrak{p}$ of the $(1 + [(\ell - 1)k_0]_{n_0}, \ell)$ th entry of the (i, i) th $n_0 \times n_0$ -block of the matrix $A''_{i-1,\ell}$, obtained from A'_{i-1} by directly applying step (2) $_{\ell}$.

For $X \in \mathrm{GL}_n(K)$, let $M(X)$ denote the $(n_0(i - 1) + 1) \times (n_0(i - 1) + 1)$ -minor of X obtained by removing from X all columns with numbers $\{j: j > n_0(i - 1) \text{ and } j \neq n_0(i - 1) + \ell\}$ and all rows with numbers $\{s: s > n_0(i - 1) \text{ and } s \neq n_0(i - 1) + 1 + [(\ell - 1)k_0]_{n_0}\}$. We compute:

$$\alpha_{i,i,\ell} \prod_{\lambda=1}^{n_0} \sigma^{\lambda-1}(m_{i-1}) \equiv \det M(A''_{i-1,\ell}) = \det M(A'_{i-1}) = \det M(g_b^{\mathrm{red}}(x)) \pmod{\mathfrak{p}}.$$

The first equality follows from the explicit form of $A''_{i-1,\ell}$ and by the induction hypothesis on the $\alpha_{i,j,\ell}$'s. The remaining equalities are true as every operation in the algorithm does not change the determinant of the matrices. On the other side, a simple calculation shows that

$$\det M(g_b^{\mathrm{red}}(x)) \equiv \frac{\sigma^\ell(m_i)}{\sigma^\ell(m_{i-1})} \prod_{\lambda=1}^{n_0} \sigma^{\lambda-1}(m_{i-1}) \pmod{\mathfrak{p}}.$$

This finishes the proof of (6.8), and thus of the proposition. $\square$

Lemma 6.16 *Let $x \in \mathcal{L}_{0,b}^{\mathrm{adm}}$ and let $\bar{x} \in \bar{V}$ denote its image. For $1 \leq i \leq n'$, let m_i denote the upper left $(i \times i)$ -minor of $\overline{g_b}(\bar{x}) \in \mathrm{GL}_{n'}(\overline{\mathbb{F}}_q)$. Then $m_i \in \overline{\mathbb{F}}_q^\times$ for all i .*

Proof Replacing $\mathbb{F}_{q^{n_0}}$ by $\mathbb{F}_q$ we may assume that $n_0 = 1$, $n' = n$. We have $\overline{g_b}(\bar{x}) = (\bar{x}_i^{q^{j-1}})_{1 \leq i, j \leq n}$ and $\det \overline{g_b}(\bar{x}) \in \overline{\mathbb{F}_q}^\times$. Clearly, $m_1 = \bar{x}_1 \neq 0$. Let $2 \leq i \leq n$. By induction we may assume that $m_{i'} \in \overline{\mathbb{F}_q}^\times$ for all $1 \leq i' < i$. Suppose $m_i = 0$. This means that the i vectors $v_j = (x_j^{q^{k-1}})_{k=1}^i \in \overline{\mathbb{F}_q}^i$ ($1 \leq j \leq i$) are linearly $\overline{\mathbb{F}_q}$ -dependent. Note that the first $i - 1$ of these vectors are $\overline{\mathbb{F}_q}$ -independent, as already the vectors $(x_j^{q^{k-1}})_{k=1}^{i-1} \in \overline{\mathbb{F}_q}^{i-1}$ ($1 \leq j \leq i - 1$) are $\overline{\mathbb{F}_q}$ -independent, which in turn follows from the induction hypothesis $m_{i-1} \neq 0$. This shows that there exist $\lambda_1, \dots, \lambda_{i-1} \in \overline{\mathbb{F}_q}$ with $\sum_{j=1}^{i-1} \lambda_j v_j = v_i$. From this we deduce two systems of linear equations which uniquely determine the λ_j 's: (1) $\sum_{j=1}^{i-1} \lambda_j (x_j^{q^{k-1}})_{k=1}^{i-1} = (x_i^{q^{k-1}})_{k=1}^{i-1}$ as well as (2) $\sum_{j=1}^{i-1} \lambda_j (x_j^{q^{k-1}})_{k=2}^i = (x_i^{q^{k-1}})_{k=2}^i$. Note that (2) is obtained from (1) by raising all coefficients to the q th power. For $1 \leq j \leq i - 1$ let $m_{i-1}^{(j)}$ denote the minor m_{i-1} , in which j th row is replaced by $(x_i^{q^{k-1}})_{k=1}^{i-1}$. Then (1) gives $\lambda_j = m_{i-1}^{-1} m_{i-1}^{(j)}$, whereas (2) gives $\lambda_j = (m_{i-1}^{-1} m_{i-1}^{(j)})^q$ for each $1 \leq j \leq i - 1$. Thus $\lambda_j \in \mathbb{F}_q$. This gives a non-trivial $\mathbb{F}_q$ -relation between the $x_1, \dots, x_i$, and hence also between the first i rows of $\overline{g_b}(\bar{x})$, i.e., $\det \overline{g_b}(\bar{x}) = 0$, contradicting the assumption. $\square$

Let

$$\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} := \mathbb{P}(\overline{V}) \setminus \bigcup_{\substack{H \subseteq \overline{V} \\ \mathbb{F}_{q^{n_0}}\text{-rational hyperplane}}} H \quad (6.9)$$

be $n' - 1$ -dimensional Drinfeld's upper half-space over $\mathbb{F}_{q^{n_0}}$.

Theorem 6.17 *Let b be the special representative with $\kappa_{\mathrm{GL}_n}(b) = \kappa$. Let $r > m \geq 0$. Then we have a decomposition of $\mathbb{F}_q$ -schemes*

$$X_{\dot{w}_r}^m(b) \cong \bigsqcup_{G/G_O} \Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} \times \mathbb{A},$$

where $\mathbb{A}$ is a finite dimensional affine space over $\mathbb{F}_q$ (with dimension depending on r, m). The morphism $\dot{X}_{\dot{w}_r}^m(b) \rightarrow X_{\dot{w}_r}^m(b)$ is a finite étale $T_w(\mathcal{O}_K/\varpi^{m+1})$ -torsor. In particular, all these schemes are smooth.

Proof To prove the first statement of the theorem it suffices, by applying Proposition 6.12, to show that $X_{\dot{w}_r}^m(b)_{\mathcal{L}_0}$ is a locally closed subset of $\mathrm{GL}_n(\check{K})/I^m$ isomorphic to $\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} \times \mathbb{A}$. By Proposition 6.15, $X_{\dot{w}_r}^m(b)_{\mathcal{L}_0} \subseteq IvD_{\kappa,n}\mu_r(\varpi)I/I^m$. So, it suffices to show that as a subset of $IvD_{\kappa,n}\mu_r(\varpi)I/I^m$, $X_{\dot{w}_r}^m(b)_{\mathcal{L}_0}$ is locally closed and with its induced reduced sub-scheme-structure isomorphic to $\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} \times \mathbb{A}$. For simplicity, we treat the case that $\kappa = 0$ or 1 , so that $v = D_{\kappa,n} = 1$. The general case is done in exactly the same way, but is slightly more technical due to the presence of the permutation

matrix v (the corresponding technical details are very similar to those appearing in the proof of Proposition 6.15, which we proved in full generality).

First, by Lemma 4.7, $I\mu_r(\varpi)I/I^m$ is isomorphic to an affine space and we fix the following coordinates on it: let α_{ij} ($1 \leq i \neq j \leq n$) denote the root of T_{diag} corresponding to (i, j) th matrix entry. Then

$$\psi: I\mu_r(\varpi)I/I^m \xrightarrow{\sim} C := \underbrace{\prod_{i=2}^n L_{[0, (i-1)r)} U_{\alpha_{i1}}}_{A:=} \times \underbrace{\prod_{n \geq i > j > 1} L_{[0, (i-j)r)} U_{\alpha_{ij}}}_{B:=} \times I/I^m \quad (6.10)$$

(the products can be taken in any fixed order; each factor – including I/I^m – is an affine space over $\overline{\mathbb{F}}_q$) is a parametrization of $I\mu_r(\varpi)I/I^m$, whose inverse sends $(a_{i1})_{i=2}^n, (a_{ij})_{i>j \neq 1}, g$ to $\prod_{i=2}^n a_{i1} \cdot \prod_{i,j} a_{ij} \cdot \mu_r(\varpi) \cdot gI^m$. It suffices to show that there is an open subset $U \subseteq A$, isomorphic to $\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} \times \mathbb{A}$, such that $X_{\dot{w}_r}(b)_{\mathcal{L}_0} \subseteq I\mu_r(\varpi)I/I^m$ is the graph of some morphism $f: U \rightarrow B \times I/I^m$. Indeed, then it follows that $X_{\dot{w}_r}(b)_{\mathcal{L}_0}$ is a locally closed subset of $I\mu_r(\varpi)I/I^m$, which (endowed with the induced reduced sub-scheme structure) gets isomorphic to U via the projection $p: C \cong A \times B \times I/I^m \rightarrow A$ to the first factor. First, we define U . Therefore, consider the natural projection

$$A \rightarrow \prod_{i=2}^{n'} L_{[0, 1)} U_{\alpha_{1+n_0(i-1), 1}} \cong \{[v] \in \mathbb{P}(\overline{V}): v = \sum_i v_i e_{1+n_0(i-1)} \in \overline{V}, v_1 \neq 0\} \quad (6.11)$$

where the latter isomorphism is $(v_i)_{i=2}^{n'} \mapsto [1 : v_2 : \dots : v_{n'}]$ and let U be the preimage of $\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1}$, which is a subspace of the right hand side via (6.9). (If $n' = 1$, $\overline{V}$ is one-dimensional, and the right hand side of (6.11) as well as $\Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1}$ is a point). It is clear that $U \cong \Omega_{\mathbb{F}_{q^{n_0}}}^{n'-1} \times \mathbb{A}$.

Next, we determine the image of $X_{\dot{w}_r}^m(b)_{\mathcal{L}_0}$ under ψ . Let $g_{b,r}(x)I^m = g_b^{\text{red}}(x)\mu_r(\varpi)I^m \in X_{\dot{w}_r}(b)_{\mathcal{L}_0}$, i.e., $x \in \mathcal{L}_0^{\text{adm}}$. This point does not change if we multiply $g_{b,r}(x)$ by an element of I^m from the right, or equivalently, if we multiply $g := g_b^{\text{red}}(x)$ by an element of

$$\mu_r(\varpi)I^m = \mu_r(\varpi)I^m \mu_r(\varpi)^{-1} = \begin{pmatrix} \mathcal{O}^\times & \mathfrak{p}^{m+1-r} & \dots & \dots & \mathfrak{p}^{m+1-(n-1)r} \\ * & \mathcal{O}^\times & \mathfrak{p}^{m+1-r} & \dots & \mathfrak{p}^{m+1-(n-2)r} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \dots & * & \mathcal{O}^\times & \mathfrak{p}^{m+1-r} \\ * & \dots & \dots & * & \mathcal{O}^\times \end{pmatrix} \quad (6.12)$$

from the right (the entries marked with $*$ are uninteresting for us). First, we multiply g by the diagonal matrix $i_1 = \text{diag}(x_1, \sigma(x_1), \dots, \sigma^{n-1}(x_1))^{-1} \in \mu_r(\varpi)I^m$ (note that

x_1 invertible by (6.6); see also Lemma 6.16), achieving that all entries of gi_1 are of the form $\sigma^i(\frac{x_j}{x_1})$ resp. $\varpi\sigma^i(\frac{x_j}{x_1})$ and, in particular, the first column of gi_1 has the entries $1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}$. Next, using that $r > m$, so that all $\mathfrak{p}^{m+1-r} \supseteq \mathcal{O}$ in (6.12), we may eliminate all $(1, j)$ th ($2 \leq j \leq n$) entries of gi_1 , i.e., we find some $i'_1 \in \mu_r(\varpi)I^m$ (with entries different from 0 and 1 only in the first row), such that $gi_1i'_1$ has the same first column as gi_1 and 0's in all entries of the first row except the first one. Also, all entries of $gi_1i'_1$ lie in $\mathcal{O}$ and are some functions in $\sigma^i(\frac{x_j}{x_1})$ (as the same is true for gi_1). Moreover, the $(2, 2)$ entry of $gi_1i'_1$ must now be in $\mathcal{O}^\times$ (otherwise $g\mu_r(\varpi)I^m \notin I\mu_r(\varpi)I/I^m$), and we can iterate the procedure: rescale the second column such that $(2, 2)$ th entry equals 1, then kill all entries $(2, j)$ with $3 \leq j \leq n$, etc. At the end we obtain a matrix $g' = gi_1i'_1i'_2i'_3 \dots i'_{n-1}$, such that $g'\mu_r(\varpi)I^m = g\mu_r(\varpi)I^m$ and

$$g' = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \frac{x_2}{x_1} & 1 & 0 & \dots & \dots & 0 \\ \frac{x_3}{x_1} & * & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{x_{n-1}}{x_1} & * & \dots & * & 1 & 0 \\ \frac{x_n}{x_1} & * & \dots & * & * & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ \frac{x_2}{x_1} & 1 & 0 & \dots & \dots & 0 \\ \frac{x_3}{x_1} & 0 & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{x_{n-1}}{x_1} & 0 & \dots & 0 & 1 & 0 \\ \frac{x_n}{x_1} & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & * & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & * & \dots & * & 1 & 0 \\ 0 & * & \dots & * & * & 1 \end{pmatrix}$$

where the entries marked by $*$ all lie in $\mathcal{O}$ and are functions of $\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}$. We can regard the first matrix in the product as an element of A (as in (6.10)), and moreover from (6.6) it follows that it lies in the open subset $U \subseteq A$. It gets clear now that with respect to the parametrization in (6.10), $\psi(X_{w_r}^m(b)_{\mathcal{L}_0})$ consists of points of the form $(u, f_0(u), 1 \cdot I^m)$ with $u \in U \subseteq A$ and $f_0(u) \in B$, where $f_0: U \rightarrow B$ is some morphism (which determines the entries $*$ in terms of $\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}$). Thus $f: U \rightarrow B \times I/I^m$, defined by $u \mapsto (f_0(u), 1 \cdot I^m)$ is the required morphism we wished to construct. This finishes the proof of the first claim of the theorem.

For the second claim in the theorem, we could repeat the above arguments with $I\mu_r(\varpi)I/\dot{I}^m$ replacing $I\mu_r(\varpi)I/I^m$. Alternatively we can argue as follows: by Theorem 4.9 (see Remark 6.3 and Lemma 6.4), $\dot{X}_{w_r}^m(b)$ is locally closed in $G(\check{K})/\dot{I}^m$, and hence a scheme locally of finite type over $\overline{\mathbb{F}}_q$ (Corollary 4.10), and by Proposition 4.2(iii) the morphism $G(\check{K})/\dot{I}^m \rightarrow G(\check{K})/I^m$ is representable and has sections étale-locally. It thus follows that $\dot{X}_{w_r}^m(b) \rightarrow X_{w_r}^m(b)$ is a honest morphism of schemes locally of finite type over $\overline{\mathbb{F}}_q$. Moreover, by the explicit description on geometric points in Theorem 6.5 it is surjective and a torsor under $T_w(\mathcal{O}_K/\varpi^{m+1})$. $\square$

Corollary 6.18 *Let $r' > m' > 0$, $r > m > 0$ be two pairs of integers with $r' \geq r$, $m' \geq m$. Then all maps $X_{\dot{w}_{r'}}^{m'}(b) \rightarrow X_{\dot{w}_r}^m(b)$, $\dot{X}_{\dot{w}_{r'}}^{m'}(b) \rightarrow \dot{X}_{\dot{w}_r}^m(b)$, $\dot{X}_{\dot{w}_{r'}}^{m'}(b) \rightarrow X_{\dot{w}_r}^m(b)$ induced by $g_{b,r'}(x) \mapsto g_{b,r}(x)$ are morphisms of schemes. In particular, $X_w^\infty(b)$ and $\dot{X}_w^\infty(b)$ are schemes over $\mathbb{F}_q$.*

Proof With respect to the coordinates on $X_{\dot{w}_r}^m(b)$, $\dot{X}_{\dot{w}_r}^m(b)$ in the proof of Theorem 6.17, these maps are simply induced by the natural projections $L_{[0,v']}U_{\alpha_{j,1}} \rightarrow L_{[0,v]}U_{\alpha_{j,1}}$ for $v' \geq v$ and $L_{[0,m'+1)}\mathbb{G}_m \rightarrow L_{[0,m+1)}\mathbb{G}_m$ for $m' \geq m$. $\square$

We are now ready to endow all objects in the diagram in Theorem 6.9 with scheme structures and compare them. The set $\mathcal{L}_{0,b}^{\text{adm, rat}}$ has an obvious scheme structure as a closed subset of the infinite dimensional affine space $\mathcal{L}_0$ over $\mathbb{F}_q$. Analogously, the natural embedding $\mathcal{L}_0^{\text{adm}}/\mathcal{O}^\times \subseteq \mathcal{L}_0/\mathcal{O}^\times = L^+\mathbb{P}(\mathcal{L}_0)(\overline{\mathbb{F}}_q)$, where $L^+\mathbb{P}(\mathcal{L}_0)$ is an infinite-dimensional $\mathbb{F}_q$ -scheme, endows $\mathcal{L}_0^{\text{adm}}/\mathcal{O}^\times$ with the structure of an open subscheme. We endow $V_b^{\text{adm, rat}}$ and $V_b^{\text{adm}}/\mathcal{O}^\times$ with the scheme structure of a disjoint union:

$$V_b^{\text{adm, rat}} = \bigsqcup_{g \in G/G_{\mathcal{O}}} g \cdot \mathcal{L}_{0,b}^{\text{adm, rat}} \quad \text{and} \quad V_b^{\text{adm}}/\mathcal{O}^\times = \bigsqcup_{g \in G/G_{\mathcal{O}}} g \cdot (\mathcal{L}_{0,b}^{\text{adm}}/\mathcal{O}^\times).$$

Since the action of ϖ on $V_b^{\text{adm}}/\mathcal{O}^\times$ just permutes the connected components, the quotient $V^{\text{adm}}/\check{K}^\times$ inherits the scheme structure $V^{\text{adm}}/\check{K}^\times = \bigsqcup_{G/Z(G)G_{\mathcal{O}}} g \cdot (\mathcal{L}_{0,b}^{\text{adm}}/\mathcal{O}^\times)$.

Corollary 6.19 *The maps of sets $V_b^{\text{adm, rat}} \xrightarrow{\sim} \dot{X}_w^\infty(b)$, $V_b^{\text{adm}}/\mathcal{O}^\times \xrightarrow{\sim} X_w^\infty(b)$ from Theorem 6.9 are isomorphisms of $\mathbb{F}_q$ -schemes. We endow $X_w^{DL}(b)$, $\dot{X}_w^{DL}(b)$ with the scheme structure via the isomorphisms in the diagram in Theorem 6.9.*

Proof To show the first isomorphism, it suffices to prove that for the special representative b , we have an isomorphism of schemes $\mathcal{L}_{0,b}^{\text{adm, rat}} \xrightarrow{\sim} \dot{X}_w^\infty(b)_{\mathcal{L}_0}$. Rescaling by an appropriate element of $\mathcal{O}^\times$, we may replace $\mathcal{L}_{0,b}^{\text{adm, rat}}$ by $\mathcal{L}_{0,b}^{\text{adm, rat}, \dot{w}_0}$. With notation as in the proof of Theorem 6.17, the coordinates on the inverse limit $\dot{X}_w^\infty(b)_{\mathcal{L}_0}$ are given by $((a_{\alpha_{j,1}})_{j=2}^n, c_1) \in L^+U_{\alpha_{j,1}} \times L^+\mathbb{G}_m$ and the map $\mathcal{L}_{0,b}^{\text{adm, rat}, \dot{w}_0} \rightarrow \dot{X}_w^\infty(b)_{\mathcal{L}_0}$ is given by $(x_i)_{i=1}^n \mapsto (\frac{x_j}{x_1})_{j=2}^n, x_1$. This is an isomorphism. The second isomorphism is proven similarly. $\square$

6.5 Example SL_2 , w Coxeter, $b = 1$

It is instructive to explicate the scheme structure on $X_w^{DL}(1)$ from Remark 3.3(ii) and compare it to the one obtained via affine Deligne–Lusztig varieties (a similar description applies in a number of further cases, in particular for GL_n or GSp_{2n} and w Coxeter). We have $\text{SL}_2/B = \mathbb{P}^1$ and

$$X_w^{DL}(1) = \mathbb{P}^1(\check{K}) \setminus \mathbb{P}^1(K).$$

It is thus given by the open condition $\det \begin{pmatrix} x & \sigma(x) \\ y & \sigma(y) \end{pmatrix} = x\sigma(y) - \sigma(x)y \neq 0$ inside

$$L\mathbb{P}_K^1(\overline{\mathbb{F}}_q) = L^+\mathbb{P}_{\mathcal{O}_K}^1(\overline{\mathbb{F}}_q) = \mathbb{P}^1(\mathcal{O}) = \{[x : y] : x, y \in \mathcal{O}, \text{ at least one of } x, y \text{ lies in } \mathcal{O}^\times\}$$

(where $[x : y] = [x' : y']$ if and only if there exists $a \in \mathcal{O}^*$ with $ax = x', ay = y'$) and

$$X_w^{DL}(1)_r = \{[x : y] \in L^+\mathbb{P}_{\mathcal{O}_K}^1(\overline{\mathbb{F}}_q) : \sigma(x)y - x\sigma(y) \not\equiv 0 \pmod{\mathfrak{p}^r}\}.$$

It is clear (from the version of Theorem 6.5 for SL_2) that if $g \in \mathrm{SL}_2(K) \setminus \mathrm{SL}_2(\mathcal{O})$, then $g \cdot X_w^{DL}(1)_1 \cap X_w^{DL}(1)_1 = \emptyset$. Moreover, $X_w^{DL}(1)_1 \subseteq X_w^{DL}(1)$ is dense open. This means that g maps a dense open subset of $X_w^{DL}(1)$ onto a subset which lies in its boundary and hence cannot be dense. Thus g cannot be an automorphism of the scheme $X_w^{DL}(1)$, and the action of G on $X_w^{DL}(1)$ with the above scheme structure is not algebraic.

The subsets Y_r of $L_r^+\mathbb{P}_{\mathcal{O}_K}^1(\overline{\mathbb{F}}_q) = \mathbb{P}^1(\mathcal{O}/\mathfrak{p}^r)$ can easily be computed to be

$$\begin{aligned} Y_1 &= \Omega_{\mathbb{F}_q} \\ Y_2 &= (\Omega_{\mathbb{F}_q} \times \mathbb{A}^1) \sqcup \bigsqcup_{\lambda \in \mathbb{P}^1(\mathbb{F}_q)} \Omega_{\mathbb{F}_q}, \\ &\vdots \\ Y_r &= (Y_{r-1} \times \mathbb{A}_{\mathbb{F}_q}^1) \sqcup \bigsqcup \Omega_{\mathbb{F}_q}, \end{aligned}$$

where the last union is taken over all hyperspecial vertices in the Bruhat–Tits building of SL_2 over K such that the minimal gallery connecting this vertex to the one stabilized by $\mathrm{SL}_2(\mathcal{O}_K)$ has length $2r - 1$. The unions are disjoint set-theoretically but not scheme-theoretically, since for example the preimage of Y_1 in Y_2 is open and not closed.

On the other hand, we can explicate the way in which $X_w^{DL}(1)$ is built from finite-dimensional pieces as dictated by Theorems 6.5, 6.17. In fact, $X_w^{DL}(1)$ is an inverse limit of the affine Deligne–Lusztig varieties of increasing level

$$X_{\dot{w}_m}^m(1) \cong \bigsqcup_{\mathrm{SL}_2(K)/\mathrm{SL}_2(\mathcal{O}_K)} \Omega_{\mathbb{F}_q}^1 \times \mathbb{A},$$

where $\Omega_{\mathbb{F}_q}^1 = \mathbb{P}_{\mathbb{F}_q}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ is the Drinfeld upper half-plane over $\mathbb{F}_q$, $\dot{w}_m$ are lifts of w whose length in the affine Weyl group has to grow with m , and $\mathbb{A}$ is some finite dimensional affine space over $\mathbb{F}_q$, whose dimension depends on m and $\dot{w}_m$ and goes to ∞ when $m \rightarrow \infty$.

7 A family of finite-type varieties X_h

In this section, we study the geometry of a family of finite-type varieties X_h for $h \geq 1$ which have natural projection maps $X_h \rightarrow X_{h-1}$. These varieties are more tractable than (components of) the affine Deligne–Lusztig varieties $\dot{X}_{\dot{w}_r}^m(b)_{\mathcal{L}_0}$, but we can see that after passing to the limit, these two families at infinite level are the same:

$$\varprojlim_{r,m: r>m} \dot{X}_{\dot{w}_r}^m(b)_{\mathcal{L}_0} = \mathcal{L}_{0,b}^{\text{adm, rat}, \dot{w}_0} \cong \mathcal{L}_{0,b}^{\text{adm, rat}} = \varprojlim_h X_h. \quad (7.1)$$

Our work in this section will prepare us for Part 3, where we will study the cohomology of X_h as representations of $G_h \times T_h$.

We remark that X_h will depend on whether we choose b to be the Coxeter-type representative or the special representative, but they are isomorphic as $\mathbb{F}_{q^n}$ -schemes for the same reason as in Corollary 6.14. The flexibility of choosing this representative b allows us to use a wide range of techniques to understand X_h and its cohomology. We will see this theme throughout Part 3.

