

A P_1 FINITE ELEMENT METHOD FOR A DISTRIBUTED ELLIPTIC OPTIMAL CONTROL PROBLEM WITH A GENERAL STATE EQUATION AND POINTWISE STATE CONSTRAINTS

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ABSTRACT. We investigate a P_1 finite element method for an elliptic distributed optimal control problem with pointwise state constraints and a state equation that includes advective/convective and reactive terms. The convergence of this method can be established for general polygonal/polyhedral domains that are not necessarily convex. The discrete problem is a strictly convex quadratic program with box constraints that can be solved efficiently by a primal-dual active set algorithm.

1. INTRODUCTION

Let Ω be a polygonal/polyhedral domain in $\mathbb{R}^n$ ($n = 2, 3$), $y_d \in L_2(\Omega)$, β be a positive constant and $g \in H^4(\Omega)$. The elliptic optimal control problem is to find

$$(1.1) \quad (\bar{y}, \bar{u}) = \operatorname{argmin}_{(y,u) \in \mathbb{K}_g} \left[\frac{1}{2} \|y - y_d\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{L_2(\Omega)}^2 \right],$$

where (y, u) belongs to $\mathbb{K}_g \subset H^1(\Omega) \times L_2(\Omega)$ if and only if

$$(1.2a) \quad a(y, z) = \int_{\Omega} uz \, dx \quad \forall z \in H_0^1(\Omega),$$

$$(1.2b) \quad y = g \quad \text{on } \partial\Omega,$$

and

$$(1.3) \quad y \leq \psi \quad \text{a.e. in } \Omega.$$

Here ψ belongs to $W^{3,p}(\Omega)$ with $p > n$ and $\psi > g$ on $\partial\Omega$, and the bilinear form $a(\cdot, \cdot)$ is defined by

$$(1.4) \quad a(y, z) = \int_{\Omega} \nabla y \cdot \nabla z \, dx + \int_{\Omega} (\boldsymbol{\zeta} \cdot \nabla y) z \, dx + \int_{\Omega} \gamma y z \, dx,$$

where the vector field $\boldsymbol{\zeta} \in [W_{\infty}^1(\Omega)]^n$ and the function $\gamma \in W_{\infty}^1(\Omega)$.

Remark 1.1. We will follow the standard notation for function spaces, norms and operators that can be found for example in [1, 12, 20].

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We assume that

(1.5) The boundary value problem (1.2) is uniquely solvable.

Remark 1.2. The assumption (1.5) holds for example under the condition that $\boldsymbol{\zeta} \in [W_\infty^1(\Omega)]^n$ and $\gamma \in L_\infty(\Omega)$ satisfy $\gamma - \frac{1}{2}\nabla \cdot \boldsymbol{\zeta} \geq 0$. The assumption that $\gamma \in W_\infty^1(\Omega)$ is needed for the derivation of the interior regularity of $\bar{y}$ (cf. Section 2 and Appendix A).

Remark 1.3. The state equation defined by (1.2) is the weak form of the following Dirichlet boundary value problem for a model second order elliptic partial differential equation with advective/convective and reactive terms:

$$(1.6a) \quad \mathcal{L}y = u \quad \text{in } \Omega,$$

$$(1.6b) \quad y = g \quad \text{on } \partial\Omega,$$

where the partial differential operator $\mathcal{L}$ is given by

$$(1.7) \quad \mathcal{L}y = -\Delta y + \boldsymbol{\zeta} \cdot \nabla y + \gamma y.$$

Let $\mathring{E}(\Delta; L_2(\Omega)) \subset H_0^1(\Omega)$ be defined by

$$(1.8) \quad \mathring{E}(\Delta; L_2(\Omega)) = \{y \in H_0^1(\Omega) : \Delta y \in L_2(\Omega)\} = \{y \in H_0^1(\Omega) : \mathcal{L}y \in L_2(\Omega)\},$$

where Δy and $\mathcal{L}y$ are understood in the sense of distributions. According to elliptic regularity results for polygonal/polyhedral domains (cf. [21, 30, 38, 40]),

$$(1.9) \quad \mathring{E}(\Delta; L_2(\Omega)) \text{ is a subspace of } H^{1+\alpha}(\Omega) \cap H_{loc}^2(\Omega) \cap H_0^1(\Omega)$$

for some $\alpha \in (\frac{1}{2}, 1]$ determined by the geometry of Ω , and

$$(1.10) \quad \|z\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|\Delta z\|_{L_2(\Omega)} \quad \forall z \in \mathring{E}(\Delta; L_2(\Omega)).$$

We note that the index α can be taken to be 1 if Ω is convex.

Remark 1.4. $\mathring{E}(\Delta; L_2(\Omega))$ is more than just a subspace of $H_{loc}^2(\Omega)$. Let Ω_δ ($\delta > 0$) be the subset of Ω consisting of points whose distances from the reentrant corners of Ω are $> \delta$. We have $z \in H^2(\Omega_\delta)$ for any $z \in \mathring{E}(\Delta; L_2(\Omega))$ and any positive δ .

It follows from (1.10) and the Sobolev inequality (cf. [1]) that

$$V_g = g + \mathring{E}(\Delta; L_2(\Omega)) \subset C(\bar{\Omega}).$$

Therefore using (1.6a) we can reformulate (1.1)–(1.3) as the following equivalent problem:

$$(1.11) \quad \text{Find } \bar{y} = \operatorname{argmin}_{y \in K_g} \left[\frac{1}{2} \|y - y_d\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|\mathcal{L}y\|_{L_2(\Omega)}^2 \right],$$

where

$$(1.12) \quad K_g = \{y \in V_g : y \leq \psi \text{ in } \Omega\}.$$

In the special case where $\boldsymbol{\zeta} = \mathbf{0}$ and $\gamma = 0$, two P_1 finite element methods for the minimization problem (1.11)–(1.12) were investigated in [8]. One of the methods exploits mass lumping so that the resulting discrete problem can be efficiently solved by a primal-dual active set algorithm. Our goal is to extend this method to the general case.

Remark 1.5. If Ω is convex, then $\mathring{E}(\Delta; L_2(\Omega))$ coincides with $H^2(\Omega) \cap H_0^1(\Omega)$ and (1.11)–(1.12) defines a fourth order obstacle problem with the boundary conditions of a simply supported plates. The idea of reformulating an elliptic optimal control problem with point-wise state constraints as a fourth order optimization problem was first introduced in [37], where the fourth order problem was solved by a Morley finite element method. This approach was extended to other finite element methods in [6, 7, 8, 10, 11, 15, 16, 17, 29], to problems with both state and control constraints in [9, 14], and the convergence analysis was refined in [13]. Compared with the traditional approach in [19, 22, 39, 33, 41], the finite element methods based on the reformulation can be efficiently solved by a primal-dual active set strategy and the convergence in the L_∞ norm for the state is a natural byproduct. Moreover, as demonstrated in [7, 8, 11] and the current paper, this new approach is also applicable to nonconvex polygonal/polyhedral domains.

