

STRONG SPACE-TIME CONVEXITY AND THE HEAT EQUATION

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ABSTRACT. We prove local strong convexity of the space-time level sets of the heat equation on convex rings for zero initial data, strengthening a result of Borell. Our proof introduces a parabolic version of a two-point maximum principle of Rosay-Rudin.

1. INTRODUCTION

A classic question in elliptic PDEs is: does the solution to a Dirichlet problem on a domain or convex ring inherit convexity properties from its boundary? Building on the well-known result that the Green's function of a convex domain in \mathbb{R}^2 has convex level curves (see [1]), this question has been studied by many authors including Gabriel, Lewis and Caffarelli-Spruck [2, 3, 4, 5, 8, 9, 10, 11, 12, 17, 19, 20, 27, 28, 29, 31, 32, 34, 35, 36, 37, 40]. One method is the “macroscopic” approach, using a globally defined function of two points; another is “microscopic”, using the principal curvatures of the level sets and a constant rank theorem. These results show that for a large class of PDEs, the superlevel sets of the solution u are convex (i.e. u is *quasiconcave*) if the boundary is convex. On the other hand, there are counterexamples to the convexity of level sets for solutions to certain semi-linear PDEs [33, 21] and the mean curvature equation [39].

The parabolic version of this problem is far less developed. The first major result is due to Borell [6] who considered the heat equation on convex rings with zero initial data and proved space-time convexity of the superlevel sets. Borell's result was extended to more general parabolic equations by Ishige-Salani [24, 25], again assuming zero initial data. For general quasiconcave initial data u_0 , Ishige-Salani had shown that quasiconcavity of the superlevel sets is in general *not* preserved [23]. Recently the authors gave counterexamples to preservation of quasiconcavity even under the additional assumption of subharmonicity of u_0 [13], which was expected to be sufficient (cf. [17]).

We describe now Borell's result more precisely. let Ω_0 and Ω_1 be bounded open convex bodies in \mathbb{R}^n with smooth boundaries and $0 \in \Omega_1 \subset\subset \Omega_0$, and define $\Omega = \Omega_0 \setminus \Omega_1$. Let u solve

$$(1.1) \quad \begin{cases} \partial u / \partial t = \Delta u, & (x, t) \in \Omega \times (0, \infty) \\ u(x, 0) = 0, & x \in \Omega \\ u(x, t) = 0, & (x, t) \in \partial\Omega_0 \times [0, \infty) \\ u(x, t) = 1, & (x, t) \in \overline{\Omega}_1 \times [0, \infty). \end{cases}$$

Borell [6] showed, using the language of probability and Brownian motion, that the level sets $\{u = c\} \subset \Omega_0 \times [0, \infty)$ are convex hypersurfaces of \mathbb{R}^{n+1} . It is said that u is

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space-time quasiconcave. Our main result is an improvement from convexity to strong convexity.

Theorem 1.1. *Let u solve (1.1). The level sets $\{u = c\}$ for $c \in (0, 1)$ are locally strongly convex hypersurfaces of \mathbb{R}^{n+1} .*

We clarify now our terminology. A smooth hypersurface S in \mathbb{R}^N is *convex* if it is contained in the boundary of a convex body in \mathbb{R}^N . It is *strongly convex* if it can be represented locally around any $p \in S$ as the graph of a function f with uniformly positive Hessian (its eigenvalues are bounded below by positive constants independent of p), and S is *locally strongly convex* if it is the union of strongly convex hypersurfaces. A convex hypersurface S is *strictly convex* if it does not contain any line segment, a weaker condition than local strong convexity. Note that we do not require Ω_0 and Ω_1 to have strongly or strictly convex boundaries.

Borell [7] introduced the notion of the *parabolic convexity* of a set as follows. We say that $E \subset \mathbb{R}^n \times [0, \infty)$ is parabolically convex if $X = (x, s), Y = (y, t) \in E$ implies that the parabolic segment

$$\lambda \mapsto P_{X,Y}(\lambda) := \left((1 - \lambda)x + \lambda y, ((1 - \lambda)\sqrt{s} + \lambda\sqrt{t})^2 \right) \quad \text{for } \lambda \in [0, 1],$$

lies entirely in E . It was shown by Ishige-Salani [24] that solutions u to (1.1), and for certain more general parabolic equations, have parabolically convex superlevel sets [24, 25]. In the course of proving our main result, we will reprove the Ishige-Salani result for the heat equation.

Our approach is different from the works above and applies the maximum principle to a parabolic version of a two-point function of Rosay-Rudin [34]. Namely, we will consider the function

$$\begin{aligned} \mathcal{C}_p((x, s), (y, t)) &= \frac{u(x, s) + u(y, t)}{2} - u\left(\frac{x+y}{2}, \left(\frac{s^{1/p} + t^{1/p}}{2}\right)^p\right) \\ (1.2) \quad &= u(x, s) - u\left(\frac{x+y}{2}, \left(\frac{s^{1/p} + t^{1/p}}{2}\right)^p\right) \end{aligned}$$

on

$\Sigma = \{((x, s), (y, t)) \in (\bar{\Omega} \times (0, \infty)) \times (\bar{\Omega} \times (0, \infty)) \mid u(x, s) = u(y, t) \text{ and } (x+y)/2 \in \bar{\Omega}\}$, and for a constant $p \in [1, 2]$. We first show that $\mathcal{C}_2 \leq 0$ on Σ . Thus if $X, Y \in \{u = c\}$ then $P_{X,Y}(1/2) \in \{u \geq c\}$ and it follows by an iterative argument that $P_{X,Y}(\lambda) \in \{u \geq c\}$ for all $0 \leq \lambda \leq 1$, in particular we obtain another proof of the parabolic convexity of superlevel sets of solutions u to (1.1) (see Theorem 4.1). We then show that $\mathcal{C}_1 \leq -c(|x_0 - y|^2 + |s_0 - t|^2)$ for all (y, t) in a neighborhood of any (x_0, s_0) in Σ for some constant $c > 0$, which in turn implies the strong convexity of the level sets of u (see for example [34, Section 3]).

A brief outline of the proof is as follows. Sections 2 and 3 develop the parabolic version of the Rosay-Rudin two-point maximum principle. A proof of the parabolic convexity of the superlevel sets using (1.2) is given in Section 4. In Section 5 we prove Theorem 1.1 and finally in Section 6 we end with some remarks and open questions.

Note. Shortly after this paper was posted on the arXiv preprint server, an updated version of an article of Chen-Ma-Salani [15] appeared, including a result related to

Theorem 1.1 proved via a constant rank theorem for the second fundamental form of the level surfaces (cf. Remark 5 in Section 6 below).

