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Einstein metrics, conformal curvature, and anti-holomorphic involutions

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Abstract
Building on previous results [17,35], we complete the classification of compact oriented Einstein 4-manifolds with $\det(W^+) > 0$. There are, up to diffeomorphism, exactly 15 manifolds that carry such metrics, and, on each of these manifolds, such metrics sweep out exactly one connected component of the corresponding Einstein moduli space.

Résumé
En s'appuyant sur les résultats de travaux antérieurs [17,35], on achève la classification des variétés d'Einstein compactes orientées de dimension 4 dont la courbure de Weyl autoduale est de déterminant strictement positif. À difféomorphisme près, il y a exactement quinze variétés qui admettent de telles métriques; et, sur chacune de ces variétés, ces métriques remplissent exactement une composante connexe de l'espace de modules des métriques d'Einstein.

Mathematics Subject Classification 53C25 (primary) ; 14J26 · 32C10 · 53C21 · 53C55 (secondary)

1 Introduction

A Riemannian metric h is said [3] to be *Einstein* if, for some real constant λ, it satisfies the *Einstein equation*

$$r = \lambda h,$$

where r is the Ricci tensor of h. Given a smooth compact n-manifold M, henceforth always assumed to be connected and without boundary, one would like to completely understand the Einstein moduli space

$$\mathscr{E}(M) = \{\text{Einstein metrics on } M\}/(\mathscr{D}\mathit{iff}(M) \times \mathbb{R}^+),$$

where the diffeomorphism group $\mathscr{D}\mathit{iff}(M)$ acts on metrics via pull-backs, and where the positive reals $\mathbb{R}^+$ act by rescaling. This moduli problem is well understood [27,28] in dimensions

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$n \leq 3$, because in these low dimensions the Einstein equation is actually equivalent to just requiring the sectional curvature to be constant. By contrast, when $n \geq 5$, the abundance of currently-available examples of "exotic" Einstein metrics on familiar manifolds [5,6,33] seems to indicate that the problem could very well turn out to be fundamentally intractable in high dimensions. On the other hand, there are certain specific 4-manifolds, such as real and complex-hyperbolic 4-manifolds, the 4-torus, and $K3$, where the Einstein moduli space $\mathscr{E}(M)$ is explicitly known, and in fact turns out to be connected [2,4,14,18]. This provides clear motivation for the intensive study of Einstein moduli spaces in dimension four.

The idiosyncratic features of 4-dimensional Riemannian geometry are generally attributable to the failure of the Lie group $\mathbf{SO}(4)$ to be simple; instead, its Lie algebra decomposes as a direct sum of proper subalgebras:

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

Because $\mathfrak{so}(4)$ and $\wedge^2\mathbb{R}^4$ can both be realized as the space of skew 4×4 matrices, this leads to a natural decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

of the bundle of 2-forms on an oriented Riemannian 4-manifold (M, h). Since the sub-bundles $\Lambda^\pm$ coincide with the (± 1)-eigenspaces of the Hodge star operator $\star : \Lambda^2 \to \Lambda^2$, sections of Λ^+ are called self-dual 2-forms, while sections of Λ^- are called anti-self-dual 2-forms. But because the Riemann curvature tensor can be naturally identified with a self-adjoint linear map

$$\mathcal{R} : \Lambda^2 \to \Lambda^2,$$

the curvature of (M^4, h) can consequently be decomposed into four pieces

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} I & \mathring{r} \\ \hline \mathring{r} & W^- + \frac{s}{12} I \end{array}\right),$$

corresponding to different irreducible representations of $\mathbf{SO}(4)$. Here s is the scalar curvature and $\mathring{r}$ is the trace-free Ricci curvature, while $W^\pm$ are by definition the trace-free pieces of the appropriate blocks. The corresponding pieces $W^{\pm a}{}_{bcd}$ of the Riemann curvature tensor are in fact both conformally invariant, and are respectively called the *self-dual* and *anti-self-dual Weyl curvature* tensors. The sum $W = W^+ + W^-$ is called the *Weyl tensor* or *conformal curvature* tensor, and vanishes if and only if the metric h is locally conformally flat. It should be emphasized that the distinction between the self-dual and anti-self-dual parts of the Weyl tensor depends on a choice of orientation; reversing the orientation of M interchanges Λ^+ and Λ^-, and so interchanges W^+ and W^-, too.

The present paper is a natural outgrowth of previous work on the Einstein moduli spaces $\mathscr{E}(M)$ for the smooth compact oriented 4-manifolds M that arise as del Pezzo surfaces. Recall that a del Pezzo surface is defined to be a compact complex surface (M^4, J) with ample anti-canonical line bundle. Up to diffeomorphism, there are exactly ten such manifolds, namely $S^2 \times S^2$ and the nine connected sums $\mathbb{CP}_2 \# m\overline{\mathbb{CP}}_2$, $m = 0, 1, \ldots, 8$. These are exactly [7] the smooth oriented compact 4-manifolds that admit both an Einstein metric with $\lambda > 0$ and an orientation-compatible symplectic structure. However, the currently-known Einstein

metrics on any of these spaces are all conformally Kähler. Indeed, on most del Pezzos, the currently-known Einstein metrics [24,29] are actually Kähler–Einstein, although there are two exceptional cases where they are instead non-trivial conformal rescalings of special extremal Kähler metrics [7,21]. Inspired in part by earlier work by Derdziński [3,9], and building upon his own results in [19,20], the author was eventually able to characterize [22] the known Einstein metrics on del Pezzo manifolds by the property that $W^+(\omega, \omega) > 0$ everywhere, where ω is a non-trivial (global) self-dual harmonic 2-form. An interesting corollary is that the known Einstein metrics on each del Pezzo 4-manifold M exactly sweep out one connected component of the corresponding Einstein moduli space $\mathscr{E}(M)$.

However, the role of a global harmonic 2-form ω in the above criterion makes it disquietingly non-local. Fortunately, Peng Wu [35] has recently discovered that these known Einstein metrics can instead be characterized by demanding that $\det(W^+)$ be positive at every point, where the self-dual Weyl curvature is considered as an endomorphism

$$W^+ : \Lambda^+ \to \Lambda^+$$

of the rank-3 bundle of self-dual 2-forms. The present author then found [17] an entirely different proof of this characterization that actually strengthens the result, while at the same time highlighting the previously-neglected point that this criterion only forces our compact oriented Einstein manifold to be a del Pezzo if we explicitly require it to be *simply connected*. In this paper, we will tackle this last issue head-on, by describing the moduli space

$$\mathscr{E}_{\det}(M) = \{\text{Einstein metrics on } M \text{ with } \det(W^+) > 0\}/(\mathcal{D}iff(M) \times \mathbb{R}^+)$$

for each compact oriented 4-manifold M where this moduli space is non-empty. Our first main result is the following:

