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A C^1 virtual element method for an elliptic distributed optimal control problem with pointwise state constraints

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We design and analyze a C^1 virtual element method for an elliptic distributed optimal control problem with pointwise state constraints. Theoretical estimates and corroborating numerical results are presented.

Keywords: Elliptic distributed optimal control problems; pointwise state constraints; C^1 virtual element method.

AMS Subject Classification 2020: 65K15, 65N30

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain, $y_d \in L^2(\Omega)$ and β be a positive constant. The elliptic optimal control problem[30] is given by

$$\min_{(y,u)\in Y_{ad}\times L^2(\Omega)} J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2(\Omega)} \tag{1.1}$$

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subject to

$$\int_\Omega \nabla y \cdot \nabla z \, dx = \int_\Omega uz \, dx \quad \forall \, z \in H_0^1(\Omega), \tag{1.2}$$

where

$$Y_{ad} = \{y \in H_0^1(\Omega) : y \le \psi \text{ a.e. in } \Omega\}. \tag{1.3}$$

We assume that

$$\psi \text{ belongs to } W^{3,p}(\Omega) \quad \text{for } p > 2 \quad \text{and} \quad \psi > 0 \text{ on } \partial\Omega. \tag{1.4}$$

Here and throughout the paper we will follow standard notation for function spaces, norms and differential operators that can be found for example in Refs. 34, 1, 20 and 10.

Our goal is to design and analyze a C^1 virtual element method for the optimal control problem (1.1)–(1.3) that is based on Refs. 27, 33 and 2. The key is to transform the optimal control problem into an equivalent fourth-order variational inequality for the state that can be solved by many numerical methods originally designed for fourth-order elliptic boundary value problems. This idea was first proposed in Ref. 48 for a nonconforming finite element method, and later it was extended to other finite element methods in Refs. 40, 24, 13, 25, 18, 15, 14, 26 and 19. A summary of this new approach can be found in Refs. 21 and 12.

Unlike elliptic optimal control problems with control constraints, the Lagrange multiplier that appears in the first-order optimality conditions for (1.1)–(1.3) is only a Borel measure (cf. (2.10)–(2.12)). Therefore, error analysis that involves the approximation of the multiplier or the adjoint state is challenging. But these complications are completely absent from the new approach which only involves the state variable. Comparing with the approaches in Refs. 36, 49, 44 and 31, where the convergence is established in the H^1 norm for the state and the L^2 norm for the control, the new approach can also obtain convergence in the L^∞ norm for the state.

The rest of the paper is organized as follows. We recall some results for the continuous problem in Sec. 2 and introduce the discrete problem in Sec. 3. We present in Sec. 4 some tools and preliminary estimates for the convergence analysis, which is carried out in Sec. 5. Numerical results are given in Sec. 6 and we end with some concluding remarks in Sec. 7.

We will use C, with or without subscript, to denote a generic positive constant that does not depend on the mesh size. We also use the symbol $A \lesssim B$ to represent $A \le (\text{constant})B$ where the positive constant is independent of the mesh size. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. The Continuous Problem

Since Ω is convex, the solution y of the state equation (1.2) belongs to $H^2(\Omega)$ by elliptic regularity.[35,41] Hence the optimal control problem (1.1)–(1.3) can be

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reformulated as

$$\min_{y\in\mathbb{K}}\left[\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\beta}{2}\|\Delta y\|^2_{L^2(\Omega)}\right], \tag{2.1}$$

where

$$\mathbb{K}=\{y\in H^2(\Omega)\cap H^1_0(\Omega):\, y\le\psi \text{ in } \Omega\}. \tag{2.2}$$

Let the bilinear form $a(\cdot,\cdot)$ be defined by

$$a(y,z)=\int_\Omega(\Delta y)(\Delta z)dx. \tag{2.3}$$

We can rewrite (2.1) as

$$\min_{y\in\mathbb{K}}\left[\frac{1}{2}\mathcal{A}(y,y)-(y_d,y)\right], \tag{2.4}$$

where

$$\mathcal{A}(y,z)=\beta a(y,z)+(y,z) \tag{2.5}$$

and

$$(y,z)=\int_\Omega yz\,dx. \tag{2.6}$$

Remark 2.1. For functions $y,z\in H^2(\Omega)\cap H^1_0(\Omega)$, the bilinear form $a(\cdot,\cdot)$ in (2.3) has an alternative expression[41] given by

$$a(y,z)=\sum_{i,j=1}^{2}\int_\Omega\left(\frac{\partial^2 y}{\partial x_i\partial x_j}\right)\left(\frac{\partial^2 z}{\partial x_i\partial x_j}\right)dx:=\int_\Omega\nabla^2 y:\nabla^2 z dx. \tag{2.7}$$

It follows from (2.7) and a Poincaré–Friedrichs inequality[50] that

$$a(v,v)=|v|^2_{H^2(\Omega)}\approx\|v\|^2_{H^2(\Omega)}\quad\forall\, v\in H^2(\Omega)\cap H^1_0(\Omega)$$

and also

$$\|v\|^2_{\mathcal{A}}=\mathcal{A}(v,v)=\beta|v|^2_{H^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\approx\|v\|^2_{H^2(\Omega)}\quad\forall\, v\in H^2(\Omega)\cap H^1_0(\Omega). \tag{2.8}$$

Since $\mathbb{K}$ is a non-empty closed convex subset of $H^2(\Omega)\cap H^1_0(\Omega)$, the minimization problem (2.4)–(2.6) has a unique solution $\bar{y}\in\mathbb{K}$ by the standard theory[46] that is characterized by the fourth-order variational inequality

$$\mathcal{A}(\bar{y},y-\bar{y})-(y_d,y-\bar{y})\ge 0\quad\forall\, y\in\mathbb{K}. \tag{2.9}$$

The variational inequality (2.9) is equivalent to the following Karush–Kuhn–Tucker conditions (cf. Refs. 30 and 21):

$$\mathcal{A}(\bar{y},z)-(y_d,z)=\int_\Omega z\,d\mu\quad\forall\, z\in H^2(\Omega)\cap H^1_0(\Omega), \tag{2.10}$$

$$\mu \text{ is a nonpositive finite Borel measure} \tag{2.11}$$

$$\int_\Omega(\psi-\bar{y})d\mu=0. \tag{2.12}$$

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Remark 2.2. The complementarity condition (2.12) is equivalent to the statement that the support of μ is the active set $\mathfrak{A} = \{x \in \Omega : \bar{y}(x) = \psi(x)\}$. Note that $\mathfrak{A}$ is a compact subset of Ω by the assumption (1.4) on ψ.