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2. A PARABOLIC ROSAY-RUDIN LEMMA

Consider u solving (1.1). We begin by proving a parabolic version of a lemma of Rosay-Rudin [34, Lemma 1.3]. Fix $T > 0$ and interior points (x_0, s_0) and (y_0, t_0) in $\Omega \times (0, T]$ with $u(x_0, s_0) = u(y_0, t_0)$ and assume that Du , the spatial derivative of u , does not vanish at these points. Let $L = (L_{ij}) \in O(n)$ satisfy

$$L(Du(x_0, s_0)) = cDu(y_0, t_0), \quad \text{for } c = \frac{|Du(x_0, s_0)|}{|Du(y_0, t_0)|}.$$

We have:

Lemma 2.1. *Assume first that $s_0, t_0 \in (0, T)$. There exists a smooth function $\psi(w, \tau) = O(|w|^3 + |\tau|^2)$ such that for all $(w, \tau) \in \mathbb{R}^n \times \mathbb{R}$ sufficiently close to the origin,*

$$u(x_0 + w, s_0 + \tau) = u(y_0 + cLw + \chi(w, \tau)\xi + \psi(w, \tau)\xi, t_0 + c^2\tau), \quad \text{where } \xi = \frac{Du(y_0, t_0)}{|Du(y_0, t_0)|},$$

for $\chi(w, \tau)$ defined by

$$\chi(w, \tau) = \frac{1}{|Du(y_0, t_0)|}(u(x_0 + w, s_0 + \tau) - u(y_0 + cLw, t_0 + c^2\tau)),$$

which satisfies the heat equation $\frac{\partial \chi}{\partial \tau} = \Delta_w \chi$.

If s_0 or t_0 is equal to T , then the same holds with the additional restriction $\tau \leq 0$.

Proof. Define a smooth real-valued function G in a neighborhood of zero in $(\mathbb{R}^n \times \mathbb{R}) \times \mathbb{R}$ by

$$G((w, \tau), \psi) = u(y_0 + cLw + \chi(w, \tau)\xi + \psi\xi, t_0 + c^2\tau) - u(x_0 + w, s_0 + \tau),$$

which satisfies $G((0, 0), 0) = 0$. Compute

$$\frac{\partial G}{\partial \psi}((0, 0), 0) = \sum_i D_i u(y_0, t_0) \xi_i = |Du(y_0, t_0)| > 0.$$

Hence by the Implicit Function Theorem, there exists a smooth $\psi = \psi(w, \tau)$ satisfying

$$G((w, \tau), \psi(w, \tau)) = 0,$$

for w, τ close to zero.

It remains to show that $\psi(w, \tau) = O(|w|^3 + |\tau|^2)$. First compute at the origin

$$\begin{aligned} 0 &= \frac{\partial G}{\partial w_j} = u_i(y_0, t_0) \left(cL_{ij} + \frac{1}{|Du(y_0, t_0)|} (u_j(x_0, s_0) - cu_k(y_0, t_0)L_{kj})\xi_i + \frac{\partial \psi}{\partial w_j} \xi_i \right) \\ &\quad - u_j(x_0, s_0) \\ &= |Du(y_0, t_0)| \frac{\partial \psi}{\partial w_j}(0, 0), \end{aligned}$$

which implies that $\frac{\partial\psi}{\partial w_j}(0,0) = 0$. Next,

$$\begin{aligned} 0 &= \frac{\partial^2 G}{\partial w_j \partial w_p} = u_{ik}(y_0, t_0) c^2 L_{kp} L_{ij} + u_i(y_0, t_0) \frac{(u_{jp}(x_0, s_0) - c^2 u_{k\ell}(y_0, t_0) L_{kj} L_{\ell p}) \xi_i}{|Du(y_0, t_0)|} \\ &\quad + u_i(y_0, t_0) \frac{\partial^2 \psi}{\partial w_p \partial w_j}(0,0) \xi_i - u_{jp}(x_0, s_0) \\ &= |Du(y_0, t_0)| \frac{\partial^2 \psi}{\partial w_p \partial w_j}(0,0), \end{aligned}$$

so that $\frac{\partial^2 \psi}{\partial w_p \partial w_j}(0,0) = 0$. Finally,

$$\begin{aligned} 0 &= \frac{\partial G}{\partial \tau} = \xi_i u_i(y_0, t_0) \frac{\partial \chi}{\partial \tau} + \xi_i u_i(y_0, t_0) \frac{\partial \psi}{\partial \tau} + c^2 u_t(y_0, t_0) - u_t(x_0, s_0) \\ &= |Du(y_0, t_0)| \frac{1}{|Du(y_0, t_0)|} (u_t(x_0, s_0) - c^2 u_t(y_0, t_0)) + |Du(y_0, t_0)| \frac{\partial \psi}{\partial \tau} \\ &\quad + c^2 u_t(y_0, t_0) - u_t(x_0, s_0) \\ &= |Du(y_0, t_0)| \frac{\partial \psi}{\partial \tau}, \end{aligned}$$

giving $\frac{\partial \psi}{\partial \tau}(0,0) = 0$, as required. \square

We end this section with another technical lemma. Using the notation of Lemma 2.1, we write

$$(x, s) = (x_0 + w, s_0 + \tau)$$

and

$$(y, t) = (y_0 + cLw + \chi(w, \tau)\xi + \psi(w, \tau)\xi, t_0 + c^2\tau).$$

Then, evaluating at (y_0, t_0) ,

$$(2.1) \quad \frac{\partial}{\partial \tau} y_i = \frac{\partial \chi}{\partial \tau} \xi_i = (\Delta_w \chi) \xi_i = \Delta_w y_i.$$

We make use of this in the following lemma.

Lemma 2.2. *With the notation above, if v is a solution to the heat equation then*

$$(\Delta_w - \partial_\tau) v(x, s) = 0, \quad \text{and} \quad (\Delta_w - \partial_\tau) v(y, t) = 0,$$

when evaluated at $(w, \tau) = (0, 0)$.

Proof. The first equation is immediate. For the second, computing at $(0, 0)$,

$$\begin{aligned} (\Delta_w - \partial_\tau) v(y, t) &= \sum_j v_{ik}(y, t) \frac{\partial y_k}{\partial w_j} \frac{\partial y_i}{\partial w_j} + \sum_j v_i(y, t) \frac{\partial^2 y_i}{\partial w_j^2} - v_i(y, t) \frac{\partial y_i}{\partial \tau} \\ &\quad - c^2 v_t(y, t) \\ &= \sum_j c^2 v_{ik}(y, t) L_{kj} L_{ij} - c^2 v_t(y, t) \\ &= c^2 \Delta v(y, t) - c^2 v_t(y, t) = 0. \end{aligned}$$

completing the proof. \square

3. MAXIMUM PRINCIPLE FOR A TWO-POINT FUNCTION

Recall the following family of functions in (1.2):

$$(3.1) \quad \mathcal{C}_p((x, s), (y, t)) = u(x, s) - u\left(\frac{x+y}{2}, \left(\frac{s^{1/p} + t^{1/p}}{2}\right)^p\right)$$

In this section we prove a parabolic maximum principle for a slight modification of these functions analogous to the function introduced by Rosay-Rudin [34]. We begin by recalling some basic properties of the solution u to (1.1).

