Learning Spatial Transformations and their Math Representations through Embodied Learning in Augmented Reality

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Abstract. In Computer Aided Design, Computer Graphics, Robotics, etc., students suffer from inefficient and non-proficient use of the 3D modeling software due to a lack of mathematical knowledge. Deficient knowledge and skills may lead students to use the modeling software through trial-and-error without understanding the algorithms and mathematics. Spatial/geometric transformation is recognized as one of the key factors in learning 3D modeling software. This paper presents a newly developed educational Augmented Reality (AR) mobile application to help students intuitively learn the geometric reasoning of transformation matrices and the corresponding trigonometric equations through play. The application, developed in primary and advanced levels, intends to facilitate the understanding of fundamentals of spatial transformations and their mathematical representations in a self-learning approach. The results of a pilot user study conducted on 7 undergraduate students for the primary level reveal that students' math scores improved after playing with the application.

Keywords: Augmented Reality; Educational Application; Spatial Transformations; Matrices; Trigonometry

1 Introduction

Despite the development of Computer Aided Design (CAD) software in learning geometry, spatial transformations, and related mathematics, many students still face challenges in solving geometric problems and rely on a trial-and-error process [20]. About 20% of students experience difficulty in problems dealing with spatial skills, such as manipulating figures in space and analyzing complex shapes [1], [2]. Specifically, using the trial-and-error technique in learning modeling software may result in understanding geometric transformations as only motions over an object rather than mapping functions on geometry's variables [3]. Computers can shape and re-mold our mathematical knowledge [4], hence students may adopt their understanding of geometric transformation concepts consistently with the strategies represented by modeling software.

The reciprocal relation between spatial reasoning and math skills [5], and the significance of math and spatial skills in Science, Technology, Engineering, and Mathematics (STEM) education and further professional career is well acknowledged in literature [6], [7]. Studying geometric transformations and the associated mathematical concepts is significant for students to consider mathematics as an interconnected discipline and eventually helps students in higher-level reasoning activities [3]. However, the difficulty of learning/teaching geometric transformations and the associated mathematics is acknowledged in the literature [3], [8].

Trigonometry as the mathematical concept inherently involved in transformation matrices is also a subject of interest in many studies. The relevance of trigonometric ideas in linking algebraic and geometric thinking is emphasized by National Council of Teachers of Mathematics (NCTM) [9]. Although learning trigonometry serves as an important prerequisite course for many college-level courses in different fields of STEM [10], students still face difficulty in solving trigonometry problems [11]. Researchers believe that students mostly memorize the formulas without understanding the concepts and the spatial reasoning behind the mathematical equations [11]. The reason may be rooted in the educational system that teaches mathematics in a number sense - as a collection of formulae - rather than emphasizing on the spatial reasonings [5]. On the other hand, researchers found that imagery skill (especially schematic imagery) is directly associated with high spatial visualization skills, meaning that students with high imagery skills outscore their peers in solving geometric-based problems [12].

Augmented Reality (AR) technology as a mediator tool with the ability to augment the physical reality with virtual information may boost learning abstract subject and facilitate understanding the spatial reasoning behind the mathematics of transformation matrices. The augmentation of abstract information besides the inherent capability of AR in perspective matching [13]–[15] may help students in improving imagery skills which could later help them solve geometry problems.

This paper demonstrates BRICKxAR/T, an educational AR mobile application developed by the authors, to facilitate learning spatial/geometric transformations and their corresponding mathematics in a “Learn through Play” environment. BRICKxAR/T intends to help students better understand the mathematical logics behind geometric modeling through playing with the parameters of transformation matrices. Employing AR features to display the dynamic relationship between physical motion and the corresponding abstract information, the authors attempt to increase the application’s ability for self-learning in an intuitive way. Using the AR environment to support embodied learning in a 3D spatial environment, we have contextualized mathematical concepts through synching them to the physical motions in real-time.

The registration of the virtual-physical models enables real-time interaction of physical and virtual objects in AR. The goal of the study is to provide a spatiotemporal experiment where students learn and apply knowledge in the same place, at the same time [16], [17].

This prototype is in the continuation of a previous research project on AR instruction for LEGO® assembly [18], using the same LEGO model as a physical manipulative for the user to interact in the AR environment.