Theorem A *There are exactly 15 diffeotypes of compact oriented* 4*-manifolds* M *that carry Einstein metrics* h *with* $\det(W^+) > 0$ *everywhere. For each such manifold, the moduli space* $\mathscr{E}_{\det}(M)$ *of these special Einstein metrics is connected, and exactly sweeps out a single connected component of the Einstein moduli space* $\mathscr{E}(M)$*.*

In order to state our second, more detailed main result, we will first need to consider two different $\mathbb{Z}_2$-actions on $S^2 \times S^2$. Let $\mathfrak{a} : S^2 \to S^2$ denote the antipodal map, and let $\mathfrak{r} : S^2 \to S^2$ denote reflection across the equator, so that

$$\mathfrak{a} = \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} \quad \text{and} \quad \mathfrak{r} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \tag{1}$$

as elements of $\mathbf{O}(3)$. Then $\mathfrak{a} \times \mathfrak{r}$ and $\mathfrak{a} \times \mathfrak{a}$ are both free, orientation-preserving involutions of $S^2 \times S^2$, and the smooth compact 4-manifolds

$$\begin{aligned} \mathscr{P} &:= (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{r} \rangle \\ \mathscr{Q} &:= (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle \end{aligned} \tag{2}$$

are therefore both orientable. Note, however, these two manifolds are not even homotopy equivalent [11, p. 101], because $\mathscr{P}$ is spin, whereas $\mathscr{Q}$ is not.

Theorem B *Let* M *be a smooth compact oriented* 4*-manifold that is* not simply connected*. Then, in the notation defined by* (2)*,* M *admits an Einstein metric* h *with* $\det(W^+) > 0$ *if and only if* M *is diffeomorphic to* $\mathscr{P}$ *or to* $\mathscr{Q}\#k\overline{\mathbb{CP}}_2$ *for some* $k = 0, 1, 2, 3$*. Moreover, whenever such an Einstein metric* h *exists, the universal cover* $(\widetilde{M}, \widetilde{h})$ *of* (M, h) *is necessarily isometric*

to a del Pezzo surface, equipped with a Kähler–Einstein metric, in such a manner that the non-trivial deck transformation becomes a free anti-holomorphic involution.

The proofs of these results are given in Sect. 2.5 below, as the culmination of a series of detailed case-by-case studies carried out in earlier parts of Sect. 2. Then, in Sect. 3, we conclude the article by generalizing these results in various ways, while also pointing pointing out some associated open problems.

2 Del Pezzos and Double Covers

We begin by carefully refining the statement of [17, Proposition 2.3] in order to emphasize a key technical fact that lay buried in the proof.

Proposition 1 *Let (M, h) be a compact oriented Einstein 4-manifold which satisfies $\det(W^+) > 0$ at every point. Then either*

(i) $\pi_1(M) = 0$, and M admits an orientation-compatible complex structure J such that (M, J) is a del Pezzo surface, and such that the conformally rescaled metric $g = |W^+|_h^{2/3} h$ is a J-compatible Kähler metric; or else,

(ii) $\pi_1(M) = \mathbb{Z}_2$, and M is doubly covered by a del Pezzo surface $(\widetilde{M}, J)$ on which the pull-back of $g = |W^+|_h^{2/3} h$ is a J-compatible Kähler metric $\widetilde{g}$, and where the non-trivial deck transformation $\sigma : \widetilde{M} \to \widetilde{M}$ is an anti-holomorphic involution of $(\widetilde{M}, J)$.

Proof The conformal rescaling of h used in [17] was actually constructed as $\alpha_h^{2/3} h$, where α_h is the top eigenvalue of $W_h^+ : \Lambda^+ \to \Lambda^+$. However, once this rescaled metric has been shown to be Kähler, it then follows that $-\alpha_h/2$ is a repeated eigenvalue of W_h^+, so that one necessarily also has $|W^+|_h^2 = \frac{3}{2}\alpha_h^2$. Thus, the Kähler metric constructed in [17] simply coincides, up to a constant factor of $\sqrt[3]{3/2}$, with the metric $g = |W^+|_h^{2/3} h$ considered above.

The proof of [17, Proposition 2.3] actually focuses on the real line-bundle $L \subset \Lambda^+$ given by the top eigenspace of W^+; this is well-defined, because the identity $\mathrm{tr}(W^+) = 0$ and the hypothesis $\det(W^+) > 0$ together imply that the top eigenvalue of W^+ has multiplicity one everywhere. If L is trivial, one can then choose a global section ω of L such that $|\omega|_g \equiv \sqrt{2}$, and a Weitzenböck argument (made possible by the fact that any Einstein metric satisfies $\delta W^+ = 0$) is then used to show that ω is parallel. If, on the other hand, L is non-trivial, $\widetilde{M} = \{\omega \in L \mid |\omega|_g = \sqrt{2}\}$ defines a double cover of M that comes equipped with a tautological self-dual 2-form ω that, by the same Weitzenböck argument as before, can then be shown to be the Kähler form of the pulled-back metric $\widetilde{g}$ with respect to a suitable complex structure J. In the latter case, the non-trivial deck transformation $\sigma : \widetilde{M} \to \widetilde{M}$ preserves $\widetilde{g}$, and sends ω to $-\omega$, and so, because $\omega = \widetilde{g}(J\cdot, \cdot)$, must send J to $-J$. Thus, in case (ii), σ is an anti-holomorphic involution of $(\widetilde{M}, J)$.

Finally, the complex surface (M, J) or $(\widetilde{M}, J)$ is automatically a del Pezzo. Indeed, since any Kähler surface satisfies $\det(W_+) = s^3/864$, where s is its scalar curvature, the assumption that $\det(W^+) > 0$ implies the scalar curvature of g or $\widetilde{g}$ must be positive everywhere. Since the Einstein metric h therefore has positive Einstein constant, and can now be rewritten as $24 s^{-2} g$, the transformation law for the Ricci curvature under conformal changes implies [19] that the $(1, 1)$-form $\rho + 2i\partial\bar{\partial} \log s$ is a positive representative of $2\pi c_1$, where ρ is the Ricci form of our Kähler surface. The Kodaira embedding theorem thus implies that the anti-canonical line-bundle K^{-1} is ample, and (M, J) or $(\widetilde{M}, J)$ is therefore a del Pezzo surface, as claimed. □

Because case (i) was thoroughly analyzed in previous papers [17,22,24], we will only need to carefully discuss case (ii) in this article. Fortunately, this part of the problem can largely be reduced to well-explored questions in real algebraic geometric. Indeed, since $(\widetilde{M}, J)$ can be embedded in a projective space $\mathbb{P}([\Gamma(\mathcal{O}(K^{-\ell})]^*)$ on which σ acts by complex conjugation, $\widetilde{M}$ can be viewed as a complex projective algebraic variety defined over $\mathbb{R}$; and because the action of σ on $\widetilde{M}$ has no fixed points, this variety automatically has empty real locus. The substantial classical and modern literature available concerning real forms of del Pezzo surfaces [12,16,25,32] has therefore paved the road ahead of us, and will make it comparatively easy to completely solve the problem.