Proposition 3.1. *We have*

- (i) $0 < u < 1$ on $\Omega \times (0, \infty)$.
- (ii) If $x \in \bar{\Omega}_1$ and $w \in \Omega$ then $(w - x) \cdot Du(w, t) < 0$ for $t \in (0, \infty)$.
- (iii) $\Delta u > 0$ on $\Omega \times (0, \infty)$.

Proof. This is well-known, as a consequence of the maximum principle (see [6, 17] for example). \square

Fix $p \in [1, 2]$ and $T \in (0, \infty)$. Let h_1, \dots, h_N be arbitrary solutions of the heat equation on $\Omega \times (0, T]$. In the later sections we will in fact only make use of $p = 1$ or $p = 2$, and we will take $N = 1$. We also fix a small constant $\delta > 0$. We consider the quantity

$$(3.2) \quad \mathcal{Q}((x, s), (y, t)) := \mathcal{C}_p((x, s), (y, t)) + \sum_{i=1}^N (h_i(x, s) - h_i(y, t))^2 - \delta s$$

on

$$\Sigma = \{((x, s), (y, t)) \in (\bar{\Omega} \times (0, \infty)) \times (\bar{\Omega} \times (0, \infty)) \mid u(x, s) = u(y, t) \text{ and } (x+y)/2 \in \bar{\Omega}\}.$$

We say that $((x, s), (y, t)) \in \Sigma$ is an *interior point* of Σ if $x, y, (x+y)/2 \in \Omega$. Note that s or t are allowed to be equal to T . The result of this section is the following maximum principle, which is a parabolic analogue of [34, Theorem 4.3].

Proposition 3.2. *\mathcal{Q} does not attain a maximum at an interior point of Σ .*

Proof. First we assume that n is even. Suppose for a contradiction that \mathcal{C} achieves a maximum at some interior point of Σ , which we will call $((x_0, s_0), (y_0, t_0))$. We will rule this out.

We apply Lemma 2.1 and use the notation there. Note that by part (ii) of Proposition 3.1, Du does not vanish at (x_0, s_0) or (y_0, t_0) . For sufficiently small $\tau \in \mathbb{R}$ and $w \in \mathbb{R}^n$, define

$$(x, s) = (x_0 + w, s_0 + \tau)$$

and

$$(y, t) = (y_0 + cLw + \chi(w, \tau)\xi + \psi(w, \tau)\xi, t_0 + c^2\tau)$$

and consider

$$F(w, \tau) = \mathcal{Q}((x, s), (y, t)).$$

Note that if one of s_0, t_0 is equal to T then we must restrict to τ to be nonpositive. Write

$$Z = \left(\frac{x+y}{2}, \left(\frac{s^{1/p} + t^{1/p}}{2} \right)^p \right).$$

Then

$$\begin{aligned} \sum_j \frac{\partial^2}{\partial w_j^2} u(Z) &= \sum_j u_{k\ell}(Z) \frac{1}{4} (\delta_{kj} + cL_{kj})(\delta_{\ell j} + cL_{\ell j}) + u_k(Z) \frac{1}{2} \Delta_w y_k \\ &= \frac{1+c^2}{4} \Delta u(Z) + \frac{c}{2} L_{k\ell} u_{k\ell}(Z) + u_k(Z) \frac{1}{2} \Delta_w y_k. \end{aligned}$$

We make an appropriate choice of L following [34, Lemma 4.1(a)], recalling our assumption that n is even. Namely, after making an orthonormal change of coordinates, we may assume, without loss of generality that $Du(x_0, s_0)/|Du(x_0, s_0)|$ is e_1 , and

$$Du(y_0, t_0)/|Du(y_0, t_0)| = \cos \theta e_1 + \sin \theta e_2,$$

for some $\theta \in [0, 2\pi)$. Here we are writing $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots)$ etc for the standard unit basis vectors in \mathbb{R}^n . Observe that

$$(3.3) \quad \cos \theta = \frac{Du(x_0, s_0)}{|Du(x_0, s_0)|} \cdot \frac{Du(y_0, t_0)}{|Du(y_0, t_0)|}, \quad c = \frac{|Du(x_0, s_0)|}{|Du(y_0, t_0)|}.$$

Then define the isometry L by

$$L(e_i) = \begin{cases} \cos \theta e_i + \sin \theta e_{i+1}, & \text{for } i = 1, 3, \dots, n-1 \\ -\sin \theta e_{i-1} + \cos \theta e_i, & \text{for } i = 2, 4, \dots, n. \end{cases}$$

In terms of entries of the matrix (L_{ij}) , this means that $L_{kk} = \cos \theta$ for $k = 1, \dots, n$ and for $\alpha = 1, 2, \dots, n/2$, we have

$$L_{2\alpha-1, 2\alpha} = -\sin \theta, \quad L_{2\alpha, 2\alpha-1} = \sin \theta,$$

with all other entries zero. Then for any point,

$$\sum_{i,k} L_{ki} u_{ki} = (\cos \theta) \Delta u.$$

Hence

$$\sum_j \frac{\partial^2}{\partial w_j^2} u(Z) = \frac{1+c^2+2c\cos\theta}{4} \Delta u(Z) + u_k(Z) \frac{1}{2} \Delta_w y_k.$$

Compute

$$\begin{aligned} \frac{\partial}{\partial \tau} u(Z) &= \frac{1}{2} \left(\left(\frac{s_0^{1/p} + t_0^{1/p}}{2} \right)^{p-1} (s_0^{\frac{1}{p}-1} + c^2 t_0^{\frac{1}{p}-1}) \right) u_t(Z) + u_k(Z) \frac{1}{2} \frac{\partial y_k}{\partial \tau} \\ &= \frac{1}{2} \left(\left(\frac{1 + (t_0/s_0)^{1/p}}{2} \right)^{p-1} + c^2 \left(\frac{1 + (s_0/t_0)^{1/p}}{2} \right)^{p-1} \right) u_t(Z) + u_k(Z) \frac{1}{2} \frac{\partial y_k}{\partial \tau} \\ &\geq \frac{1}{4} \left(1 + c^2 + (t_0/s_0)^{(p-1)/p} + c^2 (s_0/t_0)^{(p-1)/p} \right) u_t(Z) + u_k(Z) \frac{1}{2} \frac{\partial y_k}{\partial \tau}, \end{aligned}$$

where for the last line we used $u_t(Z) = \Delta u(Z) \geq 0$ from Proposition 3.1 and the concavity of the map $x \mapsto x^{p-1}$. Note that the inequality is an equality in the cases $p = 1$ and $p = 2$.

