Research Article

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Improved quantitative unique continuation for complex-valued drift equations in the plane

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Abstract: In this article, we investigate the quantitative unique continuation properties of complex-valued solutions to drift equations in the plane. We consider equations of the form $\Delta u + W \cdot \nabla u = 0$ in $\mathbb{R}^2$, where $W = W_1 + iW_2$ with each W_j being real-valued. Under the assumptions that $W_j \in L^{q_j}$ for some $q_1 \in [2, \infty]$, $q_2 \in (2, \infty]$ and that W_2 exhibits rapid decay at infinity, we prove new global unique continuation estimates. This improvement is accomplished by reducing our equations to vector-valued Beltrami systems. Our results rely on a novel order of vanishing estimate combined with a finite iteration scheme.

Keywords: Carleman estimates, elliptic systems, quantitative unique continuation

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1 Introduction

The goal of this paper is to show that under suitable hypotheses, we may establish a stronger quantification of the unique continuation properties of complex-valued solutions to drift equations in $\mathbb{R}^2$ of the form

$$-\Delta u + W \cdot \nabla u = 0. \tag{1.1}$$

Before describing our main results, we recall a few fundamental concepts in unique continuation theory. The partial differential equation (PDE) $Lu = 0$ is said to have the *unique continuation property* (UCP) if whenever u is a solution in Ω and $u \equiv 0$ in an open subset of Ω, then $u \equiv 0$ in Ω. Going further, the equation $Lu = 0$ is said to have the *strong unique continuation property* (SUCP) if whenever u is a solution in Ω and u vanishes to infinite order at some point $x_0 \in \Omega$ (in an appropriate sense), then $u \equiv 0$ in Ω. Therefore, whenever we are in a setting where the SUCP holds, it makes sense to ask the following question:
- What is the fastest rate of decay that a non-trivial solution can have?

This local quantity is referred to as the *order of vanishing* and can be interpreted as a quantification of the SUCP. A related global object is the *rate of decay at infinity*, a quantity that distinguishes between trivial and non-trivial entire solutions based on their asymptotic behavior. Other topics of study in unique continuation theory include doubling indices and nodal (zero) sets of solutions. We refer the reader to [15–17] for recent progress in these related directions. Our current work is related to Landis' conjecture, which seeks to determine the optimal rate of decay at infinity for solutions to Schrödinger equations. As briefly described above,

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order of vanishing estimates are interesting on their own, but these quantities also serve as an important tool in our study of quantitative unique continuation at infinity properties.

In the late 1960s, Landis [13] conjectured that if u is a bounded solution to

$$\Delta u - Vu = 0$$

in $\mathbb{R}^n$, where V is a bounded function and $|u(x)| \lesssim \exp(-c|x|^{1+})$, then $u \equiv 0$. This conjecture was later disproved by Meshkov [19] who constructed non-trivial functions u and V that solve $\Delta u - Vu = 0$ in $\mathbb{R}^2$, where V is bounded and $|u(x)| \lesssim \exp(-c|x|^{4/3})$. Meshkov also proved the following *qualitative unique continuation* result: If $\Delta u - Vu = 0$ in $\mathbb{R}^n$, where V is bounded and $|u(x)| \lesssim \exp(-c|x|^{4/3+})$, then necessarily $u \equiv 0$.

In their work on Anderson localization [1], Bourgain and Kenig established a quantitative version of Meshkov's result. As a first step in their proof, they used three-ball inequalities derived from Carleman estimates to establish order of vanishing estimates for local solutions to Schrödinger equations. Then, through a scaling argument, they showed that if u and V are bounded, and u is normalized so that $|u(0)| \geq 1$, then for sufficiently large values of R,

$$\inf_{|x_0|=R} \|u\|_{L^\infty(B_1(x_0))} \geq \exp\left(-CR^{\frac{4}{3}} \log R\right).$$

Since $\frac{4}{3} > 1$, the constructions of Meshkov, in combination with the qualitative and quantitative unique continuation theorems just described, indicate that Landis' conjecture cannot be true for complex-valued solutions at least in $\mathbb{R}^2$. However, Landis' conjecture still remains open in the general real-valued case.

In recent years, there has been a surge of activity surrounding Landis' conjecture in the real-valued planar setting. The breakthrough article [11] proved a quantitative form of Landis' conjecture under the assumption that the zeroth-order term satisfies $V \geq 0$ a.e. Subsequent papers established analogous results in the settings with variable coefficients [5] and singular lower-order terms [8, 12]. Then it was shown that this theorem still holds when V_- exhibits rapid decay at infinity [6], and when V_- exhibits slow decay at infinity [3]. More recently, Logunov, Malinnikova, Nadirashvili and Nazarov [18] proved Landis' theorem for real-valued solutions in the plane. That is, they established the order of vanishing estimates without having to impose any conditions on V_-.

The work in [12] focuses on quantitative Landis-type theorems for *real-valued* solutions to drift equations in the plane of the form (1.1). One of the main theorems in [12] shows that if $W \in L^q$ for some $q \in [2, \infty]$ and u is a real-valued, bounded, normalized solution to (1.1), then whenever R is sufficiently large, it holds that

$$\inf_{|z_0|=R} \|u\|_{L^\infty(B_1(z_0))} \geq \begin{cases} \exp(-CR^{1-\frac{2}{q}} \log R) & \text{if } q > 2, \\ R^{-C} & \text{if } q = 2. \end{cases} \tag{1.2}$$

In contrast, the article [9] contains quantitative Landis-type theorems for *complex-valued* solutions to elliptic equations in the plane. The related theorem in [9] for drift equations shows that if $W \in L^q$ for some $q \in (2, \infty]$ and u is a complex-valued, bounded, normalized solution to (1.1), then whenever R is sufficiently large, it holds that

$$\inf_{|z_0|=R} \|u\|_{L^\infty(B_1(z_0))} \geq \exp(-CR^2 \log R). \tag{1.3}$$

By comparing the results of (1.2) and (1.3), we see that the rate of decay significantly improves when we restrict to the real-valued setting. In particular, the presence of an imaginary part of W drastically affects the rate of decay of solutions. This current paper is motivated by our desire to understand and quantify the effect that the complex part of W has on the rate of decay at infinity.

Davey [2] and Lin and Wang [14] investigated the quantitative unique continuation properties of solutions to elliptic equations with lower-order terms that exhibit pointwise decay at infinity. The results in [2, 14] imply that if $W \in L^\infty$ exhibits rapid enough polynomial decay at infinity and u is a complex-valued, bounded, normalized solution to (1.1), then whenever $\varepsilon > 0$ and R is sufficiently large, it holds that

$$\inf_{|z_0|=R} \|u\|_{L^\infty(B_1(z_0))} \geq \exp(-R^{1+\varepsilon}). \tag{1.4}$$

We initiated this project with the belief that we could somehow combine the results described by (1.2)–(1.4). As described in Theorem 1.1 below, this is in fact true if we assume that the complex part of W exhibits significant exponential decay at infinity in an appropriate sense that we will quantify.

In order to further understand the motivation for the current setting, we will describe the techniques that led to the estimates in (1.2)–(1.4). Carleman estimate techniques were used in [9], while Carleman estimates were combined with iterative arguments in [2, 14] to prove (1.3) and (1.4), respectively. Such techniques have been used to prove many other results related to Landis' conjecture; see, for example, [1, 4, 10]. The Carleman method is applicable in any dimension and, in some cases, it gives rise to optimal bounds in the complex-valued setting. Since Carleman estimates do not distinguish between real and complex values, a different approach was used in [12] to prove (1.2), where the focus was on real-valued solutions and equations in the plane. The proofs in [3, 5, 6, 8, 11, 12] center around the relationship between second-order elliptic equations in the plane and Beltrami equations. In suitable settings, one can use a second-order PDE to generate a Beltrami equation, a first-order elliptic equation in the complex plane. The similarity principle for solutions to the Beltrami equation, along with Hadamard's three-circle theorem, leads to a three-ball inequality similar to the one derived in [1]. However, these new three-ball inequalities give the precise exponents that could not be achieved with a direct Carleman approach.

In this article, by viewing complex-valued drift equations as systems of real-valued drift equations, we have found a new way to combine many of the ideas mentioned above. First, we show that (1.1) can be realized as a system of real-valued drift equations. Then we show that such real-valued systems can be reduced to vector-valued Betrami equations. This observation is one of the main novelties of this article. Instead of invoking a similarity principle for these systems (as we did in [6]), we rely on $L^p - L^2$ Carleman estimates for the operator $\bar\partial$ (similar to those that were previously developed in [7]) to give rise to our three-ball inequalities. The three-ball inequality is then used to establish the order of vanishing result. If the complex part of the potential function decays sufficiently quickly, then a scaling argument combined with repeated applications of the order of vanishing estimate gives rise to our quantitative unique continuation at infinity estimates. This iterative argument is reminiscent of the ideas in [3, 6], which were inspired by [2, 14].

Before stating the main result of this article, we describe the kinds of potential functions that we will work with. Assume that there exist $q_1 \in [2, \infty]$, $q_2 \in (2, \infty]$ and $c_0, \delta_0 > 0$ so that $W = W_1 + iW_2$, where $W_i : \mathbb{R}^2 \to \mathbb{R}^2$ for $i = 1, 2$, and

$$\|W_1\|_{L^{q_1}(\mathbb{R}^2)} \le 1, \tag{1.5}$$

$$\|W_2\|_{L^{q_2}(B_1(z_0))} \le \exp(-c_0|z_0|^{1-\frac{2}{q_1}+\delta_0}) \quad \text{for all } z_0 \in \mathbb{R}^2. \tag{1.6}$$

In particular, the real part of W satisfies the same hypotheses as it did in [12], while the complex part of W must decay exponentially at a rate that depends on the properties of the real part of W. Since we may allow $q_1 \ge 2$ and $q_2 > 2$, we are able to analyze both the subcritical and critical (for the real part of the drift) scaling regimes.

Now we may state the main result of this article. The following theorem is a quantitative unique continuation at infinity estimate for solutions to (1.1), or a Landis-type theorem for complex-valued drift equations.

Theorem 1.1. *Assume that for some $q_1 \in [2, \infty]$, $q_2 \in (2, \infty]$ and $c_0, \delta_0 > 0$, $W = W_1 + iW_2 : \mathbb{R}^2 \to \mathbb{C}^2$ satisfies* (1.5) *and* (1.6). *Let $u : \mathbb{R}^2 \to \mathbb{C}$ be a solution to* (1.1) *that is bounded and normalized in the sense that for some $t_0 \in [1, 2]$,*

$$|u(z)| \le \exp(C_0|z|^{1-\frac{2}{q_1}}), \tag{1.7}$$

$$\|\nabla u\|_{L^{t_0}(B_1(0))} \ge 1, \tag{1.8}$$

where $t_0 < 2$ when $q_1 = 2$. Then for any $\varepsilon > 0$ and any $R \ge \tilde R(R_0, C_0, q_1, q_2, c_0, \delta_0, t_0, \varepsilon)$, it holds that

$$\inf_{|z_0|=R} \|u\|_{L^\infty(B_1(z_0))} \ge \exp(-R^{1+\varepsilon}).$$

Remark. The value R_0 that appears in this theorem belongs to $(0, \frac{1}{e})$ and is a byproduct of the Carleman estimate that is used in our proofs.

Compared to the results of [12], this rate of decay estimate is more rapid. That is, when we allow for a non-trivial complex part of the potential, even a rapidly decaying part, the order of vanishing jumps from $1 - \frac{2}{q_1}$ to any value greater than 1. On the flipside, this rate of decay is a great improvement over the results of [9] since the power is far below 2. In summary, when we consider equations with a rapidly-decaying complex part of the potential, the resulting rate of decay for solutions falls in between the rates for equations with a purely real potential and equations with a singular complex potential.

