

UNIQUENESS OF ASYMPTOTICALLY CONICAL TANGENT FLOWS

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Abstract

Singularities of the mean curvature flow of an embedded surface in \mathbb{R}^3 are expected to be modeled on self-shrinkers that are compact, cylindrical, or asymptotically conical. In order to understand the flow before and after the singular time, it is crucial to know the uniqueness of tangent flows at the singularity.

In all dimensions, assuming that the singularity is of multiplicity 1, uniqueness in the compact case has been established by the second-named author, and in the cylindrical case by Colding and Minicozzi. We show here the uniqueness of multiplicity-1 asymptotically conical tangent flows for mean curvature flow of hypersurfaces.

In particular, this implies that when a mean curvature flow has a multiplicity-1 conical singularity model, the evolving surface at the singular time has an (isolated) regular conical singularity at the singular point. This should lead to a complete understanding of how to “flow through” such a singularity.

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1. Introduction

1.1. Uniqueness of tangent flows

By work of Huisken [29], White [49], and Ilmanen [30], singularities of mean curvature flow can be modeled by self-similar shrinking solutions to the flow. For flows of embedded surfaces in \mathbb{R}^3 , Ilmanen [30] proves that self-shrinkers arising as tangent flows at the first singular time are smooth and embedded (possibly with higher multiplicity). Wang [47] has proved that such shrinkers, if noncompact, have ends that are asymptotic to a cylinder or smooth cone (cf. Definition A.4; see also [45]). Moreover, Kapouleas, Kleene, and Møller [33] and Nguyen [39]–[41] have constructed embedded, smooth self-shrinkers in \mathbb{R}^3 with (smoothly) conical ends.

An important question is to determine whether or not these tangent flows are unique. The second-named author [42] has proved in that this holds (in all dimensions and codimension) when there is a compact multiplicity-1 (smooth) tangent flow. Colding and Minicozzi [19] (cf. [16]) have proved that uniqueness holds (for hypersurfaces, in all dimensions) for multiplicity-1 cylindrical tangent flows (see also [4]).

In this work, we show that uniqueness also holds in the case of multiplicity-1 tangent flows whose self-shrinker is smoothly conical.

THEOREM 1.1 (Uniqueness of conical tangent flows)

Fix an asymptotically conical self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$. Let $\mathcal{M} = (\mu_t)_{t \in (-t_1, 0)}$ be an integral n -Brakke flow so that the self-similar shrinking multiplicity-1 Brakke flow associated to Σ , \mathcal{M}_Σ , arises as a tangent flow to \mathcal{M} at $(0, 0)$. Then \mathcal{M}_Σ is the unique tangent flow to \mathcal{M} at $(0, 0)$.

See Section 9.1 for estimates concerning the rate of convergence. We expect that the argument will extend to higher codimension with little change.

An interesting feature of our proof of Theorem 1.1 is that it shows that the Łojasiewicz–Simon approach to uniqueness of blowups can be applied in the case of a noncompact singularity model. Colding and Minicozzi’s work in [19] on the uniqueness of cylindrical tangent flows does not proceed via a reduction to the finite-dimensional Łojasiewicz inequality à la Simon, but rather proves a Łojasiewicz-type inequality by hand, using the explicit structure of the cylinder in a fundamental

way. Here the situation is different: we do not use any explicit structure of the conical shrinkers, so instead must rely on a Łojasiewicz–Simon inequality proved by “abstract” methods, after introducing relevant weighted function spaces.

This approach has the drawback that it requires much stronger “closeness” of the flow relative to the shrinker. Thus, we must develop a new “extension of closeness” mechanism that is not present in the cylindrical case (cf. Lemma 1.3, Proposition 7.2). We then must combine this mechanism with several crucial ideas of Colding and Minicozzi concerning improvement and extension of curvature estimates to overcome the noncompactness of the problem.

Our approach seems to be quite general and flexible; we expect that it will apply to the uniqueness of noncompact singularities in other geometric problems, when the singularity is “well behaved” at infinity.

1.2. The structure of the singular set around an asymptotically conical shrinker

We note that conjecturally (cf. Ilmanen’s no cylinder conjecture [31, #12]), the cylinder is the only shrinker in \mathbb{R}^3 with a cylindrical end. Combing Theorem 1.1 with [19], [42], and [47], it would follow that for the mean curvature flow of a smooth embedded surface in \mathbb{R}^3 , *all* multiplicity-1 tangent flows at the first singular time are unique.

Uniqueness of tangent flows gives important information about the singular behavior of the flow. Using their result on the uniqueness of cylindrical tangent flows, Colding and Minicozzi [20] have proved (among other things) that a mean curvature flow of hypersurfaces in \mathbb{R}^{n+1} with only multiplicity-1 cylindrical tangent flows has a space-time singular set contained in finitely many compact embedded $(n-1)$ -dimensional Lipschitz submanifolds and an $(n-2)$ -dimensional set. Moreover, in \mathbb{R}^3 they have shown that such flows are smooth for almost all times, and any connected component of the singular set is completely contained in a time-slice (see also [21]).

Similarly, Theorem 1.1 (and the pseudolocality arguments used in Lemma 9.1 below) implies the following.

COROLLARY 1.2

For \mathcal{M} and Σ as in Theorem 1.1, there is $\varepsilon > 0$ so that for all $t \in (-\varepsilon^2, 0)$, we have $\mu_t \lfloor B_\varepsilon(0) = \mathcal{H}^n \lfloor M_t$ for a smooth family M_t of embedded surfaces flowing by mean curvature in $B_\varepsilon(0)$. The surfaces M_t are diffeomorphic to Σ . Moreover, as $t \nearrow 0$, the flow $M_t \cap (B_\varepsilon(0) \setminus \{0\})$ converges in C_{loc}^∞ to a smooth surface $M_0 \subset B_\varepsilon(0) \setminus \{0\}$ with a conical singularity at 0 smoothly modeled on the asymptotic cone of Σ .¹

¹In other words, rescaling M_0 around 0 converges in $C_{\text{loc}}^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ to the asymptotic cone of Σ .

We note that Colding and Minicozzi [18] have proved that the plane, sphere, and cylinders are the unique entropy stable shrinkers. They have proposed this as a mechanism for a possible way to construct a generic mean curvature flow. Corollary 1.2 suggests that one can flow *through* points with conical tangent flows, instead of trying to perturb them away. Understanding the flow through these “nongeneric” situations will be particularly important towards understanding families of mean curvature flows. We will investigate this elsewhere.

1.3. Some recent results in singularity analysis of mean curvature flow

We remark that Brendle [11] has recently proved that the only smooth properly embedded self-shrinkers in \mathbb{R}^3 with genus zero are the plane, sphere, and cylinder; hence, a conical shrinker must have nonzero genus. Moreover Bernstein and Wang [5] have shown that the round sphere has the least entropy among any closed hypersurface (up to the singular dimension; cf. [51] and see also [17], [35]); the same authors have extended this in [6] to noncompact surfaces in \mathbb{R}^3 (see also [7]). Wang [46] has proved that two shrinkers asymptotic to the same smooth cone must be identical. Ketover [34] has recently constructed self-shrinking Platonic solids.

Brendle and Choi [12] have classified the bowl soliton as the unique strictly convex ancient solution in \mathbb{R}^3 (cf. [13], [25], [26], [48]). Moreover, Angenent, Daskalopoulos, and Sesum [1] have classified closed noncollapsed ancient solutions that are uniformly two-convex. Finally, Choi, Haslhofer, and Hershkovits [15] have proved the mean convex neighborhood conjecture in \mathbb{R}^3 , by classifying low entropy ancient solutions (see also [27]).

1.4. Idea of the proof of Theorem 1.1

The basic idea to prove Theorem 1.1 is to rely on a Łojasiewicz-type inequality (see [38], [43], [44]) to show uniqueness of the tangent flow. Indeed, this strategy was already successful in the compact (see [42]) and cylindrical (see [19]) cases. In the cylindrical and conical cases, the noncompactness of the shrinker causes serious issues (beyond simply those of a technical nature), due to the fact that one cannot write the entire flow as a graph over the shrinker.

Unlike the cylindrical case in [19], we do not exploit any specific structure of the shrinker (beyond the fact that it has conical ends). Conical ends seem to be less degenerate with regard to the uniqueness problem, allowing us to obtain very strong estimates in annular regions around the point where the singularity is forming. Because we do not assume any specific structure of the shrinker, we must prove the Łojasiewicz–Simon inequality by “abstract” methods (i.e., by a finite-dimensional reduction to Łojasiewicz’s original inequality in [38]). In Section 3, we construct weighted Hölder and Sobolev spaces in which Simon’s argument in [43] can be used

to prove a Łojasiewicz–Simon inequality for entire graphs over the shrinker (see Theorem 4.3). Roughly speaking, we consider Hölder spaces (inspired by [33]) $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ of functions $u : \Sigma \rightarrow \mathbb{R}$ so that in coordinates $(r, \omega) \in (1, \infty) \times \Gamma$ on the end of Σ (where Γ is the link of the asymptotic cone of Σ), we have

$$f(r, \omega) = c(\omega)r + O(r^{-1}),$$

where the error term is taken in $C^{2,\alpha}$ on balls of unit size. We also require the improved radial derivative estimate

$$\partial_r f(r, \omega) = c(\omega) + O(r^{-2})$$

in $C^{0,\alpha}$. Geometrically, we can think of $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ as functions whose graphs are asymptotically conical (for a different cone) and decay to their asymptotic cone at a rate $O(r^{-1})$ in $C^{2,\alpha}$.

The linearized shrinker operator maps the space $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ to $\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$; that is, $Lu = O(r^{-1})$ in $C^{0,\alpha}$ (this is where the improved radial derivative estimate is needed). We can prove Schauder estimates for the L -operator between these spaces (see Proposition 3.5). Moreover (based on ideas communicated to us by Bernstein [3]), one can also establish (see Section 3.4) regularity and existence for the L -operator (the linearized shrinker operator) between L^2 -based Sobolev spaces $L_W^2(\Sigma)$ and $H_W^2(\Sigma)$, when weighted by the Gaussian density $\rho = (4\pi)^{-\frac{n}{2}} e^{-|x|^2/4}$. Combining these facts, we find that the L -operator behaves between these spaces in essentially the same way as in the compact cases considered by Simon [43]. This yields a Łojasiewicz–Simon inequality for *entire* graphs over Σ (see Theorem 4.3); that is, if $\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)}$ is sufficiently small, then for $M = \text{graph } u$,

$$|F(M) - F(\Sigma)|^{1-\theta} \leq C \left(\int_M |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2}}. \quad (1.1)$$

Here $F(M)$ is the Gaussian area (see Definition A.1) and ϕ is the deviation from M being a shrinker (see Definition A.3).

To apply (1.1) to prove uniqueness of conical tangent flows, the basic strategy is to show that if a Brakke flow \mathcal{M} has a multiplicity-1 conical tangent flow (modeled by Σ) at $(0, 0)$, then it is possible to write part of \mathcal{M} as a graph over part of Σ , and that this graphical function extends to a function that is small in $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$. At this point (1.1) can be applied to this extended function. Applying the resulting inequality to \mathcal{M} introduces errors based on the fact that \mathcal{M} is not an entire graph over Σ . Controlling the size of these errors relative to the terms in (1.1) is a serious issue, which we now describe in some detail.

We consider the rescaled mean curvature flow around $(0, 0)$; assume that the rescaled flow consists of surfaces M_τ for $\tau \in [-1, \infty)$ and $M_{\tau_i} \rightarrow \Sigma$ in C_{loc}^∞ along

some sequence $\tau_i \rightarrow \infty$. We seek to prove by a continuity argument that for \underline{r} fixed and τ sufficiently large, $M_\tau \cap B_{\underline{r}}$ is a $C^{\ell+1}$ graph of a function with $C^{\ell+1}$ -norm bounded by b . This is (roughly) the *core graphical hypothesis* $(*_b, \underline{r})$ (see Definition 5.7). Notice that the core graphical hypothesis will not suffice to control the errors when applying the Łojasiewicz inequality. The reason for this is that we must not destroy the term

$$\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n := e^{-\frac{\mathbf{R}(M_\tau)^2}{4}}$$

on the right-hand side of (1.1). We call $\mathbf{R}(M_\tau)$ the *shrinker scale* (see Definition 5.4).² On the other hand, cutting off the Łojasiewicz–Simon inequality outside of a ball of radius R will introduce terms on the order of $o(1)e^{-\frac{R^2}{4}}$ (see Theorem 6.1). Thus, we must show that M_τ is graphical over $\Sigma \cap B_R$ for $R \sim \mathbf{R}(M_\tau)$. More precisely, we must show that there is $u : \Sigma \rightarrow \mathbb{R}$ with $\|u\|_{\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)}$ sufficiently small so that $M_\tau \cap B_R$ is contained in the graph of u . We call the largest R satisfying this property the *conical scale* (see Definition 5.6), denoted by $\mathbf{r}_\ell(M_\tau)$. We would thus like to show that the conical scale $\mathbf{r}_\ell(M_\tau)$ is comparable to the shrinker scale $\mathbf{R}(M_\tau)$.

Observe that this is far from clear: we must show that M_τ decays like $O(r^{-1})$ towards a cone (which is close to the asymptotic cone of Σ) nearly all the way to $\mathbf{R}(M_\tau)$. However, if $\mathbf{R}(M_\tau)$ is very large, then we have to transmit the graphical information contained in the core graphical hypothesis (only on a fixed compact set) essentially all the way to $\mathbf{R}(M_\tau)$, while even obtaining decay!

The way we do this has some features in common with the methods used in [19], but the argument on the whole is rather different. To obtain control on the conical scale $\mathbf{r}_\ell(M_\tau)$ we first introduce a weaker notion, the *rough conical scale* $\tilde{\mathbf{r}}_\ell(M_\tau)$ (see Definition 5.5), which is the largest radius where the curvature of M_τ behaves like the curvature along a cone. As a preliminary step, we prove that the rough conical scale improves very rapidly, as long as the core graphical hypothesis $(*_b, \underline{r})$ is satisfied.

Indeed, to control the rough conical scale $\tilde{\mathbf{r}}_\ell(M_\tau)$ we first observe that pseudolocality applied to the unrescaled flow gives curvature estimates on an annular region that persist all the way up to the singular time (using the fact that the flow is close on a large compact set to the conical shrinker). This is depicted in Figure 1 (the region where we obtain curvature estimates is shaded). When translated to the rescaled flow, this annular region will grow exponentially. This initially seems like a problem, since the inner boundary is also moving away exponentially. However, as long as the core graphical hypothesis is satisfied, we can use the pseudolocality estimates at a later time to get curvature estimates further inside. This is shown in Figure 2. The argu-

²Note that our shrinker scale differs from the definition used in [19] slightly, due to the nature of our Łojasiewicz–Simon inequality.

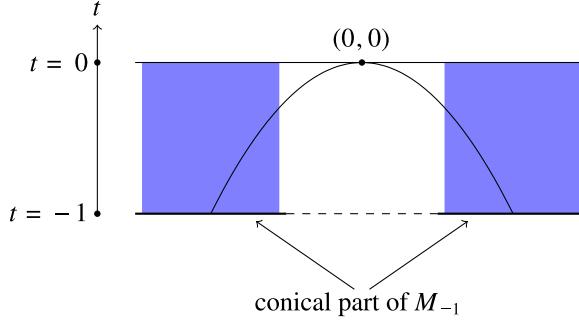


Figure 1. The conical nature of the shrinker Σ (and thus the unrescaled flow at time $t = -1$) yields—via pseudolocality—curvature estimates in the region that is shaded. Note that we can only expect (1.1) to give useful bounds *below* the parabola, since this is the set where the backwards heat kernel ρ is uniformly bounded away from zero.

ment we have just described shows that as long as the core graphical hypothesis $(*_b, \underline{L})$ applies, we have that $\tilde{r}_\ell(M_\tau) \geq C e^{\frac{\tau}{2}}$ (see Lemma 9.1).

Finally, we must show that the core graphical hypothesis $(*_b, \underline{r})$ together with the estimate we have just obtained on the rough conical scale $\tilde{r}_\ell(M_\tau)$ imply that the conical scale (i.e., the scale at which we can cut off (1.1)) is comparable to the shrinker scale $\mathbf{R}(M_\tau)$. Since the rough conical scale is improving exponentially, it basically suffices to show that the conical and shrinker scales are comparable, when the shrinker scale is much smaller than the rough conical scale, that is, $\mathbf{R}(M_\tau) \ll \tilde{r}_\ell(M_\tau)$ (see (9.1) for the case where this does not hold).

At this point, we can use the argument of Colding and Minicozzi from [19, Corollary 1.28] to argue that because $\mathbf{R}(M_\tau) \ll \tilde{r}_\ell(M_\tau)$, the function $\phi_{M_\tau} = \frac{1}{2}\langle x, v_{M_\tau} \rangle - H_{M_\tau}$ (which measures how close M_τ is to a shrinker) must be very small (see the proof of Theorem 8.1).

Finally, we show that this (along with the rough conical scale $\tilde{r}_\ell(M_\tau)$ estimates) suffices to extend the graphicality (and decay estimates) from the core B_r nearly all the way out to the shrinker scale $\mathbf{R}(M_\tau)$ (see Proposition 7.2). Because this step is delicate and forms a key part of the argument, we explain this argument in a model situation below.

