Aspect Ratio Universal Rectangular Layouts*

Stefan Felsner[1], Andrew Nathenson[2], and Csaba D. Tóth[3]

[1] Institut für Mathematik, Technische Universität Berlin, Berlin, Germany
felsner@math.tu-berlin.de
[2] University of California, San Diego, CA, USA
anathenson@ucsd.edu
[3] California State University Northridge, Los Angeles, CA, USA
csaba.toth@csun.edu

Abstract. A *generic rectangular layout* (for short, *layout*) is a subdivision of an axis-aligned rectangle into axis-aligned rectangles, no four of which have a point in common. Such layouts are used in data visualization and in cartography. The contacts between the rectangles represent semantic or geographic relations. A layout is weakly (strongly) *aspect ratio universal* if any assignment of aspect ratios to rectangles can be realized by a weakly (strongly) equivalent layout. We give a combinatorial characterization for weakly and strongly aspect ratio universal layouts, respectively. Furthermore, we describe a quadratic-time algorithm that decides whether a given graph G is the dual graph of a strongly aspect ratio universal layout, and finds such a layout if one exists.

1 Introduction

A *rectangular layout* (a.k.a. *mosaic floorplan* or *rectangulation*) is a subdivision of an axis-aligned rectangle into axis-aligned rectangle faces, it is *generic* if no four faces have a point in common. In the *dual graph* $G(\mathcal{L})$ of a layout $\mathcal{L}$, the nodes correspond to rectangular faces, and an edge corresponds to a pair of rectangles whose common boundary contains a line segment [6,28,29].

Two generic layouts are *strongly equivalent* if they have isomorphic dual graphs, and the corresponding line segments between rectangles have the same orientation (horizontal or vertical); see Fig. 1 for examples. Two generic layouts are *weakly equivalent* if there is a bijection between their horizontal and vertical segments, resp., such that the contact graphs of the segments are isomorphic plane graphs. Strong equivalence implies weak equivalence [9]; however, for example the brick layouts in Figs. 4a and 4b are weakly equivalent, but not strongly equivalent. The closures of weak (resp., strong) equivalence classes under the uniform norm extend to nongeneric layouts, and a nongeneric layout may belong to the closures of multiple equivalence classes.

Rectangular layouts have been studied for more than 40 years, originally motivated by VLSI design [21,23,34] and cartography [26], and more recently by data visualization [17]. The weak equivalence classes of layouts are in bijection with Baxter permutations [1,27,35].

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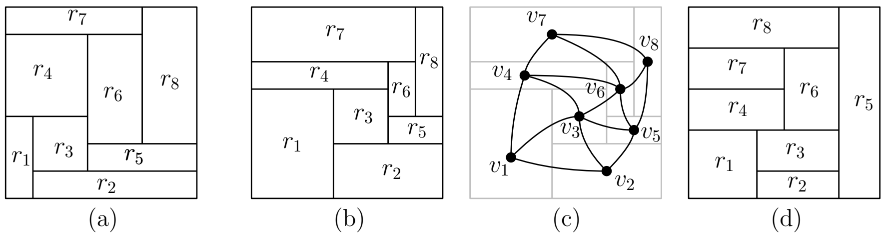


Fig. 1: (a-b) Two equivalent layouts. (c) Dual graph. (d) Another layout with the same dual graph. The layout in (d) is sliceable, none of them is one-sided.

An (abstract) graph is called a *proper graph* if it is the dual of a generic layout. Every proper graph is a near-triangulation (a plane graph where every bounded face is a triangle, but the outer face need not be a triangle). But not every near-triangulation is a proper graph [28,29]. Ungar [33] gave a combinatorial characterization of proper graphs (see also [16,31]); and they can be recognized in linear time [12,22,24,25].

In data visualization and cartography [17,26], the rectangles correspond to entities (e.g., countries or geographic regions); adjacency between rectangles represents semantic or geographic relations, and the "shape" of a rectangle represent data associated with the entity. It is often desirable to use equivalent layouts to realize different statistics associated with the same entities. A generic layout $\mathcal{L}$ is *weakly* (*strongly*) *area universal* if any area assignment to the rectangles can be realized by a layout weakly (strongly) equivalent to $\mathcal{L}$. Wimer et al. [34] showed that every generic layout is weakly area universal (see also [9, Thm. 3]). Eppstein et al. [6] proved that a layout is strongly area universal if and only if it is one-sided (defined below). However, no polynomial-time algorithm is known for testing whether a given graph G is the dual of some area-universal layout.

In some applications, the aspect ratios (rather than the areas) of the rectangles are specified. For example, in word clouds adapted to multiple languages, the aspect ratio of (the bounding box of) each word depends on the particular language. The *aspect ratio* of an axis-aligned rectangle r is height(r)/width(r). A generic layout $\mathcal{L}$ is *weakly* (*strongly*) *aspect ratio universal* (*ARU* for short) if any assignment of aspect ratios to the rectangles can be realized by a layout weakly (strongly) equivalent to $\mathcal{L}$.

Our Results. We characterize strongly and weakly aspect ratio universal layouts.