This theorem and the Landis-type results in [3, 6] all give the same bound for the rate of decay at infinity. In both [3] and [6], the setting is real-valued and the zeroth-order potential V has a negative part that decays at infinity. In [6], we assume that $V_- = \max\{-V, 0\}$ exhibits (rapid) exponential decay at infinity, quantitatively similar to the assumption that has been placed on W_2 in the current article. In both the current article and [6], we reduce our PDE to a Beltrami system of equations in which the multiplying factor is a 2×2 off-diagonal matrix. To ensure that the non-trivial entries of the matrix are small enough for our techniques to work, we assume that some part of the potential (V_- in [6], W_2 here) is exponentially small. The same unique continuation estimate was shown to hold in [3] when V_- exhibits (slow) polynomial decay at infinity. There, it is observed that if V_- decays polynomially at infinity, then a positive multiplier exists and can be used to transform the PDE into a scalar-valued Beltrami equation. By avoiding the vector-valued setting, we do not need to impose any further decay conditions on the potential functions. In the current setting, we do not see how to avoid the vector-valued setting, either with the introduction of a positive multiplier or through some other technique. As such, we impose the condition that W_2 exhibits rapid decay at infinity. It would be interesting to extend the ideas in this article to complex-valued operators with zeroth-order potentials. However, in order to understand that setting, some new ideas will be required.

To prove our global theorem, we rely on the following order of vanishing estimate. Although this theorem serves as an important tool in the proof of our first result, it also provides a quantification of the strong unique continuation property for local solutions to (1.1). Furthermore, since this theorem allows the real part of W to belong to L^2 instead of L^{2+}, this result serves as an improvement over other known results in this direction; see, for example, [9, Corollary 1]. An alternative order of vanishing theorem appears below within Section 3.

Theorem 1.2. *Let $d \in (1, 2]$. Assume that for some $q_1 \in [2, \infty]$ and $q_2 \in (2, \infty]$, $\|W_j\|_{L^{q_j}(B_d)} \le M_j$ for $j = 1, 2$. Let u be a solution to (1.1) in B_d that satisfies*

$$\|u\|_{L^\infty(B_d)} \le \hat{C}. \tag{1.9}$$

If $q_1 > 2$ and we assume that

$$\|\nabla u\|_{L^2(B_1)} \ge \hat{c}, \tag{1.10}$$

then for any $z_0 \in B_1$ and any r sufficiently small,

$$\|\nabla u\|_{L^2(B_r(z_0))} \ge r^{C_2[1+M_2^{\mu_2}\exp(C_3M_1)]+\frac{c}{\log d}\{C_1M_1+\log[\frac{C_2\hat{C}(1+M_2)}{\hat{c}\sqrt{d-1}}]\}},$$

where

$$\mu_2 = \frac{2q_2}{q_2 - 2}, \quad C_1 = C_1(R_0, q_1), \quad C_2 = C_2(R_0, q_2), \quad C_3 = C_3(R_0, q_1, q_2),$$

and c is universal.

If $q_1 = 2$ and we assume that for some $t_0 \in [1, 2)$,

$$\|\nabla u\|_{L^{t_0}(B_1)} \ge \hat{c}, \tag{1.11}$$

then for any $z_0 \in B_1$, any r sufficiently small, any $q \in (2, q_2)$, any $t \in (\max\{\frac{q}{q-1}, t_0\}, 2)$, and any $t_1 \in (t, 2]$,

$$\|\nabla u\|_{L^{t_1}(B_r(z_0))} \ge r^{C_2[1+M_2^{\mu}\exp(C_3M_1^2)]+\frac{c}{\log d}\{C_1M_1^2+\log[\frac{C_2\hat{C}(1+M_2)}{\hat{c}\sqrt{d-1}}]\}},$$

where

$$\mu = \frac{tq}{tq - q - t}, \quad C_1 = C_1(R_0, q, t_0, t, t_1), \quad C_2 = C_2(R_0, q_2, q, t), \quad C_3 = C_3(R_0, q_2, q),$$

and c is universal.

Remark. If $W_2 \equiv 0$, then $M_2 = 0$ and we recover results on the order of vanishing estimates and the decay rates at infinity (a real version of Theorem 1.1) from [12]. As such, this theorem may be interpreted as a complex perturbation of the real-valued result.

This article is organized as follows. In the next section, Section 2, three-ball inequalities for general vector-valued Beltrami systems are used to prove order of vanishing estimates for solutions to such equations. Section 3 shows how the drift equation (1.1) may be reduced to a vector-valued Beltrami equation. Using these new presentations, we prove the order of vanishing results given by Theorems 1.2 and 3.1. Section 4 shows how Theorem 1.1 follows from Theorem 1.2 through rescaling combined with iteration. When $q_1 > 2$, we must use the alternative order of vanishing estimate described by Theorem 3.1 to initiate the iterative process. As such, this section has been divided into two parts, corresponding to the proof for $q_1 > 2$ and the proof for $q_1 = 2$. The Carleman estimates that are crucial to the proof in Section 2 are presented in Section 5.

2 Estimates for general Beltrami systems

Here we use three-ball inequalities derived from Carleman estimates to prove order of vanishing estimates for solutions to 2-vector equations of the form

$$\bar{\partial}\vec{v} = G\vec{v}, \tag{2.1}$$

where $\vec{v} = (v_1, v_2)$ is some 2-vector and G is a 2×2 matrix function. This is the major tool in proving our order of vanishing estimates for drift equations. The following Carleman estimate for first-order operators is crucial to the arguments. For a very similar estimate, we refer the reader to [7, Theorem 3.1].

Theorem 2.1. *Suppose $p \in (1, 2]$. There exists $R_0 \in (0, \frac{1}{e})$ such that for any τ sufficiently large and any $u \in C_c^\infty(B_{R_0} \setminus \{0\})$, it holds that*

$$\tau^\beta \|(r \log r)^{-1} e^{-\tau\phi(r)} u\|_{L^2(B_{R_0})} \leq C \|r^{1-\frac{2}{p}} (\log r) e^{-\tau\phi(r)} \bar{\partial} u\|_{L^p(B_{R_0})},$$

where

$$\phi(r) = \log r + \tfrac{1}{2}\log(\log r)^2, \quad \beta = 1 - \frac{1}{p}, \quad C = C(p, R_0).$$

The technical proof of this theorem appears below in Section 5. For now, we use this Carleman estimate to prove the following lower bound, which is the main result of this section.

Theorem 2.2. *Let $a \in (1, 2]$. Define $v = |v_1| + |v_2|$, where $\vec{v}$ is a 2-vector solution to (2.1) in B_a with $\|G\|_{L^q(B_a)} \leq M$ for some $q \in (2, \infty]$. Assume that for some $t \in (\frac{q}{q-1}, 2]$ and some $\hat{c} \leq 1 \leq \hat{C}$,*

$$\|v\|_{L^t(B_1)} \geq \hat{c} \quad \text{and} \quad \|v\|_{L^t(B_a)} \leq \hat{C}.$$

Then for any r_0 sufficiently small and any $b \in (1, a)$, it holds that

$$\|v\|_{L^t(B_{r_0})} \geq r_0^{C(1+M^\mu) + c\log(\frac{C\hat{C}}{\hat{c}})/\log b},$$

where $\mu = \frac{tq}{tq-q-t}$, $C = C(q, t, R_0)$, and c is universal.

Remark. The theorem gives the best result (i.e. minimizes μ) when we choose $t = 2$. However, for technical reasons, there will be situations where we need $t < 2$. Therefore, we present the very general result and choose t appropriately in the proofs of our order of vanishing theorems.

Proof. Choose r_0 sufficiently small and $b \in (1, a)$. Let

$$K_1 = \left\{\frac{r_0}{2} \leq |z| \leq r_0\right\}, \quad K_2 = \{r_0 \leq |z| \leq b\}, \quad K_3 = \{b \leq |z| \leq a\}.$$

Set $K = K_1 \cup K_2 \cup K_3 \subset B_a \setminus \{0\}$ and define $\chi \in C_0^\infty(K)$, where $\chi \equiv 1$ on K_2 and $\operatorname{supp} \nabla\chi = K_1 \cup K_3$. Define $\vec{u} = \chi\vec{v}$, where $\vec{v}$ is the solution to $\bar{\partial}\vec{v} = G\vec{v}$.

Since $q \in (2, \infty]$, for any $t \in (\frac{q}{q-1}, 2]$ we have that $p := \frac{qt}{q+t} \in (1, 2]$. For each j, set $\tilde{u}_j(z) = u_j(\frac{a}{R_0} z)$ so that $\operatorname{supp} \tilde{u}_j \subset B_{R_0} \setminus \{0\}$. Then we may apply the Carleman estimate described by Theorem 2.1 with p as chosen to each $\tilde{u}_j$. With $\tilde{u} = |\tilde{u}_1| + |\tilde{u}_2|$ and $\tilde{K} = \frac{R_0}{a} K \subset B_{R_0} \setminus \{0\}$, we see that

$$
\begin{aligned}
\tau^\beta \|(r \log r)^{-1} e^{-\tau\phi(r)} \tilde{u}\|_{L^2(\tilde{K})} &\le \tau^\beta \sum_{j=1,2} \|(r \log r)^{-1} e^{-\tau\phi(r)} \tilde{u}_j\|_{L^2(\tilde{K})} \\
&\le C \sum_{j=1,2} \|r^{1-\frac{2}{p}} (\log r) e^{-\tau\phi(r)} \bar{\partial} \tilde{u}_j\|_{L^p(\tilde{K})},
\end{aligned}
$$

where $r = |z|$ and $\beta = 1 - \frac{1}{p} = 1 - \frac{1}{t} - \frac{1}{q} = \mu^{-1}$. Define $\rho(z) = \frac{R_0}{a}|z| = \frac{R_0}{a} r$. An application of Hölder (since $t \le 2$) and a change of variables shows that

$$
\tau^\beta \|(\rho \log \rho)^{-1} e^{-\tau\phi(\rho)} u\|_{L^t(K)} \le C \sum_{j=1,2} \|\rho^{1-\frac{2}{p}} (\log \rho) e^{-\tau\phi(\rho)} \bar{\partial} u_j\|_{L^p(K)}, \tag{2.2}
$$

where C depends on q, t, R_0.