LEMMA 1.3 (Model problem for the extension of the conical scale)

Fix $\beta_0 > 0$, and suppose that $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies

$$\mathcal{L}_{\frac{1}{2}} u := \Delta u - \frac{1}{2}(r \partial_r u - u) = 0$$

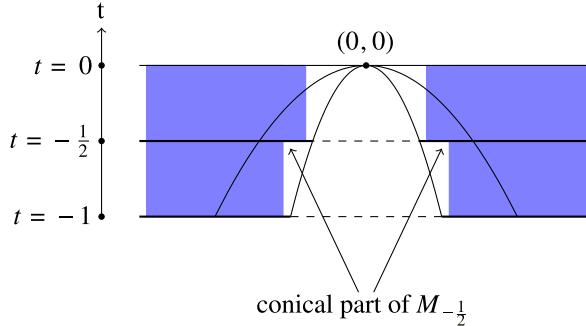


Figure 2. Assuming that we have control over M_t via (1.1) inside of the wide parabola (for $t \in [-1, -\frac{1}{2}]$), we can then use pseudolocality out of the conical region in $M_{-\frac{1}{2}}$ to gain curvature estimates on a *larger* region (still shaded). This is our first improvement/iteration mechanism.

on \mathbb{R}^2 and $|\nabla^k u| = O(r^{1-k})$ for all $k \in \mathbb{N}$. Finally, assume that $\|u\|_{C^3(B_{L+2})} \leq b$ for L sufficiently large and b sufficiently small depending³ on β_0 . Then, there is $c : \mathbb{S}^1 \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ so that outside of B_1 ,

$$u(r, \theta) = c(\theta)r + f(r, \theta)$$

$$\text{and } \|c\|_{C^0(\mathbb{S}^1)} + \|rf\|_{C^0(\mathbb{R}^2)} \leq \beta_0$$

Before proving this lemma, we explain the relationship with the full improvement/extension result (Proposition 7.2). Firstly, we have considered the simplest possible conical shrinker $\mathbb{R}^2 \subset \mathbb{R}^3$ instead of a general asymptotically conical shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$. In the full problem, we have that ϕ_{M_τ} is very small, so the part of M_τ that is graphical over Σ roughly solves the graphical shrinker equation. The $\mathcal{L}_{\frac{1}{2}}$ -operator is the linearization (at the flat plane) of the shrinker equation, so to simplify this situation we have simply assumed that $\mathcal{L}_{\frac{1}{2}}u = 0$. The higher derivative estimates on u are the analogue here of the rough conical scale estimates. Finally, the C^3 -smallness of u in B_{L+2} is analogous to the core graphical hypothesis. We have simplified the conclusion above—in Proposition 7.2 we prove full $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ estimates for u (but the result described here contains the essential ideas).

We note that a key technical difficulty present in Proposition 7.2 that does not occur in this model problem is the fact that M_τ is *not* an entire graph over Σ (and a priori is only graphical up to B_r). Thus, the argument below must be coupled with a continuity argument outwards; this necessarily complicates the argument.

³We will think of the $|\nabla^k u|$ estimates as being given a priori, so everything here is allowed to depend on the implied constants.

Proof

The beginning of the proof is very similar to the proof of [33, Theorem 8.9]. As an initial step, we treat the Laplacian in $\mathcal{L}_{\frac{1}{2}}$ as an error term, since $\Delta u = O(r^{-1})$ from the Hessian estimates on u . Thus, we find that

$$r^2 \partial_r \left(\frac{u}{r} \right) = r \partial_r u - u = O(r^{-1}). \quad (1.2)$$

Integrating this to infinity, we find that

$$c(\theta) := \lim_{r \rightarrow \infty} \frac{u(r, \theta)}{r}$$

is well defined (and continuous). Thus, we have obtained the asserted decomposition. It remains to prove the asserted estimates for c and f .

We begin by proving that $\frac{u}{r}$ is small (we have already proved that it is bounded). Integrating (1.2) from \underline{r} to r , we find that

$$\frac{u(r, \theta)}{r} - \frac{u(\underline{r}, \theta)}{\underline{r}} = O(\underline{r}^{-2} - r^{-2}). \quad (1.3)$$

In particular,

$$c(\theta) = \frac{u(\underline{r}, \theta)}{\underline{r}} + O(\underline{r}^{-2}).$$

We can arrange that the right-hand side is less than $\frac{\beta_0}{2}$ by choosing \underline{r} large (to control the second term) and b small (to control the first term). This proves the desired estimate for $c(\theta)$.

We now turn to the estimate for f . The key idea is to interpolate smallness in the C^0 -norm of u (that we have just obtained) with scale-invariant boundedness of higher derivatives: this implies that the Laplacian term in $\mathcal{L}_{\frac{1}{2}}$ is controlled with a small constant. Then, integrating the resulting ODE estimate to infinity, we obtain decay (and, more importantly,⁴ smallness) estimates for f .

First of all, we note that by (1.3), we have

$$|u(r, \theta)| \leq \delta^2 r,$$

for $r \geq \underline{r}$, where we can take δ small below (at the cost of taking \underline{r} larger and b smaller). Interpolating this (on balls of unit size) with $|D^k u| = O(r^{1-k})$, for k large, we find that

$$|\Delta u| \leq O(\delta) r^{-1},$$

⁴Note that the initial step in the proof can be used to prove decay for f , but not smallness.

for $r \geq \underline{r}$. Now, returning to $\mathcal{L}_{\frac{1}{2}}u = 0$, we have gained smallness in the constant on the right-hand side of (1.2); that is,

$$\partial_r \left(\frac{u}{r} \right) = O(\delta) r^{-3}.$$

Now, integrating this on $[\underline{r}, \infty)$, we find that

$$c(\theta) = \frac{u(\underline{r}, \theta)}{\underline{r}} + O(\delta \underline{r}^{-2}).$$

Because $u(r, \theta) = c(\theta)r + f(r, \theta)$, this gives

$$f(r, \theta) = O(\delta r^{-1}).$$

Choosing δ sufficiently small (in terms of β_0), we find that $\|rf\|_{C^0(\mathbb{R}^2 \setminus B_{\underline{r}}(0))} \leq \frac{\beta_0}{2}$. This completes the proof (since we already control u , and thus f inside of $B_{\underline{r}}(0)$). \square

At this point, we have proved that the conical scale $\mathbf{r}_\ell(M_\tau)$ is sufficiently large, so that when cutting off the Łojasiewicz–Simon inequality (1.1) at this scale, the error terms do not affect the right-hand side of the equation. At this point, we can use the now-standard uniqueness argument based on the Łojasiewicz inequality for parabolic equations (cf. [42], [43]). This completes the sketch of the proof of Theorem 1.1.

1.5. Organization of the paper

In Section 2, we prove several estimates on the geometry of asymptotically conical self-shrinkers. In Section 3, we establish the relevant linear PDE theory in weighted Hölder and Sobolev spaces. In Section 4, we apply these estimates to establish the Łojasiewicz–Simon inequality for entire graphs over a conical shrinker. So as to localize this inequality, in Section 5 we define the various scales used later. This then allows us to localize the inequality in Section 6. In Section 7, we carry out the central improvement/extension argument (cf. the model problem Lemma 1.3 above). In Section 8, we establish our final Łojasiewicz–Simon inequality. Putting this all together, we prove the uniqueness of conical tangent flows (Theorem 1.1) in Section 9. In Appendix A, we recall several standard definitions and conventions, while in Appendix B we recall some useful interpolation inequalities. Appendix C contains an analysis of normal graphs and Appendix D recalls the first and second variations of Gaussian area. Appendix E recalls an entropy-area bound estimate. Finally, we include a list of notation.

2. Geometric preliminaries

Throughout this section we fix $\Sigma^n \subset \mathbb{R}^{n+1}$ a smooth, smoothly asymptotically conical self-shrinker. We denote by

$$\mathcal{C} = \lim_{t \nearrow 0} \sqrt{-t} \Sigma$$

the asymptotic cone of Σ and assume that \mathcal{C}^n is the cone over $\Gamma^{n-1} \subset \mathbb{S}^n$. Note that the induced metric on \mathcal{C} satisfies

$$g_{\mathcal{C}} = dr \otimes dr + r^2 g_{\Gamma}$$

for $r = |x|$ the radial variable.

The following estimate is a straightforward consequence of the smooth convergence of $\sqrt{-t} \Sigma$ to \mathcal{C} combined with scaling considerations.

LEMMA 2.1

For $R > 0$ sufficiently large, the induced metric, g_{Σ} , on $\Sigma \setminus B_R(0)$ satisfies

$$g_{\Sigma} = g_{\mathcal{C}} + h$$

for h a symmetric $(0, 2)$ -tensor on $\Sigma \setminus B_R(0)$ satisfying $|\nabla^{(j)} h| = o(r^{-j})$ as $r \rightarrow \infty$, for all $j \geq 0$. The second fundamental form of Σ satisfies

$$|\nabla^{(j)} A_{\Sigma}| = O(r^{-j-1})$$

as $r \rightarrow \infty$ for $j \geq 0$.

In the sequel, we will improve these estimates based on the fact that Σ is a self-shrinker. Indeed, the shrinker equation (A.1) and second fundamental form decay in the previous lemma combine to yield decay for $\langle x, v_{\Sigma} \rangle$ that is faster than scaling.

COROLLARY 2.2

For $R > 0$ sufficiently large, we have

$$|\nabla^{(j)} \langle x, v_{\Sigma} \rangle| = O(r^{-j-1})$$

as $r \rightarrow \infty$ for $j \geq 0$.

2.1. Improved conical estimates for shrinkers

LEMMA 2.3

For $R > 0$ sufficiently large, there is $w \in C^{\infty}(\mathcal{C} \setminus B_R(0))$ so that

$$\text{graph } w := \{p + w(p)v_{\mathcal{C}}(p) : p \in \mathcal{C} \setminus B_R(0)\} \subset \Sigma$$

parameterizes Σ outside of a compact set. The function w satisfies

$$w = O(r^{-1})$$

and

$$\nabla^{(j)} w = O(r^{-1-j+\eta})$$

as $r \rightarrow \infty$ for any $\eta > 0$ and $j \geq 1$. Moreover, the radial derivatives satisfy the sharper relation $\partial_r^{(j)} w = O(r^{-1-j})$.

Proof

For $p \in \Gamma$, consider the plane $T_p \mathcal{C}$ with normal vector $v_{\mathcal{C}}(p)$. After a rotation, we can assume that $T_p \mathcal{C} = \{x^{n+1} = 0\}$ and $v_{\mathcal{C}}(p) = \pm e^{n+1}$. Define

$$\Gamma_{\varepsilon,R} := \{x \in T_p \mathcal{C} : |\langle x, p \rangle| > (1 - \varepsilon)|x|, |x| > R\}.$$

For $\varepsilon > 0$ sufficiently small and R sufficiently large, there are $u, u_{\infty} : \Gamma_{\varepsilon,R} \rightarrow \mathbb{R}$ so that

$$\begin{aligned} \text{graph } u &= \{(y, u(y)) : y \in \Gamma_{\varepsilon,R}\} \subset \Sigma, \\ \text{graph } u_{\infty} &= \{(y, u_{\infty}(y)) : y \in \Gamma_{\varepsilon,R}\} \subset \mathcal{C}. \end{aligned}$$

We have that

$$\nabla^{(j)} u(y) = \nabla^{(j)} u_{\infty}(y) + o(|y|^{1-j})$$

as $y \rightarrow \infty$.

We recall that

$$v_{\Sigma} = \pm \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}$$

Thus, by Corollary 2.2, we find that

$$\langle y, \nabla u(y) \rangle - u(y) = O(|y|^{-1}). \quad (2.1)$$

Thus, the function $v(s) = \frac{u(sp)}{s}$ satisfies $\lim_{s \rightarrow \infty} v(s) = 0$ (because $u_{\infty}(sp) = 0$) and $v'(s) = O(s^{-3})$ by (2.1). Integrating this, we find that

$$u(sp) = O(s^{-1}). \quad (2.2)$$

Thus (taking R larger if necessary), we may find $w \in C^{\infty}(\Sigma \setminus B_R(0))$ so that

$$\text{graph } w := \{q + w(q)v_{\mathcal{C}}(q) : q \in \mathcal{C} \setminus B_R(0)\} \subset \Sigma$$

parameterizes Σ outside of a compact set. From (2.2) we find that

$$|w| = O(r^{-1}).$$

This yields the first asserted decay estimate. Furthermore, scaling considerations yield

$$|\nabla^{(j)} w| = o(r^{1-j}),$$

as $r \rightarrow \infty$ for $j \geq 1$. Hence, the second assertion follows by interpolating these two estimates (cf. Lemma B.1). Finally, by differentiating (2.1) in the radial direction, the improved radial derivative estimate follows. \square

COROLLARY 2.4

For $R > 0$ sufficiently large, we have the following improved estimates on the induced metric:

$$g_\Sigma = dr \otimes dr + r^2 g_\Gamma + h$$

for h a symmetric $(0, 2)$ -tensor on $\Sigma \setminus B_R(0)$ satisfying $|h| = O(r^{-2})$ and $|\nabla^{(j)} h| = O(r^{-2-j+\eta})$ as $r \rightarrow \infty$, for all $j \geq 1$ and $\eta > 0$.

Proof

Write $F : \mathcal{C} \setminus B_R(0) \rightarrow \Sigma$, $F(p) = p + w(p)v_{\mathcal{C}}(p)$. We compute (using the fact that $A_{\mathcal{C}}(\partial_r, \cdot) = 0$)

$$\begin{aligned} \partial_r F &= \partial_r + (\partial_r w(p))v_{\mathcal{C}}(p), \\ r^{-1} \partial_{\omega_i} F &= r^{-1} \partial_{\omega_i} + r^{-1} (\partial_{\omega_i} w(p))v_{\mathcal{C}}(p) - w(p)A_{\mathcal{C}}|_p(r^{-1} \partial_{\omega_i}, \cdot). \end{aligned}$$

That $|h| = O(r^{-2})$ follows from these expressions and Lemma 2.3. The higher derivative estimates follow from interpolation, as in Lemma 2.3. \square

LEMMA 2.5

The unit normal to Σ satisfies

$$v_\Sigma(F(p)) = O(r^{-2})\partial_r + \sum_{j=1}^{n-1} O(r^{-2+\eta})r^{-1}\partial_{\omega_j} + (1 - O(r^{-4+\eta}))v_{\mathcal{C}}(p)$$

for $\eta > 0$ as $r \rightarrow \infty$.

Proof

Write

$$v_\Sigma(F(p)) = A\partial_r + \sum_{j=1}^{n-1} B_j r^{-1} \partial_{\omega_j} + C v_{\mathcal{C}}(p), \quad (2.3)$$

where

$$A^2 + \sum_{j=1}^{n-1} B_j^2 + C^2 = 1.$$

Because $\langle v_\Sigma, \partial_r F \rangle = \langle v_\Sigma, r^{-1} \partial_{\omega_i} F \rangle = 0$, we find that

$$\begin{aligned} 0 &= A + C(\partial_r w(p)), \\ 0 &= \sum_{j=1}^{n-1} B_j(\delta_{ij} + O(r^{-2})) + C(r^{-1} \partial_{\omega_i} w(p)). \end{aligned}$$

This implies the claim. \square

LEMMA 2.6

We have $|\nabla^{(j)}(r \partial_r w(p) - w(p))| = O(r^{-1-j})$ for any $j \geq 0$.

Proof

Revisiting the proof of Lemma 2.5, we find that the components of v_Σ in (2.3) satisfy

$$\begin{aligned} A &= -C(\partial_r w(p)), \\ 0 &= \sum_{j=1}^{n-1} B_j(\delta_{ij} + \tilde{b}_j) + C(r^{-1} \partial_{\omega_i} w(p)), \end{aligned}$$

where $|\tilde{b}_j| = O(r^{-2})$ and $|\nabla^{(j)} \tilde{b}_j| = O(r^{-2-j+\eta})$. Thus, we find that the expressions from the proof of Lemma 2.5 can be differentiated in the sense that

$$\begin{aligned} A &= -\partial_r w(p) + a, \\ B_i &= -r^{-1} \partial_{\omega_i} w(p) + b_i, \quad C = 1 + c, \end{aligned}$$

where $|\nabla^{(j)} a| = O(r^{-5-j+\eta})$, $|\nabla^{(j)} b_i| = |\nabla^{(j)} c| = O(r^{-4-j+\eta})$. This implies that

$$\langle F(p), v_\Sigma(F(p)) \rangle = rA + w(p)C = w(p) - r\partial_r w(p) + (ar - cw).$$

Using Corollary 2.2 and the above estimates for a, c , we conclude the proof. \square

LEMMA 2.7

The second fundamental form of Σ satisfies

$$\begin{aligned} A_\Sigma(\partial_r F, \partial_r F) &= O(r^{-3}), \\ A_\Sigma(\partial_r F, r^{-1} \partial_{\omega_i} F) &= O(r^{-3}), \\ A_\Sigma(r^{-1} \partial_{\omega_i} F, r^{-1} \partial_{\omega_j} F) &= A_C(r^{-1} \partial_{\omega_i}, r^{-1} \partial_{\omega_j}) + O(r^{-3+\eta}) \end{aligned}$$

as $r \rightarrow \infty$. Moreover, $|\nabla_{\mathcal{C}}^{(k)}(A_{\Sigma} \circ F - A_{\mathcal{C}})| = O(r^{-3-k+\eta})$ for any $\eta > 0$ and $k \geq 1$.

Proof

We compute

$$\begin{aligned} \partial_{r,r}^2 F &= (\partial_{r,r}^2 w(p))v_{\mathcal{C}}(p), \\ r^{-1}\partial_{r,\omega_i}^2 F &= r^{-2}\partial_{\omega_i} + r^{-1}(\partial_{r,\omega_i}^2 w)v_{\mathcal{C}}(p) - (\partial_r w(p))A_{\mathcal{C}}|_p(r^{-1}\partial_{\omega_i}, \cdot), \\ r^{-2}\partial_{\omega_i,\omega_j}^2 F &= A_{\mathcal{C}}|_p(r^{-1}\partial_{\omega_i}, r^{-1}\partial_{\omega_j})v_{\mathcal{C}}(p) + r^{-2}(\partial_{\omega_i,\omega_j}^2 w(p))v_{\mathcal{C}}(p) \\ &\quad - r^{-1}(\partial_{\omega_i} w(p))A_{\mathcal{C}}|_p(r^{-1}\partial_{\omega_j}, \cdot) \\ &\quad - r^{-1}(\partial_{\omega_j} w(p))A_{\mathcal{C}}|_p(r^{-1}\partial_{\omega_i}, \cdot) \\ &\quad - w(p)(\nabla_{r^{-1}\partial_{\omega_j}} A_{\mathcal{C}})|_p(r^{-1}\partial_{\omega_i}, \cdot). \end{aligned}$$

Using Lemma 2.5, the first and third equations follow immediately. For the second, we use the expression for $r^{-1}\partial_{w_i} F$ (which is orthogonal to $v_{\Sigma}(F(p))$) to write

$$\begin{aligned} r^{-1}\partial_{r,\omega_i}^2 F &= r^{-2}\partial_{\omega_i} F + r^{-2}(r\partial_{r,\omega_i}^2 w - \partial_{\omega_i} w(p))v_{\mathcal{C}}(p) \\ &\quad + r^{-1}(w(p) - r\partial_r w(p))A_{\mathcal{C}}|_p(r^{-1}\partial_{\omega_i}, \cdot). \end{aligned}$$

Using Lemmas 2.5 and 2.6, the first estimates follow. The higher derivatives follow by differentiating these expressions. \square

LEMMA 2.8

The vector field $V := \text{proj}_{T\Sigma} F(p) - r\partial_r F$ is tangent to Σ and satisfies $|V| = O(r^{-1})$, $|\nabla^{(k)} V| = O(r^{-1-k+\eta})$ for $\eta > 0$.

