



Spectral Asymptotics and Lamé Spectrum for Coupled Particles in Periodic Potentials

Ki Yeun Kim¹ · Mark Levi¹ · Jing Zhou¹

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Abstract

We make two observations on the motion of coupled particles in a periodic potential. Coupled pendula, or the space-discretized sine-Gordon equation is an example of this problem. Linearized spectrum of the synchronous motion turns out to have a hidden asymptotic periodicity in its dependence on the energy; this is the gist of the first observation. Our second observation is the discovery of a special property of the purely sinusoidal potentials: the linearization around the synchronous solution is equivalent to the classical Lamé equation. As a consequence, *all but one instability zones of the linearized equation collapse to a point for the one-harmonic potentials*. This provides a new example where Lamé's finite zone potential arises in the simplest possible setting.

1 Introduction; the Setting

In this paper we study the stability of a “binary”, i.e. of two elastically coupled particles in a periodic potential V on line (c.f. Fig. 1).

The system is described by

$$\begin{aligned}\ddot{x} + V'(x) &= \kappa(y - x) \\ \ddot{y} + V'(y) &= \kappa(x - y).\end{aligned}\tag{1}$$

The special case of $V(x) = -\cos x$ can be realized as two coupled pendula (c.f. Fig. 1):

$$\begin{aligned}\ddot{x} + \sin x &= \kappa(y - x) \\ \ddot{y} + \sin y &= \kappa(x - y).\end{aligned}\tag{2}$$

We note parenthetically that this system is in fact a discretization of the sine-Gordon equation, which arises naturally in many physical applications such as the Frenkel-Kontorova (F-K) model of electrons in a crystal lattice [2] and the arrays of Josephson junctions [7].

Furthermore, our model is related to the F-K model in the following way: the original F-K model consists of an infinite chain of particles with nearest-neighbor coupling, *in an*

✉ Jing Zhou
jingzhou@psu.edu

¹ Department of Mathematics, Penn State University, University Park, State College, PA 16802, USA

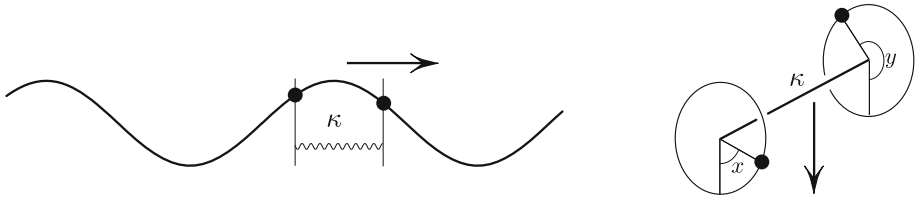


Fig. 1 Elastically coupled particles in a periodic potential V . The case $V = \cos$ corresponds to torsionally coupled pendula

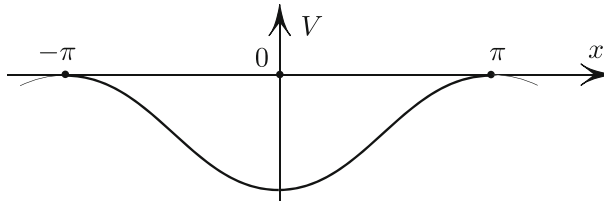


Fig. 2 Periodic potential V

equilibrium state. One can consider instead a *dynamical* Frenkel-Kontorova model:

$$\ddot{x}_n + \sin x_n = \frac{\kappa}{2}(x_{n-1} - 2x_n + x_{n+1}) \quad (3)$$

(here the potential is special: $V(x) = -\cos x$). The two-particle system (2) is a special case of (3): it governs the evolution of space-periodic solutions of (3) of period 2:

$$x_n = x_{n+2} \quad \text{for all } n \in \mathbb{Z}.$$

Indeed, substituting $x = x_{2n}$, $y = x_{2n+1}$ into (3) yields (2).

We call a solution of (1) *synchronous* if $x = y$ for all t ; the common angle $x = y \equiv p$ satisfies the single pendulum equation $\ddot{p} + V'(p) = 0$. Up to time translation, solutions are determined by the total energy

$$E = \frac{\dot{p}^2}{2} + V(p); \quad (4)$$

$E = 0$ corresponds to the unstable equilibrium and also to the heteroclinic solutions. We thus think of E as the “excess energy” (or energy deficit if $E < 0$). We study the stability of the synchronous solution $x = y \equiv p$, where p satisfies

$$\ddot{p} + V'(p) = 0, \quad (5)$$

as E varies.

Throughout the paper we impose the condition $p(0, E) = 0$, then the solution $p = p(t, E)$ is uniquely determined by the pair of initial conditions $p(0, E) = 0$, $\dot{p}(0, E) = \sqrt{2(E - V)}$.

Assumptions on V : We assume throughout the paper that (c.f. Fig. 2)

- (1) $V : \mathbb{R} \rightarrow \mathbb{R}$, $V \in C^{(2)}(\mathbb{R})$, $V(x + 2\pi) = V(x)$;
- (2) V achieves global maximum at $x = \pi \bmod 2\pi$, with

$$V(\pi) = 0 \quad \text{and} \quad V''(\pi) < 0; \quad (6)$$

- (3) There are no other global maxima, i.e. $V(x) < 0$ for all $x \neq \pi + 2\pi n$, $n \in \mathbb{Z}$;

(4) V is even: $V(x) = V(-x)$.¹

We note that the canonical example of coupled pendula (2) satisfies these assumptions.

Under these assumptions we make two observations on the coupled particle in periodic potential V . First, we show that the linearized spectrum of the synchronous motion has a hidden asymptotic periodicity in its dependence on the (logarithm of) energy. Second, we show that in the special case of sinusoidal potentials the linearization around the synchronous solution is equivalent to the classical Lamé equation. We use this fact to prove that sinusoidal potentials have a remarkable property: synchronous motions in such potentials are never hyperbolic except for one interval of energy values. The appearance of Lamé's equation as a linearization around periodic motions of particles in a Hénon-Heiles potentials in \mathbb{R}^2 has been observed earlier by Churchill, Pecelli and Rod in their extensive study [3].

This equation was first introduced by Lamé in 1837 in the separation of variables of the Laplace equation in elliptic domains [9] and later shown to arise in many situations, for instance in the study of the Korteweg-de Vries equation [12]. The double pendula example yields a yet another appearance of the Lamé equation in perhaps the most basic setting.