Note that by (2.1),

$$
\bar{\partial} u_j = \bar{\partial}\chi v_j + \chi \bar{\partial} v_j = \bar{\partial}\chi v_j + \chi \sum_{k=1,2} g_{jk} v_k = \bar{\partial}\chi v_j + \sum_{k=1,2} g_{jk} u_k.
$$

This equation combined with Hölder's inequality shows that for each $j = 1, 2$,

$$
\begin{aligned}
&\|\rho^{1-\frac{2}{p}} (\log \rho) e^{-\tau\phi(\rho)} \bar{\partial} u_j\|_{L^p(K)} \\
&\quad\le \sum_{k=1,2} \|\rho^{1-\frac{2}{p}} (\log \rho) e^{-\tau\phi(\rho)} g_{jk} u_k\|_{L^p(K)} + \|\rho^{1-\frac{2}{p}} (\log \rho) e^{-\tau\phi(\rho)} |\nabla\chi| v_j\|_{L^p(K_1 \cup K_3)} \\
&\quad\le \sum_{k=1,2} \|g_{jk}\|_{L^q(K)} \|\rho^{1-\frac{1}{p}} (\log \rho)\|^2_{L^\infty(K)} \|(\rho \log \rho)^{-1} e^{-\tau\phi(\rho)} u_k\|_{L^t(K)} \\
&\qquad\quad + \|\rho|\nabla\chi|\|_{L^\infty(K_1)} \|\rho^{-\frac{2}{q}}\|_{L^q(K_1)} \|\rho^{-\frac{2}{t}} (\log \rho) e^{-\tau\phi(\rho)} v_j\|_{L^t(K_1)} \\
&\qquad\quad + \|\nabla\chi\|_{L^\infty(K_3)} \|\rho^{1-\frac{2}{q}}\|_{L^q(K_3)} \|\rho^{-\frac{2}{t}} (\log \rho) e^{-\tau\phi(\rho)} v_j\|_{L^t(K_3)}.
\end{aligned}
$$

A computation shows that

$$
\|\rho^{1-\frac{1}{p}} (\log \rho)\|^2_{L^\infty(K)}, \quad \|\rho|\nabla\chi|\|_{L^\infty(K_1)}, \quad \|\rho^{-\frac{2}{q}}\|_{L^q(K_1)}, \quad \|\nabla\chi\|_{L^\infty(K_3)} \|\rho^{1-\frac{2}{q}}\|_{L^q(K_3)}
$$

are bounded by constants depending on R_0 and q. Combining the previous inequality with (2.2) then shows that

$$
\tau^\beta \|(\rho \log \rho)^{-1} e^{-\tau\phi(\rho)} u\|_{L^t(K)} \le CM \|(\rho \log \rho)^{-1} e^{-\tau\phi(\rho)} u\|_{L^t(K)} + C \|\rho^{-\frac{2}{t}} (\log \rho) e^{-\tau\phi(\rho)} v\|_{L^t(K_1 \cup K_3)}.
$$

If $\tau \ge (2CM)^\mu$, then the first term may be absorbed into the left to get

$$
\begin{aligned}
\|e^{-(\tau+1)\phi(\rho)} v\|_{L^t(K_2)} &\le \|e^{-(\tau+1)\phi(\rho)} \chi v\|_{L^t(K)} \\
&\le \|e^{-(\tau+1)\phi(\rho)} u\|_{L^t(K)} \\
&\le C \rho_0^{1-\frac{2}{t}} (\log \rho_0)^2 \|e^{-(\tau+1)\phi(\rho)} v\|_{L^t(K_1)} + C (\log R_0)^2 \|e^{-(\tau+1)\phi(\rho)} v\|_{L^t(K_3)},
\end{aligned}
$$

where we have used the definition of ϕ and introduced $\rho_0 := R_0 r_0 / a$. Replacing $\tau + 1$ with τ and assuming that $\tau \ge C(1 + M^\mu)$, it holds that

$$
\begin{aligned}
\|v\|_{L^t(\{r_0 \le |x| \le 1\})} &\le e^{\tau\phi(\frac{R_0}{a})} \|e^{-\tau\phi(\rho)} v\|_{L^t(K_2)} \\
&\le C e^{\tau\phi(\frac{R_0}{a})} \big[\rho_0^{1-\frac{2}{t}} (\log \rho_0)^2 \|e^{-\tau\phi(\rho)} v\|_{L^t(K_1)} + (\log R_0)^2 \|e^{-\tau\phi(\rho)} v\|_{L^t(K_3)}\big] \\
&\le C \rho_0^{1-\frac{2}{t}} (\log \rho_0)^2 \frac{e^{\tau\phi(R_0 a)}}{e^{\tau\phi(\rho_0/2)}} \|v\|_{L^t(K_1)} + C (\log R_0)^2 \frac{e^{\tau\phi(R_0/a)}}{e^{\tau\phi(R_0 b/a)}} \|v\|_{L^t(K_3)}.
\end{aligned}
$$

Adding $\|v\|_{L^t(B_{r_0})}$ to both sides of the inequality shows that

$$\|v\|_{L^t(B_1)} \le C\rho_0^{1-\frac{2}{t}}(\log \rho_0)^2 e^{\tau(\phi(\frac{R_0}{a})-\phi(\frac{\rho_0}{2}))}\|v\|_{L^t(B_{r_0})} + C(\log R_0)^2 e^{\tau(\phi(\frac{R_0}{a})-\phi(\frac{R_0 b}{a}))}\|v\|_{L^t(B_a)}.$$

Define

$$\kappa = \frac{\phi(R_0 b/a) - \phi(R_0/a)}{\phi(R_0 b/a) - \phi(\rho_0/2)}$$

and set

$$\tau_0 = \frac{\kappa}{\phi(R_0 b/a) - \phi(R_0/a)} \log\Big[\frac{(\log R_0)^2\|v\|_{L^t(B_a)}}{\rho_0^{1-2/t}(\log \rho_0)^2\|v\|_{L^t(B_{r_0})}}\Big].$$

If $\tau_0 \ge C(1+M^\mu)$, then the above computations are valid with this choice of τ and we see that

$$\|v\|_{L^t(B_1)} \le C[\rho_0^{1-\frac{2}{t}}(\log \rho_0)^2\|v\|_{L^t(B_{r_0})}]^\kappa[(\log R_0)^2\|v\|_{L^t(B_a)}]^{1-\kappa}.$$

On the other hand, if $\tau_0 < C(1+M^\mu)$, then

$$\|v\|_{L^t(B_1)} \le \|v\|_{L^t(B_a)} \le \exp\Big[C(1+M^\mu)\Big(\phi\Big(\frac{R_0 b}{a}\Big) - \phi\Big(\frac{\rho_0}{2}\Big)\Big)\Big]\rho_0^{1-\frac{2}{t}}\Big(\frac{\log \rho_0}{\log R_0}\Big)^2\|v\|_{L^t(B_{r_0})}.$$

Adding the previous two inequalities and invoking the assumptions that $\hat{c} \le \|v\|_{L^t(B_1)}$ and $\|v\|_{L^t(B_a)} \le \hat{C}$ shows that

$$\hat{c} \le \mathrm{I} + \Pi,$$

where

$$\mathrm{I} = C[\rho_0^{1-\frac{2}{t}}(\log \rho_0)^2\|v\|_{L^t(B_{r_0})}]^\kappa[(\log R_0)^2\hat{C}]^{1-\kappa}$$
$$\Pi = \exp\Big[C(1+M^\mu)\Big(\phi\Big(\frac{R_0 b}{a}\Big) - \phi\Big(\frac{\rho_0}{2}\Big)\Big)\Big]\rho_0^{1-\frac{2}{t}}\Big(\frac{\log \rho_0}{\log R_0}\Big)^2\|v\|_{L^t(B_{r_0})}.$$

On one hand, if $\mathrm{I} \le \Pi$, then $\hat{c} \le 2\Pi$ so that

$$\|v\|_{L^t(B_{r_0})} \ge \frac{\hat{c}}{2}\rho_0^{\frac{2}{t}-1}\Big(\frac{\log R_0}{\log \rho_0}\Big)^2 \exp\Big[C(1+M^\mu)\Big(\phi\Big(\frac{\rho_0}{2}\Big) - \phi\Big(\frac{R_0 b}{a}\Big)\Big)\Big]$$

Assuming that $r_0 \ll R_0$, we have

$$\phi\Big(\frac{\rho_0}{2}\Big) - \phi\Big(\frac{R_0 b}{a}\Big) \ge c\log r_0,$$

and thus

$$\|v\|_{L^t(B_{r_0})} \ge C\hat{c}(\log R_0)^2 r_0^{C(1+M^\mu)}. \tag{2.3}$$

On the other hand, if $\Pi \le \mathrm{I}$, then

$$\hat{c} \le 2C[\rho_0^{1-\frac{2}{t}}(\log \rho_0)^2\|v\|_{L^t(B_{r_0})}]^\kappa[(\log R_0)^2\hat{C}]^{1-\kappa}.$$

Raising both sides to $\frac{1}{\kappa}$ shows that

$$\|v\|_{L^t(B_{r_0})} \ge \hat{C}\rho_0^{\frac{2}{t}-1}\Big(\frac{\log R_0}{\log \rho_0}\Big)^2\Big[\frac{2C\hat{C}(\log R_0)^2}{\hat{c}}\Big]^{-\frac{1}{\kappa}}.$$

As above, for any $r_0 \ll R_0$,

$$-\frac{1}{\kappa} = \frac{\phi(\rho_0/2) - \phi(R_0 b/a)}{\phi(R_0 b/a) - \phi(R_0/a)} \ge \frac{c\log r_0}{\log b},$$

and thus

$$\|v\|_{L^t(B_{r_0})} \ge \hat{C}(\log R_0)^2 r_0^{c\log[\frac{2C\hat{C}(\log R_0)^2}{\hat{c}}]/\log b}. \tag{2.4}$$

Combining (2.3) and (2.4) leads to the conclusion of Theorem 2.2. □

3 Order of vanishing estimates

This section contains the proofs of our order of vanishing results, Theorem 1.2 and Theorem 3.1 below. The idea underlying our proofs is that we can reduce the PDE given in (1.1) to a first-order Beltrami equation. The novelty here is that the resulting equation is a vector equation instead of a scalar equation as it was in [11, 12]. More specifically, we will show that the elliptic PDE described by (1.1) is equivalent to an equation of the form (2.1).

If $u = u_1 + iu_2$, then the drift equation (1.1) is equivalent to the system

$$\begin{cases} \Delta u_1 = W_1 \cdot \nabla u_1 - W_2 \cdot \nabla u_2, \\ \Delta u_2 = W_1 \cdot \nabla u_2 + W_2 \cdot \nabla u_1. \end{cases} \tag{3.1}$$

Recall that

$$\bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \partial = \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

Using the natural association between 2-vectors and complex values, i.e. $(a, b) \sim a + ib$, we define

$$W_k(u_j) = \begin{cases} \frac{1}{4}\Big(W_k + \overline{W_k}\frac{\bar\partial u_j}{\partial u_j}\Big) & \text{if } \partial u_j \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$4W_k(u_j)\partial u_j = W_k \partial u_j + \overline{W_k}\bar\partial u_j = 2\Re W_k \partial u_j = W_k \cdot \nabla u_j.$$

Then system (3.1) may be rewritten as

$$\begin{cases} \bar\partial\partial u_1 - W_1(u_1)\partial u_1 = -W_2(u_2)\partial u_2, \\ \bar\partial\partial u_2 - W_1(u_2)\partial u_2 = W_2(u_1)\partial u_1. \end{cases}$$

If we define

$$\vec v = \begin{bmatrix} \partial u_1 \\ \partial u_2 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} W_1(u_1) & -W_2(u_2) \\ W_2(u_1) & W_1(u_2) \end{bmatrix}, \tag{3.2}$$

then the system of equations described by (3.1) is equivalent to (2.1).

The following theorem is an alternative order of vanishing estimate. Although Theorem 1.2 is our main order of vanishing estimate, we will use the following result to initiate the proof of Theorem 1.1 in the setting where $q_1 > 2$. This proof is also interesting because it demonstrates how we make use of the Beltrami representation in a simpler setting.

Theorem 3.1. *Assume that $\|W\|_{L^q(B_2)} \le M$ for some $q \in (2, \infty]$. Let u be a solution to* (1.1) *in B_2 that satisfies* (1.9) *with $d = 2$ and* (1.10). *Then for any r sufficiently small,*

$$\|\nabla u\|_{L^2(B_r)} \ge r^{C(1+M^\mu)+c\log[\frac{C\hat C(1+M)}{\hat c}]},$$

where $\mu = \frac{2q}{q-2}$ and $C = C(q, R_0)$.