The following regularity results, which are based on the assumption (1.4) on ψ and the work in Refs. 38, 39, 28 and 31, can be found in Refs. 21 and 15.

$$\bar{y} \in H^3_{\mathrm{loc}}(\Omega) \cap W^{2,\infty}_{\mathrm{loc}}(\Omega) \cap H^{2+\alpha}(\Omega), \tag{2.13}$$

$$\mu \in H^{-1}(\Omega), \tag{2.14}$$

where the index of elliptic regularity $\alpha \in (0, 1]$ is determined by the angles at the corners of Ω (cf. Refs. 9, 41 and 35).

3. The Discrete Problem

The discrete problem is defined on a H^2-conforming C^1 virtual element space associated with a triangulation of Ω.

3.1. *The triangulation $\mathcal{T}_h$*

Let $\mathcal{T}_h$ be a triangulation of Ω by polygons, i.e. $\bar{\Omega}$ is the union of the (closed) polygons from $\mathcal{T}_h$ and the intersection of two distinct (closed) polygons from $\mathcal{T}_h$ is either empty, a common vertex or a common edge.

The set of the vertices (respectively, edges) of a polygon $K \in \mathcal{T}_h$ is denoted by $\mathcal{V}_K$ (respectively, $\mathcal{E}_K$), and the diameter of K (respectively, $e \in \mathcal{E}_K$) is denoted by h_K (respectively, h_e). The mesh size of $\mathcal{T}_h$ is $h = \max_{K\in\mathcal{T}_h} h_K$.

We make the following assumptions[4] on the shape regularity of $\mathcal{T}_h$. There exists a constant $\gamma \in (0, 1)$ such that

$$K \text{ is star-shaped with respect a ball of radius } \gamma h_K \text{ for all } K \in \mathcal{T}_h, \tag{3.1a}$$

$$h_e \geq \gamma h_K \quad \text{for all } e \in \mathcal{E}_K \quad \text{and all } K \in \mathcal{T}_h. \tag{3.1b}$$

3.2. *A local virtual element space*

Let K be a polygon in $\mathcal{T}_h$ and $e \in \mathcal{E}_K$. We will denote by $\mathbb{P}_\ell(K)$ (respectively, $\mathbb{P}_\ell(e)$) the space of the restrictions of polynomials of total degree $\leq \ell$ to K (respectively, e).

For $\zeta, \eta \in H^2(K)$, we define

$$\begin{aligned}((\zeta, \eta))_K = \int_K \nabla^2\zeta : \nabla^2\eta \, dx + \left(\int_{\partial K} \zeta \, ds\right)\left(\int_{\partial K} \eta \, ds\right) \\ + \left(\int_{\partial K} \nabla\zeta \, ds\right) \cdot \left(\int_{\partial K} \nabla\eta \, ds\right).\end{aligned} \tag{3.2}$$

It follows from a Poincaré–Friedrichs inequality[50] that $((\cdot, \cdot))_K$ is an inner product on $H^2(K)$ and the norm defined by $((\cdot, \cdot))_K$ is equivalent to the standard norm of

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the Sobolev space $H^2(K)$. Therefore, we can define a projection operator $\Pi_K^\Delta : H^2(K) \to \mathbb{P}_2(K)$ by

$$(\!(\Pi_K^\Delta \zeta, q)\!)_K = (\!(\zeta, q)\!)_K \quad \forall\, q \in \mathbb{P}_2(K),$$

or equivalently (cf. Ref. 33),

$$\int_K \nabla^2(\Pi_K^\Delta \zeta) : \nabla^2 q\, dx = \int_K \nabla^2 \zeta : \nabla^2 q\, dx \quad \forall\, q \in \mathbb{P}_2(K), \tag{3.3}$$

$$\int_{\partial K} \Pi_K^\Delta \zeta\, ds = \int_{\partial K} \zeta\, ds, \tag{3.4}$$

$$\int_{\partial K} \nabla(\Pi_K^\Delta \zeta) ds = \int_{\partial K} \nabla \zeta\, ds. \tag{3.5}$$

Note that (3.3) implies

$$|\Pi_K^\Delta \zeta|_{H^2(K)} \le |\zeta|_{H^2(K)} \quad \forall\, \zeta \in H^2(K). \tag{3.6}$$

In view of the integration by parts formula

$$\int_K \nabla^2 \zeta : \nabla^2 \eta\, dx = \int_{\partial K} \nabla \zeta \cdot \left(\frac{\partial}{\partial n} \nabla \eta\right) ds - \int_{\partial K} \zeta \left(\frac{\partial}{\partial n} \Delta \eta\right) ds + \int_K \zeta (\Delta^2 \eta) dx$$

that holds for $\zeta \in H^2(K)$ and $\eta \in H^4(K)$, we have

$$\int_K \nabla^2 \zeta : \nabla^2 q\, dx = \int_{\partial K} \nabla \zeta \cdot \left(\frac{\partial}{\partial n} \nabla q\right) ds - \int_{\partial K} \zeta \left(\frac{\partial}{\partial n} \Delta q\right) ds,$$

for all $\zeta \in H^2(K)$ and $q \in \mathbb{P}_2(K)$. Therefore, we can compute $\Pi_K^\Delta \zeta$ through (3.3)–(3.5) in terms of ζ and $\nabla \zeta$ on ∂K.

We are now ready to define the local virtual element space $\mathcal{Q}(K) \subset H^2(K)$. A function $v \in H^2(K)$ belongs to the virtual element space $\mathcal{Q}(K)$ if and only if it satisfies the following conditions:

$$v|_e \in \mathbb{P}_3(e) \qquad \forall\, e \in \mathcal{E}_K, \tag{3.7}$$

$$(\partial v/\partial n)|_e \in \mathbb{P}_1(e) \qquad \forall\, e \in \mathcal{E}_K, \tag{3.8}$$

$$\Delta^2 v \in \mathbb{P}_2(K), \tag{3.9}$$

$$\Pi_K^0 v - \Pi_K^\Delta v = 0, \tag{3.10}$$

where (3.9) is understood in the sense of distributions and Π_K^0 is the L^2 projection operator from $L^2(K)$ to $\mathbb{P}_2(K)$. It is clear from the definition that $\mathbb{P}_2(K)$ is a subspace of $\mathcal{Q}(K)$.

Remark 3.1. The virtual element $\mathcal{Q}(K)$ defined by (3.7)–(3.10) is similar to the one in Ref. 2, where the projection Π_K^Δ was defined with respect to a discrete inner product instead of the inner product defined by (3.2). One can show by essentially the same arguments for Lemma 2.3 in Ref. 2 that the dimension of $\mathcal{Q}(K)$ is $3|\mathcal{V}_K|$

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and a function in $\mathcal{Q}(K)$ is uniquely determined by its derivatives up to order 1 at the vertices.