Proof

Because $\langle F(p), v_{\Sigma}(F(p)) \rangle = O(r^{-1})$, we compute

$$\begin{aligned} \text{proj}_{T\Sigma} F(p) &= F(p) - \langle F(p), v_{\Sigma}(F(p)) \rangle v_{\Sigma}(F(p)) \\ &= p + w(p)v_{\mathcal{C}}(p) + O(r^{-1}) \\ &= r\partial_r + w(p)v_{\mathcal{C}}(p) + O(r^{-1}) \\ &= r\partial_r F + O(r^{-1}). \end{aligned}$$

The higher derivatives follow similarly. \square

The function w from Lemma 2.3 gives a diffeomorphism from $\mathcal{C} \setminus B_R(0) \simeq \Gamma \times [R, \infty)$ to the noncompact part of Σ , where we recall that Γ is the link of the

asymptotic cone \mathcal{C} . We will thus parameterize points of Σ by $(r, \omega) \in \Gamma \times [R, \infty)$ below. We will write $g_{\mathcal{C}}$ for the metric on the end of Σ given by

$$g_{\mathcal{C}} = dr \otimes dr + r^2 g_{\Gamma}$$

in this parameterization. We emphasize that the coordinate r along Σ is *not* exactly equal to $d_{\mathbb{R}^3}(\cdot, 0)$ (like it is along the cone). It is useful to extend r to \tilde{r} defined on all of Σ so that $\tilde{r} \geq 1$ on Σ and $\tilde{r} = r$ outside of B_R for R as above.

LEMMA 2.9

The radial derivative satisfies

$$\vec{x} \cdot \nabla_{\Sigma} f = r \partial_r f + \alpha_3 \cdot \nabla_{g_{\mathcal{C}}} f,$$

where $|\alpha_3| = O(r^{-1})$ and $|\nabla^{(j)} \alpha_3| = O(r^{-1-j+\eta})$ for $\eta > 0$ and $j \geq 1$, as $r \rightarrow \infty$.

Proof

This follows from Lemma 2.8. □

3. Linear estimates in weighted spaces

In this section, we consider the relevant weighted function spaces which will play a role in our proof of the Łojasiewicz–Simon inequality for the conical shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$. Our choice of Hölder spaces will be heavily influenced by the work of Kapouleas, Kleene, and Møller [33] except for the complication that in [33], it was only necessary to define the spaces on a flat \mathbb{R}^2 (which is, of course, a conical shrinker), whereas here we must consider general conical shrinkers. Additionally, in various points of [33], the discrete symmetry of the problem was used in certain places, which will not be available to us here.

3.1. Weighted Hölder spaces

We now define the relevant weighted Hölder spaces. We begin with the most basic weighted space.

Definition 3.1 (Homogeneously weighted Hölder spaces)

We define a norm, for $\gamma \in \mathbb{R}$,

$$\|f\|_{0;-\gamma}^{\text{hom}} := \sup_{x \in \Sigma} \tilde{r}(x)^{\gamma} |f(x)|$$

and a seminorm

$$[f]_{\alpha;-\gamma-\alpha}^{\text{hom}} := \sup_{x,y \in \Sigma} \frac{1}{\tilde{r}(x)^{-\gamma-\alpha} + \tilde{r}(y)^{-\gamma-\alpha}} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

We thus define $C_{\text{hom},-\gamma}^{0,\alpha}(\Sigma)$ to be the set of functions $f : \Sigma \rightarrow \mathbb{R}$ so that

$$\|f\|_{0,\alpha;-\gamma}^{\text{hom}} := \|f\|_{0;-\gamma}^{\text{hom}} + [f]_{\alpha;-\gamma-\alpha}^{\text{hom}}$$

is finite. Similarly, we define $C_{\text{hom},-\gamma}^{2,\alpha}(\Sigma)$ to be the set of functions $f : \Sigma \rightarrow \mathbb{R}$ so that the norm

$$\|f\|_{2,\alpha;-\gamma}^{\text{hom}} = \sum_{j=0}^2 \|(\nabla_\Sigma)^{(j)} f\|_{0,\alpha;-\gamma}^{\text{hom}}$$

is finite.

Loosely speaking, $C_{\text{hom},-\gamma}^{2,\alpha}(\Sigma)$ is the space of $C^{2,\alpha}$ -functions whose $C^{2,\alpha}$ -norm falls off like $r^{-\gamma}$ at infinity. We now define a space which will require stronger weights in the radial direction.

Definition 3.2 (Anisotropically weighted Hölder spaces)

We define $C_{\text{an},-1}^{2,\alpha}(\Sigma)$ to be the space of $f \in C_{\text{hom},-1}^{2,\alpha}(\Sigma)$ so that

$$\|f\|_{2,\alpha;-1}^{\text{an}} := \|f\|_{2,\alpha;-1}^{\text{hom}} + \|\vec{x} \cdot \nabla_\Sigma f\|_{0,\alpha;-1}^{\text{hom}}$$

is finite.

Now, we fix a cutoff function $\chi : [0, \infty) \rightarrow [0, 1]$ so that $\text{supp } \chi \subset [R, \infty)$, $\chi \equiv 1$ in $[2R, \infty)$, and $|\nabla^j \chi| \leq CR^{-j}$ for $j \geq 1$ and C independent of R sufficiently large. This now allows us to define our primary Hölder space.

Definition 3.3 (Cone Hölder spaces)

We define $\mathcal{CS}_{-1}^{0,\alpha}(\Sigma) := C_{\text{hom},-1}^{0,\alpha}(\Sigma)$ and

$$\mathcal{CS}_{-1}^{2,\alpha}(\Sigma) := C^{2,\alpha}(\Gamma) \times C_{\text{an},-1}^{2,\alpha}(\Sigma).$$

An element $(c, f) \in \mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ will be considered as a function on Σ given by

$$u = u_{(c,f)}(r, \omega) = \chi(r)c(\omega)r + f(r, \omega)$$

for $r \geq R$, and $u = f$ otherwise. We will frequently conflate u with (c, f) . We take the norm

$$\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} := \|c\|_{C^{2,\alpha}(\Gamma)} + \|f\|_{2,\alpha;-1}^{\text{an}}.$$

Observe that an element of $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ is allowed to grow linearly at infinity, but only in a particularly prescribed manner. The remaining terms then must decay like r^{-1} . It is a standard exercise to observe that all of the above spaces are indeed Banach spaces.

3.2. Mapping properties

We observe that the cone spaces are well suited to the analysis of the $\mathcal{L}_{\frac{1}{2}}$ -operator

$$\mathcal{L}_{\frac{1}{2}}u := \Delta_{\Sigma}u - \frac{1}{2}\vec{x} \cdot \nabla_{\Sigma}u + \frac{1}{2}u$$

(see also Definition A.5) in the following sense.

LEMMA 3.4

For $a : \Sigma \rightarrow \mathbb{R}$ with $\|a\|_{C^{0,\alpha}(B_1(x))} = O(|x|^{-2})$ for $x \in \Sigma$ with $|x| \rightarrow \infty$, that is, $a \in C_{\text{hom};-2}^{0,\alpha}(\Sigma)$, we have that the operator

$$\mathcal{L}_{\frac{1}{2}} + a : \mathcal{CS}_{-1}^{2,\alpha}(\Sigma) \rightarrow \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$$

is bounded.

Proof

This follows directly from the definition of the cone spaces (after observing that the linear term $rc(\omega)$ exhibits a cancellation in the term $\frac{1}{2}(u - \vec{x} \cdot \nabla^{\Sigma}u)$; note that this fact does not hold for general \mathcal{L}_{γ} when $\gamma \neq \frac{1}{2}$). \square

3.3. Schauder estimates

In this section, we prove Schauder estimates for the \mathcal{L} -operator in the cone Hölder spaces. These estimates are essentially the generalization of [33, Proposition 8.8] to our setting, and we will closely follow their arguments, with some necessary modifications as discussed above. We note that Schauder estimates for the linearization of the expander equation on asymptotically conical self-expanders were proved by a related method in [8, Proposition 5.3].

PROPOSITION 3.5

Consider $a : \Sigma \rightarrow \mathbb{R}$ with $\|a\|_{C^{0,\alpha}(B_1(x))} = O(|x|^{-2})$ for $x \in \Sigma$ with $|x| \rightarrow \infty$, that is, $a \in C_{\text{hom};-2}^{0,\alpha}(\Sigma)$. Then there is $C = C(\Sigma, a)$ so that if $u \in C_{\text{loc}}^{2,\alpha}(\Sigma) \cap C_{\text{hom};+1}^0(\Sigma)$ has $\mathcal{L}_{\frac{1}{2}}u + au \in \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$, then $u \in \mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ and we have the estimate

$$\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} \leq C(\|u\|_{C_{\text{hom},+1}^0(\Sigma)} + \|\mathcal{L}_{\frac{1}{2}}u + au\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)}).$$

Because the \mathcal{L} -operator is related to the linearization of the shrinker equation, which is in turn a special case of the mean curvature flow (whose linearization is related to the heat equation), we might expect that such an estimate can be proved from standard parabolic Schauder estimates. This is nearly the case, except it turns out that the appropriate time parameterization of the equations will produce functions

which are not Hölder continuous (at $t = 0$) in the time variables. As such, we will require the following nonstandard parabolic Schauder estimates due to Brandt [10]. We note that these estimates were strengthened in [36] (see also [37]) but we will not make use of these stronger estimates here.

THEOREM 3.6 (Nonstandard interior Schauder estimates; see [10])

Suppose that $B_2 \subset \mathbb{R}^n$ and we are given coefficients $a_{ij}(x, t), b_i(x, t), c : B_2 \times [-2, 0] \rightarrow \mathbb{R}$ and functions $f, u : B_2 \times [-2, 0] \rightarrow \mathbb{R}$ so that u is a classical solution of

$$\frac{\partial u}{\partial t} - a_{ij} D_{ij}^2 u - b_i D_i u - cu = f.$$

Assume that the coefficients a_{ij}, b_i, c have spatial Hölder norms bounded uniformly in time, that is,

$$\sup_{t \in [-2, 0]} (\|a_{ij}(\cdot, t)\|_{C^{0,\alpha}(B_1)} + \|b_i(\cdot, t)\|_{C^{0,\alpha}(B_1)} + \|c(\cdot, t)\|_{C^{0,\alpha}(B_1)}) < \Lambda,$$

and that the equation is uniformly parabolic in the sense that

$$a_{ij}(x, t) \xi_i \xi_j \geq \lambda |\xi|^2$$

for $\lambda > 0$. Then, for $T \in (-1, 0]$,

$$\sup_{t \in [-1, T]} \|u(\cdot, t)\|_{C^{2,\alpha}(B_1)} \leq C \sup_{t \in [-2, T]} (\|u(\cdot, t)\|_{C^0(B_2)} + \|f(\cdot, t)\|_{C^{0,\alpha}(B_2)})$$

for some $C = C(n, \lambda, \Lambda)$.

We now explain how to relate the $\mathcal{L}_{\frac{1}{2}}$ -operator considered in Proposition 3.5 to a parabolic equation where we can apply Theorem 3.6.

Definition 3.7 (Intrinsic shrinker quantities)

It is useful to consider the intrinsic behavior of the shrinker Σ under the mean curvature flow. To this end, for $t \in [-1, 0)$, we define the (time-dependent) vector field $X_t = \frac{1}{2(-t)} x^T$. Here x^T is the tangential component of the position vector along Σ . For $t \in [-1, 0)$, define $\Phi_t : \Sigma \rightarrow \Sigma$ to be the family of diffeomorphisms generated by X_t (i.e., $\frac{\partial}{\partial t} \Phi_t = X_t \circ \Phi_t$) with $\Phi_{-1} = \text{Id}$. Finally, define the metric $\hat{g}_t := (-t) \Phi_t^* g_\Sigma$.

Observe that if $F : \Sigma \rightarrow \mathbb{R}^{n+1}$ is the embedding of Σ in \mathbb{R}^{n+1} , then

$$\hat{F}_t := \sqrt{-t} (F \circ \Phi_t) : \Sigma \rightarrow \mathbb{R}^{n+1}$$

is a mean curvature flow of hypersurfaces parameterized by normal speed. Moreover, we have that $\hat{g}_t = \hat{F}_t^* g_{\mathbb{R}^{n+1}}$. Thus, because the (extrinsic) blowdown of Σ is \mathcal{C} , we see that (Σ, \hat{g}_t, p) converges in the pointed C^∞ -Cheeger–Gromov sense to (the incomplete metric) $(\mathcal{C}, g_{\mathcal{C}}, p)$ for any point p sufficiently far out in the conical part of Σ . This will be useful in the sequel.

As in the proof of Corollary 2.4, we write the end of Σ via the map $F : \mathcal{C} \setminus B_r(0) \rightarrow \Sigma$, $p \mapsto F(p) + w(p)v_{\mathcal{C}}(p)$ as a normal graph over the cone \mathcal{C} with coordinates $(r, \omega) \in \Gamma \times [R, \infty)$ for R sufficiently large. We consider the induced flow of Φ_t in these coordinates, that is,

$$\tilde{\Phi}_t := F^{-1} \circ \Phi_t \circ F.$$

For $t \in [-1, 0)$, we consider the map

$$\phi_t : (R, \infty) \times \Gamma \rightarrow (R, \infty) \times \Gamma, \quad (r, \omega) \mapsto ((-t)^{-1/2}r, \omega).$$

Then we have the following estimates.

LEMMA 3.8

For $t \in [-1, 0)$, for r sufficiently large, we have

$$d_{g_{\mathcal{C}}}(\tilde{\Phi}_t(r, \theta), \phi_t(r, \theta)) \lesssim \frac{1}{\sqrt{-tr}}.$$

Moreover, in the coordinates (r, ω) we have the (nonsharp) estimate⁵

$$|D^{(j)}(\tilde{\Phi}_t - \phi_t)|(r, \theta) \lesssim \frac{1}{\sqrt{-tr^{1+j-\eta}}}$$

for $j \geq 1$ and $\eta > 0$.

Proof

We denote the ambient radius by $\underline{r}(x) := |x|$ and compute along Σ , using Lemma 2.8,

$$\begin{aligned} \frac{\partial}{\partial t}(\underline{r} \circ \Phi_t) &= (\nabla_{\dot{\Phi}_t} \underline{r}) \circ \Phi_t \\ &= \frac{1}{2(-t)} \langle x^T, \nabla \underline{r} \rangle_g \circ \Phi_t \\ &= \frac{1}{2(-t)} (r \circ \Phi_t + O((r \circ \Phi_t)^{-1})). \end{aligned}$$

⁵We emphasize that in this estimate we are *not* using the conical metric, but rather the flat cylindrical metric $dr^2 + g_{\Gamma}$ to estimate these derivatives. This avoids defining derivatives of diffeomorphisms as sections of an appropriate bundle and this estimate here suffices for our purposes.

Integrating this, we see that

$$\frac{1}{\sqrt{-t}} \left(\underline{r}(x) - \frac{c}{\underline{r}(x)} \right) \leq \underline{r}(\Phi_t(x)) \leq \frac{1}{\sqrt{-t}} \left(\underline{r}(x) - \frac{c}{\underline{r}(x)} \right). \quad (3.1)$$

Now, we have that

$$x^T = r \partial_r F + O(r^{-1})$$

by Lemma 2.8. This implies that

$$\frac{\partial}{\partial t} \tilde{\Phi}_t = \frac{1}{2(-t)} (r \partial_r + O(r^{-1}) \partial_r + O(r^{-2}) \partial_{\omega_i}), \quad (3.2)$$

where the right-hand side is evaluated at $\tilde{\Phi}_t(\cdot)$. Note that ϕ_t satisfies

$$\frac{\partial}{\partial t} \phi_t = \frac{1}{2(-t)} r \partial_r,$$

where the right-hand side is evaluated at $\phi_t(\cdot)$. In combination with (3.1), this implies that

$$\begin{aligned} \frac{\partial}{\partial t} d_{g_e}(\tilde{\Phi}_t(\cdot), \phi_t(\cdot)) &\leq \frac{1}{2(-t)} \left(d_{g_e}(\tilde{\Phi}_t(\cdot), \phi_t(\cdot)) + \frac{c}{\underline{r}(\Phi_t(\cdot))} \right) \\ &\leq \frac{1}{2(-t)} \left(d_{g_e}(\tilde{\Phi}_t(\cdot), \phi_t(\cdot)) + \frac{c}{\underline{r}(\cdot)} \sqrt{-t} \right). \end{aligned}$$

Integrating this yields

$$d_{g_e}(\tilde{\Phi}_t(\cdot), \phi_t(\cdot)) \leq \frac{c}{\sqrt{-t} \underline{r}(\cdot)}.$$

The derivative estimates follow similarly. \square

Now, assume that $\mathcal{L}_{\frac{1}{2}} u + au = E$ for some $u \in C_{\text{loc}}^{2,\alpha}(\Sigma)$ and $a : \Sigma \rightarrow \mathbb{R}$ with $\|a\|_{C^{0,\alpha}(B_1(x))} = O(|x|^{-2})$ for $x \in \Sigma$ with $|x| \rightarrow \infty$. We define

$$\begin{aligned} \hat{u}(x, t) &:= \sqrt{-t} (u(\Phi_t(x))), & \hat{E}(x, t) &:= \frac{1}{\sqrt{-t}} E(\Phi_t(x)), \\ \hat{a}(x, t) &:= \frac{1}{(-t)} a(\Phi_t(x)). \end{aligned}$$

Then we find that

$$\Delta_{\hat{g}_t} \hat{u} = \frac{1}{\sqrt{-t}} (\Delta_g u) \circ \Phi_t,$$

since the Laplacian is diffeomorphism invariant, as well as

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} &= \sqrt{-t}(\nabla_{X_t} u) \circ \Phi_t - \frac{1}{2\sqrt{-t}}u \circ \Phi_t \\ &= \frac{1}{2\sqrt{-t}}(\vec{x} \cdot \nabla_{\Sigma} u \circ \Phi_t - u \circ \Phi_t).\end{aligned}$$

We thus find that

$$\frac{\partial \hat{u}}{\partial t} - \Delta_{\hat{g}_t} \hat{u} - \hat{a} \hat{u} = \hat{E}. \quad (3.3)$$

We now use this equation in conjunction with Theorem 3.6 to prove the desired Schauder estimates. Observe that Lemma 3.8 and the presumed decay of a shows that $\hat{a}(\cdot, t)$ is uniformly bounded in $C^{0,\alpha}$ on sufficiently far out balls of unit size, allowing us to apply Theorem 3.6.