7.1 Ramified Witt vectors

Recall the schemes $\mathbb{W}, \mathbb{W}_h$ from Sect. 2 (see [18, 18.6.13, 25.3.18] for more details on the construction of ramified Witt vectors). We will need to coordinatize $\mathbb{W}$ in order to make an explicit computations about the variety X_h . If A is a perfect $\mathbb{F}_q$ -algebra, the elements of $\mathbb{W}(A)$ can be written in the form $\sum_{i \geq 0} [x_i] \varpi^i$, where $[x_i]$ is the Teichmüller lift of $x_i \in A$ if $\text{char } K = 0$ and $[x_i] = x_i$ if $\text{char } K > 0$. (Note that the perfectness assumption is only necessary when $\text{char } K = 0$.) We identify $\mathbb{W}$ with $\mathbb{A}^{\mathbb{Z}_{\geq 0}}$ and identify $\mathbb{W}_h$ with $\mathbb{A}^h$ under this choice of coordinates. We recall the following lemma about the ring structure of $\mathbb{W}$ with respect to these coordinates.

Lemma 7.1 *Let A be a perfect $\mathbb{F}_q$ -algebra.*

(i) *The coefficient of ϖ^i in $(\sum_{i \geq 0} [a_i] \varpi^i) + (\sum_{i \geq 0} [b_i] \varpi^i)$ is*

$$[a_i + b_i + c_i], \quad \text{where } c_i \in A[a_j^{1/q^N}, b_j^{1/q^N} : j < i, N \in \mathbb{Z}_{\geq 0}].$$

(ii) *The coefficient of ϖ^i in $(\sum_{i \geq 0} [a_i] \varpi^i)(\sum_{i \geq 0} [b_i] \varpi^i)$ is*

$$\left[\sum_{j=0}^i a_j b_{i-j} + c_i \right], \quad \text{where } c_i \in A[a_{i_1}^{e_1/q^N} b_{i_2}^{e_2/q^N} : i_1 + i_2 < i, e_1, e_2, N \in \mathbb{Z}_{\geq 0}].$$

In both cases, we call c_i the “minor contribution.” Note that if $\text{char } K > 0$, then the minor contribution is identically zero. In particular, for any given i , the i th minor contribution does only depend on a_j, b_j with $j < i$.

This lemma says that up to “minor contributions”, working in coordinates with the Witt vectors is the same as working in coordinates in $\mathbb{F}_q[[t]]$. This allows us to

uniformly perform calculations in the mixed and equal characteristic settings. We will implicitly use Lemma 7.1 in Sects. 7.4 and 9.

Remark 7.2 Note in particular that by Lemma 7.1(ii), the coefficient of ϖ^i in the product $(\sum_{i \geq 0} [a_i] \varpi^i)(\sum_{i \geq 0} [b_i] \varpi^i)$ is of the form $[a_0 b_i + \mathbf{e}_i]$ where $\mathbf{e}_i$ is independent of b_i . For this reason, the minor contributions never play a role in our formulae as we study X_h as a subvariety of $X_{h-1} \times \mathbb{A}^N$ (for some N), and so the minor contributions only contribute to unspecified “constant” terms (see c in Proposition 7.6).

We also point out a case where the minor contribution vanishes (this is used in Proposition 7.7). Let A be a perfect $\mathbb{F}_q$ -algebra and pick an integer $h \geq 1$. Then the product $(1 + \sum_{i \geq h} [a_i] \varpi^i)(\sum_{i \geq 0} [b_i] \varpi^i) \in [b_0] + [b_1] \varpi + \cdots + [b_{h-1}] \varpi^{h-1} + [a_h + a_0 b_h] \varpi^h + \varpi^{h+1} \mathbb{W}(A)$. Indeed, it suffices to compute modulo ϖ^{h+1} (that is, in the $(h+1)$ -truncated ramified Witt vectors $\mathbb{W}_{h+1}(A)$), where we have

$$(1 + \varpi^h [a_h]) \cdot \sum_{i=0}^h [b_i] \varpi^i = \sum_{i=0}^{h-1} [b_i] \varpi^i + \varpi^h ([b_h] + [b_0 a_h]) = \sum_{i=0}^{h-1} [b_i] \varpi^i + [b_h + b_0 a_h] \varpi^h.$$

7.2 The scheme X_h

Fix a $0 \leq \kappa < n$ and let b be either the Coxeter-type or special representative with $\kappa_G(b) = \kappa$ as in Sect. 5.2. Define the $\mathcal{O}$ -submodule of $\mathcal{L}_0$,

$$\mathcal{L}_0^{(h)} := \bigoplus_{\substack{1 \leq i \leq n \\ i \equiv 1 \pmod{n_0}}} \varpi^h \mathcal{L}_0 \oplus \bigoplus_{\substack{1 \leq i \leq n \\ i \not\equiv 1 \pmod{n_0}}} \varpi^{h-1} \mathcal{L}_0.$$

Under the conventions set in Sect. 7.1, any $x \in \mathcal{L}_0 / \mathcal{L}_0^{(h)}$ can be written as

$$x = \sum_{\substack{1 \leq i \leq n \\ i \equiv 1 \pmod{n_0}}} \sum_{\ell=0}^{h-1} [x_{i,\ell}] \varpi^\ell \mathbf{e}_i + \sum_{\substack{1 \leq i \leq n \\ i \not\equiv 1 \pmod{n_0}}} \sum_{\ell=0}^{h-2} [x_{i,\ell}] \varpi^\ell \mathbf{e}_i \quad (x_{i,\ell} \in \overline{\mathbb{F}}_q). \quad (7.2)$$

This identifies $\mathcal{L}_0 / \mathcal{L}_0^{(h)}$ with $\mathbb{A}_{\mathbb{F}_{q^n}}^{n(h-1)+n'}$. Observe that if b is Coxeter-type, then although $\mathcal{L}_0^{(h)}$ is stable under $(b\sigma)^{n_0} \varpi^{-k_0}$, the $\mathbb{F}_{q^{n_0}}$ -rational structure given by this Frobenius on $\mathcal{L}_0 / \mathcal{L}_0^{(h)}$ does not agree with the $\mathbb{F}_{q^{n_0}}$ -rational structure on $\mathbb{A}^{n(h-1)+n'}$ given by the standard $\mathbb{F}_{q^{n_0}}$ -Frobenius.

Definition 7.3 For $h \geq 1$, define

$$X_h(\overline{\mathbb{F}}_q) := \mathcal{L}_{0,b}^{\text{adm, rat}} / \mathcal{L}_0^{(h)} = \text{image of } \mathcal{L}_{0,b}^{\text{adm, rat}} \text{ in } \mathcal{L}_0 / \mathcal{L}_0^{(h)}$$

and let $X_h \subset \mathbb{A}^{n(h-1)+n'}$ be the $\mathbb{F}_{q^n}$ -subscheme whose $\mathbb{F}_{q^n}$ -rational structure comes from the standard $\mathbb{F}_{q^n}$ -Frobenius on $\mathbb{A}^{n(h-1)+n'}$.

As $\det(g_b^{\text{red}}(\cdot)): X_h \rightarrow (\mathcal{O}_K/\varpi^h)^\times$ is a morphism onto a discrete scheme, we have the scheme-theoretic disjoint decomposition

$$X_h = \bigsqcup_{a \in (\mathcal{O}_K/\varpi^h)^\times} g_a \cdot X_h^{\det=1}, \quad (7.3)$$

where $X_h^{\det=1}$ consists of all $x \in X_h$ with $\det g_b^{\text{red}}(x) \equiv 1 \pmod{\varpi^h}$, and $g_a \in G_h$ is any matrix with determinant a .

Proposition 7.4 *X_h is a smooth affine scheme of dimension $(n-1)(h-1) + (n'-1)$.*

Proof The proof is very similar to that of [12, Proposition 3.10]. Choose b to be the Coxeter-type or special representative. Write $\det(g_b^{\text{red}}(x_1, \dots, x_n)) = [g_s]_{0 \leq s \leq h-1}$. It is enough to prove the assertions for the open and closed subset $X_h^{\det=1}$, which is defined by the equations $g_0 = 1$ and $g_s = 0$ for $1 \leq s \leq h-1$.

To prove that $X_h^{\det=1}$ is a smooth affine scheme of dimension $(n-1)(h-1) + n' - 1$, it suffices to show that for any point $X_h^{\det=1}$, there exists a nonsingular $h \times h$ submatrix of the Jacobian matrix J . First let $g_b^{\text{red}}(x_1, \dots, x_n) \in X_h^{\det=1}$. Then for some x_r , the determinant of the $(n-1) \times (n-1)$ minor obtained by deleting the 1st column and the r th row is nonzero modulo ϖ —denote by $d[x_r]$ the reduction of this determinant. From (6.5), observe that $x_{r,i}$ only contributes to g_s if $i \leq s$,

$$g_s = d[x_r]x_{r,s} + (\text{terms w/ } q\text{th powers of } x_{r,s}, \text{ and } x_{i,j} \text{ for } (i, j) \neq (r, s)).$$

Reorder the rows of J so that the first h rows correspond to the coordinates $x_{r,0}, \dots, x_{r,h-1}$ of $x_r \in \mathbb{W}_h$. Since we are working in characteristic p , we have

$$\frac{\partial g_s}{\partial x_{1,i}} = \begin{cases} d[x_r] \neq 0 & \text{if } i = s, \\ 0 & \text{if } i > s, \\ ? & \text{if } i < s. \end{cases}$$

This submatrix of J is an upper triangular matrix with nonzero determinant. Hence we have shown that $X_h^{\det=1}$ is a smooth complete intersection of dimension $(n-1)(h-1) + n' - 1$. $\square$

7.3 Relation to classical Deligne–Lusztig varieties

Recall that for $\bar{V} = \mathcal{L}_0/\mathcal{L}_0^{(1)}$ we have that $\mathbb{G}_1 = \text{Res}_{\mathbb{F}_{q^{n_0}}/\mathbb{F}_q} \text{GL}(\bar{V})$ (see Sects. 5.3, 5.6). The scheme X_1 is a classical Deligne–Lusztig variety corresponding to the maximal nonsplit torus $\mathbb{F}_{q^n}^\times$ in $\mathbb{G}_1(\mathbb{F}_q) = \text{GL}_{n'}(\mathbb{F}_{q^{n_0}})$. We get a commutative diagram

$$\begin{array}{ccccccc} \mathcal{L}_0 & \longrightarrow & \mathcal{L}_0/\mathcal{L}_0^{(h)} & \longrightarrow & \bar{V} & \longleftarrow & \bar{V} \setminus \{0\} \longrightarrow \mathbb{P}(\bar{V}) \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{L}_{0,b}^{\text{adm, rat}} & \longrightarrow & X_h & \longrightarrow & X_1 & \longrightarrow & \Omega_{\bar{V}} \end{array}$$

where $\Omega_{\overline{V}}$ is isomorphic to the Drinfeld upper half-space $\mathbb{P}(\overline{V}) \setminus \mathbb{P}(\overline{V})(\mathbb{F}_{q^{n_0}})$ and X_1 is a $\mathbb{F}_{q^n}^\times$ -torsor over $\Omega_{\overline{V}}$. (If b is the special representative, $\Omega_{\overline{V}}$ is literally the Drinfeld upper half-space.)

For $v \in \overline{V}$ define $\overline{g_b}(v)$ to be the $(n' \times n')$ -matrix whose i th column is $\overline{\sigma_b}^{i-1}(v)$ (written with respect to the basis $\{\mathbf{e}_i\}_{i \equiv 1 \pmod{n_0}}$ of $\overline{V}$ from Lemma 5.8). Then

$$X_1 = \{v \in \overline{V} : \det \overline{g_b}(v) \in \mathbb{F}_{q^{n_0}}^\times\}.$$

Example 7.5 If $\kappa = 0$, then $\overline{V} = \mathcal{L}_0/\varpi \mathcal{L}_0$, $\overline{\sigma_b}^{i-1} = b\sigma$ and X_1 is the Deligne–Lusztig variety for $\mathrm{GL}_n(\mathbb{F}_q)$ associated to the maximal nonsplit torus $\mathbb{F}_{q^n}^\times$. If κ, n are coprime, then $\overline{V}$ is one-dimensional and X_1 is a finite set of points and can be identified with $\mathbb{F}_{q^n}^\times$.

7.4 The projection $X_h \rightarrow X_{h-1}$ and its fibers

Let $h \geq 2$. We will actually work with an intermediate scheme: $X_h \twoheadrightarrow X_{h-1}^+ \twoheadrightarrow X_{h-1}$. By Sects. 7.1, 7.2, the quotient $\mathcal{L}_0/\varpi^{h-1} \mathcal{L}_0$ can be identified with the affine space $\mathbb{A}^{n(h-1)}$. Define X_{h-1}^+ to be the $\mathbb{F}_{q^n}$ -subscheme of $\mathbb{A}^{n(h-1)}$ defined by

$$X_{h-1}^+(\overline{\mathbb{F}}_q) := \mathcal{L}_{0,b}^{\mathrm{adm}, \mathrm{rat}}/\varpi^{h-1} \mathcal{L}_0 = \text{image of } \mathcal{L}_{0,b}^{\mathrm{adm}, \mathrm{rat}} \text{ in } \mathcal{L}_0/\varpi^{h-1} \mathcal{L}_0.$$

Observe that

$$X_{h-1}^+ = X_{h-1} \times \mathbb{A}^{n-n'}, \quad (7.4)$$

since the coordinates $x_{i,h-2}$ for $i \not\equiv 1 \pmod{n_0}$ do not contribute to $\det(g_b^{\mathrm{red}}(x))$ modulo ϖ^{h-1} . Furthermore, X_h is a closed subscheme of $X_{h-1}^+ \times \mathbb{A}^{n'}$, and under this embedding

$$X_h \hookrightarrow X_{h-1}^+ \times \mathbb{A}^{n'},$$

we may write $x = (\tilde{x}, x_{1,h-1}, x_{n_0+1,h-1}, \dots, x_{n_0(n'-1)+1,h-1})$ for $x \in X_h$ and its image $\tilde{x} \in X_{h-1}^+$. More precisely, we have the following technical proposition, which will be used in Sect. 9.

Proposition 7.6 *Let $h \geq 2$.*

(i) *X_h is the closed subscheme of $X_{h-1}^+ \times \mathbb{A}^{n'}$ cut out by the polynomial*

$$P := P_0^q - P_0,$$

where $[P_0]$ is the coefficient of ϖ^{h-1} in $\det(g_b^{\mathrm{red}}(\cdot))$.

(ii) Let b be the special representative. Then

$$P_0(x) = c(\tilde{x}) + \sum_{i=0}^{n_0-1} P_1(x)^{q^i}$$

where $\sum_{i=0}^{n_0-1} P_1^{q^i}$ exactly consists of all terms of P_0 that depend on the coordinates $x_{1,h-1}, x_{n_0+1,h-1}, \dots, x_{n_0(n'-1)+1,h-1}$ and c is a morphism $X_{h-1}^+ \rightarrow \mathbb{A}^1$. In particular, X_h is the closed subscheme of $X_{h-1}^+ \times \mathbb{A}^{n'}$ cut out by the equation

$$P_1^{q^{n_0}} - P_1 = c - c^q.$$

(iii) Let b be the special representative. Explicitly, the polynomial in (ii) is given by

$$P_1 = \sum_{1 \leq i, j \leq n'} m_{ji} x_{1+n_0(i-1), h-1}^{q^{(j-1)n_0}},$$

where $m := (m_{ji})_{j,i}$ is the adjoint matrix of $\overline{g_b}(\bar{x})$ and $\bar{x}$ denotes the image of x in $\overline{V} = \mathcal{L}_0 / \mathcal{L}_0^{(1)}$. Explicitly, $m \cdot \overline{g_b}(\bar{x}) = \det \overline{g_b}(\bar{x}) \cdot 1_{n'}$ and the (j, i) th entry of m is $(-1)^{i+j}$ times the determinant of the $(n' - 1) \times (n' - 1)$ matrix obtained from $\overline{g_b}(\bar{x})$ by deleting the i th row and j th column.

Proof An explicit calculation shows that $P_0 = c + \sum_{i=0}^{n_0-1} \sigma^i(P_1)$, with P_1 as claimed if b is the special representative. Note that in the mixed characteristic setting, we use Lemma 7.1(ii) to see that minor contributions only appear in the $c(\tilde{x})$ term. From this the proposition easily follows. $\square$

7.5 Level compatibility on the cohomology of X_h

Proposition 7.7 Let $h \geq 2$. The action of $\ker(T_h \rightarrow T_{h-1}) = \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ on X_h preserves each fiber of the map $X_h \rightarrow X_{h-1}$, the induced morphism $X_h / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \rightarrow X_{h-1}$ is smooth, and each of its fibers is isomorphic to $\mathbb{A}^{n-1}$.

Proof Let b be the special representative and let $x \in X_h$ be coordinatized as in Sect. 7.2. Then $x_{i,0} \neq 0$ for $i \equiv 1 \pmod{n_0}$. By (a slight variant of) Proposition 7.6, X_h is the closed subscheme of $X_{h-1} \times \mathbb{A}^n$ given by $P = 0$, where $\mathbb{A}^n$ has the coordinates $\{y_i\}_{i=1,\dots,n}$, where $y_i = x_{i,h-1}$ if $i \equiv 1 \pmod{n_0}$ and $y_i = x_{i,h-2}$ if $i \not\equiv 1 \pmod{n_0}$. Note that the natural $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ -action on X_h extends to the action on $X_{h-1} \times \mathbb{A}^n$ over X_{h-1} given by

$$1 + [\lambda]\varpi^{h-1} : \begin{cases} x_{i,h-1} \mapsto x_{i,h-1} + x_{i,0}\lambda & \text{if } i \equiv 1 \pmod{n_0}, \\ x_{i,h-2} \mapsto x_{i,h-2} & \text{otherwise,} \end{cases}$$

where $\lambda \in \mathbb{F}_{q^n}$. (Note that we use Remark 7.2 here.) Consider the morphism

$$f: X_{h-1} \times \mathbb{A}^n \rightarrow X_{h-1} \times \mathbb{A}^n, \quad y_i \mapsto \begin{cases} \left(\frac{y_1}{x_{1,0}}\right)^{q^n} - \frac{y_1}{x_{1,0}} & \text{if } i = 1, \\ y_i - \frac{x_{i,0}y_1}{x_{1,0}} & \text{if } i > 1, i \equiv 1 \pmod{n_0}, \\ y_i & \text{if } i \not\equiv 1 \pmod{n_0}. \end{cases}$$

This morphism factors through the surjection $X_{h-1} \times \mathbb{A}^n \rightarrow X_{h-1} \times \mathbb{A}^n / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ so that it is a composition

$$X_{h-1} \times \mathbb{A}^n \rightarrow X_{h-1} \times \mathbb{A}^n / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \xrightarrow{\sim} X_{h-1} \times \mathbb{A}^n,$$

where the second map must in fact be an isomorphism. Since $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ is a p -group, [11, Proposition 3.6] implies that $P((y_i)_{i=1,\dots,n}) = P'(f(y_i)_{i=1,\dots,n})$ for some $P': X_{h-1} \times \mathbb{A}^n \rightarrow \mathbb{A}^1$. Now $X_h / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ is the closed subscheme of $X_h \times \mathbb{A}^n / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ defined by $P' = 0$. We therefore have a commutative diagram

$$\begin{array}{ccccc} X_{h-1} \times \mathbb{A}^n & \longrightarrow & X_{h-1} \times \mathbb{A}^n / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) & \xrightarrow{\sim} & X_{h-1} \times \mathbb{A}^n \\ \uparrow & & \uparrow & & \uparrow \\ X_h & \longrightarrow & X_h / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) & \xrightarrow{\sim} & \{P' = 0\}, \end{array} \quad (7.5)$$

Since P is a degree- q^n polynomial in $x_{1,h-1}$, we know that P' must be at most degree one in y_1 . A calculation shows that the coefficient of y_1 is the function on X_{h-1} given by $x \mapsto \det \overline{g_b}(\bar{x})$, where $\bar{x}$ is the image of $x \in X_{h-1}$ in X_1 (notation as in Sect. 7.3). This function is constant on connected components of X_{h-1} , taking values in $\mathbb{F}_q^\times$. In particular, the coefficient of y_1 in P' over any point in X_{h-1} is nonzero, so it follows that each fiber of $X_h / \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \rightarrow X_{h-1}$ is isomorphic to $\mathbb{A}^{n-1}$. $\square$

Corollary 7.8 *There is a natural isomorphism*

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)^{\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})} \cong H_c^{i+2(n-1)}(X_{h-1}, \overline{\mathbb{Q}}_\ell)(n-1),$$

where $(n-1)$ denotes the Tate twist.

This corollary allows to define a *direct* limit of the *homology* groups for X_h (see Sect. 11).

7.6 X_h as a subscheme of $\mathbb{G}_h$

Let b be a Coxeter-type representative. Let $1 \leq l \leq n$ be an integer satisfying $e_{\kappa,n}l \equiv 1 \pmod{n}$. For $x = \sum_{i=1}^n x_i \mathbf{e}_i \in \mathcal{Z}_0$ where $x_i \in \mathcal{O}$, define

$$\lambda(x) := \sum_{i=1}^n \frac{1}{\varpi^{\lfloor k_0(i-1)/n_0 \rfloor}} \cdot b^{i-1} \cdot D(x_i),$$

where $D(a) = \text{diag}(a, \sigma^l(a), \dots, \sigma^{[(n-1)l]}(a))$. Let γ be the inverse of the permutation of the set $\{1, 2, \dots, n\}$ defined by $1 \mapsto 1$ and $i \mapsto [(i-1)e_{\kappa,n}] + 1$ for $2 \leq i \leq n$. Let $\gamma \in \text{GL}_n(K)$ also denote the matrix given by $\gamma(e_i) = e_{\gamma(i)}$.

Lemma 7.9 *We have*

$$\lambda(x) = g_b^{\text{red}}(\gamma^{-1}(x)) \cdot \gamma.$$

In particular, $\det \lambda(x) = \det g_b^{\text{red}}(x)$. Moreover, we have $\gamma b_0^{e_{\kappa,n}} \gamma^{-1} = b_0$.

Proof This is a direct computation. $\square$

Example 7.10 (i) For $n = 3, \kappa = e_{\kappa,n} = 1$, we have $b = \begin{pmatrix} 0 & 0 & \varpi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

$$\lambda(x) = g_b^{\text{red}}(x) = g_b(x) = \begin{pmatrix} x_1 & \varpi \sigma(x_3) & \varpi \sigma^2(x_2) \\ x_2 & \sigma(x_1) & \varpi \sigma^2(x_3) \\ x_3 & \sigma(x_2) & \sigma^2(x_1) \end{pmatrix}.$$

We have $F(\lambda(x)) \neq \lambda(\sigma(x))$. Thus λ is not an $\mathbb{F}_q$ -morphism.

(ii) For $n = 3, \kappa = 2, e_{\kappa,n} = 2$, we have $b = \begin{pmatrix} 0 & \varpi & 0 \\ 0 & 0 & \varpi \\ 1 & 0 & 0 \end{pmatrix}$ and

$$g_b^{\text{red}}(x) = \begin{pmatrix} x_1 & \varpi \sigma(x_2) & \varpi \sigma^2(x_3) \\ x_2 & \varpi \sigma(x_3) & \sigma^2(x_1) \\ x_3 & \sigma(x_1) & \sigma^2(x_2) \end{pmatrix} \quad \text{and} \quad \lambda(x) = \begin{pmatrix} x_1 & \varpi \sigma^2(x_2) & \varpi \sigma(x_3) \\ x_3 & \sigma^2(x_1) & \varpi \sigma(x_2) \\ x_2 & \sigma^2(x_3) & \sigma(x_1) \end{pmatrix} \in \check{G}_{\mathbf{x},0}.$$

Proposition-Definition 7.11 *The assignment λ defines an embedding,*

$$\mathcal{L}_0 \hookrightarrow M_{n'}(\mathcal{O}_{D_{k_0/n_0}}),$$

which restricts to

$$\lambda: \mathcal{L}_{0,b}^{\text{adm}} \hookrightarrow \check{G}_{\mathbf{x},0},$$

Moreover, $\det(\lambda(x)) \in \mathcal{O}_K^\times$ if and only if $x \in \mathcal{L}_{b,0}^{\text{adm, rat}}$. The reduction modulo ϖ^h of λ induces an $\mathbb{F}_{q^n}$ -rational embedding

$$\mathcal{L}_{0,b}^{\text{adm, rat}} / \mathcal{L}_0^{(h)} = X_h \hookrightarrow \mathbb{G}_h.$$

We denote its image again by X_h . This is an $\mathbb{F}_{q^n}$ -subscheme of $\mathbb{G}_h$.

Proof It is easy to see that $\lambda(e_i) \in M_{n'}(\mathcal{O}_{D_{k_0/n_0}})$ for $i = 1, \dots, n$. This implies that $\lambda(\mathcal{L}_0) \subseteq M_{n'}(\mathcal{O}_{D_{k_0/n_0}})$. By Lemma 7.9 it is immediate that $\det(\lambda(x)) \in \mathcal{O}^\times$ if and only if $\det(g_b^{\text{red}}(x)) \in \mathcal{O}^\times$ and similarly $\det(\lambda(x)) \in \mathcal{O}_K^\times$ if and only if $\det(g_b^{\text{red}}(x)) \in \mathcal{O}_K^\times$. Finally, note that λ is a $\mathbb{F}_{q^n}$ -morphism since $\lambda(\sigma^n(x)) = \sigma^n(\lambda(x)) = F^n(\lambda(x))$. $\square$

The natural $(\check{G}_{\mathbf{x},0}^F \times \mathcal{O}_L^\times)$ -action on $\mathcal{L}_{0,b}^{\text{adm, rat}}$ induces a left action of $(G_h \times T_h)$ -action on $X_h \subseteq \mathbb{G}_h$, given by left-multiplication by $G_h = \mathbb{G}_h(\mathbb{F}_q)$ and right-multiplication by $T_h = \mathbb{T}_h(\mathbb{F}_q)$:

$$(g, t) \cdot x := gxt, \quad \text{for } g \in G_h, t \in T_h, x \in X_h.$$

7.7 Relation to Deligne–Lusztig varieties for finite rings

Let b be the Coxeter-type representative. The following proposition gives a description of X_h reminiscent of Deligne–Lusztig varieties for reductive groups over finite rings [26, 31]. Let U_{up} and U_{low} denote the $\check{K}$ -subgroups of upper and lower triangular unipotent matrices in J_b . Consider the unipotent radicals $U := \gamma^{-1}U_{up}\gamma$, $U^- = \gamma^{-1}U_{low}\gamma$ of opposite Borels (over $\check{K}$) in J_b containing the diagonal torus T . Let $\mathbb{U}$ and $\mathbb{U}^-$ denote the smooth subgroup schemes of $\mathbb{G}$ whose $\overline{\mathbb{F}}_q$ -points are $U(\check{K}) \cap \check{G}_{\mathbf{x},0}$ and $U^-(\check{K}) \cap \check{G}_{\mathbf{x},0}$, and let $\mathbb{U}_h$ and $\mathbb{U}_h^-$ be the corresponding subgroups of $\mathbb{G}_h$.