Since $\mathbb{P}_2(K)$ is invariant under Π_K^0, we have, by the shape regularity assumption (3.1) and the Bramble–Hilbert lemma,[11,37]

$$\|\zeta - \Pi_K^0\zeta\|_{L^2(K)} + h_K|\zeta - \Pi_K^0\zeta|_{H^1(K)} \le Ch_K^2|\zeta|_{H^2(K)} \qquad \forall\,\zeta \in H^2(K), \quad (3.11)$$

$$\|\zeta - \Pi_K^0\zeta\|_{L^\infty(K)} \le Ch_K|\zeta|_{H^2(K)} \qquad \forall\,\zeta \in H^2(K), \quad (3.12)$$

$$\|\zeta - \Pi_K^0\zeta\|_{H^2(K)} \le Ch_K^\alpha|\zeta|_{H^{2+\alpha}(K)} \quad \forall\,\zeta \in H^{2+\alpha}(K), \quad (3.13)$$

where α is the index of elliptic regularity in (2.13).

3.3. *The discrete variational inequality*

We need a discrete analog of (2.7) in order to define the discretization of (2.9). For $K \in \mathcal{T}_h$, we define $a^K(\cdot,\cdot)$ by

$$a^K(v,w) = \int_K \nabla^2 v : \nabla^2 w\,dx \quad \forall\,v,w \in H^2(K). \qquad (3.14)$$

Then, we construct a symmetric positive definite bilinear form $S^K(v,w)$ with the following stabilization property:

$$C_0a^K(v,v) \le S^K(v,v) \le C_1a^K(v,v) \quad \forall\,v \in \mathcal{Q}(K) \quad \text{such that } \Pi_K^\Delta v = 0 \qquad (3.15)$$

and set the discrete analog of (2.5) and (2.7) on K to be

$$\mathcal{A}_h^K(v,w) = \beta a_h^K(v,w) + \int_K (\Pi_K^0 v)(\Pi_K^0 w)dx, \qquad (3.16)$$

$$a_h^K(v,w) = a^K(\Pi_K^\Delta v, \Pi_K^\Delta w) + S^K(v - \Pi_K^\Delta v, w - \Pi_K^\Delta w), \qquad (3.17)$$

for all $v,w \in \mathcal{Q}(K)$.

Note that (3.3), (3.14), (3.15) and (3.17) imply

$$a_h^K(v,v) \approx a^K(v,v) \quad \forall\,v \in \mathcal{Q}(K)$$

and hence, in view of (3.11), (3.14) and (3.16),

$$\mathcal{A}_h^K(v,v) \approx \mathcal{A}^K(v,v) \quad \forall\,v \in \mathcal{Q}(K), \qquad (3.18)$$

where

$$\mathcal{A}^K(v,w) = \beta a^K(v,w) + \int_K vw\,dx. \qquad (3.19)$$

Remark 3.2. There are many variants of $S^K(\cdot,\cdot)$. The one we use (cf. Sec. 4.4 of Ref. 27) is defined by

$$S^K(v_h,w_h) = \sum_{p\in\mathcal{V}_K} h_p^{-2}\big\{v_h(p)\cdot w_h(p) + [h_p\nabla v_h(p)]\cdot[h_p\nabla w_h(p)]\big\},$$

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where $v_h(p)$ and $h_p\nabla v_h(p)$ are the degrees of freedom, and h_p is the average of the lengths of the edges of $\mathcal{T}_h$ that share p as a common vertex.

Let the global virtual element space associated with $\mathcal{T}_h$ be defined by

$$V_h = \{v \in H^2(\Omega) \cap H_0^1(\Omega) : v\big|_K \in \mathcal{Q}(K)\ \forall\, K \in \mathcal{T}_h\}$$

and the discrete constraint set $\mathbb{K}_h$ be given by

$$\mathbb{K}_h = \{y_h \in V_h : y_h(p) \le \psi(p)\ \forall\, p \in \mathcal{V}_h\}, \tag{3.20}$$

where $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$.

The discrete problem for (2.9) is to find $\bar{y}_h \in \mathbb{K}_h$ such that

$$\mathcal{A}_h(\bar{y}_h, y_h - \bar{y}_h) - \big(y_d, \Pi_h^0(y_h - \bar{y}_h)\big) \ge 0 \quad \forall\, y_h \in \mathbb{K}_h, \tag{3.21}$$

where

$$\mathcal{A}_h(y_h, z_h) = \sum_{K\in\mathcal{T}_h} \mathcal{A}_h^K(y_h, z_h) = \sum_{K\in\mathcal{T}_h} \beta a_h^K(y_h, z_h) + (\Pi_h^0 y_h, \Pi_h^0 z_h) \tag{3.22}$$

and Π_h^0 is the projection operator from $L^2(\Omega)$ onto the space $\mathbb{P}_2(\Omega;\mathcal{T}_h)$ of piecewise quadratic polynomial functions defined by

$$(\Pi_h^0 v)\big|_K = \Pi_K^0(v\big|_K) \quad \forall\, K \in \mathcal{T}_h. \tag{3.23}$$

4. Preliminary Estimates

In this section, we introduce some tools and preliminary estimates in preparation for the convergence analysis in Sec. 5.

4.1. *A mesh-dependent norm*

Let $H^2(\Omega;\mathcal{T}_h)$ be the space of piecewise H^2 functions with respect to $\mathcal{T}_h$. The piecewise version of the energy norm $\|\cdot\|_{\mathcal{A}}$ is defined by

$$\|v\|_h^2 = \sum_{K\in\mathcal{T}_h} \mathcal{A}^K(v,v) = \sum_{K\in\mathcal{T}_h} \big(\beta|v|_{H^2(K)}^2 + \|v\|_{L^2(K)}^2\big) \quad \forall\, v \in H^2(\Omega;\mathcal{T}_h). \tag{4.1}$$

It follows from (3.18) that

$$\sum_{K\in\mathcal{T}_h} \mathcal{A}_h^K(v,v) \approx \|v\|_h^2 \quad \forall\, v \in V_h + \mathbb{P}_2(\Omega;\mathcal{T}_h). \tag{4.2}$$

4.2. *The interpolation operator Π_h*

The interpolation operator $\Pi_h : H^{2+\alpha}(\Omega)\cap H_0^1(\Omega) \to V_h$ is defined by the condition that ζ and $\Pi_h\zeta$ share the same degrees of freedom, i.e.

$$\Pi_h\zeta(p) = \zeta(p) \qquad \forall\, p \in \mathcal{V}_h, \tag{4.3}$$

$$\nabla\Pi_h\zeta(p) = \nabla\zeta(p) \quad \forall\, p \in \mathcal{V}_h, \tag{4.4}$$

where $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$.