2 Literature Review

The close interrelation between math problem solving and spatial thinking has been reported in studies [5], [19], [20]. Although the textbooks in literature provide essential information to learn the mathematics behind spatial transformations they do not support a practical context to learn and apply the gained knowledge concurrently. Since the advent of CAD methods in the early 1980s, most US engineering schools shifted to CAD systems to teach geometry-related courses [21]. Studies show the superiority of Computer-Assisted Learning compared to the conventional text-books and lectures [2], [21], [22]. However, many studies, including longitudinal research, reveal that mere 3D modeling software may not improve students' spatial skills by itself, and hands-on experiences are still a matter of significance [1], [21], [23]. Theoretical studies in cognitive neuroscience demonstrate that STEM pedagogy specifically benefits from hands-on activities for learning abstract information where there is a lack of real-world referent [24]. Furthermore, the perspective view in 3D desktop software is not the natural user's view; imposing inevitable mental load to interpret any transformation into one's natural perspective view. Finally, CAD workshops may not be interesting enough for students as they often drop the course before ending [25].

2.1 Related Work

AR has been studied in the educational fields like physics, mathematics, chemistry, and biomedical sciences, especially when students cannot feasibly achieve real-world experiences on certain subjects. In these studies, AR is used as a tool to help students embody the real-world perception of objects or abstractions to ease the learning process [26]–[29]. The results of the studies on the application of AR in education reveal that students' learning efficacy increase when the relevant information is spatially/temporally attached to the real-world experiments [28], [30]. This technology can provide a context in which the learners get involved in the experiment physically/interactively, which could benefit the learners in embodied learning [31]. Supporting a physical learning environment, AR technology helps students better integrate new subjects with prior knowledge and experience [32]. Appropriate integration of body movements with the learning content can significantly improve memory retrieval and knowledge retention [33].

Several AR applications have been developed for learning descriptive geometry and mathematics [34]–[38] which demonstrate positive impacts of AR intervention in geometry perception. A couple of studies used AR applications as visualization tools, displaying 3D geometries and different representations (for example, images of unfolded geometry) in a spatial environment, to help students' spatial visualization skills in an engineering graphic course [36], [37]. Their studies showed significant

Improvement of experimental group over the students who used traditional textbooks in understanding geometry and geometric relations.

In a quasi-experimental study, Dünser et al. (2006) have developed and evaluated an AR application as a geometric construction tool, Construct3D, to improve students' spatial skills of 3D perception and enhance learning mathematical concepts [26], [34]. In their study they compared the AR intervention with a 2D CAD software tool through a user study. Assessing the learning effect through multiple spatial ability tests, they found positive but insignificant differences between the treatment and the control groups [34]. In their experiment student's interaction with the virtual model is realized through a pencil and a panel and no direct physical/virtual model interaction is provided. Furthermore, their application does not display the relation between algebraic equations and the corresponding geometric transformations.

GeoGebra AR is one of the recent AR applications in learning geometry and algebra [35]. It has been explored in a couple of studies [39]–[41] demonstrating positive and significant impact of the application in spatial visualization and learning mathematics. However, GeoGebra AR, which can display created virtual objects in a physical environment, does not provide a physical interaction/interplay of physical and virtual objects through tracking the physical objects and visualizing the spatial relationship between the physical and virtual objects. This lack of physical – virtual object interplay limits the application's capability of allowing learners' physical manipulation and embodied learning in AR.

With the aim to enhance AR-based learning and overcome the found limitations of existing AR tools used in learning, BRICKxAR/T is designed to facilitate the physical-virtual object interplay and to strengthen AR's unique power of integrating embodied learning and visualization of abstract information.

3 Methodology

BRICKxAR/T app consists of two prototypes for learning geometric transformations and the corresponding mathematics in primary and advanced levels. The prototypes have been developed based on the progressive learning method introduced in literature for learning spatial transformations [42] in three levels of 'motions, mappings and functions' [42]. We have leveraged the AR technology to realize this process for spatiotemporal experiments as follows:

- Motions: AR supports physical motions, physical/virtual models interplay, and embodied learning.
- Mappings: AR supports visualizing graphics to illustrate spatial mapping operations.
- Functions: AR supports synchronized visualization of mathematical functions and their relations with physical motions.

The current app is programmed in C# and developed for iOS devices using the Unity game engine. AR Foundation and Apple's ARKit image tracking method are adopted to register the models. This paper includes: 1) a brief description of prototype 1 and the corresponding result of a pilot user study which can be found in detail in our previous publications [43], [44] , 2) a detailed description of prototype 2 development, and 3) the design of future user study corresponding to prototype 2.