Proposition 7.12 *The subgroup $\mathbb{U}_h^- \cap F(\mathbb{U}_h) \subseteq \mathbb{G}_h$ consists of matrices with 1's along the main diagonal and 0's outside the first column. We have*

$$\begin{aligned} X_h(\overline{\mathbb{F}}_q) &= \left\{ g \in \mathbb{G}_h(\overline{\mathbb{F}}_q) : g^{-1}F(g) \in \mathbb{U}_h^- \cap F(\mathbb{U}_h) \right\} \\ &= \left\{ g \in \mathbb{G}_h(\overline{\mathbb{F}}_q) : g^{-1}F(g) \in \mathbb{U}_h^- \right\} / (\mathbb{U}_h^- \cap F^{-1}(\mathbb{U}_h^-)). \end{aligned}$$

Proof Using $\gamma b_0^{e_{\kappa,n}} \gamma^{-1} = b_0$ and $\gamma t_{\kappa,n} \gamma^{-1} = t_{\kappa,n}$ from Lemma 7.9, we compute

$$\begin{aligned} U^- \cap F(U) &= \gamma^{-1}U_{low}\gamma \cap F(\gamma^{-1}U_{up}\gamma) = \gamma^{-1}U_{low}\gamma \cap b_0^{e_{\kappa,n}} t_{\kappa,n} \gamma^{-1} U_{up} t_{\kappa,n}^{-1} b_0^{-e_{\kappa,n}} \gamma \\ &= \gamma^{-1}(U_{low} \cap b_0 U_{up} b_0^{-1}) \gamma \end{aligned}$$

and (using $\gamma(e_1) = e_1$) the claim about $\mathbb{U}_h^- \cap F(\mathbb{U}_h)$ follows easily. For any $a \in \mathbb{W}_h(\overline{\mathbb{F}}_q)$, we have that

$$F(\text{diag}(a, \sigma^{[l]}(a), \dots, \sigma^{[(n-1)l]}(a))) = \text{diag}(\sigma^n(a), \sigma^{[l]}(a), \dots, \sigma^{[(n-1)l]}(a)).$$

Thus for any $v = (v_i)_{i=1}^n$ with $v_i \in \mathbb{W}_h(\overline{\mathbb{F}}_q)$ ($i \equiv 1 \pmod{n_0}$) and $v_i \in \mathbb{W}_{h-1}(\overline{\mathbb{F}}_q)$ ($i \not\equiv 1 \pmod{n_0}$),

$$F(\lambda(x)) = \sum_{i=1}^n \frac{1}{\varpi^{\lfloor k_0(i-1)/n_0 \rfloor}} \cdot b^{i-1} \cdot \text{diag}(\sigma^n(x_i), \sigma^{[l]}(x_i), \dots, \sigma^{[(n-1)l]}(x_i))$$

differs from $\lambda(x)$ in only the first column. Thus for $x \in \mathcal{L}_0$, we see that $\lambda(x)^{-1}F(\lambda(x))$ can differ from an element of $\mathbb{U}_h^- \cap F(\mathbb{U}_h)$ only in the left upper entry, and this entry

is equal to $\det(\lambda(x)^{-1}F(\lambda(x))) = \det(g_b^{\text{red}}(x)^{-1}\sigma(g_b(x)))$ (Lemma 7.9). Now for $x \in \mathcal{L}_0^{\text{adm, rat}}$, $\det g_b^{\text{red}}(x) \in \mathcal{O}_K^\times$. This proves

$$X_h \subset \left\{ g \in \mathbb{G}_h(\overline{\mathbb{F}}_q) : g^{-1}F(g) \in \mathbb{U}_h^- \cap F(\mathbb{U}_h) \right\}.$$

To see the other inclusion, observe that if $F(g) = g \cdot u$ for some $u \in \mathbb{U}_h^- \cap F(\mathbb{U}_h)$, then comparing the j th column for $j \geq 2$ shows that g must necessarily be of the form $\lambda(v)$ for some $v \in \mathcal{L}_0^{\text{adm}}$. The determinant condition then follows from $\det(u) = 1$. The last equality in the proposition follows from Lemma 7.13. $\square$

Lemma 7.13 *The morphism*

$$(\mathbb{U}_h^- \cap F^{-1}\mathbb{U}_h^-) \times (\mathbb{U}_h^- \cap F\mathbb{U}_h) \rightarrow \mathbb{U}_h^-, \quad (x, g) \mapsto x^{-1}gF(x).$$

is an isomorphism.

Proof We can consider the $\overline{\mathbb{F}}_q$ -scheme $\gamma\mathbb{G}\gamma^{-1}$, whose $\overline{\mathbb{F}}_q$ -points are $\gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1}$, together with a Frobenius isomorphism

$$F_0: \gamma\mathbb{G}\gamma^{-1} \xrightarrow{\sim} \gamma\mathbb{G}\gamma^{-1}, \quad F_0(x) = b_0(\gamma t_{\kappa, n}\gamma^{-1})\sigma(x)(b_0(\gamma t_{\kappa, n}\gamma^{-1}))^{-1}$$

By Lemma 7.9, $\gamma b_0^{e_{\kappa, n}} t_{\kappa, n} \gamma^{-1} = b_0(\gamma t_{\kappa, n} \gamma^{-1})$. Thus if $c_\gamma: \mathbb{G} \xrightarrow{\sim} \gamma\mathbb{G}\gamma^{-1}$, $x \mapsto \gamma x \gamma^{-1}$ denotes the conjugation by γ , we have $c_\gamma \circ F = F_0 \circ c_\gamma$ (this in particular shows that F_0 is an isomorphism).

We will first show that $(\mathbb{U}^- \cap F^{-1}\mathbb{U}^-) \times (\mathbb{U}^- \cap F\mathbb{U}) \rightarrow \mathbb{U}^-$, $(x, g) \mapsto x^{-1}gF(x)$ is bijective. We have $\mathbb{U}^-(\overline{\mathbb{F}}_q) = \gamma^{-1}(\check{U}_{\text{low}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1})\gamma$ and $\mathbb{U}(\overline{\mathbb{F}}_q) = \gamma^{-1}(\check{U}_{\text{up}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1})\gamma$. Applying c_γ , we thus have to show that the map

$$\begin{aligned} & ((\check{U}_{\text{low}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1}) \cap F_0(\check{U}_{\text{low}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1})) \\ & \times ((\check{U}_{\text{low}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1}) \cap F_0(\check{U}_{\text{up}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1})) \\ & \rightarrow \check{U}_{\text{low}} \cap \gamma\mathbb{G}(\overline{\mathbb{F}}_q)\gamma^{-1}, \end{aligned} \quad (7.6)$$

$(x, g) \mapsto x^{-1}gF_0(x)$ is bijective. We first show that the following is an isomorphism:

$$(\check{U}_{\text{low}} \cap b_0^{-1}\check{U}_{\text{low}}b_0) \times (\check{U}_{\text{low}} \cap b_0\check{U}_{\text{up}}b_0^{-1}) \rightarrow \check{U}_{\text{low}}, \quad (x, g) \mapsto x^{-1}gF_0(x). \quad (7.7)$$

To do this, it is equivalent to prove that given any $A \in \check{U}_{low}$, there exists a unique element $(x, g) \in (\check{U}_{low} \cap b_0^{-1}\check{U}_{low}b_0) \times (\check{U}_{low} \cap b_0\check{U}_{up}b_0^{-1})$ such that $xA = gF_0(x)$. We now compute explicitly and write

$$x = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ b_{21} & 1 & 0 & \cdots & \cdots & 0 \\ b_{31} & b_{32} & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-2} & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ c_{n-1} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $\gamma t_{\kappa,n} \gamma^{-1} = \text{diag}(t_1, t_2, \dots, t_n)$ so that we have

$$b_0 t_{\kappa,n} \sigma(x) t_{\kappa,n}^{-1} b_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma(b_{21})t_2/t_1 & 1 & 0 & & 0 \\ 0 & \sigma(b_{31})t_3/t_1 & \sigma(b_{32})t_3/t_2 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \sigma(b_{n-1,1})t_{n-1}/t_1 & \sigma(b_{n-1,2})t_{n-1}/t_2 & \cdots & \sigma(b_{n-1,n-2})t_{n-1}/t_{n-2} & 1 \end{pmatrix}.$$

We therefore see that the (i, j) th entry of $gF_0(x)$ is

$$(gF_0(x))_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ c_{i-1} & \text{if } i > j = 1, \\ \sigma(b_{i-1,j-1})t_{i-1}/t_{j-1} & \text{if } i > j > 1. \end{cases} \quad (7.8)$$

We also compute the (i, j) th entry of xA when $A = (a_{i,j})_{i,j} \in \check{U}_{low}$:

$$(xA)_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ b_{ij} + \sum_{k=j+1}^{i-1} b_{ik}a_{kj} + a_{ij} & \text{if } j < i \leq n-1, \\ a_{nj} & \text{if } j < i = n. \end{cases} \quad (7.9)$$

We now have n^2 equations given by (7.8) = (7.9), viewed as equations in the variables b_{ij} and c_i . First look at the equations corresponding to $(n, 2), (n, 3), \dots, (n, n-1)$. This gives

$$\sigma(b_{n-1,j-1})t_{n-1}/t_{j-1} = a_{nj} \quad \text{for } 1 < j < n.$$

which uniquely determines $b_{n-1,1}, b_{n-1,2}, \dots, b_{n-1,n-2}$. Proceeding inductively, let $1 < i \leq n-1$, and suppose that all $b_{i',j}$ for all $i' \geq i$ and $1 < j \leq i'$ are uniquely

determined. Then look at the equations corresponding to $(i-1, 2), (i-1, 3), \dots, (i-1, i-2)$. This gives

$$\sigma(b_{i-1,j-1})t_{i-1}/t_{j-1} = b_{i,j} + \sum_{k=j+1}^{i-1} b_{i,k}a_{kj} + a_{i,j} \quad \text{for } 1 < j < i, \quad (7.10)$$

which uniquely determines $b_{i-1,1}, b_{i-2,2}, \dots, b_{i-1,i-2}$. This uniquely determines x . Finally, by looking at the equations corresponding to $(2, 1), (3, 1), \dots, (n, 1)$, it is immediately clear that the c_i 's are also uniquely determined, so g is as well. This shows the isomorphism (7.7).

Now we deduce (7.6) from this. Using the same notation as above, assume that $A \in \check{U}_{low} \cap \gamma \mathbb{G}(\overline{\mathbb{F}}_q) \gamma^{-1}$. Let $\tau_i := \text{ord}(t_i)$ ($1 \leq i \leq n$) and $\lambda_{i,j}$ denote the minimum of valuations of all elements of $\gamma \mathbb{G}(\overline{\mathbb{F}}_q) \gamma^{-1} \cap \check{U}_{\alpha_{i,j}}$, where $\check{U}_{\alpha_{i,j}}$ is the root subgroup corresponding to the (i, j) th entry ($1 \leq i \neq j \leq n$). Then $\tau_i, \lambda_{ij} \in \{0, 1\}$ for all i, j . Moreover, the fact that F_0 is an isomorphism shows

$$\lambda_{i,j} = \lambda_{i-1,j-1} + \tau_{i-1} - \tau_{j-1}. \quad (7.11)$$

To establish (7.6), we have to show that for all $2 \leq j < i \leq n$, we have $\text{ord}(b_{i-1,j-1}) \geq \lambda_{i-1,j-1}$ and $\text{ord}(c_{i-1}) \geq \lambda_{i-1,1}$.

We first prove the assertion about the b 's. As in the proof of (7.7) above, we may proceed inductively on i : assuming that the assertion holds for all $i' > i$, we will show that the assertion holds for i . (The basic induction step $i = n$ follows from the same argument as below.) Observe that if $\tau_{i-1} = 0$ and $\tau_{j-1} = 1$, then we are done by formula (7.10).

Assume that $\tau_{i-1} = \tau_{j-1}$. If $\lambda_{i-1,j-1} = 0$ then by (7.10) there is again nothing to show. Thus we may assume $\lambda_{i-1,j-1} = 1$. By (7.10) we have to check that $\lambda_{i,j} = 1$ and that for each $j+1 \leq k \leq i-1$, either $\lambda_{i,k} = 1$ or $\lambda_{k,j} = 1$. First, $\lambda_{i,j} = 1$ follows from (7.11). Second, $\alpha_{i,k} + \alpha_{k,j} = \alpha_{i,j}$ ($\alpha_{i,j}$ is the root of the diagonal torus of GL_n corresponding to (i, j) th entry). Thus the fact that $\gamma \mathbb{G}(\overline{\mathbb{F}}_q) \gamma^{-1}$ is a group implies that $\lambda_{i,k} + \lambda_{k,j} \geq \lambda_{i,j}$ (for all k), so $\lambda_{i,k} = 1$ or $\lambda_{k,j} = 1$.

Finally, assume that $\tau_{i-1} = 1, \tau_{j-1} = 0$. Then (7.11) implies $\lambda_{i-1,j-1} = 0$ and $\lambda_{i,j} = 1$. Then by (7.10) we have to show that $\lambda_{i,j} = 1$ (which we already know) and that for each $j+1 \leq k \leq i-1$, we have $\lambda_{i,k} = 1$ or $\lambda_{k,j} = 1$ (which holds for the same reason as above). This completes the proof of the assertion about the b 's.

Analogously, one proves the assertion about the c_{i-1} 's. Since (7.10) and the equations corresponding to $(2, 1), (3, 1), \dots, (n, 1)$ uniquely determine the b 's and the c 's, this establishes bijectivity of (7.6).

To finish the proof of the lemma, it suffices to check that if $A_1, A_2 \in \gamma \mathbb{G}(\overline{\mathbb{F}}_q) \gamma^{-1}$ differ by some element in the normal subgroup $\gamma \ker(\mathbb{G}(\overline{\mathbb{F}}_q) \rightarrow \mathbb{G}_h(\overline{\mathbb{F}}_q)) \gamma^{-1}$, then the corresponding pairs (x_1, g_1) and (x_2, g_2) with $x_i^{-1} g_i F_0(x_i) = A_i$ ($i = 1, 2$) satisfy $x_1^{-1} x_2, g_1^{-1} g_2 \in \ker(\mathbb{G}(\overline{\mathbb{F}}_q) \rightarrow \mathbb{G}_h(\overline{\mathbb{F}}_q))$. Let $\lambda_{i,j}^h \in \{h-1, h\}$ be the smallest possible valuation of an element in $\gamma \ker(\mathbb{G}(\overline{\mathbb{F}}_q) \rightarrow \mathbb{G}_h(\overline{\mathbb{F}}_q)) \gamma^{-1} \cap \check{U}_{\alpha_{i,j}}$. As F_0 induces an

isomorphism of $\mathbb{G}_h$, we again have a formula

$$\lambda_{i,j}^h = \lambda_{i-1,j-1}^h + \tau_{i-1} - \tau_{j-1}.$$

We can once again proceed inductively to deduce that the $b_{i-1,j-1}$ and c_{i-1} are uniquely determined as elements in $\mathfrak{p}^{\lambda_{i-1,j-1}}/\mathfrak{p}^{\lambda_{i-1,j-1}^h}$ by the elements $a_{i,j} \in \mathfrak{p}^{\lambda_{i,j}}/\mathfrak{p}^{\lambda_{i,j}^h}$. $\square$

Part 3. Alternating sum of cohomology of X_h

In this part, we study the virtual G_h -representations

$$R_{T_h}^{G_h}(\theta) := \sum_{i \geq 0} (-1)^i H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta],$$

where θ is a character of $T_h \cong \mathcal{O}_L^\times / U_L^h = \mathbb{W}_h^\times(\mathbb{F}_{q^n})$.

In Sect. 8, we prove that if θ is *primitive*, then $R_{T_h}^{G_h}(\theta)$ is (up to a sign) an irreducible G_h -representation (Theorem 8.1). Our strategy is to extend ideas of Lusztig, who proves the analogous result in the context of division algebras [25] and split groups [26] (see [31] for the mixed characteristic analogue). This is done in Sect. 8.4. We note that the main result there, Proposition 8.7, is more general than Theorem 8.1 in that it works for any Frobenius F on $\mathbb{G}_h$ and the F -fixed points of any F -stable maximal torus in $\mathbb{G}_h$. For example, if we take F to be the twisted Frobenius coming from the Coxeter-type representative, then the F -fixed points of the diagonal torus forms the group $\mathbb{W}_h^\times(\mathbb{F}_{q^n})$, which exactly gives Theorem 8.1. On the other hand, if we take F to be the twisted Frobenius coming from the special representative, then the F -fixed points of the diagonal torus forms the n' -fold product of $\mathbb{W}_h^\times(\mathbb{F}_{q^{n_0}})$, which corresponds to the maximally split unramified torus in G_h .

In Sect. 8, we also give a character formula for $R_{T_h}^{G_h}(\theta)$ on certain elements of T_h (Proposition 8.3) and give a geometric interpretation of determinant-twisting on the cohomology groups (Lemma 8.4). Keeping in mind the remarks in the preceding paragraph, the methods in Sect. 8 primarily use the Coxeter-type representative b (Sect. 5.2).

In Sect. 9, we prove an analogue of a cuspidality result for $R_{T_h}^{G_h}(\theta)$ when θ is primitive (Theorem 9.1). To do this, we perform a character calculation using the geometry of X_h . Our approach is a (far-reaching) generalization of the proof in [21] in the special case $G = \mathrm{GL}_2(K)$. We use the special representative b (Sect. 5.2) as F -stable parahoric subgroups are more well-behaved for this choice. We note that although there is no notion of cuspidality for G_h -representations, we will see later that Theorem 9.1 implies the supercuspidality of the corresponding G -representation (Theorem 12.5).

8 Deligne–Lusztig varieties for Moy–Prasad quotients for GL_n

We say that a character $\theta: T_h \cong \mathbb{W}_h^\times(\mathbb{F}_{q^n}) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is *primitive* if the restriction of θ to $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ does not factor through any nontrivial norm maps $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \rightarrow \mathbb{W}_h^{h-1}(\mathbb{F}_{q^r})$ for $r \mid n, r < n$.

8.1 Irreducibility of $R_{T_h}^{G_h}(\theta)$

Theorem 8.1 *Let $\theta, \theta': T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be two characters and assume θ is primitive. Then*

$$\left\langle R_{T_h}^{G_h}(\theta), R_{T_h}^{G_h}(\theta') \right\rangle_{G_h} = \begin{cases} 1 & \text{if } \theta = \theta', \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the virtual G_h -representation $R_{T_h}^{G_h}(\theta)$ is (up to a sign) irreducible.

Let $\mathbb{U}_h, \mathbb{U}_h^- \subseteq \mathbb{G}_h$ be as in Sect. 7.7. Put

$$S_h := \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h^-\}.$$

This has an action of $G_h \times T_h$ by $(g, t): x \mapsto gxt$. Recalling from Proposition 7.12 that $X_h = \{x \in \mathbb{G}_h : x^{-1}F(x) \in \mathbb{U}_h^-\}/(\mathbb{U}_h^- \cap F(\mathbb{U}_h^-))$, we immediately have the following lemma.

Lemma 8.2 *The morphism $X_h \times (\mathbb{U}_h^- \cap F\mathbb{U}_h^-) \xrightarrow{\sim} S_h$ given by $(x, h) \mapsto xh$ is a $(G_h \times T_h)$ -equivariant isomorphism, where the action on the left-hand side is given by $(g, t): (x, h) \mapsto (gxt, t^{-1}ht)$. As $\mathbb{U}_h^- \cap F\mathbb{U}_h^-$ is isomorphic to an affine space, for any character θ of T_h , we have $R_{T_h}^{G_h}(\theta) = \sum_i (-1)^i H_c^i(S_h, \overline{\mathbb{Q}}_\ell)[\theta]$ as virtual G_h -representations.*

We show how to reduce Theorem 8.1 to a calculation of the cohomology of

$$\Sigma := \{(x, x', y) \in \mathbb{U}_h^- \times \mathbb{U}_h^- \times \mathbb{G}_h : xF(y) = yx'\},$$

and postpone the study of Σ to Sect. 8.4. Taking for granted Proposition 8.7, we give the proof of the main theorem:

Proof of Theorem 8.1 Let F be the twisted Frobenius given by the Coxeter-type representative b of Sect. 5.2. Consider the action of $G_h \times T_h \times T_h$ on $S_h \times S_h$ given by $(g, t_1, t_2): (x_1, x_2) \mapsto (gx_1t_1, gx_2t_2)$. The map

$$(g, g') \mapsto (x, x', y), x = g^{-1}F(g), x' = g'^{-1}F(g'), y = g^{-1}g'$$

defines an $T_h \times T_h$ -equivariant isomorphism $G_h \backslash S_h \times S_h \cong \Sigma$. We denote by $H_c^i(S_h \times S_h)_{\theta^{-1}, \theta'}$ and $H_c^i(\Sigma)_{\theta^{-1}, \theta'}$ the subspace where $T_h \times T_h$ acts by $\theta^{-1} \otimes \theta'$. We have

$$\begin{aligned} \langle R_{T_h}^{G_h}(\theta), R_{T_h}^{G_h}(\theta') \rangle_{G_h} &= \sum_{i, i' \in \mathbb{Z}} (-1)^{i+i'} \dim(H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H_c^{i'}(X_h, \overline{\mathbb{Q}}_\ell)[\theta'])^{G_h} \\ &= \sum_{i, i' \in \mathbb{Z}} (-1)^{i+i'} \dim(H_c^i(S_h, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H_c^{i'}(S_h, \overline{\mathbb{Q}}_\ell)[\theta'])^{G_h} \quad (\text{by Lemma 8.2}) \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(G_h \backslash (S_h \times S_h), \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} \\ &= \sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\Sigma, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} \\ &= \#\{\gamma \in \text{Gal}(L/K) : \theta \circ \gamma = \theta'\} \quad (\text{by Proposition 8.7}) \end{aligned}$$

where in the final equality, we use the fact that θ is primitive if and only if θ is regular in the sense of Lusztig [26, 1.5] with respect to the F coming from the Coxeter-type representative b . Finally, since the primitivity of θ implies that the stabilizer of θ in $\text{Gal}(L/K)$ is trivial, the desired conclusion of Theorem 8.1 now follows. $\square$

8.2 Traces of very regular elements

In Part 4, where we study $R_{T_h}^{G_h}(\theta)$ from the perspective of automorphic induction, we will need to know the trace of *very regular elements* of $\mathcal{O}_L^\times$; i.e. elements $x \in \mathcal{O}_L^\times$ whose image in the residue field generates the multiplicative group $\mathbb{F}_{q^n}^\times$. In fact, we can explicate the character on elements of $\mathcal{O}_L^\times$ whose image in the residue field has trivial $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ -stabilizer.

Proposition 8.3 *Let $\theta: T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be any character. Then for any element $x \in \mathcal{O}_L^\times/U_L^h \cong T_h$ in G_h whose image in the residue field has trivial $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ -stabilizer,*

$$\text{Tr}\left(x^*; R_{T_h}^{G_h}(\theta)\right) = \sum_{\gamma \in \text{Gal}(L/K)[n']} \theta^\gamma(x),$$

where $\text{Gal}(L/K)[n']$ is the unique order n' subgroup of $\text{Gal}(L/K)$.

Proof Let $\zeta_1, \zeta_2 \in T_h$ be $(q^n - 1)$ th roots of unity, let $t_1, t_2 \in T_h^1$, and assume that the image of ζ_1 modulo ϖ has trivial $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ -stabilizer. Note that $(\zeta_1 t_1, \zeta_2 t_2) \in G_h \times T_h$ and therefore acts on X_h . By Proposition 7.4, X_h is a separated, finite-type scheme over $\mathbb{F}_{q^n}$. Since $(\zeta_1, \zeta_2) = (\zeta_1 t_1, \zeta_2 t_2)^{q^{n(h-1)}}$ has order prime-to- p and $(t_1, t_2) = (\zeta_1 t_1, \zeta_2 t_2)^N$ (where $N \equiv 1 \pmod{q^{n(h-1)}}$ and $(q^n - 1) \mid N$) has order a power of p , by the Deligne–Lusztig fixed-point formula [13, Theorem 3.2],

$$\sum_i (-1)^i \text{Tr}\left((\zeta_1 t_1, \zeta_2 t_2)^*; H_c^i(X_h, \overline{\mathbb{Q}}_\ell)\right) = \sum_i (-1)^i \text{Tr}\left((t_1, t_2)^*; H_c^i(X_h^{(\zeta_1, \zeta_2)}, \overline{\mathbb{Q}}_\ell)\right).$$

By definition, if $\lambda(x) \in X_h$ corresponds to $x = (x_1, \dots, x_n) \in \mathcal{L}_{0,b}^{\text{adm, rat}} / \mathcal{L}_0^{(h)}$, then $(\zeta_1, \zeta_2)x$ corresponds to the tuple $(\zeta_1 \zeta_2 x_1, \sigma^l(\zeta_1) \zeta_2 x_2, \dots, \sigma^{(n-1)l}(\zeta_1) \zeta_2 x_n)$, where l is the inverse of $e_{\kappa, n} \pmod{n}$. In particular, we see that if ζ_1 has trivial stabilizer in $\text{Gal}(\mathbb{F}_{q^n} / \mathbb{F}_q)$, then the set $X_h^{(\zeta_1, \zeta_2)}$ is nonzero if and only if ζ_2^{-1} is one of the n distinct elements $\zeta_1, \sigma(\zeta_1), \dots, \sigma^{n-1}(\zeta_1)$.