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Since $\mathbb{P}_2(K)$ is invariant under the restriction of Π_h to $H^{2+\alpha}(K)$, we have

$$\sum_{\ell=0}^{2} h_K^{\ell}|\zeta - \Pi_h\zeta|_{H^{\ell}(K)} \lesssim h_K^{2+\alpha}|\zeta|_{H^{2+\alpha}(K)} \quad \forall\, K \in \mathcal{T}_h \tag{4.5}$$

by the shape regularity assumption (3.1) and the Bramble–Hilbert lemma.[11,37]

Note that

$$\Pi_h \bar{y} \text{ belongs to } \mathbb{K}_h \tag{4.6}$$

by (2.2), (3.20) and (4.3). In particular $\mathbb{K}_h$ is a non-empty closed convex subset of V_h and (3.21) has a unique solution $\bar{y}_h \in \mathbb{K}_h$ by the standard theory.[47]

4.3. *The interpolation operator* I_h

Let $W_h \subset H_0^1(\Omega)$ be the lowest-order H^1-conforming virtual element space associated with $\mathcal{T}_h$ (cf. Ref. 4). The functions in W_h are harmonic functions on each $K \in \mathcal{T}_h$ and linear polynomials on the edges of $\mathcal{T}_h$. The interpolation operator $I_h : H^2(\Omega) \cap H_0^1(\Omega) \to W_h$ is defined by

$$I_h\zeta(p) = \zeta(p) \quad \forall\, p \in \mathcal{V}_h.$$

In view of the maximum principle for harmonic functions, we can rewrite (3.20) as

$$\mathbb{K}_h = \{y_h \in V_h : I_h y_h \le I_h \psi\}. \tag{4.7}$$

Note that $\mathbb{P}_1(K)$ is invariant under the restriction of I_h to $H^2(K)$. Therefore we have, by the shape regularity assumption (3.1) and the Bramble–Hilbert lemma,[11,20]

$$\|\zeta - I_h\zeta\|_{L^\infty(K)} \lesssim h_K^2|\zeta|_{W^{2,\infty}(K)} \quad \forall\, \zeta \in W^{2,\infty}(K), \tag{4.8}$$

$$\|\zeta - I_h\zeta\|_{H^1(K)} \lesssim h_K|\zeta|_{H^2(K)} \qquad \forall\, \zeta \in H^2(K). \tag{4.9}$$

More details can be found in Refs. 6, 16, 22, 32 and 29.

4.4. *A preliminary estimate*

We are now ready to derive an estimate that reduces the convergence analysis to the continuous level.

Lemma 4.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2 \le C\big(\big[\mathcal{A}(\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (y_d, \Pi_h\bar{y} - \bar{y}_h)\big] + h^{2\alpha} + h^{\alpha}\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}\big). \tag{4.10}$$

Proof. Using (2.8) and (4.5), we can estimate the difference between the solution $\bar{y}$ of (2.9) and the solution $\bar{y}_h$ of (3.21) by

$$\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2 \le 2\big(\|\bar{y} - \Pi_h\bar{y}\|_{\mathcal{A}}^2 + \|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2\big) \lesssim h^{2\alpha} + \|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2,$$

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and we have, by (2.8), (3.11), (3.21), (4.1), (4.2) and (4.6),

$$
\begin{aligned}
\|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2 &= \|\Pi_h\bar{y} - \bar{y}_h\|_h^2 \\
&\lesssim \mathcal{A}_h(\Pi_h\bar{y} - \bar{y}_h, \Pi_h\bar{y} - \bar{y}_h) \\
&= \mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - \mathcal{A}_h(\bar{y}_h, \Pi_h\bar{y} - \bar{y}_h) \\
&\leq \mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (y_d, \Pi_h^0(\Pi_h\bar{y} - \bar{y}_h)) \\
&= \mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (y_d, \Pi_h\bar{y} - \bar{y}_h) \\
&\quad + (y_d, (\Pi_h\bar{y} - \bar{y}_h) - \Pi_h^0(\Pi_h\bar{y} - \bar{y}_h)) \\
&\lesssim \mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (y_d, \Pi_h\bar{y} - \bar{y}_h) + h^2\|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}}.
\end{aligned}
$$

It follows that

$$
\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2 \lesssim \mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) - (y_d, \Pi_h\bar{y} - \bar{y}_h) + h^{2\alpha} + h^2\|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}}. \tag{4.11}
$$

Next we observe that the bilinear form $\mathcal{A}_h^K$ satisfies the consistency relation

$$
\mathcal{A}_h^K(q, v_h) = \mathcal{A}^K(q, v_h) \quad \forall\, v_h \in \mathcal{Q}(K) \quad \text{and} \quad q \in \mathbb{P}_2(K),
$$

which follows from (3.16), (3.17), (3.19) and the fact that Π_K^0 and Π_K^Δ are projections onto $\mathbb{P}_2(K)$. We can therefore rewrite the first term on the right-hand side of (4.11) as follows:

$$
\begin{aligned}
\mathcal{A}_h(\Pi_h\bar{y}, \Pi_h\bar{y} - \bar{y}_h) &= \sum_{K\in\mathcal{T}_h} \big(\mathcal{A}_h^K(\Pi_h\bar{y} - \Pi_h^0\bar{y}, \Pi_h\bar{y} - \bar{y}_h) + \mathcal{A}_h^K(\Pi_h^0\bar{y}, \Pi_h\bar{y} - \bar{y}_h)\big) \\
&= \sum_{K\in\mathcal{T}_h} \big(\mathcal{A}_h^K(\Pi_h\bar{y} - \Pi_h^0\bar{y}, \Pi_h\bar{y} - \bar{y}_h) + \mathcal{A}^K(\Pi_h^0\bar{y}, \Pi_h\bar{y} - \bar{y}_h)\big) \\
&= \sum_{K\in\mathcal{T}_h} \big(\mathcal{A}_h^K(\Pi_h\bar{y} - \Pi_h^0\bar{y}, \Pi_h\bar{y} - \bar{y}_h) + \mathcal{A}^K(\Pi_h^0\bar{y} - \bar{y}, \Pi_h\bar{y} - \bar{y}_h)\big) \\
&\quad + \mathcal{A}(\bar{y}, \Pi_h\bar{y} - \bar{y}_h).
\end{aligned} \tag{4.12}
$$

We can use (3.13), (4.1), (4.2) and (4.5) to estimate the sum on the right-hand side of (4.12)