Hence, using (2.1),

$$\left(\Delta_w - \frac{\partial}{\partial \tau} \right) u(Z) \leq \frac{1}{4} \left(2c \cos \theta - (t_0/s_0)^{(p-1)/p} - c^2 (s_0/t_0)^{(p-1)/p} \right) \Delta u(Z).$$

Putting this together, we obtain at $(w, \tau) = (0, 0)$, using Lemma 2.2 and (3.3),

$$\begin{aligned} \left(\Delta_w - \frac{\partial}{\partial \tau} \right) F &\geq \frac{1}{4} \left(-2c \cos \theta + (t_0/s_0)^{(p-1)/p} + c^2 (s_0/t_0)^{(p-1)/p} \right) \Delta u(Z) \\ &\quad + 2 \sum_{j=1}^n \sum_{i=1}^N \left(\frac{\partial}{\partial w_j} (h_i(x, s) - h_i(y, t)) \right)^2 + \delta \\ &\geq \frac{1}{4|Du(y_0, t_0)|^2} \left| \left(\frac{s_0}{t_0} \right)^{\frac{p-1}{2p}} Du(x_0, s_0) - \left(\frac{t_0}{s_0} \right)^{\frac{p-1}{2p}} Du(y_0, t_0) \right|^2 \Delta u(Z) + \delta \\ &> 0. \end{aligned}$$

This contradicts the fact that F attains a maximum at this point.

Finally, we deal with the case when n is odd, making modifications analogous to those in [34]. Namely, define L to be an isometry of \mathbb{R}^{n+1} satisfying $L(Du(x_0, s_0), 0) = (c(Du)(y_0, t_0), 0)$ and in Lemma 2.1 we consider $w \in \mathbb{R}^{n+1}$. Writing π for the projection $(w_1, \dots, w_{n+1}) \mapsto (w_1, \dots, w_n)$ the statement of Lemma 2.1 becomes

$$u(x_0 + \pi(w), s_0 + \tau) = u(y_0 + c\pi(Lw) + \chi(w, \tau)\xi + \psi(w, \tau)\xi, t_0 + c^2\tau),$$

for the same ξ and with

$$\chi(w, \tau) = \frac{1}{|Du(y_0, t_0)|} (u(x_0 + \pi(w), s_0 + \tau) - u(y_0 + c\pi(Lw), t_0 + c^2\tau)),$$

which satisfies the heat equation in a neighborhood of the origin in $\mathbb{R}^{n+1} \times \mathbb{R}$. The rest of the proof then goes through with the obvious changes. \square

4. PARABOLIC CONVEXITY

In this section we give a proof of a result of Ishige-Salani [25] that the superlevel sets of u solving (1.1) are parabolically convex. Our proof is somewhat different, and uses the following two point function from (1.2):

$$\mathcal{C}_2((x, s), (y, t)) = u(x, s) - u\left(\frac{x+y}{2}, \left(\frac{\sqrt{s} + \sqrt{t}}{2}\right)^2\right)$$

on

$$\Sigma = \{((x, s), (y, t)) \in (\bar{\Omega} \times (0, \infty)) \times (\bar{\Omega} \times (0, \infty)) \mid u(x, s) = u(y, t) \text{ and } (x+y)/2 \in \bar{\Omega}\}.$$

Theorem 4.1. *We have $\mathcal{C}_2 \leq 0$ on Σ . Equivalently, the superlevel sets of u are parabolically convex.*

Proof. Fix $T \in (0, \infty)$ and a small $\delta > 0$ and consider the quantity

$$\mathcal{Q}((x, s), (y, t)) = \mathcal{C}_2((x, s), (y, t)) - \delta s.$$

We will prove $\mathcal{Q} \leq 0$ on $\Sigma_T = \Sigma \cap \{0 \leq s, t \leq T\}$ and the result will follow from letting $\delta \rightarrow 0$ and $T \rightarrow \infty$.

From Proposition 3.2, we only need to consider the case when (x, s) or (y, t) are boundary points of Σ_T . If s and t are both positive, there are four cases:

- (1) If x or y lie on $\partial\Omega_1$, by convexity of Ω_1 and the definition of Σ_T , the only possibility is that x, y and $(x+y)/2$ all lie on $\partial\Omega_1$ and by the boundary conditions for u we have $\mathcal{Q} \leq 0$.
- (2) If x or y lie on $\partial\Omega_0$ then $u(x, s) = 0 = u(y, t)$ and $\mathcal{Q} \leq 0$.
- (3) If $(x+y)/2 \in \partial\Omega_0$ then we must have x and y in $\partial\Omega_0$ by convexity of Ω_0 and we are in case (2).
- (4) If $(x+y)/2 \in \partial\Omega_1$ then $u((x+y)/2, (\sqrt{s}/2 + \sqrt{t}/2)^2) = 1$ and $\mathcal{Q} \leq 0$.

It remains to deal with the case when s or t (or both) tend to zero. A difficulty here is that u is discontinuous at $t = 0$ at the boundary of Ω_1 . Assume we have a sequence of points $X_i = (x_i, s_i)$ and $Y_i = (y_i, t_i)$ in Σ_T for which $\mathcal{Q}(X_i, Y_i) \geq \eta$ for a positive constant $\eta > 0$. Define $Z_i = (z_i, r_i)$, where $z_i = (x_i + y_i)/2$ and $r_i = (\sqrt{s_i}/2 + \sqrt{t_i}/2)^2$. Assume that

$$X_i \rightarrow X = (x, s), \quad Y_i \rightarrow Y = (y, t), \quad Z_i \rightarrow Z = (z, r)$$

for $z = (x+y)/2$ and $r = (\sqrt{s}/2 + \sqrt{t}/2)^2$.

We also assume, without loss of generality, that $s \leq t$. There are two cases.

(i) The case when $t > 0$ and $s = 0$. We must have $x \in \partial\Omega_1$ since otherwise this would contradict the inequality $\mathcal{Q}(X_i, Y_i) \geq \eta$. By the same reasoning as in (1)-(4) above, we may also assume that y and $z = (x+y)/2$ lie in Ω . We have the following lemma:

Lemma 4.2. *Suppose that $0 \leq s < t$ and $x \in \partial\Omega_1$, $y \in \Omega$. Then*

$$(4.1) \quad \frac{d}{d\lambda} u((1-\lambda)x + \lambda y, ((1-\lambda)\sqrt{s} + \lambda\sqrt{t})^2) \leq 0,$$

whenever $\lambda \in (0, 1]$ and $(1-\lambda)x + \lambda y \in \Omega$.

Proof. We recall a differential inequality of Borell [6, (2.1)]. If $x \in \partial\Omega_1$ and $w \in \Omega$, we have

$$(4.2) \quad (w - x) \cdot Du(w, t) + 2tu_t(w, t) \leq 0, \text{ for } t > 0,$$

where Du is the spatial derivative of u . In fact, Borell used probabilistic methods to derive a sharper inequality, but for our purposes, (4.2) suffices. For convenience of the reader, we include here the brief proof of (4.2), following [24, Lemma 4.4]. Assume without loss of generality that x is the origin. Consider for $\sigma \in [0, 1]$ the quantity

$$W(\zeta, t) = u(\sigma\zeta, \sigma^2 t) - u(\zeta, t)$$

on the set where $\sigma\zeta, \zeta \in \Omega$. On the boundary of its domain, W is nonnegative, and W vanishes at $t = 0$. Since W solves the heat equation, the weak maximum principle implies that $W \geq 0$. Differentiating with respect to σ and evaluating at $\sigma = 1$ gives (4.2).