Since traditional approaches to the subject emphasize the *degree* $c_1^2 > 0$ of a del Pezzo surface, it will be important for us to relate the degree of $\widetilde{M}$ to the topology of $M = \widetilde{M}/\langle\sigma\rangle$. For this purpose, it is useful to remember that any almost-complex 4-manifold satisfies $c_1^2 = 2\chi + 3\tau$, where χ is the Euler characteristic and $\tau = b_+ - b_-$ is the signature. For the del Pezzo surface $\widetilde{M}$, however, the Todd genus $\mathbf{Td} = h^{0,0} - h^{0,1} + h^{0,2} = (\chi + \tau)/4$ must equal 1, since $h^{0,1} = h^{0,2} = 0$ by the Kodaira vanishing theorem. It therefore follows that

$$c_1^2(\widetilde{M}) = 8 + \tau(\widetilde{M}) = 8 + 2\tau(M),$$

where in the last step we have recalled that the signature τ is multiplicative under finite covers. On the other hand, $b_+(M) = 0$, since the Kähler form ω spans the self-dual harmonic forms on $(\widetilde{M}, \widetilde{g})$, but is σ-anti-invariant. Hence $\tau(M) = -b_-(M) = -b_2(M)$, and $c_1^2(\widetilde{M}) = 2[4 - b_2(M)]$. As a consequence, the only possibilities are $b_2(M) = 0, 1, 2$ or 3. We will now proceed by discussing each of these cases separately.

2.1 The $b_2(M) = 0$ Case

When $b_2(M) = 0$, the double cover $\widetilde{M}$ must have signature zero. Since this covering space is therefore a del Pezzo surface of degree 8, classification [8,10] tells us that $\widetilde{M}$ is diffeomorphic to either $S^2 \times S^2$ or $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$. Now, it is a classical fact [25,32] that any anti-holomorphic involution of the one-point blow-up of $\mathbb{CP}_2$ must have a fixed point. But, as we will now observe, this is actually preordained by a more general topological result. Although elementary, the proof is worth recounting here in some detail, as doing so will eventually save us needless extra work in Sect. 3.

Lemma 1 *No smooth* orientable 4*-manifold M with $\pi_1 \neq 0$ has a covering space homeomorphic to $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$.*

Proof Let us proceed by contradiction, and assume there exists a covering map $\varpi : N \to M$, where M is a smooth oriented non-simply-connected 4-manifold, and where N is homeomorphic (but perhaps not diffeomorphic) to $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$. Notice that $M = \varpi(N)$ is automatically compact, and that the simply connected manifold N is automatically its universal cover. We now give N the orientation lifted from M, so that the degree ≥ 2 of ϖ then equals $|\pi_1(M)|$. Since this in particular means that $\pi_1(M)$ is finite,

$$H^1(M, \mathbb{R}) = \mathrm{Hom}(\pi_1(M), \mathbb{R}) = 0,$$

and Poincaré duality for the oriented 4-manifold M therefore implies

$$b_3(M) = b_1(M) = 0 \quad \text{and} \quad b_4(M) = b_0(M) = 1,$$

where b_j denotes the j^{th} Betti number with $\mathbb{R}$ coefficients. The Euler characteristic of M is therefore given by

$$\chi(M) = \sum_{j=0}^{4} (-1)^j b_j(M) = 2 + b_2(M) \geq 2.$$

However, because the Euler characteristic χ is multiplicative under finite coverings, we also have

$$\chi(M) = \chi(\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2)/|\pi_1(M)| = 4/|\pi_1(M)| \leq 2.$$

It therefore follows that $\chi(M) = 2$, and that $b_2(M) = 0$. In particular, $H^2(M, \mathbb{Z})$ has trivial free part, and so consists entirely of torsion elements.

On the other hand, any smooth, orientable 4-manifold is spin^c. Thus, there exists [13,34] an integral cohomology class $a \in H^2(M, \mathbb{Z})$ satisfying

$$\varrho(a) = w_2(M) := w_2(TM),$$

where

$$\varrho : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$$

denotes the natural homomorphism induced by mod-2 reduction $\mathbb{Z} \to \mathbb{Z}_2$. However, since ϖ is a smooth submersion, $\varpi_* : TN \cong \varpi^* TM$. Thus, the naturality of Stiefel-Whitney classes with respect to pull-backs and the commutativity of the diagram

$$\begin{array}{ccc} H^2(N, \mathbb{Z}) & \stackrel{\varrho}{\to} & H^2(N, \mathbb{Z}_2) \\ \uparrow_{\varpi^*} & & \uparrow_{\varpi^*} \\ H^2(M, \mathbb{Z}) & \stackrel{\varrho}{\to} & H^2(M, \mathbb{Z}_2) \end{array}$$

together guarantee that

$$\begin{aligned} \varrho\left(\varpi^*(a)\right) &= \varpi^*(\varrho\,(a)) = \varpi^*(w_2(TM)) \\ &= w_2(\varpi^* TM) = w_2(TN) \\ &= w_2(N) \in H^2(N, \mathbb{Z}_2). \end{aligned}$$

On the other hand, since $a \in H^2(M, \mathbb{Z})$ is a torsion element, it follows that $\varpi^* a \in H^2(N, \mathbb{Z})$ is a torsion element, too. But

$$H^2(N, \mathbb{Z}) \cong H^2(\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

is a free Abelian group, so this implies that $\varpi^* a = 0$. Hence

$$w_2(N) = \varrho\left(\varpi^*(a)\right) = 0.$$

But this is absurd, because $N \approx \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ has odd intersection form, and so is not spin. It follows that the oriented 4-manifold M cannot exist, as claimed. □

In our context, this simple fact has a striking consequence:

Theorem 1 *Let (M, h) be a compact oriented* non-simply-connected *Einstein 4-manifold that satisfies $\det(W^+) > 0$ at every point. Then M is doubly covered by a del Pezzo surface $(\widetilde{M}, J)$ on which the pull-back $\widetilde{h}$ of h is a J-compatible* Kähler–Einstein *metric.*

Proof Case (ii) of Proposition 1 tells us that the Einstein manifold $(\widetilde{M}, \widetilde{h})$ is conformally Kähler. However, by [20, Theorem A], $\mathbb{CP}_2\#\overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2\#2\overline{\mathbb{CP}}_2$ are the only two compact 4-manifolds that carry Einstein metrics that are conformally Kähler, but not Kähler–Einstein. But neither of these is the double cover of an oriented 4-manifold; the second is prohibited because its signature is odd, while the first is ruled out by Lemma 1. □