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} \big(\mathcal{A}_h^K(\Pi_h\bar{y} - \Pi_h^0\bar{y}, \Pi_h\bar{y} - \bar{y}_h) + \mathcal{A}^K(\Pi_h^0\bar{y} - \bar{y}, \Pi_h\bar{y} - \bar{y}_h)\big) \\
&\quad \lesssim \|\Pi_h\bar{y} - \Pi_h^0\bar{y}\|_h\|\Pi_h\bar{y} - \bar{y}_h\|_h + \|\Pi_h^0\bar{y} - \bar{y}\|_h\|\Pi_h\bar{y} - \bar{y}_h\|_h \\
&\quad \lesssim (\|\Pi_h\bar{y} - \bar{y}\|_h + \|\Pi_h^0\bar{y} - \bar{y}\|_h)\|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}} \\
&\quad \lesssim h^\alpha\|\Pi_h\bar{y} - \bar{y}_h\|_{\mathcal{A}}.
\end{aligned} \tag{4.13}
$$

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Finally, we conclude from (2.8), (4.5) and (4.11)–(4.13) that

$$\begin{aligned}\|\bar{y}-\bar{y}_h\|_{\mathcal{A}}^2 &\lesssim \big[\mathcal{A}(\bar{y},\Pi_h\bar{y}-\bar{y}_h)-(y_d,\Pi_h\bar{y}-\bar{y}_h)\big]+h^{2\alpha}+h^{\alpha}\|\Pi_h\bar{y}-\bar{y}_h\|_{\mathcal{A}}\\ &\lesssim \big[\mathcal{A}(\bar{y},\Pi_h\bar{y}-\bar{y}_h)-(y_d,\Pi_h\bar{y}-\bar{y}_h)\big]+h^{2\alpha}\\ &\quad +h^{\alpha}\big(\|\Pi_h\bar{y}-\bar{y}\|_{\mathcal{A}}+\|\bar{y}-\bar{y}_h\|_{\mathcal{A}}\big)\\ &\lesssim \big[\mathcal{A}(\bar{y},\Pi_h\bar{y}-\bar{y}_h)-(y_d,\Pi_h\bar{y}-\bar{y}_h)\big]+h^{2\alpha}+h^{\alpha}\|\bar{y}-\bar{y}_h\|_{\mathcal{A}}. \qquad \square\end{aligned}$$

Remark 4.1. It only remains to estimate the first term on the right-hand side of (4.10), which no longer involves the discrete bilinear form $\mathcal{A}_h(\cdot,\cdot)$.

5. Convergence Analysis

In view of (2.10) and (2.12), we can write

$$\begin{aligned}&\mathcal{A}(\bar{y},\Pi_h\bar{y}-\bar{y}_h)-(y_d,\Pi_h\bar{y}-\bar{y}_h)\\ &\quad=\int_\Omega(\Pi_h\bar{y}-\bar{y}_h)d\mu\\ &\quad=\int_\Omega\Big[(\Pi_h\bar{y}-\bar{y})+(\psi-I_h\psi)+(I_h\psi-I_h\bar{y}_h)+(I_h\bar{y}_h-\bar{y}_h)\Big]d\mu, \end{aligned}\tag{5.1}$$

and the first three terms on the right-hand side of (5.1) can be estimated as follows:

$$\int_\Omega(\Pi_h\bar{y}-\bar{y})d\mu\le\|\mu\|_{H^{-1}(\Omega)}\|\Pi_h\bar{y}-\bar{y}\|_{H^1(\Omega)}\lesssim h^{1+\alpha}\tag{5.2}$$

by (2.14) and (4.5),

$$\int_\Omega(\psi-I_h\psi)d\mu\le|\mu(\Omega)|\|\psi-I_h\psi\|_{L^\infty(\Omega)}\lesssim h^2\tag{5.3}$$

by (1.4), (2.11) and (4.8),

$$\int_\Omega(I_h\psi-I_h\bar{y}_h)d\mu\le 0\tag{5.4}$$

by (2.11) and (4.7).

For the fourth term on the right-hand side of (5.1), we have a decomposition

$$\int_\Omega(I_h\bar{y}_h-\bar{y}_h)d\mu=\int_\Omega[I_h(\bar{y}_h-\bar{y})-(\bar{y}_h-\bar{y})]d\mu+\int_\Omega(I_h\bar{y}-\bar{y})d\mu\tag{5.5}$$

and

$$\begin{aligned}\int_\Omega[I_h(\bar{y}_h-\bar{y})-(\bar{y}_h-\bar{y})]d\mu&\le\|\mu\|_{H^{-1}(\Omega)}\|I_h(\bar{y}_h-\bar{y})-(\bar{y}_h-\bar{y})\|_{H^1(\Omega)}\\ &\lesssim h\|\bar{y}-\bar{y}_h\|_{\mathcal{A}}\end{aligned}\tag{5.6}$$

by (2.8), (2.14) and (4.9),

$$\int_\Omega(I_h\bar{y}-\bar{y})d\mu\le|\mu(\mathfrak{A})|\|I_h\bar{y}-\bar{y}\|_{L^\infty(\mathfrak{A})}\lesssim h^2\tag{5.7}$$

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by (2.11), Remark 2.2, (2.13) and (4.8), where $\mathfrak{A}$ is the active set introduced in Remark 2.2.

Combining (5.1)–(5.7), we arrive at

$$\mathcal{A}(\bar{y}, \Pi_h \bar{y} - \bar{y}_h) - (y_d, \Pi_h \bar{y} - \bar{y}_h) \lesssim h^{1+\alpha} + h\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}. \tag{5.8}$$

Theorem 5.1. *There exists a positive constant C independent of h such that*

$$\|\bar{y} - \bar{y}_h\|_{\mathcal{A}} \leq Ch^{\alpha},$$

where α is the index of elliptic regularity in (2.13).

Proof. It follows from (4.10) and (5.8) that

$$\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}^2 \lesssim h^{2\alpha} + h^{\alpha}\|\bar{y} - \bar{y}_h\|_{\mathcal{A}}$$

and the proof is completed by an application of the inequality of arithmetic and geometric means. □

We can approximate $\bar{u} = -\Delta\bar{y}$ by $\bar{u}_h = -\Delta\bar{y}_h$.

Corollary 5.1. *There exists a positive constant C independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + |\bar{y} - \bar{y}_h|_{H^1(\Omega)} + \|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq Ch^{\alpha},$$

where α is the index of elliptic regularity in (2.13).

Proof. The estimate for the control and the L^2 and H^1 estimates for the state follow immediately from (2.8) and Theorem 5.1. The L^∞ estimate for the state is a consequence of the Sobolev embedding[1] $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and Theorem 5.1. □

We can also derive convergence results for the computable approximation $\bar{y}_h^c = \Pi_h^0 \bar{y}_h \in \mathbb{P}_2(\Omega; \mathcal{T}_h)$.

Theorem 5.2. *There exists a positive constant C independent of h such that*

$$\left(\sum_{K\in\mathcal{T}_h} |\bar{y} - \bar{y}_h^c|_{H^2(K)}^2\right)^{\frac{1}{2}} \leq Ch^{\alpha},$$

where α is the index of elliptic regularity in (2.13).