3.1 Prototype 1

The preliminary prototype is an RTS (Rotation, Translation, and Scale) game that helps students understand mathematical notions of spatial transformations along with the mathematical components of transformations, such as variables, parameters, and functions. The spatial visualization of graphical representations (distance lines with dimensions and rotation arcs with angles) that are matched with physical motions in real-time, assists students to perceive the geometric reasonings underlying transformation matrices. In the AR device (iPad) user interface, the first- and second-row matrices correspond to the transformations of the physical LEGO model and the virtual model, respectively. Students can play with the models, i.e., translate, rotate, and scale (for the virtual model only) in x-, y-, and z-axes, and follow these matrices to observe the synchronized mathematical functions of the geometric transformations in real-time (Figure 1).

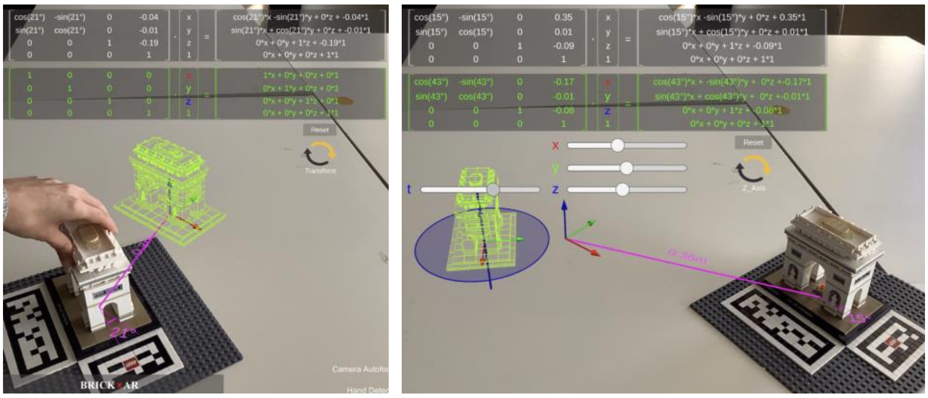


Fig. 1. Left: student playing with the physical model in AR; Right: student playing with the matrix parameters to apply transformations to the virtual model.

User Study

We conducted a pilot user study (TAMU IRB2020-1213M) on a group of 7 undergraduate students (3 females and 4 males; 6 from the College of Engineering and 1 from the College of Architecture). The students had at least college-level knowledge in algebra and geometry and were somehow familiar with AR technology. The students who participated were a convenience sample as they previously worked with the researchers for a related research project, however, spatial transformations and their mathematical representations (matrices) were not learned or used in the students' project tasks.

In the pre-sessions students took online tests, including PVRT (Purdue Visualization of Rotations Test) [45] and a math test on transformation matrices, design based on learning materials of Khan Academy [46], using their own computing devices. Tests answers were not disclosed after the pre-session tests. Due to the COVID-19 pandemic, we hosted the post-sessions and workshops virtually through Zoom meetings with a guidance and Q&A in a 1- to 5-week time interval between the pre- and post-sessions. In the post-sessions, each student was provided with a LEGO set and an iPad with the prototype installed. The students played with the prototype for 30 to 40 minutes and took the PVRT and math tests afterward. During the play we asked students to follow the step-by-step instructions provided in the prototype and record their screens during the play for our further behavior-based analysis. We evaluated the effectiveness of our application through measuring the learning gain of students in pre- and post-sessions. The results, analyzed using a non-parametric statistical method, show that students' scores improved in the post-sessions in most samples. The mean scores of the students in the PVRT ($mean_{pre\text{-}test}$ =75.7, $mean_{post\text{-}test}$ = 77.9) and math test ($mean_{pre\text{-}test}$ = 63.74, $mean_{post\text{-}test}$ = 81.32) increased after playing with the app, and more substantially in the math tests (27.59%) than PVRT (2.8%).

We measured the prototype task load using NASA_TLX survey [47]. The results showed that students on average rated the prototype a low demanding task for learning the targeted material. Also, the results from a motivation questionnaire, adopted from MSLQ [48], revealed that students were interested in playing with the prototype and thought that the app was useful for learning transformation matrices. Specifically, all students agreed that what they learned in the workshop helped them in answering the tests in the post sessions and 5 out of the 7 students claimed that they used visual imagery, visualizing what they learned in the workshop, to answer the math tests.

3.2 Prototype 2

The second prototype covers more advanced concepts and applications of geometric transformations such as trigonometric equations behind transformation matrices. In this prototype students will get introduced to the conception of object detection as an example application of geometric transformations in practice, applied in many recent studies [49]–[51]. Besides physical and virtual interaction and augmentation of abstract information, this prototype intends to leverage AR features to enhance embodied learning by encouraging students to move and rotate physically around the physical model and explore the transformations of virtual camera objects through the physical camera view observing the physical model. The authors are practically adopting this methodology through simulations in parametric modeling to generate the training data of machine learning for brick detection in the LEGO set assembly process in the future development of the application. In this prototype, abstract information and spatial graphics, such as coordinate systems, rotation arcs, distance and projection lines, and notations are displayed in the AR environment and synchronized with physical motions to visualize the geometric reasoning of transformations and the corresponding trigonometry equations behind the transformations.