This paper is structured as follows. In Sect. 2 we describe the infinitely repeating loss and gain of strong stability with the change of energy, and the asymptotic periodicity of the linearized spectrum as a function of the logarithm of the energy, in the limit of small energies. In Sect. 3 we show that for large energies synchronous solutions are linearly stable for all periodic (sufficiently smooth) potentials. Finally, in the last Sect. 4 we (i) show that the linearization of synchronous solution of coupled pendula is a special case of Lamé's equation, (ii) show that only one interval of instability survives, i.e. that sinusoidal potentials are "exceptionally stable", and (iii) give an explicit expression for the interval of unstable energy values.

2 Asymptotic Spectral Periodicity

In this section we show that the Floquet spectrum of the synchronous solution changes periodically as the function of the logarithm of the energy, asymptotically for small energies. We first make a topological observation in Sect. 2.1 that the synchronous solution oscillates between stability and instability zones infinitely many times as the energy E approaches zero. In Sect. 2.2 we then describe the asymptotically periodic spectral dependence on the energy E for small energies.

2.1 A Topological Observation

We first note a fact based on a topological argument: as the energy E crosses a neighborhood of $E = 0$, the synchronous solution (5) loses and regains strong stability infinitely many times. To make this statement more precise, consider the linearization of (2) around the synchronous solution (5):

$$\begin{aligned}\ddot{\xi} + V''(p)\xi + \kappa(\xi - \eta) &= 0 \\ \ddot{\eta} + V''(p)\eta + \kappa(\eta - \xi) &= 0.\end{aligned}$$

¹ This symmetry assumption is added merely to avoid uninteresting technicalities; our proofs do not make essential use of these assumptions.

By setting $u = \xi + \eta$, $w = \xi - \eta$ we decouple this system into

$$\ddot{u} + V''(p)u = 0 \quad (7)$$

$$\ddot{w} + (2\kappa + V''(p))w = 0. \quad (8)$$

The decoupled linear system with periodic coefficients can be written in the Hamiltonian form and thus the spectrum of the associated Floquet matrix is symmetric with respect to the unit circle. Two of these eigenvalues are $\lambda_1 = \lambda_2 = 1$; these eigenvalues correspond to (7). The remaining two eigenvalues λ_3, λ_4 correspond to (8). These determine stability of the synchronous solution; we will call this solution *strongly stable* if λ_3, λ_4 lie on the unit circle and differ from ± 1 .

As E decreases to 0 (the heteroclinic value), strong stability of the synchronous solution is lost and regained infinitely many times, according to the following theorem.

Theorem 1 Assume that V satisfies the conditions (1)-(3) above, and let $2\kappa > -V''(\pi)$. Then there exists a monotone decreasing sequence of disjoint segments $[E_{2n}, E_{2n-1}]$ clustering at 0:

$$E_1 \geq E_2 > E_3 \geq E_4 > \dots \downarrow 0 \quad (9)$$

such that for some $E \in [E_{2n}, E_{2n-1}]$ the synchronous solution of (2) with energy E is not strongly stable, i.e. the eigenvalues $\lambda_3, \lambda_4 = \lambda_3^{-1}$ of the linearization (8) are real. Outside these intervals, i.e. for $E \in (E_{2n+1}, E_{2n})$ and for $E \in (E_1, \infty)$ the linearization is strongly stable.

Proof The angle $\theta = \arg(w + i\dot{w})$ in the phase plane of (8) satisfies

$$\dot{\theta} = -\sin^2 \theta - (2\kappa + V''(p)) \cos^2 \theta, \quad (10)$$

where $p = p(t; E)$ increases monotonically from $-\pi$ to π over the period

$$T(E) \equiv 2\tau(E) = \frac{1}{\sqrt{2}} \int_{-\pi}^{\pi} \frac{dx}{\sqrt{E - V(x)}}, \quad (11)$$

obtained from (4).

It suffices to show that for any solution θ of (10) we have $\theta(T(E)) \rightarrow -\infty$ as $E \downarrow 0$ - indeed, by the argument in [10] this implies that the Keller-Maslov index of (8) goes to infinity as $E \downarrow 0$, i.e. that the Floquet matrix executes infinitely many turns around the “hole” in $SL(2, \mathbb{R})$ in Fig. 5 thus passing repeatedly through the elliptic regions (shaded) in the symplectic group.

To prove that $\theta(T(E)) \rightarrow -\infty$ we will show that for E small enough $\dot{\theta}$ is bounded away from 0 by a constant for a long time - the time spent by p near π . Let $\delta > 0$ be such that

$$2\kappa + V''(x) > \frac{1}{2}(2\kappa + V''(\pi)) > 0 \text{ for } |\pi - x| < \delta; \quad (12)$$

such a δ exists by continuity of V'' and by the assumption $2\kappa + V''(\pi) > 0$. Let $M > 0$ be arbitrarily large (used to bound the change in τ), and choose $E_\delta > 0$ such that

$$\tau_\delta(E) = \int_{\pi-\delta}^{\pi} \frac{dx}{\sqrt{E - V(x)}} > M$$

for all $0 < E < E_\delta$; such E_δ exists since for any fixed δ the integral approaches infinity as $E \downarrow 0$.

For all $0 < E < E_\delta$ and for all $t \in [\tau(E) - \tau_\delta(E), \tau(E)]$ we therefore have (with $p = p(t; E)$):

$$\begin{aligned}\dot{\theta} &= -\sin^2 \theta - (2\kappa + V''(p)) \cos^2 \theta < -\sin^2 \theta - \frac{1}{2}(2\kappa + V''(\pi)) \cos^2 \theta \\ &< -\min(1, \kappa + V''(\pi)/2) < 0.\end{aligned}$$

Since the duration $\tau_\delta(E) > M$ the angle changes by

$$\theta(t) \Big|_{t=\tau(E)-\tau_\delta(E)}^{\tau(E)} < -\min(1, \kappa + V''(\pi)/2) \cdot M,$$

showing that for a part of the interval $[0, \tau(E)]$ the angle changes by an arbitrarily large amount. To show that there is no counter-rotation during the preceding time $[0, \tau_\delta - \tau]$, we observe that the counterclockwise rotation is limited by π : for any $t > 0$

$$\theta(t) \Big|_0^t < \pi$$

– indeed, the θ -interval $-\pi/2 < \theta < \pi/2$ (modulo 2π) cannot be crossed to the right: $\dot{\theta} = -1$ for $\theta = \pm\pi/2$. This completes the proof. \square

2.2 The Normal Form

In this section we show the asymptotic periodicity of the spectrum of the trace of the Floquet matrix of (8) as E approaches 0.