Proof of Proposition 3.5

We can choose R sufficiently large such that the normal evolution of $\Sigma_t := \sqrt{t} \cdot \Sigma$ for $t \in [-2, 0)$ is almost orthogonal to x outside of $B_{R/4}$. Applying Theorem 3.6 to (3.3) we find that (where the implied constant is independent of R sufficiently large)

$$\begin{aligned}&\sup_{t \in [-1, 0)} \|D_x^2 \hat{u}(\cdot, t)\|_{C^0(\Sigma_t \cap (B_{R+2}(0) \setminus B_{R+1}(0)))} \\ &+ \sup_{t \in [-1, 0)} \|D_x \hat{u}(\cdot, t)\|_{C^0(\Sigma_t \cap (B_{R+2}(0) \setminus B_{R+1}(0)))} \\ &+ \sup_{t \in [-1, 0)} [D_x^2 \hat{u}(\cdot, t)]_{\alpha; \Sigma_t \cap (B_{R+2}(0) \setminus B_{R+1}(0)))} \\ &\lesssim \sup_{t \in [-2, 0)} \|\hat{u}(\cdot, t)\|_{C^0(\Sigma_t \cap (B_{R+3}(0) \setminus B_R(0)))} \\ &+ \sup_{t \in [-2, 0)} [\hat{E}(\cdot, t)]_{\alpha; \Sigma_t \cap (B_{R+3}(0) \setminus B_R(0)))}.\end{aligned}$$

On the other hand, Lemma 3.8 implies that for R sufficiently large, we can estimate the Hölder norms of \hat{u} in terms of weighted norms of u as follows:

$$\begin{aligned}&\sup_{t \in [-1, 0)} \|D_x^2 \hat{u}(\cdot, t)\|_{C^0(\Sigma \cap (B_{R+2}(0) \setminus B_{R+1}(0)))} \\ &+ \sup_{t \in [-1, 0)} \|D_x \hat{u}(\cdot, t)\|_{C^0(\Sigma \cap (B_{R+2}(0) \setminus B_{R+1}(0)))} \\ &+ \sup_{t \in [-1, 0)} [D_x^2 \hat{u}(\cdot, t)]_{\alpha; \Sigma \cap (B_{R+2}(0) \setminus B_{R+1}(0))} \\ &\gtrsim \sup_{x \in \Sigma \setminus B_{R+1}(0)} r(x) |D^2 u(x)|\end{aligned}$$

$$+ \sup_{x,y \in \Sigma \setminus B_{R+1}(0)} \frac{1}{r(x)^{-1-\alpha} + r(y)^{-1-\alpha}} \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha}.$$

Arguing similarly for the other terms, we thus rewrite the above parabolic Schauder estimates as weighted elliptic estimates:

$$\begin{aligned} & \sup_{x \in \Sigma \setminus B_{R+1}(0)} r(x) |D^2u(x)| \\ & + \sup_{x,y \in \Sigma \setminus B_{R+1}(0)} \frac{1}{r(x)^{-1-\alpha} + r(y)^{-1-\alpha}} \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha} \\ & \lesssim \sup_{x \in \Sigma \setminus B_{R/2}(0)} r(x)^{-1} |u(x)| + \sup_{x \in \Sigma \setminus B_{R/2}(0)} r(x) |E(x)| \\ & + \sup_{x,y \in \Sigma \setminus B_{R/2}(0)} \frac{1}{r(x)^{-1-\alpha} + r(y)^{-1-\alpha}} \frac{|E(x) - E(y)|}{|x - y|^\alpha}. \end{aligned}$$

This implies that

$$\begin{aligned} & \|D^2u\|_{\mathcal{C}^{0,\alpha}_{-1}(\Sigma \setminus B_{R+1}(0))} \\ & = \sup_{x \in \Sigma \setminus B_{R+1}(0)} r(x) |D^2u(x)| \\ & + \sup_{x,y \in \Sigma \setminus B_{R+1}(0)} \frac{1}{r(x)^{-1-\alpha} + r(y)^{-1-\alpha}} \frac{|D^2u(x) - D^2u(y)|}{|x - y|^\alpha} \\ & \lesssim \sup_{x \in \Sigma \setminus B_{R/2}(0)} r(x)^{-1} |u(x)| + \sup_{x \in \Sigma \setminus B_{R/2}(0)} r(x) |E(x)| \\ & + \sup_{x,y \in \Sigma \setminus B_{R/2}(0)} \frac{1}{r(x)^{-1-\alpha} + r(y)^{-1-\alpha}} \frac{|E(x) - E(y)|}{|x - y|^\alpha} \\ & \lesssim \|u\|_{C^0_{\text{hom};+1}(\Sigma \setminus B_{R/2}(0))} + \|E\|_{\mathcal{C}^{0,\alpha}_{-1}(\Sigma \setminus B_{R/2}(0))}. \end{aligned}$$

Arguing similarly for Du and combining all of this with standard interior (elliptic) Schauder theory, we thus find that

$$\|Du\|_{\mathcal{C}^{0,\alpha}_{-1}(\Sigma)} + \|D^2u\|_{\mathcal{C}^{0,\alpha}_{-1}(\Sigma)} \lesssim \|u\|_{C^0_{\text{hom};+1}(\Sigma)} + \|E\|_{\mathcal{C}^{0,\alpha}_{-1}(\Sigma)}. \quad (3.4)$$

Note that we can combine this inequality with an interpolation between u in C^0 and C^1 to find that

$$\|u\|_{C^{0,\alpha}_{\text{hom};+1}(\Sigma)} \lesssim \|u\|_{C^0_{\text{hom};+1}(\Sigma)} + \|E\|_{\mathcal{C}^{0,\alpha}_{-1}(\Sigma)}.$$

This allows us to bound au in $\mathcal{C}^{0,\alpha}_{-1}(\Sigma)$ in the sequel.

We now argue that u can be decomposed as $u(r, \omega) = \chi(r)c(\omega)r + f(r, \omega)$ making u into an element of $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$. We have that

$$w := r\partial_r u - u = -2E + 2\Delta u + 2au - (\vec{x} \cdot \nabla_{\Sigma} u - r\partial_r u).$$

Combining (3.4) with Lemma 2.8, we see that $w \in \mathcal{CS}_{-1}^{0,\alpha}$ with

$$\|w\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)} \lesssim \|u\|_{C_{\text{hom};+1}^0(\Sigma)} + \|E\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)}.$$

Now, we define

$$c(\omega) := \frac{u(R, \omega)}{R} + \int_R^\infty \frac{w(s, \omega)}{s^2} ds, \quad (3.5)$$

where R is chosen large above (we emphasize that this expression is independent of the choice of R and that the integral is finite, thanks to the fact that $w \in \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$).

We note that the functions

$$\omega \mapsto \frac{u(r, \omega)}{r}$$

have uniformly bounded $C^{2,\alpha}(\Gamma)$ -norm for r sufficiently large. On the one hand, they converge in $C^{0,\alpha}(\Gamma)$ to $c(\omega)$ by the previous analysis. On the other hand, by Arzelà–Ascoli, they converge in $C^{2,\beta}(\Gamma)$ (for any $\beta < \alpha$) to $c(\omega) \in C^{2,\alpha}(\Gamma)$, and we find that (by lower semicontinuity of the Hölder norm in this situation)

$$\|c\|_{C^{2,\alpha}(\Gamma)} \leq \|D^2 u\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)} \lesssim \|u\|_{C_{\text{hom};+1}^0(\Sigma)} + \|E\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)},$$

where we again used (3.4) in the second inequality. Now, defining

$$f(r, \omega) = \chi(r)c(\omega)r - u(r, \omega),$$

we see that $f \in C_{\text{loc}}^{2,\alpha}(\Sigma)$. Note that for r sufficiently large we have from (3.5) that

$$f(r, \omega) = r \int_r^\infty \frac{w(s, \omega)}{s^2} ds,$$

which implies that

$$\|f\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)} \lesssim \|u\|_{C_{\text{hom};+1}^0(\Sigma)} + \|E\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)}.$$

Moreover, using the estimates for $D^2 u$ (and for $D^2(\chi(r)rc(\omega))$) which are easily derived from the $C^{2,\alpha}$ estimate for c , along with interpolation, we find that

$$\|f\|_{C_{\text{hom},-1}^{2,\alpha}(\Sigma)} \lesssim \|u\|_{C_{\text{hom};+1}^0(\Sigma)} + \|E\|_{\mathcal{CS}_{-1}^{0,\alpha}(\Sigma)}.$$

Finally, it remains to estimate $\vec{x} \cdot \nabla_{\Sigma} f \in C_{\text{hom}, -1}^{0, \alpha}(\Sigma)$. However, this follows from

$$r \partial_r f = f - w$$

and Lemma 2.8. This completes the proof. \square

3.4. Weighted Sobolev spaces

In this section, we combine the Hölder space theory developed above, with integral estimates and a Fredholm alternative to establish existence results for the $\mathcal{L}_{\frac{1}{2}}$ -operator. The way to use these weighted Sobolev spaces to prove the Fredholm alternative (cf. Theorem 3.14 below) was explained to us by Bernstein [3].

We denote by L_W^2 the space of measurable functions $f : \Sigma \rightarrow \mathbb{R}$ with

$$\|f\|_W^2 := \int_{\Sigma} f^2 \rho d\mathcal{H}^n < \infty.$$

We then define the Sobolev norm

$$\|f\|_{W,k}^2 := \sum_{j=0}^k \|(\nabla_{\Sigma})^j f\|_W^2.$$

It is easy to see that the associated Sobolev space $H_W^k(\Sigma)$ is precisely the closure of $C_0^{\infty}(\Sigma)$ under this norm.

We recall the following Sobolev inequality due to Ecker [22, p. 109] (see also [6, Lemma B.1].

PROPOSITION 3.9

For $f \in H_W^1(\Sigma)$, we have

$$\int_{\Sigma} f^2 |x|^2 \rho d\mathcal{H}^n \leq 4 \int_{\Sigma} (nf^2 + 4|\nabla_{\Sigma} f|^2) \rho d\mathcal{H}^n.$$

Proof

Assume first that $f \in C_0^{\infty}(\Sigma)$. Consider the vector field $V := f^2 \rho \vec{x}$ in the (Euclidean) first variation formula along Σ . We obtain

$$\int_{\Sigma} \left(nf^2 + 2f \vec{x} \cdot \nabla_{\Sigma} f - \frac{1}{2} f^2 |x^T|^2 \right) \rho d\mathcal{H}^n = \int_{\Sigma} f^2 H \langle x, v_{\Sigma} \rangle \rho d\mathcal{H}^n.$$

Using the shrinker equation, we thus find that

$$\int_{\Sigma} (nf^2 + 2f \vec{x} \cdot \nabla_{\Sigma} f) \rho d\mathcal{H}^n = \frac{1}{2} \int_{\Sigma} f^2 |x|^2 \rho d\mathcal{H}^n.$$

Thus, we find that

$$\begin{aligned} \frac{1}{2} \int_{\Sigma} f^2 |x|^2 \rho d\mathcal{H}^n &\leq \int_{\Sigma} (nf^2 + 2f \vec{x} \cdot \nabla_{\Sigma} f) \rho d\mathcal{H}^n \\ &\leq \int_{\Sigma} \left(nf^2 + 4|\nabla_{\Sigma} f|^2 + \frac{1}{4}|x|^2 f^2 \right) \rho d\mathcal{H}^n. \end{aligned}$$

Now let $f \in H_W^1(\Sigma)$, and choose $f_i \in C_0^\infty(\Sigma)$ such that $f_i \rightarrow f$ in $H_W^1(\Sigma)$. The above estimate yields, for any $R > 0$,

$$\int_{\Sigma \cap B_R(0)} f_i^2 |x|^2 \rho d\mathcal{H}^n \leq 4 \int_{\Sigma} (nf_i^2 + 4|\nabla_{\Sigma} f_i|^2) \rho d\mathcal{H}^n.$$

Letting $i \rightarrow \infty$ and then $R \rightarrow \infty$ yields the statement. \square

COROLLARY 3.10

The map $\mathcal{L}_\gamma : H_W^2(\Sigma) \rightarrow L_W^2(\Sigma)$ is bounded.

Proof

Apply Ecker's Sobolev inequality to the gradient of f to bound $\vec{x} \cdot \nabla_{\Sigma} f \in L_W^2(\Sigma)$. \square

LEMMA 3.11 (cf. [9, Proposition 3.4])

For $f \in H_W^2(\Sigma)$,

$$\|f\|_{W,1}^2 \leq \|\mathcal{L}_0 f\|_W \|f\|_W.$$

Proof

It suffices to prove this for $f \in C_0^\infty(\Sigma)$. Note that \mathcal{L}_0 is self-adjoint with respect to the Gaussian area. Thus,

$$0 = \int_{\Sigma} \mathcal{L}_0(f^2) \rho d\mathcal{H}^n = 2 \int_{\Sigma} (|\nabla f|^2 + f \mathcal{L}_0 f) \rho d\mathcal{H}^n.$$

This proves the claim. \square

LEMMA 3.12 ([6, Proposition B.2])

The inclusion $H_W^1 \subset L_W^2$ is compact.

Proof

For $f_j \in H_W^1$ with $\|f_j\|_{H_W^1} \leq C$, the classical Rellich compactness theorem applied to an exhaustion of Σ shows that (after passing to a subsequence) there is $f \in H_W^1$ so that $f_j \rightarrow f$ in L_{loc}^2 . That $f_j \rightarrow f$ follows easily from Ecker's Sobolev inequality, which implies that

$$\int_{\Sigma \setminus B_\lambda(0)} (f_j - f)^2 \rho d\mathcal{H}^n \lesssim \frac{C}{\lambda^2}.$$

This concludes the proof. \square

LEMMA 3.13 (cf. [9, Proposition 3.4])

For $f \in H_W^2(\Sigma)$, we have

$$\|f\|_{W,2}^2 \leq C(\|\mathcal{L}_0 f\|_W^2 + \|f\|_W^2).$$

Proof

It suffices to prove this for $f \in C_0^\infty(\Sigma)$. Using the Bochner identity and the Gauss equations, we find (using $|A_\Sigma| = O(1)$) that

$$\begin{aligned} & \frac{1}{2} \mathcal{L}_0 |\nabla_\Sigma f|^2 \\ &= |\nabla^2 f|^2 + \langle \nabla_\Sigma \Delta_\Sigma f, \nabla_\Sigma f \rangle + \text{Ric}_\Sigma(\nabla_\Sigma f, \nabla_\Sigma f) - \frac{1}{4} \langle x, \nabla |\nabla f|^2 \rangle \\ &= |\nabla^2 f|^2 + \langle \nabla_\Sigma \Delta_\Sigma f, \nabla_\Sigma f \rangle - \frac{1}{4} \langle x, \nabla |\nabla f|^2 \rangle \\ & \quad + H_\Sigma \cdot A_\Sigma(\nabla_\Sigma f, \nabla_\Sigma f) - (A_\Sigma)^2(\nabla_\Sigma f, \nabla_\Sigma f) \\ &= |\nabla^2 f|^2 + \langle \nabla_\Sigma \mathcal{L}_0 f, \nabla_\Sigma f \rangle + \frac{1}{2} \langle \nabla_\Sigma(\vec{x} \cdot \nabla_\Sigma f), \nabla_\Sigma f \rangle - \frac{1}{4} \langle x, \nabla |\nabla f|^2 \rangle \\ & \quad + H_\Sigma \cdot A_\Sigma(\nabla_\Sigma f, \nabla_\Sigma f) - (A_\Sigma)^2(\nabla_\Sigma f, \nabla_\Sigma f) \\ &= |\nabla^2 f|^2 + \langle \nabla_\Sigma \mathcal{L}_0 f, \nabla_\Sigma f \rangle + \frac{1}{2} |\nabla_\Sigma f|^2 \\ & \quad + H_\Sigma \cdot A_\Sigma(\nabla_\Sigma f, \nabla_\Sigma f) - (A_\Sigma)^2(\nabla_\Sigma f, \nabla_\Sigma f) \\ &= |\nabla^2 f|^2 + \langle \nabla_\Sigma \mathcal{L}_0 f, \nabla_\Sigma f \rangle + O(|\nabla_\Sigma f|^2). \end{aligned}$$

Integrating this and using that \mathcal{L}_0 is self-adjoint with respect to the Gaussian area, the conclusion follows (after integrating by parts the second term on the right-hand side, and using Lemma 3.11 to control the H_W^1 -norm of f). \square

This suffices to establish an existence theory for the L -operator (cf. [9, Proposition 3.4]), where

$$L := \mathcal{L}_{\frac{1}{2}} + |A_\Sigma|^2 = \Delta_\Sigma - \frac{1}{2}(\vec{x} \cdot \nabla_\Sigma - 1) + |A_\Sigma|^2.$$

Define

$$B_\gamma(u, v) := \int_{\Sigma} \left(\langle \nabla_{\Sigma} u, \nabla_{\Sigma} v \rangle + \left(\gamma - |A_{\Sigma}|^2 - \frac{1}{2} \right) u v \right) \rho d\mathcal{H}^n,$$

the bilinear form naturally associated to $L + \gamma$. For γ sufficiently large so that $\gamma \geq \max_{\Sigma} |A_{\Sigma}|^2 + \frac{3}{2}$, we see that

$$\|u\|_{W,1}^2 \leq B_\gamma(u, v),$$

so B_γ is coercive on $H_W^1(\Sigma)$. It is clearly bounded, so applying the Lax–Milgram theorem, and applying the standard Fredholm alternative to this setting (combining Lemma 3.13 with Lemma 3.12), we have the following result.