Theorem 1. *A generic layout is weakly aspect ratio universal if and only if it is sliceable.*

Theorem 2. *For a generic layout $\mathcal{L}$, the following properties are equivalent:*

(i) *$\mathcal{L}$ is strongly aspect ratio universal;*
(ii) *$\mathcal{L}$ is one-sided and sliceable;*
(iii) *the extended dual $G^*(\mathcal{L})$ of $\mathcal{L}$, admits a unique transversal structure.*

The terms in Theorems 1–2 are defined below. It is not difficult to show that one-sided sliceable layouts are strongly aspect ratio universal; and admit a unique transversal structure. Proving the converses, however, is more involved.

Algorithmic results. In some applications, the rectangular layout is not specified, and we are only given the dual graph of a layout (i.e., a proper graph). This raises the following problem: Given a proper graph G with n vertices, find a strongly (resp., weakly) ARU layout $\mathcal{L}$ such that $G \simeq G(\mathcal{L})$ or report that none exists. Using structural properties of one-sided sliceable layouts that we develop here, we present an algorithm for recognizing the duals of strongly ARU layouts.

Theorem 3. *We can decide in $O(n^2)$ time whether a given graph G with n vertices is the dual of a one-sided sliceable layout.*

Thomassen [31] gave a linear-time algorithm to recognize proper graphs if the nodes corresponding to corner rectangles are specified, using combinatorial characterizations of layouts [33]. Kant and He [13,15] described a linear-time algorithm to test whether a given graph G^* is the extended dual of a layout, using transversal structures. Later, Rahman et al. [12,22,24,25] showed that proper graphs can be recognized in linear time (without specifying the corners). However, a proper graph may have exponentially many nonequivalent realizations, and prior algorithms may not find a one-sided sliceable realization even if one exists. Currently, no polynomial-time algorithm is known for recognizing the duals of sliceable layouts [5,18,36] (i.e., weakly ARU layouts); or one-sided layouts [6].

Background and Terminology. A *rectangular layout* (for short, *layout*) is a rectilinear graph in which each face is a rectangle, the outer face is also a rectangle, and the vertex degree is at most 3. A *sublayout* of a layout $\mathcal{L}$ is a subgraph of $\mathcal{L}$ which is a layout. A layout is *irreducible* if it does not contain any nontrivial sublayout. A *rectangular arrangement* is a 2-connected subgraph of a layout in which bounded faces are rectangles (the outer face need not be a rectangle).

One-Sided Layouts. A *segment* of a layout $\mathcal{L}$ is a path of collinear inner edges of $\mathcal{L}$. A segment of $\mathcal{L}$ that is not contained in any other segment is maximal. In a *one-sided* layout, every maximal line segment s must be a side of at least one rectangle R; in particular, any other segment orthogonal to s with an endpoint in the interior of s lies in a halfplane bounded by s, and points away from R.

Sliceable Layouts. A maximal line segment subdividing a rectangle or a rectangular union of rectangular faces is called a *slice*. A *sliceable layout* (a.k.a. *slicing floorplan* or *guillotine rectangulation*) is one that can be obtained through recursive subdivision with vertical or horizontal lines; see Fig 1(d). The recursive subdivision can be represented by a *binary space partition tree* (*BSP-tree*), which is a binary tree where each vertex is associated with either a rectangle with a slice, or just a rectangle if it is a leaf [3]. For a nonleaf vertex, the two subrectangles on each side of the slice are associated with the two children. The number of (equivalence classes of) sliceable layouts with n rectangles is known to be the nth Schröder number [35]. One-sided sliceable layouts are in bijection with certain pattern-avoiding permutations, closed formulas for their number has been given by Asinowski and Mansour [2]; see also [20] and OEIS A078482 in the on-line encyclopedia of integer sequences (https://oeis.org/) for further references.

A *windmill* in a layout is a set of four pairwise noncrossing maximal line segments, called *arms*, which contain the sides of a central rectangle, and each arm has an endpoint on the interior of another (e.g., the maximal segments around r_3 or r_6 in Fig. 2 (a)). We orient each arm from the central rectangle to the other endpoint. A windmill is either *clockwise* or *counterclockwise*. It is well known that a layout is sliceable if and only if it does not contain a windmill [1].

Transversal Structure. The dual graph $G(\mathcal{L})$ of a layout $\mathcal{L}$ encodes adjacency between faces, but does not specify the relative positions between faces (above-below or left-right). The transversal structure (a.k.a. regular edge-labelling) was introduced by He [13,15] for the efficient recognition of proper graphs, and later used extensively for counting and enumerating (equivalence classes of) layouts [11]. The *extended dual graph* $G^*(\mathcal{L})$ is the contact graph of the rectangular faces *and* the four edges of the bounding box of $\mathcal{L}$; it is a triangulation in an outer 4-cycle without separating triangles; see Fig. 2.