Assume $\zeta_2^{-1} = \sigma^{jl}(\zeta_1)$ with $0 \leq j \leq n-1$, then the elements of $X_h^{(\zeta_1, \zeta_2)}$ correspond to vectors of the shape $x = (0, \dots, 0, x_{j+1}, 0, \dots, 0)$. If n_0 does not divide j , then $\det \lambda(x) \equiv 0 \pmod{\varpi}$, which contradicts $\det \lambda(x) \in \mathcal{O}_K^\times$. Thus in this case we have $X_h^{(\zeta_1, \zeta_2)} = \emptyset$. Assume n_0 divides j . Then $x = (0, \dots, 0, x_{j+1}, 0, \dots, 0)$ with $x_{j+1} \in \mathbb{W}_h(\overline{\mathbb{F}}_q)$ lies in X_h if and only if $\det \lambda(x) = \prod_{i=0}^{n-1} \sigma^i(x_{j+1}) \in (\mathcal{O}_K / \varpi^h)^\times$. Thus $X_h^{(\zeta_1, \zeta_2)} = \{x = (0, \dots, 0, x_{j+1}, 0, \dots, 0) : x_{j+1} \in (\mathcal{O}_L / \varpi^h)^\times = T_h\}$ is zero-dimensional, and the action of (t_1, t_2) is given by $x_{j+1} \mapsto \sigma^{jl}(t_1)t_2 x_{j+1}$. Thus

$$\text{Tr}((t_1, t_2)^*, H_c^0(X_h^{(\zeta_1, \zeta_2)})) = \begin{cases} \#T_h & \text{if } t_2 = \sigma^{jl}(t_1)^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

From this, we see that

$$\begin{aligned} \text{Tr}((\zeta_1 t_1, 1)^*, R_{T_h}^{G_h}(\theta)) &= \frac{1}{\#T_h} \sum_{\zeta_2 \in \mathbb{F}_{q^n}^\times} \sum_{t_2 \in T_h^1} \theta(\zeta_2)^{-1} \theta(t_2)^{-1} \text{Tr}((t_1, t_2)^*, H_c^0(X_h^{(\zeta_1, \zeta_2)}, \overline{\mathbb{Q}}_\ell)) \\ &= \sum_{\substack{0 \leq j \leq n-1 \\ n_0 | j}} \theta(\sigma^{jl}(\zeta_1)) \theta(\sigma^{jl}(t_1)) = \sum_{\gamma \in \text{Gal}(L/K)[n']} \theta^\gamma(\zeta_1 t_1). \end{aligned}$$

□

8.3 Behavior under twisting of θ

Lemma 8.4 *Let $\theta : T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character with trivial $\text{Gal}(L/K)$ -stabilizer and let $\chi : \mathbb{W}_h^\times(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be any character. Then as G_h -representations,*

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta \otimes (\chi \circ \text{Nm})] \cong H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta] \otimes (\chi \circ \det), \quad \text{for all } i \geq 0.$$

Proof Let Γ_h denote the kernel of the natural homomorphism $G_h \times T_h \rightarrow \mathbb{W}_h^\times(\mathbb{F}_q)$ given by $(g, t) \mapsto \det(t) \text{Nm}(t)$. Recall from (7.3) that we have a scheme-theoretic morphism $X_h \rightarrow \mathbb{W}_h^\times(\mathbb{F}_q)$. Write $X_h^{\det=1}$ for the preimage of the identity. First observe that as $G_h \times T_h$ -representations,

$$\bigoplus_{\substack{\theta' : T_h \rightarrow \overline{\mathbb{Q}}_\ell \\ \theta'|_{T_h^\circ} = \theta|_{T_h^\circ}}} H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta'] \cong \text{Ind}_{\Gamma_h}^{G_h \times T_h} \left(H_c^i(X_h^{\det=1}, \overline{\mathbb{Q}}_\ell)[\theta|_{T_h^\circ}] \right).$$

Since the number of summands on the left-hand side is equal to the index of Γ_h in $G_h \times T_h$, it follows that as representations of Γ_h ,

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta'] \cong H_c^i(X_h^{\det=1}, \overline{\mathbb{Q}}_\ell)[\theta|_{T_h^\circ}] \quad (8.1)$$

for any $\theta': T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$ with $\theta'|_{T_h^\circ} = \theta|_{T_h^\circ}$. In particular, as Γ_h -representations,

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta \otimes (\chi \circ \text{Nm})] \cong H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta].$$

Now observe that the subgroup of $G_h \times T_h$ generated by Γ_h and $1 \times T_h$ is the whole group. For any $g \in G_h$, let $t_g \in T_h$ be any element such that $\det(g) \text{Nm}(t_g) = 1$. Then $(g, t_g) \in \Gamma_h$, and we have

$$\begin{aligned} & \text{Tr}((g, 1)^*; H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta \otimes (\chi \circ \text{Nm})]) \\ &= \text{Tr}((g, t_g)^*; H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta \otimes (\chi \circ \text{Nm})]) \cdot \theta(t_g^{-1}) \cdot \chi(\text{Nm}(t_g^{-1})) \\ &= \text{Tr}((g, t_g)^*; H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta]) \cdot \theta(t_g^{-1}) \cdot \chi(\text{Nm}(t_g^{-1})) \\ &= \text{Tr}((g, 1)^*; H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta]) \cdot \theta(t_g) \cdot \theta(t_g^{-1}) \cdot \chi(\text{Nm}(t_g^{-1})) \\ &= \text{Tr}((g, 1)^*; H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta]) \cdot \chi(\det(g)). \end{aligned}$$

□

Observe that by Lemma 8.4, we have that $R_{T_h}^{G_h}(\theta)$ is (up to sign) irreducible if and only if $R_{T_h}^{G_h}(\theta \otimes (\chi \circ \text{Nm}))$ is, where $\chi: \mathbb{W}_h(\mathbb{F}_q)^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Recall that by Proposition 7.7, if θ is a character of T_h that factors through the natural surjection $T_h \rightarrow T_{h'}$ for some $h' < h$, then $R_{T_h}^{G_h}(\theta) = R_{T_{h'}}^{G_{h'}}(\theta)$. Thus we can strengthen Theorem 8.1 to obtain that $R_{T_h}^{G_h}(\theta \otimes (\chi \circ \text{Nm}))$ is (up to sign) irreducible for any primitive $\theta: T_{h'} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ and any $\chi: \mathbb{W}_h(\mathbb{F}_q)^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Such characters exactly correspond to *minimal admissible* characters of $L^\times$ of level h (see Part 4). This argument will be appear again in the proof of Theorem 12.5.

8.4 Lusztig's theorem

This is a generalization of [26,31] to non-reductive groups over $\mathcal{O}$. The Iwahori case (which corresponds to the division algebra setting over K) was done in [25] (see also [12, Section 6.2]) and is a simpler incarnation of these ideas. We keep our notation as close as possible to that of [26,31] as most of the arguments are the same.

8.4.1 Set-up

Let T, T' be two maximal F -stable tori of J_b , split over $\check{K}$ and let (U, U^-) and (U', U'^-) be two pairs of (possibly not F -stable) unipotent radicals of opposite Borels containing T and T' , respectively. (Outside Sect. 8.4 T always denotes a maximal

elliptic torus of G , but here we want the notation to coincide with [26]). Consider the intersections of $\check{K}$ -points of T, T', U, U^-, U', U'^- with $\check{G}_{\mathbf{x},0}$ (Sect. 5.3) and denote the corresponding subgroup schemes in $\mathbb{G}_h$ by $\mathbb{T}_h, \mathbb{T}'_h, \mathbb{U}_h, \mathbb{U}_h^-, \mathbb{U}'_h, \mathbb{U}'_h^-$. For $1 \leq a \leq h$, let $\mathbb{G}_h^a := \ker(\mathbb{G}_h \rightarrow \mathbb{G}_a)$ be the kernel of the natural projection, and analogously define $\mathbb{T}_h^a, \mathbb{U}_h^a$, and so forth. We set $\mathbb{G}_h^{a,*} = \mathbb{G}_h^a \setminus \mathbb{G}_h^{a+1}$, and analogously for $\mathbb{T}_h^{a,*}, \mathbb{U}_h^{a,*}$, and so forth. We use the shorthand $\mathcal{T} := \mathbb{T}_h^{h-1}$.

Let $N(\mathbb{T}, \mathbb{T}') = \{g \in \check{G}_{\mathbf{x},0} : g^{-1}\mathbb{T}g = \mathbb{T}'\}$ and $N(\mathbb{T}_h, \mathbb{T}'_h) = \{g \in \mathbb{G}_h : g^{-1}\mathbb{T}_h g = \mathbb{T}'_h\}$, and define

$$W(T, T') := \mathbb{T} \backslash N(\mathbb{T}, \mathbb{T}') = \mathbb{T}_h \backslash N(\mathbb{T}_h, \mathbb{T}'_h).$$

Observe that $W(T, T')$ is a principal homogeneous space under the Weyl group of the torus $\mathbb{T}_1$ in the reductive quotient $\mathbb{G}_1$ of $G_{\mathcal{O}}$.

8.4.2 Roots and regularity

Let $\Phi = \Phi(T, J_b)$ denote the set of roots of T in J_b . It carries a natural action of F . For $\alpha \in \Phi$, let $\mathbb{G}_h^\alpha$ denote the subgroup of $\mathbb{G}_h$ coming from the root subgroup of $J_b(\check{K}) = \mathrm{GL}_n(\check{K})$ corresponding to α . For $\alpha \in \Phi$, let $T^\alpha \subseteq T$ be the image of the coroot of T in $\mathrm{GL}_n(\check{K})$ corresponding to α . It is an one-dimensional subtorus of T . We denote by $\mathbb{T}_h^\alpha$ the corresponding subgroup of $\mathbb{G}_h$. We write $T^\alpha \subseteq \mathcal{T}$ for the one-dimensional subgroup $(\mathbb{T}_h^\alpha)^{h-1}$ of $\mathbb{T}_h^\alpha$.

Following [26, 1.5], a character $\chi : \mathcal{T}^F \rightarrow \overline{\mathbb{Q}_\ell}^\times$ is called *regular* if for any $\alpha \in \Phi$ and any $m \geq 1$ such that $F^m(T^\alpha) = T^\alpha$, the restriction of $\chi \circ N_F^{F^m} : \mathcal{T}^{F^m} \rightarrow \overline{\mathbb{Q}_\ell}^\times$ to $(T^\alpha)^{F^m}$ is non-trivial. Here, $N_F^{F^m} : \mathcal{T}^{F^m} \rightarrow \mathcal{T}^F$ is the map $t \mapsto tF(t) \cdots F^{m-1}(t)$. A character χ of $\mathbb{T}_h^F$ is called *regular* if its restriction $\chi|_{\mathcal{T}^F}$ is regular.

Remark 8.5 In our situation, when b is a Coxeter-type representative and T is the elliptic diagonal torus of J_b , let χ be a character of $T(K) \cong L^\times$ of level h . Then the restriction of χ to $\mathcal{O}_L^\times$ can be viewed as a character χ_h of $\mathbb{T}_h^F \cong (\mathcal{O}_L/\varpi^h)^\times$. A straightforward computation shows: χ_h is regular in the above sense if and only if it is *primitive*, i.e. the restriction of χ_h to $\mathcal{T}^F \cong \mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})$ does not factor through any of the norm maps $\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n}) \rightarrow \mathbb{W}_h^{h-1}(\mathbb{F}_{q^r})$ for $r \mid n, r < n$. We use this in the proof of Theorem 8.1. $\diamond$

8.4.3 Bruhat decomposition

For each $w \in W(T, T')$ choose a representative $\dot{w} \in N(T, T')$. We have the Bruhat decomposition $\mathbb{G}_1 = \bigsqcup_{w \in W(T, T')} \mathbb{G}_{1,w}$ of the reductive quotient, where $\mathbb{G}_{1,w} = \mathbb{U}_1 \dot{w} \mathbb{T}_1 \mathbb{U}'_1$. Define $\mathbb{G}_{h,w}$ to be the pullback of $\mathbb{G}_{1,w}$ along the natural projection $\mathbb{G}_h \rightarrow \mathbb{G}_1$. Thus $\mathbb{G}_h = \bigsqcup_{w \in W(T, T')} \mathbb{G}_{h,w}$. Let $\mathbb{K}_h := \mathbb{U}_h^- \cap \dot{w} \mathbb{U}'_h \dot{w}^{-1}$ and $\mathbb{K}_h^1 := \mathbb{K}_h \cap \mathbb{G}_h^1$.

Lemma 8.6 $\mathbb{G}_{h,w} = \mathbb{U}_h \mathbb{K}_h^1 \dot{w} \mathbb{T}'_h \mathbb{U}'_h$.

Proof Indeed, we compute

$$\begin{aligned}\mathbb{G}_{h,w} &= \mathbb{U}_h \dot{w} \mathbb{T}'_h \mathbb{G}_h^1 \mathbb{U}'_h = \mathbb{U}_h \dot{w} \mathbb{T}'_h \left((\mathbb{G}_h^1 \cap \mathbb{T}'_h) (\mathbb{G}_h^1 \cap \mathbb{U}'_h) (\mathbb{G}_h^1 \cap \mathbb{U}'_h) \right) \mathbb{U}'_h \\ &= \mathbb{U}_h \dot{w} \mathbb{T}'_h (\mathbb{G}_h^1 \cap \mathbb{U}'_h) \mathbb{U}'_h = \mathbb{U}_h \left(\dot{w} (\mathbb{G}_h^1 \cap \mathbb{U}'_h) \dot{w}^{-1} \right) \dot{w} \mathbb{T}'_h \mathbb{U}'_h \\ &= \mathbb{U}_h \left(\mathbb{U}_h^- \cap \dot{w} (\mathbb{G}_h^1 \cap \mathbb{U}'_h) \dot{w}^{-1} \right) \dot{w} \mathbb{T}'_h \mathbb{U}'_h = \mathbb{U}_h \mathbb{K}_h^1 \dot{w} \mathbb{T}'_h \mathbb{U}'_h.\end{aligned}$$

□

8.4.4 The scheme Σ

Define

$$\begin{aligned}\Sigma &= \{(x, x', y) \in F(\mathbb{U}_h) \times F(\mathbb{U}'_h) \times \mathbb{G}_h : xF(y) = yx'\} \\ \Sigma_w &= \{(x, x', y) \in F(\mathbb{U}_h) \times F(\mathbb{U}'_h) \times \mathbb{G}_h : xF(y) = yx', y \in \mathbb{G}_{h,w}\} \subseteq \Sigma,\end{aligned}$$

for $w \in W(T, T')$. Set-theoretically, Σ is the disjoint union of the locally closed subschemes Σ_w . The group $\mathbb{T}_h^F \times \mathbb{T}_h'^F$ acts on Σ by $(t, t') : (x, x', y) \mapsto (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$ and Σ_w is stable under this action for any $w \in W(T, T')$.

Proposition 8.7 *Let θ and θ' be characters of $\mathbb{T}_h^F$ and $\mathbb{T}_h'^F$ respectively and assume that θ is regular. Then*

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\Sigma, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} = \#\{w \in W(T, T')^F : \theta \circ \text{Ad}(\dot{w}) = \theta'\}.$$

Proof Using $\Sigma = \bigcup_w \Sigma_w$, it is enough to show that $\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\Sigma, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'}$ is 1 if $w \in W(T, T')^F$ and $\theta \circ \text{Ad}(\dot{w}) = \theta'$, and is 0 otherwise. Fix a $w \in W(T, T')$. Let

$$\begin{aligned}\widehat{\Sigma}_w &= \{(x, x', u, u', z, \tau') \in F(\mathbb{U}_h) \times F(\mathbb{U}'_h) \times \mathbb{U}_h \times \mathbb{U}'_h \times \mathbb{K}_h^1 \times \mathbb{T}'_h : \\ &\quad xF(uz\dot{w}\tau'u') = uz\dot{w}\tau'u'x'\}.\end{aligned}$$

We have the morphism $\widehat{\Sigma}_w \rightarrow \Sigma_w$, $(x, x', u, u', z, \tau') \mapsto (x, x', uz\dot{w}\tau'u')$, which by Lemma 8.6 is surjective. Moreover, this map is $\mathbb{T}_h^F \times \mathbb{T}_h'^F$ -equivariant, when we endow $\widehat{\Sigma}_w$ with the $\mathbb{T}_h^F \times \mathbb{T}_h'^F$ -action

$$(t, t') : (x, x', u, u', z, \tau') \mapsto (txt^{-1}, t'x't'^{-1}, tut^{-1}, t'u't'^{-1}, tz\tau^{-1}, \dot{w}^{-1}t\dot{w}\tau't'^{-1}). \quad (8.2)$$

As the projection $\widehat{\Sigma}_w \rightarrow \Sigma_w$ is locally trivial fibration, the cohomology does not change if we pass from Σ_w to $\widehat{\Sigma}_w$. Thus to finish the proof the proposition it is enough

to show that

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\widehat{\Sigma}, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} = \begin{cases} 1 & \text{if } w \in W(T, T')^F \text{ and } \theta \circ \text{Ad}(\dot{w}) = \theta', \\ 0 & \text{otherwise.} \end{cases} \quad (8.3)$$

We make the change of variables replacing $xF(u)$ by x and $x'F(u')^{-1}$ by x' , and rewrite $\widehat{\Sigma}_w$ as

$$\widehat{\Sigma}_w = \{(x, x', u, u', z, \tau') \in F(\mathbb{U}_h) \times F(\mathbb{U}'_h) \times \mathbb{U}_h \times \mathbb{U}'_h \times \mathbb{K}_h^1 \times \mathbb{T}'_h : xF(z\dot{w}\tau') = uz\dot{w}\tau'u'x'\},$$

and the torus action is still given by (8.2). Define a partition $\widehat{\Sigma}_w = \widehat{\Sigma}'_w \sqcup \widehat{\Sigma}''_w$ by

$$\begin{aligned} \widehat{\Sigma}'_w &= \{(x, x', u, u', z, \tau') \in \widehat{\Sigma}_w : z \neq 1\}, \\ \widehat{\Sigma}''_w &= \{(x, x', u, u', z, \tau') \in \widehat{\Sigma}_w : z = 1\}. \end{aligned}$$

Both subsets are stable under the $\mathbb{T}_h^F \times \mathbb{T}'_h{}^F$ -action. By Sect. 8.4.5,

$$\begin{aligned} \sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\widehat{\Sigma}''_w, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} \\ = \begin{cases} 1 & \text{if } w \in W(T, T')^F \text{ and } \theta \circ \text{Ad}(\dot{w}) = \theta', \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (8.4)$$

and by Sect. 8.4.8, under the assumption that θ is regular,

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\widehat{\Sigma}'_w, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} = 0, \quad (8.5)$$

so (8.3) holds. $\square$

8.4.5 Cohomology of $\widehat{\Sigma}''_w$

We prove (8.4). This works exactly as in [26] (see the proof of Lemma 1.9, specifically the proof of claim (b) in *op. cit.* beginning on page 8). For convenience of the reader, we recall the arguments. Consider the closed subgroup

$$\widetilde{H} = \{(t, t') \in \mathbb{T}_h \times \mathbb{T}'_h : tF(t)^{-1} = F(\dot{w})t'F(t')^{-1}F(\dot{w}^{-1})\} \subseteq \mathbb{T}_h \times \mathbb{T}'_h.$$

Note that $\widetilde{H}$ contains $\mathbb{T}_h^F \times \mathbb{T}'_h{}^F$ and (8.2) containing $\mathbb{T}_h^F \times \mathbb{T}'_h{}^F$. The action of $\mathbb{T}_h^F \times \mathbb{T}'_h{}^F$ on $\widehat{\Sigma}''_w$ extends to an action of $\widetilde{H}$, still given by (8.2). Let $\mathbb{T}_{h,*}$ and $\mathbb{T}'_{h,*}$ be the reductive part of $\mathbb{T}_h$ and $\mathbb{T}'_h$ respectively. Set $\widetilde{H}_* := \widetilde{H} \cap (\mathbb{T}_{h,*} \times \mathbb{T}'_{h,*})$ and let $\widetilde{H}_*^0$ be the connected component of $\widetilde{H}_*$. Then $\widetilde{H}_*^0$ is a torus acting on $\widehat{\Sigma}''_w$. By [14, 4.5 (and

11.2) and 10.15] (compare the similar computation in the proof of [31, Theorem 3.1]), we have

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i(\widehat{\Sigma}_w'', \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} = \sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i((\widehat{\Sigma}_w'')^{\widetilde{H}_*^0}, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'}.$$

Let $(x, x', u, u', 1, \tau') \in (\widehat{\Sigma}_w'')^{\widetilde{H}_*^0}$. By Lang's theorem, $\widetilde{H}_* \rightarrow \mathbb{T}_{h,*}$ is surjective and hence (as $\mathbb{T}_{h,*}$ is connected) also $\widetilde{H}_*^0 \rightarrow \mathbb{T}_{h,*}$ is surjective. Similarly, $\widetilde{H}_*^0 \rightarrow \mathbb{T}'_{h,*}$ is surjective. Thus for any $t \in \mathbb{T}_{h,*}$, $t' \in \mathbb{T}'_{h,*}$, we have

$$tx t^{-1} = x, \quad t'x't'^{-1} = x', \quad tut^{-1} = u, \quad t'u't'^{-1} = u'.$$

This implies $x = x' = u = u' = 1$ since $\mathbb{T}_{h,*}$ acts non-trivially on all affine roots subgroups contained in $\mathbb{U}_h$ (and similarly for $\mathbb{T}'_h, \mathbb{U}'_h$). Thus

$$(\widehat{\Sigma}_w'')^{\widetilde{H}_*^0} \subseteq \{(1, 1, 1, 1, 1, \tau') : \tau' \in \mathbb{T}'_h, F(\dot{w}\tau') = \dot{w}\tau'\},$$

and we deduce

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim H_c^i((\widehat{\Sigma}_w'')^{\widetilde{H}_*^0}, \overline{\mathbb{Q}}_\ell)_{\theta^{-1}, \theta'} = \begin{cases} 1 & \text{if } F(w) = w \text{ and } \theta \circ \text{Ad}(\dot{w}) = \theta', \\ 0 & \text{otherwise.} \end{cases}$$

8.4.6 Some preparations

In the next two sections, we make the necessary preparations in order to carry out Lusztig's argument for (8.5) in Sect. 8.4.8. Let N, N^- be unipotent radicals of opposite Borel subgroups of $J_b(\check{K}) = \text{GL}_n(\check{K})$ containing T , and for $h \geq 1$, let $\mathbb{N}_h, \mathbb{N}_h^-$ be the corresponding subgroups of $\mathbb{G}_h$. Let $\Phi^+ = \{\alpha \in \Phi : \mathbb{G}_h^\alpha \subseteq \mathbb{N}_h\}$ and $\Phi^- = \Phi \setminus \Phi^+ = \{\alpha \in \Phi : \mathbb{G}_h^\alpha \subseteq \mathbb{N}_h^-\}$. For $\alpha \in \Phi^+$ let $\text{ht}(\alpha)$ denote the largest integer $m \geq 1$, such that $\alpha = \sum_{i=1}^m \alpha_i$ with $\alpha_i \in \Phi^+$.

We call the roots $\alpha \in \Phi$ for which $\mathbb{G}_1^\alpha \neq 1$ *reductive* and the other roots *non-reductive*. Equivalently, a root $\alpha \in \Phi$ is reductive if and only if $\langle \alpha, \mathbf{x} \rangle \in \mathbb{Z}$, where $\mathbf{x}$ is as in Sect. 5.3.

To make explicit calculations, we may assume that T is the diagonal torus in $\text{GL}_n(\check{K})$. For $1 \leq i \neq j \leq n$, let $\alpha_{i,j}$ denote the root corresponding to the (i, j) th entry of an $n \times n$ matrix. For $1 \leq i \leq n$, let $1 \leq [i]_{n_0} \leq n_0$ denote its residue modulo n_0 . Define $\text{ht}_{n_0}(\alpha_{i,j}) := [i]_{n_0} - [j]_{n_0}$. Then $\alpha \in \Phi$ is reductive if and only if $\text{ht}_{n_0}(\alpha) = 0$. If $\text{ht}_{n_0}(\alpha) > 0$ (resp. $\text{ht}_{n_0}(\alpha) < 0$), we call α *non-reductive of type 1* (resp. *of type 2*). For any $\alpha = \alpha_{i,j} \in \Phi$ and $1 \leq a \leq h$, we have

$$(\mathbb{G}_h^\alpha)^a \cong \begin{cases} \mathfrak{p}^{a-1}/\mathfrak{p}^{h-1} & \text{if } \text{ht}_{n_0}(\alpha) > 0, \\ \mathfrak{p}^a/\mathfrak{p}^h & \text{if } \text{ht}_{n_0}(\alpha) \leq 0, \end{cases} \quad (8.6)$$

in the sense that $(\mathbb{G}_h^\alpha)^a$ consists of $n \times n$ matrices with 1's on the main diagonal, an element of the subgroup $\mathfrak{p}^{a-1}/\mathfrak{p}^{h-1}$ (resp. $\mathfrak{p}^a/\mathfrak{p}^h$) sitting in the (i, j) th entry, and 0's everywhere else.