THEOREM 3.14

The space $\ker L \subset H_W^1$ of weak solutions to $Lu = 0$ is finite-dimensional. For $f \in L_W^2(\Sigma)$, $Lu = f$ has a weak solution in $H_W^1(\Sigma)$ if and only if f is L_W^2 -orthogonal to $\ker L$. Moreover, if u is orthogonal to $\ker L$ and satisfies $Lu = f$, then we have the estimate $\|u\|_{H_W^2(\Sigma)} \leq C \|f\|_{L_W^2(\Sigma)}$.

To complete this section, we now show that for $f \in \mathcal{CS}_{-1}^{\alpha}(\Sigma)$ perpendicular to $\ker L$, we can solve $Lu = f$. It remains to check that a solution of $Lu = f$ with $f \in \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$ satisfies $u \in C_{\text{hom};+1}^0(\Sigma)$ a priori.

LEMMA 3.15

For $f \in L_W^2(\Sigma) \cap C^0(\Sigma)$, if $Lu = f$ for $u \in H_W^1(\Sigma)$, then $u \in C_{\text{hom};+1}^0(\Sigma)$ and for R sufficiently large,

$$\|u\|_{C_{\text{hom};+1}^0(\Sigma)} \lesssim \|f\|_{C^0(\Sigma)} + \|u\|_{C^0(\Sigma \cap B_R(0))}.$$

Proof

For $\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, we compute

$$\begin{aligned} L\varphi &= \Delta_{\Sigma}\varphi - \frac{1}{2}(\vec{x} \cdot \nabla_{\Sigma}\varphi - \varphi) + |A_{\Sigma}|^2\varphi \\ &= \Delta_{\mathbb{R}^{n+1}}\varphi - D^2\varphi(v_{\Sigma}, v_{\Sigma}) - H_{\Sigma}\langle v_{\Sigma}, \nabla_{\mathbb{R}^{n+1}}\varphi \rangle - \frac{1}{2}(\vec{x} \cdot \nabla_{\Sigma}\varphi - \varphi) + |A_{\Sigma}|^2\varphi \\ &= \Delta_{\mathbb{R}^{n+1}}\varphi - D^2\varphi(v_{\Sigma}, v_{\Sigma}) - \frac{1}{2}\langle \vec{x}, \nabla_{\mathbb{R}^{n+1}}\varphi \rangle + \frac{1}{2}\varphi + |A_{\Sigma}|^2\varphi. \end{aligned}$$

We consider $\varphi(x) = \alpha|x| - \beta$. Then,

$$\begin{aligned} L\varphi &= \alpha \left(\frac{n-1}{|x|} - \frac{\langle \vec{x}, v_{\Sigma} \rangle^2}{|x|^3} \right) - \frac{1}{2}\beta + O(|x|^{-2})(\alpha|x| - \beta) \\ &\leq -\frac{1}{2}(1 + O(|x|^{-2}))\beta + O(|x|^{-1})\alpha. \end{aligned}$$

Thus, $v = u - \varphi$ satisfies

$$Lv \geq f + \frac{1}{2}(1 + O(|x|^{-2}))\beta - O(|x|^{-1})\alpha.$$

We fix $R \geq \underline{R}(\Sigma)$ and set

$$\alpha = 2 \sup_{\Sigma} |f| + 2R^{-1} \sup_{\Sigma \cap \partial B_R(0)} |u|,$$

$$\beta = 4 \sup_{\Sigma} |f| + R^{-1} \sup_{\Sigma \cap \partial B_R(0)} |u|.$$

This yields

$$Lv \geq (1 - O(R^{-1})) \sup_{\Sigma} |f| + \frac{1}{2} R^{-1} (1 + O(R^{-1})) \sup_{\Sigma \cap \partial B_R(0)} |u| > 0,$$

for R sufficiently large. Moreover, we find that

$$\sup_{\Sigma \cap \partial B_R} v \leq -(1 - R^{-1}) \sup_{\Sigma \cap \partial B_R} |u| - 2(R - 2) \sup_{\Sigma} |f| < 0$$

as long as R is sufficiently large. Thus, we have arranged that $v \leq 0$ in a neighborhood of $\Sigma \cap \partial B_R(0)$. We now argue that $v \leq 0$ on $\Sigma \setminus B_R(0)$. Because $v^+ \in H_W^1$, we find that

$$\begin{aligned} - \int_{\Sigma \setminus B_R(0)} |A_{\Sigma}|^2 (v^+)^2 \rho d\mathcal{H}^n &\leq \int_{\Sigma \setminus B_R(0)} v^+ \mathcal{L}_{\frac{1}{2}} v \rho d\mathcal{H}^n \\ &= \int_{\Sigma \setminus B_R(0)} \left(-|\nabla v^+|^2 + \frac{1}{2} (v^+)^2 \right) \rho d\mathcal{H}^n. \end{aligned}$$

Thus, using Ecker's Sobolev inequality, Proposition 3.9, we find that

$$R^2 \int_{\Sigma \setminus B_R(0)} (v^+)^2 \rho d\mathcal{H}^n \leq (8 + 4n + O(R^{-2})) \int_{\Sigma \setminus B_R(0)} (v^+)^2 \rho d\mathcal{H}^n.$$

For R sufficiently large, we thus see that $v^+ \equiv 0$. Thus, $u \leq \varphi$ on $\Sigma \setminus B_R(0)$. Applying the same reasoning to $-u$ completes the proof. \square

Combining this estimate with Proposition 3.5 we arrive at the following.

COROLLARY 3.16

For $f \in \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$, if $u \in H_W^1(\Sigma)$ satisfies $Lu = f$ weakly, then $u \in \mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ and for $R > 0$ fixed sufficiently large,

$$\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} \lesssim \|u\|_{C^0(\Sigma \cap B_R(0))} + \|f\|_{\mathcal{CS}^{0,\alpha}(\Sigma)}.$$

Combined with Theorem 3.14, we thus see that the following standard solvability condition continues to hold in our setting.

COROLLARY 3.17

If $f \in \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$, then we can find $u \in \mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ solving $Lu = f$ if and only if f is L_W^2 -orthogonal to $\ker L \subset H_W^1(\Sigma)$.

4. The Łojasiewicz–Simon inequality for entire graphs

We now show that the weighted Hölder and Sobolev spaces considered in the previous section (along with the solvability criteria proved for L) provides a framework to prove the Łojasiewicz–Simon inequality following the arguments in the compact case (cf. [42]–[44], [50]). By the Fredholm alternative, Theorem 3.14, $\ker L \subset H_W^1(\Sigma)$ is finite-dimensional and we can define $\Pi : L_W^2(\Sigma) \rightarrow L_W^2(\Sigma) \cap \mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$, the projection onto $\ker L$.

Recall (see Appendix D) that the Euler–Lagrange equation (with respect to the L_W^2 -inner product) is

$$\begin{aligned} \mathcal{M}(v) &= \Pi_{T^\perp \Sigma} \left(\vec{H}_M + \frac{x^\perp}{2} \right) \Big|_{x=y+v(y)v_\Sigma(y)} J(y, v, \nabla_\Sigma v) \\ &\quad \times \rho(y + v(y)v_\Sigma) \rho(y)^{-1}, \end{aligned} \tag{4.1}$$

where $\Pi_{T^\perp \Sigma}$ is the projection onto the normal bundle to Σ and

$$J(y, v, \nabla_\Sigma v) = \text{Jac}(D \exp_y(v(y)v_\Sigma(y)))$$

is the area element.

We now observe that \mathcal{M} is a well-behaved map between the weighted Hölder spaces considered in the previous section.

LEMMA 4.1

For β sufficiently small depending on Σ , we have a continuous map

$$\mathcal{M} : \mathcal{CS}_{-1}^{2,\alpha}(\Sigma) \cap \{\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} < \beta\} \rightarrow \mathcal{CS}_{-1}^{0,\alpha}(\Sigma).$$

Moreover, \mathcal{M} is Fréchet differentiable with derivative at 0 given by L .

Proof

Fix $v \in \mathcal{CS}_{-1}^{2,\alpha}(\Sigma) \cap \{\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} < \beta\}$. Note that

$$\begin{aligned} J(y, v, \nabla_\Sigma v) \rho(y + v(y)v_\Sigma) \rho(y)^{-1} \\ = J(y, v, \nabla_\Sigma v) \exp\left(-\frac{2v(y)\langle y, v_\Sigma \rangle + (v(y))^2}{4}\right) \end{aligned} \tag{4.2}$$

and this is easily seen to be uniformly bounded in $C^{0,\alpha}(\Sigma \cap B_1(y))$ as $y \in \Sigma \rightarrow \infty$. Thus, it remains to check the first term. Observe that the mean curvature term is uniformly bounded in $C^{0,\alpha}(\Sigma \cap B_1(y))$ by c/r as $y \in \Sigma \rightarrow \infty$. Recall that differentiating the shrinker equation yields (or see Lemmas 2.7 and 2.8)

$$A(x^T, \cdot) = O(r^{-2}).$$

Combining this with (C.1) and the shrinker equation for Σ we get for the other term that

$$\langle x, v \rangle = (1 + |(\text{Id} - vS)^{-1}(\nabla^M v)|^2)^{-\frac{1}{2}}(v - \langle y, \nabla_\Sigma v \rangle) + O(|y|^{-1})$$

in $C^{0,\alpha}(\Sigma \cap B_1(y))$ as $y \rightarrow \infty$. Observing that $v \mapsto v - \langle y, \nabla_\Sigma v \rangle$ is a bounded map $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma) \rightarrow \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$ we obtain the first assertion. The second follows similarly. \square

We define

$$\mathcal{N} := \mathcal{M} + \Pi$$

which has the same mapping properties as \mathcal{M} . Moreover, \mathcal{N} is Fréchet differentiable with derivative at 0 given by $L + \Pi$ (which is bijective as a linear map $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma) \rightarrow \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$). Thus, the implicit function theorem allows us to find open neighborhoods of 0,

$$W_1 \subset \mathcal{CS}_{-1}^{2,\alpha}(\Sigma) \cap \{\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} < \beta\},$$

$$W_2 \subset \mathcal{CS}_{-1}^{0,\alpha}(\Sigma)$$

so that $\mathcal{N} : W_1 \rightarrow W_2$ is bijective with inverse $\Psi : W_2 \rightarrow W_1$. Moreover (cf. [44, Section 3.13] and [42, p. 168]), \mathcal{N} and Ψ are holomorphic, after tensoring with \mathbb{C} (and possibly shrinking W_1 , W_2).

We now prove that \mathcal{M} is continuous as a map $H_W^2 \cap W_1 \rightarrow L_W^2$ and that Ψ is continuous as a map $L_W^2 \cap W_2 \rightarrow H_W^2$.

LEMMA 4.2

Shrinking W_1 , W_2 if necessary, there is $C > 0$ so that

$$\|\mathcal{M}(u_1) - \mathcal{M}(u_2)\|_{L_W^2(\Sigma)} \leq C \|u_1 - u_2\|_{H_W^2(\Sigma)}$$

for $u_1, u_2 \in W_1$, and, moreover,

$$\|\Psi(f_1) - \Psi(f_2)\|_{H_W^2(\Sigma)} \leq C \|f_1 - f_2\|_{L_W^2(\Sigma)}$$

for $f_1, f_2 \in W_2$.

Proof

We claim that

$$\begin{aligned} \mathcal{M}(u_1) - \mathcal{M}(u_2) &= L(u_1 - u_2) + A \cdot \nabla^2(u_1 - u_2) \\ &\quad + B \cdot \nabla(u_1 - u_2) + C(u_1 - u_2), \end{aligned} \quad (4.3)$$

where

$$\sup_{\Sigma}(|A| + |B| + |C|) \lesssim \|u_1\|_{\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)} + \|u_2\|_{\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)}. \quad (4.4)$$

This follows from using (4.1), (C.2), (C.3), and (C.4) together with the shrinker equation along Σ to write

$$\mathcal{M}(u) = Lu + Q(p, u, \nabla u, \nabla^2 u)$$

and interpolating $Q(p, u, \nabla u, \nabla^2 u)$ in the standard way between u_1 and u_2 . Combined with Corollary 3.10, this proves the first assertion. The second claim now follows from standard arguments (cf. [44, Section 3.12]) given (4.3), (4.4), and Theorem 3.14. \square

At this point, we can follow the arguments in [44, Sections 3.11–3.13] essentially verbatim (except we use Corollary 3.17, Theorem 3.14, Lemma 4.1, and Lemma 4.2 in place of their standard counterparts in the compact case) to prove the following.

THEOREM 4.3 (Łojasiewicz–Simon inequality for entire graphs)

There is $\beta_0 > 0$ sufficiently small, $\theta \in (0, \frac{1}{2})$, and $C > 0$, all depending on Σ , so that if M is the graph over Σ of a function in $u \in \mathcal{CS}^{2,\alpha}_{-1}(\Sigma)$ with $\|u\|_{\mathcal{CS}^{2,\alpha}_{-1}(\Sigma)} < \beta_0$, then

$$|F(M) - F(\Sigma)|^{1-\theta} \leq C \| \mathcal{M}(u) \|_{L^2_W(\Sigma)} \leq C \left(\int_M |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2}}.$$

We note that the second inequality here follows a similar reasoning to (4.2) (so as to control the change in ρ when evaluated along M and as opposed to Σ).

5. Defining the relevant scales

In order to apply the inequality obtained in Theorem 4.3, we must understand the various geometric scales involved.

5.1. Pseudolocality and the scale of the core of the shrinker

These definitions are relevant to the pseudolocality-based improvement argument in Lemma 9.1.

PROPOSITION 5.1 (Pseudolocality [32, Theorem 1.5])

Given $\delta > 0$, there exists $\gamma > 0$ and a constant $\rho = \rho(n, \delta) \in (0, \infty)$ such that if a mean curvature flow $\{M_t\}_{t \in [-1, 0]}$ satisfies that $M_{-1} \cap B_\rho(0)$ is a Lipschitz graph over the plane $\{x_{n+1} = 0\}$ with Lipschitz constant less than γ and $0 \in M_{-1}$, then $M_t \cap B_\rho(0)$ intersects $B_\delta(0)$ and $M_t \cap B_\delta(0)$ remains a Lipschitz graph over $\{x_{n+1} = 0\}$ with Lipschitz constant less than δ for all $t \in [-1, 0]$.

Definition 5.2 (Fixing the pseudolocality constants)

We will fix $\delta = 10^{-2}$ in the preceding pseudolocality result. We denote the corresponding γ by γ_* and $\rho = \rho_*$. For consistency, we also write $\delta_* = \delta$. We will always assume that $\rho_* \geq 1$.

Definition 5.3 (Scale of the core of the conical shrinker)

For an asymptotically conical self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$, we choose $\underline{R}(\Sigma)$ so that for $x \in \Sigma \setminus B_{\underline{R}(\Sigma)}(0)$, we have that $\Sigma \cap B_{2\rho_*}(x)$ is a Lipschitz graph over $T_x \Sigma$ with Lipschitz constant less than $\gamma_*/2$. Furthermore, we require that the map from the end of \mathcal{C} described in Lemma 2.3 be defined outside of $B_{\underline{R}(\Sigma)-1}$.

It is clear that for an asymptotically conical shrinker, we may take $\underline{R}(\Sigma) < \infty$.

5.2. Scales of hypersurfaces near the shrinker

The definitions here are relevant to the radius at which one can apply a cutoff version of Theorem 4.3.

Definition 5.4 (Shrinker scale)

For $M^n \subset \mathbb{R}^{n+1}$, we define the shrinker scale $\mathbf{R}(M)$ by

$$e^{-\frac{\mathbf{R}(M)^2}{4}} := |\nabla_M F|^2 = \int_M |\phi|^2 \rho d\mathcal{H}^n. \quad (5.1)$$

Definition 5.5 (Rough conical scale)

For $M^n \subset \mathbb{R}^{n+1}$, $\ell \in \mathbb{N}$, and $C_\ell > 0$, we define the rough conical scale $\tilde{\mathbf{r}}_\ell(M)$ to be the largest radius so that $M^n \cap B_{\tilde{\mathbf{r}}_\ell(M)}(0)$ is smooth and

$$|\nabla^{(k)} A_M| \leq C_\ell (1+r)^{-1-k}$$

for $k \in \{0, \dots, \ell+1\}$.

Definition 5.6 (Conical scale)

Fix an asymptotically conical self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$, and choose $\beta_0 = \beta_0(\Sigma) > 0$ as in Theorem 4.3. For a hypersurface $M^n \subset \mathbb{R}^{n+1}$, we define the conical scale

$\mathbf{r}_\ell(M)$ to be the largest radius in $[R(\Sigma), \tilde{\mathbf{r}}_\ell(M)]$ so that there is $u : \Sigma \rightarrow \mathbb{R}$ with

$$\text{graph } u|_{\Sigma \cap B_{\mathbf{r}_\ell(M)}} \subset M \quad \text{and} \quad M \cap B_{\mathbf{r}_\ell(M)-1} \subset \text{graph } u,$$

where $u \in \mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$ with $\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} < \beta_0$.

Definition 5.7 (Core graphical hypothesis)

We say that M satisfies the *core graphical hypothesis*, denoted by $(*_b, \underline{r})$, if $\tilde{\mathbf{r}}_\ell(M) \geq \underline{r}$ and there is $u : \Sigma \cap B_{\underline{r}}(0) \rightarrow \mathbb{R}$ so that

$$\text{graph } u \subset M \quad \text{and} \quad M \cap B_{\underline{r}-1} \subset \text{graph } u$$

with $\|u\|_{C^{\ell+1}(B_{\underline{r}}(0))} \leq b$.

We will always assume that $\underline{r} > \sqrt{2n}$ (so that $\partial B_{\underline{r}}(0)$ expands under the rescaled mean curvature flow).

We fix $b > 0$ to be very small (e.g., $b \ll \beta_0$) in Proposition 7.2.

6. Localizing the Łojasiewicz–Simon inequality

We now localize Theorem 4.3 to hypersurfaces that are not entire graphs over Σ . For the definition of $\lambda(M)$ see Definition A.1.