The rest of the paper is organized as follows. We recall some properties of the continuous problem in Section 2 and present the finite element method in Section 3. Instead of repeating arguments similar to the ones in [8], we keep these two sections concise and provide details of the modifications necessitated by the general state equation (1.2) in Appendix A and Appendix B. Numerical results are presented in Section 4 and we end with some concluding remarks in Section 5.

Throughout the paper we will use C with or without subscript to denote a generic positive constant that is independent of the meshes.

2. THE CONTINUOUS PROBLEM

Let $\bar{z} = \bar{y} - g$. Then the minimization problem defined by (1.11)–(1.12) is equivalent to find

$$(2.1) \quad \bar{z} = \operatorname{argmin}_{z \in \tilde{K}} \left[\frac{1}{2} \|z - (y_d - g)\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|\mathcal{L}(z + g)\|_{L_2(\Omega)}^2 \right],$$

where

$$(2.2) \quad \tilde{K} = \{z \in \mathring{E}(\Delta; L_2(\Omega)) : z \leq \tilde{\psi} = \psi - g \text{ in } \Omega\}.$$

Note that $\mathring{E}(\Delta; L_2(\Omega))$ is a Hilbert space under the inner product $((\cdot, \cdot))$ defined by

$$((z, q)) = (z, q)_{L_2(\Omega)} + (\mathcal{L}z, \mathcal{L}q)_{L_2(\Omega)},$$

and $\tilde{K}$ is a nonempty closed convex subset of $\mathring{E}(\Delta; L_2(\Omega))$. Therefore we can conclude from the classical theory of calculus of variations (cf. [24, 35]) that (2.1)–(2.2) has a unique solution characterized by the variational inequality

$$(2.3) \quad \beta \left[(\mathcal{L}\bar{z}, \mathcal{L}(z - \bar{z}))_{L_2(\Omega)} + (g, \mathcal{L}(z - \bar{z}))_{L_2(\Omega)} \right] + (\bar{z} - z_d, z - \bar{z})_{L_2(\Omega)} \geq 0 \quad \forall z \in \tilde{K},$$

where $z_d = y_d - g$, and it follows from the interior regularity theory in [18, 26, 27] for fourth order variational inequalities that

$$(2.4) \quad \bar{z} \in H_{loc}^3(\Omega) \cap W_{\infty, loc}^2(\Omega).$$

(Details for the derivation of (2.4) can be found in Appendix A.)

The unique solvability of (1.11)–(1.12) follows from the unique solvability of (2.1)–(2.2), and we can translate the variational inequality (2.3) into

$$(2.5) \quad (\bar{y} - y_d, y - \bar{y})_{L_2(\Omega)} + \beta(\mathcal{L}\bar{y}, \mathcal{L}(y - \bar{y}))_{L_2(\Omega)} \geq 0 \quad \forall y \in K_g.$$

Since $\bar{y} = \bar{z} + g$ and $g \in H^4(\Omega)$, we have, by (1.9) and (2.4),

$$(2.6) \quad \bar{y} \in H^{1+\alpha}(\Omega) \cap H_{loc}^3(\Omega) \cap W_{\infty,loc}^2(\Omega).$$

It follows from (2.5) and (2.6) that

$$(2.7) \quad \int_{\Omega} \left[(\bar{y} - y_d)z + \beta(\mathcal{L}\bar{y})(\mathcal{L}z) \right] dx = \int_{\Omega} z \, d\mu \quad \forall z \in \mathring{E}(\Delta; L_2(\Omega)),$$

where

$$(2.8) \quad \mu \text{ is a non-positive regular Borel measure,}$$

and the following complementarity condition holds:

$$(2.9) \quad \int_{\Omega} (\psi - \bar{y}) d\mu = 0.$$

Moreover, we have

$$(2.10) \quad \left| \int_{\Omega} z \, d\mu \right| \leq C \|z\|_{H^1(\Omega)} \quad \forall z \in \mathring{E}(\Delta; L_2(\Omega)).$$

Remark 2.1. Condition (2.9) is equivalent to the statement that μ is supported on the active set $\mathfrak{A} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$. Note that $\mathfrak{A}$ is a compact subset of Ω because $\psi > g$ on $\partial\Omega$.

The properties of $\bar{y}$ in (2.6), the optimality conditions (2.7)–(2.9) and the regularity of μ in (2.10) are key ingredients for the convergence analysis of the finite element method. We omit the derivations of (2.7)–(2.10) since they can be found in [8, 13].

3. THE DISCRETE PROBLEM

Let $\mathcal{T}_h$ be a shape regular simplicial triangulation of Ω , $V_h \subset H_0^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}_h$ (cf. [12, 20]), and $\mathring{V}_h = V_h \cap H_0^1(\Omega)$. The diameter of $T \in \mathcal{T}_h$ is denoted by h_T and $h = \max_{T \in \mathcal{T}_h} h_T$ is the mesh diameter.

Let the mass-lumping inner product $(\cdot, \cdot)_h$ (cf. [42] and [47, Chapter 15]) be defined by

$$(3.1) \quad (v, w)_h = \sum_{p \in \mathcal{V}_h} \left(\sum_{T \in \mathcal{T}_p} \frac{|T|}{n+1} \right) v(p) w(p) \quad \forall v, w \in V_h,$$

where $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$, $\mathcal{T}_p$ is the set of the elements in $\mathcal{T}_h$ that share p as a common vertex, and $|T|$ is the area ($n=2$) or volume ($n=3$) of T .

The linear map $\mathcal{L}_h : H^1(\Omega) \rightarrow V_h$ is then defined by

$$(3.2) \quad (\mathcal{L}_h w, v_h)_h = a(w, v_h) \quad \forall v_h \in \mathring{V}_h.$$

Let K_h^g be the affine subspace of V_h defined by

$$(3.3) \quad K_h^g = \{y_h \in I_h g + \mathring{V}_h : y_h \leq I_h \psi \text{ in } \Omega\},$$

where $I_h : C(\bar{\Omega}) \rightarrow V_h$ is the nodal interpolation operator. The P_1 finite element method for (1.11)–(1.12) is to find

$$(3.4) \quad \bar{y}_h = \operatorname{argmin}_{y_h \in K_h^g} \left[\frac{1}{2} (y_h - y_d, y_h - y_d)_{L_2(\Omega)} + \frac{\beta}{2} (\mathcal{L}_h y_h, \mathcal{L}_h y_h)_h \right].$$

Remark 3.1. The minimization problem (3.4) is a strictly convex quadratic program with box constraints. Due to mass lumping the system matrix for this quadratic program is readily available. Therefore (3.4) can be solved efficiently by a primal-dual active set method (cf. [4, 5, 32, 34]).