Theorem 2 *Assume that V satisfies the assumptions stated at the end of the previous section, and introduce*

$$\lambda = \sqrt{-V''(\pi)}, \quad (13)$$

the positive eigenvalue of the saddle $(\pi \bmod 2\pi, 0)$ in the phase plane of $\ddot{x} + V'(x) = 0$. Assume also that $2\kappa + V''(\pi) \equiv 2\kappa - \lambda^2 > 0$ and define the frequency ω via²

$$\omega^2 = 2\kappa - \lambda^2 > 0. \quad (14)$$

There exist constants $a \geq 2$ and φ depending on the potential V and on κ such that the Floquet matrix of (8) satisfies

$$\operatorname{tr} F_E = a \cos\left(\frac{\omega}{\lambda} \ln E - \varphi\right) + o(E^0), \quad (15)$$

where the $o(E^0) \equiv o(1) \rightarrow 0$ as $E \rightarrow 0$; here and below we write $o(E^0)$ instead of $o(1)$ to indicate that E is the argument.

In Sect. 2.2.1 we state the key lemmas and give a brief outline of the proof of Theorem 2; the details of the proof are given in Sect. 2.2.2, with the key lemmas assumed; these lemmas are proven in Sect. 2.2.3 (Fig. 3).

² ω is the frequency of “internal oscillations” of the “binary” near the maximum of the potential.

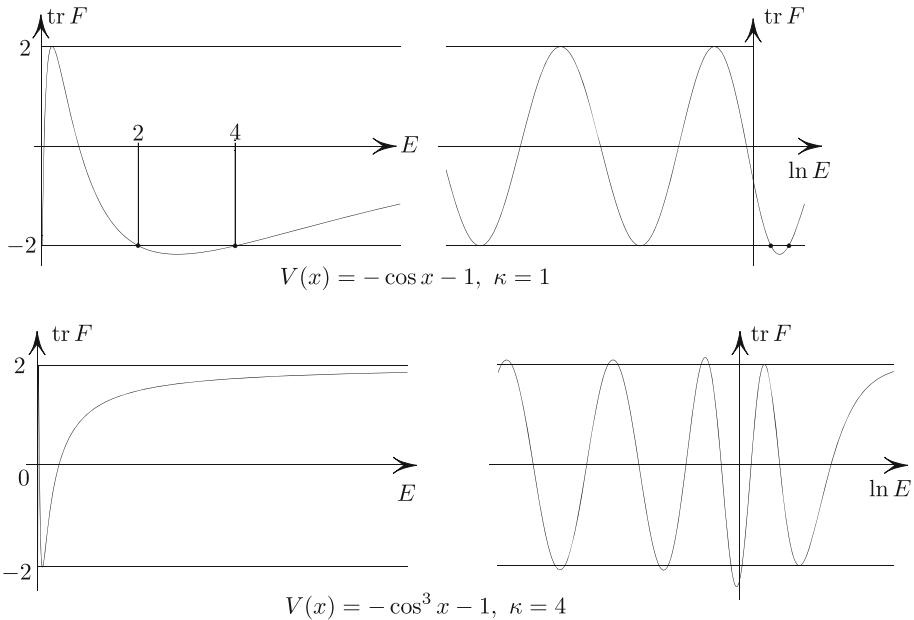


Fig. 3 For $V(x) = -\cos x - 1$ all but one resonance intervals collapse; for $V(x) = -\cos^3 x - 1$ the intervals open up

2.2.1 Key Lemmas

The linearized equation

$$\ddot{w} + (2\kappa + V''(p(t; E)))w = 0 \quad (16)$$

has the periodic coefficient of period $T(E)$ given by (11) - the time it takes for p to change from $-\pi$ to π .

Our goal is to prove (15) for the $T(E)$ - advance matrix $F = F_E$ of the linear system associated with (16):

$$\begin{cases} \dot{w} = u \\ \dot{u} = -(2\kappa + V''(p(t; E)))w. \end{cases} \quad (17)$$

We state the following three key lemmas, proven later in Sect. 2.2.3.

Lemma 1 Assume that V is 2π -periodic and has a unique non-degenerate maximum at $x = \pi \pmod{2\pi}$, with $V(\pi) = 0$ (as assumed throughout). There exists a constant $K > 0$ depending on V such that for small $E > 0$

$$T(E) \equiv 2\tau(E) = \frac{1}{\sqrt{2}} \int_{-\pi}^{\pi} \frac{dx}{\sqrt{-V(x) + E}} = \frac{1}{\lambda} \ln \frac{1}{E} + K + o(E^0), \quad (18)$$

where $\lambda = \sqrt{-V''(\pi)}$.

Lemma 2 ("exponential death") Assume that the coefficient matrix of the matrix ODE $\dot{C} = G(t)C$ decays exponentially in both future and the past: there exist $c > 0$ and $\lambda > 0$ such that

$$\|G(t)\| \leq ce^{-\lambda|t|} \text{ for all } t \in (-\infty, \infty). \quad (19)$$

Then there exists a constant matrix N and a constant b depending only on G such that any fundamental solution matrix C satisfies

$$\|C(t)C^{-1}(-t) - N\| \leq be^{-\lambda t} \text{ for all } t > 0. \quad (20)$$

In other words, the time-advance map $C(t)C^{-1}(-t)$ from $-t$ to t approaches N exponentially as $t \rightarrow \infty$.

Lemma 3 *There exists some constant $c > 0$ such that*

$$|p(t) - p_0(t)| \leq c\sqrt{E} \text{ for } |t| \leq \tau(E), \quad (21)$$

where $p(t) = p(t; E)$ and $p_0(t) = p(t; 0)$.

Based on the three key lemmas whose proofs are presented in Sect. 2.2.3, the proof of Theorem 2 proceeds in two steps:

Step 1 - outline. We replace $p(t, E)$ in (16) with $p_0(t) = p(t, 0)$, the heteroclinic solution (corresponding to $E = 0$) which approaches $\pm\pi$ as $t \rightarrow \pm\infty$:

$$\ddot{w} + (2\kappa + V''(p_0(t)))w = 0 \quad (22)$$

and consider first the time advance matrix F_E^0 for the modified system (22) but over the period $T(E)$ of the unmodified system (16). We note that F_E^0 depends on E only through $\tau = \tau(E)$, whereas F_E depends on E in one additional way, namely through the dependence of coefficient matrix on $p(t; E)$. We will prove (15) for F_E^0 . A crucial use will be made of the fact that the coefficient in (22) approaches a positive constant at a sufficiently fast exponential rate (c.f. Lemma 2).