Remark. An application of the Caccioppoli inequality as in (3.3) below allows us to replace the L^2-norm of the gradient on the left-hand side with the L^∞-norm of the function itself. After such a reduction, this result is essentially the same as the order of vanishing result from [10, Corollary 1]. The proof that we present here is different.

Remark. Consider the case with $q = \infty$. Then $\mu = 2$ and we obtain the well-known order of vanishing estimate for drift equations; see, for example, [2].

Remark. This theorem differs from Theorem 1.2 and, at first glance, it may appear that this theorem is stronger because of the absence of an exponential dependence in the bound. However, this theorem does not cover the case of $q_1 = 2$. Moreover, if $M_2 \ll M_1$, then the bound that we obtain in Theorem 1.2 is better than this one. In a sense, our new result may be interpreted as a perturbation of the order of vanishing results for real-valued solutions to drift equations that appeared in [12]. This theorem holds for complex-valued equations.

Proof. If we define $\vec{v}$ and G as in (3.2), then equation (2.1) holds in B_2. With $v = |v_1| + |v_2|$, we see that $v \sim |\nabla u|$. Therefore, it follows from (1.10) that $\|v\|_{L^2(B_1)} \gtrsim \hat{c}$. By the assumption on W and the fact that $|W_j(u_k)(z)| \le |W_j(z)|$ for all z, we see that $\|G\|_{L^q(B_2)} \le CM$. A standard integration by parts argument shows that whenever $\Delta u = W \cdot \nabla u$ in B_R, where $W \in L^q(B_R)$ for some $q \in [2, \infty]$, we have

$$\|\nabla u\|_{L^2(B_r)} \le C\Big[\Big(1 - \frac{r}{R}\Big)^{-\frac{1}{2}} + R^{1-\frac{2}{q}}\|W\|_{L^q(B_R)}\Big]\|u\|_{L^\infty(B_R)}. \tag{3.3}$$

Combining (3.3) with (1.9) then implies that $\|v\|_{L^2(B_{3/2})} \le \hat{C}(1 + M)$. An application of Theorem 2.2 with $t = 2$ and $a = \frac{3}{2}$ shows that

$$\|\nabla u\|_{L^2(B_r(x_0))} \gtrsim \|v\|_{L^2(B_r(x_0))} \ge r^{C(1+M^\mu)+c\log[\frac{C\hat{C}(1+M)}{\hat{c}}]},$$

as required. □

The Cauchy–Pompeiu operator on B_d is defined by

$$T_{B_d}\omega(z) = \frac{1}{\pi}\int_{B_d} \frac{\omega(\xi)}{\xi - z}\,d\xi, \tag{3.4}$$

where $\omega \in L^q(B_d)$ for some $q \in [2, \infty]$. Returning to the Beltrami system from (3.2) and the preceding line, we take an alternative approach and define

$$v_j = \partial u_j e^{-T(W_1(u_j))} \quad \text{for each } j = 1, 2, \tag{3.5}$$

where we use the notation $T = T_{B_d}$. Then

$$\begin{aligned}\bar{\partial} v_j &= \bar{\partial}(\partial u_j e^{-T(W_1(u_j))})\\ &= [\bar{\partial}\partial u_j - W_1(u_j)\partial u_j]e^{-T(W_1(u_j))}\\ &= (-1)^j W_2(u_{\hat{j}})\partial u_{\hat{j}} e^{-T(W_1(u_j))}\\ &= (-1)^j W_2(u_{\hat{j}}) e^{T[W_1(u_{\hat{j}})-W_1(u_j)]} v_{\hat{j}},\end{aligned}$$

where $\hat{j} = j \pm 1$. If we introduce the vector notation

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & -W_2(u_2)e^{T[W_1(u_2)-W_1(u_1)]} \\ W_2(u_1)e^{-T[W_1(u_2)-W_1(u_1)]} & 0 \end{bmatrix}, \tag{3.6}$$

then (2.1) holds. This is the representation that will be used in the proof of our order of vanishing estimate described by Theorem 1.2.

Before proving that theorem, we establish an L^q-bound for the matrix G given in (3.6). To do this, we have to recall some properties of T. Let $\omega \in L^q$ for some $q \in [2, \infty]$ satisfy $\|\omega\|_{L^q(B_d)} \le M$. The Cauchy–Pompeiu transform of ω is defined as in (3.4). If $q > 2$, then $T(\omega) \in L^\infty$ with $\|T\omega\|_{L^\infty(B_d)} \le CM$, where C depends on q and d. Otherwise, if $q = 2$, then $T(\omega) \in W^{1,2}$ with

$$\|T\omega\|_{W^{1,2}(B_d)} = \|T\omega\|_{L^2(B_d)} + \|\nabla T\omega\|_{L^2(B_d)} \le CM.$$

For further analysis of $T\omega$ in the setting where $q = 2$, we recall the following lemma from [12].

Lemma 3.2 (cf. [12, Lemma 3.3]). *Set $h = T\omega$ for some $\omega \in L^2(B_d)$ with $\|\omega\|_{L^2(B_d)} \le M$. For $s > 0$ and $0 < r \le d$, it holds that*

$$\fint_{B_r} \exp(s|h|) \le Cr^{-sCM}\exp(sCM + s^2CM^2),$$

where we denote

$$\fint_{B_r} f = |B_r|^{-1}\int_{B_r} f.$$

Now we can show that G is bounded in L^q for some $q \in (2, q_2]$.

Lemma 3.3. *Assume that $d \in (1, 2]$ and for some $q_1 \in [2, \infty]$ and $q_2 \in (2, \infty]$, $\|W_j\|_{L^{q_j}(B_d)} \leq M_j$ for $j = 1, 2$. Define the matrix function G as in (3.6). Set $q = q_2$ if $q_1 > 2$, and otherwise choose $q \in (2, q_2)$. Then*

$$\|G\|_{L^q(B_d)} \lesssim M_2 \exp(CM_1^\alpha),$$

where $\alpha = 1$ if $q_1 > 2$, and $\alpha = 2$ otherwise.

Proof. Recall that

$$G_{jj} = 0 \quad \text{and} \quad G_{j\hat{j}} = (-1)^j W_2(u_{\hat{j}}) e^{(-1)^{\hat{j}} T[W_1(u_2) - W_1(u_1)]}.$$

Since $|W_j(u_k)(z)| \leq |W_j(z)|$ for all z, $W_j \in L^{q_j}$ implies that $W_j(u_k) \in L^{q_j}$ as well with the same norm.

If $q_1 > 2$, then

$$\|T[W_1(u_2) - W_1(u_1)]\|_{L^\infty(B_d)} \leq CM_1,$$

and then

$$\|G\|_{L^{q_2}(B_d)} \leq M_2 \exp(CM_1).$$

If $q_1 = 2$, choose $q \in (2, q_2)$ and set $s = \frac{qq_2}{q_2 - q}$. An application of the Hölder inequality shows that

$$\begin{aligned}
\|G_{j\hat{j}}\|_{L^q(B_d)} &= \|W_2(u_{\hat{j}}) e^{(-1)^{\hat{j}} T[W_1(u_2) - W_1(u_1)]}\|_{L^q(B_d)} \\
&\leq \|W_2\|_{L^{q_2}(B_d)} \|e^{T[W_1(u_2) - W_1(u_1)]}\|_{L^s(B_d)} \\
&\leq C_s d^{\frac{2}{s}} M_2 \Big(\fint_{B_d} \exp(s|T[W_1(u_2) - W_1(u_1)]|) \Big)^{\frac{1}{s}} \\
&\leq C_s d^{-CM_1} M_2 \exp(CM_1 + sCM_1^2),
\end{aligned}$$

where the last step invokes Lemma 3.2. The conclusion follows. □

Now we prove the new order of vanishing estimate described by Theorem 1.2.

Proof of Theorem 1.2. Define $\vec{v}$ and G as in (3.5) and (3.6) so that equation (2.1) holds in B_d. Choose $1 < b < a < d$ so that $b - 1 \simeq a - b \simeq d - a$. Then $\log b \simeq \log d$ and $a - b \simeq d - 1$. Set $v = |v_1| + |v_2|$. In order to keep track of the dependencies in the constants, we'll use a subscript notation within this proof.

Assume first that $q_1 > 2$. We see from (1.10) and Hölder's inequality that

$$\begin{aligned}
\hat{c} &\leq \|\nabla u\|_{L^2(B_1)} \\
&\leq \|\nabla u_1\|_{L^2(B_1)} + \|\nabla u_2\|_{L^2(B_1)} \\
&= \|e^{T(W_1(u_1))} v_1\|_{L^2(B_1)} + \|e^{T(W_1(u_2))} v_2\|_{L^2(B_1)} \\
&\leq \|e^{T(W_1(u_1))}\|_{L^\infty(B_1)} \|v_1\|_{L^2(B_1)} + \|e^{T(W_1(u_2))}\|_{L^\infty(B_1)} \|v_2\|_{L^2(B_1)} \\
&\leq \exp(C_{q_1} M_1) \|v\|_{L^2(B_1)}.
\end{aligned}$$

It follows that

$$\|v\|_{L^2(B_1)} \geq \hat{c} \exp(-C_{q_1} M_1).$$

Similarly,

$$\begin{aligned}
\|v\|_{L^2(B_a)} &\leq \|e^{-T(W_1(u_1))} \nabla u_1\|_{L^2(B_a)} + \|e^{-T(W_1(u_2))} \nabla u_2\|_{L^2(B_a)} \\
&\leq \exp(C_{q_1} M_1) \|\nabla u\|_{L^2(B_a)} \\
&\leq \Big(\sqrt{\frac{d}{d - a}} + C_{q_1} M_1 + C_{q_2} M_2 \Big) \exp(C_{q_1} M_1) \|u\|_{L^\infty(B_d)} \\
&\leq \frac{\hat{C}(1 + C_{q_2} M_2)}{\sqrt{d - 1}} \exp(C_{q_1} M_1),
\end{aligned}$$

where we have applied the interior estimate described by (3.3) and the upper bound from (1.9). Since Lemma 3.3 shows that

$$\|G\|_{L^{q_2}(B_d)} \leq M_2 \exp(C_{q_1} M_1),$$

an application of Theorem 2.2 with $t = 2$ shows that

$$\|v\|_{L^2(B_r(x_0))} \geq r^{C_{q_2}\{1+[M_2 \exp(C_{q_1} M_1)]^{\mu_2}\}+\frac{c}{\log d}\{C_{q_1} M_1 + \log[\frac{C\hat{C}(1+C_{q_2} M_2)}{\hat{c}\sqrt{d-1}}]\}}.$$

Since

$$\|v\|_{L^2(B_r)} \leq \exp(C_{q_1} M_1)\|\nabla u\|_{L^2(B_r)},$$

we can rearrange to reach the conclusion of the theorem for the case $q_1 > 2$.