Proof. Let $K \in \mathcal{T}_h$ be arbitrary. We begin with

$$\begin{aligned}|\bar{y} - \bar{y}_h^c|_{H^2(K)} &= |(\bar{y} - \Pi_K^0\bar{y}) - \Pi_K^0(\bar{y}_h - \Pi_K^0\bar{y})|_{H^2(K)} \\ &\leq |\bar{y} - \Pi_K^0\bar{y}|_{H^2(K)} + |\Pi_K^0(\bar{y}_h - \Pi_K^0\bar{y})|_{H^2(K)}\end{aligned}$$

and observe that

$$\begin{aligned}|\Pi_K^0(\bar{y}_h - \Pi_K^0\bar{y})|_{H^2(K)} &= |\Pi_K^{\Delta}(\bar{y}_h - \Pi_K^0\bar{y})|_{H^2(K)} \leq |\bar{y}_h - \Pi_K^0\bar{y}|_{H^2(K)} \\ &\leq |\bar{y} - \bar{y}_h|_{H^2(K)} + |\bar{y} - \Pi_K^0\bar{y}|_{H^2(K)}\end{aligned}$$

by (3.6) and (3.10).

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It follows that

$$|\bar{y} - \bar{y}_h^c|_{H^2(K)} \leq 2|\bar{y} - \Pi_K^0 \bar{y}|_{H^2(K)} + |\bar{y} - \bar{y}_h|_{H^2(K)}$$

and we can complete the proof by invoking (3.13) and Theorem 5.1. □

We can also approximate $\bar{u} = -\Delta \bar{y}$ by the computable $\bar{u}_h^c = -\Delta_h \Pi_h^0 \bar{y}_h$, where Δ_h is the piecewise defined Laplacian.

Corollary 5.2. *There exists a positive constant C independent of h such that*

$$\|\bar{u} - \bar{u}_h^c\|_{L^2(\Omega)} + \|\bar{y} - \bar{y}_h^c\|_{L^2(\Omega)} + \left(\sum_{K \in \mathcal{T}_h} |\bar{y} - \bar{y}_h^c|^2_{H^1(K)} \right)^{\frac{1}{2}} + \|\bar{y} - \bar{y}_h^c\|_{L^\infty(\Omega)} \leq C h^\alpha,$$

where α is the index of elliptic regularity in (2.13).

Proof. The estimate for the control follows immediately from (4.1) and Theorem 5.2.

Let $K \in \mathcal{T}_h$ be arbitrary. We have

$$\begin{aligned} \|\bar{y} - \bar{y}_h^c\|_{L^2(K)} &= \|(\bar{y} - \Pi_K^0 \bar{y}) - \Pi_K^0(\bar{y}_h - \bar{y})\|_{L_2(K)} \\ &\leq \|\bar{y} - \Pi_K^0 \bar{y}\|_{L_2(K)} + \|\bar{y}_h - \bar{y}\|_{L_2(K)}, \end{aligned}$$

which together with (3.11) and Corollary 5.1 implies the L^2 estimate for the state.

Similarly, we have, by (3.11),

$$\begin{aligned} |\bar{y} - \bar{y}_h^c|_{H^1(K)} &\leq |\bar{y} - \Pi_K^0 \bar{y}|_{H^1(K)} + |\Pi_K^0(\bar{y}_h - \bar{y})|_{H^1(K)} \\ &\leq |\bar{y} - \Pi_K^0 \bar{y}|_{H^1(K)} + |\bar{y}_h - \bar{y}|_{H^1(K)} + |(\bar{y}_h - \bar{y}) - \Pi_K^0(\bar{y}_h - \bar{y})|_{H^1(K)} \\ &\lesssim |\bar{y} - \Pi_K^0 \bar{y}|_{H^1(K)} + |\bar{y}_h - \bar{y}|_{H^1(K)} + h_K |\bar{y}_h - \bar{y}|_{H^2(K)}, \end{aligned}$$

which together with (3.11), Theorem 5.1 and Corollary 5.1 implies the nonconforming H^1 estimate for the state.

The proof of the L^∞ estimate for the state can be established in the same way after (3.11) is replaced by (3.12). □

Remark 5.1. Numerical results in Sec. 6 indicate that the error estimates in Corollary 5.2 for the state are not sharp.

6. Numerical Results

We have tested our method on four examples from the literature. The domains are squares and hence the index of elliptic regularity $\alpha = 1$ in all the examples. The discrete variational inequalities are solved by a primal-dual active set method.[7,8,42,45]

For a problem whose exact solution is not available, we use nested meshes consisting of general polygons (cf. Fig. 1, left), and we use Voronoi meshes (cf.

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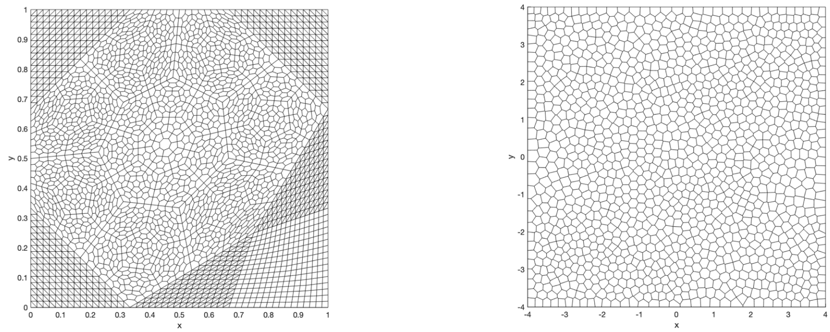

A nested polygonal mesh | A Voronoi mesh

Fig. 1. Polygonal decompositions of the domain.

Fig. 1, right) if the exact solution is known. The Voronoi meshes are generated by `PolyMesher`.[51]

The errors of the computable approximation $\bar{y}_h^c = \Pi_h^0 \bar{y}_h$ in various norms are denoted by

$$e_{0,h} = \|\bar{y} - \bar{y}_h^c\|_{L^2(\Omega)}, \quad e_{1,h} = \left(e_{0,h}^2 + \sum_{K \in \mathcal{T}_h} |\bar{y} - \bar{y}_h^c|_{H^1(K)}^2 \right)^{\frac{1}{2}},$$

$$e_{2,h} = \left(e_{1,h}^2 + \sum_{K \in \mathcal{T}_h} |\bar{y} - \bar{y}_h^c|_{H^2(K)}^2 \right)^{\frac{1}{2}}, \quad e_{\infty,h} = \max_{p \in \mathcal{V}_h} |\bar{y}(p) - \bar{y}_h(p)|,$$

where $\mathcal{V}_h$ is the set of the vertices of $\mathcal{T}_h$.