As an immediate consequence, any compact oriented Einstein 4-manifold (M, h) with $\det(W^+) > 0$ and $b_2 = 0$ must be doubly covered by $\mathbb{CP}_1 \times \mathbb{CP}_1$, equipped with a Kähler–Einstein metric. However, a theorem of Matsushima [23, Théorème 1] implies that any Kähler–Einstein metric on $\mathbb{CP}_1 \times \mathbb{CP}_1$ must be invariant under a maximal compact subgroup $\cong \mathbf{SO}(3) \times \mathbf{SO}(3)$ of the identity component $\mathbf{PSL}(2, \mathbb{C}) \times \mathbf{PSL}(2, \mathbb{C})$ of the complex automorphism group. Thus, the universal cover $(\widetilde{M}, \widetilde{h})$ of (M, h) must be homothetic to the homogeneous Einstein manifold $(S^2, g_0) \times (S^2, g_0)$, where g_0 is the "round" unit-sphere metric on $S^2 = \mathbb{CP}_1$. This allows us to deduce the following:

Proposition 2 *Modulo constant rescalings, any compact oriented Einstein* 4*-manifold* (M, h) *with* $\det(W^+) > 0$ *and* $b_2 = 0$ *is isometric to exactly one of the Riemannian quotients described by* (2)*. Since the two* 4*-manifolds* $\mathscr{P}$ *and* $\mathscr{Q}$ *are not diffeomorphic, it thus follows that the moduli spaces* $\mathscr{E}_{\det}(\mathscr{P})$ *and* $\mathscr{E}_{\det}(\mathscr{Q})$ *each consist of a single point.*

Proof With respect to the product metric $g_0 \oplus g_0$, the sectional curvature $K(\Pi)$ of a 2-plane $\Pi \subset T(S^2 \times S^2)$ belongs to $[0, 1]$, and satisfies $K(\Pi) = 1$ iff Π is tangent to an S^2 factor. Thus, any isometry of $(S^2 \times S^2, g_0 \oplus g_0)$ must send each 2-sphere $S^2 \times \{pt\}$ or $\{pt\} \times S^2$ to a 2-sphere of one of these two types. On the other hand, because the orientation-preserving isometric involution $\sigma : S^2 \times S^2 \to S^2 \times S^2$ must not have fixed points, the Lefschetz fixed-point theorem tells us that its Leftschetz number must vanish. That is,

$$0 = L(\sigma) = \sum_j (-1)^j \operatorname{tr}\left(\sigma_*|_{H_j(S^2\times S^2)}\right) = 2 + \operatorname{tr}\left(\sigma_*|_{H_2(S^2\times S^2)}\right),$$

where σ_* is the induced map on homology with $\mathbb{R}$ coefficients. Since $(\sigma_*)^2 = I$ and $\operatorname{tr}\left(\sigma_*|_{H_2(S^2\times S^2)}\right) = -2$, it follows that $\sigma_* = -I$ on $H_2(S^2 \times S^2, \mathbb{R})$. Hence each sphere $S^2 \times \{pt\}$ must be sent isometrically by σ to a sphere of the same kind, in an orientation-reversing manner; and the same conclusion similarly applies to spheres of the form $\{pt\} \times S^2$. Since the projection of $S^2 \times S^2$ to either factor is a Riemannian submersion, it therefore follows that σ must be the product of two isometric, orientation-reversing involutions of (S^2, g_0). However, any such involution is diagonalizable, with eigenvalues ± 1. Up to conjugation, the only candidates for these maps of S^2 are therefore the involutions $\mathfrak{a}$ and $\mathfrak{r}$ described by (1). However, $\mathfrak{r} \times \mathfrak{r}$ can be excluded as a candidate for σ, since it has fixed points. Thus, after interchanging factors if necessary, the only remaining possibilities for σ are the free anti-holomorphic involutions $\mathfrak{a} \times \mathfrak{r}$ and $\mathfrak{a} \times \mathfrak{a}$ of $S^2 \times S^2 = \mathbb{CP}_1 \times \mathbb{CP}_1$.

It therefore only remains to show that $\mathscr{P} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{r} \rangle$ is not diffeomorphic to $\mathscr{Q} := (S^2 \times S^2)/\langle \mathfrak{a} \times \mathfrak{a} \rangle$. To see this, first notice that $w_2(\mathscr{Q}) \neq 0$, since the diagonal $S^2 \subset S^2 \times S^2$ projects to an $\mathbb{RP}^2 \subset \mathscr{Q}$ that has normal bundle $\cong T\mathbb{RP}^2$, and so has self-intersection $\chi(\mathbb{RP}^2) = 1 \bmod 2$. By contrast, $H_2(\mathscr{P}, \mathbb{Z}_2)$ is generated by the $\mathbb{RP}^2$-image of $S^2 \times \{(1, 0, 0)\}$ and the S^2-image of $\{pt\} \times S^2$; and since each of these submanifolds has small perturbations that do not intersect it, both have self-intersection zero, and it follows that $w_2(\mathscr{P}) = 0$. Thus, the 4-manifolds $\mathscr{P}$ and $\mathscr{Q}$ certainly aren't diffeomorphic, because one is spin, while the other isn't. □

2.2 The $b_2(M) = 1$ case

When $b_2(M) = 1$, the del Pezzo surface $(\widetilde{M}, J)$ has degree $c_1^2 = 6$. Because this complex surface has K^{-1} ample, surface classification easily allows one to show [8,10] that it must exactly be the blow-up of $\mathbb{CP}_2$ at three non-collinear points, which we may take to be $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. By adjusting our coordinates if necessary, the free anti-holomorphic involution $\sigma : \widetilde{M} \to \widetilde{M}$ can moreover then be identified [32, p. 60] with the map

$$\Upsilon : \mathbb{CP}_2\#3\overline{\mathbb{CP}}_2 \to \mathbb{CP}_2\#3\overline{\mathbb{CP}}_2$$

given by the conjugated Cremona transformation

$$[z_1 : z_2 : z_3] \longmapsto \left[\frac{1}{\bar{z}_1} : -\frac{1}{\bar{z}_2} : \frac{1}{\bar{z}_3}\right].$$

This last uniqueness assertion might come as something of a surprise. For instance, if we blow up $\mathbb{CP}_1 \times \mathbb{CP}_1$ at a generic pair of distinct points that are interchanged by $\mathfrak{a} \times \mathfrak{r}$, the anti-holomorphic involution thereby induced on the blow-up is actually isomorphic to the one we would have produced had we instead started with $\mathfrak{a} \times \mathfrak{a}$; for although identifying the two-point blow-up of $\mathbb{CP}_1 \times \mathbb{CP}_1$ with the three-point blow up $\mathbb{CP}_2$ in the standard way produces two anti-holomorphic involutions that look superficially different, these actually turn out to simply differ by a Cremona transformation [25]. In particular, it follows that the non-spin 4-manifolds $\mathscr{P}\#\overline{\mathbb{CP}}_2$ and $\mathscr{Q}\#\overline{\mathbb{CP}}_2$ are both diffeomorphic to $(\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2)/\langle\Upsilon\rangle$.