Now we consider $q_1 = 2$. Choose $q \in (2, q_2)$ and $t \in (\max\{\frac{q}{q-1}, t_0\}, 2)$, and then define $t' < \infty$ to satisfy $\frac{1}{t_0} = \frac{1}{t} + \frac{1}{t'}$. It follows from the lower bound in (1.11) and Hölder's inequality that

$$\begin{aligned}
\hat{c} &\leq \|\nabla u\|_{L^{t_0}(B_1)} \\
&\leq \|\nabla u_1\|_{L^{t_0}(B_1)} + \|\nabla u_2\|_{L^{t_0}(B_1)} \\
&\leq \|e^{T(W_1(u_1))}\|_{L^{t'}(B_1)}\|v_1\|_{L^t(B_1)} + \|e^{T(W_1(u_2))}\|_{L^{t'}(B_1)}\|v_2\|_{L^t(B_1)} \\
&\leq \exp(C_{t'} M_1^2)\|v\|_{L^t(B_1)},
\end{aligned}$$

where we have applied Lemma 3.2. Similarly,

$$\begin{aligned}
\|v\|_{L^t(B_a)} &\leq \|e^{-T(W_1(u_1))}\nabla u_1\|_{L^t(B_a)} + \|e^{-T(W_1(u_2))}\nabla u_2\|_{L^t(B_a)} \\
&\leq \exp(C_t M_1^2)\|\nabla u\|_{L^2(B_a)} \\
&\leq \left(\sqrt{\frac{d}{d-a}} + C_2 M_1 + C_{q_2} M_2\right) \exp(C_t M_1^2)\|u\|_{L^\infty(B_d)} \\
&\leq \frac{\hat{C}(1 + C_{q_2} M_2)}{\sqrt{d-1}} \exp(C_t M_1^2).
\end{aligned}$$

Since Lemma 3.3 implies that

$$\|G\|_{L^q(B_d)} \leq M_2 \exp(C_{q,q_2} M_1^2),$$

an application of Theorem 2.2 with our choice of t shows that

$$\|v\|_{L^t(B_r(x_0))} \geq r^{C_{q,t}\{1+[M_2 \exp(C_{q,q_2} M_1^2)]^{\mu}\}+\frac{c}{\log d}\{C_{t,t_0} M_1^2 + \log[\frac{C_{q,t}\hat{C}(1+C_{q_2} M_2)}{\hat{c}\sqrt{d-1}}]\}},$$

where $\mu = \frac{tq}{tq-q-t}$. Since

$$\|v\|_{L^t(B_r)} \leq \exp(C_{t,t_1} M_1^2)\|\nabla u\|_{L^{t_1}(B_r)}$$

for any $t_1 > t$, we reach the conclusion of the theorem after further simplifications. □

4 Unique continuation at infinity estimates

Here we use Theorem 1.2 combined with an iterative argument to prove Theorem 1.1. Our arguments are similar to those that appear in [3, 6], which were inspired by the work of [2, 14]. We prove the theorem for $q_1 > 2$ and $q_1 = 2$ in slightly different ways, and therefore divide this section accordingly.

4.1 The case of $q_1 > 2$

The proof of the theorem relies on an iteration scheme. Therefore, we begin by presenting two propositions that are instrumental to this argument. The first proposition gives the initial estimate, while the second gives the iterative step. The initial estimate is as follows.

Proposition 4.1 (Initial estimate). *Assume that for some $q_1, q_2 \in (2, \infty]$ and $c_0, \delta_0 > 0$,*

$$W = W_1 + iW_2 : \mathbb{R}^2 \to \mathbb{C}^2$$

satisfies (1.5) *and* (1.6). *Let* $u : \mathbb{R}^2 \to \mathbb{C}$ *be a solution to* (1.1) *for which* (1.7) *and* (1.8) *hold. For any* $\varepsilon_0 > 0$ *and any* $S \geq S_b(R_0, C_0, c_0, q_1, q_2, \delta_0, t_0, \varepsilon_0)$, *it holds that*

$$\inf_{|z_0|=S} \|\nabla u\|_{L^2(B_{1/2}(z_0))} \geq \exp(-S^\alpha), \tag{4.1}$$

where $\alpha = \frac{2\hat{q}(\check{q}-2)}{\check{q}(\hat{q}-2)} + \varepsilon_0$ *with* $\hat{q} = \min\{q_1, q_2\}$ *and* $\check{q} = \max\{q_1, q_2\}$.

Proof. Let $\varepsilon_0 > 0$ be given. Assume that S is sufficiently large with respect to $R_0, C_0, c_0, q_1, q_2, \delta_0, t_0, \varepsilon_0$ as we will specify below. Choose $z_0 \in \mathbb{R}^2$ so that $|z_0| = S - 1$. Define

$$\begin{aligned} \tilde{u}(z) &= u(z_0 + Sz), \\ \widetilde{W}(z) &= SW(z_0 + Sz). \end{aligned}$$

Then $\Delta \tilde{u} - \widetilde{W} \cdot \nabla \tilde{u} = 0$ in B_2. Assumption (1.5) implies that

$$\|\widetilde{W}_1\|_{L^{q_1}(B_2)} \leq S\Big(\int_{\mathbb{R}^2} |W_1(z_0 + Sz)|^{q_1}\,dz\Big)^{\frac{1}{q_1}} = S^{1-\frac{2}{q_1}},$$

while (1.6) implies that $\|W_2\|_{L^{q_2}(\mathbb{R}^2)} \leq A(c_0, \delta_0)$, from which it follows that

$$\|\widetilde{W}_2\|_{L^{q_2}(B_2)} \leq S\Big(\int_{\mathbb{R}^2} |W_2(z_0 + Sz)|^{q_2}\,dz\Big)^{\frac{1}{q_2}} \leq AS^{1-\frac{2}{q_2}}.$$

We see that

$$\begin{aligned} \|\widetilde{W}\|_{L^{\hat{q}}(B_2)} &\leq \|\widetilde{W}_1\|_{L^{\hat{q}}(B_2)} + \|\widetilde{W}_2\|_{L^{\hat{q}}(B_2)} \\ &\leq C_{\hat{q},q_1}\|\widetilde{W}_1\|_{L^{q_1}(B_2)} + C_{\hat{q},q_2}\|\widetilde{W}_2\|_{L^{q_2}(B_2)} \\ &\leq C_{\hat{q},q_1} S^{1-\frac{2}{q_1}} + C_{\hat{q},q_2} AS^{1-\frac{2}{q_2}}. \end{aligned}$$

Moreover,

$$\|\tilde{u}\|_{L^\infty(B_2)} \leq \exp[C_0(3S)^{1-\frac{2}{q_1}}],$$

and from (1.8) we have

$$c_{t_0}\|\nabla \tilde{u}\|_{L^2(B_1)} \geq \|\nabla \tilde{u}\|_{L^{t_0}(B_1)} \geq S\|\nabla u\|_{L^{t_0}(B_1(0))} \geq S.$$

Observe that

$$\log\Big\{\exp[C_0(3S)^{1-\frac{2}{q_1}}]\frac{1+C_{\hat{q},q_1}S^{1-\frac{2}{q_1}} + C_{\hat{q},q_2}AS^{1-\frac{2}{q_2}}}{S}\Big\} \leq CS^{1-\frac{2}{q_1}}.$$

Since $\hat{q} > 2$, an application of Theorem 3.1 shows that

$$\|\nabla u\|_{L^2(B_{1/2}(z_0))} = \frac{1}{S}\|\nabla \tilde{u}\|_{L^2(B_{1/2S})} \geq \Big(\frac{1}{2S}\Big)^{C(C_{q,q_1}S^{\frac{q_1-2}{q_1}} + C_{q,q_2}AS^{\frac{q_2-2}{q_2}})^{\frac{2\hat{q}}{\hat{q}-2}}} \geq \exp(-CS^{\frac{2\hat{q}(\check{q}-2)}{\check{q}(\hat{q}-2)}} \log S),$$

where we have assumed that S is large with respect to C_0, q_1, q_2, and A. Assuming further that S is so large that $C \log S \leq S^{\varepsilon_0}(1 - \frac{1}{S})^\alpha$, we see that (4.1) holds, as required. □

Now we present the proposition which will be repeatedly applied in the proof of Theorem 1.1 when $q_1 > 2$.

Proposition 4.2 (Iterative estimate). *Assume that for some* $q_1, q_2 \in (2, \infty]$ *and* $c_0, \delta_0 > 0$,

$$W = W_1 + iW_2 : \mathbb{R}^2 \to \mathbb{C}^2$$

satisfies (1.5) *and* (1.6). *Let* $u : \mathbb{R}^2 \to \mathbb{C}$ *be a solution to* (1.1) *for which* (1.7) *holds. Let*

$$\varepsilon > 0, \quad \varepsilon_1 \in \Big(0, \frac{\delta_0}{1 - \frac{2}{q_1} + \delta_0}\Big).$$

Suppose that for any $S \geq S_r(R_0, C_0, c_0, q_1, q_2, \delta_0, \varepsilon_1, \varepsilon)$ there exists an $\alpha > 1 + \varepsilon$ so that

$$\inf_{|z_0|=S} \|\nabla u\|_{L^2(B_{1/2}(z_0))} \geq \exp(-S^\alpha). \tag{4.2}$$

With $R = S + (\frac{S}{2})^{\frac{1}{1-\varepsilon_1}} - \frac{1}{2}$ and

$$\beta = \begin{cases} \alpha - \dfrac{\alpha - 1}{2}\varepsilon_1 & \text{if } \alpha(1-\varepsilon_1) > 1 - \dfrac{2}{q_1}, \\ 1 - \dfrac{2}{q_1} + 2\varepsilon_1 & \text{otherwise,} \end{cases}$$

it holds that

$$\inf_{|z_1|=R} \|\nabla u\|_{L^2(B_{1/2}(z_1))} \geq \exp(-R^\beta). \tag{4.3}$$

Proof. Define $T = (\frac{S}{2})^{1/(1-\varepsilon_1)}$ and set $d = 1 + \frac{S}{2T}$. Let $z_1 \in \mathbb{R}^2$ be such that $|z_1| = S + T - \frac{1}{2} = R$. Define

$$\begin{aligned} \tilde{u}(z) &= u(z_1 + Tz), \\ \widetilde{W}(z) &= TW(z_1 + Tz). \end{aligned}$$

Then $\Delta \tilde{u} - \widetilde{W} \cdot \nabla \tilde{u} = 0$ in B_d. Assumption (1.5) implies that

$$\|\widetilde{W}_1\|_{L^{q_1}(B_d)} \leq T\Big(\int_{\mathbb{R}^2} |W_1(z_1 + Tz)|^{q_1}\, dz \Big)^{\frac{1}{q_1}} = T^{1-\frac{2}{q_1}},$$

while

$$\|\widetilde{W}_2\|_{L^{q_2}(B_d)} = T\Big(\int_{B_d} |W_2(z_1 + Tz)|^{q_2}\, dz \Big)^{\frac{1}{q_2}} = T^{1-\frac{2}{q_2}} \Big(\int_{B_{Td}(z_1)} |W_2(z)|^{q_2}\, dz \Big)^{\frac{1}{q_2}}.$$

We may cover $B_{Td}(z_1)$ with $N \sim T^2$ balls of radius 1, so it follows from condition (1.6) that

$$\begin{aligned} \|\widetilde{W}_2\|_{L^{q_2}(B_d)} &\leq T^{1-\frac{2}{q_2}} \Big(\sum_{j=1}^{N} \int_{B_1(z_j)} |W_2(z)|^{q_2}\, dz \Big)^{\frac{1}{q_2}} \\ &\leq T^{1-\frac{2}{q_2}} \Big[\sum_{j=1}^{N} \exp(-q_2 c_0 |z_j|^{1-\frac{2}{q_1}+\delta_0}) \Big]^{\frac{1}{q_2}} \\ &\leq T^{1-\frac{2}{q_2}} \Big\{ cT^2 \exp\Big[-q_2 c_0 \Big(\frac{S-1}{2} \Big)^{1-\frac{2}{q_1}+\delta_0} \Big] \Big\}^{\frac{1}{q_2}} \\ &\leq \exp(-\tilde{c}_0 S^{1-\frac{2}{q_1}+\delta_0}), \end{aligned}$$

where we have used that each ball is centered at a distance of at least $\frac{S-1}{2}$ from the origin. Moreover,

$$\|\tilde{u}\|_{L^\infty(B_d)} \leq \exp\Big[C_0 \Big(\frac{3}{2}S + 2T \Big)^{1-\frac{2}{q_1}} \Big] \leq \exp(5^{1-\frac{2}{q_1}} C_0 T^{1-\frac{2}{q_1}}) = \exp(\tilde{C}_0 T^{1-\frac{2}{q_1}}),$$

and from (4.2) we see that with $z_0 := S\frac{z_1}{|z_1|}$,

$$\|\nabla \tilde{u}\|_{L^2(B_1)} \geq T \|\nabla u\|_{L^2(B_{1/2}(z_0))} \geq \exp(-cS^\alpha).$$