Step 2 - outline. We will show that $\|F_E - F_E^0\| = o(E^0)$; together with Step 1 this would imply (15) thus completing the proof of the theorem.

In the next Sect. 2.2.2, we present the details of the proof of Theorem 2 based on the three key lemmas stated in this section. The proofs of these key lemmas can be found in Sect. 2.2.3.

2.2.2 Proof of Theorem 2

Based on the three key lemmas in Sect. 2.2.1, we now prove Theorem 2.

Proof Following the idea of Step 1 outlined above, let us write the system (22) in vector form, splitting the coefficient matrix into a constant part and the part that decays at infinity:

$$\dot{z} = (A + B(t))z, \quad (23)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ -\lambda^2 - V''(p_0(t)) & 0 \end{pmatrix},$$

where ω and λ were defined in the statement of the theorem. Now $0 < \pi - |p_0(t)| < ce^{-\lambda|t|}$ for some $c > 0$ and for all t since $-\lambda$ is the stable eigenvalue of the saddle in the phase plane of $\ddot{x} + V'(x) = 0$. Thus $|V''(\pi) - V''(p_0(t))| \leq ce^{-\lambda|t|}$ and hence the (say) Frobenius norm of B decays at infinity:

$$\|B(t)\| \leq ce^{-\lambda|t|} \text{ for all } t \in \mathbb{R}. \quad (24)$$

As stated in the outline, we define the Floquet matrix F_E^0 as the $T(E)$ -advance map of the linearization (23) around the heteroclinic solution, i.e. one corresponding to $E = 0$, from

time $t = -\tau$ to $t = \tau$, where $\tau = T(E)/2$. Letting X be the fundamental solution matrix of (23) we have

$$F_E^0 = X(\tau)X^{-1}(-\tau) \quad (25)$$

Indeed, assuming $X(0) = I$, $X(-\tau)$ propagates an initial condition vector from $t = 0$ to $t = -\tau$; thus $X^{-1}(-\tau)$ propagates from $-\tau$ to 0. And $X(\tau)$ propagates from 0 to τ ; this yields (25). To estimate this product, we strip off the elliptic-rotational part of X by introducing the matrix function $C(t)$ via

$$X = e^{At}C. \quad (26)$$

Substitution into (23) shows that C satisfies the ODE

$$\dot{C} = e^{-At}B(t)e^{At}C.$$

Note that e^{At} is an elliptic rotation (since $\omega^2 = 2\kappa + V''(\pi) > 0$), and thus is bounded, together with its inverse, for all t , so that the coefficient matrix decays at infinity:

$$\|e^{-At}B(t)e^{At}\| \leq ce^{-\lambda|t|},$$

as follows from (24).

According to Lemma 2 this property implies existence of a constant matrix N such that

$$C(\tau)C^{-1}(-\tau) \stackrel{(20)}{=} N + O(e^{-\lambda\tau});$$

here and throughout $O(f(t))$ denotes a function whose absolute value is bounded by $cf(t)$ for all t , for some constant c independent of t (and E). Substituting (26) into (25) and using the last estimate we obtain

$$F_E^0 = e^{A\tau}Ne^{A\tau} + O(e^{-\lambda\tau});$$

where we again used the boundedness of e^{At} for all $t \in \mathbb{R}$.

A simple calculation shows that

$$\operatorname{tr}(e^{A\tau}Ne^{A\tau}) = (n_{11} + n_{22})\cos 2\omega\tau + \left(\frac{1}{\omega}n_{21} - \omega n_{12}\right)\sin 2\omega\tau, \quad (27)$$

where n_{ij} are the elements of N , so that

$$\operatorname{tr} F_E^0 = a\cos(2\omega\tau - c_1) + O(e^{-\lambda\tau}). \quad (28)$$

We observe that $\det N = 1$, as follows from the Hamiltonian character of our linear systems. Using this in (27) implies that the amplitude $a \geq 2$. We have

$$2\omega\tau = \omega T \stackrel{\text{Lemma 1}}{=} \omega\left(\frac{1}{\lambda}\ln\frac{1}{E} + K + o(E^0)\right),$$

so that (28) becomes

$$\operatorname{tr} F_E^0 = a\cos\left(\frac{\omega}{\lambda}\ln E - K\right) + o(E^0).$$

In accordance with Step 2 mentioned before, to complete the proof of Theorem 2 it remains to show that $\|F_E - F_E^0\| = o(E^0)$.

We show that Lemma 3 implies $\|F_E - F_E^0\| = o(E^0)$. Recall that F_E is expressed in terms of the fundamental solution matrix $X_E(t)$ of $\dot{z} = L(t; E)z$ with

$$L(t; E) = \begin{pmatrix} 0 & 1 \\ -2\kappa - V''(p(t; E)) & 0 \end{pmatrix}$$

via $F_E = X_E(\tau)X_E^{-1}(\tau)$. Similarly, $F_E^0 = X_0(\tau)X_0^{-1}(\tau)$ (this is just a repetition of (25)). It therefore suffices to prove that

$$\sup_{t \in [0, \tau]} \|X_E(t) - X_0(t)\| = o(E^0). \quad (29)$$

To that end we subtract $\dot{X}_0 = L_0 X_0$ from $\dot{X}_E = L_E X_E$ (where we abbreviated $L(t; E)$ as L_E) and obtain

$$\frac{d}{dt}(X_E - X_0) = L_0(X_E - X_0) + (L_E - L_0)X_E,$$

or

$$X_E(t) - X_0(t) = \int_0^t \underbrace{X_0(s)X_0^{-1}(s)}_{\leq M} \underbrace{(L_E(s) - L_0(s))X_E(s)}_{\leq c\sqrt{E}} ds,$$

for all $t \in [0, \tau]$, where M is a constant independent of E and in the above estimate of $L_E - L_0$ we have applied Lemma 3. This implies that for all sufficiently small E

$$|X_E - X_0|_{C^0[0, \tau]} \leq \tau M c \sqrt{E} |X_E|_{C^0[0, \tau]} \leq c_1 E^{1/4} |X_E|_{C^0[0, \tau]}$$

where we used $\tau = O(\ln E^{-1})$. This implies (29), and hence conclude the proof. \square

2.2.3 Proof of Key Lemmas

In this section we prove the key lemmas stated in Sect. 2.2.1.