Our discussion thus far has revealed that any compact oriented Einstein manifold (M^4, h) with $\pi_1 \neq 0, b_2 = 1$, and $\det(W^+) > 0$ must be diffeomorphic to $\mathscr{Q}\#\overline{\mathbb{CP}}_2$. We will now show that, conversely, this possibility actually arises, and that it does so moreover in an essentially unique way:

Proposition 3 *There is an Einstein metric h on $\mathscr{Q}\#\overline{\mathbb{CP}}_2$ that satisfies $\det(W^+) > 0$ at every point. Moreover, any compact oriented Einstein manifold (M^4, h') with $\pi_1 \neq 0$, $b_2 = 1$, and $\det(W^+) > 0$ is isometric to $(\mathscr{Q}\#\overline{\mathbb{CP}}_2, \mathsf{a}h)$ for some positive constant a. As a consequence, the restricted Einstein moduli space $\mathscr{E}_{\rm det}(\mathscr{Q}\#\overline{\mathbb{CP}}_2)$ therefore consists of exactly one point.*

Proof Siu [26, p. 621] proved that $\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2$ admits a J-compatible Kähler–Einstein metric g with Einstein constant 1 that is invariant under the compact group of automorphisms generated by the permutations

$$\alpha_1 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix},$$

along with the action of the 2-torus

$$\mathbb{T}^2 := \mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1)) = \left\{ \begin{bmatrix} e^{i\theta} & & \\ & e^{i\phi} & \\ & & e^{-i(\theta+\phi)} \end{bmatrix} \right\},$$

lifted in the obvious way to act on the three-point blow-up. In point of fact, Matsushima's theorem [23] tells us that invariance under the torus action is automatic here, because $\mathbb{T}^2/\mathbb{Z}_3$ is actually the unique maximal compact subgroup of the identity component $(\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_3$ of the complex automorphism group of $\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2$. By contrast, its invariance with respect to the specific finite group $\mathfrak{S}_3$ generated by the $\{\alpha_j\}$, together with the normalization of choosing the Einstein constant to be 1, *uniquely* picks out Siu's Kähler–Einstein metric g.

Indeed, the Bando–Mabuchi uniqueness theorem [1] tells us that any other J-compatible Kähler–Einstein metric $\widehat{g}$ on $\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2$ with Einstein constant 1 must be obtained from g by moving it by an element of the connected component of $(\mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{Z}_3$ of the complex automorphism group. However, any such rival Einstein metric $\widehat{g} \neq g$ is then invariant under a different representation of $\mathfrak{S}_3$, where the generators $\widehat{\alpha}_j = A^{-1} \circ \alpha_j \circ A$ have been conjugated by a diagonal matrix A of determinant 1 whose eigenvalues do not all have norm 1. If $\widehat{g}$ were also invariant under the original α_j, it would then be invariant $\alpha_j \circ A^{-1} \circ \alpha_j \circ A \in \mathbb{C}^\times \times \mathbb{C}^\times$ for each $j = 1, 2, 3$, and the powers of at least one such diagonal matrix will then diverge in $\mathbb{C}^\times \times \mathbb{C}^\times$. But this is a contradiction, since the isometry group of any compact Riemannian manifold is compact. This proves that Siu's Kähler–Einstein metric is uniquely determined by its $\mathfrak{S}_3$-invariance, together with our (arbitrarily chosen) normalization of its Einstein constant.

Now notice that

$$\Upsilon \circ \alpha_j = \beta_j \circ \alpha_j \circ \Upsilon, \quad j = 1, 2, 3,$$

where the $\beta_j \in \mathbb{T}^2$ are defined by

$$\beta_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \beta_3 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}.$$

Since the Kähler–Einstein metric g is compatible with both J and $-J$, and since Υ just interchanges these two integrable complex structures, it follows that $\widehat{g} := \Upsilon^* g$ is a J-compatible $\lambda = +1$ Kähler–Einstein metric. But, using the invariance of g under α_j and β_j, we now see that

$$\begin{aligned} \alpha_j^* \widehat{g} &= \alpha_j^* \Upsilon^* g = (\Upsilon \circ \alpha_j)^* g \\ &= (\beta_j \circ \alpha_j \circ \Upsilon)^* g = \Upsilon^* (\beta_j \circ \alpha_j)^* g \\ &= \Upsilon^* g = \widehat{g}. \end{aligned}$$

This shows that $\widehat{g}$ is another $\lambda = +1$ Kähler–Einstein metric that is invariant under the action of $\mathfrak{S}_3$ generated by the $\{\alpha_j\}$. But since Siu's metric is uniquely characterized by these properties, we must have $g = \widehat{g} = \Upsilon^* g$. Thus, the free anti-holomorphic involution Υ is an isometry of $(\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2, g)$, and g therefore descends to $\mathscr{Q}\#\overline{\mathbb{CP}}_2 = (\mathbb{CP}_2\#3\overline{\mathbb{CP}}_2)/\langle\Upsilon\rangle$ as an Einstein metric h with $\det(W^+) > 0$ everywhere. Moreover, since there is only one del Pezzo surface of degree 6, Proposition 1 and the Bando–Mabuchi uniquess theorem together guarantee that any other compact oriented Einstein manifold (M^4, h') with $\pi_1 \neq 0$, $b_2 = 1$, and $\det(W^+) > 0$ must be isometric to a rescaled version of this Einstein manifold $(\mathscr{Q}\#\overline{\mathbb{CP}}_2, h)$. □

2.3 The $b_2(M) = 2$ case

When $b_2(M) = 2$, the del Pezzo surface $(\widetilde{M}, J)$ has degree $c_1^2 = 4$. Because this complex surface has K^{-1} ample, the Riemann–Roch–Hirzebruch and Kodaira vanishing theorems immediately tell us that $h^0(\widetilde{M}, \mathcal{O}(K^{-1})) = 5$, and $h^0(\widetilde{M}, \mathcal{O}(K^{-2})) = 13$. On the other hand, surface classification tells us that $\widetilde{M}$ must be obtained by blowing up $\mathbb{CP}_2$ at five distinct points, no three of which are collinear. Using these facts, one can then deduce [8,10] that the anti-canonical system embeds $\widetilde{M}$ in $\mathbb{P}([H^0(\mathcal{O}(K^{-1})]^*) \cong \mathbb{CP}_4$, and that the image of $(\widetilde{M}, J)$ is actually the transverse intersection of two non-singular quadrics in $\mathbb{CP}_4$. In our

case, though, we also have an anti-hololomorphic involution $\sigma : \widetilde{M} \to \widetilde{M}$, and this then induces a complex-anti-linear involution

$$\sigma^* : [H^0(\mathcal{O}(K^{-1}))]^* \to [H^0(\mathcal{O}(K^{-1}))]^*$$

that looks like component-wise complex conjugation in $\mathbb{C}^5$. Obviously, the image of $\widetilde{M}$ is automatically invariant under the involution of $\mathbb{CP}_4$ induced by σ^*, and this involution moreover restricts to $\widetilde{M}$ as the given anti-holomorphic involution σ. In addition, there is an induced complex-anti-linear involution