We are now in a position to apply Theorem 1.2 to the function $\tilde{u}$. Doing so yields

$$\|\nabla \tilde{u}\|_{L^2(B_{1/2T}(0))} \geq \Big(\frac{1}{2T} \Big)^{C_2[1+\exp(C_3 T^{1-\frac{2}{q_1}} - \tilde{c}_0 \mu_2 S^{1-\frac{2}{q_1}+\delta_0})] + \frac{2cT}{S}[\tilde{C}_1 T^{1-\frac{2}{q_1}} + cS^\alpha + \exp(-\tilde{c}_0 S^{1-\frac{2}{q_1}+\delta_0}) + \log(C_2\sqrt{\frac{2T}{S}})]},$$

where $\tilde{C}_1 = \tilde{C}_0 + C_1$, $\mu_2 = \frac{2q_2}{q_2-2}$ and all of the new constants depend on R_0, q_1, and q_2. If S is sufficiently large in the sense that

$$\tilde{c}_0 \mu_2 S^{1-\frac{2}{q_1}+\delta_0} \geq C_3 \Big(\frac{S}{2} \Big)^{\frac{1-2/q_1}{1-\varepsilon_1}}$$

(which is always possible because of the relationship between ε_1 and δ_0), then

$$\|\nabla u\|_{L^2(B_{1/2}(z_1))} = \frac{1}{T}\|\nabla \tilde{u}\|_{L^2(B_{1/2T}(0))} \geq \left(\frac{1}{2T}\right)^{2C_2 + \frac{2\tilde{c}T}{S}(\tilde{C}_1 T^{1-\frac{2}{q_1}} + cS^{\alpha})}.$$

If $\alpha(1-\varepsilon_1) > 1 - \frac{2}{q_1}$, then $S^{\alpha} > T^{1-\frac{2}{q_1}}$, and thus

$$\|\nabla u\|_{L^2(B_{1/2}(z_1))} \geq \exp(-CT^{\alpha-(\alpha-1)\varepsilon_1}\log T).$$

If S is sufficiently large in the sense that

$$\left(\frac{S}{2}\right)^{\frac{\varepsilon_1\varepsilon}{2(1-\varepsilon_1)}} \geq \frac{C}{1-\varepsilon_1}\log\left(\frac{S}{2}\right),$$

then $R^{\beta} \geq CT^{\alpha-(\alpha-1)\varepsilon_1}\log T$ and it follows that

$$\|\nabla u\|_{L^2(B_{1/2}(z_1))} \geq \exp(-R^{\beta}). \tag{4.4}$$

On the other hand, if $\alpha(1-\varepsilon_1) \leq 1 - \frac{2}{q_1}$, then the first term is dominant and

$$\|\nabla u\|_{L^2(B_{1/2}(z_1))} \geq \exp(-CT^{1-\frac{2}{q_1}+\varepsilon_1}\log T).$$

If S is large enough so that

$$\left(\frac{S}{2}\right)^{\frac{\varepsilon_1}{1-\varepsilon_1}} \geq \frac{C}{1-\varepsilon_1}\log\left(\frac{S}{2}\right),$$

then we again see that (4.4) holds. Since $z_1 \in \mathbb{R}^2$ with $|z_1| = R$ was arbitrary, (4.3) has been shown. □

Now we use Proposition 4.1 followed by repeated applications of Proposition 4.2 to prove Theorem 1.1.

The proof of Theorem 1.1 for $q_1 > 2$. Let $\varepsilon > 0$ be given. Then choose

$$\varepsilon_1 \in \left(0, \min\left\{\frac{\delta_0}{1-\frac{2}{q_1}+\delta_0}, \frac{\frac{2}{q_1}+\frac{\varepsilon}{2}}{1+\frac{\varepsilon}{2}}\right\}\right)$$

and $\varepsilon_0 > 0$. Choose

$$S_0 \geq \max\left\{S_b(R_0, C_0, c_0, q_1, q_2, \delta_0, t_0, \varepsilon_0), S_r\left(R_0, C_0, c_0, q_1, q_2, \delta_0, \varepsilon_1, \frac{\varepsilon}{2}\right)\right\},$$

where S_b and S_r are as given in Propositions 4.1 and 4.2, respectively. Define

$$\alpha_0 = \frac{2\hat{q}(\check{q}-2)}{\check{q}(\hat{q}-2)} + \varepsilon_0,$$

where $\hat{q} = \min\{q_1, q_2\}$ and $\check{q} = \max\{q_1, q_2\}$. An application of Proposition 4.1 shows that

$$\inf_{|z|=S_0}\|\nabla u\|_{L^2(B_{1/2}(z))} \geq \exp(-S_0^{\alpha_0}).$$

By assumption, we have that

$$1 + \frac{\varepsilon}{2} > \frac{1-\frac{2}{q_1}}{1-\varepsilon_1}.$$

Assuming that $\alpha_k > 1 + \frac{\varepsilon}{2}$ for $k = 0, 1, \dots$, we are in the first case of the choice for β from Proposition 4.2, so we recursively define

$$\alpha_{k+1} = \alpha_k - \frac{\alpha_k - 1}{2}\varepsilon_1,$$
$$S_{k+1} = S_k + \left(\frac{S_k}{2}\right)^{\frac{1}{1-\varepsilon_1}} - \frac{1}{2}.$$

Then, for each such k, an application of Proposition 4.2 shows that

$$\inf_{|z|=S_{k+1}}\|\nabla u\|_{L^2(B_{1/2}(z))} \geq \exp(-S_{k+1}^{\alpha_{k+1}}).$$

Observe that $|\alpha_k - \alpha_{k+1}| > \frac{\varepsilon\varepsilon_1}{4}$. Therefore, there exists $M \in \mathbb{N}$ with $M \le N := \lceil 4(\alpha_0 - 1 - \frac{\varepsilon}{2})/\varepsilon\varepsilon_1 \rceil$ so that $\alpha_M > 1 + \frac{\varepsilon}{2}$, while $\alpha_{M+1} \le 1 + \frac{\varepsilon}{2}$. In particular, for any $R \ge S_{N+1} \ge S_{M+1}$, it holds that

$$\inf_{|z|=R} \|\nabla u\|_{L^2(B_{1/2}(z))} \ge \exp(-R^{\alpha_{M+1}}) \ge \exp(-R^{1+\frac{\varepsilon}{2}}).$$

An application of the Caccioppoli inequality described by (3.3) shows that

$$\|\nabla u\|_{L^2(B_{1/2}(z))} \le C(1 + \|W_1\|_{L^{q_1}} + \|W_2\|_{L^{q_2}})\|u\|_{L^\infty(B_1(z))} \le C\|u\|_{L^\infty(B_1(z))} \le \exp(R^{\frac{\varepsilon}{2}})\|u\|_{L^\infty(B_1(z))},$$

assuming that R is sufficiently large with respect to C. Combining the previous two inequalities leads to the conclusion of the theorem. □

Remark. The careful reader may wonder why we have avoided using the second case of the choice for β, i.e. $\beta = 1 - \frac{2}{q_1} + 2\varepsilon_1$, from Proposition 4.2 in our iteration scheme. As the initial exponent is greater than 2, we must always start in the first case. Each repeated application of Proposition 4.2 will produce an exponent that is greater than 1. Therefore, the only way to move into the second case of β is by choosing ε_1 so that $\alpha(1 - \varepsilon_1) \le 1 - \frac{2}{q_1}$. Doing so implies that $\varepsilon_1 > \frac{2}{q_1}$, and then the resulting exponent is given by

$$\beta = 1 - \frac{2}{q_1} + 2\varepsilon_1 > 1 + \varepsilon_1,$$

which still exceeds 1. In other words, the second case of β does not lead to any improvements, so we have chosen to avoid using this case.

4.2 The case of $q_1 = 2$

Now we consider the case where W_1 belongs to the threshold space L^2. In contrast to the previous cases where $q_1 > 2$, here we only need to run the iteration process twice.

The proof of Theorem 1.1 for $q_1 = 2$. Choose $q \in (2, q_2)$. With

$$\nu = \frac{1}{4}\Big(2 - \max\Big\{\frac{q}{q-1}, t_0\Big\}\Big) > 0,$$

define $t_j = t_0 + j\nu$ for $j = 1, 2, 3$. Define

$$\alpha > \Big(1 - \frac{2}{q_2}\Big)\frac{t_1 q}{t_1 q - q - t_1} > 2.$$

For $\varepsilon \in (0, 1)$ as given, define $\varepsilon_0 = \frac{\varepsilon}{2(\alpha-1)}$.

Assume that S is sufficiently large with respect to $R_0, C_0, q_2, c_0, \delta_0, t_0, \varepsilon$, as well as q, t_1, t_2, t_3, α (which depend on the other terms), as we will specify below. Choose $z_0 \in \mathbb{R}^2$ so that $|z_0| = S - 1$. Define

$$u_0(z) = u(z_0 + Sz),$$
$$W_0(z) = SW(z_0 + Sz).$$

Then $\Delta u_0 - W_0 \cdot \nabla u_0 = 0$ in B_2. Assumption (1.5) implies that

$$\|W_{0,1}\|_{L^2(B_2)} \le S\Big(\int_{\mathbb{R}^2} |W_1(z_0 + Sz)|^2 dz\Big)^{\frac{1}{2}} = 1,$$

while (1.6) implies that $\|W_2\|_{L^{q_2}(\mathbb{R}^2)} \le A(c_0, \delta_0)$, from which it follows that

$$\|W_{0,2}\|_{L^{q_2}(B_2)} = S\Big(\int_{\mathbb{R}^2} |W_2(z_0 + Sz)|^{q_2} dz\Big)^{\frac{1}{q_2}} \le AS^{1-\frac{2}{q_2}}.$$

Moreover, $\|u_0\|_{L^\infty(B_2)} \le e^{C_0}$ and from (1.8) we see that

$$\|\nabla u_0\|_{L^{t_0}(B_1)} \ge S\|\nabla u\|_{L^{t_0}(B_1(0))} \ge S.$$

An application of Theorem 1.2 with $d = 2$ shows that

$$\begin{aligned}\|\nabla u\|_{L^{t_2}(B_{1/2}(z_0))} &= \frac{1}{S}\|\nabla u_0\|_{L^{t_2}(B_{1/2S})}\\ &\geq \Big(\frac{1}{2S}\Big)^{C_2[1+(AS^{1-\frac{2}{q_2}})^{\frac{t_1 q}{t_1 q-q-t_1}} e^{C_3}]+cC_1+c\log[\frac{C_2 e^{C_0}}{S}(1+AS^{1-\frac{2}{q_2}})]}\\ &\geq \exp(-CS^{(1-\frac{2}{q_2})\frac{t_1 q}{t_1 q-q-t_1}}\log S),\end{aligned}$$

where we have assumed that S is large enough to absorb all of the other terms into the dominant one by making the constant larger. Assuming further that S is so large that

$$C\log S \leq S^{\alpha-(1-\frac{2}{q_2})\frac{t_1 q}{t_1 q-q-t_1}}\Big(1-\frac{1}{S}\Big)^{\alpha},$$

we see that

$$\|\nabla u\|_{L^{t_2}(B_{1/2}(z_0))} \geq \exp(-|z_0|^{\alpha}) \quad \text{whenever } |z_0| \gg 1. \tag{4.5}$$