We now prove the lemma. Writing $w = (1 - \lambda)x + \lambda y$ and $\rho = (1 - \lambda)\sqrt{s} + \lambda\sqrt{t}$ we have

$$\begin{aligned}\frac{d}{d\lambda}u(w, \rho^2) &= (y - x) \cdot Du(w, \rho^2) + 2\rho(\sqrt{t} - \sqrt{s})u_t(w, \rho^2) \\ &= \frac{1}{\lambda}(w - x) \cdot Du(w, \rho^2) + \frac{2\rho^2}{\lambda}u_t(w, \rho^2) - \frac{2\rho\sqrt{s}}{\lambda}u_t(w, \rho^2) \leq 0.\end{aligned}$$

Indeed (4.2) implies that sum of the first two terms is nonpositive, and the last term is nonpositive since $u_t > 0$. \square

The points X, Z and Y have coordinates $((1 - \lambda)x + \lambda y, ((1 - \lambda)\sqrt{s} + \lambda\sqrt{t})^2)$ with $\lambda = 0, 1/2$ and 1 respectively. Since $x \in \partial\Omega_1$ and $y, z \in \Omega$ it follows that the line segment $(1 - \lambda)x + \lambda y$ for $\lambda \in [1/2, 1]$, which goes from z to y , lies completely in Ω . Lemma 4.2 implies that $u(Y) \leq u(Z)$ and by the continuity of u at Y and Z we see that $u(X_i) = u(Y_i) \leq u(Z_i) + \eta/2$ for i sufficiently large, contradicting $\mathcal{Q}(X_i, Y_i) \geq \eta$.

(ii) The case when s and t are both zero. Our line of reasoning in this case is analogous to the probabilistic argument of [6, Section 3]. The points x, y and z must lie on the boundary $\partial\Omega_1$. Now for each i , we can find an affine transformation $T_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the function

$$u_i(w, t) := u(T_i^{-1}w, t)$$

still solves the boundary value problem (1.1), but on the transformed domain $T_i(\Omega)$ in the coordinates w_1, \dots, w_n , and:

- (a) $T_i(\Omega_1)$ is tangent to the hyperplane $w_1 = 0$ at the origin, and lies in the half space $w_1 \leq 0$;
- (b) $T_i(z_i)$ lies on the w_1 axis.

Let $v(w_1, \dots, w_n, t)$ be defined on $\mathbb{R}^n \times [0, \infty)$ as being identically 1 when $w_1 \leq 0$ and otherwise given by the solution of the heat equation on the half space $w_1 \geq 0$ with initial condition $v = 0$ when $w_1 > 0$, and boundary condition $v = 1$ on $w_1 = 0$. We can write down v explicitly as

$$(4.3) \quad v(w, t) = \Psi\left(\frac{w_1^2}{t}\right), \quad \Psi(\lambda) = \int_0^{1/\lambda} (4\pi\sigma^3)^{-1/2} \exp(-1/(4\sigma)) d\sigma.$$

In particular, note that the level sets of v are given by $t = cw_1^2$ for $c > 0$ from which it is straightforward to show that the superlevel sets of v are parabolically convex. Moreover, the maximum principle implies that $u_i(w, t) \leq v(w, t)$ on $T_i(\Omega) \cap \{w_1 \geq 0\}$. We have the following claim.

Claim. For compact subsets $K \subset (\{w_1 \geq 0\} \times (0, T])$, and any positive sequence $a_i \rightarrow 0$,

$$u_i(a_i w, a_i^2 t) \rightarrow v(w, t), \quad \text{as } i \rightarrow \infty,$$

uniformly for $(w, t) \in K$.

Proof of Claim. This follows from the fact that the function $\tilde{u}_i(w, t) = u_i(a_i w, a_i^2 t)$ solves the heat equation on $(1/a_i)T_i(\Omega)$, and as $i \rightarrow \infty$ the boundary conditions of \tilde{u}_i approach those of v . To make this more precise, assume K lies in $B_R \cap \{w_1 \geq 0\} \times [\delta, T]$ for $\delta > 0$, where B_R is a ball in \mathbb{R}^n of radius $R > 0$ centered at the origin. Fix $\varepsilon > 0$.

For a small $\beta > 0$, we define v_β to be the translate of v in the negative w_1 direction by the amount β , namely $v_\beta(w, t) = \Psi((w_1 + \beta)^2/t)$. Pick β sufficiently small so that on the compact set K ,

$$|v - v_\beta| = \int_{t/(w_1 + \beta)^2}^{t/w_1^2} (4\pi\sigma^3)^{-1/2} \exp(-1/(4\sigma)) d\sigma < \varepsilon.$$

The function v_β solves the heat equation on the set $\{w_1 > -\beta\}$ with zero initial data and boundary condition $v_\beta = 1$ on $\{w_1 = -\beta\}$.

Next for $S > R > 0$, define a function $\varphi_S(w, t)$ to be a solution to the heat equation on $\{w_1 > -\beta\} \cap B_S$ with zero initial data and boundary condition given by

$$\varphi_S(w, t) = \begin{cases} 0, & \text{for } w \in B_S \cap \{w_1 = -\beta\} \\ 1, & \text{for } w \in \partial(B_S) \cap \{w_1 > -\beta\} \end{cases}$$

We choose S sufficiently large so that $\varphi_S \leq \varepsilon$ on K .

Now choose i sufficiently large, depending on S , so that $((1/a_i)T_i(\Omega)) \cap B_S$ lies entirely in the set $\{w_1 > -\beta\}$, or in other words $B_S \cap \{w_1 \leq -\beta\}$ is contained in $(1/a_i)T_i(\Omega_1)$. We may also assume without loss of generality that the boundary of $T_i(\Omega_0)$ lies outside B_S . Now the function $\tilde{u}_i + \varphi_S - v_\beta$ solves the heat equation on $((1/a_i)T_i(\Omega)) \cap B_S$ for $t \in [0, a_i^{-2}T]$ with zero initial data, and strictly positive boundary condition by construction. Indeed on the part of the boundary which coincides with the boundary of $T_i(\Omega_1)$ we have $\tilde{u}_i = 1 \geq v_\beta$, and on the rest of the boundary we have $\varphi_S = 1 \geq v_\beta$. Hence $\tilde{u}_i + \varphi_S - v_\beta \geq 0$, and hence on K we have

$$\tilde{u}_i \geq v - 2\varepsilon.$$

Since we have $\tilde{u}_i \leq v$ by the maximum principle this completes the proof of the claim. \square

Recall that we have a sequence $X_i = (x_i, s_i)$, $Y_i = (y_i, t_i)$ with $\mathcal{C}(X_i, Y_i) \geq \eta > 0$. Writing $\tilde{X}_i = (\tilde{x}_i, s_i) = (T_i(x_i), s_i)$ and similarly for \tilde{Y}_i and \tilde{Z}_i we have

$$(4.4) \quad u_i(\tilde{Z}_i) + \eta \leq u_i(\tilde{X}_i) \leq \min(v(\tilde{X}_i), v(\tilde{Y}_i)) \leq v(\tilde{Z}_i).$$

Here the second inequality follows from $u_i(\tilde{X}_i) = u_i(\tilde{Y}_i)$ and $u_i \leq v$, while the third inequality follows from the parabolic quasiconcavity of v .