$$\sigma_* : H^0(\mathcal{O}(K^{-2})) \to H^0(\mathcal{O}(K^{-2}))$$

that is compatible with the one induced by σ^* on the 15-dimensional space $\odot^2 H^0(\mathcal{O}(K^{-1}))$ of homogeneous quadratic polynomials. The 2-dimensional kernel of the restriction map $\odot^2 H^0(\mathcal{O}(K^{-1})) \to H^0(\mathcal{O}(K^{-2}))$ therefore also carries an induced complex conjugation map. Taking a generic real basis for this space, we thus see that $\widetilde{M} \subset \mathbb{CP}_4$ is actually the transverse intersection of two non-singular quadrics *with real coefficients*, but with disjoint real loci. By choosing a suitable basis for the real homogeneous polynomials vanishing on $\widetilde{M}$, and then altering our homogeneous coordinates by the action of $\mathbf{GL}(5, \mathbb{R})$, we may thus arrange for $\widetilde{M}$ to be cut out [31,32] by the equations

$$0 = \sum_{j=1}^{5} z_j^2 = \sum_{j=1}^{5} a_j z_j^2$$

where $a_1, \ldots, a_5$ are distinct real numbers. Conversely, any such choice of the coefficients a_j defines a degree-four del Pezzo surface $\widetilde{M}$ with free anti-holomorphic involution σ; the requirement that the coefficients a_j be distinct is exactly equivalent to requiring that intersection of the given quadrics be smooth. Replacing these quadrics with linear combinations and then rescaling our coordinates has the effect of replacing $a_1, \ldots, a_5$ with their images under a fractional linear transformation of $\mathbb{R}$, so we may further refine our normal form so that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $3 < a_4 < a_5$. This not only shows that the moduli space of smooth degree-four del Pezzo surfaces with free anti-holomorphic involution is *connected* [25,32], but also reveals that this moduli space has real dimension 2.

Now, every smooth degree-four del Pezzo surface admits a J-compatible Kähler–Einstein metric [24,30]. Moreover, since there are no non-trivial holomorphic vector fields on such a del Pezzo, the uniqueness theorem of Bando-Mabuchi guarantees that this J-compatible Kähler–Einstein metric g is completely *unique* once we exclude non-trivial constant rescalings by, for example, normalizing the Einstein constant. However, if g is a Kähler–Einstein metric, then $\sigma^* g$ is also Kähler–Einstein. Moreover, since g is compatible with the two integrable almost-complex structures $\pm J$, the same is true of $\sigma^* g$, since the anti-holomorphic involution σ exactly interchanges J and $-J$. Since the Einstein metrics g and $\sigma^* g$ also have the same Einstein constant, it thus follows that $g = \sigma^* g$. Since that the Einstein metric g is therefore σ-invariant. it pushes down to a unique Einstein metric h on $M = \widetilde{M}/\langle\sigma\rangle$. We have thus arranged for g to become the pull-back $\widetilde{h}$ of an Einstein metric h on M with $\det(W^+) > 0$. To summarize:

Proposition 4 *Any compact oriented, Einstein manifold (M^4, h) with $\pi_1 \neq 0$, $b_2 = 2$, and $\det(W^+) > 0$ is orientedly diffeomorphic to $\mathcal{Q}\#2\overline{\mathbb{CP}}_2$, and is doubly covered by a degree-four del Pezzo surface equipped with a fixed-point-free free anti-holomorphic involution. Moreover, the moduli space $\mathcal{E}_{\rm det}(\mathcal{Q}\#2\overline{\mathbb{CP}}_2)$ of these special Einstein metrics is non-empty, connected, and of real dimension* 2.

2.4 The $b_2(M) = 3$ case

We finally come to the case where $b_2(M) = 3$, and where $(\widetilde{M}, J)$ is a del Pezzo surface of degree $c_1^2 = 2$. This time, Riemann–Roch–Hirzebruch and Kodaira vanishing tell us that $h^0(\widetilde{M}, \mathcal{O}(K^{-1})) = 3$, while the classification of rational surfaces tells us that $(\widetilde{M}, J)$ is obtained from $\mathbb{CP}_2$ by blowing up 7 points, with no three of them collinear, and no six on a conic. This can then be used [8,10] to show that the anti-canonical system is base-point free, and so defines a degree-2 holomorphic map

$$\widetilde{M} \to \mathbb{P}([H^0(\mathcal{O}(K^{-1})]^*) \cong \mathbb{CP}_2;$$

further use of the ampleness of K^{-1} then reveals that $(\widetilde{M}, J)$ is therefore a branched double cover of the projective plane, with branch locus a smooth quartic curve. Thus, $\widetilde{M}$ is biholomorphic to the subvariety of $\mathcal{O}(2) \to \mathbb{CP}_2$, given by $\zeta^2 = -f(z_1, z_2, z_3)$, where $[z_1, z_2, z_3] \in \mathbb{CP}_2$, the fiber-coordinate ζ is homogeneous of degree 2 in (z_1, z_2, z_3), and where $f \in H^0(\mathbb{CP}_2, \mathcal{O}(4))$ vanishes along a smooth quartic plane curve Σ.

However, in our case, we also have a fixed-point-free anti-holomorphic involution $\sigma : \widetilde{M} \to \widetilde{M}$, and the induced anti-holomorphic action of this involution on the line bundle $K^{-1} \to \widetilde{M}$ then induces a standard complex conjugation map on $[H^0(\mathcal{O}(K^{-1})]^* \cong \mathbb{C}^3$. The induced anti-holomorphic action on $\mathbb{CP}_2$ then preserves the branch locus, and acts on Σ without fixed points. We may thus take the defining equation f of Σ to be real, and everywhere positive on $\mathbb{RP}^2 \subset \mathbb{CP}_2$. Fortunately, the moduli space of such smooth real quartics without real points has been studied extensively, and is know to be connected. Indeed, it can be naturally identified [12] with a specific arithmetic quotient of hyperbolic 6-space.