Recalling that

$$\varepsilon_0 = \frac{\varepsilon}{2(\alpha-1)},$$

define $T = (\frac{S}{2})^{1/\varepsilon_0}$ and set $d = 1 + \frac{S}{2T}$. Let $z_1 \in \mathbb{R}^2$ be such that $|z_1| = S + T - \frac{1}{2} = R$. With

$$\begin{aligned}\tilde{u}(z) &= u(z_1 + Tz),\\ \widetilde{W}(z) &= TW(z_1 + Tz),\end{aligned}$$

we see that $\Delta\tilde{u} - \widetilde{W}\cdot\nabla\tilde{u} = 0$ in B_d. As in the previous proof, assumption (1.5) implies that $\|\widetilde{W}_1\|_{L^2(B_d)} \leq 1$, while

$$\|\widetilde{W}_2\|_{L^{q_2}(B_d)} = T\Big(\int_{B_d}|W_2(z_1+Tz)|^{q_2}\,dz\Big)^{\frac{1}{q_2}} = T^{1-\frac{2}{q_2}}\Big(\int_{B_{Td}(z_1)}|W_2(z)|^{q_2}\,dz\Big)^{\frac{1}{q_2}}.$$

We may cover $B_{Td}(z_1)$ with $N \sim T^2$ balls of radius 1, so it follows from condition (1.6) that

$$\begin{aligned}\|\widetilde{W}_2\|_{L^{q_2}(B_d)} &\leq T^{1-\frac{2}{q_2}}\Big(\sum_{j=1}^{N}\int_{B_1(z_j)}|W_2(z)|^{q_2}\,dz\Big)^{\frac{1}{q_2}}\\ &\leq T^{1-\frac{2}{q_2}}\Big[\sum_{j=1}^{N}\exp(-q_2 c_0|z_j|^{\delta_0})\Big]^{\frac{1}{q_2}}\\ &\leq T^{1-\frac{2}{q_2}}\Big\{cT^2\exp\Big[-q_2 c_0\Big(\frac{S-1}{2}\Big)^{\delta_0}\Big]\Big\}^{\frac{1}{q_2}}\\ &\leq \exp(-\tilde{c}_0 S^{\delta_0}),\end{aligned}$$

where we have used that each ball is centered at a distance of at least $\frac{S-1}{2}$ from the origin. Moreover,

$$\|\tilde{u}\|_{L^\infty(B_d)} \leq e^{C_0},$$

and from (4.5) we see that, with $z_0 := S\frac{z_1}{|z_1|}$,

$$\|\nabla\tilde{u}\|_{L^{t_1}(B_1)} \geq T\|\nabla u\|_{L^{t_1}(B_{1/2}(z_0))} \geq \exp(-cS^{\alpha}).$$

Now we apply the order of vanishing estimate described by Theorem 1.2 again. With t_3 as defined above and $\mu = \frac{t_3 q}{t_3 q-q-t_3}$, we have

$$\begin{aligned}\|\nabla u\|_{L^2(B_{1/2}(z_1))} &= \frac{1}{T}\|\nabla\tilde{u}\|_{L^2(B_{1/2T})}\\ &\geq \Big(\frac{1}{2T}\Big)^{C_2[1+\exp(C_3-\tilde{c}_0\mu S^{\delta_0})]+\frac{2cT}{S}[C_1+C_0+cS^{\alpha}+\exp(-\tilde{c}_0 S^{\delta_0})+\log(C_2\sqrt{\frac{2T}{S}})]}\\ &\geq \exp(-CT^{1+(\alpha-1)\varepsilon_0}\log T),\end{aligned}$$

where we have used that S is large enough to absorb all other terms into the dominant one. Further, assuming that

$$\log\Big(\frac{S}{2}\Big) \le \frac{\varepsilon_0}{C}\Big(\frac{S}{2}\Big)^{\frac{\varepsilon}{4\varepsilon_0}} = \frac{\varepsilon}{2C(\alpha-1)}\Big(\frac{S}{2}\Big)^{\frac{\alpha-1}{2}}$$

shows that $C \log T \le T^{\varepsilon/4}$, from which it follows that

$$CT^{1+(\alpha-1)\varepsilon_0}\log T \le R^{1+\frac{3\varepsilon}{4}}.$$

As in the previous proof, if R is sufficiently large, then an application of the Caccioppoli inequality shows that

$$\|\nabla u\|_{L^2(B_{1/2}(z_1))} \le C(1 + \|W_1\|_{L^{q_1}} + \|W_2\|_{L^{q_2}})\|u\|_{L^\infty(B_1(z_1))} \le C\|u\|_{L^\infty(B_1(z_1))} \le \exp(R^{\frac{\varepsilon}{4}})\|u\|_{L^\infty(B_1(z_1))}.$$

It follows that

$$\|u\|_{L^\infty(B_1(z_1))} \ge \exp(-R^{1+\varepsilon}).$$

Since z_1 was an arbitrary point of sufficient distance to the origin, the conclusion of the theorem follows. □

5 Carleman estimates

In this section, we prove the Carleman estimate given by Theorem 2.1. To do this, we rewrite the operator in polar coordinates and then use an eigenvalue decomposition to establish our stated bounds. The techniques used here are very similar to those that appeared in [3, 7, 9, 10] and the references therein.

We use standard polar coordinates in $\mathbb{R}^2 \setminus \{0\}$ by setting $x = r\cos\theta$ and $y = r\sin\theta$, where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(\frac{y}{x})$. With the new coordinate $t = \log r$, we see that

$$\partial_x = e^{-t}\Big(\cos\theta\frac{\partial}{\partial t} - \sin\theta\frac{\partial}{\partial\theta}\Big), \quad \partial_y = e^{-t}\Big(\sin\theta\frac{\partial}{\partial t} + \cos\theta\frac{\partial}{\partial\theta}\Big),$$

so that

$$\mathcal{L} := 2e^{t-i\theta}\bar\partial = \partial_t + i\partial_\theta. \tag{5.1}$$

The eigenvalues of ∂_θ are ik, $k \in \mathbb{Z}$, with corresponding eigenspace $E_k = \operatorname{span}\{e_k\}$, where $e_k = \frac{1}{\sqrt{2\pi}}e^{ik\theta}$ so that $\|e_k\|_{L^2(S^1)} = 1$. For any $v \in L^2(S^1)$, let $P_k v = v_k$ denote the projection of v onto E_k. We remark that the projection operator P_k acts only on the angular variables. In particular, $P_k v(t,\theta) = P_k v(t,\cdot)(\theta)$. We may then rewrite the operator $\mathcal{L}$ as

$$\mathcal{L} = \partial_t - \sum_{k\in\mathbb{Z}} kP_k. \tag{5.2}$$

By changing to the variable $t = \log|z|$, the weight function is given by

$$\varphi(t) = t + \tfrac{1}{2}\log t^2.$$

Since our result applies to functions that are supported in $B_{R_0} \setminus \{0\}$, in terms of the new coordinate t, we study the case when t is sufficiently close to $-\infty$. By a slight modification to the result described by [9, Lemma 2] (see also [7, Lemma 5.1]), we get the following lemma. For the proof of this result, we refer the reader to either [9] or [7].

Lemma 5.1. *Let $M, N \in \mathbb{N}$ and let $\{c_k\}$ be a sequence of numbers such that $|c_k| \le 1$ for all k. For any $v \in L^2(S^1)$ and every $p \in [1,2]$, we have that*

$$\Big\|\sum_{k=N}^{M} c_k P_k v\Big\|_{L^2(S^1)} \le C\Big(\sum_{k=N}^{M}|c_k|^2\Big)^{\frac{1}{p}-\frac{1}{2}}\|v\|_{L^p(S^1)}, \tag{5.3}$$

where $C = C(p)$.

The following proposition is crucial to the proof of Theorem 2.1.

Proposition 5.2. *Let $p \in (1,2]$. There exists a $t_0 < 0$ such that for any $\tau \gg 1$ and any $u \in C_c^\infty((-\infty, t_0) \times S^1)$, it holds that*

$$\|t^{-1} e^{-\tau\varphi(t)} u\|_{L^2(dt\,d\theta)} \leq C\tau^{-1+\frac{1}{p}} \|t e^{-\tau\varphi(t)} \mathcal{L} u\|_{L^p(dt\,d\theta)}, \tag{5.4}$$

where $C = C(p, t_0)$.

Proof. To prove this lemma, we introduce the conjugated operator $\mathcal{L}_\tau$ of $\mathcal{L}$, defined by

$$\mathcal{L}_\tau v = e^{-\tau\varphi(t)} \mathcal{L}(e^{\tau\varphi(t)} v).$$

With $u = e^{\tau\varphi(t)} v$, inequality (5.4) is equivalent to

$$\|t^{-1} v\|_{L^2(dt\,d\theta)} \leq C\tau^{-1+\frac{1}{p}} \|t \mathcal{L}_\tau v\|_{L^p(dt\,d\theta)}. \tag{5.5}$$

From (5.1) and (5.2), the operator $\mathcal{L}_\tau$ takes the form

$$\mathcal{L}_\tau = \sum_{k\in\mathbb{Z}} (\partial_t + \tau\varphi'(t) - k) P_k = \sum_{k\in\mathbb{Z}} (\partial_t + \tau + \tau t^{-1} - k) P_k. \tag{5.6}$$

We first consider $p = 2$. Since $\mathcal{L}_\tau v = \partial_t v + \tau(1 + t^{-1}) v - \sum_k k v_k$, an integration by parts shows that

$$\begin{aligned}
\|\mathcal{L}_\tau v\|^2_{L^2(dt\,d\theta)} &= \iint \Big|\partial_t v + \tau(1 + t^{-1}) v - \sum_{k\in\mathbb{Z}} k v_k\Big|^2 \,dt\,d\theta \\
&= \iint |\partial_t v|^2 \,dt\,d\theta + \iint \sum_k [\tau(1 + t^{-1}) - k]^2 |v_k|^2 \,dt\,d\theta \\
&\quad + \iint \tau(1 + t^{-1}) \partial_t |v|^2 \,dt\,d\theta - \iint \sum_{k\in\mathbb{Z}} k \partial_t |v_k|^2 \,dt\,d\theta \\
&\geq \tau \|t^{-1} v\|^2_{L^2(dt\,d\theta)},
\end{aligned}$$

which implies (5.5) when $p = 2$.

Now we consider all $p \in (1,2)$. Since $\sum_{k\in\mathbb{Z}} P_k v = v$, we split the sum into three parts. Let $M = \lceil 2\tau \rceil$ and define

$$P_\tau^h = \sum_{k>M} P_k, \quad P_\tau^l = \sum_{k=0}^{M} P_k, \quad P_\tau^n = \sum_{k<0} P_k.$$

In order to prove (5.5), it suffices to show that for any $p \in (1,2)$ and any $v \in C_c^\infty((-\infty, t_0) \times S^1)$,

$$\|t^{-1} P_\tau^\square v\|_{L^2(dt\,d\theta)} \leq C\tau^{-1+\frac{1}{p}} \|t \mathcal{L}_\tau v\|_{L^p(dt\,d\theta)} \tag{5.7}$$

for $\square = h, l, n$. The sum of all three inequalities will yield (5.5), which implies (5.4).