Now $v(\tilde{Z}_i) = v(\tilde{z}_i, r_i) \geq \eta > 0$ and it follows that $\rho_i := (w_1(\tilde{z}_i))^2/r_i \leq C$ for a uniform constant C , since $\Psi(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Here we are writing $w_1(\tilde{z}_i)$ for the w_1 coordinate of $\tilde{z}_i = T_i(z_i)$. After passing to a subsequence, we may assume that $\rho_i \rightarrow \rho < \infty$.

Then using the above, and recalling the properties of the transformation T_i we have

$$(4.5) \quad u_i(\tilde{Z}_i) = u_i(w_1(\tilde{z}_i), 0, \dots, 0, r_i) = u_i(\sqrt{\rho_i}\sqrt{r_i}, 0, \dots, 0, r_i) \rightarrow v(\sqrt{\rho}, 0, \dots, 0, 1),$$

as $i \rightarrow \infty$, using the Claim with $a_i = \sqrt{r_i}$. But

$$(4.6) \quad v(\tilde{Z}_i) = v(\sqrt{\rho_i}\sqrt{r_i}, 0, \dots, 0, r_i) = v(\sqrt{\rho_i}, 0, \dots, 0, 1) \rightarrow v(\sqrt{\rho}, 0, \dots, 0, 1),$$

as $i \rightarrow \infty$ which contradicts (4.4). \square

5. PROOF OF THEOREM 1.1

In this section, we give the proof of Theorem 1.1. Fix $T \in (0, \infty)$ and $0 < \mu < 1$. Let $S_\mu = \{(x, t) \in \Omega \times (0, T] \mid u(x, t) = \mu\}$, which is convex from Borell's result [6] (or, from Theorem 4.1). We wish to show strong convexity of S_μ on compact subsets. We will do this using the two point function

$$(5.1) \quad \mathcal{C}_1((x, s), (y, t)) = u(x, s) - u\left(\frac{x+y}{2}, \left(\frac{s+t}{2}\right)\right)$$

from (1.2). For α, β with $0 < \alpha < \mu < \beta < 1$ we define the space-time region

$$\Xi = \{(x, t) \in \Omega \times (0, T] \mid \alpha \leq u(x, t) \leq \beta\},$$

bounded by $\{t = T\}$ and “inner boundary” S_β and “outer boundary” S_α , which are defined in the same way as S_μ . The following lemma is the key result of this section.

Lemma 5.1. *Fix (x_0, t_0) in S_μ and a unit vector $V = (V_1, \dots, V_{n+1}) \in \mathbb{R}^{n+1}$ with $V_{n+1} \geq 0$. Then there exist α and β with $0 < \alpha < \mu < \beta < 1$ and a smooth function h on Ξ satisfying the heat equation in the interior of Ξ such that:*

(i) *The function*

$$(5.2) \quad \mathcal{Q}((x, s), (y, t)) = \mathcal{C}_1((x, s), (y, t)) + (h(x, s) - h(y, t))^2,$$

defined on

$$\Sigma' = \{((x, s), (y, t)) \in \Xi \mid u(x, s) = u(y, t), ((x+y)/2, (s+t)/2) \in \Xi\},$$

is nonpositive.

(ii) *We have*

$$\nabla_V h(x_0, t_0) \neq 0.$$

Here we are using $\nabla_V h$ to denote the space-time directional derivative

$$\sum_{i=1}^n V_i D_i h + V_{n+1} \partial_t h.$$

Given the lemma we can complete the proof of the main theorem.

Proof of Theorem 1.1. This is an almost immediate consequence of Lemma 5.1. Fix (x_0, t_0) in S_μ . By compactness of the unit sphere in \mathbb{R}^{n+1} we obtain in a neighborhood of (x_0, t_0) ,

$$\mathcal{C}_1((x, s), (y, t)) + c(|x - y|^2 + |s - t|^2) \leq 0,$$

for a uniform constant $c > 0$. It follows that any compact subset of S_μ is strongly convex (see for example [34, Section 3]) as required. \square

It remains then to prove the lemma.

Proof of Lemma 5.1. Fix (x_0, t_0) and a unit vector V as in the statement of the lemma. We first make the following claim.

Claim. There exists $0 < \alpha < \mu < \beta < 1$, a strongly convex open set $E_{\alpha,\beta}$ of $S_\alpha \cap \{t > t_0/2\}$ and a smooth compactly supported $f : E_{\alpha,\beta} \rightarrow (0, \beta - \alpha)$ such that

(a) There exists a unique solution $h(x, t)$ to the heat equation $\partial h / \partial t = \Delta h$ on Ξ with boundary conditions

$$(5.3) \quad \begin{cases} h(x, t) = 0, & (x, t) \in (S_\alpha \setminus E_{\alpha,\beta}) \cup S_\beta \\ h(x, t) = f(x, t), & (x, t) \in E_{\alpha,\beta} \end{cases}$$

and initial data $h(x, 0) = 0$.

(b) $\nabla_V h(x_0, t_0) \neq 0$.

Proof of Claim. We first prove part (a). In particular we show the existence of a solution $h(x, t)$ as in (a) given any $0 < \alpha < \mu < \beta < 1$, $E_{\alpha,\beta}$ and f as in the hypothesis of the claim. To deal with the fact that the boundary is changing in time, we consider $\Xi_0 := \Xi \cap \{t \geq t_0/2\}$ which is diffeomorphic to the cylinder $\overline{\Omega} \times [t_0/2, T]$. Indeed, there is a diffeomorphism $\Psi_{\alpha,\beta} : \overline{\Xi}_0 \rightarrow \overline{\Omega} \times [t_0/2, T]$ satisfying

- (i) $\Psi_{\alpha,\beta}$ is the identity in the t factor and maps each time slice $\overline{\Xi}_0 \cap \{t = t'\}$ diffeomorphically to $\overline{\Omega} \setminus \Omega_1$, and is a diffeomorphism of the boundary $S_\alpha \cap \{t = t'\}$ to $\partial\Omega_0$ and $S_\beta \cap \{t = t'\}$ to $\partial\Omega_1$.
- (ii) $\Psi_{\alpha,\beta}^{-1}$ converges smoothly uniformly to the identity on $\overline{\Omega} \times [t_0/2, T]$ as $(\alpha, \beta) \rightarrow (0, 1)$.