Let τ be defined by

$$(3.5) \quad \tau = \begin{cases} \alpha & \text{if } d = 2 \text{ or } 3 \text{ and } \mathcal{T}_h \text{ is quasi-uniform,} \\ 1 & \text{if } d = 2 \text{ and } \mathcal{T}_h \text{ is graded around the reentrant corners.} \end{cases}$$

Here $\alpha \in (\frac{1}{2}, 1]$ is the index of elliptic regularity in (1.10).

Theorem 3.2. *Let $\bar{y}$ be the solution of (1.11), $\bar{y}_h$ be the solution of (3.4) and $\bar{u}_h = \mathcal{L}_h \bar{y}_h$. We have*

$$(3.6) \quad \|\bar{u} - \bar{u}_h\|_{L_2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L_2(\Omega)} + |\bar{y} - \bar{y}_h|_{H^1(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau),$$

where the positive constant C is independent of h and τ is given by (3.5).

We also have an L_∞ error estimate in terms of the P_1 finite element approximation $R_h \bar{y}$ of $\bar{y}$ defined by

$$R_h \bar{y} \in V_h^g = I_h g + \mathring{V}_h$$

and

$$(3.7) \quad a(R_h \bar{y}, z) = a(\bar{y}, z) \quad \forall z \in \mathring{V}_h.$$

Remark 3.3. $R_h \bar{y}$ is well-defined for h sufficiently small under assumption (1.5) (cf. [12, Section 5.6] and [44]).

Theorem 3.4. *Let $\bar{y}$ be the solution of (1.11) and $\bar{y}_h$ be the solution of (3.4). We have, for h sufficiently small,*

$$(3.8) \quad \|\bar{y} - \bar{y}_h\|_{L_\infty(\Omega)} \leq C(|\ln h|^{\frac{1}{2}} h + h^\tau) + \|\bar{y} - R_h \bar{y}\|_{L_\infty(\Omega)},$$

where the positive constant C is independent of h and τ is given by (3.5).

The proofs of these theorems are based on the properties of the continuous problem stated in (2.6)–(2.10) and results from finite element analysis, in particular the estimates

$$(3.9) \quad \|\bar{y} - R_h \bar{y}\|_{H^1(\Omega)} \leq C h^\tau,$$

$$(3.10) \quad \|\bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \leq C h^{2\tau},$$

$$(3.11) \quad \|\bar{y} - R_h \bar{y}_h\|_{L_\infty(G)} \leq C(|\ln h| h^2 + h^{2\tau}),$$

that are valid for h sufficiently small. Here $G \subset \Omega$ is an open neighborhood of the active set $\mathfrak{A}$ such that $\bar{G}$ is a compact subset of Ω . (A derivation of (3.9)–(3.11) is provided in Appendix B.)

We omit the proofs of Theorem 3.2 and Theorem 3.4 since they follow the same lines as in [8].

4. NUMERICAL RESULTS

In this section we present numerical examples on three different domains, where for simplicity we take $\beta = \gamma = 1$ and ζ to be a constant vector. Note that for such choices of γ and ζ we have

$$(4.1) \quad \mathcal{L}^t \mathcal{L} y = \Delta^2 y - \zeta \cdot \nabla (\zeta \cdot \nabla y) - 2\Delta y + y$$

where the operator $\mathcal{L}^t$ is the transpose of $\mathcal{L}$.

The discrete problems are solved by a primal-dual active set algorithm (cf. [4, 5, 32, 34]).

Example 4.1 (Disk Active Set). In this example we take $\Omega = [-4, 4]^2$, $\zeta = [1, 0]^t$ (first test) and $[2, 2]^t$ (second test), $\psi = |x|^2 - 1$ and $g = 0$.

Following the ideas behind [8, Example 7.1], we define the exact solution by

$$(4.2) \quad \bar{y} = \begin{cases} |x|^2 - 1 & |x| \leq 1 \\ v(|x|) + (1 - \phi(|x|))w(x) & 1 \leq |x| \leq 3 \\ w(x) & 3 \leq |x| \end{cases},$$

where

$$(4.3a) \quad v(|x|) = (|x|^2 - 1)(1 - \frac{|x| - 1}{2})^4 + \frac{1}{4}(|x| - 1)^2(|x| - 3)^4,$$

$$(4.3b) \quad \phi(|x|) = (1 + 4\frac{|x| - 1}{2} + 10(\frac{|x| - 1}{2})^2 + 20(\frac{|x| - 1}{2})^3)(1 - \frac{|x| - 1}{2})^4,$$

$$(4.3c) \quad w(x) = 2 \sin(\frac{\pi}{8}(x_1 + 4))^3 \sin(\frac{\pi}{8}(x_2 + 4))^3.$$

Then we take (cf. (4.1))

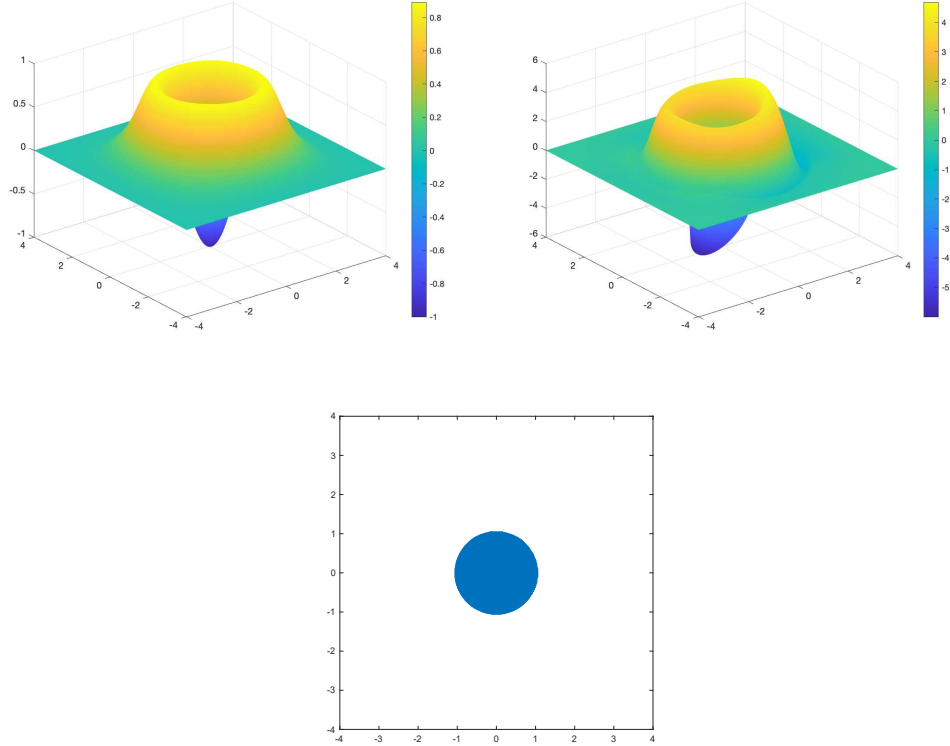
$$y_d = \begin{cases} \mathcal{L}^t \mathcal{L} \bar{y} + \bar{y} & |x| > 1 \\ \mathcal{L}^t \mathcal{L} \bar{y} + \bar{y} + 2 & |x| \leq 1 \end{cases}.$$