Example 8.8 Let $n = 4$, $\kappa = 2$. Then if $\mathcal{A}$ is the apartment of $\mathcal{B}^{\text{red}}(\text{GL}_4, \check{K})$ corresponding to the diagonal torus, then $\mathbf{x}$ is the unique fixed point under the action of $b = b_0 \cdot \text{diag}(1, \varpi, 1, \varpi) = \begin{pmatrix} 1 & & & \\ & \varpi & & \\ & & 1 & \\ & & & \varpi \end{pmatrix}$. Computing, the matrix of inner products for $\alpha_{i,j} \in \Phi$ is

$$(\langle \alpha_{i,j}, \mathbf{x} \rangle)_{1 \leq i,j \leq 4} = \begin{pmatrix} * & -\frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & * & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & * & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & * \end{pmatrix}.$$

Hence for $h \geq 1$, we have

$$\check{G}_{\mathbf{x},0} = \begin{pmatrix} \mathcal{O} & \mathfrak{p} & \mathcal{O} & \mathfrak{p} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathfrak{p} & \mathcal{O} & \mathfrak{p} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}^\times \rightarrow \mathbb{G}_h(\overline{\mathbb{F}}_q) = \begin{pmatrix} \mathcal{O}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h & \mathcal{O}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h \\ \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^h & \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^h \\ \mathcal{O}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h & \mathcal{O}/\mathfrak{p}^h & \mathfrak{p}/\mathfrak{p}^h \\ \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^h & \mathcal{O}/\mathfrak{p}^{h-1} & \mathcal{O}/\mathfrak{p}^h \end{pmatrix}^\times,$$

where the $\times$ superscript means the group of invertible matrices, and for $1 \leq a \leq h$,

$$\mathbb{G}_h^a(\overline{\mathbb{F}}_q) = \begin{pmatrix} 1+\mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^a/\mathfrak{p}^h \\ \mathfrak{p}^{a-1}/\mathfrak{p}^{h-1} & 1+\mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^{a-1}/\mathfrak{p}^{h-1} & \mathfrak{p}^a/\mathfrak{p}^h \\ \mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^a/\mathfrak{p}^h & 1+\mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^a/\mathfrak{p}^h \\ \mathfrak{p}^{a-1}/\mathfrak{p}^{h-1} & \mathfrak{p}^a/\mathfrak{p}^h & \mathfrak{p}^{a-1}/\mathfrak{p}^{h-1} & 1+\mathfrak{p}^a/\mathfrak{p}^h \end{pmatrix}.$$

For two elements $z, \xi \in \mathbb{G}_h$, we write $[\xi, z] = \xi^{-1}z^{-1}\xi z$.

Lemma 8.9 Let $\alpha \in \Phi$. Let $1 \leq a \leq h-1$.

- (i) If α is non-reductive, then $[\mathbb{G}_h^{a+1}, (\mathbb{G}_h^\alpha)^{h-a}] = 1$.
- (ii) If α is reductive, then $[\mathbb{G}_h^a, (\mathbb{G}_h^\alpha)^{h-a}] = 1$.

Proof The computation to show (i) and (ii) is nearly the same. We prove (i). It suffices to check that $[\mathbb{T}_h^{a+1}, (\mathbb{G}_h^\alpha)^{h-a}] = 1$ and that $[(\mathbb{G}_h^\beta)^{a+1}, (\mathbb{G}_h^\alpha)^{h-a}] = 1$ for any $\beta \in \Phi$. This is an immediate computation using the explicit description of $\mathbb{G}_h$ and (8.6). The only critical case is when α, β are both non-reductive of type 1. Here, it suffices to observe that if $\alpha + \beta$ is again a root, then it is again non-reductive of type 1. $\square$

Let $(\mathbb{N}_h^1)^{\leq 0}$ denote the subgroup of $\mathbb{N}_h^1$ generated by $\mathbb{N}_h^2$ and all $(\mathbb{G}_h^\beta)^1$ with $\beta \in \Phi^+$ satisfying $\text{ht}_{n_0}(\beta) \leq 0$. Obviously $\mathbb{N}_h^2 \subseteq (\mathbb{N}_h^1)^{\leq 0} \subseteq \mathbb{N}_h^1$.

Lemma 8.10 Let $1 \leq a \leq h-1$ and $z \in \mathbb{N}_h^{a,*}$. Write $z = \prod_{\beta \in \Phi^+} x_\beta^z$ with $x_\beta^z \in (\mathbb{G}_h^\beta)^a$ for a fixed (but arbitrary) order on Φ^+ . For $\beta \in \Phi^+$, let $a \leq a(\beta, z) \leq h$ be the integer such that $x_\beta^z \in (\mathbb{G}_h^\beta)^{a(\beta,z),*}$.

(i) If $z \in \mathbb{N}_h^{a,*} \cap (\mathbb{N}_h^1)^{\leq 0}$, then the set

$$A_z := \{\beta \in \Phi^+ : a(\beta, z) = a\}$$

is independent of the chosen order on Φ^+ .

(ii) If $z \in \mathbb{N}_h^1 \setminus (\mathbb{N}_h^1)^{\leq 0}$, then the set

$$A_z := \{\beta \in \Phi^+ : \text{ht}_{n_0}(\beta) \text{ is minimal among those with } \text{ht}_{n_0} > 0 \text{ and } a(\beta, z) = 1\}$$

does not depend on the chosen order on Φ^+ .

Proof (i) First let $2 \leq a \leq h-1$. From the explicit description of the root subgroups it follows that the quotient $\mathbb{N}_h^a/\mathbb{N}_h^{a+1}$ is abelian (for $a=2$ one needs to use that the sum of two non-reductive roots of type 1 is again of type 1 if it is a root), thus its elements are simply tuples $(x_\beta)_{\beta \in \Phi^+}$ with $x_\beta \in (\mathbb{G}_h^\beta)^a/(\mathbb{G}_h^\beta)^{a+1}$ with entry-wise multiplication. If $\bar{z} = (\bar{x}_\beta^z)$ is the image of z in this quotient, then A_z identifies with the set of those β for which $\bar{x}_\beta^z \neq 1$ (which is obviously independent of the order). Now let $a=1$. Then $z \in (\mathbb{N}_h^1)^{\leq 0} \setminus \mathbb{N}_h^2$ and the same arguments apply to the abelian quotient $(\mathbb{N}_h^1)^{\leq 0}/\mathbb{N}_h^2$.

(ii) The group $\mathbb{N}_h^1/(\mathbb{N}_h^1)^{\leq 0}$ is not abelian, but is generated by its subgroups $(\mathbb{G}_h^\beta)^1/(\mathbb{G}_h^\beta)^2$ for $\beta \in \Phi^+$ non-reductive of type 1. For $m \geq 1$, let H_m be the subgroup generated by all $(\mathbb{G}_h^\beta)^1/(\mathbb{G}_h^\beta)^2$ with $\text{ht}_{n_0}(\beta) \geq m$. Since the function ht_{n_0} is additive on Φ , the H_m form a filtration of $\mathbb{N}_h^1/(\mathbb{N}_h^1)^{\leq 0} = H_1$ with abelian quotients $H_m/H_{m+1} \cong \prod_{\substack{\beta \text{ non-red. type 1} \\ \text{ht}_{n_0}(\beta)=m}} (\mathbb{G}_h^\beta)^1/(\mathbb{G}_h^\beta)^2$. Since $z \notin (\mathbb{N}_h^1)^{\leq 0}$, there is an $m \geq 1$ such

that the image of z in $\mathbb{N}_h^1/(\mathbb{N}_h^1)^{\leq 0}$ lies in $H_m \setminus H_{m+1}$. Denote by $\bar{z} = (\bar{x}_\beta^z)_{\substack{\beta \text{ non-red. type 1} \\ \text{ht}_{n_0}(\beta)=m}}$ the image of z in H_m/H_{m+1} . Now A_z is the set of all $\beta \in \Phi^+$ non-reductive of type 1 with $\text{ht}_{n_0}(\beta) = m$ such that $\bar{x}_\beta^z \neq 1$. This does not depend on the chosen order. $\square$

8.4.7 Stratification of $\mathbb{K}_h^1$

Lemma 8.11 Let $1 \leq a \leq h-1$, $z \in \mathbb{N}_h^{a,*}$ and A_z as in Lemma 8.10.

(i) If A_z contains a non-reductive root, let $-\alpha \in A_z$ be a non-reductive root of maximal height and $\alpha \in \Phi^-$ its opposite. Then for any $\xi \in (\mathbb{G}_h^\alpha)^{h-a}$, we have $[\xi, z] \in \mathcal{T}^\alpha(\mathbb{N}_h^-)^{h-1}$. Moreover, projecting $[\xi, z]$ into $\mathcal{T}^\alpha$ induces an isomorphism

$$\lambda_z : (\mathbb{G}_h^\alpha)^{h-a}/(\mathbb{G}_h^\alpha)^{h-a+1} \xrightarrow{\sim} \mathcal{T}^\alpha$$

(ii) If A_z contains only reductive roots, let $-\alpha \in A_z$ be a root of maximal height and $\alpha \in \Phi^-$ its opposite. Then for any $\xi \in (\mathbb{G}_h^\alpha)^{h-a-1}$, we have $[\xi, z] \in \mathcal{T}^\alpha(\mathbb{N}_h^-)^{h-1}$. Moreover, projecting $[\xi, z]$ into $\mathcal{T}^\alpha$ induces an isomorphism

$$\lambda_z : (\mathbb{G}_h^\alpha)^{h-a-1}/(\mathbb{G}_h^\alpha)^{h-a} \xrightarrow{\sim} \mathcal{T}^\alpha$$

Proof We first prove (i) when $z \in (\mathbb{N}_h^1)^{\leq 0}$. Assume first that A_z contains a non-reductive root and let $-\alpha$ be such a root of maximal height and $\alpha \in \Phi^-$ its opposite. By Lemma 8.9 (applied three times), the commutator map $\mathbb{N}_h^a \times (\mathbb{G}_h^\alpha)^{h-a} \rightarrow \mathbb{G}_h$ induces a pairing of abelian groups,

$$\mathbb{N}_h^a / \mathbb{N}_h^{a+1} \times (\mathbb{G}_h^\alpha)^{h-a} / (\mathbb{G}_h^\alpha)^{h-a+1} \rightarrow \mathbb{G}_h^{h-1}, \quad \bar{x}, \bar{\xi} \mapsto [\bar{\xi}, \bar{x}].$$

(If $a = 1$, one has to replace $\mathbb{N}_h^a / \mathbb{N}_h^{a+1}$ by $(\mathbb{N}_h^1)^{\leq 0} / \mathbb{N}_h^2$.) This is bilinear in $\bar{x}$: if $x_1, x_2 \in \mathbb{N}_h^a$, then

$$\begin{aligned} [\bar{\xi}, x_1 x_2] &= \xi^{-1} x_2^{-1} x_1^{-1} \xi x_1 x_2 = \xi^{-1} x_1^{-1} x_2^{-1} \xi x_2 x_1 \\ &= \xi^{-1} x_1^{-1} \xi [\bar{\xi}, x_2] x_1 = [\bar{\xi}, x_1] [\bar{\xi}, x_2], \end{aligned}$$

where the second equality follows from Lemma 8.9 and $\mathbb{N}_h^a / \mathbb{N}_h^{a+1}$ (resp. $(\mathbb{N}_h^1)^{\leq 0} / \mathbb{N}_h^2$ if $a = 1$) being abelian, and the fourth follows from Lemma 8.9 as $[\bar{\xi}, x_2] \in \mathbb{G}_h^{h-1}$.

Now let $\bar{\xi} \in (\mathbb{G}_h^\alpha)^{h-a} / (\mathbb{G}_h^\alpha)^{h-a+1}$ and $\bar{z} \in \mathbb{N}_h^a / \mathbb{N}_h^{a+1}$ be the images of ξ and z respectively. Write

$$\bar{z} = \bar{x}_{-\alpha}^z \prod_{\beta \in \Phi^+ \text{ red.}} \bar{x}_\beta^z \cdot \prod_{\substack{\beta \in \Phi^+ \text{ non-red., } \beta \neq -\alpha \\ \text{ht}(\beta) \leq \text{ht}(-\alpha)}} \bar{x}_\beta^z.$$

Then $[\bar{\xi}, \bar{z}]$ is the product of $[\bar{\xi}, \bar{x}_{-\alpha}]$ with all the $[\bar{\xi}, \bar{x}_\beta^z]$ in any order. Let $\bar{x}_\beta^z$ be any lift of $\bar{x}_\beta^z$ to $(\mathbb{G}_h^\beta)^a$. If β is reductive and α is (non-reductive) of type 1, then either ξ, x_β^z commute anyway or $\alpha + \beta$ is again a root (necessarily non-reductive of type 1) and (8.6) shows that $[\bar{\xi}, x_\beta^z] = 1$. If β is reductive and α is (non-reductive) of type 2, then (8.6) shows that ξ, x_β^z commute. If $\beta \neq -\alpha$ is non-reductive, then by assumption $\text{ht}(\beta) \leq \text{ht}(-\alpha)$. Then $[\bar{\xi}, x_\beta^z] = 1$ unless $\alpha + \beta$ is a root, in which case $[\bar{\xi}, x_\beta^z] \in (\mathbb{G}_h^{\alpha+\beta})^{h-1}$ by (8.6). But the height condition implies that $\alpha + \beta \in \Phi^-$. Following this case-by-case examination, the claim about λ_z in (i) when $z \in (\mathbb{N}_h^1)^{\leq 0}$ is then established once we make the following observation: If ξ has $[y]\varpi^{h-a}$ (resp. $[y]\varpi^{h-a-1}$, if $a = 1$) and $x_{-\alpha}^z$ has $[u]\varpi^{a-1}$ (resp. $[u]\varpi^a$) in their only non-trivial entries, then $[\bar{\xi}, x_{-\alpha}^z]$ is a diagonal matrix with only two nontrivial entries: $1 \pm [uy]\varpi^{h-1}$.

In (ii), it is automatic that $z \in (\mathbb{N}_h^1)^{\leq 0}$, and this case can be proven in exactly the same way as above (and is slightly easier) and we omit the details.

It remains to prove (i) in the case that $z \in \mathbb{N}_h^1 \setminus (\mathbb{N}_h^1)^{\leq 0}$. In particular, $\xi \in (\mathbb{G}_h^\alpha)^{h-1}$ since $a = 1$. By construction, A_z consists of non-reductive roots of type 1, so α must be non-reductive of type 2. Modulo $(\mathbb{N}_h^1)^{\leq 0}$ (which commutes with ξ) we may write

$$z = \left(\prod_{\substack{\gamma \in \Phi^+ \\ \text{ht}_{n_0}(\gamma) > \text{ht}_{n_0}(-\alpha)}} x_\gamma^z \right) \left(\prod_{\beta \in A_z \setminus \{-\alpha\}} x_\beta^z \right) x_{-\alpha}^z.$$

Recall that $A_z \setminus \{-\alpha\}$ consists of (necessarily non-reductive, type 1) roots with $\text{ht}_{n_0}(\beta) = \text{ht}_{n_0}(-\alpha)$. By construction $\text{ht}(\gamma), \text{ht}(\beta) \leq \text{ht}(-\alpha)$, and so in particular,

$$s := \prod_{\beta} [\xi^{-1}, (x_{\beta}^z)^{-1}] \in (\mathbb{N}_h^-)^{h-1}.$$

We claim:

$$\begin{aligned} \xi z &= \xi \left(\prod_{\gamma} x_{\gamma}^z \right) \left(\prod_{\beta} x_{\beta}^z \right) x_{-\alpha}^z \\ &= \left(\prod_{\gamma} x_{\gamma}^z \right) \xi \left(\prod_{\beta} x_{\beta}^z \right) x_{-\alpha}^z \end{aligned} \quad (8.7)$$

$$\begin{aligned} &= \left(\prod_{\gamma} x_{\gamma}^z \right) \left(\prod_{\beta} [\xi^{-1}, (x_{\beta}^z)^{-1}] x_{\beta}^z \right) x_{-\alpha}^z \xi[\xi, x_{-\alpha}^z] \\ &= \left(\prod_{\gamma} x_{\gamma}^z \right) \left(\prod_{\beta} x_{\beta}^z \right) x_{-\alpha}^z s \xi[\xi, x_{-\alpha}^z] \end{aligned} \quad (8.8)$$

$$\begin{aligned} &= z s \xi[\xi, x_{-\alpha}^z] \\ &= z \xi[\xi, x_{-\alpha}^z] s \end{aligned} \quad (8.9)$$

Here (8.7) holds as $\alpha + \gamma$ (if it is a root) must be non-reductive of type 1, and hence ξ and x_{γ}^z commute by (8.6). To justify (8.8), let $\beta \in A_z \setminus \{-\alpha\}$. If $\alpha + \beta$ is not a root, then $[\xi, x_{\beta}^z] = 1$. If $\alpha + \beta$ is a root, then $\alpha + \beta$ is reductive (since $\text{ht}_{n_0}(\beta) = \text{ht}_{n_0}(-\alpha)$) and $[\xi, x_{\beta}^z] \in (\mathbb{G}_h^{\alpha+\beta})^{h-1} \subseteq (\mathbb{N}_h^-)^{h-1}$ (since $\text{ht}(\beta) \leq \text{ht}(-\alpha)$ by definition of α). But every $\beta' \in A_z$ is non-reductive of type 1, so we must also have $\text{ht}_{n_0}(\beta' + (\alpha + \beta)) > 0$, and (8.6) shows that $[\xi, x_{\beta'}^z]$ commute with $x_{\beta'}^z$ for all $\beta' \in A_z$. Finally, (8.9) follows from the fact that $s \in (\mathbb{N}_h^-)^{h-1}$ commutes with $\xi \in (\mathbb{N}_h^-)^{h-1}$ and with $[\xi, x_{-\alpha}^z] \in \mathcal{T}^{\alpha}$. But now we have shown $[\xi, z] = [\xi, x_{-\alpha}^z] s \in \mathcal{T}^{\alpha} (\mathbb{N}_h^-)^{h-1}$, which finishes the proof of the last remaining assertion of the lemma. $\square$

Let $\mathbb{K}_h = \mathbb{U}_h^- \cap \mathbb{N}_h$. Let $\Phi' = \{\beta \in \Phi^+ : \mathbb{G}_h^{\beta} \in \mathbb{K}_h\}$. Let $\mathcal{X}$ denote the set of all non-empty subsets $I \subseteq \Phi'$, on which $\text{ht} : \Phi^+ \rightarrow \mathbb{Z}_{>0}$ is constant. To $z \in \mathbb{K}_h^1 \setminus \{1\}$ we attach a pair (a_z, I_z) with $1 \leq a_z \leq h-1$ and $I_z \in \mathcal{X}$. Define a_z by $z \in \mathbb{K}_h^{a_z,*}$. If A_z contains a non-reductive root, let $I_z \subseteq A_z$ be the subset of all non-reductive roots of maximal height. (Note that if $a = 1$, then I_z contains only roots of type 1 if $z \notin (\mathbb{N}_h^1)^{\leq 0}$ and only contains roots of type 2 if $z \in (\mathbb{N}_h^1)^{\leq 0}$.) If A_z contains only reductive roots, let $I_z \subseteq A_z$ be the subset of all roots of maximal height. We have a

stratification into locally closed subsets

$$\mathbb{K}_h^1 \setminus \{1\} = \bigsqcup_{a,I} \mathbb{K}_h^{a,*,I} \quad \text{where } \mathbb{K}_h^{a,*,I} = \{z \in \mathbb{K}_h^1 \setminus \{1\} : (a_z, I_z) = (a, I)\}. \quad (8.10)$$

8.4.8 Cohomology of $\widehat{\Sigma}'_w$

We are now ready to prove (8.5) using the same arguments as in the proof of [26, Lemma 1.9]. To do this, it is enough to show that $H_c^j(\widehat{\Sigma}'_w)_{\theta, \theta'} = 0$ for all $j \geq 0$. For a $\mathcal{T}'^F$ -module M and a character χ of $\mathcal{T}'^F$, write $M_{(\chi)}$ for the χ -isotypic component of M . Note that $\mathcal{T}'^F$ acts on $\widehat{\Sigma}'_w$ by

$$t' : (x, x', u, u', z, \tau') \mapsto (x, t'x't'^{-1}, u, t'u't'^{-1}, z, \tau't'^{-1}).$$

Hence $H_c^j(\widehat{\Sigma}'_w)$ is a $\mathcal{T}'^F$ -module. It is enough to show that $H_c^j(\widehat{\Sigma}'_w)_{(\chi)} = 0$ for any regular character χ of $\mathcal{T}'^F$. Fix such a χ . Define $N = \dot{w}U'\dot{w}^{-1}$, $N^- = \dot{w}U'\dot{w}^{-1}$. Then with notation as in Sects. 8.4.6 and 8.4.7, the stratification of $\mathbb{K}_h^1 \setminus \{1\}$ given in (8.10) induces a stratification of $\widehat{\Sigma}'_w$ indexed by $1 \leq a \leq h-1$ and $I \in \mathcal{X}$:

$$\widehat{\Sigma}'_w = \bigsqcup_{a,I} \widehat{\Sigma}'_w^{a,I} \quad \text{where } \widehat{\Sigma}'_w^{a,I} = \{(x, x', u, u', z, \tau') \in \widehat{\Sigma}'_w : z \in \mathbb{K}_h^{a,*,I}\}.$$

Note that each $\widehat{\Sigma}'_w^{a,I}$ is stable under $\mathcal{T}'^F$. Thus to show (8.5), it is enough to show

$$H_c^j(\widehat{\Sigma}'_w^{a,I}, \overline{\mathbb{Q}}_\ell)_{(\chi)} = 0 \quad \text{for any fixed } a, I. \quad (8.11)$$

Choose a root α such that $-\alpha \in I$. Then $\mathbb{G}_h^\alpha \subseteq \mathbb{U}_h \cap \dot{w}\mathbb{U}'_h\dot{w}^{-1}$. For any $z \in \mathbb{K}_h^{a,*,I}$, Lemma 8.11 grants us an isomorphism

$$\begin{aligned} \lambda_z : (\mathbb{G}_h^\alpha)^{h-a} / (\mathbb{G}_h^\alpha)^{h-a+1} &\xrightarrow{\sim} \mathcal{T}^\alpha, & \text{if } \alpha \text{ is non-reductive,} \\ \lambda_z : (\mathbb{G}_h^\alpha)^{h-a-1} / (\mathbb{G}_h^\alpha)^{h-a} &\xrightarrow{\sim} \mathcal{T}^\alpha, & \text{if } \alpha \text{ is reductive.} \end{aligned}$$

Let π denote the natural projection $(\mathbb{G}_h^\alpha)^{h-a} \rightarrow (\mathbb{G}_h^\alpha)^{h-a} / (\mathbb{G}_h^\alpha)^{h-a+1}$ if α is non-reductive and the natural projection $(\mathbb{G}_h^\alpha)^{h-a-1} \rightarrow (\mathbb{G}_h^\alpha)^{h-a-1} / (\mathbb{G}_h^\alpha)^{h-a}$ if α is reductive. Let ψ be a section to π such that $\pi\psi = 1$ and $\psi(1) = 1$. Let

$$\mathcal{H}' := \{t' \in \mathcal{T}' : t'^{-1}F(t') \in \dot{w}^{-1}\mathcal{T}^\alpha\dot{w}\}.$$

This is a closed subgroup of $\mathcal{T}'$. For any $t' \in \mathcal{H}'$ define $f_{t'} : \widehat{\Sigma}'_w^{a,I} \rightarrow \widehat{\Sigma}'_w^{a,I}$ by

$$f_{t'}(x, x', u, u', z, \tau') = (xF(\xi), \hat{x}', u, F(t')^{-1}u'F(t'), z, \tau'F(t')),$$

where

$$\xi = \psi \lambda_z^{-1} (\dot{w} F(t')^{-1} t' \dot{w}^{-1}) \in (\mathbb{G}_h^\alpha)^{h-a-1} \subseteq \mathbb{U}_h \cap \dot{w} \mathbb{U}_h \dot{w}^{-1}$$

$((\mathbb{G}_h^\alpha)^{h-a-1})$ should be replaced by $(\mathbb{G}_h^\alpha)^{h-a}$ if α is non-reductive), and $\hat{x}' \in \mathbb{G}_h$ is defined by the condition that

$$x F(\xi z \dot{w} \tau' F(t')) \in u z \dot{w} \tau' F(t') F(t')^{-1} u' F(t') \hat{x}'.$$

To check that this is well-defined one needs to show $\hat{x}' \in F(\mathbb{U}_h')$. This is done with exactly the same computation as in the proof of [26, Lemma 1.9], and we omit this. It is clear that $f_{t'}: \widehat{\Sigma}_w'^{a,I} \rightarrow \widehat{\Sigma}_w'^{a,I}$ is an isomorphism for any $t' \in \mathcal{H}'$. Moreover, since $T'^F \subseteq \mathcal{H}'$ and since for any $t' \in T'^F$ the map $f_{t'}$ coincides with the action of t' in the T'^F -action on $\widehat{\Sigma}_w'^{a,I}$ (we use $\psi(1) = 1$ here), it follows that we have constructed an action f of $\mathcal{H}'$ on $\widehat{\Sigma}_w'^{a,I}$ extending the T'^F -action.