THEOREM 6.1 (The local Łojasiewicz–Simon inequality)

For $M^n \subset \mathbb{R}^{n+1}$ with $\lambda(M) \leq \lambda_0$, $\gamma \in (1, 2)$, and $R \in [1, \mathbf{r}_\ell(M) - 1]$, we have that

$$|F(M) - F(\Sigma)| \leq C \left(\left(\int_{M \cap B_R(0)} |\phi|^2 \rho \, d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta)}} + R^{\frac{n-4}{2(1-\theta)}} e^{-\frac{R^2}{8(1-\theta)}} + e^{-\frac{R^2}{4\gamma}} \right)$$

for $C = C(\Sigma, \lambda_0, \alpha, \gamma)$. Here $\theta \in (0, \frac{1}{2})$ depends on Σ and the Hölder coefficient α ; θ is fixed in Theorem 4.3.

Proof

By the definition of $\mathbf{r}_\ell(\Sigma)$ (Definition 5.6), there is $u : \Sigma \rightarrow \mathbb{R}$ with

$$\text{graph } u|_{\Sigma \cap B_R(0)} \subset M \quad \text{and} \quad M \cap B_R(0) \subset \text{graph } u$$

with $\|u\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} < \beta_0$. We may thus apply Theorem 4.3 to $\text{graph } u$ to obtain (allowing the constant C to change from line to line as usual)

$$\begin{aligned} |F(M) - F(\Sigma)| &= \left| \int_M \rho \, d\mathcal{H}^n - F(\Sigma) \right| \\ &\leq \left| \int_{M \cap B_R(0)} \rho \, d\mathcal{H}^n - F(\Sigma) \right| + C e^{-\frac{R^2}{4\gamma}} \end{aligned}$$

$$\begin{aligned} &\leq \left| \int_{\text{graph } u} \rho d\mathcal{H}^n - F(\Sigma) \right| + Ce^{-\frac{R^2}{4\gamma}} \\ &\leq C \left(\int_{\text{graph } u} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta)}} + Ce^{-\frac{R^2}{4\gamma}}. \end{aligned}$$

It remains to argue that we can restrict the first integral to $\Sigma \cap B_R(0)$. It is easy to see that $r|\phi_{\text{graph } u}| \leq C\beta_0$ by the definition of $\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)$. Using

$$\int_R^\infty r^{n-3} e^{-\frac{r^2}{4}} dr \lesssim R^{n-4} e^{-\frac{R^2}{4}},$$

we thus obtain

$$\begin{aligned} |F(M) - F(\Sigma)| &\leq C \left(\int_{(\text{graph } u) \cap B_R(0)} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta)}} \\ &\quad + CR^{\frac{n-4}{2(1-\theta)}} e^{-\frac{R^2}{8(1-\theta)}} + Ce^{-\frac{R^2}{4\gamma}}. \end{aligned}$$

This completes the proof. \square

7. Approximate shrinkers up to the rough conical scale

For θ fixed in Theorem 4.3, define

$$\Theta = \left(\frac{1 - \frac{\theta}{2}}{1 - \theta} \right)^{\frac{1}{4}} \in \left(1, \left(\frac{3}{2} \right)^{\frac{1}{4}} \right].$$

Definition 7.1

For $R \geq \underline{r}$, we say that $M^n \subset \mathbb{R}^{n+1}$ is a *roughly conical approximate shrinker up to scale R* if

- (1) we have $\Theta R \leq \tilde{r}_\ell(M)$,
- (2) M satisfies the core graphical hypothesis $(*_b, \underline{r})$, and
- (3) $|\phi| + (1 + |x|)|\nabla \phi| \leq s(1 + |x|)^{-1}$ on $M \cap B_{\Theta R}(0)$.

We will fix s, b sufficiently small in the following proposition giving a lower bound on the conical scale.

PROPOSITION 7.2

Taking ℓ sufficiently large, there are constants $b, s > 0$ sufficiently small, depending on the shrinker Σ , the conical scale constant β_0 , the rough conical scale constant C_ℓ , and the entropy bound λ_0 with the following property. If $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \lambda_0$ and is a roughly conical approximate shrinker up to scale R in the sense of Definition 7.1, then there is a function $u : \Sigma \rightarrow \mathbb{R}$ with

$$\text{graph } u|_{\Sigma \cap B_R(0)} \subset M \quad \text{and} \quad M \cap B_{R-1}(0) \subset \text{graph } u$$

and $\|u\|_{\mathcal{CS}^{2,\alpha}(\Sigma)} \leq \beta_0$. Equivalently, the conical scale satisfies $\mathbf{r}_\ell(M) \geq R$.

Certain aspects of the following proof are inspired by the proof of [33, Theorem 8.9].

Proof

We claim that for b, s sufficiently small, the conclusion eventually holds for any $R \geq \underline{r}$. As such, we will take $b, s \rightarrow 0$ and prove that for any given (sequence of) $R \geq \underline{r}$, the conclusion eventually holds for R . We may assume that $R \rightarrow \infty$ (the subsequent argument is easily modified to the case where R is bounded). It is clear that M converges to Σ in C^ℓ in $B_{\underline{r}-1}$ with multiplicity 1. Moreover, M converges in $C_{\text{loc}}^\ell(\mathbb{R}^{n+1})$ to⁶ M' , which satisfies $\phi \equiv 0$, and is thus a properly⁷ embedded shrinker. Unique continuation implies that⁸ $M' = \Sigma$. Finally, it is clear that M converges to Σ in C^ℓ with multiplicity 1 everywhere by connectedness of Σ and the multiplicity-1 convergence on $B_{\underline{r}-1}$.

Hence, if we let $\hat{R} \in [\underline{r}, R]$ denote the largest radius (depending on b, s) so that the conclusion holds with \hat{R} (in the place of R), then it is clear that $\hat{R} \rightarrow \infty$. We will prove that the proposition holds up to $\tilde{R} := \frac{1}{2}(1 + \Theta)\hat{R}$ (note that this is a fixed factor less than $\Theta\hat{R}$). This will then imply the claim by a straightforward contradiction argument.

First of all, we can assume that $R/\hat{R} \rightarrow \lambda \in [1, \infty]$. Observe that $(\hat{R})^{-1}M$ converges in $C_{\text{loc}}^\ell(B_{\Theta\lambda}(0) \setminus \{0\})$ to a cone $\hat{\mathcal{C}}$ which is a C^ℓ -graph over the original cone \mathcal{C} . Moreover, because we have assumed that the proposition holds up to \hat{R} , we see that the cones are close in the sense that $d_H(\mathcal{C}, \hat{\mathcal{C}}) = O(\beta_0)$.⁹ Thus, we can find a C^ℓ -function $u : \Sigma \cap B_{\tilde{R}}(0) \rightarrow \mathbb{R}$ with

$$\text{graph } u \subset M \quad \text{and} \quad M \cap B_{\tilde{R}-1} \subset \text{graph } u.$$

Moreover, $r^{-1}|u| \leq O(\beta_0)$ on $\Sigma \cap (B_{\tilde{R}}(0) \setminus B_{\underline{r}}(0))$ by the above observation that the blowdown cones are $O(\beta_0)$ -close. Furthermore, the second fundamental form estimates coming from the rough conical scale estimate $\tilde{\mathbf{r}}_\ell(M) \geq \Theta\tilde{R}$ yield

$$|D^{2+k}u| = O(r^{-1-k}) \tag{7.1}$$

⁶By Lemma E.1, M' is a properly embedded hypersurface.

⁷Properness of M' follows from the entropy bounds, which imply local area bounds by Lemma E.1.

⁸Note that there cannot be more than one component of M' . One way to see this is that it would have to lie outside of $B_{\underline{r}}(0)$ and we chose $\underline{r} > \sqrt{2n}$; this would contradict the maximum principle. Alternatively, this follows from the Frankel property for shrinkers (i.e., two properly embedded shrinkers must intersect); see [14, Corollary C.4] for Ilmanen's proof of this fact.

⁹We have written d_H for the Hausdorff distance in \mathbb{S}^{n-1} between the two links of the cones.

on $\Sigma \cap (B_{\Theta\tilde{R}}(0) \setminus B_{\underline{R}}(0))$, for $k \in \{0, \dots, \ell - 1\}$. Finally, because M converges in $C_{\text{loc}}^{\ell}(\mathbb{R}^{n+1})$ to Σ (as $b, s \rightarrow 0$), for $\delta \in (0, \underline{r}^{-1})$ fixed sufficiently small depending only on β_0 (this will be made explicit in the last line of the proof), we can assume that $\|u\|_{C^3(\Sigma \cap B_{2\delta^{-1}}(0))} \leq \delta^3$.

We now relate the smallness condition on ϕ to decay properties of u . These computations are similar to those considered in Section 2.1 for an exact shrinker (except we are now parameterizing M over the shrinker Σ , rather than parameterizing the end of Σ over the cone \mathcal{C} ; this complicates certain aspects of the subsequent computation).

We write $F(p) = p + u(p)v_{\Sigma}(p)$ for the function parameterizing (part of) M over $\Sigma \cap B_{\tilde{R}}(0)$. The computations below will hold for $p \in \Sigma$ with $|p| \in [\underline{R}(\Sigma), \tilde{R}]$, with error terms uniform with respect to $b, s \rightarrow 0$. Recall that we have fixed coordinates (r, ω) on $\Sigma \setminus B_{\underline{R}(\Sigma)}$ in Section 2. In particular, the vector fields ∂_r and ∂_{ω_j} are tangent to Σ .¹⁰

We write

$$v_M(F(p)) = A\partial_r + \sum_{j=1}^{n-1} B_j r^{-1} \partial_{\omega_j} + C v_{\Sigma}(p),$$

where

$$A^2 + \sum_{j=1}^{n-1} B_j^2 + C^2 = 1 + O(r^{-2}) \quad (7.2)$$

by Corollary 2.4 (we emphasize that (r, ω) are the coordinates induced on the end of Σ by the parameterization over \mathcal{C} constructed in Lemma 2.3).

Moreover, for $p \in \Sigma$ with $|p|$ sufficiently large (assuming that ω_j are normal coordinates at ω for $p = (r, \omega)$), we find that

$$\begin{aligned} 0 &= A(1 + O(r^{-2}) - u(p)A_{\Sigma}|_p(\partial_r, \partial_r)) \\ &\quad + \sum_{j=1}^{n-1} B_j(O(r^{-2}) - u(p)A_{\Sigma}|_p(\partial_r, r^{-1}\partial_{\omega_j})) + C(\partial_r u(p)), \\ 0 &= A(O(r^{-2}) - u(p)A_{\Sigma}|_p(\partial_r, r^{-1}\partial_{\omega_i})) \\ &\quad + \sum_{j=1}^{n-1} B_j(\delta_{ij} + O(r^{-2}) - u(p)A_{\Sigma}|_p(r^{-1}\partial_{\omega_i}, r^{-1}\partial_{\omega_j})) + C(r^{-1}\partial_{\omega_i} u(p)). \end{aligned}$$

Now, using Lemma 2.7, we find that

¹⁰In particular, we reiterate that the vector field ∂_r is *not* the Euclidean radial vector field!

$$0 = A(1 + O(r^{-2})) + \sum_{j=1}^{n-1} B_j(O(r^{-2})) + C(\partial_r u(p)) \quad (7.3)$$

and

$$\begin{aligned} 0 = A(O(r^{-2})) + \sum_{j=1}^{n-1} B_j(\delta_{ij} - u(p)A_{\mathcal{C}}|_p(r^{-1}\partial_{\omega_i}, r^{-1}\partial_{\omega_j}) + O(r^{-2+\eta})) \\ + C(r^{-1}\partial_{\omega_i}u(p)). \end{aligned} \quad (7.4)$$

Observe that (7.3) yields (since $A, B, C = O(1)$)

$$A + C(\partial_r u) = O(r^{-2}).$$

Moreover, as long as β_0 is sufficiently small so that $r^{-1}|u(p)| \sup_{\Gamma} |A_{\Gamma}| \leq \frac{1}{2}$, we see that $C^{-1} = O(1)$, that is, C is not tending to zero.¹¹

We now compute

$$\begin{aligned} \langle F(p), v_M(F(p)) \rangle &= \left\langle p + u(p)v_{\Sigma}(p), A\partial_r + \sum_{j=1}^{n-1} B_j r^{-1}\partial_{\omega_j} + C v_{\Sigma}(p) \right\rangle \\ &= A\langle p, \partial_r \rangle + \sum_{j=1}^{n-1} B_j \langle p, r^{-1}\partial_{\omega_j} \rangle + C u(p) + C \langle p, v_{\Sigma}(p) \rangle \\ &= A(r + O(r^{-1})) + C u(p) + 2C H_{\Sigma}(p) + O(r^{-1}) \\ &= C(u(p) - (r + O(r^{-1}))\partial_r u(p)) + O(r^{-1}) \\ &= C(u(p) - r\partial_r u(p)) + O(r^{-1})\partial_r u(p) + O(r^{-1}). \end{aligned} \quad (7.5)$$

We begin by analyzing this expression (below, we will repeat the above derivation to yield more precise estimates). We have that

$$\langle F(p), v_M(F(p)) \rangle = 2\phi(F(p)) + 2H_M(F(p)) = O(r^{-1}).$$

Thus, (7.5) (and $C^{-1} = O(1)$) gives

$$r\partial_r u(p) - u(p) = O(r^{-1})\partial_r u(p) + O(r^{-1}).$$

Thus,

¹¹Indeed, if $C \rightarrow 0$, then this condition on β_0 combined with (7.4) yields $B_i \rightarrow 0$ as well; returning to (7.3) yields $A \rightarrow 0$, which contradicts (7.2).

$$\begin{aligned}
\partial_r \frac{u(p)}{r} &= r^{-2} (r \partial_r u(p) - u(p)) \\
&= O(r^{-3}) \partial_r u(p) + O(r^{-3}) \\
&= O(r^{-2}) \partial_r \frac{u(p)}{r} - O(r^{-4}) u(p) + O(r^{-3}).
\end{aligned}$$

Thus, using $r^{-1}u = O(1)$, we conclude that

$$\partial_r \frac{u(p)}{r} = O(r^{-3}).$$

We integrate this from δ^{-1} to $r \in (\delta^{-1}, \tilde{R}]$ to find that

$$\frac{u(r, \omega)}{r} = \delta u(\delta^{-1}, \omega) + O(\delta^2) + O(r^{-2}) = O(\delta^2) + O(r^{-2}),$$

using the fact that $\|u\|_{C^3(\Sigma \cap B_{\delta^{-1}}(0))} \leq \delta^3$. Thus,

$$u = O(\delta^2)r + O(r^{-1}). \quad (7.6)$$

Note that we immediately get

$$\partial_r u(p) = r^{-1}u(p) + O(r^{-2}) = O(\delta^2) + O(r^{-2}).$$

We now interpolate (7.6) (on balls of radius 1) with the higher derivative estimates from (7.1), using Lemma B.1. This yields

$$\begin{aligned}
|D^2 u| &\lesssim (O(\delta^2)r + O(r^{-1}))^{1-\frac{2}{\ell}} r^{(1-\ell)\frac{2}{\ell}} \\
&= O(\delta^{2-\frac{4}{\ell}})r^{-1} + O(r^{\frac{4}{\ell}-3}).
\end{aligned}$$

Similarly, we can obtain an estimate for the full gradient

$$\begin{aligned}
|Du| &\lesssim (O(\delta^2)r + O(r^{-1}))^{1-\frac{1}{\ell}} r^{(1-\ell)\frac{1}{\ell}} \\
&= O(\delta^{2-\frac{2}{\ell}}) + O(r^{\frac{2}{\ell}-2}).
\end{aligned}$$

Now we return to (7.5) and use this improved decay for the derivatives to derive a sharper equation. Firstly, we note that as long as β_0 is small, as above, using the gradient estimate for u , (7.4), together with (7.6), implies that

$$B_i = O(\delta^{2-\frac{2}{\ell}}),$$

for $r \geq \delta^{-1}$. Finally, using this, $A + C(\partial_r u) = O(r^{-2})$, and (7.2), we find that

$$C = 1 + O(\delta^{2-\frac{2}{\ell}}),$$

for $r \geq \delta^{-1}$.

Now, repeating the derivation used in (7.5), with this additional information on B_j and C , we find that

$$\begin{aligned}
& \langle F(p), v_M(F(p)) \rangle \\
&= \left\langle p + u(p)v_\Sigma(p), A\partial_r + \sum_{j=1}^{n-1} B_j r^{-1} \partial_{\omega_j} + C v_\Sigma(p) \right\rangle \\
&= A\langle p, \partial_r \rangle + \sum_{j=1}^{n-1} B_j \langle p, r^{-1} \partial_{\omega_j} \rangle + C u(p) + C \langle p, v_\Sigma(p) \rangle \\
&= A(r + O(r^{-1})) + C u(p) + 2H_\Sigma(p) + O(\delta^{2-\frac{2}{\ell}} r^{-1}) \\
&= C(u(p) - (r + O(r^{-1}))\partial_r u(p)) + 2H_\Sigma(p) + O(\delta^{2-\frac{2}{\ell}} r^{-1}) \\
&= C(u(p) - r\partial_r u(p)) + O(r^{-1})\partial_r u(p) + 2H_\Sigma(p) + O(\delta^{2-\frac{2}{\ell}} r^{-1}). \quad (7.7)
\end{aligned}$$

We thus have

$$\begin{aligned}
& 2\phi(F(p)) + 2H_M(F(p)) \\
&= C(u(p) - r\partial_r u(p)) + O(r^{-1})\partial_r u(p) + 2H_\Sigma(p) + O(\delta^{2-\frac{2}{\ell}} r^{-1}).
\end{aligned}$$

Moreover, we have that (for ℓ sufficiently large)

$$\begin{aligned}
H_M(F(p)) - H_\Sigma &= (1 + O(|\nabla u|^2)) O(|D^2 u|) \\
&= O(\delta)r^{-1} + O(r^{-2}) = O(\delta)r^{-1},
\end{aligned}$$

since $r \geq \delta^{-1}$. Thus, we find that

$$\partial_r \frac{u(p)}{r} = O(\delta)r^{-3}$$

(assuming that $s \ll \delta$, which can be arranged since we have fixed δ independently of the value of s).