By construction $\bar{y} \leq \psi$ and the active set $\mathfrak{A}$ is the closed disk $D = \{x : |x| \leq 1\}$. The choices of y_d , $\bar{y}$ and ψ lead to the optimality conditions (2.7)–(2.9) with the measure μ defined by

$$(4.4) \quad \int_{\Omega} z d\mu = -2 \int_D z dx - 42 \int_{\partial D} z ds.$$

The numerical results for the first test where $\zeta = [1, 0]^t$ are reported in Table 4.1. The state, the control and the active set computed at level 9 are shown in Figure 4.1. The convergence in the H^1 norm for the state is approaching $O(h)$, which agrees with the estimate in Theorem 3.2. We also observe that the orders of convergence in the L_2 and L_{∞} norm for the state and the L_2 norm for the control are higher than the ones predicted by Theorem 3.2 and Theorem 3.4. This is likely due to the additional regularity enjoyed by the exact solution $\bar{y}$ defined by (4.2). Indeed $\bar{y}$ is a piecewise C^{∞} function that belongs to $C^2(\bar{\Omega})$ and hence has higher regularity than as indicated by (2.6).

k	$\ \bar{y} - \bar{y}_h\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_h _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_h\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_h\ _{L_\infty(\Omega)}$	Order
1	6.68e+00	-	1.17e+01	-	1.43e+01	-	2.29e+00	-
2	6.05e+00	0.14	9.15e+00	0.35	1.68e+01	-0.23	2.52e+00	-0.14
3	4.11e-01	3.88	1.96e+00	2.22	4.28e+00	1.97	1.81e-01	3.80
4	6.41e-01	-0.64	1.13e+00	0.80	1.91e+00	1.16	1.58e-01	0.20
5	2.30e-01	1.48	5.16e-01	1.13	6.97e-01	1.45	5.79e-02	1.45
6	8.68e-02	1.40	2.44e-01	1.08	2.69e-01	1.37	2.32e-02	1.32
7	3.08e-02	1.49	1.17e-01	1.06	9.82e-02	1.45	9.29e-03	1.32
8	7.18e-03	2.10	5.53e-02	1.08	3.23e-02	1.61	2.95e-03	1.65
9	3.10e-03	1.21	2.46e-02	1.17	1.17e-02	1.46	1.10e-03	1.43

TABLE 4.1. Convergence results for Example 4.1 with $\zeta = [1, 0]^t$.FIGURE 4.1. State, control and active set at level 9 for Example 4.1 with $\zeta = [1, 0]^t$

The numerical results for the second test where $\zeta = [2, 2]^t$ are reported in Table 4.2. We observe similar behavior as in the first test case.

k	$\ \bar{y} - \bar{y}_h\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_h _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_h\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_h\ _{L_\infty(\Omega)}$	Order
1	8.42e+00	-	1.83e+01	-	1.84e+01	-	3.91e+00	-
2	5.24e+00	0.68	8.30e+00	1.14	1.62e+01	0.18	2.24e+00	0.80
3	3.47e-01	3.92	2.00e+00	2.05	4.33e+00	1.91	1.60e-01	3.81
4	4.99e-01	-0.53	1.07e+00	0.90	1.84e+00	1.23	1.27e-01	0.33
5	1.82e-01	1.46	4.98e-01	1.10	6.64e-01	1.47	5.09e-02	1.32
6	6.87e-02	1.41	2.38e-01	1.07	2.54e-01	1.39	2.02e-02	1.33
7	2.47e-02	1.47	1.15e-01	1.05	9.20e-02	1.46	7.98e-03	1.34
8	5.91e-03	2.06	5.50e-02	1.07	3.04e-02	1.60	2.48e-03	1.69
9	2.51e-03	1.23	2.45e-02	1.17	1.10e-02	1.46	9.30e-04	1.41

TABLE 4.2. Convergence results for Example 4.1 with $\zeta = [2, 2]^t$.

Example 4.2 (L-shaped Domain with Singular Function). In this example Ω is the L-shaped domain $\Omega = [-8, 8]^2 \setminus ([0, 8] \times [-8, 0])$, $\zeta = [2, 1]^t$ and $g = 10$.

Let the singular function ψ_s be defined by

$$\begin{aligned} \mathcal{L}\psi_s &= 0 \quad \text{in } \Omega, \\ \psi_s &= 1 \quad \text{on } \partial\Omega. \end{aligned}$$

We take the exact solution to be

$$\bar{y} = \tilde{y} + 10\psi_s,$$

where

$$\tilde{y} = \begin{cases} |x - x_*|^2 - 1 & |x - x_*| \leq 1 \\ v(|x - x_*|) + (1 - \phi(|x - x_*|))w(x - x_*) & 1 \leq |x - x_*| \leq 3 \\ w(x - x_*) & 3 \leq |x - x_*| \end{cases},$$

$x_* = (-4, 4)$, and the functions v , ϕ and w are defined in (4.3).

The constraint function ψ is given by

$$\psi = |x - x_*|^2 - 1 + 10\psi_s$$

and the desired state is

$$y_d = \begin{cases} \mathcal{L}^t \mathcal{L} \bar{y} + \bar{y} + 10\psi_s & |x - x_*| > 1 \\ \mathcal{L}^t \mathcal{L} \bar{y} + \bar{y} + 2 + 10\psi_s & |x - x_*| \leq 1 \end{cases}.$$

By construction the active set of this problem is the closed unit disc D centered at x_* , and (4.4) remains valid for all $z \in \mathring{E}(\Delta; L_2(\Omega))$. We solve the discrete problem on uniform meshes and graded meshes (cf. [28]). The results are reported in Table 4.3 and Table 4.4. We can clearly observe improvements in the orders of convergence in the H^1 and L_∞ norms for the state on graded meshes, which agrees with Theorem 3.2 and Theorem 3.4.