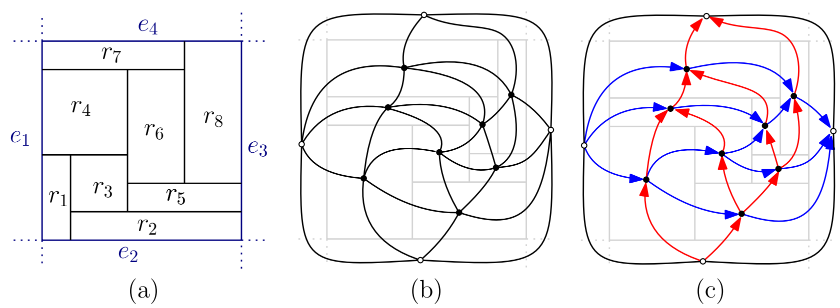


Fig. 2: (a) A layout $\mathcal{L}$ bounded by $e_1, \ldots, e_4$. (b) Extended dual graph $G^*(\mathcal{L})$ with an outer 4-cycle $(e_1, \ldots, e_4)$. (c) Transversal structure.

A layout $\mathcal{L}$ is encoded by a *transversal structure* that comprises $G^*(\mathcal{L})$ and an orientation and bicoloring of the inner edges of $G^*(\mathcal{L})$, where red (resp., blue) edges correspond to above-below (resp., left-to-right) relation between two objects in contact. An (abstract) *transversal structure* is defined as a graph G^*, which is a 4-connected triangulation of an outer 4-cycle (S, W, N, E), together with a bicoloring and orientation of the inner edges of G^* such that all the inner edges incident to S, W, N, and E, respectively, are outgoing red, outgoing blue, incoming red, and incoming blue; and at each inner vertex the counterclockwise rotation of incident edges consists of four nonempty blocks of outgoing red, outgoing blue, incoming red, and incoming blue edges; see Fig. 2(c).

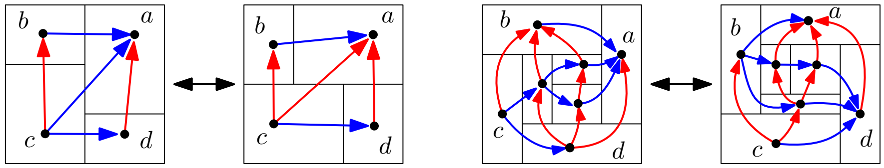


Fig. 3: A flip of an empty (left) and a nonempty (right) alternating cycle.

Flips and Alternating 4-Cycles. It is known that transversal structures are in bijection with the strong equivalence classes of generic layouts [8,11,15]. Furthermore, a sequence of flip operations can transform any transversal structure with n inner vertices into any other [7,11]. Each *flip* considers an *alternating 4-cycle* C, which comprises red and blue edges alternatingly, and changes the color of every edge in the interior of C; see Fig. 3. If, in particular, there is no vertex in the interior of C, then the flip changes the color of the inner diagonal of C. Furthermore, every flip operation yields a valid transversal structure on $G^*(\mathcal{L})$, hence a new generic layout $\mathcal{L}'$ that is strongly non-equivalent to $\mathcal{L}$. We can now establish a relation between geometric and combinatorial properties.

Lemma 1. *A layout $\mathcal{L}$ is one-sided and sliceable if and only if $G^*(\mathcal{L})$ admits a unique transversal structure.*

Proof. Assume that $\mathcal{L}$ is a layout where $G^*(\mathcal{L})$ admits two or more transversal structures. Consider a transversal structure of $G^*(\mathcal{L})$. Since any two transversal structures are connected by a sequence of flips, there exists an alternating 4-cycle. Any alternating 4-cycle with no interior vertex corresponds to a segment in $\mathcal{L}$ that is two-sided. Any alternating 4-cycle with interior vertices corresponds to a windmill in $\mathcal{L}$. Consequently, $\mathcal{L}$ is not one-sided or not sliceable.

Conversely, if $\mathcal{L}$ is not one-sided (resp., sliceable), then the transversal structure of $G^*(\mathcal{L})$ contains an alternating 4-cycle with no interior vertex (resp., with interior vertices). Consequently, we can perform a flip operation, and obtain another transversal structure for $G^*(\mathcal{L})$. $\square$

2 Aspect Ratio Universality

An *aspect ratio assignment* to a layout $\mathcal{L}$ is a function that maps a positive real to each rectangle in $\mathcal{L}$. An aspect ratio assignment to $\mathcal{L}$ is *realizable* if there exists an equivalent layout $\mathcal{L}'$ with the required aspect ratios (a *realization*). A layout is *aspect ratio universal* (*ARU*) if every aspect ratio assignment is realizable. In this section, we characterize weakly and strongly ARU layouts (Theorems 1–2). We start with an easy observation. (Omitted proofs are in the full paper [10].)

Lemma 2. *Let $\mathcal{L}$ be a sliceable layout. If an aspect ratio assignment for $\mathcal{L}$ is realizable, then there is a unique realization up to scaling and translation. Furthermore, for every $\alpha > 0$ there exists a realizable aspect ratio assignment for which the bounding box of the realization has aspect ratio α.*

Corollary 1. *If $\mathcal{L}$ is one-sided and sliceable, then it is strongly ARU.*

Corollary 2. *If $\mathcal{L}$ is sliceable, then it is weakly ARU.*

2.1 Sliceable and One-Sided Layouts

Next we show that any sliceable layout that is strongly ARU must be one-sided. We present two types of simple layouts that are not aspect ratio universal, and then show that all other layouts that are not one-sided or not sliceable can be reduced to these prototypes.