We now define

$$c(\omega) := \frac{u(\tilde{R}, \omega)}{\tilde{R}}$$

and observe that by interpolation of (7.6) with (7.1), we have $\|c\|_{C^{2,\alpha}(\Gamma)} = O(\delta)$. Then we set

$$f(r, \omega) = u(r, \omega) - c(\omega)r.$$

We have that $f(\tilde{R}, \cdot) = 0$ and

$$\partial_r \frac{f(r, \omega)}{r} = O(\delta) r^{-3}.$$

Thus,

$$rf(r, \omega) = O(\delta)(1 - r^2 \tilde{R}^{-2}) = O(\delta).$$

These two expressions imply that $\partial_r f = O(\delta)r^{-2}$.

Moreover, we easily see that $|D^k f| = O(r^{1-k})$ for $k \in \{2, \dots, \ell\}$. Interpolating this (and discarding some unnecessary decay with respect to r), we find that $\|f\|_{C^{2,\alpha}} = O(\delta^{1-\frac{2+\alpha}{\ell}})r^{-1}$, where the Hölder norm is taken on balls of unit size.

These estimates provide $C_{\text{an},-1}^{2,\alpha}$ estimates on f , so it remains to extend f to all of Σ while only increasing these norms by a fixed factor (we can trivially extend $c(\omega)r$). Before we do this, we must obtain improved estimates for $\partial_r^2 f$. Using $C^1 \cap C^\ell \subset C^2$ interpolation applied to the 1-dimensional function $r \mapsto f(r, \omega)$ (for ω fixed but arbitrary), on a unit interval, we see that

$$|\partial_r^2 f(\tilde{R}, \omega)| \lesssim (O(\delta)\tilde{R}^{-2})^{1-\frac{1}{\ell-1}} \tilde{R}^{(1-\ell)\frac{1}{\ell-1}} = O(\delta^{1-\frac{1}{\ell-1}})\tilde{R}^{\frac{2}{\ell-1}-3}.$$

Thus, taking ℓ sufficiently large, we see that

$$\tilde{R}|f(\tilde{R}, \omega)| + \tilde{R}^2|\partial_r f(\tilde{R}, \omega)| + \tilde{R}^2|\partial_r^2 f(\tilde{R}, \omega)| = O(\delta^\mu) \quad (7.8)$$

for some absolute constant $\mu > 0$. In particular, we emphasize that the third term in (7.8) is better than the $C_{\text{an},-1}^{2,\alpha}$ -norm requires (we need this improved estimate when we extend f to all of Σ).

We now define

$$\tilde{f}(r, \omega) := \begin{cases} f(r, \omega) & r \leq \tilde{R}, \\ \partial_r f(\tilde{R}, \omega)(r - \tilde{R}) + \frac{1}{6}\partial_r^2 f(\tilde{R}, \omega)(\tilde{R} + 3 - r)(r - \tilde{R})^2 & r > \tilde{R}. \end{cases}$$

(Recall that $f(\tilde{R}, \cdot) = 0$.) We then fix a cutoff function ζ with $\zeta \equiv 1$ on $(-\infty, 0)$ and $\zeta \equiv 0$ on $(1, \infty)$. Then we set $\hat{f}(r, \omega) = \tilde{f}(r, \omega)\zeta(r - \tilde{R})$. Using (7.8), we easily see that

$$\|\tilde{f}\|_{C_{\text{an},-1}^{2,\alpha}(\Sigma)} = O(\delta^\mu).$$

Thus,

$$\|(c, \tilde{f})\|_{\mathcal{CS}_{-1}^{2,\alpha}(\Sigma)} = O(\delta^\mu).$$

Taking δ sufficiently small depending on β_0 , this concludes the proof. \square

8. The final localized Łojasiewicz–Simon inequality and the rough conical scale

We now show that the error terms in the localized Łojasiewicz–Simon inequality (Theorem 6.1) are small, under the assumption that the rough conical scale is larger than the shrinker scale.

THEOREM 8.1 (The final localized Łojasiewicz–Simon inequality)

Assume that $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \lambda_0$ and $\mathbf{R}(M)$ sufficiently large depending on Σ . Assume that M additionally satisfies the core graphical hypothesis $(*_b, \underline{r})$ and that $\mathbf{R}(M) \leq \tilde{\mathbf{r}}_\ell(M) - 1$. Then,

$$|F(M) - F(\Sigma)| \leq C \left(\int_M |\phi|^2 \rho \, d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta/3)}}$$

for $C = C(\Sigma, \lambda_0, \alpha)$. Note that θ is fixed in Theorem 4.3.

Proof

We first claim that M is a roughly conical approximate shrinker up to scale $R = \Theta^{-2} \mathbf{R}(M)$ in the sense of Definition 7.1. We have already assumed that the first two conditions hold, so it remains to check that

$$|\phi| + (1 + |x|)|\nabla \phi| \leq s(1 + |x|)^{-1}$$

on $M \cap B_{\Theta R}(0)$. We will do this by modifying the proof of [19, Corollary 1.28].

Pick $z \in M \cap B_{\Theta R}(0)$. Set $r_z = (1 + |z|)^{-1}$, so that the Gaussian weight ρ has uniformly bounded oscillation in $B_{r_z}(z)$. Set

$$\psi(z) := \left(\int_{M \cap B_{r_z}(z)} |\phi_M|^2 \rho \, d\mathcal{H}^n \right)^{\frac{1}{2}} \leq e^{-\frac{\mathbf{R}(M)^2}{8}}.$$

Hölder's inequality yields

$$\int_{B_{r_z}(z)} |\phi| \, d\mathcal{H}^n \lesssim r_z^{\frac{n}{2}} e^{\frac{|z|^2}{8}} \left(\int_{B_{r_z}(z)} |\phi|^2 \rho \, d\mathcal{H}^n \right)^{\frac{1}{2}} = r_z^{\frac{n}{2}} e^{\frac{|z|^2}{8}} \psi.$$

Because $1 + \Theta R \leq \tilde{\mathbf{r}}_\ell(M)$, we have that (see Definition 5.5)

$$|\nabla^k \phi| \leq C_\ell (1 + |z|)^{1-k}$$

on $M \cap B_{r_z}(z)$, for $k \in \{1, \dots, \ell\}$ and $z \in M \cap B_{\Theta R}(0)$. Now, by the $L^1 \cap C^\ell \subset C^0$ interpolation inequality described in Lemma B.2, we have that

$$\begin{aligned} (1 + |z|) \sup_{B_{r_z}(z)} |\phi| \\ \leq C \left(r_z^{-1-\frac{n}{2}} e^{\frac{|z|^2}{8}} \psi + (r_z^{\frac{n}{2}} e^{\frac{|z|^2}{8}} \psi)^{a_{\ell,n}} (1 + |z|)^{(1-\ell)(1-a_{\ell,n})-1} \right) \end{aligned}$$

$$\begin{aligned}
&\leq C \left((1 + |z|)^{\frac{n}{2} + 1} e^{\frac{|z|^2}{8}} \psi \right. \\
&\quad \left. + (1 + |z|)^{-a_{\ell,n} \frac{n}{2}} (e^{\frac{|z|^2}{8}} \psi)^{a_{\ell,n}} (1 + |z|)^{(1-\ell)(1-a_{\ell,n})} \right) \\
&\leq C \left(\mathbf{R}(M)^{\frac{n}{2} + 1} e^{-\frac{(1-\Theta-2)\mathbf{R}(M)^2}{8}} + e^{-\frac{a_{\ell,n}(1-\Theta-2)\mathbf{R}(M)^2}{8}} \right).
\end{aligned}$$

The negative powers in the exponentials allow us to arrange that this is smaller than s , as long as $\mathbf{R}(M)$ is sufficiently large. A similar argument can be used to bound $|\nabla \phi|$.

Thus, we see that M is a roughly conical approximate shrinker up to scale R . Proposition 7.2 implies that the strong conical scale satisfies $\mathbf{r}_\ell(M) \geq R$. Thus, we can apply the localized Łojasiewicz–Simon inequality from Theorem 6.1 to find that

$$\begin{aligned}
&|F(M) - F(\Sigma)| \\
&\leq C \left(\left(\int_{M \cap B_R(0)} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta)}} + R^{\frac{n-4}{2(1-\theta)}} e^{-\frac{R^2}{8(1-\theta)}} + e^{-\frac{R^2}{4\gamma}} \right) \\
&\leq C \left(\left(\int_M |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta)}} + R^{\frac{n-4}{2(1-\theta)}} e^{-\frac{R^2}{8(1-\theta)}} + e^{-\frac{R^2}{4\gamma}} \right).
\end{aligned}$$

Note that

$$\begin{aligned}
R^{\frac{n-4}{2(1-\theta)}} e^{-\frac{R^2}{8(1-\theta)}} &= (\Theta^{-2} \mathbf{R}(M))^{\frac{n-4}{2(1-\theta)}} e^{-\frac{\mathbf{R}(M)^2}{8\Theta^4(1-\theta)}} \\
&= (\Theta^{-2} \mathbf{R}(M))^{\frac{n-4}{2(1-\theta)}} e^{-\frac{\mathbf{R}(M)^2}{8(1-\theta/2)}} \\
&\leq C (e^{-\frac{\mathbf{R}(M)^2}{4}})^{\frac{1}{2(1-\theta/3)}} \\
&= C \left(\int_M |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta/3)}}
\end{aligned}$$

and

$$e^{-\frac{R^2}{4\gamma}} = \left(\int_M |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{\Theta^4\gamma}},$$

so choosing $\gamma = 2\Theta^{-4}(1 - \theta/3) \in (1, 2)$, we conclude the proof. \square

9. The uniqueness of conical tangent flows: Proof of Theorem 1.1

Fix \underline{r} sufficiently large in terms of the scale of the core of the conical shrinker $\underline{R}(\Sigma)$, and the pseudolocality radius ρ_* (this choice will be made explicit in Lemma 9.1 below).

Now, consider $\varepsilon = \varepsilon(\Sigma, \underline{r}) > 0$, which will be chosen sufficiently small below. Suppose that $\{M_\tau\}_{\tau \in [-1, \infty)}$ is a rescaled mean curvature flow (Brakke flow) on $[-1, \infty) \times \mathbb{R}^{n+1}$ so that there is

$$u : (\Sigma \cap B_{\varepsilon^{-1}}(0)) \times [-1, \varepsilon^{-2}) \rightarrow \mathbb{R}$$

with

- (1) $\text{graph } u(\cdot, \tau) \subset M_\tau$,
- (2) $M_\tau \cap B_{\varepsilon^{-1}-1} \subset \text{graph } u(\cdot, \tau)$,
- (3) $\|u\|_{C^{\ell+1}(\Sigma \cap B_{\varepsilon^{-1}}(0))} \leq \varepsilon$, and
- (4) $F(M) - F(\Sigma) \leq \varepsilon$.

Here, $\ell \in \mathbb{N}$ controls the number of derivatives in the definition of $\tilde{\mathbf{r}}_\ell$. It has been fixed in Proposition 7.2. We additionally fix λ_0 so that $\lambda(M_0) \leq \lambda_0$ (which implies that $\lambda(M_\tau) \leq \lambda_0$). Finally, we assume that there is a sequence of times $s_k \rightarrow \infty$ so that M_{s_k} converges smoothly on compact subsets of \mathbb{R}^{n+1} to Σ (with multiplicity 1).

Recall that the core graphical hypothesis $(*_b, \underline{r})$ has been defined in Definition 5.7. Define the *graphical time* $\bar{\tau}$ by

$$\bar{\tau} := \sup\{\hat{\tau} \in [-1, \infty) : M_\tau \text{ satisfies } (*_{b, \underline{r}}) \text{ for all } \tau \in [-1, \hat{\tau}]\}.$$

Our first goal is to show that $\bar{\tau} = \infty$. Note that by taking ε sufficiently small (depending on b , \underline{r} , Σ), we can assume that $\bar{\tau}$ is arbitrarily large.

LEMMA 9.1 (The rough conical scale improves rapidly)

There is $\underline{r}_0(\Sigma, \underline{R}(\Sigma), \rho_*)$ sufficiently large so that taking $\underline{r} \geq \underline{r}_0$, $\varepsilon_0 = \varepsilon_0(\Sigma, \underline{r})$ sufficiently small, and fixing $C_\ell = C_\ell(\Sigma, \underline{r})$ sufficiently large in the definition of the rough shrinker scale, we have that $\tilde{\mathbf{r}}_\ell(M_\tau) \geq \frac{1}{2}e^{\frac{\tau}{2}}\underline{r}$ for all $\tau \in [0, \bar{\tau}]$.

Moreover, we can find $u : \Sigma \cap B_{4\underline{r}}(0) \times [0, \bar{\tau}) \rightarrow \mathbb{R}$ with $u(\cdot, \tau)$ uniformly bounded in $C^{\ell+2}$ and with

$$\text{graph } u(\cdot, \tau) \subset M_\tau \quad \text{and} \quad M_\tau \cap B_{4\underline{r}-1} \subset \text{graph } u(\cdot, \tau).$$

Proof

Consider $\tau_0 \in [0, \bar{\tau})$. Note that $t \mapsto \sqrt{-t}M_{(\tau_0 - \log(-t))} := \hat{M}_t^{(\tau_0)}$ is a mean curvature flow for $t \in [-e^{\tau_0}, 0)$ and $\hat{M}_{-1}^{(\tau_0)} = M_{\tau_0}$. Take b sufficiently small in the core graphical hypothesis. Then, by the definition of the pseudolocality scale ρ_* , the scale $\underline{R}(\Sigma)$ of the core of the shrinker and the core graphical scale \underline{r} , we can ensure that for

$$x \in \hat{M}_{-1}^{(\tau_0)} \cap (\overline{B_{\underline{r}-2\rho_*}(0)} \setminus B_{\underline{R}(\Sigma)}(0)),$$

there is some plane Π_x through x so that $\hat{M}_{-1}^{(\tau_0)} \cap B_{\rho_*}(x)$ is a Lipschitz graph over Π_x with Lipschitz constant at most γ_* . Thus, by pseudolocality (Proposition 5.1), $\hat{M}_t^{(\tau_0)} \cap B_{\delta_*}(x)$ is nonempty, and a δ_* -Lipchitz graph over Π_x for all $t \in [-1, 0)$.

We can patch these graphs together to find a family of domains $\hat{\Omega}_t^{(\tau_0)} \subset \Sigma$ with

$$(\Sigma \cap (\overline{B_{\underline{r}-3\rho_*}(0)} \setminus B_{\underline{R}(\Sigma)+1}(0))) \subset \hat{\Omega}_t^{(\tau_0)}$$

and a function $\hat{v}_t^{(\tau_0)} : \Omega_t \rightarrow \mathbb{R}$ so that $\text{graph } \hat{v}_t^{(\tau_0)} \subset \hat{M}_t^{(\tau_0)}$. Using a shrinking sphere as a barrier, we can see that for $t \in [-1, 0)$, this graph describes all of $\hat{M}_{-1}^{(\tau_0)} \cap (\overline{B_{\underline{r}-4\rho_*}(0)} \setminus B_{\underline{R}(\Sigma)+2}(0))$. The shrinking sphere of radius $\underline{r} - 4\rho_*$ at $t = -1$ still contains $B_{\underline{r}-3\rho_*}(0)$ as long as we choose \underline{r} sufficiently large so that

$$(\underline{r} - 4\rho_*)^2 \leq (\underline{r} - 3\rho_*)^2 - 1 \Leftrightarrow \frac{1}{2\rho_*}(5\rho_*^2 + 1) \leq \underline{r}.$$

Now, for $\omega \in (0, 1)$, by applying interior estimates in [23] (cf. [2, Corollary 8.4]) for graphical mean curvature flow, we find that

$$|\nabla^{(k)} A_{\hat{M}_t^{(\tau_0)}}|(x) \leq C = C(\Sigma, \lambda_0, \omega)$$

for $x \in \hat{M}_{-t}^{(\tau_0)} \cap (\overline{B_{\underline{r}-5\rho_*}(0)} \setminus B_{\underline{R}(\Sigma)+3}(0))$, $t \in [-1 + \omega, 0)$, and $k \in \{0, \dots, \ell\}$.

By the definition of the core graphical scale, $\tilde{r}_\ell(M_\tau) \geq \underline{r}$, so the desired curvature estimates hold on $M_\tau \cap B_{\underline{r}}(0)$. Moreover, by taking the parameter ε sufficiently small, we can ensure that the desired estimates hold for $\tau \in [-1, 1]$. On the other hand, for $\tau \in [1, \bar{\tau}]$ and

$$x \in M_\tau \cap (\overline{B_{e^{\frac{\tau}{2}}(\underline{r}-5\rho_*)}(0)} \setminus B_{\underline{r}}(0)),$$

we choose

$$\tau_0 = \tau + 2 \log(|x|^{-1}(\underline{r} - 5\rho_*)) \in [0, \tau).$$

Then,

$$t = -e^{\tau_0-\tau} = -|x|^{-2}(\underline{r} - 5\rho_*)^2 \in [-1 + \omega, 0)$$

for $\omega = \omega(n, \underline{r}) \in (0, 1)$ fixed by

$$\omega := 1 - (1 - 5\underline{r}^{-1}\rho_*)^2 \in (0, 1).$$

Now, we find that the point x is rescaled to

$$\hat{x} := \sqrt{-t}x \in \hat{M}_{-t}^{(\tau_0)} \cap \partial B_{\underline{r}-5\rho_*}(0),$$

so the curvature estimates established above yield

$$|\hat{x}|^{-1-k} |\hat{x}|^{1+k} |\nabla^{(k)} A_{\hat{M}_t^{(\tau_0)}}|(\hat{x}) = |\nabla^{(k)} A_{\hat{M}_t^{(\tau_0)}}|(\hat{x}) \leq C = C(\Sigma, \lambda_0, \underline{r}).$$

Unwinding this, we find that

$$|x|^{1+k} |\nabla^{(k)} A_{M_\tau}|(x) \leq C(\underline{r} - 5\rho_*)^{1+k},$$

for $k \in \{0, \dots, \ell\}$. Thus, by choosing $C_\ell = C_\ell(\Sigma, \lambda_0, \underline{r})$ sufficiently large, we find that $\tilde{\mathbf{r}}_\ell(M_\tau) \geq e^{\frac{\tau}{2}}(\underline{r} - 5\rho_*)$, as claimed. As such, the asserted curvature estimates follow by requiring that $\underline{r} \geq 10\rho_*$.