If a connected group acts on a scheme, the induced action in the cohomology is constant. Thus for any $t' \in \mathcal{H}'^0$, the induced map $f_{t'}^*: H_c^j(\widehat{\Sigma}_w'^{a,I}, \overline{\mathbb{Q}}_\ell) \rightarrow H_c^j(\widehat{\Sigma}_w'^{a,I}, \overline{\mathbb{Q}}_\ell)$ is constant when t' varies in $\mathcal{H}'^0$. Hence the restriction of the T'^F -action on $H_c^j(\widehat{\Sigma}_w'^{a,I}, \overline{\mathbb{Q}}_\ell)$ to $T'^F \cap \mathcal{H}'^0$ is trivial.

Now we can find some $m \geq 1$ such that $F^m(\dot{w}^{-1} T^\alpha \dot{w}) = \dot{w}^{-1} T^\alpha \dot{w}$. Then

$$t' \mapsto t' F(t') F^2(t') \cdots F^{m-1}(t')$$

defines a morphism $\dot{w}^{-1} T^\alpha \dot{w} \rightarrow \mathcal{H}'$. Since T^α is connected, its image is also connected and hence contained in $\mathcal{H}'^0$. If $t' \in (\dot{w}^{-1} T^\alpha \dot{w})^{F^m}$, then $N_F^{F^m}(t') \in T'^F$ and hence also $N_F^{F^m}(t') \in T'^F \cap \mathcal{H}'^0$. Thus the action of $N_F^{F^m}(t') \in T'^F$ on $H_c^j(\widehat{\Sigma}_w'^{a,I})$ is trivial for any $t' \in (\dot{w}^{-1} T^\alpha \dot{w})^{F^m}$.

Finally, observe that if $H_c^j(\widehat{\Sigma}_w'^{a,I}, \overline{\mathbb{Q}}_\ell)_{(\chi)} \neq 0$, then the above shows that $t' \mapsto \chi(N_F^{F^m}(t'))$ is the trivial character, which contradicts the regularity assumption on χ . This establishes (8.11), which establishes (8.5), which was the last outstanding claim in the proof of Proposition 8.7.

9 Cuspidality

The next theorem (proved in Sect. 9.1) concerns the “cuspidality” of the representation $R_{T_h}^{G_h}(\theta)$ for primitive θ . This is the higher-level analogue of Deligne–Lusztig’s theorem [13, Theorem 8.3] required to prove that the induced representation $\text{c-Ind}_{Z \cdot G_O}^G(|R_{T_h}^{G_h}(\theta)|)$ is irreducible and supercuspidal (Theorem 12.5). A proof that this induced representation is irreducible supercuspidal when $h = 1$ can be found in [27, Proposition 6.6], and when $G = \text{GL}_2(K)$ and h arbitrary it was done by the first author in [21].

We work with a special representative b as in Sect. 5.2.2.

Let N' be the unipotent radical of any standard parabolic subgroup of $\text{GL}_{n'}$ and let $\check{N}$ denote the subgroup of $\text{GL}_n(\check{K})$ consisting of unipotent matrices such that

any $(n_0 \times n_0)$ -block consists of a diagonal matrix and the (i, j) th block can have nonzero entries if and only if the (i, j) th entry of an element of N' is nonzero. For each $h \geq 1$, let $\check{N}_h$ denote the image of $\check{N} \cap G_{\mathbf{x},0}$ in $\mathbb{G}_h(\overline{\mathbb{F}}_q)$. Define $N_h := \check{N}_h^F$ and $N_h^{h-1} := \ker(N_h \rightarrow N_{h-1})$.

Theorem 9.1 Assume $\theta: T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is primitive. Then the restriction of $|R_{T_h}^{G_h}(\theta)|$ to N_h^{h-1} does not contain the trivial representation.

9.1 Proof of Theorem 9.1

9.1.1. We retain notation as in the statement of the Theorem and the set-up directly preceding it. Let $J := \{\alpha = (i, j) : U_\alpha \subset N'\}$ be the set of roots of the diagonal torus in $\mathrm{GL}_{n'}$ occurring in N' . Let l be the inverse of k_0 modulo n_0 and let $[a]_{n_0}$ denote the residue of $a \in \mathbb{Z}$ in $1 \leq [a] \leq n_0$. The elements of N_h^{h-1} consist of $n \times n$ -matrices, whose (i, j) th $(n_0 \times n_0)$ -block is the identity matrix if $i = j$, is zero if $i \neq j$ and $(i, j) \notin J$, and is of the form $\mathrm{diag}(\varpi^{h-1}u, \varpi^{h-1}\sigma^{[l]n_0}(u), \varpi^{h-1}\sigma^{[2l]n_0}(u), \dots, \varpi^{h-1}\sigma^{[(n_0-1)l]n_0}(u))$ for some $u \in \mathbb{F}_{q^{n_0}}$ if $(i, j) \in J$. Observe that it is sufficient to show that the theorem holds under the assumption that N' is the unipotent radical of a maximal proper parabolic; that is,

$$J = \{(i, j) : 1 \leq i \leq n' - \ell, n' - \ell + 1 \leq j \leq n, \} \quad \text{for some } 1 \leq \ell \leq n'.$$

(This will be used only in the proof of Lemma 9.6.)

9.1.2. Our main tool will be a close variant of [5, Lemma 2.12]. The set-up and proof of Lemma 9.2 is nearly the same as the proof of *op. cit.* verbatim. Assume that X is a separated scheme of finite type over $\mathbb{F}_q$ and we are given an automorphism φ of X and a right action of a finite group A on X that commute with φ . For each character $\chi: A \rightarrow \overline{\mathbb{Q}}_\ell^\times$, we write $H_c^i(X, \overline{\mathbb{Q}}_\ell)[\chi]$ for the subspace of $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ on which A acts by χ . Note that this subspace is invariant under the action of $\varphi^*: H_c^i(X, \overline{\mathbb{Q}}_\ell)[\chi] \xrightarrow{\cong} H_c^i(X, \overline{\mathbb{Q}}_\ell)[\chi]$.

Lemma 9.2 Let $\chi: A \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character. Assume that Fr_q acts on $\sum_i (-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)[\chi]$ by a scalar λ . Then

$$\begin{aligned} & \mathrm{Tr} \left(\varphi^*; \sum_i (-1)^i (X, \overline{\mathbb{Q}}_\ell)[\chi] \right) \\ &= \frac{1}{\lambda \cdot \#A} \cdot \sum_{a \in A} \chi(a) \cdot \#\{x \in X(\overline{\mathbb{F}}_q) : \varphi(\mathrm{Fr}_q(x)) = x \cdot a\}. \end{aligned}$$

Proof For each $a \in A$, let $\rho_a: X \rightarrow X$ denote the automorphism $x \mapsto x \cdot a$ and write $\varphi_a = \varphi \circ \rho_a$. Then ρ_a is a finite-order automorphism of X and (as in the proof of [13,

Proposition 3.3])

$$\sum_i (-1)^i \operatorname{Tr}(\operatorname{Fr}_q \circ \varphi_a^*; H_c^i(X, \overline{\mathbb{Q}}_\ell)) = \#\{x \in X(\overline{\mathbb{F}}_q) : \varphi(\operatorname{Fr}_q(x)) = x \cdot a^{-1}\}.$$

Hence averaging over $\chi^{-1}(a)$, we have

$$\begin{aligned} & \frac{1}{\#A} \cdot \sum_{a \in A} \chi^{-1}(a) \cdot \#\{x \in X(\overline{\mathbb{F}}_q) : \varphi(\operatorname{Fr}_q(x)) = x \cdot a^{-1}\} \\ &= \sum_i (-1)^i \operatorname{Tr}(\operatorname{Fr}_q \circ \varphi^*; H_c^i(X, \overline{\mathbb{Q}}_\ell)[\chi]) \\ &= \lambda \operatorname{Tr}\left(\varphi^*; \sum_i (-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)[\chi]\right). \end{aligned}$$

□

9.1.3. Now fix a character $\theta: T_h \rightarrow \overline{\mathbb{Q}}_\ell^\times$ as in the theorem. Recall from (7.3) that

$$\begin{aligned} X_h &= \bigsqcup_{a \in (\mathcal{O}_K/\varpi^h)^\times} g_a \cdot X_h^{\det \equiv 1}, \quad \text{where } X_h^{\det \equiv 1} \\ &= \{x \in X_h : \det g_b^{\text{red}}(x) \equiv 1 \pmod{\varpi^h}\}. \end{aligned}$$

Note that T_h transitively permutes the components $g_a \cdot X_h^{\det \equiv 1}$ ($a \in (\mathcal{O}_K/\varpi^h)^\times$) and let $T_h^\circ \subseteq T_h$ denote the stabilizer of a (any) component. Since the composition $H_c^i(X_h^{\det \equiv 1})[\theta|_{T_h^\circ}] \hookrightarrow H_c^i(X_h)[\theta|_{T_h^\circ}] \rightarrow H_c^i(X_h)[\theta]$ is bijective, it must be an isomorphism of N_h^{h-1} -representations (see also (8.1)). Hence to show the theorem, it is enough to show that the trivial character of N_h^{h-1} does not occur in $\sum_i (-1)^i H_c^i(X_h^{\det \equiv 1})[\theta|_{T_h^\circ}]$; that is,

$$\begin{aligned} & \left\langle \operatorname{triv}, \sum_i (-1)^i H_c^i(X_h^{\det \equiv 1}, \overline{\mathbb{Q}}_\ell)[\theta|_{T_h^\circ}] \right\rangle_{N_h^{h-1}} \\ &= \frac{1}{\#N_h^{h-1}} \sum_{g \in N_h^{h-1}} \operatorname{Tr}\left(g; \sum_i (-1)^i H_c^i(X_h^{\det \equiv 1}, \overline{\mathbb{Q}}_\ell)[\theta|_{T_h^\circ}]\right) = 0. \end{aligned} \quad (9.1)$$

We now apply Lemma 9.2 to the $\mathbb{F}_{q^n}$ -scheme $X_h^{\det \equiv 1}$ with $A = T_h^\circ$ and $\varphi: X_h^{\det \equiv 1} \rightarrow X_h^{\det \equiv 1}$ given by $x \mapsto g \cdot x$ for some $g \in N_h^{h-1}$. We see that to show (9.1), we must show

$$\sum_{g \in N_h^{h-1}} \sum_{t \in T_h^\circ} \theta(t) \cdot \#S_{g,t} = 0, \quad \text{where } S_{g,t} := \{x \in X_h^{\det \equiv 1}(\overline{\mathbb{F}}_q) : g \cdot \operatorname{Fr}_{q^n}(x) = x \cdot t\}.$$

Lemma 9.3 *Let $g \in N_h^{h-1}$ and $t \in T_h^\circ$ such that $S_{g,t} \neq \emptyset$. Then $t \equiv (-1)^{n'-1} \pmod{\varpi^{h-1}}$ and $\sigma^n(x) \equiv (-1)^{n'-1}x \pmod{\varpi^{h-1}}$ for all $x \in S_{g,t}$.*

Proof An element $y \in \mathcal{L}_0$ lies in $\mathcal{L}_{0,b}^{\text{adm, rat}}$ if and only if $\det g_b^{\text{red}}(y) \in \mathcal{O}_K^\times$, or equivalently, $\text{ord} \det g_b(y) = \text{ord} \det(D_{\kappa,n}) =: c$ and $\sigma(\det g_b(y)) = \det g_b(y)$. Multiplying by b on both sides, we see that these conditions are equivalent to

$$\det(b\sigma(y) \mid (b\sigma)^2(y) \mid \dots \mid (b\sigma)^n(y)) = \det(b) \det(g_b(y)) \in \varpi^{c+\kappa} \mathcal{O}^\times.$$

As b is the special representative, $\det(b) = ((-1)^{(n_0-1)k_0} \varpi^{k_0})^{n'} = (-1)^{\kappa(n_0-1)} \varpi^\kappa$, and moreover, $b^n = \varpi^\kappa$. Thus the above is equivalent to

$$\varpi^\kappa \det(b\sigma(y) \mid (b\sigma)^2(y) \mid \dots \mid \sigma^n(y)) = (-1)^{\kappa(n_0-1)} \varpi^\kappa \det(g_b(y)) \in \varpi^{c+\kappa} \mathcal{O}^\times,$$

An elementary computation shows $(-1)^{n'-1-\kappa(n_0-1)} = (-1)^{n'-1}$, thus the above is equivalent to

$$(-1)^{n'-1} \det(\sigma^n(y) \mid b\sigma(y) \mid (b\sigma)^2(y) \mid \dots \mid (b\sigma)^{n-1}(y)) = \det(g_b(y)) \in \varpi^c \mathcal{O}^\times. \quad (9.2)$$

Let now $x \in S_{g,t} \subseteq X_h$. Denote by $y \in \mathcal{L}_0^{\text{adm, rat}}$ a lift of x . As $g \equiv 1 \pmod{\varpi^{h-1}}$, we by assumption have $\sigma^n(y) \equiv yt \pmod{\varpi^{h-1}}$. Thus replacing in (9.2) $\sigma^n(y)$ by $ty + \varpi^{h-1}*$ for some $*$ in $\mathcal{L}_0$, using the linearity of the determinant in the first column, and the fact that each entry of the i th column ($2 \leq i \leq n$) of the matrix on the left hand side of (9.2) is in $\mathcal{O}$ divisible by $\varpi^{\lfloor \frac{(i-1)\kappa_0}{n_0} \rfloor}$ (and $\sum_{i=2}^n \lfloor \frac{(i-1)\kappa_0}{n_0} \rfloor = c$), we deduce that

$$(-1)^{n'-1} t \equiv 1 \pmod{\varpi^{h-1}}.$$

If $\tilde{x} \in X_{h-1}^+$ denotes the image of x modulo ϖ^{h-1} , we obtain $\sigma^n(\tilde{x}) = (-1)^{n'-1} \tilde{x}$. $\square$

Thus for g, t as in the lemma, $S_{g,t} \neq \emptyset$ implies

$$t \in (-1)^{n'-1} T_h^{h-1} \cap T_h^\circ = \{(-1)^{n'-1} (1 + \varpi^{h-1}[a]) : a \in \mathbb{F}_{q^n}, \text{tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a) = 0\}.$$

so that, after factoring out the constant $\theta(-1)^{n'-1}$, it remains to show:

$$\sum_{g \in N_h^{h-1}} \sum_{a \in \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_q)} \theta(1 + \varpi^{h-1}[a]) \cdot \#S_{g, (-1)^{n'-1} (1 + \varpi^{h-1}[a])} = 0. \quad (9.3)$$

9.1.4. Before we can prove (9.3), we need some preparations. Recall from Sect. 7.4 that one has an intermediate scheme $X_h \rightarrow X_{h-1}^+ \rightarrow X_{h-1}$. Define $X_{h-1}^{+, \det \equiv 1}$ to be

the subscheme of X_{h-1}^+ consisting of $x \in X_{h-1}^+$ with $\det(g_b^{\text{red}}(x)) \equiv 1$ modulo ϖ^{h-1} . Then we have a surjection

$$f: X_h^{\det \equiv 1} \rightarrow X_{h-1}^{+, \det \equiv 1}$$

and by Proposition 7.6, $X_h^{\det \equiv 1} \hookrightarrow X_{h-1}^{+, \det \equiv 1} \times \mathbb{A}^{n'}$ is the (relative) hypersurface given by

$$\sum_{i=0}^{n_0-1} \sigma^i(P_1) + c = 0,$$

where $c: X_h \rightarrow \mathbb{A}^1$ factors through f and P_1 is a polynomial over X_{h-1}^+ in the variables $x_{i,h-1}$ for $i \equiv 1 \pmod{n_0}$.

9.1.5. By Lemma 9.3, for $g \in N_h^{h-1}$ and $t \in (-1)^{n'-1}(T_h^{h-1} \cap T_h^\circ)$ with $S_{g,t} \neq \emptyset$, we have $S_{g,t} \subseteq f^{-1}(S_{h-1})$, where

$$S_{h-1} := \left\{ \tilde{x} \in X_{h-1}^{+, \det \equiv 1} : \sigma^n(\tilde{x}) = (-1)^{n'-1} \tilde{x} \right\} \subset X_{h-1}^{+, \det \equiv 1}$$

is a finite set of points. Regard S_{h-1} as a (zero-dimensional, reduced) subscheme of $X_{h-1}^{+, \det \equiv 1}$. Consider the S_{h-1} -morphism $S_{h-1} \times \mathbb{A}^{n'} \xrightarrow{\sim} S_{h-1} \times \mathbb{A}^{n'}$, which is the linear change of variables defined by

$$(x_{1,h-1}, x_{n_0+1,h-1}, \dots, x_{n_0(n'-1)+1,h-1})^\top = \overline{g_b}(\tilde{x})(z_1, z_2, \dots, z_{n'})^\top,$$

where $\tilde{x}$ is the image of $\tilde{x} \in S_{h-1}$ in X_1 and $\overline{g_b}(\tilde{x})$ is as in Sect. 7.3.

Claim $X_h^{\det \equiv 1}$ is a (relative) hypersurface over $X_{h-1}^{+, \det \equiv 1}$ defined by an equation $\sum_{i=0}^{n-1} z_1^{q^i} = c$.

It is enough to show that in the new coordinates $z_1, \dots, z_{n'}$, the polynomial P_1 as in Proposition 7.6 takes the form $P_1 = \sum_{i=0}^{n'-1} z_1^{q^{n_0 i}}$. We prove this now.

Recall the n' -dimensional $\mathbb{F}_{q^{n_0}}$ -vector space $\overline{V}$ with its distinguished basis $\{\tilde{e}_{n_0(i-1)+1}\}_{1 \leq i \leq n'}$ and the $\mathbb{F}_{q^{n_0}}$ -linear morphism $\overline{\sigma}_b$ of $\overline{V}$ from Lemma 5.8. To simplify notation, we write $\tilde{x}_j$ instead of $x_{n_0(j-1)+1,0}$ for $x \in X_h$ and $1 \leq j \leq n' - 1$ in what follows (i.e., the image of $x \in X_h$ in $\overline{V}$ is $\tilde{x} = (\tilde{x}_i)_{i=1}^{n'}$). Recall from Sect. 7.3 that for $\tilde{x} \in \overline{V}$ the i th column of the $(n' \times n')$ -matrix $\overline{g_b}(\tilde{x})$ is $\overline{\sigma}_b^{i-1}(\tilde{x})$. Let m_i denote

the i th row of the adjoint matrix (m_{ij}) of $\overline{g_b}(\bar{x})$. Then the above change of variables gives

$$\begin{aligned} P_1 = & (m_1 \cdot \bar{x})z_1 + (m_2 \cdot \bar{\sigma}_b(\bar{x}))z_1^{q^{n_0}} + (m_3 \cdot \bar{\sigma}_b^2(\bar{x}))z_1^{q^{2n_0}} + \cdots + (m_{n'} \cdot \bar{\sigma}_b^{n'-1}(\bar{x}))z_1^{q^{n_0(n'-1)}} \\ & + (m_1 \cdot \bar{\sigma}_b(\bar{x}))z_2 + (m_2 \cdot \bar{\sigma}_b^2(\bar{x}))z_2^{q^{n_0}} + (m_3 \cdot \bar{\sigma}_b^3(\bar{x}))z_2^{q^{2n_0}} + \cdots + (m_{n'} \cdot \bar{\sigma}_b^{n'}(\bar{x}))z_2^{q^{n_0(n'-1)}} \\ & + \cdots + (m_1 \cdot \bar{\sigma}_b^{n'-1}(\bar{x}))z_{n'} + (m_2 \cdot \bar{\sigma}_b^{n'}(\bar{x}))z_{n'}^{q^{n_0}} + \cdots + (m_n \cdot \bar{\sigma}_b^{2n'-2}(\bar{x}))z_{n'}^{q^{n_0(n'-1)}}. \end{aligned}$$

(Here $\cdot$ denotes the matrix product.) But $\sigma^n(\bar{x}) = (-1)^{n'-1}\bar{x}$ (where σ^n is applied entry-wise), and hence from the explicit form of $\bar{\sigma}_b$ we deduce that $\bar{\sigma}_b^{n'}(\bar{x}) = (-1)^{n'-1}\bar{x}$. As (m_{ij}) is adjoint to $\overline{g_b}(\bar{x})$ and $\det(\overline{g_b}(\bar{x})) = 1 \in \mathbb{F}_q^\times$, we have

$$m_i \cdot \bar{\sigma}_b^j(\bar{x}) = \begin{cases} 1 & \text{if } j = i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that all coefficients are equal to 1 in the first line of the above expression and vanish in lines 2, $\dots$, n' . This completes the proof of the claim.

9.1.6. Note that $N_h^{h-1} \times (T_h^{h-1} \cap T_h^\circ)$ stabilizes S_{h-1} and acts trivially on it. We describe the action of $N_h^{h-1} \times (T_h^{h-1} \cap T_h^\circ)$ on the new coordinates $z_1, \dots, z_{n'}$.

Let $g \in N_h^{h-1}$ and for $(i, j) \in J$ as in Sect. 9.1.1, let $[u_{i,j}] \varpi^{h-1}$ denote the upper left entry of the (i, j) th $n_0 \times n_0$ -block of g . Recall that the action of g on $f^{-1}(S_{h-1})$ in the old coordinates $x_{1,h-1}, \dots, x_{n_0(n'-1)+1,h-1}$ is given by

$$g \cdot (x_{n_0(i-1),h-1})_{i=1}^{n'} = \left(x_{n_0(i-1),h-1} + \sum_{\substack{1 \leq j \leq n' \\ (i,j) \in J}} u_{i,j} \bar{x}_j \right)_{i=1}^{n'}.$$

Since $\det(\overline{g_b}(\bar{x})) = 1$ by assumption, the adjoint matrix $(m_{ij})_{ij}$ of $\overline{g_b}(\bar{x})$ is in fact the inverse, so that $z_i = \sum_{j=1}^{n'} m_{ij} x_{n_0(j-1)+1,h-1}$. Thus the action of g on the new coordinates is given by

$$z_i \mapsto z_i + \sum_{\substack{1 \leq k, j \leq n' \\ (k,j) \in J}} m_{ik} u_{kj} \bar{x}_j.$$

We now describe the action of $(T_h^{h-1} \cap T_h^\circ) = \{1 + [a] \varpi^{h-1} : \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a) = 0\}$. For $a \in \ker(\text{Tr} : \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q)$, the action of $1 + [a] \varpi^{h-1}$ on the old coordinates is given by

$$x_{n_0(i-1)+1,h-1} \mapsto x_{n_0(i-1)+1,h-1} + a \bar{x}_i.$$

Since $\sum_{k=1}^n m_{ik} \bar{x}_k$ is equal to 1 when $i = 1$ and equal to 0 when $i > 1$, the action of $1 + [a]\varpi^{h-1}$ on the new coordinates is given by

$$z_i \mapsto \begin{cases} z_1 + a & \text{if } i = 1, \\ z_i & \text{if } i = 2, \dots, n'. \end{cases}$$

Moreover, $x \mapsto \sigma^n(x) \cdot (-1)^{n'-1}$ defines an isomorphism of each fiber $f^{-1}(\tilde{x})$, and one computes that in coordinates z_i it is given by $z_i \mapsto \sigma^n(z_i)$. Thus for $t = (-1)^{n'-1}(1 + \varpi^{h-1}[a]) \in (-1)^{n'-1}(T_h^{h-1} \cap T_h^\circ)$, the assignment $x \mapsto \sigma^n(x) \cdot t$ defines an isomorphism of $f^{-1}(\tilde{x})$ which in the coordinates z_i is given by $z_1 \mapsto \sigma^n(z_1) + a$, $z_i \mapsto \sigma^n(z_i)$ for $2 \leq i \leq n'$.

9.1.7. We next claim that $c(\tilde{x}) \in \mathbb{F}_q$ for $\tilde{x} \in S_{h-1}$. Consider the “extension by zero” morphism $\mathbb{W}_{h-1} \rightarrow \mathbb{W}$ given by $\sum_{i=0}^{h-2} [a_i] \varpi^i \mapsto \sum_{i=0}^{h-2} [a_i] \varpi^i$. It defines a map $\mathcal{L}_0/\varpi^{h-1} \mathcal{L}_0 \rightarrow \mathcal{L}_0$, $y \mapsto [y, 0]$. To show the claim it is sufficient to show that $[\tilde{x}, 0]$ lies in $\mathcal{L}_{0,b}^{\text{adm}, \text{rat}}$. Obviously, $\det g_b^{\text{red}}([\tilde{x}, 0]) \in \mathcal{O}$. Now, note that as $\tilde{x} \in X_{h-1}^+$, there exists some lift $z \in \mathcal{L}_{0,b}^{\text{adm}, \text{rat}}$ of $\tilde{x}$. This gives in particular $\det g_b^{\text{red}}(z) \equiv \det g_b^{\text{red}}([\tilde{x}, 0]) \pmod{\varpi^{h-1}}$. We deduce $\det g_b^{\text{red}}([\tilde{x}, 0]) \in \mathcal{O}^\times$. It remains to show that $\det g_b^{\text{red}}([\tilde{x}, 0]) \in K$. To do this, it suffices to prove that $\det g_b([\tilde{x}, 0]) \in K$, as $\det g_b^{\text{red}}(\cdot)$ and $\det g_b(\cdot)$ differ only by a power of ϖ . But as $\sigma^n(\tilde{x}) = (-1)^{n'-1}(\tilde{x})$, we have $\sigma^n([\tilde{x}, 0]) = (-1)^{n'-1}[\tilde{x}, 0]$. Using this and $\det(b) = (-1)^{\kappa(n_0-1)} \varpi^\kappa$ we compute:

$$\begin{aligned} & (-1)^{\kappa(n_0-1)} \varpi^\kappa \sigma(\det g_b([\tilde{x}, 0])) \\ &= \det b \sigma(g_b([\tilde{x}, 0])) \\ &= \det \left(b \sigma([\tilde{x}, 0]) \mid (b \sigma)^2([\tilde{x}, 0]) \mid \dots \mid \varpi^k \sigma^n([\tilde{x}, 0]) \right) \\ &= (-1)^{(n-1)+(n'-1)} \varpi^\kappa \det g_b([\tilde{x}, 0]). \end{aligned}$$

But as in the proof of Lemma 9.3, we have $(-1)^{\kappa(n_0-1)} = (-1)^{(n-1)+(n'-1)}$. This shows the claim.