k	$\frac{\ \bar{y}-\bar{y}_h\ _{L_2(\Omega)}}{\ \bar{y}\ _{L_2(\Omega)}}$	Order	$\frac{ \bar{y}-\bar{y}_h _{H^1(\Omega)}}{ \bar{y} _{H^1(\Omega)}}$	Order	$\frac{\ \bar{u}-\bar{u}_h\ _{L_2(\Omega)}}{\ \bar{u}\ _{L_2(\Omega)}}$	Order	$\frac{\ \bar{y}-\bar{y}_h\ _{L_\infty(\Omega)}}{\ \bar{y}\ _{L_\infty(\Omega)}}$	Order
1	3.89e-01	-	7.61e-01	-	8.71e-01	-	4.98e-01	-
2	1.79e-01	1.12	5.05e-01	0.59	8.33e-01	0.06	3.12e-01	0.67
3	3.90e-02	2.19	2.82e-01	0.84	2.14e-01	1.96	1.68e-01	0.89
4	1.35e-02	1.53	1.52e-01	0.90	9.14e-02	1.23	9.06e-02	0.89
5	4.35e-03	1.63	7.98e-02	0.92	3.30e-02	1.47	5.25e-02	0.79
6	1.52e-03	1.52	4.24e-02	0.91	1.26e-02	1.39	3.22e-02	0.71
7	5.34e-04	1.51	2.30e-02	0.89	4.56e-03	1.46	1.96e-02	0.71
8	1.30e-04	2.04	1.26e-02	0.86	1.50e-03	1.61	1.16e-02	0.76

TABLE 4.3. Convergence results on uniform meshes for Example 4.2.

k	$\frac{\ \bar{y}-\bar{y}_h\ _{L_2(\Omega)}}{\ \bar{y}\ _{L_2(\Omega)}}$	Order	$\frac{ \bar{y}-\bar{y}_h _{H^1(\Omega)}}{ \bar{y} _{H^1(\Omega)}}$	Order	$\frac{\ \bar{u}-\bar{u}_h\ _{L_2(\Omega)}}{\ \bar{u}\ _{L_2(\Omega)}}$	Order	$\frac{\ \bar{y}-\bar{y}_h\ _{L_\infty(\Omega)}}{\ \bar{y}\ _{L_\infty(\Omega)}}$	Order
1	4.31e-01	-	7.91e-01	-	1.31e+00	-	6.33e-01	-
2	2.11e-01	1.03	5.72e-01	0.47	6.52e-01	1.01	4.47e-01	0.50
3	8.65e-02	1.29	3.63e-01	0.66	4.46e-01	0.54	2.46e-01	0.86
4	1.90e-02	2.19	2.03e-01	0.84	1.33e-01	1.74	1.00e-01	1.30
5	7.60e-03	1.32	1.06e-01	0.94	6.38e-02	1.06	3.29e-02	1.61
6	2.56e-03	1.57	5.38e-02	0.98	2.61e-02	1.29	9.49e-03	1.79
7	9.40e-04	1.45	2.68e-02	1.00	1.04e-02	1.33	3.09e-03	1.62
8	2.39e-04	1.98	1.34e-02	1.01	3.34e-03	1.64	1.76e-03	0.81

TABLE 4.4. Convergence results on graded meshes for Example 4.2.

Example 4.3 (Three Dimensional Analog of Example 4.1). In this example Ω is the cubic domain $(-4, 4)^3$, $\psi = |x|^2 - 1$, $\zeta = [1, 1, 1]^t$ and $g = 0$. We define $\bar{y}$ again by (4.2), where we modify w as follows:

$$w(x) = 2 \sin\left(\frac{\pi}{8}(x_1 + 4)\right)^3 \sin\left(\frac{\pi}{8}(x_2 + 4)\right)^3 \sin\left(\frac{\pi}{8}(x_3 + 4)\right)^3.$$

As in Example 4.1, the active set is the closed ball $\mathfrak{A} = \{x : |x| \leq 1\}$. The convergence results are shown in Table 4.5. Again we observe $O(h)$ convergence in the H^1 norm for the state, which agrees with the estimate in Theorem 3.2. We also observe better convergence than the ones predicted by Theorem 3.2 and Theorem 3.4 in the L_2 and L_∞ norms for the state and the L_2 norm for the control, which can be explained by the fact that the exact solution $\bar{y}$ is a piecewise C^∞ function that belongs to $C^2(\bar{\Omega})$.

k	$\ \bar{y} - \bar{y}_h\ _{L_2(\Omega)}$	Order	$ \bar{y} - \bar{y}_h _{H^1(\Omega)}$	Order	$\ \bar{u} - \bar{u}_h\ _{L_2(\Omega)}$	Order	$\ \bar{y} - \bar{y}_h\ _{L_\infty(\Omega)}$	Order
1	1.01e+01	0.00	2.65e+01	0.00	2.76e+01	0.00	2.29e+00	0.00
2	1.15e+01	-0.18	1.47e+01	0.85	2.84e+01	-0.04	1.48e+00	0.62
3	1.01e+00	3.51	3.85e+00	1.93	7.22e+00	1.98	1.92e-01	2.95
4	5.63e-01	0.84	1.86e+00	1.05	2.60e+00	1.47	8.50e-02	1.18
5	1.11e-01	2.35	8.41e-01	1.14	7.98e-01	1.70	2.15e-02	1.99
6	4.30e-02	1.37	3.72e-01	1.18	2.78e-01	1.52	6.92e-03	1.63

TABLE 4.5. Convergence results for Example 4.3.

5. CONCLUDING REMARKS

We have extended the P_1 finite element method based on mass lumping in [8] to elliptic optimal control problems with a general state equation. Since we do not use any stabilization technique in the discretization of the state equation, the performance of our method depends on ζ . For instance, if the vector field ζ in Example 4.1 is taken to be $[1000, 1000]^t$, then the resulting errors at the beginning refinement levels are two orders higher than the corresponding ones in Table 4.2. It is both interesting and challenging to construct more sophisticated $\mathcal{L}_h$ in order to obtain methods that are robust with respect to ζ . We note that in the case of control constraints this was investigated in [31, 36].

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APPENDIX A. INTERIOR REGULARITY OF $\bar{z}$

We will establish (2.4) by relating (2.3) to fourth order variational inequalities analyzed in [18, 26, 27].

It follows from (2.3) and the Riesz representation theorem for non-negative functionals (cf. [25, 43, 46]) that

$$(A.1) \quad \beta \left[(\mathcal{L}\bar{z}, \mathcal{L}z)_{L_2(\Omega)} + (g, \mathcal{L}z)_{L_2(\Omega)} \right] + (\bar{z} - z_d, z)_{L_2(\Omega)} = \int_{\Omega} z \, d\nu \quad \forall z \in \mathring{E}(\Delta; L_2(\Omega)),$$

where ν is a non-positive regular Borel measure. Moreover,

$$(A.2) \quad \nu \text{ is supported on } \mathfrak{A} = \{x \in \Omega : \bar{z} = \tilde{\psi}(x)\} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$$

by the principle of virtual work.