App Development

Opening the corresponding scene of prototype 2, the physical LEGO model (with image marker attached) get registered in the AR environment and a digital coordinate system will be superimposed on the center pivot point of the model, representing the World Coordinate System (WCS) in this experiment. The real-time registration allows students to move or rotate the LEGO model in any step of the experiment. Then, virtual models and spatial graphics (axes, distance lines with dimensions, arcs with angles, etc. drawn in the 3D space of AR for annotations) will be updated seamlessly according to the new position and rotation of the registered model.

Clicking on the “Circle of Altitude”, a hemisphere wireframe model is displayed embracing the LEGO model with a highlighted circle drawn by default on the largest circle of the hemisphere perimeter. The mathematical equations corresponding to the calculations of circle radius and height will be displayed on a 2D panel on the screen (Figure 2).

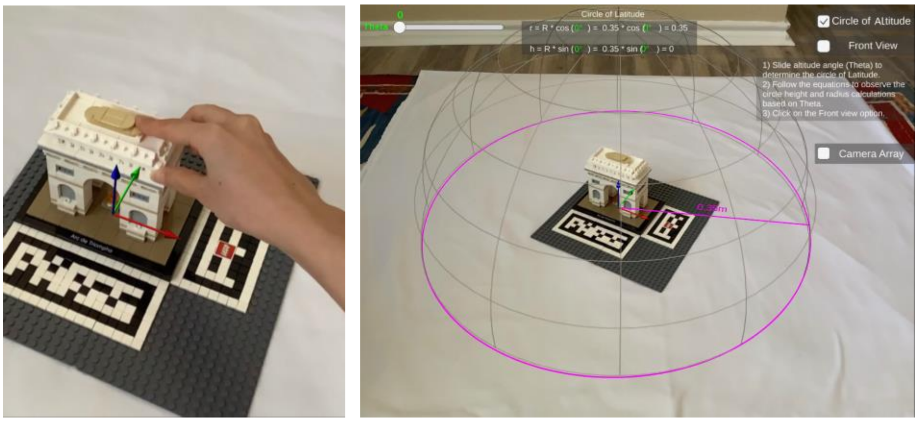


Fig. 2. Left: the physical model attached to the image marker get registered; Right: a circle will be highlighted (in pink color) on a hemisphere embracing the LEGO model.

Then, students can play with the altitude angle parameter of the equations to update the circle, with a new height and radius, that is continuously placed on the hemisphere. Changing from 0º to 90º altitude angles, the circle will be re-drawn seamlessly in the space from maximum radius and minimum height to the maximum height and minimum radius on the hemisphere. The spatial graphics representing the height, base, and hypotenuse of the associated right triangle used in the calculations of the circle height and radius will be drawn in real-time. The corresponding measure notations matched with the numbers in the equations are also spatially adjusted in the AR environment respecting the associated lines (Figure 3).

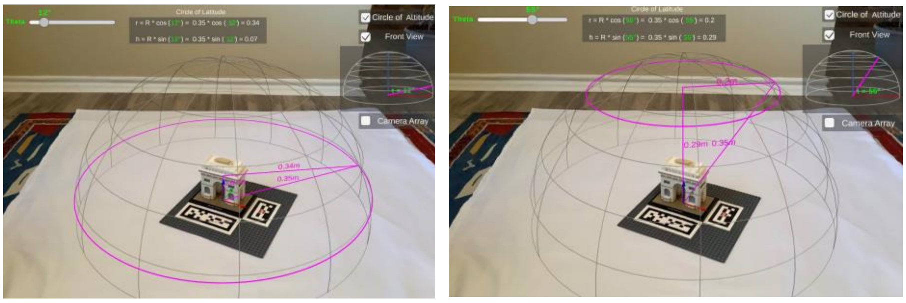


Fig. 3. Using angle parameter, a student can parametrically update the circle of the hemisphere, with different altitudes, heights, and radius.

Students can turn on the front view option to visualize the parametric changes of altitude angle in a sub-window. The optional sub-windows (top- and front-views) are designed in this experiment as those views are common features in desktop modeling software; however, the necessity and impact of those features in the spatial AR environment need to be investigated through user studies in the future.