9.1.8. Fix $\tilde{x} \in S_{h-1}$ and $t = (-1)^{n'-1}(1 + \varpi^{h-1}[a])$ with $\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a) = 0$ (as in Eq. (9.3)). We see that a point $x \in f^{-1}(\tilde{x})$ with coordinates $(z_i)_{i=1}^{n'}$ as in Sect. 9.1.5 lies in $S_{g,t} \cap f^{-1}(\tilde{x})$ if and only if

$$g \cdot \sigma^n(x) \cdot t^{-1} = x \quad \text{and} \quad z_1 + z_1^q + \dots + z_1^{q^{n-1}} = c(\tilde{x}).$$

By Sect. 9.1.6, the first equation is equivalent to (use that $\sigma^n(m_{i,k}) = -m_{i,k}, \sigma^n(\bar{x}_j) = (-1)^{n'-1}\bar{x}_j$)

$$z_1^{q^n} + \sum_{(k,j) \in J} m_{1k} u_{kj} \bar{x}_j = z_1 + a,$$

along with similar equations for the $(z_i)_{i=2}^{n'}$ (of the form $z_i^{q^n} + (\text{sum of terms}) = z_i$). Since $c(\tilde{x}) \in \mathbb{F}_q$ by Sect. 9.1.7, the second equation implies $z_1 = z_1^{q^n}$, which eliminates z_1 from the first equation. Therefore $S_{g,t} \cap f^{-1}(\tilde{x}) \neq \emptyset$ if and only if

$$\psi(\tilde{x}, g) = a, \quad \text{where } \psi(\tilde{x}, g) := \sum_{(k,j) \in J} m_{1k} u_{kj} \bar{x}_j. \quad (9.4)$$

Moreover, since the $n' - 1$ equations for $(z_i)_{i=2}^{n'}$ is a separable polynomial in z_i , each gives precisely q^n choices for z_i , $2 \leq i \leq n'$, with no further conditions. Thus

$$\begin{aligned} S_{g,t} \cap f^{-1}(\tilde{x}) \neq \emptyset &\iff (9.4) \text{ holds} \\ &\iff \#(S_{g,t} \cap f^{-1}(\tilde{x})) = \underbrace{q^{n-1}}_{\text{for } z_1} \left(\underbrace{q^n}_{\text{for } z_i, 2 \leq i \leq n'} \right)^{n'-1} = q^{nn'-1}. \end{aligned}$$

This shows the following lemma.

Lemma 9.4 For $g \in N_h^{h-1}$, $t = (-1)^{n'-1}(1 + \varpi^{h-1}[a])$ with $\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a) = 0$, and $\tilde{x} \in S_{h-1}$,

$$\#S_{g,t} \cap f^{-1}(\tilde{x}) = \begin{cases} q^{nn'-1} & \text{if } \psi(g, \tilde{x}) = a, \\ 0 & \text{otherwise.} \end{cases}$$

For $a \in \ker(\text{Tr}: \mathbb{F}_{q^n} \rightarrow \mathbb{F}_q)$, put

$$B_{g,a} := \{\tilde{x}: \psi(g, \tilde{x}) = a\} \subseteq S_{h-1}.$$

As $S_{g,t} = \bigsqcup_{\tilde{x} \in S_{h-1}} S_{g,t} \cap f^{-1}(\tilde{x})$, Lemma 9.4 implies that

$$\#S_{g,(-1)^{n'-1}(1+\varpi^{h-1}[a])} = q^{nn'-1} \cdot \#B_{g,a}.$$

Thus the left hand side of (9.3) is

$$q^{nn'-1} \cdot \sum_{a \in \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_q)} \sum_{g \in N_h^{h-1}} \theta(1 + \varpi^{h-1}[a]) \cdot \#B_{g,a}. \quad (9.5)$$

9.1.9. We have the following lemma.

Lemma 9.5 *Let $g \in N_h^{h-1}$ and let $t = (-1)^{n'-1}(1 + \varpi^{h-1}[a])$ with $\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a) = 0$. We have $S_{g,t} = \emptyset$, unless $\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(a) = 0$.*

Proof It is enough to show that if $\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(a) \neq 0$, then $S_{g,t} \cap f^{-1}(\tilde{x}) = \emptyset$ for all $\tilde{x} \in S_{h-1}$. By Lemma 9.4, it is enough to show that for all $g \in N_h^{h-1}$ and $\tilde{x} \in S_{h-1}$, we have $\psi(\tilde{x}, g) \in \ker(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}})$. Fix such g and $\tilde{x}$ and let $\varpi^{h-1}u_{k,j}$ ($((k, j) \in J)$) denote the entries of g (as in beginning of Sect. 9.1.6). As $u_{k,j} \in \mathbb{F}_{q^{n_0}}$, and as $k \neq j$ holds for all pairs $(k, j) \in J$, it suffices to show that $m_{1,k}\tilde{x}_j \in \ker(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}})$ if $k \neq j$. Since $\tilde{x} \in S_{h-1}$, one computes $m_{k,l} = m_{k-1,l}^{q^{n_0}}$. Thus $\mathrm{Tr}(m_{1,k}\tilde{x}_j)$ is precisely the (j, k) th entry of the matrix $\overline{g_b}(\tilde{x}) \cdot m$, which is equal 0. $\square$

By Lemma 9.5 and (9.5), we have reduced showing (9.3) to showing

$$\sum_{a \in \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}})} \sum_{g \in N_h^{h-1}} \theta(1 + \varpi^{h-1}[a]) \cdot \#B_{g,a} = 0 \quad (9.6)$$

9.1.10. For $\tilde{x} \in S_{h-1}$, consider the $\mathbb{F}_{q^{n_0}}$ -vector subspace $W(\tilde{x}) := \langle m_{1,i}\tilde{x}_j | (i, j) \in J \rangle \subseteq \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}})$. The left hand side of (9.6) is

$$\begin{aligned} & \sum_{a \in \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}})} \sum_{g \in N_h^{h-1}} \theta(1 + \varpi^{h-1}[a]) \cdot \#B_{g,a} \\ &= \sum_{a \in \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}})} \sum_{\tilde{x} \in S_{h-1}} \theta(1 + \varpi^{h-1}[a]) \cdot \#\{g \in N_h^{h-1} : \psi(g, \tilde{x}) = a\} \\ &= \sum_{a \in \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}})} \sum_{W \subseteq \ker(\mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}})} \sum_{\tilde{x} : W(\tilde{x}) = W} \theta(1 + \varpi^{h-1}[a]) \cdot \#\{g \in N_h^{h-1} : \psi(g, \tilde{x}) = a\}. \end{aligned} \quad (9.7)$$

Now fix some W and $\tilde{x} \in S_{h-1}$ such that $W(\tilde{x}) = W$. Then $\{m_{1,i}\tilde{x}_j\}_{(i,j) \in J}$ span the $\mathbb{F}_{q^{n_0}}$ -vector space W , and from the explicit form (9.4) of $\psi(g, a)$, it is clear that

$$\#\{g \in N_h^{h-1} : \psi(g, \tilde{x}) = a\} = \begin{cases} q^{n_0(\#J - \dim W)} & \text{if } a \in W, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\#\{g \in N_h^{h-1} : \psi(g, \tilde{x}) = a\}$ depends only on $W(\tilde{x})$ and not on $\tilde{x}$ itself. Thus, if we set $S_{h-1,W} := \{\tilde{x} \in S_{h-1} : W(\tilde{x}) = W\}$, then (9.7) is equal to

$$\sum_W \#S_{h-1,W} \cdot q^{n_0(\#J - \dim W)} \cdot \sum_{a \in W} \theta(1 + \varpi^{h-1}[a]).$$

But as θ is assumed to be primitive in the theorem, this expression is equal to 0 once we show the following lemma:

Lemma 9.6 *Let $\tilde{x} \in S_{h-1}$. Then there is some $r \mid n'$, $r < n'$ such that $W(\tilde{x}) = \ker(\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0r}}})$.*

Proof Write $W = W(\tilde{x})$. Consider the perfect symmetric $\mathbb{F}_{q^{n_0}}$ -bilinear trace pairing

$$\mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \rightarrow \mathbb{F}_{q^{n_0}}, \quad (x, y) \mapsto \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(xy).$$

It is an immediate computation that $\ker(\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0r}}})^\perp = \mathbb{F}_{q^{n_0r}}$ for any divisor r of n' , so we need to show that $W^\perp$ is of the form $\mathbb{F}_{q^{n_0r}}$ for some $r < n'$. For this, it suffices to show that $W^\perp$ is an $\mathbb{F}_{q^{n_0}}$ -algebra, which is properly contained in $\mathbb{F}_{q^n}$.

First of all note that $W^\perp$ contains 1 since for all $(i, j) \in J$, $m_{1,i}\tilde{x}_j \in \ker(\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}})$ (as in the proof of Lemma 9.5). Since $W^\perp$ is an $\mathbb{F}_{q^{n_0}}$ -vector space and contains 1, it must contain $\mathbb{F}_{q^{n_0}}$. It remains to show that $W^\perp$ is closed under multiplication. We now use that J is of the form

$$J = \{(i, j) : 1 \leq i \leq n' - \ell, n' - \ell + 1 \leq j \leq n, \}$$

for some $1 \leq \ell \leq n'$ (see Sect. 9.1.1). For a fixed $n' - \ell + 1 \leq j \leq n'$, let

$$L_j := \text{span}_{\mathbb{F}_{q^{n_0}}} \langle m_{1,i}\tilde{x}_j : 1 \leq i \leq n' - \ell \rangle^\perp.$$

Observe that the $m_{1,i}$ are all $\mathbb{F}_{q^{n_0}}$ -linearly independent (since b is the special representative) and hence L_j has dimension $n' - \ell$. For $1 \leq i \leq n' - \ell$ and $n' - \ell + 1 \leq i' \leq n'$ we have

$$\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(m_{1,i}\tilde{x}_j \cdot \frac{\tilde{x}_{i'}}{\tilde{x}_j}) = \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(m_{1,i}\tilde{x}_{i'}) = 0,$$

as in the proof of Lemma 9.5. This implies the inclusion “ $\supseteq$ ” in the formula

$$L_j = \text{span}_{\mathbb{F}_{q^{n_0}}} \left\langle \frac{\tilde{x}_{i'}}{\tilde{x}_j} : n' - \ell + 1 \leq i' \leq n' \right\rangle.$$

The other inclusion follows by dimension reasons. As W is generated by all $L_j^\perp$ ($n' - \ell + 1 \leq j \leq n'$), we have $W^\perp = \bigcap_{j=n'-\ell+1}^n L_j$. Let $v, w \in W^\perp$. We need to show that $vw \in W^\perp$, i.e., that for all $(i_0, j_0) \in J$ we have $\text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(m_{1,i_0}\tilde{x}_{j_0}vw) = 0$. As $v \in L_{j_0}$, we may write $v = \sum_{(a,j_0) \in J} v_a \cdot \frac{\tilde{x}_i}{\tilde{x}_{j_0}}$ with $v_a \in \mathbb{F}_{q^{n_0}}$. Then

$$\begin{aligned} \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(m_{1,i_0}\tilde{x}_{j_0}vw) &= \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}\left(m_{1,i_0}\tilde{x}_{j_0}\left(\sum_{(a,j_0) \in J} v_a \cdot \frac{\tilde{x}_a}{\tilde{x}_{j_0}}\right)w\right) \\ &= \sum_{(a,j_0) \in J} v_a \cdot \text{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^{n_0}}}(m_{1,i_0}\tilde{x}_a w) = 0, \end{aligned}$$

where the last equality holds since $w \in W^\perp$ is orthogonal to each $L_a^\perp$. □

Part 4. Automorphic induction and the Jacquet–Langlands correspondence

In this part, we use the results of Parts 2 and 3 to study the ℓ -adic homology groups of the semi-infinite Deligne–Lusztig variety $\dot{X}_w^{DL}(b)$, which by Theorem 6.9 along with Corollary 6.19 is isomorphic to the affine Deligne–Lusztig variety at infinite level $\dot{X}_w^\infty(b)$ constructed in Sect. 6. In Sect. 10, we recall methods of Henniart characterizing certain representations by considering the action of very regular elements. In Sect. 11, we define the homology of $\dot{X}_w^\infty(b) \cong X_w^{DL}$ and give a representation-theoretic description of

$$R_T^G(\theta) := \sum_{i \geq 0} (-1)^i H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta] \quad \text{for } \theta: T = L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times \text{ smooth}$$

in terms of the cohomology of the finite-type variety X_h studied in the previous two parts of the paper. Using methods of Henniart as reviewed in Sect. 10, we prove Theorem 11.3: if $|R_T^G(\theta)|$ is irreducible supercuspidal, then the assignment $\theta \mapsto |R_T^G(\theta)|$ realizes automorphic induction. To finish, we prove in Sect. 12 that when $\theta: L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is *minimal admissible*, then $|R_T^G(\theta)|$ is irreducible supercuspidal.

We now give some basic definitions which we will use throughout the next few sections. Recall that for any smooth character $\theta: L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, there exists an integer $h \geq 1$ such that θ is trivial on $U_L^h = 1 + \varpi^h \mathcal{O}_L$. We call the smallest such h the *level* of θ . We say that θ is *in general position* if its stabilizer in $\text{Gal}(L/K)$ is trivial. Let $\mathcal{X}$ denote the set of such characters.

We say that an element x of $L^\times$ is *very regular* if $x \in \mathcal{O}_L^\times$ and its image in the residue field $\mathbb{F}_{q^n}$ generates its multiplicative group $\mathbb{F}_{q^n}^\times$.

We say that a virtual representation is a *genuine representation* if it is a nonnegative linear combination of irreducible representations. If R is a virtual representation that is $\pm\pi$, where π is a genuine representation, we write $|R| = \pi$.

10 Results of Henniart on the local Langlands correspondence

In this section, we review the methods of Henniart [19, 20] characterizing certain cases of automorphic induction by considering the action of very regular elements. We give a generalization of the discussions of [6] to all inner forms of $\text{GL}_n(K)$. There are no technical difficulties in doing this, but we provide it for completeness of our paper.

Fix a character ϵ of $K^\times$ with $\ker(\epsilon) = \text{Nm}_{L/K}(L^\times)$, and let $\mathcal{G}_K^\epsilon(n)$ denote the set of irreducible n -dimensional representations σ of the Weil group $\mathcal{W}_K$ such that $\sigma \cong \sigma \otimes (\epsilon \circ \text{rec}_K^{-1})$, where $\text{rec}_K: K^\times \rightarrow \mathcal{W}_K^{ab}$ is the reciprocity isomorphism from local class field theory. It is known that every element of $\mathcal{G}_K^\epsilon(n)$ is of the form $\text{Ind}_{\mathcal{W}_L}^{\mathcal{W}_K}(\theta)$ for some character $\theta \in \mathcal{X}$. However, it is also known that automorphic induction is not compatible with induction on Weil groups in the sense that the Langlands parameter may have a twist by a rectifying character. Hence the approach we take is via the χ -datum of Langlands–Shelstad [24, Section 2.5]. Because L/K is unramified, there

is a canonical choice of χ -datum, and this gives rise to a bijection

$$\mathcal{X} / \text{Gal}(L/K) \rightarrow \mathcal{G}_K^\epsilon(n), \quad \theta \mapsto \sigma_\theta.$$

See [11, Section 7.2] for an exposition and an explicit discussion of the unramified setting. Note that σ_θ differs from the notation of [6] by a rectifying character.

Let $\mathcal{A}_K^\epsilon(\text{GL}_n)$ denote the set of isomorphism classes of irreducible supercuspidal representations π of $\text{GL}_n(K)$ such that $\pi \cong \pi \otimes (\epsilon \circ \det)$. There is a canonical bijection

$$\mathcal{G}_K^\epsilon(n) \xrightarrow{\text{LLC}} \mathcal{A}_K^\epsilon(\text{GL}_n), \quad \sigma_\theta \mapsto \pi_\theta$$

satisfying certain properties. By work of Henniart, the character of π_θ is very nicely behaved on certain elements of $\text{GL}_n(K)$.

Now let G be an inner form of $\text{GL}_n(K)$ so that $G \cong \text{GL}_{n'}(D_{k_0/n_0})$, where D_{k_0/n_0} is the division algebra of dimension n_0^2 over K with Hasse invariant k_0/n_0 . Let $\mathcal{A}_K^\epsilon(G)$ denote the set of isomorphism classes of irreducible supercuspidal representations π' of G such that $\pi' \cong \pi' \otimes (\epsilon \circ \det)$. By the Jacquet–Langlands correspondence, there is a canonical bijection

$$\mathcal{A}_K^\epsilon(\text{GL}_n) \xrightarrow{\text{JLC}} \mathcal{A}_K^\epsilon(G), \quad \pi \mapsto \pi' := \text{JL}(\pi)$$

such that the central characters of π and π' match and such that their characters on regular semisimple elements differs by $(-1)^{n-n'}$.

Remark 10.1 We remark that the notation π_θ agrees with the $\pi(\theta)$ of [20], but with the $\pi'(\theta)$ (rather than the $\pi(\theta)$) of [19]. When n is odd, there is no discrepancy, but when n is even, our π_θ is the representation $\pi_{\theta\omega} = \pi'(\theta)$ in [19], where ω is the unique unramified character of $L^\times$ of order 2. $\diamond$

The following theorem can be found in [19, Section 3.14].

Theorem 10.2 (Henniart) *For each $\theta \in \mathcal{X}$, there exists a constant $c_\theta = \pm 1$ such that*

$$\text{Tr JL}(\pi_\theta)(x) = c_\theta \cdot \sum_{\gamma \in \text{Gal}(L/K)} \theta^\gamma(x)$$

for every very regular element $x \in L^\times \subset \text{GL}_n(K)$.

As we will see momentarily, one can even go the other direction: the trace of $\pi \in \mathcal{A}_K^\epsilon(\text{GL}_n)$ on very regular elements of $L^\times$ characterizes π . Furthermore, c_θ can be pinpointed for $\text{GL}_n(K)$ by [19, Theorem 3.14] and extended to any inner form of $\text{GL}_n(K)$ via the character condition of the Jacquet–Langlands correspondence. For each positive integer r and each $\theta \in \mathcal{X}$, consider the subgroup of $\text{Gal}(L/K)$ given by $\mathcal{G}_{\theta,r} := \{\gamma \in \text{Gal}(L/K) : a(\gamma) \leq r\}$, where $a(\gamma)$ is the level of θ/θ^γ .

Theorem 10.3 (Henniart) *The constant c_θ of Theorem 10.2 satisfies*

$$(-1)^{n-n'} c_\theta = \begin{cases} +1 & \text{if } n \text{ is odd,} \\ +1 & \text{if } n \text{ is even and } s \text{ is even,} \\ -1 & \text{if } n \text{ is even and } s \text{ is odd,} \end{cases}$$

where s is such that $\mathcal{G}_{\theta,s} \setminus \mathcal{G}_{\theta,s-1}$ contains the unique element of order 2 in $\text{Gal}(L/K)$.

Lemma 10.4 (Henniart) *Let $\theta \in \mathcal{X}$ and suppose that there exists a character θ' of $L^\times$ (a priori, not necessarily in $\mathcal{X}$) such that $\theta(\varpi) = \theta'(\varpi)$ and*

$$c \cdot \sum_{\gamma \in \text{Gal}(L/K)} \theta^\gamma(x) = c' \cdot \sum_{\gamma \in \text{Gal}(L/K)} \theta'^\gamma(x) \quad (10.1)$$

for all very regular elements of $x \in L^\times$. Assume in addition that $c = c'$ in the special case $n = 2$, $q = 3$, and $\theta|_{U_L^1}$ factors through the norm $U_L^1 \rightarrow U_K^1$ (i.e. $\theta \in \mathcal{X}^0$ with notation from Sect. 12). Then

$$\theta' = \theta^\gamma \quad \text{for some } \gamma \in \text{Gal}(L/K).$$

Proof We provide the proof in the case that $\theta|_{U_L^1}$ has trivial $\text{Gal}(L/K)$ -stabilizer, following [20, Section 5.3] (see also [6, Lemma 1.7]). This is the simplest setting. In [20, Section 5.3], Henniart proves the lemma for $\theta \in \mathcal{X}$ in the case $[L : K]$ is prime by essentially the arguments presented here. A significantly more involved incarnation of these arguments is used in [19, Identity (2.5), Sections 2.6–2.12] to prove the lemma in full generality as stated.

We first show that the conclusion holds on U_L^1 . Fix a very regular element $x \in L^\times$. Since every element of $xU_L^1 \subset L^\times$ is a very regular element, the assumption implies that we have an equation of linear dependence between the $2n$ characters of U_L^1 given by the restrictions of the $\text{Gal}(L/K)$ -translates of θ and θ' . Explicitly: on U_L^1 , we have

$$\theta' = c'^{-1} \theta'(x)^{-1} \cdot \left(\sum_{\gamma \in \text{Gal}(L/K)} c \theta^\gamma(x) \cdot \theta^\gamma - \sum_{1 \neq \gamma \in \text{Gal}(L/K)} \theta'(x)^{-1} \theta'^\gamma(x) \cdot \theta'^\gamma \right).$$

Considering the character inner product of θ' with θ'^γ on U_L^1 for some fixed $\gamma' \in \text{Gal}(L/K)$, we have:

$$\langle \theta', \theta'^\gamma \rangle = \frac{c \cdot \theta'^\gamma(x)}{c' \cdot \theta'(x)} - c' \cdot \sum_{1 \neq \gamma \in \text{Gal}(L/K)} \theta'^\gamma(x) \langle \theta'^\gamma, \theta'^\gamma \rangle.$$

If $\langle \theta'^\gamma, \theta'^\gamma \rangle = 1$ for some $1 \neq \gamma \in \text{Gal}(L/K)$, then we are done. Otherwise, we must have $c' \theta'(x) = c \theta'^\gamma(x)$ and $\theta' = \theta'^\gamma$ on U_L^1 since θ, θ' agree on $K^\times$.

We have now shown that there exists a $\gamma \in \text{Gal}(L/K)$ such that $\theta'(x) = \theta^\gamma(x)$ for any very regular element $x \in L^\times$. But now it follows that $\theta' = \theta^\gamma$ on $\mathcal{O}_L^\times$ since any very regular element together with U_L^1 generate $\mathcal{O}_L^\times$. The desired conclusion now follows by the assumption $\theta(\pi) = \theta'(\pi)$ since $\langle \varpi \rangle \cdot \mathcal{O}_L^\times = L^\times$. $\square$

From Lemma 10.4, we obtain the following result:

Proposition 10.5 (Henniart, Boyarchenko–Weinstein) *Let $\theta \in \mathcal{X}$ and let G be any inner form of $\text{GL}_n(K)$. Assume that π is an irreducible supercuspidal representation of G with central character $\theta|_{K^\times}$ satisfying:*

- (i) $\pi \cong \pi \otimes (\epsilon \circ \det)$,
- (ii) *there exists a constant $c \neq 0$ satisfying $\text{Tr } \pi(x) = c \cdot \sum_{\gamma \in \text{Gal}(L/K)} \theta^\gamma(x)$ for each very regular element $x \in L^\times$.*

If $n = 2$, $q = 3$, and $\theta|_{U_L^1}$ factors through the norm $U_L^1 \rightarrow U_K^1$ (i.e. $\theta \in \mathcal{X}^0$ with notation from Sect. 12), assume in addition that $c = \begin{cases} -1 & \text{if } G \cong \text{GL}_2(K) \\ +1 & \text{if } G \cong D_{1/2}^\times \end{cases}$. Then π corresponds to θ under automorphic induction and the Jacquet–Langlands correspondence:

$$\pi \cong \text{JL}(\pi_\theta).$$

Proof This is [6, Proposition 1.5] (combined with the remarks of Section 1.4 of *op. cit.*) when $G \cong \text{GL}_n(K)$ or $G \cong D_{1/n}^\times$. The proof extends to the general situation with no complications. $\square$

11 Homology of affine Deligne–Lusztig varieties at infinite level

We explain how the results of Part 3 on the cohomology of the finite-type of $\mathbb{F}_{q^n}$ -schemes X_h for $h \geq 1$ (Proposition-Definition 7.11, Proposition 7.12) allows one to define and determine homology groups of the schemes $X_w^\infty(b)$.

11.1 Definition of the homology groups

Following [25], for any smooth $\overline{\mathbb{F}}_q$ -scheme S of pure dimension d , we set

$$H_i(S, \overline{\mathbb{Q}}_\ell) := H_c^{2d-i}(S, \overline{\mathbb{Q}}_\ell)(d),$$

where (d) denotes the d th Tate twist. Recall from Proposition 7.4 that for any $h \geq 1$, the $\mathbb{F}_{q^n}$ -scheme X_h is smooth of pure dimension $(n-1)(h-1) + (n'-1)$.