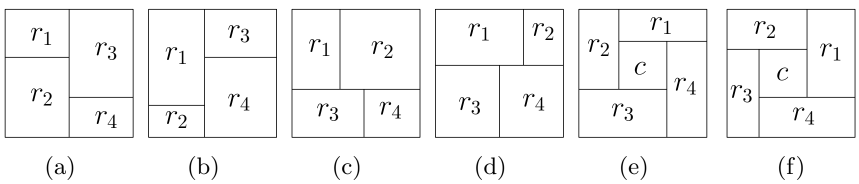


Fig. 4: Prototype layouts that are not aspect ratio universal: (a)–(d) brick layouts are sliceable but not one-sided; (e)–(f) windmills are one-sided but not sliceable.

Lemma 3. *The brick layouts in Figs. 4a–4d are not strongly ARU; the windmill layouts in Figs. 4e–4f are neither strongly nor weakly ARU.*

Proof. Suppose w.l.o.g. that a brick layout $\mathcal{L}_0$ in Fig. 4a is strongly ARU. Then there exists a strongly equivalent layout $\mathcal{L}$ for the aspect ratio assignment $\alpha(r_2) = \alpha(r_3) = 1$ and $\alpha(r_1) = \alpha(r_4) = 2$. Since $\text{width}(r_1) = \text{width}(r_2)$ and $\alpha(r_1) = 2\alpha(r_2)$, then $\text{height}(r_1) = 2\,\text{height}(r_2)$, and the left horizontal slice is below the median of $r_1 \cup r_2$. Similarly, $\text{width}(r_3) = \text{width}(r_4)$ and $\alpha(r_4) = 2\alpha(r_2)$ imply that the right horizontal slice is above the median of $r_3 \cup r_4$. Consequently, r_1 and r_4 are in contact, and $\mathcal{L}$ is not equivalent to $\mathcal{L}_0$, which is a contradiction.

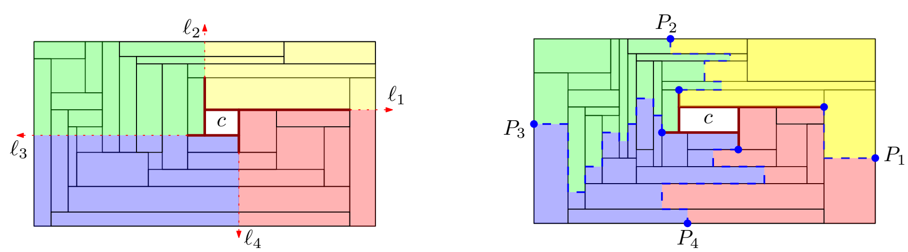
Suppose w.l.o.g. that the windmill layout $\mathcal{L}_1$ in Fig. 4e is weakly ARU. Then there exists a weakly equivalent layout $\mathcal{L}$ for the aspect ratio assignment $\alpha(c) = \alpha(r_1) = \alpha(r_2) = \alpha(r_3) = \alpha(r_4) = 1$. In particular, $r_1, \ldots, r_4$ are squares; denote their side lengths by s_i, for $i = 1, \ldots, 4$. Note that one side of r_i strictly contains a side of r_{i-1} for $i = 1, \ldots, 4$ (with arithmetic modulo 4). Consequently, $s_1 < s_2 < s_3 < s_4 < s_1$, which is a contradiction. □

Lemma 4. *If a layout is sliceable but not one-sided, then it is not strongly ARU.*

Proof. To show that a layout is not strongly ARU, it is sufficient to show that any of its sublayouts are not strongly ARU, because any nonrealizable aspect ratio assignment for a sublayout can be expanded arbitrarily to an aspect ratio assignment for the entire layout.

Let $\mathcal{L}$ be a sliceable but not one-sided layout. We claim that $\mathcal{L}$ contains a sublayout strongly equivalent to a layout in Figs. 4a–4d. Because $\mathcal{L}$ is not one-sided, it contains a maximal line segment ℓ which is not the side of any rectangle. Because $\mathcal{L}$ is sliceable, every maximal line segment in it subdivides a larger rectangle into two smaller rectangles. We may assume w.l.o.g. that ℓ is vertical. Because ℓ is not the side of any rectangle, the rectangles on the left and right of ℓ must be subdivided horizontally in the recursion. Let ℓ_{left} and ℓ_{right} be the first maximal horizontal line segments on the left and right of ℓ, respectively. Assume that they each subdivide a rectangle adjacent to ℓ into r_1 and r_2 (on the left) and r_3 and r_4 on the right. These rectangles comprise a layout equivalent to the one in Figs. 4a–4d; but they may be further subdivided recursively. By Lemma 3, there exists an aspect ratio assignment to $\mathcal{L}$ not realizable by a strongly equivalent layout. $\square$