For each such real quartic curve, we conversely obtain a unique degree-two del Pezzo surface $(\widetilde{M}, J)$ given by $\zeta^2 = -f(z_1, z_2, z_3)$, and which is equipped with a fixed-point-free anti-holomorphic involution $\sigma : \widetilde{M} \to \widetilde{M}$ given by $(z_1, z_2, z_3, \zeta) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{\zeta})$. But any degree-two del Pezzo admits [24] a J-compatible Kähler–Einstein metric g, and pulling back this metric by our anti-holomorphic involution then gives a second Kähler–Einstein metric $\sigma^* g$ with the same Einstein constant. But the del Pezzo surface $\widetilde{M}$ is also biholomorphic to a blow-up of $\mathbb{CP}_2$ at seven points in general position, it carries no holomorphic vector fields. Thus, the Bando–Mabuchi uniqueness theorem tells us that $\sigma^* g = g$. It therefore follows that g descends to the quotient $M = \widetilde{M}/\langle\sigma\rangle$ as a uniquely defined Einstein metric h with $\det(W^+) > 0$, thereby making g equal its pull-back $\widetilde{h}$. We have thus proved the following:

Proposition 5 *Any compact oriented, Einstein manifold (M^4, h) with $\pi_1 \neq 0$, $b_2 = 3$, and $\det(W^+) > 0$ is orientedly diffeomorphic to $2\#3\overline{\mathbb{CP}}_2$, and is doubly covered by a degree-two del Pezzo surface equipped with a fixed-point-free free anti-holomorphic involution. Moreover, the moduli space $\mathscr{E}_{\det}(2\#2\overline{\mathbb{CP}}_2)$ of these special Einstein metrics is non-empty, connected, and of real dimension 6.*

2.5 Proofs of the main theorems

By putting together the above results, it is now straightforward to prove our main theorems. For the sake of clarity, we will do so in reverse order.

Proof of Theorem B. If (M, h) is a compact oriented Einstein 4-manifold with $\pi_1 \neq 0$ and $\det(W^+) > 0$, case (ii) of Proposition 1 tells us that $M = \widetilde{M}/\langle\sigma\rangle$, where $\widetilde{M}$ is a del Pezzo surface, and σ is a fixed-point-free anti-holomorphic involution; moreover, Theorem 1 tells us that the pull-back $\widetilde{h}$ to $\widetilde{M}$ is actually Kähler–Einstein.

Because $4 - b_2(M) = \frac{1}{2}c_1^2(\widetilde{M}) > 0$, the only possible values of $b_2(M)$ are 0, 1, 2, or 3, and we have thoroughly analyzed each of these possibilities. When $b_2(M) = 0$, Proposition 2 tells us that M must be diffeomorphic to $\mathscr{P}$ or $\mathscr{Q}$; both cases actually arise, and they are topologically distinct, because only one of them is spin. When $b_2(M) = 1$, Proposition 3 tells us that M must be diffeomorphic to $\mathscr{Q}\#\overline{\mathbb{CP}}_2$, and that this manifold actually carries an Einstein metric of the required type. When $b_2(M) = 2$, M must instead be diffeomorphic to $\mathscr{Q}\#2\overline{\mathbb{CP}}_2$ by Proposition 4, which also tells us that this manifold actually carries a family of Einstein metrics with the required property. Finally, when $b_2(M) = 3$, Proposition 5 tells us that M is necessarily diffeomorphic to $\mathscr{Q}\#3\overline{\mathbb{CP}}_2$, and that this manifold actually carries a family of such Einstein metrics. □

Proof of Theorem A. By Theorem B, there are exactly five diffeotypes of *non-simply connected* compact oriented 4-manifolds M that carry Einstein manifolds with $\det(W^+) > 0$ everywhere; namely, these are $\mathscr{P}$ and $\mathscr{Q}\#k\overline{\mathbb{CP}}_2$ for $k = 0, 1, 2$, and 3. Moreover, Propositions 2, 3, 4, and 5 tell us that in each case the moduli space $\mathscr{E}_{\rm det}(M)$ of these special Einstein metrics is actually connected. In addition, there are [17] exactly ten *simply connected* diffeotypes of compact oriented 4-manifolds that carry such metrics, corresponding to the ten deformation types of del Pezzo surfaces; for each such diffeotype, our moduli space $\mathscr{E}_{\rm det}(M)$ of special Einstein metrics is connected, because it is in fact exactly the (connected) moduli space of del Pezzo complex structures, modulo the $\mathbb{Z}_2$-action induced by $J \rightsquigarrow -J$. Taken together, this means there are exactly 15 possible diffeotypes, and that in each case the moduli space $\mathscr{E}_{\rm det}(M)$ of these special Einstein metrics is both non-empty and connected.

Finally, the moduli space $\mathscr{E}_{\rm det}(M)$ of these special Einstein metrics is always both open and closed as a subset of the moduli space $\mathscr{E}(M)$ of *all* Einstein metrics. Indeed, by [17], these special Einstein metrics are characterized among all Einstein metrics by the open condition $\det(W^+) > 0$; whereas results of Derdziński [9, Theorem 2] and Hitchin [3, Theorem 13.30] instead characterize them, among Einstein metrics on these spaces, by the pair of closed conditions $\det(W^+) = \frac{1}{3\sqrt{6}}|W^+|^3$ and $s \geq 0$. Thus, for each of these fifteen 4-manifolds M, the connected space $\mathscr{E}_{\rm det}(M)$ is precisely a single connected component of the Einstein moduli space $\mathscr{E}(M)$. □

3 Related results

For clarity and simplicity, we have supposed throughout this article that the Einstein metrics h under investigation satisfied Wu's condition $\det(W^+) > 0$. However, by [17, Theorem C], we could have actually reached exactly the same conclusions if we had merely imposed an ostensibly weaker hypothesis:

Theorem 2 *Let (M, h) be a compact oriented Einstein 4-manifold. If*

$$W^+ \neq 0 \quad \textit{and} \quad \det(W^+) \geq -\frac{5\sqrt{2}}{21\sqrt{21}}|W^+|^3 \tag{3}$$

at every point of M, then (M, h) actually satisfies $\det(W^+) > 0$ everywhere. Consequently, by Theorem A, there are exactly 15 diffeotypes of 4-manifolds M that carry such Einstein metrics, and their moduli space $\mathscr{E}_{\rm det}(M)$ is in each case exactly a connected component of the Einstein moduli space $\mathscr{E}(M)$.

In fact, the results of [17] apply more generally to oriented Riemannian 4-manifolds that satisfy $\delta W^+ = 0$ and (3). This led there to a complete diffeomorphism classification of all

such manifolds with $b_+ \neq 0$. Regarding the $b_+ = 0$, case, we can now at least say the following:

Theorem 3 *Let (M, h) be a compact oriented Riemannian 4-manifold with $\delta W^+ = 0$, and suppose that $b_+(M) = 0$. If h satisfies (3) at every point, then M admits a double cover $\widetilde{M}$ that is diffeomorphic to $(\Sigma_{\mathfrak{g}} \times S^2)\#2k\overline{\mathbb{CP}_2}$, where k and $\mathfrak{g}$ are non-negative integers, and where $\Sigma_{\mathfrak{g}}$ is the compact oriented surface of genus $\mathfrak{g}$. Conversely, each of these possibilities occurs: for every pair $(k, \mathfrak{g})$ of non-negative integers, there is a compact oriented 4-manifold (M, h) with $b_+ = 0$, $\delta W^+ = 0$, and $\det(W^+) > 0$ that is doubly covered by a manifold $\widetilde{M}$ diffeomorphic to $(\Sigma_{\mathfrak{g}} \times S^2)\#2k\overline{\mathbb{CP}_2}$.*