Proof of Lemma 1 Let us replace V by its leading Taylor term and consider the resulting change in the integral, considering first the interval $[0, \pi]$:

$$\int_0^\pi \left(\frac{1}{\sqrt{-V(x) + E}} - \frac{1}{\sqrt{\frac{1}{2}\lambda^2(x - \pi)^2 + E}} \right) dx. \quad (30)$$

As $E \downarrow 0$, the integrand converges pointwise on $[0, \pi]$ to the function

$$f(x) = \frac{1}{\sqrt{-V(x)}} - \frac{\sqrt{2}}{\lambda(\pi - x)}, \quad x \in [0, \pi], \quad (31)$$

defined at $x = \pi$ by continuity; f is *bounded and continuous* (singularities cancel at $x = \pi$, the key point of the proof), and convergence is monotone at every x (increasing or decreasing depending on the sign of the Taylor remainder). We conclude that (30) converges to $\int_0^\pi f \, dx$. Now the integral of the second term in the integrand of (30) is computed explicitly as

$$\frac{\sqrt{2}}{\lambda} \left(\frac{1}{2} \ln \frac{1}{E} + \ln(\sqrt{2}\pi\lambda) \right) + O(E),$$

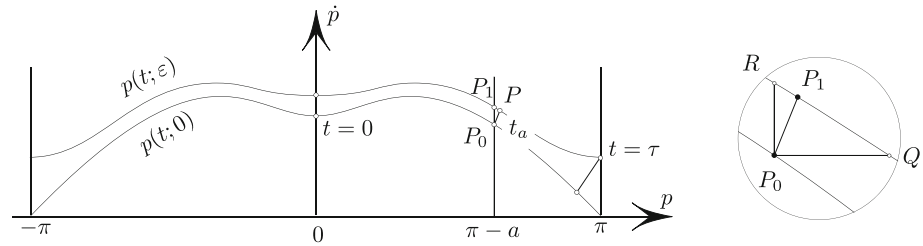


Fig. 4 Proof of (21)

and therefore convergence of (30) to $\int_0^\pi f \, dx$ implies

$$\int_0^\pi \frac{1}{\sqrt{-V(x) + E}} = \frac{\sqrt{2}}{\lambda} \left(\frac{1}{2} \ln \frac{1}{E} + \ln(\sqrt{2\pi}\lambda) \right) + \int_0^\pi f(x) dx + O(E^0).$$

The integral over $[-\pi, 0]$ is estimated similarly; adding the two estimates yields (18), with

$$K = \frac{2\sqrt{2}}{\lambda} \ln(\sqrt{2}\pi\lambda) + \int_{-\pi}^{\pi} f(x) dx,$$

where f was defined above in terms of V .

Proof of Lemma 2 Since $C(t)C^{-1}(t)$ is independent of the choice of the fundamental solution matrix, we lose no generality by assuming $C(0) = I$.

$$\left| \frac{d}{dt} \|C\| \right| \leq \|\dot{C}\| = \|GC\| \leq ce^{-\lambda|t|} \|C\|;$$

this proves boundedness: $\|C(t)\| \leq c/\lambda$ for all $t \in \mathbb{R}$. Now

$$C(t) = I + \int_0^t G(s)C(s) ds. \quad (32)$$

Since C is bounded and G decays exponentially, the improper integral in

$$I + \int_0^\infty G(s)C(s) \, ds \stackrel{\text{def}}{=} N_+$$

converges; and

$$\|N_+ - C(t)\| = \left\| \int_t^\infty G(s)C(s) ds \right\| \leq \int_t^\infty ce^{-\lambda s} \frac{c}{\lambda} ds = \frac{c^2}{\lambda^2} e^{-\lambda t}. \quad (33)$$

Similarly one shows that there exists a constant matrix N_- such that $C^{-1}(-t) \rightarrow N_-$ as $t \rightarrow \infty$ at the same exponential rate λ . Together with (33) this implies (20) with $N = N_+ N_-$ and completes the proof of the lemma. \square \square

Proof of Lemma 3 **1.** We first note that $p(t) > p_0(t)$ for all $t > 0$; indeed, $p(0) = p_0(0)$ and p, p_0 satisfy

$$\dot{p} = \sqrt{2(E - V(p))}, \quad \dot{p}_0 = \sqrt{2(-V(p_0))}, \quad (34)$$

and the claim follows from the comparison theorem for the first order ODEs.

2. From $V'(\pi) = 0$ and $V''(\pi) < 0$ we conclude that V' is monotone increasing in some interval $[\pi - a, \pi]$, $a > 0$. Define t_a by $p_0(t_a) = \pi - a$ (Fig. 4). First we prove that (21)

holds for $t \in [0, t_a]$. Subtracting the second equation in (34) from the first we obtain, using the mean value theorem:

$$\left| \frac{d}{dt}(p - p_0) \right| \leq \alpha(p - p_0) + \beta E,$$

where α and β are the bounds on the partial derivatives of $\sqrt{2(H - V(x))}$ with respect to H and x over $x \in [0, \pi - a/2]$, $H \in [0, E]$; a short calculation gives

$$\alpha = \sup_{x \in [0, \pi - a/2]} \frac{|V'(x)|}{\sqrt{-2V(x)}}, \quad \beta = \sup_{x \in [0, \pi - a/2]} \frac{1}{\sqrt{-2V(x)}}$$

We conclude that

$$|p(t) - p_0(t)| \leq \int_0^{t_a} e^{\alpha(t-s)} \beta E ds < cE \quad \text{for all } t \in [0, t_a], \quad (35)$$

where $c = \beta(e^{\alpha t_a} - 1)/\alpha$. It remains to prove (21) for the remaining time $t \in [t_a, \tau]$.

3. Consider the time-shifted solution $p_1(t) = p(t - \delta)$ where δ is defined by $p(t_a - \delta) = p_0(t_a) = \pi - a$. Note that $\delta = O(E)$ as follows from $|p(t_a) - p(t_a - \delta)| = |p(t_a) - p_0(t_a)| = O(E)$ (according to (35)) and the fact that \dot{p} is bounded away from 0 in the region in question. From $\delta = O(E)$ it follows that $|p(t) - p_1(t)| < cE$ for $t \in [t_a, \tau]$ since \dot{p} is bounded.