For the examples with unknown exact solutions solved on nested meshes, we use the same notation $e_{0,h}$, $e_{1,h}$, $e_{2,h}$, $e_{\infty,h}$ to denote the errors which are computed by replacing $\bar{y}$ with the solution obtained after the mesh is refined.

The numerical results from all four examples indicate that the H^2 convergence of the state and the L^2 convergence of the control is 1, which agree with Theorem 5.2 and Corollary 5.2. The convergence for the state in the L^2, H^1 and L^∞ norms is better than the convergence predicted by Corollary 5.2.

Example 6.1. The domain Ω is the unit square $(0,1)^2$. The data are given by

$$y_d = 10(\sin(2\pi x_1) + x_2), \quad \psi = 0.01 \quad \text{and} \quad \beta = 0.1.$$

This example is from Ref. 43 and it has also been tested in Refs. 48, 24 and 13.

The exact solution of this problem is unknown and the discrete variational inequality (3.21) is solved on nested polygonal meshes. The results are presented in Tables 1 and 2. The optimal state, optimal control and active set computed on a mesh with $h = 1.1108 \times 10^{-2}$ are displayed in Fig. 2. They match the results in Refs. 43, 48, 24 and 13.

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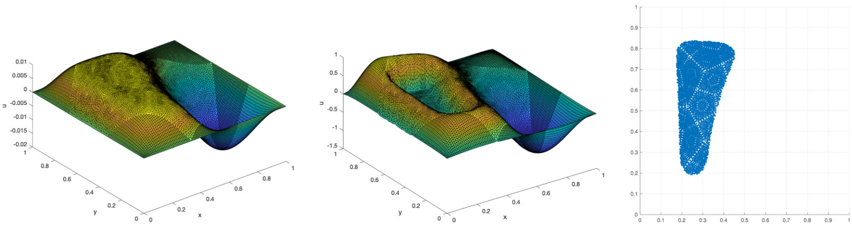
Table 1. Errors and orders of convergence for the state (Example 6.1).

h	$e_{2,h}$	Order	$e_{1,h}$	Order	$e_{0,h}$	Order	$e_{\infty,h}$	Order
4.3800e-1	1.6560e0	—	3.2952e-1	—	6.0428e-2	—	3.1742e-2	—
1.8222e-1	2.9137e-1	1.98	3.7752e-2	2.47	4.2256e-3	3.03	1.3074e-2	1.01
8.8407e-2	1.4995e-1	0.92	7.6856e-3	2.20	8.9746e-4	2.14	3.0611e-3	2.01
4.4303e-2	6.9608e-2	1.11	2.2298e-3	1.79	2.7516e-4	1.71	9.0366e-4	1.77
2.2193e-2	3.1081e-2	1.17	5.1339e-4	2.12	6.4203e-5	2.11	2.0829e-4	2.12
1.1108e-2	1.3864e-2	1.17	1.1049e-4	2.22	1.3721e-5	2.23	4.4367e-5	2.23

Table 2. Errors and orders of convergence for the control (Example 6.1).

h	4.3800e-1	1.8222e-1	8.8407e-2	4.4303e-2	2.2193e-2	1.1108e-2
$\|\bar{u} - \bar{u}_h^c\|_{L^2(\Omega)}$	1.3399e0	2.7089e-1	1.3896e-1	6.6671e-2	2.9876e-2	1.3308e-2
Order	—	1.82	0.92	1.06	1.16	1.17



(a) The optimal state (b) The optimal control (c) The active set

Fig. 2. Optimal state, optimal control and active set of Example 1 ($h = 1.1108 \times 10^{-2}$).

Example 6.2. The domain Ω is the unit square $(0,1)^2$. The data are given by

$$y_d = \sin(2\pi x_1 x_2), \quad \psi = 0.1 \quad \text{and} \quad \beta = 10^{-3}.$$

This example is from Ref. 8 and it has also been tested in Refs. 48, 24, 13 and 25.

The exact solution of this problem is unknown and the discrete variational inequality (3.21) is solved on nested polygonal meshes. The results are presented in Tables 3 and 4. The optimal state, optimal control and active set computed on a mesh with $h = 1.1108 \times 10^{-2}$ are displayed in Fig. 3. They match the results in Refs. 8, 48, 24, 13 and 25.

Table 3. Errors and orders of convergence for the state (Example 6.2).

h	$e_{2,h}$	Order	$e_{1,h}$	Order	$e_{0,h}$	Order	$e_{\infty,h}$	Order
4.3800e-1	5.7491e0	—	8.9377e-1	—	1.9290e-1	—	0.0000e0	—
1.8222e-1	2.3643e0	1.01	3.2736e-1	1.15	3.1017e-2	2.08	3.8640e-2	—
8.8407e-2	1.0050e0	1.18	4.9625e-2	2.61	3.2090e-3	3.14	9.3601e-3	1.96
4.4303e-2	4.7210e-1	1.09	1.2696e-2	1.97	7.7692e-4	2.05	2.5240e-3	1.90
2.2193e-2	2.1292e-1	1.15	3.3270e-3	1.94	1.9385e-4	2.01	5.7181e-4	2.15
1.1108e-2	9.5519e-2	1.16	8.3215e-4	2.00	4.2860e-5	2.18	1.4404e-4	1.99

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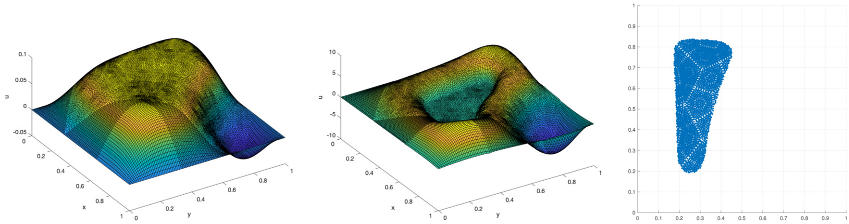
Table 4. Errors and orders of convergence for the control (Example 6.2).

h	4.3800e-1	1.8222e-1	8.8407e-2	4.4303e-2	2.2193e-2	1.1108e-2
$\lVert\bar{u}-\bar{u}_h^c\rVert_{L^2(\Omega)}$	5.3237e0	2.5665e0	9.4045e-1	4.4387e-1	2.0078e-1	9.0244e-2
Order	—	0.83	1.39	1.09	1.15	1.16



(a) The optimal state (b) The optimal control (c) The active set

Fig. 3. Optimal state, optimal control and active set of Example 6.2 ($h = 1.1108 \times 10^{-2}$).

Example 6.3. The domain Ω is the unit square $(0,1)^2$. The data are given by

$$y_d = \sin(4\pi x_1 x_2) + 1.5, \quad \psi = 1 \quad \text{and} \quad \beta = 10^{-4}.$$

This example is from Ref. 8 and it has also been tested in Refs. 48, 24, 13 and 25.