Proof By [17, Proposition 3.2], any such manifold (M, h) is double-covered by a complex surface $(\widetilde{M}, J)$ on which the pull-back $\widetilde{h}$ of h becomes conformal to a J-compatible Kähler metric g of positive scalar curvature s. By [36], this implies that $(\widetilde{M}, J)$ has Kodaira dimension $-\infty$, and so is rational or ruled. Since the signature $\tau(\widetilde{M}) = 2\tau(M)$ must be even, it follows that $\widetilde{M}$ is diffeomorphic to either $(\Sigma_{\mathfrak{g}} \times S^2)\#2k\overline{\mathbb{CP}_2}$ for some $\mathfrak{g}, k \geq 0$, where $\Sigma_{\mathfrak{g}}$ denotes the compact oriented surface of genus $\mathfrak{g}$, or to the non-spin oriented 2-sphere bundle $\Sigma_{\mathfrak{g}} \rtimes S^2$ over some $\Sigma_{\mathfrak{g}}$. However, the latter is excluded here by a variant of the proof of Lemma 1; indeed, since the putative oriented 4-manifold $M = [\Sigma_{\mathfrak{g}} \rtimes S^2]/\mathbb{Z}_2$ would have signature $\tau = 0$ and $b_+ = 0$, its second cohomology group $H^2(M, \mathbb{Z})$ would be finite, and, since $\operatorname{Tor} H^2(\Sigma_{\mathfrak{g}} \rtimes S^2, \mathbb{Z}) = \operatorname{Tor} H_1(\Sigma_{\mathfrak{g}} \rtimes S^2, \mathbb{Z}) = \operatorname{Tor} H_1(\Sigma_{\mathfrak{g}}, \mathbb{Z}) = 0$, pulling back a spin^c structure on M would then yield a spin structure on the non-spin manifold $\widetilde{M} \approx \Sigma_{\mathfrak{g}} \rtimes S^2$. This contradiction therefore shows that $\widetilde{M}$ must instead be diffeomorphic to $(\Sigma_{\mathfrak{g}} \times S^2)\#2k\overline{\mathbb{CP}_2}$ for some $\mathfrak{g}, k \geq 0$.

Conversely, let $\breve{\Sigma} = \#_{\mathfrak{g}+1}\mathbb{RP}^2$ be the connected sum of $\mathfrak{g}+1$ copies of the real projective plane, and let $\breve{g}_1$ be a smooth Riemannian metric on $\breve{\Sigma}$ that has positive Gauss curvature on some non-empty open set $\breve{U} \subset \breve{\Sigma}$. Let $\Sigma \to \breve{\Sigma}$ be the oriented double cover of $\breve{\Sigma}$, let g_1 be the pull-back of $\breve{g}_1$ to Σ, and let $j : T\Sigma \to T\Sigma$ be the integrable almost-complex complex structure on Σ induced by g_1 and the orientation of Σ. The non-trivial deck transformation $\mathcal{A} : \Sigma \to \Sigma$ now becomes a fixed-point-free anti-holomorphic involution of the genus-$\mathfrak{g}$ compact complex curve (Σ, j), while g_1 becomes a j-compatible Kähler metric on Σ that has Gauss curvature $\kappa > 0$ on the non-empty $\mathcal{A}$-invariant region U that is the pre-image of $\breve{U}$. If g_0 is the usual unit-sphere metric on $S^2 = \mathbb{CP}_1$, then the Riemannian product $(\Sigma, g_1) \times (S^2, \varepsilon g_0)$ will have positive scalar curvature provided we take $\varepsilon > 0$ to be small enough so that $\varepsilon^{-1} > -\min \kappa$. Moreover, the product Kähler metric $g_1 \oplus \varepsilon g_0$ on $\Sigma \times \mathbb{CP}_1$ has positive holomorphic section curvature on the open subset $U \times \mathbb{CP}_1$. Let us now endow $\Sigma \times \mathbb{CP}_1$ with the fixed-point-free anti-holomorphic involution $\mathcal{U} := \mathcal{A} \times \mathfrak{a}$, where $\mathfrak{a} : S^2 \to S^2$ is once again the antipodal map of (1), and, for a given $k \geq 0$, choose $2k$ distinct points $p_1, \ldots, p_{2k} \in U \times \mathbb{CP}_1$ such that $\mathcal{U}(p_{2j-1}) = p_{2j}$ for $j = 1, \ldots, k$. Letting $\widetilde{M}$ denote the blow-up of $\Sigma \times \mathbb{CP}_1$ at $p_1, \ldots, p_{2k}$, there is then a fixed-point-free anti-holomorphic involution $\sigma : \widetilde{M} \to \widetilde{M}$ induced by $\mathcal{U}$, and, using a result of Hitchin [15], we will now construct a σ-invariant Kähler metric $\widetilde{g}$ of positive scalar curvature on $\widetilde{M}$. The recipe only calls for changing the previously constructed Kähler metric $g_1 \oplus \varepsilon g_0$ within a disjoint union of the Riemannian balls of radius ϵ about the p_j, and modifies the metric within these balls by adding $i\mathbf{t}\partial\bar{\partial} f(\mathsf{r})$ to the Kähler form, where r is the Riemannian distance from the center p_j, $f(\mathsf{r})$ is a gently modified version of $\log \mathsf{r}$ that is constant for $\mathsf{r} > \epsilon$, and the parameter $\mathbf{t}$ is any sufficiently small positive constant. For $\mathbf{t} > 0$ sufficiently small, Hitchin's computation then shows that this modified Kähler metric $\widetilde{g}$ has positive scalar curvature everywhere, because our background metric has positive holomorphic sectional curvature in

the modification region, and positive scalar curvature everywhere else. On the other hand, since we have carefully arranged for this Kähler metric $\widetilde{g}$ on the blow-up to be σ-invariant, our $\widetilde{g}$ is automatically the pull-back of a unique Riemannian metric g on the oriented 4-manifold $M := [(\Sigma \times \mathbb{CP}_1)\#2k\overline{\mathbb{CP}}_2]/\langle\sigma\rangle$. However, by construction, g has scalar curvature $s > 0$, and is moreover everywhere locally isometric to a Kähler metric $\widetilde{g}$. An observation of Derdziński [3,9] thus shows that the conformally rescaled metric $h = s^{-2}g$ therefore has $\det(W^+) > 0$ and $\delta W^+ = 0$ at every point of M. □

While the above enumeration of the possibilities for $\widetilde{M}$ is similar in spirit to our discussion of the Einstein case, Theorem 3 is certainly far weaker than our main results. First of all, we have not tried to classify the possible $\mathbb{Z}_2$-actions that arise, although it seems clear that that there must be many of them. Second, in stark contrast to the Einstein case, the moduli spaces of solutions to the weaker equation $\delta W^+ = 0$ consistently turn out to be *infinite dimensional* in the present context, and nothing substantial seems to be known concerning whether or not they are connected. We leave these open questions for the reader's further consideration, in the hope that this will stimulate further research, and eventually lead to definitive answers.

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