The above proof also shows that there is a function $u : (\Sigma \cap B_{4\underline{r}}) \times [0, \bar{\tau}) \rightarrow \mathbb{R}$ with

$$\text{graph } u(\cdot, \tau) \subset M_\tau \quad \text{and} \quad M_\tau \cap B_{4\underline{r}-1} \subset \text{graph } u(\cdot, \tau),$$

and so that $u(\cdot, \tau)$ is uniformly bounded in $C^{\ell+2}$. Note that this u agrees with the function in the definition of the core graphical hypothesis, on their common domain of definition. \square

First, suppose that τ is such that $\mathbf{R}(M_\tau) \leq \tilde{\mathbf{r}}_\ell(M_\tau) - 1$. By Theorem 8.1, we have that for $\theta' = \theta/3$,

$$F(M_\tau) - F(\Sigma) \leq C \left(\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2(1-\theta')}},$$

so

$$\begin{aligned} -\frac{d}{d\tau} (F(M_\tau) - F(\Sigma))^{\theta'} &= \theta' (F(M_\tau) - F(\Sigma))^{\theta'-1} \int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \\ &\geq C \left(\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, suppose that τ is such that $\mathbf{R}(M_\tau) > \tilde{\mathbf{r}}_\ell(M_\tau) - 1 \geq \frac{1}{2}e^{\frac{\tau}{2}}\underline{r} - 1$ (by Lemma 9.1). The following coarse estimate will suffice in this case:

$$\left(\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2}} = e^{-\frac{\mathbf{R}(M_\tau)^2}{8}} \leq C e^{-\tau}. \quad (9.1)$$

Thus, we can conclude that for all $\tau \in [0, \bar{\tau})$,

$$C \left(\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2}} \leq -\frac{d}{d\tau} (F(M_\tau) - F(\Sigma))^{\theta'} + e^{-\tau}.$$

Integrating this, we find that for $\tau_0 \in [0, \bar{\tau})$,

$$\begin{aligned} \int_{\tau_0}^{\bar{\tau}} \left(\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \right)^{\frac{1}{2}} d\tau &\lesssim (F(M_{\tau_0}) - F(\Sigma))^{\theta'} + e^{-\tau_0} \\ &\lesssim \varepsilon^{\theta'} + e^{-\tau_0}. \end{aligned} \quad (9.2)$$

For the function $u : (\Sigma \cap B_{2r}) \times [0, \bar{\tau}) \rightarrow \mathbb{R}$ described in Lemma 9.1, we have that

$$\int_{M_\tau} |\phi|^2 \rho d\mathcal{H}^n \geq C \left\| \frac{\partial u}{\partial \tau} \right\|_{L^2(\Sigma \cap B_{4r}(0))}^2,$$

so we see that

$$\sup_{\tau \in [\tau_0, \bar{\tau}]} \|u(\cdot, \tau) - u(\cdot, \tau_0)\|_{L^2(\Sigma \cap B_{4r}(0))} \lesssim \varepsilon^{\theta'} + e^{-\tau_0}.$$

Because $u(\cdot, \tau)$ is uniformly bounded in $C^{\ell+2}$ by Lemma 9.1, by taking ε sufficiently small and $\tau_0 = \frac{1}{2}\varepsilon^{-2}$, we have that

$$\|u(\cdot, \tau_0)\|_{C^{\ell+1}(\Sigma \cap B_{2r}(0))} \leq \frac{b}{4}$$

and

$$\sup_{\tau \in [\tau_0, \bar{\tau}]} \|u(\cdot, \tau) - u(\cdot, \tau_0)\|_{C^{\ell+1}(\Sigma \cap B_{2r}(0))} \leq \frac{b}{4}.$$

Thus, we see that $\|u(\cdot, \tau)\|_{C^{\ell+1}(\Sigma \cap B_{2r}(0))} \leq \frac{b}{2}$ for $\tau \in [0, \bar{\tau})$. This (combined with pseudolocality and interior estimates) implies that we can extend the graphical hypothesis slightly beyond $\bar{\tau}$, a contradiction.

Thus, $\bar{\tau} = \infty$. Now, returning to (9.2), we have that (recall that $s_k \rightarrow \infty$ are so that $M_{s_k} \rightarrow \Sigma$)

$$\begin{aligned} \sup_{\tau \in [s_k, \infty)} \|u(\cdot, \tau) - u(\cdot, s_k)\|_{L^2(\Sigma \cap B_{4r}(0))} &\lesssim \int_{s_k}^{\infty} \left\| \frac{\partial u}{\partial \tau} \right\|_{L^2(\Sigma \cap B_{4r}(0))} d\tau \\ &\lesssim (F(M_{s_k}) - F(\Sigma))^{\theta'} + e^{-s_k}. \end{aligned}$$

Since $u(\cdot, s_k) \rightarrow 0$ in $L^2(\Sigma \cap B_{4r}(0))$, we thus see that $u(\cdot, \tau) \rightarrow 0$ in $L^2(\Sigma \cap B_{4r}(0))$ as $\tau \rightarrow \infty$, and thus in $C^{\ell+1}(\Sigma \cap B_{2r}(0))$.

From this, it is clear that M_τ converges on compact sets to Σ as $\tau \rightarrow \infty$. This completes the proof of Theorem 1.1.

9.1. Rate of convergence

Here, we observe that similar arguments can yield a rate of convergence of M_τ towards Σ . Arguing as above, we have that

$$\begin{aligned} \frac{d}{d\tau}(F(M_\tau) - F(\Sigma)) &= - \int_{M_\tau} |\phi|^2 \rho \, d\mathcal{H}^n \\ &\leq -C(F(M_\tau) - F(\Sigma))^{2(1-\theta')} + Ce^{-2\tau} \end{aligned}$$

for all $\tau \in [0, \infty)$. We claim that

$$F(M_\tau) - F(\Sigma) \leq D(1 + \tau)^{-\frac{1}{1-2\theta'}}$$

for D sufficiently large in terms of M_0 and Σ . Indeed, letting $\tilde{\tau}$ denote the first time this fails, since $e^{-2x} \lesssim (1 + x)^{-\alpha}$ for all $x > 0$, we have that

$$(F(M_{\tilde{\tau}}) - F(\Sigma))^{2(1-\theta')} = D^{2(1-\theta')}(1 + \tilde{\tau})^{-\frac{2(1-\theta')}{1-2\theta'}} \geq cD^{2(1-\theta')}e^{-2\tilde{\tau}}.$$

Thus, as long as D is sufficiently large, we find that

$$\begin{aligned} -\frac{D}{1-2\theta'}(1 + \tilde{\tau})^{-\frac{2(1-\theta')}{1-2\theta'}} &\leq \frac{d}{d\tau}(F(M_\tau) - F(\Sigma))|_{\tau=\tilde{\tau}} \\ &\leq -C(F(M_{\tilde{\tau}}) - F(\Sigma))^{2(1-\theta')} \\ &\leq -CD^{2(1-\theta')}(1 + \tilde{\tau})^{-\frac{2(1-\theta')}{1-2\theta'}}. \end{aligned}$$

Taking D larger if necessary, this yields a contradiction. Thus, we have that for any R fixed,

$$\begin{aligned} \|u(\cdot, \tau)\|_{L^2(\Sigma \cap B_{2R})} &\lesssim \int_\tau^\infty \left(\int_{M_\tau} |\phi|^2 \rho \, d\mathcal{H}^n \right)^{\frac{1}{2}} d\tau \\ &\lesssim (F(M_\tau) - F(\Sigma))^{\theta'} + e^{-\tau} \\ &\lesssim (1 + \tau)^{-\frac{\theta'}{1-2\theta'}}. \end{aligned}$$

Interpolating yields

$$\|u(\cdot, \tau)\|_{C^k(\Sigma \cap B_R)} \lesssim (1 + \tau)^{-\frac{\theta'}{1-2\theta'} + \eta}$$

for any k , R , and $\eta > 0$, as $\tau \rightarrow \infty$.

9.2. Proof of Corollary 1.2

Note that the proof of Theorem 1.1 implies that there exists $\varepsilon > 0$ such that the surfaces $M_t \cap B_\varepsilon(0)$ for $t \in (-\varepsilon^2, 0)$ are smooth graphs over $\sqrt{t} \cdot \Sigma$. Even more, one also

sees that $(M_0 \cap B_\varepsilon(0)) \setminus \{0\}$ is a smooth normal graph over the asymptotic cone C of Σ with curvature bounded by c/r (plus all corresponding higher order derivative estimates). Note that the tangent flow \mathcal{M}_Σ has as the time zero slice the cone C . Thus by taking rescaling limits of the flow, including time zero, we see that the uniqueness of the tangent flow implies that rescalings of M_0 converge smoothly on compact subsets of $\mathbb{R}^n \setminus \{0\}$ to C .

Appendix A. Standard definitions

We recall the following definitions and conventions.

Definition A.1

For $M^n \subset \mathbb{R}^{n+1}$ with polynomial area growth, the *Gaussian area* of M is

$$F(M) = \int_M \rho d\mathcal{H}^n,$$

where $\rho = (4\pi)^{-\frac{n}{2}} e^{-|x|^2/4}$. Recall that the entropy $\lambda(M)$ is defined as the supremum of the Gaussian area over all centers and scales (see [18]).

Definition A.2

A hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is a *self-shrinker* if $\sqrt{-t} \cdot \Sigma$ is a solution to mean curvature flow for $t \in (-\infty, 0)$. This is equivalent to

$$H_\Sigma = \frac{1}{2} \langle x, v_\Sigma \rangle. \quad (\text{A.1})$$

Definition A.3

For a general hypersurface $M^n \subset \mathbb{R}^{n+1}$, we define the function

$$\phi = \phi_M := \frac{1}{2} \langle x, v_M \rangle - H_M.$$

Note that Σ is a self-shrinker if and only if $\phi_\Sigma \equiv 0$.

Definition A.4

A smooth self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$ is (smoothly) *asymptotically conical* if

$$\lim_{t \nearrow 0} \sqrt{-t} \cdot \Sigma = \mathcal{C}$$

in $C_{\text{loc}}^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ with multiplicity 1, where \mathcal{C} is a cone over a smooth closed hypersurface $\Gamma^{n-1} \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$.

Definition A.5

We define the following operators along Σ :

$$\begin{aligned}\mathcal{L}_\gamma u &:= \Delta_\Sigma u - \frac{1}{2} \vec{x} \cdot \nabla_\Sigma u + \gamma u, \\ Lu &:= \mathcal{L}_{\frac{1}{2}} u + |A_\Sigma|^2 u \\ &= \Delta_\Sigma u - \frac{1}{2} (\vec{x} \cdot \nabla_\Sigma u - u) + |A_\Sigma|^2 u.\end{aligned}$$

Note that L is the full second variation of Gaussian area along Σ . Moreover, \mathcal{L}_0 and $\mathcal{L}_{\frac{1}{2}}$ will be particularly relevant in the sequel.

Appendix B. Interpolation inequalities

We recall the following standard interpolation inequalities in multiplicative form.

LEMMA B.1

Suppose that $u \in C^k(B_2)$. Then, for $j < k$,

$$\|D^j u\|_{C^0(B_1)} \leq C \|u\|_{C^0(B_2)}^{1-\frac{j}{k}} \|D^k u\|_{C^0(B_2)}^{\frac{j}{k}}$$

for $C = C(n, k)$. Similarly, if $u \in C^{k,\alpha}(B_2)$, then for $j + \beta < k + \alpha$,

$$[D^j u]_{\beta; B_1} \leq C \|u\|_{C^0(B_2)}^{1-\frac{j+\beta}{k+\alpha}} [D^k u]_{\alpha; B_2}^{\frac{j+\beta}{k+\alpha}}$$

for $C = C(n, k, \alpha, \beta)$.

These follow in a similar manner to the linear inequalities given in [24, Lemma 6.32], except that in the proof one should optimize with respect to the parameter μ rather than just choosing μ sufficiently small. Alternatively, see [28, Lemma A.2].

We will also need the following interpolation inequality.

LEMMA B.2 (cf. [19, Lemma B.1])

If u is a C^k function on $B_{2r} \subset \mathbb{R}^n$, then

$$\begin{aligned}\|u\|_{L^\infty(B_r)} &\leq C (r^{-n} \|u\|_{L^1(B_{2r})} + \|u\|_{L^1(B_{2r})}^{a_{k,n}} \|\nabla^k u\|_{L^\infty(B_{2r})}^{1-a_{k,n}}), \\ r \|\nabla u\|_{L^\infty(B_r)} &\leq C (r^{-n} \|u\|_{L^1(B_{2r})} + \|u\|_{L^1(B_{2r})}^{b_{k,n}} \|\nabla^k u\|_{L^\infty(B_{2r})}^{1-b_{k,n}})\end{aligned}$$

for $C = C(k, n)$, $a_{k,n} = \frac{k}{k+n}$, and $b_{k,n} = \frac{k-1}{k+n}$.

Appendix C. Geometry of normal graphs

We consider hypersurfaces M, N in \mathbb{R}^{n+1} such that N can be locally written as a normal graph over M with height function v , where we assume that the C^1 -norm of u is sufficiently small (depending on the geometry of M). Let $p \in M$, and choose a local parameterization F , parameterizing an open neighborhood U of p in M such that $F(0) = p$. We can assume that $g_{ij} = \langle \partial_i F, \partial_j F \rangle$ satisfies

$$g_{ij}|_{x=0} = \delta_{ij} \quad \text{and} \quad \partial_k g_{ij}|_{x=0} = 0.$$

For simplicity we can furthermore assume that the second fundamental form (h_{ij}) is diagonalized at p with eigenvalues $\lambda_1, \dots, \lambda_n$. A direct calculation (see [46, (2.27)]) yields that the normal vector $v_N(q)$, where $q = p + u(p)v_M(q)$, is colinear to the vector

$$N = - \sum_{i=1}^n \frac{\partial_i u}{1 - \lambda_i u} \partial_i F|_{x=0} + v_M(p).$$

Denoting the shape operator by $S = (h_{ij}^i)$ we see that thus in coordinate-free notation

$$v_N(q) = v^{-1}(-(\text{Id} - uS)^{-1}\nabla^M u + v_M)(p), \quad (\text{C.1})$$

where $v := (1 + |(\text{Id} - uS)^{-1}(\nabla^M u)|^2)^{\frac{1}{2}}$. This implies that

$$\langle q, v_N(q) \rangle = v^{-1}(u + \langle p, v_M(p) \rangle - \langle p, (\text{Id} - uS)^{-1}\nabla^M u \rangle). \quad (\text{C.2})$$

For the induced metric \tilde{g} one obtains in the above coordinates at p (again see [46, (2.32)]),

$$\tilde{g}_{ij} = (1 - \lambda_i u)(1 - \lambda_j u)\delta_{ij} + \partial_i u \partial_j u,$$

which implies that

$$\tilde{g}^{ij} = \frac{\delta^{ij}}{(1 - \lambda_i u)(1 - \lambda_j u)} - v^{-2} \frac{\partial_i u}{(1 - \lambda_i u)^2} \frac{\partial_j u}{(1 - \lambda_j u)^2}. \quad (\text{C.3})$$

Furthermore, from [46, (2.30)] we have

$$\begin{aligned} \tilde{h}_{ij} &= \langle \partial_{ij}^2 F, v_N \rangle \\ &= v^{-1} \left(\frac{\lambda_i}{1 - \lambda_i u} \partial_i u \partial_j u + \frac{\lambda_j}{1 - \lambda_j u} \partial_i u \partial_j u \right. \\ &\quad \left. + \sum_k \frac{u}{1 - \lambda_k u} \partial_k u \partial_i h_{jk} + h_{ij} - \lambda_i \lambda_j u \delta_{ij} + \partial_{ij}^2 u \right), \end{aligned} \quad (\text{C.4})$$

which yields a closed expression for the mean curvature \tilde{H} of N , since $\tilde{H}(p) = \tilde{g}^{ij}(p)\tilde{h}_{ij}(p)$.

Appendix D. Variations of Gaussian area and the Euler–Lagrange equation

Suppose that $M^n \subset \mathbb{R}^{n+1}$ is a normal graph of $v : \Sigma \rightarrow \mathbb{R}$ for a fixed shrinker Σ^n . Recall that the Gaussian area is defined as

$$F(M) := \int_M \rho d\mathcal{H}^n.$$

For \tilde{v} a variation of v (i.e., a variation in the normal direction to Σ), the first variation of F in the direction of \tilde{v} satisfies (see [42])

$$\begin{aligned} \delta_{\tilde{v}} F(M) = & - \int_{\Sigma} \Pi_{T^\perp \Sigma} \left(\vec{H}_M + \frac{x^\perp}{2} \right) \Big|_{x=y+v(y)v_{\Sigma}(y)} \tilde{v}(y) J(y, v, \nabla_{\Sigma} v) \\ & \times \rho(y + v_{\Sigma}(y)) \rho(y)^{-1} \rho(y) d\mathcal{H}^n, \end{aligned}$$

where $\Pi_{T^\perp \Sigma}$ is the projection onto the normal bundle to Σ and

$$J(y, v, \nabla_{\Sigma} v) = \text{Jac}(D \exp_y(v(y)v_{\Sigma}(y)))$$

is the area element.

Hence, the Euler–Lagrange operator \mathcal{M} (with respect to the weighted space L^2_W) satisfies

$$\begin{aligned} \mathcal{M}(v) = & \Pi_{T^\perp \Sigma} \left(\vec{H}_M + \frac{x^\perp}{2} \right) \Big|_{x=y+v(y)v_{\Sigma}(y)} J(y, v, \nabla_{\Sigma} v) \\ & \times \rho(y + v(y)v_{\Sigma}(y)) \rho(y)^{-1}. \end{aligned}$$

It is well known that the linearization of \mathcal{M} at $v = 0$ is the L -operator (cf. [19, Lemma 4.3]).

Appendix E. Area growth bounds from Gaussian area estimates

The following is a well-known fact.

LEMMA E.1

For $M^n \subset \mathbb{R}^{n+1}$ with $\lambda(M) \leq \lambda_0$, there is $C = C(\lambda_0, n)$ so that

$$\mathcal{H}^n(M \cap B_R(x)) \leq CR^n$$

for all $R > 0$ and $x \in \mathbb{R}^{n+1}$.

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