In the next step, students can parametrically generate a spherical array of virtual camera objects on the drawn circle, using a slider that determines the number of the objects from 1 to 15. Each generated virtual camera object contains a coordinate system which demonstrates its position and orientation respecting the LEGO model with the WCS in this experiment (Figure 4).

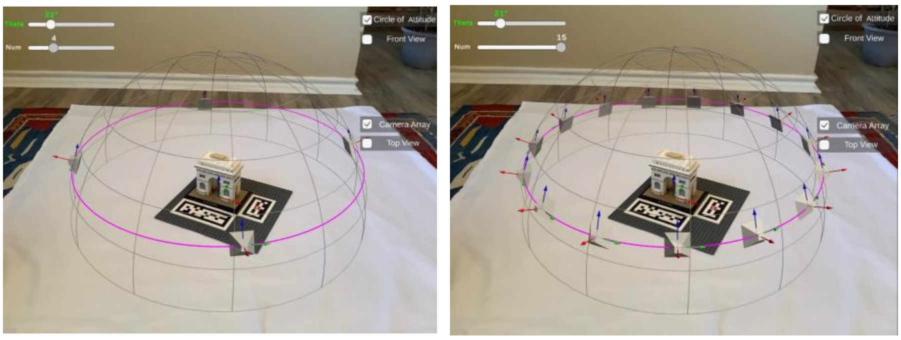


Fig. 4. Parametrically generating virtual camera objects on the drawn circle (pink color) in a polar array fashion.

In this prototype, students can play with the sliders in any step to explore a new scenario. The relative positions and the distances between the generated camera objects get updated seamlessly when the student parametrically changes the circle through the altitude angle parameter. Also, the rotations of the camera objects adjust dynamically so they continuously point to the LEGO model's pivot. Figure 5 shows the automatic transformations of virtual camera objects updated with the new re-drawn circle in real-time.

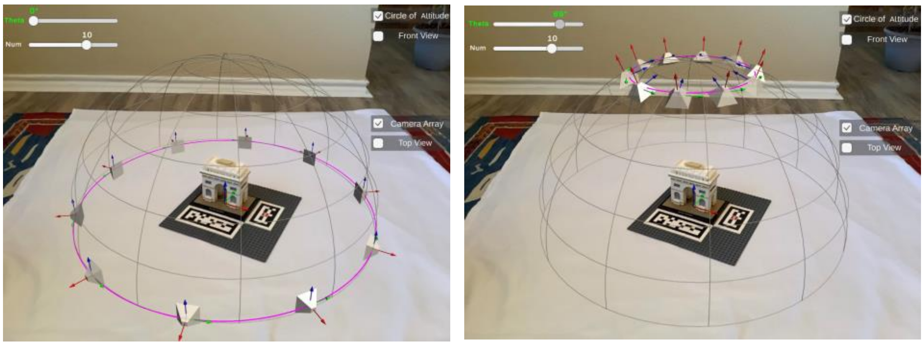


Fig. 5. Student can update the (pink color) circle's altitude and then the generated camera objects get transformed accordingly.

In the next step, as shown in Figure 6, the student can select any virtual camera object by touching the screen. The selected virtual camera gets highlighted and its corresponding transformation matrices along with the associated trigonometry functions will be displayed on the AR screen. The matrices include three 4×4 matrices of rotations, representing rotations around x-, y-, and z-axes, and one 4×4 matrix of translation along x-, y-, and z-axes (Figure 6 bottom). The vector represents a point variable on the camera object that will be multiplied by the translation and rotation matrices for transforming the point from the WCS (the LEGO model's pivot) to a new position (the current position of the selected camera object).

The numbers in the matrices (for example, rotation angles and translations in x, y, and z) match in color with the graphics and notations in the AR environment to catch the student's visual attention in exploring the geometric relations of math functions.

The other sets of mathematical statements placed on top of the matrices show the trigonometry equations behind the calculation of the new position of the selected camera object and in relation to the elements in the transformation matrices. The lines and arcs (with colors matched with the corresponding variables) visualized in AR could help students intuitively perceive the geometric reasoning of the trigonometry equations in the matrices.