We write Ψ for $\Psi_{\alpha,\beta}$. Define $F : \Psi(E_{\alpha,\beta}) \rightarrow (0, \beta - \alpha)$ by $F(w, t) = f(\Psi^{-1}(w, t))$. Now let $H(x, t)$ be the solution of the parabolic equation

$$(5.4) \quad H_t = (\Psi_i^k \Psi_i^j) H_{jk} + (\Psi_{ii}^j - \Psi_t^j) H_j$$

on the space-time cylinder $\Omega \times (t_0/2, T]$, with zero initial data $H = 0$ at $t = t_0/2$ and boundary conditions

$$(5.5) \quad \begin{cases} H(w, t) = F(w, t), & (w, t) \in \Psi(E_{\alpha,\beta}) \\ H(w, t) = 0, & \text{otherwise.} \end{cases}$$

Note that (5.4) is a strictly parabolic equation in $\Omega \times (t_0/2, T]$ with smooth coefficients. Now define $h(x, t) = H(\Psi(x, t))$ on Ξ_0 and extend to Ξ by setting it to be zero for $0 \leq t \leq t_0/2$. Then h is the required solution for (a). Uniqueness is a consequence of the maximum principle.

We now turn to part (b) of the claim.

Assertion 1. If $m > 0$ is sufficiently large then for any open $E \subset \partial\Omega_0$ there exists $(w, c) \in E \times (t_0 - 1/m, t_0)$ such that $\nabla_V P_{w,c}(x_0, t_0) \neq 0$. Here $P_{w,c}(x, t)$ is the solution to the heat equation on $\Omega \times (0, T]$ with zero initial data and boundary data $f = \delta_{(w,c)}$, for $\delta_{w,c}$ the delta function on $\partial\Omega \times (0, T]$ supported at (w, c) .

Assertion 1 follows by essentially the same proof as [34, Postscript] adapted to our parabolic setting. Indeed, recall the formula $P_{w,c}(x, t) = \partial_{\nu_w} q(x, w, t - c)$ for all $t > c$ (see for example [26, (4.3.28)]) where q is the Dirichlet heat kernel for Ω and ∂_{ν_w} is the derivative in the w variable in the direction of the inward facing unit normal. If the assertion is false then for all m we have $\nabla_V \partial_{\nu_w} q(x, w, t_0 - c) = \partial_{\nu_w} \nabla_V q(x, w, t_0 - c) = 0$ for all $(w, c) \in E \times (t_0 - 1/m, t_0)$, where ∇_V is the derivative in the first and third variables. Hence $H(w, t) := \nabla_V q(x_0, w, t) = 0$ on $\partial\Omega \times (0, 1/m)$ while $\partial_{\nu_w} H(w, t) = 0$

on $E \times (0, 1/m)$ and it follows by a unique extension result for solutions to the heat equation [30] that $H(w, t) = 0$ on $\Omega \times (0, 1/m)$, which we show in the following cannot be true, thus establishing Assertion 1 by contradiction.

For any w in Ω we observe that (see for example [38, Theorem 1.1] and [18, Section 3.2])

$$\varepsilon^n q(w + \varepsilon(x - w), w, \varepsilon^2 t) \rightarrow K(x, w, t), \quad \text{as } \varepsilon \rightarrow 0,$$

smoothly uniformly for (x, t) in compact subsets of $(\mathbb{R}^n \setminus \{w\}) \times (0, \infty)$, where

$$K(x, w, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-w|^2/4t}$$

is the heat kernel on $\mathbb{R}^n \times (0, \infty)$. From the formula for $K(x, w, t)$ it follows that for any fixed w in Ω there are positive constants c, C and $T \in (0, 1/m)$ such that in the annulus A_w given by $c < |x - w| < C$ and for $t \in (T/2, T)$ we have:

$$\frac{\partial q}{\partial t}(x, w, t) > 0, \quad \text{and} \quad \sum_{i=1}^n W^i \frac{\partial q}{\partial x_i}(x, w, t) > 0$$

where $(W_1, \dots, W_n) = w - x$ is the vector from x to w .

Now let $x_0 \in \Omega$ and $V = (V_1, \dots, V_n, V_{n+1})$ be the unit vector as above, recalling that $V_{n+1} \geq 0$. We may then choose $w \in \Omega$ such that $x_0 \in A_w$ and $(V_1, \dots, V_n) = \lambda(w - x_0)$ for some $\lambda \geq 0$. It follows then that $H(w, t) := \nabla_V q(x_0, w, t) > 0$ for some $t \in (0, 1/m)$.

This completes the proof of Assertion 1 by contradiction.

Assertion 2. Let E be a strongly convex open subset of $\partial\Omega_0$ and $(a, b) \subset (0, T)$. Then $\Psi_{\alpha, \beta}^{-1}(E \times (a, b))$ contains a strongly convex open subset $E_{\alpha, \beta}$ for all α, β sufficiently close to 0, 1.

By the strong convexity of E and property (ii) of the map $\Psi_{\alpha, \beta}$, the set $V_{\alpha, \beta} = \Psi_{\alpha, \beta}^{-1}(E \times (a, b))$ is strongly convex in the spatial directions as long as (α, β) is sufficiently close to $(0, 1)$. We now show that $V_{\alpha, \beta}$ is *strictly convex*. Take any pair $(x, s), (y, t) \in V_{\alpha, \beta}$ and consider the line segment L joining them (by shrinking E if necessary, we may assume L is contained entirely in $\Omega \times (0, T]$). We now show that no interior point of L lies in $V_{\alpha, \beta}$. By the strong convexity of $V_{\alpha, \beta}$ in the space directions, we may assume that $s \neq t$, in which case the interior of L lies strictly above (i.e. has strictly larger time component) than the interior of the parabolic segment

$$\lambda \mapsto ((1 - \lambda)x + \lambda y, ((1 - \lambda)\sqrt{s} + \lambda\sqrt{t})^2), \quad \lambda \in [0, 1],$$

connecting (x, s) to (y, t) . Since $\{u \geq \alpha\}$ is parabolically convex we have $u \geq \alpha$ on this parabolic segment, and since $u_t > 0$ in Ξ we have $u > \alpha$ on the interior of L and hence no such point can lie in $V_{\alpha, \beta}$. We have thus far shown that $V_{\alpha, \beta}$ is a strictly convex subset of S_α . Assertion 2 follows from the fact that every open strictly convex hypersurface in \mathbb{R}^{n+1} contains an open subset which is strongly convex. Indeed, after a coordinate rotation we may write such a hypersurface locally as a graph $x_{n+1} = f(x_1, \dots, x_n)$ over a ball $B \subset \mathbb{R}^n$ such that f attains a minimum value at the center B and is strictly positive on ∂B . By comparing with a quadratic function and applying the maximum principle we obtain that f and hence the hypersurface is strongly convex at some point. Thus Assertion 2 holds.