In the remainder of this section, we prove that if a layout is not sliceable, then it contains a sublayout similar, in some sense, to a prototype in Figs. 4e–4f. In a nutshell, our proof goes as follows: Consider an arbitrary windmill in a nonsliceable layout $\mathcal{L}$. We subdivide the exterior of the windmill into four *quadrants*, by extending the arms of the windmill into rays $\ell_1, \ldots, \ell_4$ to the bounding box; see Fig. 5. Each rectangle of $\mathcal{L}$ lies in a quadrant or in the union of two consecutive quadrants. We assign aspect ratios to the rectangles based on which quadrant(s) it lies in. If these aspect ratios can be realized by a layout $\mathcal{L}'$ weakly equivalent to $\mathcal{L}$, then the rays $\ell_1, \ldots, \ell_4$ will be "deformed" into x- or y-monotone paths that subdivide $\mathcal{L}'$ into the center of the windmill and four arrangements of rectangles, each incident to a unique corner of the bounding box. We assign the aspect ratios for the rectangles in $\mathcal{L}'$ so that these arrangements can play the same role as rectangles $r_1, \ldots, r_4$ in the prototype in Figs. 4e–4f. We continue with the details.



(a) A nonsliceable layout, a windmill, where rays $\ell_1, \ldots, \ell_4$ define quadrants.

(b) An equivalent layout, where four paths define rectangular arrangements.

Fig. 5: A rays $\ell_1, \ldots, \ell_4$ deform into monotone paths in an equivalent layout.

We clarify what we mean by a "deformation" of a (horizontal) ray ℓ.

Lemma 5. *Let a ray ℓ be the extension of a horizontal segment in a layout $\mathcal{L}$ such that ℓ does not contain any other segment and it intersects the rectangles*

$r_1, \dots, r_k$ in this order. Suppose that in a weakly equivalent layout $\mathcal{L}'$, the corresponding rectangles $r'_1, \dots, r'_k$ are sliced by horizontal segments $s_1, \dots, s_k$. Then there exists an x-monotone path comprised of horizontal edges $s_1, \dots, s_k$, and vertical edges along vertical segment of the layout $\mathcal{L}'$.

Proof. Assume w.l.o.g. that ℓ points to the right. Since ℓ does not contain any other segment and it intersects the rectangles $r_1, \dots, r_k$ in this order, then r_i and r_{i+1} are on opposite sides of a vertical segment for $i = 1, \dots, k-1$. The same holds for r'_i and r'_{i+1} as $\mathcal{L}'$ is weakly equivalent to $\mathcal{L}$. In particular, the right endpoint of s_i and the left endpoint of s_{i+1} are on the same vertical segment in $\mathcal{L}'$, for all $i = 1, \dots k-1$. ◻

The next lemma allows us to bound the aspect ratio of the bounding box of a rectangular arrangement in terms of the aspect ratios of individual rectangles.

Lemma 6. *If every rectangle in a rectangular arrangement has aspect ratio αm, where m is the number of rectangles in the arrangement, then the aspect ratio of the bounding box of the arrangement is at least α and at most αm^2.*

Proof. Consider an arrangement A with m rectangles and a bounding box R. Let w be the maximum width of a rectangle in A. Then, $\text{width}(R) \leq mw$. A rectangle of width w has height αmw, and so $\text{height}(R) \geq \alpha mw$. The aspect ratio of R is $\text{height}(R)/\text{width}(R) \geq (\alpha mw)/(mw) = \alpha$.

Similarly, let h be the maximum height of rectangle in A. Then $\text{height}(R) \leq mh$. A rectangle of height h has width $\frac{h}{\alpha m}$, and so $\text{width}(R) \geq \frac{h}{\alpha m}$. The aspect ratio of R is $\text{height}(R)/\text{width}(R) \leq mh/(\frac{h}{\alpha m}) = \alpha m^2$, as claimed. ◻

We can now complete the characterization of aspect ratio universal layouts.

Lemma 7. *If a layout $\mathcal{L}$ is not sliceable, it is not weakly ARU.*

Proof. Let R be a nonslicable layout of n rectangles in a bounding box of $\mathcal{L}$. We may assume that $\mathcal{L}$ is irreducible, otherwise we can choose a minimal nonsliceable sublayout $\mathcal{L}^*$ from $\mathcal{L}$, and replace each maximal sublayout of $\mathcal{L}^*$ with a rectangle to obtain an irreducible layout. By Lemma 2, a suitable aspect ratio assignment to each sliceable sublayout of $\mathcal{L}^*$ can generate any aspect ratio for the replacement rectangle.

In particular, $\mathcal{L}$ thus contains no slices, as any slice would create two smaller sublayouts. Every nonsliceable layout contains a windmill. Consider an arbitrary windmill in $\mathcal{L}$, assume w.l.o.g. that it is clockwise (cf. Fig. 4e). and let c be its central rectangle. By extending the arms of the windmill into rays, $\ell_1, \dots, \ell_4$, we subdivide $R \setminus c$ into four *quadrants*, denoted by $Q_1, \dots, Q_4$ in counterclockwise order starting with the top-right quadrant.

Note that at most one ray intersects the interior of a rectangle in $\mathcal{L}$. Indeed, any two points in two different rays, $p_i \in \ell_i$ and $p_j \in \ell_j$, span an axis-parallel rectangle that intersects the interior of c. Consequently, p_i and p_j cannot be in the same rectangle in $R \setminus c$. It follows that every rectangle of $\mathcal{L}$ in $R \setminus c$ lies in one quadrant or in the union of two consecutive quadrants.