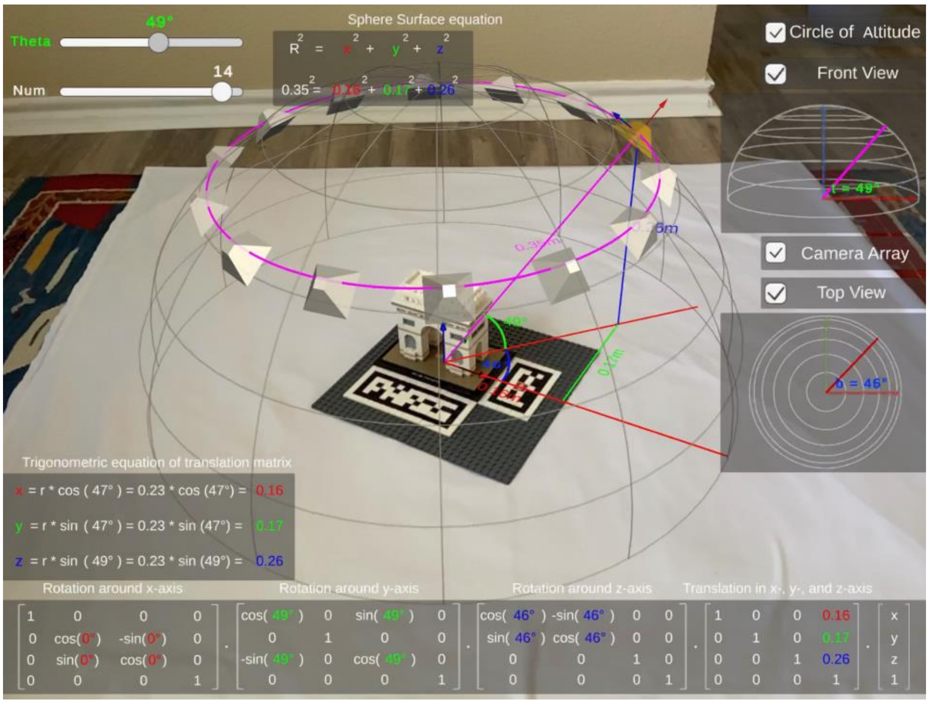


Fig. 6. Students can select any virtual camera object to visualize its transformation matrices and trigonometry equations along with the graphics of dimensions and angles.

The real-time model registration allows students to interact with the physical LEGO at any step of the experiment while all spatial graphics and mathematical representations update seamlessly respecting the WCS (Figure 7).

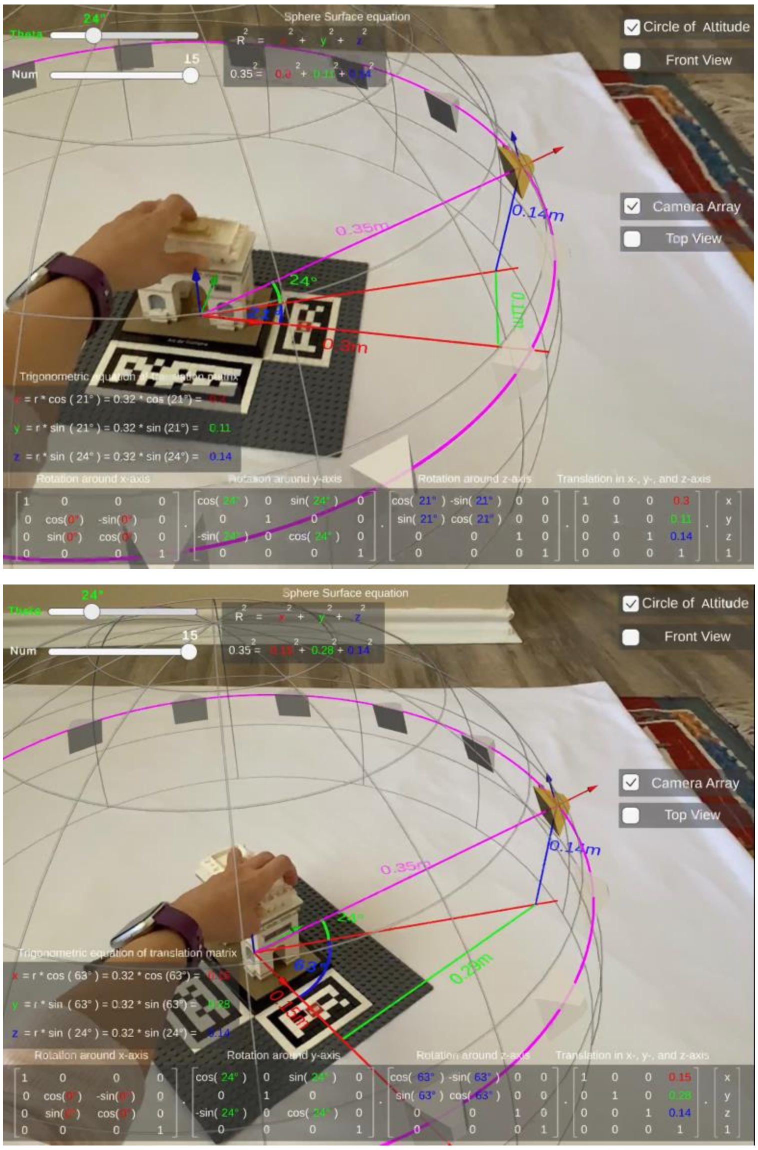


Fig. 7. While student transforms the physical LEGO model from pre-image on the top to image on bottom, the spatial graphics and math representations update in real time accordingly.