Let ϕ be any C^∞ function with compact support in Ω such that

$$(A.3) \quad \phi = 1 \quad \text{on an open neighborhood of } \mathfrak{A}.$$

We will show that $\tilde{z} = \phi\bar{z}$ belongs to $H^3(\Omega) \cap W_\infty^2(\Omega)$, which then implies (2.4).

Given any $z \in \mathring{E}(\Delta; L_2(\Omega))$, we have, in view of (1.7),

$$\begin{aligned} \mathcal{L}(\phi z) &= -\Delta(\phi z) + \zeta \cdot \nabla(\phi z) + \gamma(\phi z) \\ (A.4) \quad &= -(\Delta\phi)z - 2\nabla\phi \cdot \nabla z - \phi(\Delta z) + (\zeta \cdot \nabla\phi)z + \phi(\zeta \cdot \nabla z) + \phi(\gamma z) \\ &= \phi\mathcal{L}z + \mathcal{M}z, \end{aligned}$$

where $\mathcal{M} : \mathring{E}(\Delta; L_2(\Omega)) \rightarrow H_0^1(\Omega)$ is defined by

$$(A.5) \quad \mathcal{M}z = -2\nabla\phi \cdot \nabla z + (\zeta \cdot \nabla\phi - \Delta\phi)z.$$

Here $\mathcal{M}z \in H_0^1(\Omega)$ because $\mathring{E}(\Delta; L_2(\Omega))$ is a subspace of $H_{loc}^2(\Omega)$.

Note that $\tilde{z}$ belongs to

$$(A.6) \quad K_{\dagger} = \{z_{\dagger} \in H_0^2(\Omega) : z_{\dagger} \leq \phi\tilde{\psi} \text{ on } \Omega\},$$

and we have, in view of (A.2) and (A.3),

$$(A.7) \quad \int_{\Omega} \phi(z_{\dagger} - \tilde{z})d\nu = \int_{\mathfrak{A}} (z_{\dagger} - \phi\tilde{\psi})d\nu \geq 0 \quad \forall z_{\dagger} \in K_{\dagger}.$$

It follows from (A.1), (A.4) and (A.7) that

$$\begin{aligned} (\mathcal{L}\tilde{z}, \mathcal{L}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} &= (\mathcal{L}\tilde{z}, \phi\mathcal{L}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} + (\mathcal{M}\tilde{z}, \mathcal{L}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &= (\mathcal{L}\tilde{z}, \mathcal{L}(\phi(z_{\dagger} - \tilde{z})))_{L_2(\Omega)} - (\mathcal{L}\tilde{z}, \mathcal{M}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &\quad + (\mathcal{M}\tilde{z}, \mathcal{L}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &\geq -\beta^{-1}(\tilde{z} - z_d, \phi(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} - (g, \mathcal{L}(\phi(z_{\dagger} - \tilde{z})))_{L_2(\Omega)} \\ &\quad - (\mathcal{L}\tilde{z}, \mathcal{M}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} + (\mathcal{M}\tilde{z}, \mathcal{L}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)}, \end{aligned}$$

which together with (1.7) implies

$$\begin{aligned} &(\Delta\tilde{z}, \Delta(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &\geq -\beta^{-1}(\tilde{z} - z_d, \phi(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} - (\mathcal{L}\tilde{z}, \mathcal{M}(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &\quad - (g, -\Delta(\phi(z_{\dagger} - \tilde{z})) + (\zeta \cdot \nabla)(\phi(z_{\dagger} - \tilde{z})) + \gamma\phi(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ (A.8) \quad &+ (\mathcal{M}\tilde{z}, -\Delta(z_{\dagger} - \tilde{z}) + \zeta \cdot \nabla(z_{\dagger} - \tilde{z}) + \gamma(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &+ (\Delta\tilde{z}, \zeta \cdot \nabla(z_{\dagger} - \tilde{z}) + \gamma(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \\ &\quad - (\zeta \cdot \nabla\tilde{z} + \gamma\tilde{z}, -\Delta(z_{\dagger} - \tilde{z}) + \zeta \cdot \nabla(z_{\dagger} - \tilde{z}) + \gamma(z_{\dagger} - \tilde{z}))_{L_2(\Omega)}. \end{aligned}$$

Since $\tilde{z} = \phi\bar{z}$ belongs to $H_0^2(\Omega)$, $\mathcal{M}\bar{z}$ belongs to $H_0^1(\Omega)$, g belongs to $H^4(\Omega)$, ζ belongs to $[W_{\infty}^1(\Omega)]^n$ and γ belongs to $W_{\infty}^1(\Omega)$, we can use (A.5) and integration by parts to rewrite (A.8) in the form of

$$(A.9) \quad (\Delta\tilde{z}, \Delta(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} \geq \sum_{i=1}^n (f_i, \partial_i(z_{\dagger} - \tilde{z}))_{L_2(\Omega)} + (f_0, z_{\dagger} - \tilde{z})_{L_2(\Omega)} \quad \forall z_{\dagger} \in K_{\dagger},$$

where $f_i \in L_2(\Omega)$ for $0 \leq i \leq n$.

Note that (A.6) and (A.9) define a biharmonic variational inequality treated in [26]. Therefore we can apply the interior regularity result there to conclude that $\tilde{z} \in H_{loc}^3(\Omega)$, and hence $\tilde{z} \in H^3(\Omega)$ because $\tilde{z}$ is compactly supported in Ω . We can also conclude that $\tilde{z} \in H_{loc}^3(\Omega)$.

According to the Sobolev embedding theorem, we have $H^1(\Omega) \hookrightarrow L_6(\Omega)$ and $W_{6/5}^1(\Omega) \hookrightarrow L_2(\Omega)$ in both two and three dimensions. Hence we can use (A.4), the facts that $\zeta \in [W_\infty^1(\Omega)]^n$, $\gamma \in W_\infty^1(\Omega)$, $\tilde{z} \in H_{loc}^3(\Omega)$ together with integration by parts to rewrite (A.8) in the form of

$$(A.10) \quad (\Delta \tilde{z}, \Delta(z_\dagger - \tilde{z}))_{L_2(\Omega)} \geq F(z_\dagger - \tilde{z}),$$

where $F \in W_6^{-1}(\Omega)$.