By Proposition 6.12, Corollary 6.19 and (7.1), we have

$$\dot{X}_w^\infty(b) = \bigsqcup_{g \in G/G_O} \varprojlim_{r > m \geq 0} g \cdot \dot{X}_{w_r}^m(b)_{\mathcal{L}_0} = \bigsqcup_{g \in G/G_O} \varprojlim_h X_h.$$

By Proposition 7.7, we have the natural inclusion

$$H_i(X_{h-1}, \overline{\mathbb{Q}}_\ell) = H_i(X_h, \overline{\mathbb{Q}}_\ell)^{\mathbb{W}_h^{h-1}(\mathbb{F}_{q^n})} \subseteq H_i(X_h, \overline{\mathbb{Q}}_\ell).$$

We may therefore define

$$\begin{aligned} H_i(\dot{X}_w^\infty(b)_{\mathcal{L}_0}, \overline{\mathbb{Q}}_\ell) &= H_i\left(\varprojlim_h X_h, \overline{\mathbb{Q}}_\ell\right) := \varprojlim_h H_i(X_h, \overline{\mathbb{Q}}_\ell), \\ H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell) &= \bigoplus_{G/G_{\mathcal{O}}} H_i(g \cdot \dot{X}_w^\infty(b)_{\mathcal{L}_0}, \overline{\mathbb{Q}}_\ell). \end{aligned}$$

Recall that in Theorem 6.9 we extended the action of $\mathcal{O}_L^\times$ on $\dot{X}_w^\infty(b)$ to an action of $T = L^\times$.

Definition 11.1 For any (smooth) character $\theta: T \rightarrow \overline{\mathbb{Q}}_\ell^\times$, define the virtual G -representation

$$R_T^G(\theta) := \sum_{i \geq 0} (-1)^i H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta],$$

where $[\theta]$ denotes the subspace where T acts by θ .

Let Z denote the center of G .

Theorem 11.2 Let $\theta: T = L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character of level $h \geq 1$. Then as G -representations,

$$R_T^G(\theta) \cong \mathrm{c}\text{-Ind}_{Z \cdot G_{\mathcal{O}}}^G \left(R_{T_h}^{G_h}(\theta) \right), \quad (11.1)$$

where we view the (virtual) G_h -representation $R_{T_h}^{G_h}(\theta)$ as a $G_{\mathcal{O}}$ -representation by pulling back along the natural surjection $G_{\mathcal{O}} \rightarrow G_h$ and then extend to Z by letting ϖ act by $\theta(\varpi)$. Furthermore, for any very regular element $x \in L^\times$,

$$\mathrm{Tr}\left(x^*; R_T^G(\theta)\right) = \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x).$$

Proof The stabilizer of $\dot{X}_w^\infty(b)_{\mathcal{L}_0} \subseteq \dot{X}_w^\infty(b)$ in G is $G_{\mathcal{O}}$. Let $T_{\mathcal{O}}$ be the preimage of $\mathcal{O}_L^\times$ under $T \cong L^\times$. It is easy to see that the stabilizer of $\dot{X}_w^\infty(b)_{\mathcal{L}_0} \subseteq \dot{X}_w^\infty(b)$ in $G \times T$ is the subgroup Γ generated by $G_{\mathcal{O}} \times T_{\mathcal{O}}$ and (ϖ, ϖ^{-1}) . Hence as representations of $G \times T$, we have

$$\sum (-1)^i H_i(\dot{X}_w(b), \overline{\mathbb{Q}}_\ell) \cong \mathrm{c}\text{-Ind}_{\Gamma}^{T \times G} \left(\sum (-1)^i H_i(\dot{X}_w(b)_{\mathcal{L}_0}, \overline{\mathbb{Q}}_\ell) \right).$$

Now let $\tilde{\Gamma}$ be the subgroup of $G \times T$ generated by Γ and $\{1\} \times T$. Note that $\tilde{\Gamma} \cong ZG_{\mathcal{O}} \times T$. The isomorphism (11.1) follows from the above together with the definition

of the homology groups of $\dot{X}_w(b) \mathcal{L}_0$ in terms of the cohomology of X_h (remembering that θ has level h by assumption).

It remains to determine the character on very regular elements of $L^\times$. We use (11.1) together with the corresponding character formula result for X_h (Proposition 8.3). By Lemma 5.7, we know that for each $\varphi \in \text{Gal}(L/K)$, there exists an element $g_\varphi \in N_G(G_{\mathcal{O}})$ satisfying $g_\varphi x g_\varphi^{-1} = \varphi(x)$ for all $x \in L^\times$ and that if $\varphi \in \text{Gal}(L/K)[n']$, one can choose $g_\varphi \in G_{\mathcal{O}}$. By Sect. 5.5, we know that $N_G(G_{\mathcal{O}})/G_{\mathcal{O}} \cong \mathbb{Z}/n_0\mathbb{Z}$, and therefore using the fact that

$$\text{Tr}\left(x^*; R_{T_h}^{G_h}(\theta)\right) = \sum_{\varphi \in \text{Gal}(L/K)[n']} \theta(g_\varphi x g_\varphi^{-1})$$

by Proposition 8.3, we have:

$$\text{Tr}\left(x^*; R_T^G(\theta)\right) = \sum_{\substack{g \in G/ZG_{\mathcal{O}} \\ gxg^{-1} \in ZG_{\mathcal{O}}}} \text{Tr}\left(x^*; R_{T_h}^{G_h}(\theta)\right) = \sum_{\varphi \in \text{Gal}(L/K)} \theta(g_\varphi x g_\varphi^{-1}).$$

□

Theorem 11.3 *Let $\theta \in \mathcal{X}$. If $|R_T^G(\theta)|$ is irreducible supercuspidal, then the assignment $\theta \mapsto |R_T^G(\theta)|$ is a geometric realization of automorphic induction and the Jacquet–Langlands correspondence. That is,*

$$|R_T^G(\theta)| \cong \text{JL}(\pi_\theta),$$

where JL denotes the Jacquet–Langlands transfer of the $\text{GL}_n(K)$ -representation π_θ to the (possibly split) inner form G of $\text{GL}_n(K)$. Moreover, writing $|R_T^G(\theta)| = c'_\theta R_T^G(\theta)$ for $c'_\theta \in \{\pm 1\}$, we have $c'_\theta = c_\theta$.

Remark 11.4 If $|R_T^G(\theta)|$ is irreducible supercuspidal, then $R_T^G(\theta) = (-1)^{r_\theta} \pi$, where π is an irreducible supercuspidal representation occurring in $H_{r_\theta}(\dot{X}_w^\infty(b), \overline{\mathbb{Q}_\ell})[\theta]$ for some $r_\theta \in \mathbb{Z}$. (There may be other degrees where π contributes, but they all cancel out. In particular, there may be more than one choice of r_θ , but the parity of r_θ is invariant.) Then by Theorem 11.3 implies that $c_\theta = (-1)^{r_\theta}$, which gives a geometric interpretation of Henniart's sign c_θ in terms of the surviving cohomological degree in the alternating sum $R_T^G(\theta)$. $\diamond$

Proof Write $|R_T^G(\theta)| = c'_\theta R_T^G(\theta)$ for some $c'_\theta = \pm 1$. If we can show that $|R_T^G(\theta)|$ satisfies the hypotheses of Proposition 10.5, then we are done. By assumption, $|R_T^G(\theta)|$ is an irreducible cuspidal representation and by definition of the $G \times T$ action on $R_T^G(\theta)$, the central character of $|R_T^G(\theta)|$ must be $\theta|_{K^\times}$.

To see that (i) of Proposition 10.5 holds, note that since L/K is unramified we have $\langle \varpi^n \rangle \mathcal{O}_K^\times = \text{Nm}_{L/K}(L^\times) = \ker(\epsilon)$. In particular, we see that $\epsilon \circ \det$ is trivial on $ZG_{\mathcal{O}}$ and so by Theorem 11.2, we have $|R_T^G(\theta)| \cong |R_T^G(\theta)| \otimes (\epsilon \circ \det)$.

We now establish (ii) of Proposition 10.5 and the additional assumption in the special case $n = 2$ and $q = 3$. By Theorem 11.2, we have that for any very regular element $x \in L^\times$,

$$\mathrm{Tr}(x^*; |R_T^G(\theta)|) = c'_\theta \cdot \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x).$$

If $n = 2$, $q = 3$, and $\theta \in \mathcal{X}^0$, then by Theorem 12.2, we know in addition that

$$c'_\theta = (-1)^{n'-1} = \begin{cases} -1 & \text{if } G \cong \mathrm{GL}_2(K), \\ +1 & \text{if } G \cong D_{1/2}^\times. \end{cases}$$

We have now established all the conditions required by Proposition 10.5 to conclude that $c'_\theta = c_\theta$ and $|R_T^G(\theta)| \cong \mathrm{JL}(\pi_\theta)$. $\square$

12 A geometric realization of automorphic induction and Jacquet–Langlands

In this section, we write down the cases in which we can prove Theorem 11.3 unconditionally. To this end, we consider the following two subsets of $\mathcal{X}$:

$$\begin{aligned} \mathcal{X}^0 &:= \{\theta \in \mathcal{X} : \theta|_{U_L^1} \text{ factors through the norm map } U_L^1 \rightarrow U_K^1\} \\ \mathcal{X}^{\min} &:= \{\theta \in \mathcal{X} : \theta \text{ is minimal admissible}\} \\ &= \{\theta \in \mathcal{X} : \text{the } \theta/\theta^\gamma \text{ have the same level for any } 1 \neq \gamma \in \mathrm{Gal}(L/K)\} \end{aligned}$$

Note that $\mathcal{X}^0 \subseteq \mathcal{X}^{\min}$ is the “depth zero” part of $\mathcal{X}^{\min}$.

Remark 12.1 Let $\theta \in L^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$ be a smooth character with trivial $\mathrm{Gal}(L/K)$ -stabilizer. Then its restriction to $\mathcal{O}_L^\times$ must have trivial $\mathrm{Gal}(L/K)$ -stabilizer. For the reader’s convenience, we summarize the relation between minimal admissibility and similar notions in the literature:

- θ is minimal admissible if and only if $(L/K, \theta)$ form a minimal admissible pair, which happens if and only if θ has only one “jump” in the sense of Bushnell–Henniart [9, Section 1.1].
- θ is minimal admissible if and only if it can be written in the form $\theta_{\mathrm{prim}} \cdot (\chi \circ \mathrm{Nm}_{L/K})$ for some smooth $\chi: K^\times \rightarrow \overline{\mathbb{Q}}_\ell^\times$, where θ_{prim} is *primitive* in the sense of Boyarchenko–Weinstein [7, Section 7.1] (see also Sect. 8 of the present paper).
- Let h be such that $\theta|_{U_L^h} = 1$ and $\theta|_{U_L^{h-1}} \neq 1$. Then θ is primitive if and only if θ is *regular* as a character of $\mathcal{O}_L^\times/U_L^h$ in the sense of Lusztig [26, Section 1.5], when $\mathcal{O}_L^\times/U_L^h$ is the F -fixed points of a maximal torus (see Remark 8.5). $\diamond$

12.1 Depth zero representations

In this section we only consider characters $\theta \in \mathcal{X}^0$ and give a nonvanishing result for the individual cohomology groups $H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta]$. Since each $\theta \in \mathcal{X}^0$ is of the form $\theta_0 \cdot (\chi \circ \text{Nm}_{L/K})$, where $\theta_0 \in \mathcal{X}^0$ and $\theta_0|_{U_L^1} = 1$, determining when $H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta] \neq 0$ can be reduced to the corresponding question for the cohomology of classical Deligne–Lusztig varieties. Recall from Proposition 7.4 that $\dim X_h = (n-1)(h-1) + (n'-1)$.

Theorem 12.2 Fix $\theta \in \mathcal{X}^0$ of level h and write $\theta = \theta_0 \cdot (\chi \circ \text{Nm}_{L/K})$ for some $\theta_0 \in \mathcal{X}^0$ of level 1 and some character χ of $K^\times$ of level h . Then:

(i) the cohomology groups $H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta]$ are concentrated in a single degree and

$$|R_{T_h}^{G_h}(\theta)| = H_c^{2(n-1)(h-1)+n'-1}(X_h, \overline{\mathbb{Q}}_\ell)[\theta] \cong H_c^{n'-1}(X_1, \overline{\mathbb{Q}}_\ell)[\theta_0] \otimes (\chi \circ \det)$$

(ii) the homology groups $H_i(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta]$ are concentrated in a single degree and

$$|R_T^G(\theta)| = H_{n'-1}(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta] \cong \text{c-Ind}_{Z \cdot \text{GL}_{n'}(\mathcal{O}_{D_{k_0/n_0}})}^{\text{GL}_{n'}(D_{k_0/n_0})}(\rho_\theta) \quad (12.1)$$

is an irreducible supercuspidal representation of G . Here ρ_θ is the extension of the $\text{GL}_{n'}(\mathcal{O}_{D_{k_0/n_0}})$ -representation $H_c^{n'-1}(X_1, \overline{\mathbb{Q}}_\ell)[\theta_0] \otimes (\chi \circ \det)$ obtained by letting $\varpi \in Z = K^\times$ act by $\theta(\varpi)$.

Moreover, $|R_{T_h}^{G_h}(\theta)| = (-1)^{n'-1} R_{T_h}^{G_h}(\theta)$ and $|R_T^G(\theta)| = (-1)^{n'-1} R_T^G(\theta)$.

Proof By Lemma 8.4 and Proposition 7.7, we have, as G_h -representations

$$H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta_0 \circ (\chi \circ \text{Nm}_{L/K})] \cong H_c^i(X_h, \overline{\mathbb{Q}}_\ell)[\theta_0] \otimes (\chi \circ \det) \cong H_c^{i-2(n-1)(h-1)}(X_1, \overline{\mathbb{Q}}_\ell)[\theta_0]$$

for all $i \geq 2(n-1)(h-1)$. This reduces the cohomology calculation to a statement about X_1 , which is a classical Deligne–Lusztig variety attached to the maximal torus $\mathbb{F}_{q^n}^\times$ in $\text{GL}_{n'}(\mathbb{F}_{q^{n_0}})$. By [13, Corollary 9.9],

$$H_c^i(X_1, \overline{\mathbb{Q}}_\ell)[\theta] \neq 0 \iff i = n' - 1.$$

This proves (i). Since $\dim X_h = (n-1)(h-1) + n' - 1$ by Proposition 7.4, we now also see the nonvanishing assertion of (ii) and $H_{n'-1}(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta]$ has the form 12.1 by Theorem 11.2. It is well known that this representation is irreducible and supercuspidal (see also Theorem 12.5). For example, one can show by hand (by the first part of the proof of Theorem 12.5) that the induction to the normalizer of $ZG_\mathcal{O}$ is irreducible, and then the conclusion follows from [27, Proposition 6.6]. $\square$

Theorem 12.3 For $\theta \in \mathcal{X}^0$, the assignment $\theta \mapsto H_{n'-1}(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta]$ is a geometric realization of automorphic induction and the Jacquet–Langlands correspondence. That is,

$$H_{n'-1}(\dot{X}_w^\infty(b), \overline{\mathbb{Q}}_\ell)[\theta] = (-1)^{n'-1} R_T^G(\theta) = |R_T^G(\theta)| \cong \text{JL}(\pi_\theta).$$

Proof By Theorem 12.2, we know that $|R_T^G(\theta)| = (-1)^{n'-1} R_T^G(\theta)$ is an irreducible supercuspidal representation, and by Theorem 11.2, we know that for any very regular element $x \in L^\times$,

$$\mathrm{Tr}(x^*; |R_T^G(\theta)|) = (-1)^{n'-1} \cdot \sum_{\gamma \in \mathrm{Gal}(L/K)} \theta^\gamma(x).$$

By definition ϵ is a finite-order character of $K^\times$ with $\ker(\epsilon) = \mathrm{Nm}_{L/K}(K^\times)$. Since L/K is unramified, $\ker(\epsilon)$ contains $\mathcal{O}_K^\times$, and therefore $\epsilon \circ \det$ is trivial on $Z \cdot \mathrm{GL}_{n'}(\mathcal{O}_{D_{k_0/n_0}})$. Hence $|R_T^G(\theta)| \otimes (\epsilon \circ \det) \cong |R_T^G(\theta)|$. We can now apply Proposition 10.5, noting that in the case $n = 2$, $q = 3$, we have the correct sign c_θ (compare with Theorem 10.3) as required by the proposition. $\square$

Remark 12.4 Observe that as in Remark 11.4, the nonvanishing degree $n' - 1$ of the homology of $\dot{X}_w^\infty(b)$ gives a geometric interpretation of Henniart’s sign c_θ from Theorem 10.3. $\diamond$

12.2 Representations corresponding to minimal admissible characters

We now prove the supercuspidality of $|R_T^G(\theta)|$ for $\theta \in \mathcal{X}^{\min}$. The main technical inputs are the irreducibility of $|R_{T_h}^{G_h}(\theta)|$ (Sect. 8) and a “cuspidality” result for $|R_{T_h}^{G_h}(\theta)|$ (Theorem 9.1).

Theorem 12.5 *If $\theta \in \mathcal{X}^{\min}$, then $|R_T^G(\theta)|$ is irreducible supercuspidal.*

Proof We first establish some notation. If $\pi: H \rightarrow \mathrm{GL}(V)$ is a representation of a subgroup $H \subset G$, then for any $\gamma \in G$, we define ${}^\gamma\pi: \gamma H \gamma^{-1} \rightarrow \mathrm{GL}(V)$ by ${}^\gamma\pi(g) := \pi(\gamma^{-1} g \gamma)$. Assume that θ is minimal admissible of level h . By definition, we can write $\theta = \theta' \otimes (\chi \circ \mathrm{Nm})$, where θ' is a primitive character of $L^\times$ of level $h' \leq h$, χ is any character of $K^\times$ of level h , and $\mathrm{Nm}: L^\times \rightarrow K^\times$ is the usual norm. Denoting by θ, θ', χ the corresponding restrictions to the unit groups, by Proposition 7.7 and Lemma 8.4, we have

$$R_{T_h}^{G_h}(\theta) = R_{T_h}^{G_h}(\theta' \otimes (\chi \circ \mathrm{Nm})) \cong R_{T_h}^{G_h}(\theta') \otimes (\chi \circ \det) \cong R_{T_{h'}}^{G_{h'}}(\theta') \otimes (\chi \circ \det).$$

In particular, by Theorem 11.2, we see that

$$R_T^G(\theta) \cong R_T^G(\theta') \otimes (\chi \circ \det).$$

Since twists of irreducible supercuspidal representations are again irreducible supercuspidal, it suffices to prove the theorem for primitive characters θ .

Assume now that θ is a primitive character of level h . By Theorem 8.1, $|R_{T_h}^{G_h}(\theta)|$ is irreducible. Recall that there is a natural surjection $G_{\mathcal{O}} \rightarrow G_h$ so that we may view $|R_{T_h}^{G_h}(\theta)|$ as a representation of $G_{\mathcal{O}}$. We extend this to a representation of $Z \cdot G_{\mathcal{O}} =$

$\langle \varpi \rangle \cdot G_{\mathcal{O}}$ by letting ϖ act on $|R_{T_h}^{G_h}(\theta)|$ by $\theta(\varpi)$. We first claim that

$$\rho_{\theta} := \text{c-Ind}_{Z \cdot G_{\mathcal{O}}}^{Z \cdot N_G(G_{\mathcal{O}})} \left(|R_{T_h}^{G_h}(\theta)| \right)$$

is irreducible. Recall from Sect. 5.5 that $\#N_G(G_{\mathcal{O}})/G_{\mathcal{O}} = n_0$ and let $\{1, \varphi_2, \dots, \varphi_{n_0}\}$ denote a complete set of coset representatives of $\text{Gal}(L/K)/\text{Gal}(L/K)[n']$. By Lemma 5.7, there exists $g_{\varphi_i} \in N_G(G_{\mathcal{O}})$ such that $g_{\varphi_i}^{-1} x g_{\varphi_i} = \varphi_i(x)$ for all $x \in \mathcal{O}_L^{\times}$. By Mackey's irreducibility criterion, it suffices to show that

$$\text{Hom}_{G_{\mathcal{O}}} \left(|R_{T_h}^{G_h}(\theta)|, {}^{g_{\varphi_i}} |R_{T_h}^{G_h}(\theta)| \right) = 0, \quad \text{for } i = 2, \dots, n_0. \quad (12.2)$$

Fix some i with $2 \leq i \leq n_0$. By Proposition 8.3, for any very regular element $x \in \mathcal{O}_L^{\times}$,

$$\text{Tr} \left(x^*; {}^{g_{\varphi_i}} R_{T_h}^{G_h}(\theta) \right) = \text{Tr} \left((g_{\varphi_i}^{-1} x g_{\varphi_i})^*; R_{T_h}^{G_h}(\theta) \right) = \sum_{\gamma \in \text{Gal}(L/K)[n']} \theta^{\gamma}(\varphi_i(x)).$$

Applying Lemma 10.4 to the case when $\theta' = \theta^{\varphi_i}$ and the base field K is replaced by the unique subfield of index n' in L containing K , we see that

$$\text{Tr} \left(x^*; {}^{g_{\varphi_i}} R_{T_h}^{G_h}(\theta) \right) \neq \text{Tr} \left(x^*; R_{T_h}^{G_h}(\theta) \right).$$

But now ${}^{g_{\varphi_i}} R_{T_h}^{G_h}(\theta)$ and $R_{T_h}^{G_h}(\theta)$ are irreducible representations of $G_{\mathcal{O}}$ whose characters differ from each other, and so necessarily (12.2) holds and ρ_{θ} is irreducible.

We now fix $\gamma \in G \setminus N_G(Z \cdot G_{\mathcal{O}})$. Once again by Mackey's criterion, to complete the proof we must show that

$$\text{Hom}_{\gamma Z G_{\mathcal{O}} \gamma^{-1} \cap Z G_{\mathcal{O}}} \left(|R_{T_h}^{G_h}(\theta)|, {}^{\gamma} |R_{T_h}^{G_h}(\theta)| \right) = 0. \quad (12.3)$$

At this point, let b be a special representative. By Sect. 5.5, we may assume that $\gamma = \Pi_0^v$, where $v = (v_1, \dots, v_1, v_2, \dots, v_2, \dots, v_{n'}, \dots, v_{n'})$ (each v_i repeated n_0 times) for $0 = v_1 \leq v_2 \leq \dots \leq v_{n'}$, and Π_0^v is the block-diagonal matrix whose i th $n_0 \times n_0$ block is given by $\begin{pmatrix} 0 & \varpi \\ 1_{n_0-1} & 0 \end{pmatrix}^{v_i}$. Observe that if $(A_{i,j})_{1 \leq i, j \leq n'} \in \text{GL}_n(\check{K})$, where each $A_{i,j}$ is a $(n_0 \times n_0)$ -matrix, then

$$\Pi_0^{-v} \cdot (A_{i,j})_{1 \leq i, j \leq n'} \cdot \Pi_0^v = (\Pi_0^{-v_i} A_{i,j} \Pi_0^{v_j})_{1 \leq i, j \leq n'}. \quad (12.4)$$

For a parabolic subgroup P' of $\text{GL}_{n'}$ containing the upper triangular matrices, let $\check{N}_{P'}$ be its unipotent radical. Let $\check{N}_P$ denote the subgroup of $\text{GL}_n(\check{K})$ such that each $(n_0 \times n_0)$ -block consists of a diagonal matrix and the (i, j) th block is nonzero if and only if the (i, j) th entry of an element of $\check{N}_{P'}$ is nonzero. Write $N_P = \check{N}_P^F \cap G_{\mathcal{O}}$. For $h \geq 1$ let $N_P^h = N_P \cap \ker(G_{\mathcal{O}} \rightarrow G_h)$.

We claim that there exists a parabolic $P' \subseteq \mathrm{GL}_{n'}$ as above, such that $\Pi_0^{-v} N_P^{h-1} \Pi_0^v \subset \ker(G_{\mathcal{O}} \rightarrow G_h)$. Let $1 \leq i_0 \leq n'$ be the last $v_{i_0} = 0$ so that $v_{i_0} < v_{i_0+1}$, and let P' be the minimal parabolic corresponding to the partition $i_0 + (n' - i_0)$. Let $(A_{i,j})_{1 \leq i,j \leq n'} \in N_P^{h-1}$ so that each $A_{i,j}$ is a diagonal $n_0 \times n_0$ matrix whose entries all lie in $\mathbb{W}_h^{h-1}(\overline{\mathbb{F}}_q)$. By (12.4), we see that the (i, j) th block of $\Pi_0^{-v} \cdot (A_{i,j})_{1 \leq i,j \leq n'} \Pi_0^v$ is $\Pi_0^{-v_i} A_{i,j} \Pi_0^{v_j}$, so that in particular, if $1 \leq i \leq i_0$ and $i_0 + 1 \leq j \leq n'$, then $v_j - v_i > 0$. By definition of G_h (Sect. 5.3), we now have that $\Pi_0^{-v} \cdot (A_{i,j})_{1 \leq i,j \leq n'} \Pi_0^v \in \ker(G_{\mathcal{O}} \rightarrow G_h)$.

The above implies that the restriction of ${}^{\Pi_0} |R_{T_h}^{G_h}(\theta)|$ to N_P^{h-1} is trivial. On the other hand, by Theorem 9.1, the restriction of $|R_{T_h}^{G_h}(\theta)|$ to N_P^{h-1} does not contain the trivial representation. Therefore:

$$\begin{aligned} & \dim \mathrm{Hom}_{\gamma Z G_{\mathcal{O}} \gamma^{-1} \cap Z G_{\mathcal{O}}}(|R_{T_h}^{G_h}(\theta)|, {}^{\gamma} |R_{T_h}^{G_h}(\theta)|) \\ & \leq \dim \mathrm{Hom}_{N_P^{h-1}}(|R_{T_h}^{G_h}(\theta)|, {}^{\gamma} |R_{T_h}^{G_h}(\theta)|) \\ & \leq \dim \mathrm{Hom}_{N_P^{h-1}}(|R_{T_h}^{G_h}(\theta)|, \mathrm{triv}) = 0. \end{aligned}$$

□

Combining Theorems 11.3 and 12.5 proves:

Theorem 12.6 *If $\theta \in \mathcal{X}^{\min}$, then the assignment $\theta \mapsto |R_T^G(\theta)|$ is a geometric realization of automorphic induction and the Jacquet–Langlands correspondence.*

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