From (5.6), we have the first-order differential equation

$$P_k \mathcal{L}_\tau v = (\partial_t + \tau\varphi'(t) - k) P_k v.$$

For $v \in C_c^\infty((-\infty, t_0) \times S^1)$, solving the first-order differential equation gives that

$$P_k v(t,\theta) = -\int_t^\infty e^{k(t-s)+\tau(\varphi(s)-\varphi(t))} P_k \mathcal{L}_\tau v(s,\theta)\,ds = \int_{-\infty}^t e^{k(t-s)+\tau(\varphi(s)-\varphi(t))} P_k \mathcal{L}_\tau v(s,\theta)\,ds. \tag{5.8}$$

We first establish (5.7) with $\square = h$ using the first line of (5.8). For $k > M \geq 2\tau$, if $-\infty < t \leq s \leq t_0 < 0$, then

$$k(t-s) + \tau(\varphi(s) - \varphi(t)) = -(k - \tau)|t - s| + \frac{\tau}{2} \log\Big(\frac{s^2}{t^2}\Big) \leq -\frac{k}{2}|t - s|.$$

Taking the $L^2(S^1)$-norm in (5.8) and using this bound gives that

$$\|P_k v(t,\cdot)\|_{L^2(S^1)} \leq \int_{-\infty}^\infty e^{-\frac{1}{2}k|t-s|} \|P_k \mathcal{L}_\tau v(s,\cdot)\|_{L^2(S^1)}\,ds.$$

With the aid of (5.3), we get

$$\|P_k v(t,\cdot)\|_{L^2(S^1)} \leq C \int_{-\infty}^{\infty} e^{-\frac{1}{2}k|t-s|} \|\mathcal{L}_\tau v(s,\cdot)\|_{L^p(S^1)}\, ds$$

for any $1 \leq p \leq 2$. Applying Young's inequality for convolution then yields

$$\|P_k v\|_{L^2(dt\,d\theta)} \leq C\Big(\int_{-\infty}^{\infty} e^{-\frac{\sigma}{2}k|z|} dz\Big)^{\frac{1}{\sigma}} \|\mathcal{L}_\tau v\|_{L^p(dt\,d\theta)} \leq C k^{\frac{1}{p}-\frac{3}{2}} \|\mathcal{L}_\tau v\|_{L^p(dt\,d\theta)},$$

where $\frac{1}{\sigma} = \frac{3}{2} - \frac{1}{p}$. Squaring and summing up $k > M$ gives that

$$\sum_{k>M} \|P_k v\|^2_{L^2(dt\,d\theta)} \leq C \sum_{k>M} k^{-3+\frac{2}{p}} \|\mathcal{L}_\tau v\|^2_{L^p(dt\,d\theta)} = C\tau^{-2+\frac{2}{p}} \|\mathcal{L}_\tau v\|^2_{L^p(dt\,d\theta)},$$

where we have used that $p > 1$ to conclude that the series converges. An application of orthogonality shows that

$$\|P_\tau^h v\|_{L^2(dt\,d\theta)} \leq C\tau^{-1+\frac{1}{p}} \|\mathcal{L}_\tau v\|_{L^p(dt\,d\theta)},$$

which implies (5.7) with $\square = h$.

Now we prove (5.7) for $\square = n$ using the second line of (5.8). For $k < 0$, if $-\infty < s \leq t \leq t_0$, then

$$k(t-s) + \tau(\varphi(s) - \varphi(t)) = -(\tau - k)|t-s| + \tau \log\Big(1 + \frac{|s-t|}{|t|}\Big) \leq -\Big(\frac{\tau}{2} - k\Big)|t-s|,$$

where we have performed a Taylor expansion. Repeating the arguments from above shows that for $k < 0$,

$$\|P_k v\|_{L^2(dt\,d\theta)} \leq C\Big(\frac{\tau}{2} - k\Big)^{\frac{1}{p}-\frac{3}{2}} \|\mathcal{L}_\tau v\|_{L^p(dt\,d\theta)}.$$

Squaring and summing up $k < 0$ gives that

$$\sum_{k<0} \|P_k v\|^2_{L^2(dt\,d\theta)} \leq C\tau^{-2+\frac{2}{p}} \|\mathcal{L}_\tau v\|^2_{L^p(dt\,d\theta)},$$

where we have again used that $p > 1$ to conclude that the series converges. As in the previous setting, inequality (5.7) holds with $\square = n$.

Fix $t \in (-\infty, t_0)$ and set $N = \lceil \tau\varphi'(t) \rceil$. Recalling that $\varphi(t) = t + \frac{1}{2}\log t^2$, an application of Taylor's theorem shows that for all $s, t \in (-\infty, t_0)$,

$$\varphi(s) - \varphi(t) = \varphi'(t)(s-t) + \frac{1}{2}\varphi''(s_0)(s-t)^2,$$

where s_0 is some number between s and t. If $s > t$, then

$$k(t-s) + \tau(\varphi(s) - \varphi(t)) \leq -(k-N)|t-s| - \frac{\tau}{2t^2}(s-t)^2. \tag{5.9}$$

Alternatively, if $s \leq t$, then

$$k(t-s) + \tau(\varphi(s) - \varphi(t)) \leq -(N-1-k)|t-s| - \frac{\tau}{2s^2}(s-t)^2. \tag{5.10}$$

For this reason, we split the sum corresponding to $\square = l$ and use both representations from (5.8).

First, we consider the values $N \leq k \leq M$. From the first equality in (5.8), we sum over k and use the bound from (5.9) to get

$$\Big\|\sum_{k=N}^{M} P_k v(t,\cdot)\Big\|_{L^2(S^1)} \leq \int_{-\infty}^{\infty} \Big\|\sum_{k=N}^{M} e^{-(k-N)|t-s|-\frac{\tau}{2t^2}(s-t)^2} P_k \mathcal{L}_\tau v(s,\cdot)\Big\|_{L^2(S^1)}\, ds.$$

With

$$c_k = e^{-(k-N)|t-s|-\frac{\tau}{2t^2}(s-t)^2},$$

it is clear that $|c_k| \le 1$. Therefore, Lemma 5.1 is applicable, so we may apply estimate (5.3) to obtain

$$\Big\|\sum_{k=N}^{M} e^{-(k-N)|t-s|-\frac{\tau}{2t^2}(s-t)^2} P_k\mathcal{L}_\tau v(s,\cdot)\Big\|_{L^2(S^1)} \le C\Big(\sum_{k=N}^{M} e^{-(k-N)|t-s|-\frac{\tau}{2t^2}(s-t)^2}\Big)^{\frac{1}{p}-\frac{1}{2}} \|\mathcal{L}_\tau v(s,\cdot)\|_{L^p(S^1)}$$

for all $1 \le p \le 2$. Since

$$\sum_{k=N}^{M} e^{-2(k-N)|t-s|} \le \sum_{k=0}^{\infty} e^{-2k|t-s|} \le 1 + |t-s|^{-1},$$

we have

$$\Big\|\sum_{k=N}^{M} P_k v(t,\cdot)\Big\|_{L^2(S^1)} \le C\int_{-\infty}^{\infty} e^{-\frac{\alpha\tau}{2t^2}(s-t)^2}(1+|t-s|^{-\alpha})\|\mathcal{L}_\tau v(s,\cdot)\|_{L^p(S^1)}\,ds,$$

where $\alpha = \frac{2-p}{2p}$. Given that

$$e^{\frac{\alpha\tau}{2t^2}(s-t)^2} \ge \sqrt{1+\frac{\alpha\tau}{t^2}(s-t)^2} \ge C(t_0)|t|^{-1}(1+\tau^{\frac{1}{2}}|s-t|),$$

since $\alpha > 0$, it follows that

$$e^{-\frac{\alpha\tau}{2t^2}(s-t)^2} \lesssim |t|(1+\tau^{\frac{1}{2}}|s-t|)^{-1}.$$

We see that

$$\Big\|\sum_{k=N}^{M} P_k v(t,\cdot)\Big\|_{L^2(S^1)} \le C\int_{-\infty}^{\infty} \frac{(1+|t-s|^{-\alpha})|t|\|\mathcal{L}_\tau v(s,\cdot)\|_{L^p(S^1)}}{(1+\tau^{1/2}|s-t|)}\,ds. \tag{5.11}$$

For $0 \le k \le N-1$, we use the second line of (5.8), and then sum over k and use the bound from (5.10) to get

$$\Big\|\sum_{k=0}^{N-1} P_k v(t,\cdot)\Big\|_{L^2(S^1)} \le \int_{-\infty}^{\infty}\Big\|\sum_{k=0}^{N-1} e^{-(N-1-k)|t-s|-\frac{\tau}{2s^2}(s-t)^2} P_k\mathcal{L}_\tau v(s,\cdot)\Big\|_{L^2(S^1)}\,ds.$$

Arguing as before, we similarly conclude that

$$\Big\|\sum_{k=0}^{N-1} P_k v(t,\cdot)\Big\|_{L^2(S^1)} \le C\int_{-\infty}^{\infty} \frac{(1+|t-s|^{-\alpha})|s|\|\mathcal{L}_\tau v(s,\cdot)\|_{L^p(S^1)}}{(1+\tau^{1/2}|s-t|)}\,ds. \tag{5.12}$$

Combining (5.11) and (5.12) shows that

$$\|t^{-1}P_\tau^l v(t,\cdot)\|_{L^2(S^1)} \le C\int_{-\infty}^{\infty} \frac{(1+|t-s|^{-\alpha})\|s\mathcal{L}_\tau v(s,\cdot)\|_{L^p(S^1)}}{(1+\tau^{1/2}|s-t|)}\,ds.$$

Applying Young's inequality for convolution, we get

$$\|t^{-1}P_\beta^l v\|_{L^2(dt\,d\theta)} \le C\Big[\int_{-\infty}^{\infty}\Big(\frac{1+|z|^{-\alpha}}{1+\tau^{1/2}|z|}\Big)^{\sigma} dz\Big]^{\frac{1}{\sigma}} \|t\mathcal{L}_\tau v\|_{L^p(S^1)},$$

where $\frac{1}{\sigma} = \frac{3}{2} - \frac{1}{p}$. A direct calculation then shows that

$$\Big[\int_{-\infty}^{\infty}\Big(\frac{1+|z|^{-\alpha}}{1+\tau^{1/2}|z|}\Big)^{\sigma} dz\Big]^{\frac{1}{\sigma}} \le C\tau^{-\frac{1}{2\sigma}+\frac{\alpha}{2}}.$$

Since

$$-\frac{1}{2\sigma}+\frac{\alpha}{2} = \frac{1}{2p}-\frac{3}{4}+\frac{1}{2p}-\frac{1}{4} = -1+\frac{1}{p},$$

we have shown (5.7) with $\square = l$, thereby completing the proof of the proposition. □

We now present the proof of Theorem 2.1.

Proof of Theorem 2.1. Since $e^{2t}\,dt\,d\theta = dz$, we have

$$\|t^{-1}e^{-\tau\varphi(t)}u\|_{L^2(dt\,d\theta)} = \|t^{-1}e^{-\tau\varphi(t)-t}ue^t\|_{L^2(dt\,d\theta)} = \|(r\log r)^{-1}e^{-\tau\phi(r)}u\|_{L^2(dz)},$$

$$\|te^{-\tau\varphi(t)}\mathcal{L}u\|_{L^p(dt\,d\theta)} = \|te^{-\tau\varphi(t)-\frac{2t}{p}}2e^{t-i\theta}\bar\partial ue^{\frac{2t}{p}}\|_{L^p(dt\,d\theta)} = 2\|r^{1-\frac{2}{p}}(\log r)e^{-\tau\phi(r)}\bar\partial u\|_{L^p(dz)},$$

and the result follows from applying Proposition 5.2. □

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