We may now complete the proof of part b) of the claim. Fix a strongly convex open subset E of $\partial\Omega_0$ (every smooth convex hypersurface contains such a subset, see for example [34, p. 104]) and an interval $(a, b) \subset (t_0 - 1/m, t_0)$. By Assertion 1 and shrinking E and (a, b) if necessary, we may assume $|\nabla_V P_{(w,c)}(x_0, t_0)| > C > 0$ for some $C > 0$ and all $(w, c) \in E \times (a, b)$. Now let $P_{(w,c)}^{\alpha,\beta}(x, t)$ be the solution to (5.4) on $\Omega \times (0, T]$ with zero initial data and boundary data $\delta_{(w,c)}$. By property (ii) of the map $\Psi_{\alpha,\beta}$, it follows that for all α, β sufficiently close to 0, 1 we have $|\nabla_V P_{(w,c)}^{\alpha,\beta}(x_0, t_0)| > C/2 > 0$ for all $(w, c) \in E \times (a, b)$. The claim then follows by using smooth compactly supported approximations of $\delta_{(w,c)}$, the fact that $H \circ \Psi_{\alpha,\beta}$ solves the standard heat equation on Ξ if H solves (5.4) on $\Omega \times (0, T]$, and Assertion 2. \square

Returning to the proof of Lemma 5.1, let $h(x, t)$ be a solution to (5.3) as in the Claim with boundary data $f : E_{\alpha,\beta} \rightarrow (0, \beta - \alpha)$. Thus $h(x, t)$ satisfies condition (ii) in the Lemma and it remains only to prove condition (i).

By Proposition 3.2, it suffices to show that \mathcal{Q} is nonpositive at the boundary points of Σ' . First suppose that $X = (x, s)$, $Y = (y, t)$ or $(X + Y)/2$ lies in a boundary point of S_α or S_β , and $s, t > 0$. There are several cases to consider.

- (1) If X and Y lie in S_β then since h vanishes on S_β there is nothing to prove since we already know that S_β is convex.
- (2) If X and Y lie in $S_\alpha \setminus E_{\alpha,\beta}$ then h vanishes at X and Y and we conclude as in Case (1).
- (3) If X and Y lie in $E_{\alpha,\beta}$ then by the strong convexity of $E_{\alpha,\beta}$ we have

$$\frac{u(x, s) + u(y, t)}{2} - u\left(\frac{x+y}{2}, \frac{s+t}{2}\right) + c(|x-y|^2 + |s-t|)^2 \leq 0$$

for $c > 0$ sufficiently small. But $|h(x, s) - h(y, t)|^2 = |f(x, s) - f(y, t)|^2 \leq C(|x-y|^2 + |s-t|^2)$ for a uniform C depending only on $E_{\alpha,\beta}$ and f . Condition (i) follows by replacing h with a sufficiently small multiple of itself.

- (4) If $X \in E_{\alpha,\beta}$ and $Y \in S_\alpha \setminus E_{\alpha,\beta}$, then we may assume X lies in the support of f as otherwise $h(X) = h(Y) = 0$ and we may conclude as in case (2). Under this assumption, that $E_{\alpha,\beta}$ is a strongly convex neighborhood of S_α implies that $\frac{1}{2}(X+Y)$ is not in S_α and hence $u\left(\frac{X+Y}{2}\right) \geq \alpha + d$ for a uniform constant $d > 0$ while $u(X) = u(Y) = \alpha$, and so $\mathcal{Q}(X, Y) \leq 0$ after replacing h with a sufficiently small multiple of itself if necessary. We argue similarly if the roles of X, Y are reversed.
- (5) If $(X + Y)/2$ lies in S_α then by convexity of S_α the points X and Y lie in S_α and this reduces to one of the cases above.
- (6) If $(X + Y)/2$ lies in S_β then $u(x, s) = u(y, t) = \beta - r$ for some $r \in [0, 1)$. Then note that $h \leq \beta - u$ by the maximum principle so that after replacing h with a sufficiently small multiple of itself if necessary we have

$$\begin{aligned} (h(x, s) - h(y, t))^2 &\leq r^2 \leq r = \beta - \frac{u(x, s) + u(y, t)}{2} \\ &= u\left(\frac{x+y}{2}, \frac{s+t}{2}\right) - \frac{u(x, s) + u(y, t)}{2} \end{aligned}$$

as required.

It remains to deal with the case when s or t tends to zero. Notice that if *both* s, t are less than $t_0/2$ then $h(X) = h(Y) = 0$ and $\mathcal{Q} \leq 0$ by the weak convexity of superlevel sets of u already proved. Hence we may assume without loss of generality that we have a sequence of points $X_i = (x_i, s_i) \rightarrow (x, s) = X$ and $Y_i = (y_i, t_i) \rightarrow (y, t) = Y$ with $s = 0, t \geq t_0/2$ and

$$(5.6) \quad \mathcal{Q}(X_i, Y_i) \geq \varepsilon$$

for some $\varepsilon > 0$. We may assume that x lies in $\partial\Omega_1$. Since $(y, t) \in \Xi$ with $t \geq t_0/2$ it follows that $|y - x|$ is bounded below uniformly away from zero.

Using these facts, Lemma 4.2 and fact that $u_t > 0$ in $\Omega \times (0, T]$ we may conclude:

$$\begin{aligned} u(Y) - u\left(\frac{X+Y}{2}\right) \\ = (u(y, t) - u((x+y)/2, t/4)) + (u((x+y)/2, t/4) - u((x+y)/2, t/2)) \\ \leq (u((x+y)/2, t/4) - u((x+y)/2, t/2)) \\ < -c \end{aligned}$$

for some constant c depending only on α, β, t_0 . Indeed, the first inequality follows from Lemma 4.2 while the second inequality follows from the fact that $u_t > 0$ in $\Omega \times (0, T]$ and that $\text{dist}((x+y)/2, \partial\Omega)$ is bounded uniformly away from zero depending on α, β, t_0 . Thus and after replacing h with a sufficiently small multiple of itself if necessary we obtain $\mathcal{Q}(X, Y) \leq 0$, contradicting (5.6). This completes the proof of the lemma. \square

6. REMARKS AND OPEN QUESTIONS

Finally, we end with some remarks and open questions related to the results of this paper.

- (1) We expect that our proof should carry over to more general parabolic equations (cf. [24]).
- (2) It would be interesting to know whether superlevel sets of u are *strictly parabolically convex* (with the obvious definition).
- (3) In view of the explicit solution (4.3) of the heat equation on the half line which is exactly parabolically convex, we expect that the convexity of the superlevel sets of u cannot be sharpened to p -convexity for $p > 2$ (as defined by taking the functional (1.2) with $p > 2$).
- (4) We used here the parabolic analogue of the two-point function of Rosay-Rudin [34]. A related two-point function was introduced in [40] and we expect this also to have a parabolic version.
- (5) By analogy to the elliptic case, it would be interesting to know whether parabolic “microscopic” techniques (cf. [14, 16, 22]), analyzing the principal curvatures of the space-time level sets, yield a different proof of Theorem 1.1.
- (6) A well-known open problem, mentioned in the introduction, is to extend Borell’s result to initial data that is not identically zero.

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