Finally, one of the major intentions of this prototype is to encourage students to learn the targeted subject through embodied learning using body gestures. In this experiment, students can practically experience the view of the selected virtual camera through the physical camera of the AR device by physically tracking and aligning the physical camera with the selected virtual camera and taking a screenshot, which can be compared with the renderings from the corresponding virtual camera (Figure 8). As mentioned before, the renderings taken from different angles by simulated virtual cameras can be used as training data for a separate machine learning project on object detection – detecting LEGO bricks using Convolutional Neural Network (CNN). Aligning the virtual and physical cameras, the matrices could be representative of the transformation matrix of the real camera, i.e., the user's view relative to the WCS in the experiment. This exploration offers an embodied learning through a spatiotemporal experience to understand the geometric transformations in practice.

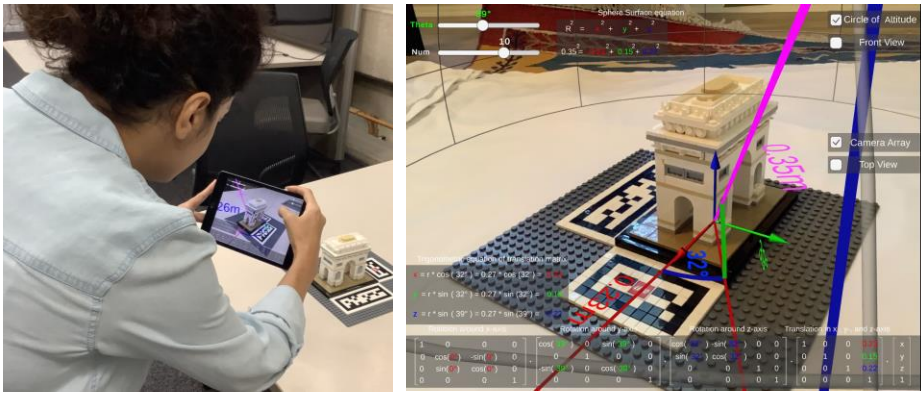


Fig. 8. Left: A student tracking the selected virtual camera object physically using the physical camera on the AR device (iPad); Right: Screenshots showing the camera view from the physical camera/student's view.

In this experiment, while virtual model information (such as the coordinate system, the hemisphere and virtual camera objects) are imported and rendered in the application, the spatial graphics (such as lines, circles, and arcs) are drawn in the 3D space based on the registered model and trigonometric calculation. Although the LEGO model pivot is hypothetically assumed as the origin of the WCS in the experiment, we had to convert the actual transformations of the models and spatial graphics in the calculations respecting the AR camera's WCS. For example, Figure 9-right depicts the conceptual calculation conducted in the equations of finding P_3 to draw the green line in the 3D space (Figure 9-left).

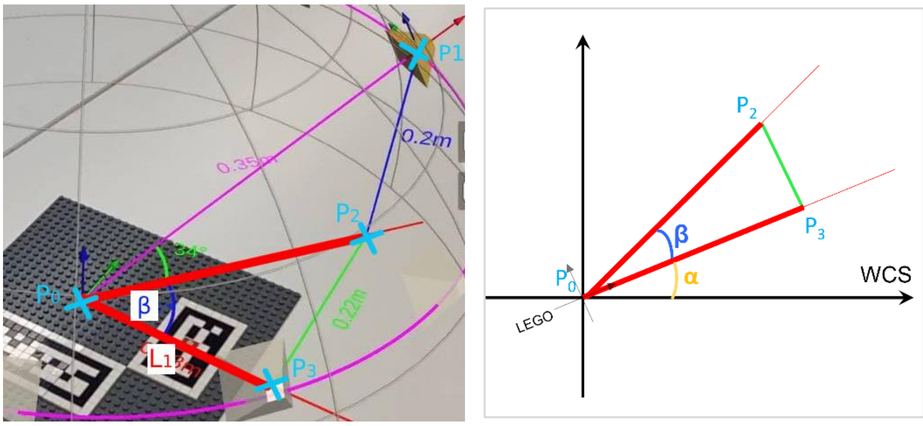


Fig. 9. Left: screenshot from the application (some graphics and notations are added manually for explanation in this paper); Right: conceptual geometric representations to calculate the spatial position of P_3 in the AR environment respecting AR camera WCS.