Because of this proximity of p and p_1 it suffices to prove (21) with p replaced by p_1 . To that end we consider the segment P_0P_1 (Fig. 4) connecting the two phase points $P_0 = (p_0, \dot{p}_0)$ and $P_1 = (p_1, \dot{p}_1)$, and its slope

$$s = \frac{\dot{p}_1 - \dot{p}_0}{p_1 - p_0}. \quad (36)$$

We show that $s(t) > 0$ for all $t \in (t_a, \tau]$ by observing that the segment P_0P_1 is trapped for all $t \in (t_a, \tau]$ in the moving sector QP_0R formed by two rays through P_0 , one horizontal and another vertical (Fig. 4). Indeed, the horizontal shear in the vector field $\dot{x} = y$, $\dot{y} = -V'(x)$ shows that P_1 cannot leave through the vertical ray. And we now show that P_1 cannot cross the horizontal segment because V' in $[\pi - a, \pi]$ is monotonically decreasing. Indeed, assuming the contrary, and let $t^* \in (t_a, \tau]$ be the first time when P_0P_1 is horizontal, i.e. when $\dot{p}_1(t^*) = \dot{p}_0(t^*)$. Then

$$\dot{s}(t^*) = \frac{(\ddot{p}_1 - \ddot{p}_0)(p_1 - p_0) - (\dot{p}_1 - \dot{p}_0)^2}{(p_1 - p_0)^2} = \frac{(-V'(p_1) + V'(p_0))(p_1 - p_0)}{(p_1 - p_0)^2} > 0,$$

using the monotonicity of V' and the fact that $p_1 > p_0$ for $t > t_a$. But this contradicts $\dot{s}(t^*) \leq 0$, and proves that $s > 0$ for all $t \in (t_a, \tau]$. And this positivity of s implies that $|p_1 - p_0| = O(\sqrt{E})$ for $t \in (t_a, \tau]$. This completes the proof of (21). \square

3 Stability for Large Energies

Numerical evidence in Fig. 3 suggests that the “binary”, i.e. the synchronous solution of (2), is stable for large energies. In this section we prove that this is indeed the case for any C^2 periodic potential.

Theorem 3 *Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary periodic potential of class $C^2(\mathbb{R})$ (with no further assumption). For any κ , there exists $E_\kappa \gg 1$ such that the linearized equation (8) is stable for all large energies $E > E_\kappa$: the associated Floquet matrix F_E of (8) is stable*

for any $E > E_k$, where $T = T(E)$ is the period of the synchronized solution $p = p(t, E)$ defined in (11).

Proof Wirting the linearized equation (8) as the first-order linear system (17), or more compactly, as

$$\dot{z} = L(t)z$$

where $z = (w, \dot{w})$ and

$$L(t) = \begin{pmatrix} 0 & 1 \\ -(2\kappa + V''(p(t))) & 0 \end{pmatrix},$$

we conclude that

$$|z(T)| \leq |z(0)|e^{lT},$$

where $l = \max_{t \in [0, T]} \|L(t)\|$ is independent of E . Here $\|\cdot\|$ denotes the matrix norm generated by the Euclidean norm $|\cdot|$. Now T is small for large E : $T(E) = O(1/\sqrt{E})$, as follows from 11. Thus for $t \in [0, T]$ there is little variation in z :

$$z(t) = z(0) + \int_0^t L(s)z(s)ds = z(0) + r_1(t),$$

where $r_1(t) = O(1/\sqrt{E})$. Thus

$$\begin{aligned} z(T) &= z(0) + \int_0^T L(s)z(s)ds \\ &= z(0) + \int_0^T L(s)z(0)ds + \int_0^T L(s)r_1(s)ds \\ &= (I + \bar{L}T)z(0) + r_2(T) \end{aligned}$$

where $\bar{L} = \frac{1}{T} \int_0^T L(s)ds$ and $r_2(T) = o(1/\sqrt{E})$, and the Floquet matrix is therefore

$$F_E = I + \bar{L}T + o(1/\sqrt{E}).$$

To prove that F_E is stable for large E we compute \bar{L} whose form turns to guarantee stability for large E . From 4 we conclude that p grows nearly linearly for large E :

$$\dot{p}(t) = \sqrt{2(E - V)} = \sqrt{2E} + O(1/\sqrt{E}),$$

so that

$$\begin{aligned} \bar{L} &= \frac{1}{T} \int_0^T (2\kappa + V''(p(s)))ds = 2\kappa + \frac{1}{T} \int_0^{2\pi} \frac{V''(p)}{\dot{p}} dp \\ &= 2\kappa + \frac{1}{T} \int_0^{2\pi} \frac{V''(p)}{\sqrt{2E}} dp + o(1/\sqrt{E}) \\ &= 2\kappa + o(1/\sqrt{E}), \end{aligned}$$

using periodicity of V' in the last step. Therefore the Floquet matrix is of the form

$$F_E = \begin{pmatrix} 1 & T \\ -2\kappa T & 1 \end{pmatrix} + o(1/\sqrt{E}).$$

It is not clear from the last expression whether the stability condition $|\operatorname{tr} F| < 2$ is satisfied without knowing more about the remainder. Interestingly, this knowledge is not necessary:

we will show that the leading term guarantees the absence of real eigenvectors and thus ellipticity. Absence of real eigenvectors amounts to showing that $F_{Eu} \cdot u^\perp \neq 0$ for any nonzero vector $u = (u_1, u_2) \in \mathbb{R}^2$, which we do now:

$$F_{Eu} \cdot u^\perp = T(2\kappa u_1^2 + u_2^2) + o(1/\sqrt{E}) \geq T \min(1, 2\kappa) + o(1/\sqrt{E}) > 0$$

for sufficiently large E , since the remainder is small relative to $T = T(E) = O(1/\sqrt{E})$. \square

4 Collapsed Resonances in Sinusoidal Potentials and Lamé's Equation

So far we described the properties common to general periodic potentials. Remarkably, in the presence of only one harmonic: $V(x) = -(\cos x + 1)$ all instability intervals, except for the first one, collapse to a point: $E_{2n} = E_{2n-1}$ for all $n \geq 2$.³ The underlying reason for this collapse of instability intervals is the fact that linearization around the synchronous solution is a disguised Lamé's equation.

The result of this section implies that sinusoidal potentials are the most stable ones for traveling "binaries", i.e. that the traveling solutions are never hyperbolically unstable except for one specific interval of energies.

In Sect. 4.1 we show that the linearization of the coupled pendula system is a Lamé equation in disguise and consequently in Sect. 4.2 we show that only one interval of instability survives.

4.1 The Coupled Pendula and the Lamé Equation

In this section we reveal the fact that the linearization of the coupled pendula system is a special case of the Lamé equation.