We define an aspect ratio assignment α as follows: Let $\alpha(c) = 1$. If $r \subseteq Q_1$ or $r \subseteq Q_3$, let $\alpha(r) = 6n$; and if $r \subseteq Q_2$ or $r \subseteq Q_4$, let $\alpha(r) = (6n^2)^{-1}$. For a rectangle r split by a ray, we set $\alpha(r) = 6n + (6n^2)^{-1}$ if r is split by a horizontal ray ℓ_1 or ℓ_3; and $\alpha(r) = ((6n)^{-1} + (6n^2))^{-1}$ if split by a vertical ray ℓ_2 or ℓ_4.

Suppose that a layout $\mathcal{L}'$ weakly equivalent to $\mathcal{L}$ realizes α. Split every rectangle of aspect ratio $6n + (6n^2)^{-1}$ in $\mathcal{L}'$ horizontally into two rectangles of aspect ratios $6n$ and $(6n^2)^{-1}$. Similarly, split every rectangle of aspect ratio $((6n)^{-1}+(6n^2))^{-1}$ vertically into two rectangles of aspect ratios $6n$ and $(6n^2)^{-1}$; see Fig. 5b. By Lemma 5, there are four x- or y-monotone paths $P_1, \dots, P_4$ from the four arms of the windwill to four distinct sides of the bounding box that pass through the slitting segments. The paths $P_1, \dots, P_4$ subdivide the exterior of the windmill into four arrangements of rectangles, $A_1, \dots, A_4$ that each contain a unique corner of the bounding box. By construction, every rectangle in A_1 and A_3 has aspect ratio $6n$, and every rectangle in A_2 and A_4 has aspect ratio $(6n^2)^{-1}$.

Let $R_1, \dots, R_4$ be the bounding boxes of $A_1, \dots, A_4$, respectively. By Lemma 6, both R_1 and R_3 have aspect ratios at least 6, and both R_2 and R_4 have aspect ratios at most $\frac{1}{6}$. By construction, the arrangements $A_1, \dots, A_4$ each contain an arm of the windmill. This implies that $\text{width}(c) < \min\{\text{width}(R_1), \text{width}(R_3)\}$ and $\text{height}(c) < \min\{\text{height}(R_2), \text{height}(R_4)\}$. Consider the arrangement comprised of A_1, c, and A_3. It contains two opposite corners of R, and so its bounding box is R. Furthermore, $\text{height}(R) \geq \max\{\text{height}(R_1), \text{height}(R_3)\}$, and

$$\begin{aligned}\text{width}(R) &\leq \text{width}(R_1) + \text{width}(c) + \text{width}(R_3) < 3\,\max\{\text{width}(R_1), \text{width}(R_3)\}\\ &\leq 3\,\max\left\{\frac{\text{height}(R_1)}{6}, \frac{\text{height}(R_3)}{6}\right\} = \frac{\max\{\text{height}(R_1), \text{height}(R_3)\}}{2},\end{aligned}$$

and so the aspect ratio of R is at least 2. Similarly, the bounding box of the arrangement comprised of A_2, c, and A_3 is also R, and an analogous argument implies that its aspect ratio must be at most $\frac{1}{2}$. We have shown that the aspect ratio of R is at least 2 and at most $\frac{1}{2}$, a contradiction. Thus the aspect ratio assignment α is not realizable, and so $\mathcal{L}$ is not weakly aspect ratio universal. $\square$

This completes the proof of both Theorems 1 and 2. Specifically, Corollary 2 and Lemma 7 imply Theorem 1. For Theorem 2, we need to show that properties (i)–(iii) are equivalent: By Lemma 1, (ii) and (iii) are equivalent; Corollary 1 states that (ii) implies (i); and the converse follows from Lemmata 4 and 7.

2.2 Unique Transversal Structure

Subdividing a square into squares has fascinated humanity for ages [4,14,32]. For example, a *perfect square tiling* is a tiling with squares with distinct integer side lengths. Schramm [30] (see also [19, Chap. 6]) proved that every near triangulation with an outer 4-cycle is the extended dual of a (possibly degenerate or nongeneric) subdivision of a rectangle into squares. The result generalizes to rectangular faces of arbitrary aspect ratios (rather than squares):

Theorem 4. *(Schramm [30, Thm. 8.1]) Let $T = (V, E)$ be near triangulation with an outer 4-cycle, and $\alpha : V^* \to \mathbb{R}^+$ a function on the set V^* of the inner vertices of T. Then there exists a unique (but possibly degenerate or nongeneric) layout $\mathcal{L}$ such that $G^*(\mathcal{L}) = T$, and for every $v \in V^*$, the aspect ratio of the rectangle corresponding to v is $\alpha(v)$.*

The caveat in Schramm's result is that all rectangles in the interior of every separating 3-cycle must degenerate to a point, and rectangles in the interior of some of the separating 4-cycles may also degenerate to a point. We only use the *uniqueness* claim under the assumption that a nondegenerate and generic realization exists for a given aspect ratio assignment.