In Figure 9, P_0 and P_1 are the WCS origin and the pivot point of the selected virtual camera object, respectively. However, other points need to be calculated respecting the AR camera's WCS. The notations and calculations corresponding to P_3 are as follows:

$$L_1 = P_0P_2 \cdot \cos(\beta) \tag{1}$$

$$x = L_1 \cdot \cos(\alpha + \gamma) \tag{2}$$

$$y = L_1 \cdot \sin(\alpha + \gamma) \tag{3}$$

$$P_2 = P_0 + (x, y, P_0.\mathrm{z}) \tag{4}$$

where: P_0 is the LEGO pivot; P_1 is the selected virtual camera pivot; P_2 is P_1 projected; α is the angle between the registered model in the very beginning (pre-image) and the AR camera WCS; and γ is the angle between the transformed LEGO model (image) and the primary registered model (pre-image).

User Study Design

We will measure the effectiveness of our application through a user study on undergraduate students between control (non-AR) and experimental (AR intervention) groups. The non-AR control group will play the same app with similar learning functions and graphics, but with the AR registration and tracking turned off. The study includes pre- and post-sessions as follows:

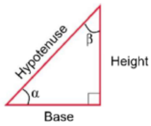
❖ Pre-session:
  - Demographic questionnaire
  - The Purdue Visualization of Rotations Test (PVRT) [45]
  - A Math Test (MT) on transformation matrices and trigonometry equations, designed based on [46], [52]

❖ Post-session:

- o Workshop of playing with the prototype (AR and non-AR by the experimental and control groups, respectively)
- o PVRT
- o MT
- o NASA TLX survey [47]
- o Motivation questionnaire, adopted from MSLQ [48]

Figure 8 shows two examples of the MT.

Answer questions 1 to 4 based on the bellow diagrams of a right triangle:



angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

4. If hypotenuse=10, $\alpha = 60^0$, what will be the base?

a) We need to know β

b) We need to know the height

c) base = 5

d) base = $\frac{10 \times \sqrt{3}}{2}$

12. Which is the best description for the transformation given by the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & \cos(t) & -\sin(t) & 2 \\ 0 & \sin(t) & \cos(t) & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} ?$$

a) Combined matrix of rotation and move in 3D space

b) Combined matrix of rotation and reflection in 3D space

c) Combined matrix of scale and move in 3D space

d) Combined matrix of scale and rotation in 3D space

Fig. 10. Two example questions of the MT, designed by the authors

Without disclosure of the tests' answers, students are going to participate in the pre- and post-sessions with a 1- to 5-week time interval to avoid Testing and History Threat [53]. We will evaluate students learning gains in the visualization of rotation and math skills through pre- and post-tests and compare students' scores in the non-AR versus AR groups to assess the effectiveness of BRICKxAR/T with AR intervention.

We will measure the task load of our application using the NASA-TLX survey. The survey assesses the application task load based on six dimensions (mental demand, physical demand, temporal demand, effort, frustration, and performance) through rating and pair-wise questions [47]. The result of the survey will demonstrate how tense/uncomfortable vs. light/convenient our application is for students to play. The motivation questionnaire will measure students motivations based on three categories of intrinsic-value, task-anxiety, and self-efficacy [48]. The results will help us infer how confusing versus easy/user-friendly BRICKxAR/T is to play and learn the targeted subject for students.

4 Conclusion and future work

In this paper, we have presented an AR educational application, named BRICKxAR/T, which is developed in two levels of primary and advanced, to help students learn the transformation matrices and the incorporated trigonometry equations behind geometric transformations through spatiotemporal experiments. The primary level (prototype 1) involves learning the fundamentals of spatial transformations and their corresponding matrices along with understanding the components of a geometric transformations in 3D modeling software. The advanced level (prototype 2) involves learning spatial transformation matrices and the corresponding trigonometry equations. BRICKxAR/T has the potential to enable (1) embodied learning through hand and body movements based on students' interaction with the physical manipulative in the immersive AR environment and (2) integrated visualization of geometric and algebraic concepts, synchronized with the movements.

As prior literature suggested improvement of the memory retrieval and knowledge retention through the use of AR, this project specifically aims to enhance students learning geometric transformations and their mathematical representations (matrices and vectors) leveraging AR. The completed pilot user study for prototype 1 is promising, and a new user study for prototype 2 is designed to be conducted with undergraduate students from different fields.

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