We recall that the general Lamé equation has the form

$$\ddot{W} + [\lambda - n(n+1)k^2 \operatorname{sn}^2(t, k)] W = 0 \quad (37)$$

where n is a positive integer and where $\operatorname{sn}(t, k)$ is Jacobi's elliptic function.⁴

Theorem 4 *Let w be a solution of the linearization*

$$\ddot{w} + (2\kappa + \sin p) w = 0 \quad (38)$$

around the synchronous solution p ($\ddot{p} + \sin p = 0$) with the "energy surplus" E , as in (4), i.e. with $\dot{p}^2/2 - (1 + \cos p) = E$. The rescaled function

$$W(\tau) = w(k\tau), \quad k^2 = \frac{2}{2+E} \quad (39)$$

³ under the assumption $\kappa > \frac{1}{2}$ which is necessary for the existence of an infinite sequence E_n .

⁴ Recall the definition of the "snoidal" function $\operatorname{sn}(t, k)$: given $t \in \mathbb{R}$ and $k \in [0, 1)$, one defines $\operatorname{sn}(t, k)$ via

$$t = \int_0^{\operatorname{sn}(t, k)} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}.$$

Equivalently, by substituting $u = \sin \theta$, one can define x via $t = \int_0^x \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}$ and set $\operatorname{sn}(t, k) \stackrel{\text{def}}{=} \sin x$. It is this latter form of the definition that we will use.

satisfies the Lamé equation corresponding to $n = 1$:

$$W'' + [\lambda - 2k^2 \operatorname{sn}^2(\tau, k)] W = 0, \quad ' = \frac{d}{d\tau}; \quad (40)$$

and with

$$\lambda = (2\kappa + 1)k^2 \quad \text{and} \quad k^2 = 2/(2 + E). \quad (41)$$

Proof To establish a connection between the coefficients in (8) and (40) we express $\cos p$ in terms of sn . From the energy conservation (4), and using the trigonometric identity $1 + \cos p = 2 - 2 \sin^2(p/2)$ we obtain the implicit expression for p ⁵:

$$t = \int_0^{p/2} \frac{d\theta}{\sqrt{1 + \frac{\epsilon}{2} - \sin^2(\theta)}},$$

or, setting $k^2 = \frac{1}{1 + \frac{\epsilon}{2}}$,

$$\frac{t}{k} = \int_0^{p/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

By the definition of sn , this gives

$$\operatorname{sn}\left(\frac{t}{k}, k\right) = \sin \frac{p(t)}{2}.$$

Squaring both sides and using $2 \sin^2(p/2) = 1 - \cos p$, we obtain after a short manipulation:

$$\cos p = 1 - 2 \operatorname{sn}^2\left(\frac{t}{k}, k\right). \quad (42)$$

Substituting this into (8) we obtain an equivalent form

$$\ddot{w} + \left(2\kappa + 1 - 2 \operatorname{sn}^2\left(\frac{t}{k}, k\right)\right) w = 0.$$

Finally, rescaling $\tau = t/k$ results in (40) and completes the proof of Theorem 4. \square

4.2 Collapse of Instability Intervals

It is a remarkable and well known fact that for the Lamé equation (37) all but the first n spectral gaps are collapsed: for fixed k and λ as the parameter, the trace of the Floquet matrix exceeds 2 in absolute value for precisely n intervals of λ [6]. In particular, for the $n = 1$ case, the first gap is open and all the others are closed. However, our case (40) λ is not an independent parameter but rather both it and k vary with the independent parameter E ; it is thus unclear whether the same statement about gaps in E holds true. Theorem 5 states that it does.

Theorem 5 *The synchronous solution of (2) (which exists iff $E > 0$) is linearly unstable if and only if*

$$E \in (4\kappa - 2, 4\kappa). \quad (43)$$

⁵ The formula for p can be found in classical literature, for example Formula (22.19.3) in [4]

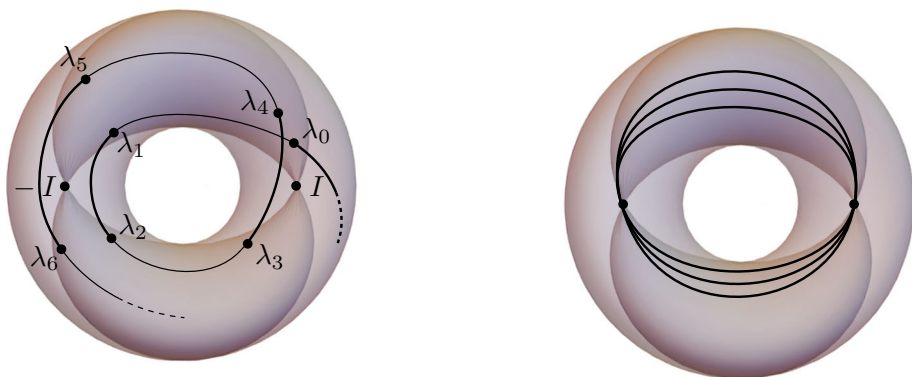


Fig. 5 The path of the Floquet matrix in $SL(2, \mathbb{R})$ for a generic potential as a function of the eigenvalue parameter λ (left) and for the Lamé potential as the function of the energy E (right). All the E -intervals except for $[E_2, E_1]$ are collapsed

For all other values of E the solution is linearly stable, Fig. 5. In particular, for $\kappa > 1/2$ the first two terms in the sequence $E_m \downarrow 0$ are $E_1 = 4\kappa$, $E_2 = 4\kappa - 2$. At all other resonant energies E_3, E_4, \dots linearization around the synchronous solutions is neutrally stable, i.e. the Floquet matrix of the linearization is identity.

Remark 1 The authors would like to thank the referee for pointing out that the proof of Theorem 5 (given below) could be shortened by using the expressions of Lamé polynomials on P.205 in [1] for the case of $n = 1$ of Lamé equations. These would directly give the expressions for the eigenvalues $\lambda'_1 = 1$ and $\lambda'_2 = 1 + k^2$, but we still present the proof for completeness.

Since Equation (40) is the $n = 1$ case of the general Lamé equation, study of the stability of the linearized system (7)–(8) reduces to the study of Lamé equation.

The Lamé equation is a special class of the Hill's equations. By the Oscillation Theorem for the Hill operator (Theorem 2.1 in [11]), there exist two sequences of real numbers $\{\lambda_i\}$ and $\{\lambda'_i\}$ tending to infinity such that

$$\lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 < \dots,$$

Figure 5 (left). Here λ_k correspond to the case when 1 is an eigenvalue of the Floquet matrix, and thus (37) has a periodic solution of the same period $2K$ as the potential; similarly, λ'_k correspond the eigenvalue -1 , i.e. to $2K$ -antiperiodic solutions of (37). The half-period $K = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}$.