Let $\rho \in H_0^2(\Omega)$ be defined by

$$(\Delta \rho, \Delta v)_{L_2(\Omega)} = F(v) \quad \forall v \in H_0^2(\Omega).$$

Then ρ belongs to $W_{6,loc}^3(\Omega) \subset W_{\infty,loc}^2(\Omega)$ by interior elliptic regularity (cf. [2, section 14]) and the Sobolev embedding theorem, and (A.10) becomes the variational inequality

$$(A.11) \quad (\Delta z_*, \Delta(z_\# - z_*))_{L_2(\Omega)} \geq 0 \quad \forall z_\# \in K_\#,$$

where

$$K_\# = \{z_\# \in H_0^2(\Omega) : z_\# \leq \phi\tilde{\psi} - \rho\}$$

and $z_* = \tilde{z} - \rho \in K_\#$.

We can now apply the interior regularity results in [18, 27] to the biharmonic variational inequality (A.11) to conclude that $z_* \in W_{\infty,loc}^2(\Omega)$, and hence $\tilde{z} = z_* + \rho \in W_\infty^2(\Omega)$ because $\tilde{z}$ is compactly supported in Ω .

APPENDIX B. ESTIMATES FOR $R_h \bar{y}$

It follows from the assumptions on ζ and γ that we have

$$(B.1) \quad a(y, z) \leq C \|y\|_{H^1(\Omega)} \|z\|_{H^1(\Omega)} \quad \forall y, z \in H^1(\Omega),$$

and also the following Gårding inequality (cf. [12, Theorem 5.6.8])

$$(B.2) \quad a(z, z) + \kappa \|z\|_{L_2(\Omega)}^2 \geq \frac{1}{2} \|z\|_{H^1(\Omega)}^2 \quad \forall z \in H^1(\Omega),$$

where κ is a positive constant.

Recall $I_h : C(\bar{\Omega}) \rightarrow V_h$ is the nodal interpolation operator and there is a standard estimate (cf. [12, 20, 23])

$$(B.3) \quad |\zeta - I_h \zeta|_{H^s(T)} \leq Ch_T^{t-s} |\zeta|_{H^t(T)}$$

that holds for $t > n/2$, $0 \leq s \leq t$, $\zeta \in H^t(T)$ and $T \in \mathcal{T}_h$.

In view of (1.10) and (B.3), we have the following interpolation error estimate (cf. [3, 12, 20, 23, 30]):

$$(B.4) \quad \|z - I_h z\|_{L_2(\Omega)} + h \|z - I_h z\|_{H^1(\Omega)} \leq Ch^{1+\tau} \|\Delta z\|_{L_2(\Omega)} \quad \forall z \in \mathring{E}(\Delta; L_2(\Omega)),$$

where τ is defined in (3.5). It follows from (B.4) that

$$(B.5) \quad \|\bar{y} - I_h \bar{y}\|_{L_2(\Omega)} + h \|\bar{y} - I_h \bar{y}\|_{H^1(\Omega)} \leq Ch^{1+\tau}$$

because $\bar{y} \in g + \mathring{E}(\Delta; L_2(\Omega))$ and $g \in H^4(\Omega)$.

As mentioned in Remark 3.3, the finite element approximation $R_h \bar{y}$ is well-defined for h sufficiently small.

Since the function $I_h \bar{y} - R_h \bar{y}$ belongs to $\mathring{V}_h \subset H_0^1(\Omega)$, we have

$$\begin{aligned} \frac{1}{2} \|I_h \bar{y} - R_h \bar{y}\|_{H^1(\Omega)}^2 &\leq a(I_h \bar{y} - R_h \bar{y}, I_h \bar{y} - R_h \bar{y}) + \kappa \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)}^2 \\ (B.6) \quad &= a(I_h \bar{y} - \bar{y}, I_h \bar{y} - R_h \bar{y}) + \kappa \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)}^2 \\ &\leq Ch^\tau \|I_h \bar{y} - R_h \bar{y}\|_{H^1(\Omega)} + \kappa \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)}^2 \end{aligned}$$

by (3.7), (B.1) and (B.5).

Let $\phi \in H_0^1(\Omega)$ be defined by

$$(B.7) \quad a(z, \phi) = (z, I_h \bar{y} - R_h \bar{y})_{L_2(\Omega)} \quad \forall z \in H_0^1(\Omega).$$

Then ϕ belongs to $\mathring{E}(\Delta; L_2(\Omega))$ and we have

$$(B.8) \quad \|\phi\|_{H^{1+\alpha}(\Omega)} \leq C \|\Delta \phi\|_{L_2(\Omega)} \leq C \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)}$$

by elliptic regularity.

It follows from (3.7) and (B.7) that

$$(B.9) \quad \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)}^2 = a(I_h \bar{y} - R_h \bar{y}, \phi - I_h \phi) + a(I_h \bar{y} - \bar{y}, I_h \phi),$$

and we can use (B.1), (B.4) and (B.8) to estimate the first term on the right-hand side of (B.9) by

$$(B.10) \quad a(I_h \bar{y} - R_h \bar{y}, \phi - I_h \phi) \leq Ch^\tau \|I_h \bar{y} - R_h \bar{y}\|_{H^1(\Omega)} \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)}.$$

According to (1.4), the second term on the right-hand side of (B.9) is given by

$$\begin{aligned} (B.11) \quad a(I_h \bar{y} - \bar{y}, I_h \phi) &= \int_{\Omega} \nabla(I_h \bar{y} - \bar{y}) \cdot \nabla(I_h \phi) dx + \int_{\Omega} [\zeta \cdot \nabla(I_h \bar{y} - \bar{y})] I_h \phi dx \\ &\quad + \int_{\Omega} \gamma(I_h \bar{y} - \bar{y}) I_h \phi dx, \end{aligned}$$

and we have

$$\begin{aligned} &\int_{\Omega} [\zeta \cdot \nabla(I_h \bar{y} - \bar{y})] I_h \phi dx + \int_{\Omega} \gamma(I_h \bar{y} - \bar{y}) I_h \phi dx \\ (B.12) \quad &= - \int_{\Omega} (I_h \bar{y} - \bar{y}) \zeta \cdot \nabla(I_h \phi) dx + \int_{\Omega} (\gamma - \nabla \cdot \zeta)(I_h \bar{y} - \bar{y}) I_h \phi dx \\ &\leq Ch^{1+\tau} \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \\ &\leq Ch^{2\tau} \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \end{aligned}$$

by (B.4) and (B.8).

It only remains to estimate the first term on the right-hand side of (B.11), which can be rewritten through integration by parts as

$$(B.13) \quad \int_{\Omega} \nabla(I_h \bar{y} - \bar{y}) \cdot \nabla(I_h \phi) dx = \sum_{\sigma \in \mathcal{S}_h} \int_{\sigma} (I_h \bar{y} - \bar{y}) [[\partial(I_h \phi - \phi)/\partial n]] dS,$$

where $\mathcal{S}_h$ is the set of all the sides, $[[\partial(I_h\phi - \phi)/\partial n]]$ is the jump of the normal derivative of $(I_h\phi - \phi)$ across σ , and dS denotes the infinitesimal length ($n = 2$) or infinitesimal area ($n = 3$).