Lemma 8. *If a layout $\mathcal{L}$ is strongly ARU, then its extended dual $G^*(\mathcal{L})$ admits a unique transversal structure.*

Proof. Consider the extended dual graph $T = G^*(\mathcal{L})$ of a strongly ARU layout $\mathcal{L}$. As noted above, T is a 4-connected inner triangulation of a 4-cycle. If T admits two different transversal structures, then there are two strongly nonequivalent layouts, $\mathcal{L}$ and $\mathcal{L}'$, such that $T = G^*(\mathcal{L}) = G^*(\mathcal{L}')$, which in turn yield two aspect ratio assignments, α and α', on the inner vertices of T. By Theorem 4, the (nondegenerate) layouts $\mathcal{L}$ and $\mathcal{L}'$, that realize α and α', are unique. Consequently, neither of them can be strongly aspect ratio universal. □

Lemma 8 readily shows that Theorem 2(i) implies Theorem 2(iii), and provides an alternative proof for the geometric arguments in Lemmata 4 and 7.

3 Recognizing Duals of Aspect Ratio Universal Layouts

We describe an algorithm that, for a given graph G, either finds a one-sided sliceable layout $\mathcal{L}$ whose dual graph is G, or reports that no such layout exists. We can decide in $O(n)$ time whether a given graph is proper [12,22,24,25]. Every proper graph is a connected plane graph in which all bounded faces are triangles.

Problem Formulation. The input of our recursive algorithm will be an *instance* $I = (G, C, P)$, where $G = (V, E)$ is a near-triangulation, $C : V(G) \to \mathbb{N}_0$ is a *corner count*, and P is a set of ordered pairs (u, v) of vertices on the outer face of G. An instance $I = (G, C, P)$ is *realizable* if there exists a one-sided sliceable layout $\mathcal{L}$ such that G is the dual graph of $\mathcal{L}$, every vertex $v \in V$ corresponds to a rectangle in $\mathcal{L}$ incident to *at least* $C(v)$ corners of $\mathcal{L}$, and every pair $(a, b) \in P$ corresponds to a pair of rectangles in $\mathcal{L}$ incident to two ccw *consecutive* corners. When we have no information about corners, then $C(v) = 0$ for all $v \in V$, and $P = \emptyset$. In the full paper [10], we establish the following structural result.

Lemma 9. *Assume that (G, C, P) admits a realization $\mathcal{L}$ and $|V(G)| \geq 2$. Then G contains a vertex v with one of the following (mutually exclusive) properties.*

(I) Vertex v is a cut vertex in G. Then r_v is bounded by two parallel sides of R and by two parallel slices; and $C(v) = 0$.

(II) Rectangle r_v is bounded by three sides of R and a slice; and $0 \leq C(v) \leq 2$.

Based on property (II), a vertex v of G is a *pivot* if there exists a one-sided sliceable layout $\mathcal{L}$ with $G \simeq G(\mathcal{L})$ in which r_v is bounded three sides of R and a slice. If we find a cut vertex or a pivot v in G, then at least one side of r_v is a slice, so we can remove v and recurse on the connected components of $G - v$. We describe an analyze our algorithm for an instance I in the full paper [10].

4 Conclusions

We have shown that a layout $\mathcal{L}$ is weakly (strongly) ARU if and only if $\mathcal{L}$ is sliceable (one-sided and sliceable); and we can decide in $O(n^2)$-time whether a given graph G on n vertices is the dual of a one-sided sliceable layout. An immediate open problem is whether the runtime can be improved. Cut vertices and 2-cuts play a crucial role in our algorithm. We can show (in Section 4 of the full paper [10]) that the duals of one-sided sliceable layouts have vertex cuts of size at most 3. Perhaps 3-cuts can be utilized to speed up our algorithm. Recall that no polynomial-time algorithm is currently known for recognizing the duals of sliceable layouts [5,18,36] and one-sided layouts [6]. It remains an open to settle the computational complexity of these problems.

References

1. Ackerman, E., Barequet, G., Pinter, R.Y.: A bijection between permutations and floorplans, and its applications. Discret. Appl. Math. **154**(12), 1674–1684 (2006)
2. Asinowski, A., Mansour, T.: Separable d-permutations and guillotine partitions. Annals of Combinatorics **14**, 17–43 (2010)
3. de Berg, M., Cheong, O., van Kreveld, M., Overmars, M.: Binary space partitions. In: Computational Geometry: Algorithms and Applications, chap. 12, pp. 259–281. Springer (2008)
4. Brooks, R.L., Smith, C.A.B., Stone, A.H., Tutte, W.T.: The dissection of rectangles into squares. Duke Math. J. **7**(1), 312–340 (1940)
5. Dasgupta, P., Sur-Kolay, S.: Slicible rectangular graphs and their optimal floorplans. ACM Trans. Design Autom. Electr. Syst. **6**(4), 447–470 (2001)
6. Eppstein, D., Mumford, E., Speckmann, B., Verbeek, K.: Area-universal and constrained rectangular layouts. SIAM J. Comput. **41**(3), 537–564 (2012)
7. Felsner, S.: Lattice structures from planar graphs. Electron. J. Comb. **11**(1) (2004)
8. Felsner, S.: Rectangle and square representations of planar graphs. In: Pach, J. (ed.) Thirty Essays on Geometric Graph Theory, pp. 213–248. Springer (2013)
9. Felsner, S.: Exploiting air-pressure to map floorplans on point sets. J. Graph Algorithms Appl. **18**(2), 233–252 (2014)
10. Felsner, S., Nathenson, A., Tóth, C.D.: Aspect ratio universal rectangular layouts. Preprint (2021), https://arxiv.org/abs/2112.03242
11. Fusy, É.: Transversal structures on triangulations: A combinatorial study and straight-line drawings. Discret. Math. **309**(7), 1870–1894 (2009)
12. Hasan, M.M., Rahman, M.S., Karim, M.R.: Box-rectangular drawings of planar graphs. J. Graph Algorithms Appl. **17**(6), 629–646 (2013)