Floquet matrix of Lamé's equation (37) is hyperbolic for $\lambda \in (\lambda_i, \lambda_{i+1})$ and for $\lambda \in (\lambda'_i, \lambda'_{i+1})$. The instability zone collapses: $\lambda_i = \lambda_{i+1}$ iff the Floquet matrix is I , i.e. iff two two linearly independent solutions of period $2K$ coexist. Similarly, the collapse $\lambda'_i = \lambda'_{i+1}$ corresponds to the Floquet matrix being $-I$ (Figure 5), i.e. to the existence of two independent antiperiodic solutions of antiperiod $2K$.

Ince [8] showed that there are at most n open intervals of instability for the Lamé equation. Erdélyi [5] proved that Ince's estimate is exact: there are exactly n open intervals of instability for the Lamé equation. Later Haese-Hill, Hallnäs and Veselov [6] pointed out the position of the instability intervals: the first n instability intervals are open.

The Lamé equation (37) corresponds to the case $n = 1$ and hence there is only interval of instability (λ'_1, λ'_2) by the foregoing discussion. Now we prove Theorem 5.

Proof of Theorem 5 We tailor the techniques in [6, 11] and, [8] and present the computation of instability intervals specifically for our case.

To begin our proof, we first discuss the coexistence problem for a more general class called Ince's equation

$$(1 + a \cos 2u)\psi'' + b(\sin 2u)\psi' + (c + d \cos 2u)\psi = 0, \quad (44)$$

where $a, b, c, d \in \mathbb{R}$ and $|a| < 1$. Then we specialize the analysis to the Lamé equation (37) via the transformation $u = \text{am}(\tau, k)$ where am is the Jacobi amplitude defined by

$$\frac{du}{d\tau} = \sqrt{1 - k^2 \sin^2 u},$$

and consequently

$$a = \frac{k^2}{2 - k^2}, \quad b = -\frac{k^2}{2 - k^2}, \quad c = \frac{2\lambda - n(n+1)k^2}{2 - k^2}, \quad d = \frac{n(n+1)k^2}{2 - k^2}.$$

By Lemma 7.3 in [11], if the Ince's equation (44) has two linearly independent π -antiperiodic solutions, then its fundamental solutions ψ_1, ψ_2 take the forms

$$\psi_1 = \sum_{n=0}^{\infty} A_{2m+1} \cos(2m+1), \quad \psi_2 = \sum_{n=0}^{\infty} B_{2m+1} \sin(2m+1). \quad (45)$$

Substituting (45) into (44), we obtain the following recurrence relations

$$\begin{cases} (Q(-\frac{1}{2}) + \Lambda(\frac{1}{2})) A_1 + Q(-\frac{3}{2}) A_3 = 0 \\ Q(m - \frac{1}{2}) A_{2m-1} + \Lambda(m + \frac{1}{2}) A_{2m+1} + Q(-(m + \frac{3}{2})) A_{2m+3} = 0, \quad m \geq 1 \end{cases} \quad (46)$$

and

$$\begin{cases} (-Q(-\frac{1}{2}) + \Lambda(\frac{1}{2})) B_1 + Q(-\frac{3}{2}) B_3 = 0 \\ Q(m - \frac{1}{2}) B_{2m-1} + \Lambda(m + \frac{1}{2}) B_{2m+1} + Q(-(m + \frac{3}{2})) B_{2m+3} = 0, \quad m \geq 1 \end{cases} \quad (47)$$

where $Q(m) = 2am^2 - bm - \frac{d}{2}$, $\Lambda(m) = 4m^2 - c$.

We observe that for the Lamé equation (37) we have $Q(n - \frac{1}{2}) = 0$. Given this observation, by examining the recurrence relations (46)(47), the Ince's equation (44) will have two linearly independent π -antiperiodic solutions if we know one of them has finite order larger than n or infinite order (Theorem 7.3 in [11]), thus closing the instability intervals. The only way to create one but not two linearly independent π -antiperiodic solution is to find solutions of order smaller than n (Theorem 7.6 in [11]). As a consequence, we consider the following two finite order linear systems

$$\begin{pmatrix} Q(-\frac{1}{2}) + \Lambda(\frac{1}{2}) & Q(-\frac{3}{2}) \\ Q(\frac{1}{2}) & \Lambda(\frac{3}{2}) & Q(-\frac{5}{2}) \\ & \dots & \\ Q(n - \frac{3}{2}) & \Lambda(n - \frac{1}{2}) \end{pmatrix} \begin{pmatrix} A_1 \\ A_3 \\ \dots \\ A_{2n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (48)$$

and

$$\begin{pmatrix} -Q(-\frac{1}{2}) + \Lambda(\frac{1}{2}) & Q(-\frac{3}{2}) \\ Q(\frac{1}{2}) & \Lambda(\frac{3}{2}) & Q(-\frac{5}{2}) \\ & \dots & \\ Q(n - \frac{3}{2}) & \Lambda(n - \frac{1}{2}) \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ \dots \\ B_{2n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (49)$$

We can find nontrivial solutions $(A_1, A_3, \dots, A_{2n-1}), (B_1, B_3, \dots, B_{2n-1})$ to the above linear systems (48)(49) by making the coefficient matrices singular, i.e.

$$\det \begin{vmatrix} Q(-\frac{1}{2}) + \Lambda(\frac{1}{2}) & Q(-\frac{3}{2}) \\ Q(\frac{1}{2}) & \Lambda(\frac{3}{2}) & Q(-\frac{5}{2}) \\ & \dots & \\ & & Q(n - \frac{3}{2}) & \Lambda(n - \frac{1}{2}) \end{vmatrix} = 0 \quad (50)$$

and

$$\det \begin{vmatrix} -Q(-\frac{1}{2}) + \Lambda(\frac{1}{2}) & Q(-\frac{3}{2}) \\ Q(\frac{1}{2}) & \Lambda(\frac{3}{2}) & Q(-\frac{5}{2}) \\ & \dots & \\ & & Q(n - \frac{3}{2}) & \Lambda(n - \frac{1}{2}) \end{vmatrix} = 0. \quad (51)$$

Since we only need the result for $n = 1$ Lamé equation, we present the computation for $n = 1$ here.

In our case,

$$Q(-\frac{1}{2}) = -\frac{k^2}{2 - k^2}, \quad \Lambda(\frac{1}{2}) = 1 - \frac{2\lambda - 2k^2}{2 - k^2},$$

thus (50) gives $\lambda'_1 = 1$ and (51) gives $\lambda'_2 = 1 + k^2$. We conclude the proof by substituting the relations $k^2 = \frac{2}{2+E}$ and $\lambda = (2\kappa + 1)k^2$ from Theorem 4. \square

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