Lemma B.1. *We have*

$$(B.14) \quad \int_{\Omega} \nabla(I_h\bar{y} - \bar{y}) \cdot \nabla(I_h\phi) dx \leq Ch^{2\tau} \|I_h\bar{y} - R_h\bar{y}\|_{L_2(\Omega)}.$$

Proof. Let σ be a side (edge if $n = 2$ and face if $n = 3$) of the element T . By the trace theorem with scaling, we have, for $1/2 < s \leq 1$,

$$(B.15) \quad \|\zeta\|_{L_2(\sigma)} \leq C[h_T^{-1/2}\|\zeta\|_{L_2(T)} + h_T^{s-(1/2)}|\zeta|_{H^s(T)}].$$

In the case of quasi-uniform meshes, we can use (B.3), (B.8), (B.13) and (B.15) to obtain

$$\begin{aligned} \int_{\Omega} \nabla(I_h\bar{y} - \bar{y}) \cdot \nabla(I_h\phi) dx &\leq C \sum_{T \in \mathcal{T}_h} h_T^{\alpha/2} |I_h\bar{y} - \bar{y}|_{H^{(1+\alpha)/2}(T)} h_T^{\alpha-(1/2)} |\phi|_{H^{1+\alpha}(T)} \\ &\leq Ch^{2\alpha} \|I_h\bar{y} - R_h\bar{y}\|_{L_2(\Omega)}, \end{aligned}$$

which is (B.14) for quasi-uniform meshes.

The case of graded meshes in two dimensions is more involved. Let $c_1, \dots, c_L$ be the corners of Ω and ω_ℓ be the interior angle at c_ℓ . We take α_ℓ to be a number $< \pi/\omega$ ($\alpha_\ell = 1$ if $\omega < \pi$) so that the index of elliptic regularity $\alpha = \min_{1 \leq \ell \leq L} \alpha_\ell$.

We can use (B.13) and (B.15) to obtain

$$(B.16) \quad \begin{aligned} \int_{\Omega} \nabla(I_h\bar{y} - \bar{y}) \cdot \nabla(I_h\phi) dx &\leq C \sum_{\ell=1}^L \sum_{T \in \mathcal{T}_{h,\ell}} h_T^{\alpha_\ell/2} |I_h\bar{y} - \bar{y}|_{H^{(1+\alpha_\ell)/2}(T)} h_T^{\alpha_\ell-(1/2)} |\phi|_{H^{1+\alpha_\ell}(T)} \\ &\quad + C \sum_{T \in \tilde{\mathcal{T}}_h} h_T^{1/2} |I_h\bar{y} - \bar{y}|_{H^1(T)} h_T^{1/2} |\phi|_{H^2(T)}, \end{aligned}$$

where $\mathcal{T}_{h,\ell}$ is the set of the triangles in $\mathcal{T}_h$ that touch the corner c_ℓ , and $\tilde{\mathcal{T}}_h = \mathcal{T}_h \setminus \left(\bigcup_{\ell=1}^L \mathcal{T}_{h,\ell}\right)$.

Note that $\bar{y}$ and ϕ belong to $H^2(T)$ for $T \in \tilde{\mathcal{T}}_h$ (cf. Remark 1.4).

The first sum on the right-hand side of (B.16) is bounded by

$$\begin{aligned} &\sum_{\ell=1}^L \sum_{T \in \mathcal{T}_{h,\ell}} h_T^{\alpha_\ell/2} |I_h\bar{y} - \bar{y}|_{H^{(1+\alpha_\ell)/2}(T)} h_T^{\alpha_\ell-(1/2)} |\phi|_{H^{1+\alpha_\ell}(T)} \\ &\leq C \sum_{\ell=1}^L \sum_{T \in \mathcal{T}_{h,\ell}} h_T^{2\alpha_\ell} |\bar{y}|_{H^{1+\alpha_\ell}(T)} |\phi|_{H^{1+\alpha_\ell}(T)} \leq Ch^2 \|I_h\bar{y} - R_h\bar{y}\|_{L_2(\Omega)}, \end{aligned}$$

where we have used (B.3), (B.8) and the fact that on the graded mesh we have $h_T \approx h^{1/\alpha_\ell}$ if $T \in \mathcal{T}_{h,\ell}$ (cf. [3, Section 4] and [30, Section 8.4.1]).

Finally the second sum on the right-hand side of (B.16) is bounded by

$$\begin{aligned}
\sum_{T \in \tilde{\mathcal{T}}_h} h_T^{1/2} |I_h \bar{y} - \bar{y}|_{H^1(T)} h_T^{1/2} |\phi|_{H^2(T)} &\leq C \sum_{T \in \tilde{\mathcal{T}}_h} h_T^2 |\bar{y}|_{H^2(T)} |\phi|_{H^2(T)} \\
&\leq Ch^2 \left(\sum_{T \in \tilde{\mathcal{T}}_h} (h_T/h)^2 |\bar{y}|_{H^2(T)}^2 \right)^{\frac{1}{2}} \left(\sum_{T \in \tilde{\mathcal{T}}_h} (h_T/h)^2 |\phi|_{H^2(T)}^2 \right)^{\frac{1}{2}} \\
&\leq Ch^2 \|\Delta \bar{y}\|_{L_2(\Omega)} \|\Delta \phi\|_{L_2(\Omega)} \\
&\leq Ch^2 \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)},
\end{aligned}$$

where we have used (B.3), (B.8), the fact that on the graded mesh we have

$$(h_T/h) \approx (\text{distance between } T \text{ and the closest corner } c_\ell)^{1-\alpha_\ell} \quad \text{if } T \in \tilde{\mathcal{T}}_h,$$

together with the nature of the singularity at a reentrant corner of Ω (cf. [3, Section 4] and [30, Section 8.4.1]). $\square$

Putting (B.9)–(B.12) and (B.14) together, we arrive at the estimate

$$(B.17) \quad \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \leq Ch^\tau (h^\tau + \|I_h \bar{y} - R_h \bar{y}\|_{H^1(\Omega)}).$$

It follows from (B.6) and (B.17) that

$$\|I_h \bar{y} - R_h \bar{y}\|_{H^1(\Omega)} \leq Ch^\tau \quad \text{and} \quad \|I_h \bar{y} - R_h \bar{y}\|_{L_2(\Omega)} \leq Ch^{2\tau},$$

which together with (B.4) imply (3.9) and (3.10).

Finally the estimate (3.11) follows from (2.6), (3.10) and the interior maximum norm estimate in [45, Eq. (0.8)].

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