13. He, X.: On finding the rectangular duals of planar triangular graphs. SIAM J. Comput. **22**(6), 1218–1226 (1993)
14. Henle, F.V., Henle, J.M.: Squaring the plane. Am. Math. Mon. **115**(1), 3–12 (2008)
15. Kant, G., He, X.: Regular edge labeling of 4-connected plane graphs and its applications in graph drawing problems. Theor. Comput. Sci. **172**(1-2), 175–193 (1997)
16. Koźmiński, K., Kinnen, E.: Rectangular duals of planar graphs. Networks **15**(2), 145–157 (1985)
17. Kreveld, M.J., Speckmann, B.: On rectangular cartograms. Comput. Geom. **37**(3), 175–187 (2007)
18. Kusters, V., Speckmann, B.: Towards characterizing graphs with a sliceable rectangular dual. In: Proc. 23rd Sympos. Graph Drawing and Network Visualization (GD). LNCS, vol. 9411, pp. 460–471. Springer (2015)
19. Lovász, L.: Graphs and Geometry, Colloquium Publications, vol. 65. AMS, Providence, RI (2019)
20. Merino, A.I., Mütze, T.: Efficient generation of rectangulations via permutation languages. In: 37th International Symposium on Computational Geometry (SoCG). LIPIcs, vol. 189, pp. 54:1–54:18. Schloss Dagstuhl (2021)
21. Mitchell, W.J., Steadman, J.P., Liggett, R.S.: Synthesis and optimization of small rectangular floor plans. Environ. Plann. B Plann. Des. **3**(1), 37–70 (1976)
22. Nishizeki, T., Rahman, M.S.: Rectangular drawing algorithms. In: Handbook on Graph Drawing and Visualization, pp. 317–348. Chapman and Hall/CRC (2013)
23. Otten, R.H.J.M.: Automatic floorplan design. In: Proc. 19th Design Automation Conference (DAC). pp. 261–267. ACM/IEEE (1982)
24. Rahman, M.S., Nakano, S., Nishizeki, T.: Rectangular grid drawings of plane graphs. Comput. Geom. **10**(3), 203–220 (1998)
25. Rahman, M.S., Nakano, S., Nishizeki, T.: Rectangular drawings of plane graphs without designated corners. Comput. Geom. **21**(3), 121–138 (2002)
26. Raisz, E.: The rectangular statistical cartogram. Geogr. Review **24**(2), 292–296 (1934)
27. Reading, N.: Generic rectangulations. Eur. J. Comb. **33**(4), 610–623 (2012)
28. Rinsma, I.: Existence theorems for floorplans. Ph.D. thesis, University of Canterbury, Christchurch, New Zealand (1987)
29. Rinsma, I.: Nonexistence of a certain rectangular floorplan with specified areas and adjacency. Environ. Plann. B Plann. Des. **14**(2), 163–166 (1987)
30. Schramm, O.: Square tilings with prescribed combinatorics. Israel Journal of Mathematics **84**, 97–118 (1993)
31. Thomassen, C.: Plane representations of graphs. In: Progress in Graph Theory, pp. 43–69. Academic Press Canada (1984)
32. Tutte, W.T.: Squaring the square. Scientific American **199**, 136–142 (1958), in Gardner's 'Mathematical Games' column. Reprinted with addendum and bibliography in the US in M. Gardner, *The 2nd Scientific American Book of Mathematical Puzzles & Diversions*, Simon and Schuster, New York (1961), pp. 186—209
33. Ungar, P.: On diagrams representing maps. Journal of the London Mathematical Society **s1-28**(3), 336–342 (1953)
34. Wimer, S., Koren, I., Cederbaum, I.: Floorplans, planar graphs, and layouts. IEEE Transactions on Circuits and Systems **35**(3), 267–278 (1988)
35. Yao, B., Chen, H., Cheng, C., Graham, R.L.: Floorplan representations: Complexity and connections. ACM Trans. Design Autom. Electr. Syst. **8**(1), 55–80 (2003)
36. Yeap, G.K.H., Sarrafzadeh, M.: Sliceable floorplanning by graph dualization. SIAM J. Discret. Math. **8**(2), 258–280 (1995)