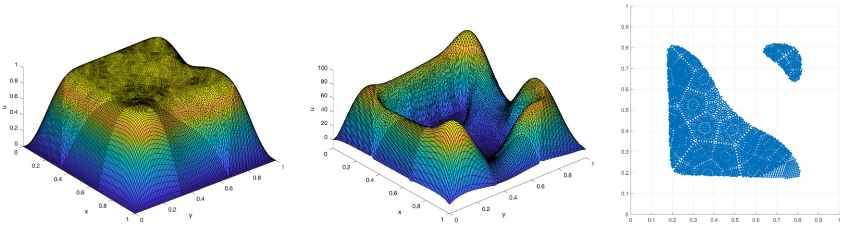
The exact solution of this problem is unknown and the discrete variational inequality (3.21) is solved on nested polygonal meshes. The results are presented in Tables 5 and 6. The optimal state, optimal control and active set computed on a mesh with $h = 1.1108 \times 10^{-2}$ are displayed in Fig. 4. They match the results in Refs. 8, 48, 24, 13 and 25.

Table 5. Errors and orders of convergence for the state (Example 6.3).

h	$e_{2,h}$	Order	$e_{1,h}$	Order	$e_{0,h}$	Order	$e_{\infty,h}$	Order
4.3800e-1	3.3514e1	—	3.8719e0	—	5.7631e-1	—	0.0000e0	—
1.8222e-1	2.4068e1	0.38	1.8534e0	0.84	1.7390e-1	1.37	2.7611e-1	—
8.8407e-2	1.4709e1	0.68	6.7202e-1	1.40	4.1621e-2	1.98	1.4707e-1	0.87
4.4303e-2	7.3367e0	1.01	1.9826e-1	1.77	9.5251e-3	2.13	3.4246e-2	2.11
2.2193e-2	3.3743e0	1.12	5.1706e-2	1.94	2.6323e-3	1.86	1.0660e-2	1.69
1.1108e-2	1.5379e0	1.14	1.3112e-2	1.98	6.7255e-4	1.97	3.0619e-3	1.80

Table 6. Errors and orders of convergence for the control (Example 6.3).

h	4.3800e-1	1.8222e-1	8.8407e-2	4.4303e-2	2.2193e-2	1.1108e-2
$\lVert\bar{u}-\bar{u}_h^c\rVert_{L^2(\Omega)}$	2.9119e1	2.2103e1	1.3501e1	6.9196e0	3.2090e0	1.4699e0
Order	—	0.31	0.68	0.97	1.11	1.13



(a) The optimal state (b) The optimal control (c) The active set

Fig. 4. Optimal state, optimal control and active set of Example 6.3 ($h = 1.1108 \times 10^{-2}$).

Example 6.4. This example is from Ref. 15. The domain Ω is the square $(-4,4)^2$. The data are given by $\beta = 1$, $\psi = |x|^2 - 1$ and

$$y_d = \begin{cases} \Delta^2 \bar{y} + \bar{y} & \text{if } |x| > 1, \\ \Delta^2 \bar{y} + \bar{y} + 2 & \text{if } |x| < 1 \end{cases}$$

and the exact optimal state $\bar{y}$ is given by

$$\bar{y}(x) = \begin{cases} |x|^2 - 1 & \text{if } |x| \le 1, \\ v(|x|) + [1 - \phi(|x|)]w(x) & \text{if } 1 \le |x| \le 3, \\ w(x) & \text{if } |x| \ge 3, \end{cases}$$

where

$$v(t) = (t^2 - 1)\left(1 - \frac{t-1}{2}\right)^4 + \frac{1}{4}(t-1)^2(t-3)^4,$$

$$\phi(t) = \left[1 + 4\left(\frac{t-1}{2}\right) + 10\left(\frac{t-1}{2}\right)^2 + 20\left(\frac{t-1}{2}\right)^3\right]\left(1 - \frac{t-1}{2}\right)^4$$

and

$$w(x) = 2\sin\left(\frac{\pi}{8}(x_1 + 4)\right)\sin\left(\frac{\pi}{8}(x_2 + 4)\right).$$

The exact active set is the disc $\{x : |x| \le 1\}$ and the exact optimal control is $\bar{u} = -\Delta\bar{y}$.

The discrete variational inequality (3.21) is solved on Voronoi meshes. The results are presented in Tables 7 and 8.

Table 7. Errors and orders of convergence for the state (Example 6.4).

h	$e_{2,h}$	Order	$e_{1,h}$	Order	$e_{0,h}$	Order	$e_{\infty,h}$	Order
2.6054e0	2.9231e1	—	2.4131e1	—	1.3487e1	—	3.8025e0	—
1.3270e0	7.7532e0	1.97	3.7478e0	2.76	2.0435e0	2.80	1.2321e0	1.67
7.8544e-1	3.9787e0	1.27	9.9106e-1	2.54	3.0289e-1	3.64	2.0243e-1	3.44
3.5849e-1	2.0335e0	0.86	2.8188e-1	1.60	8.3854e-2	1.64	3.0817e-2	2.40
1.8057e-1	1.0590e0	0.95	8.7947e-2	1.70	3.0596e-2	1.47	9.3315e-3	1.74
9.7469e-2	5.3274e-1	1.11	2.6411e-2	1.95	1.0359e-2	1.76	2.8191e-3	1.94

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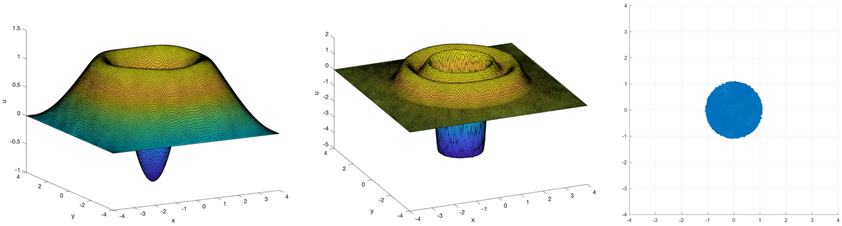
Table 8. Errors and orders of convergence for the control (Example 6.4).

h	2.6054e0	1.3270e0	7.8544e-1	3.5849e-1	1.8057e-1	9.7469e-2
$\Vert\bar{u}-\bar{u}_h^c\Vert_{L^2(\Omega)}$	1.9111e1	7.1613e0	3.9383e0	2.0404e0	1.0473e0	5.2275e-1
Order	—	1.45	1.14	0.84	0.97	1.13



(a) The optimal state (b) The optimal control (c) The active set

Fig. 5. Optimal state, optimal control and active set of Example 6.4 ($h = 9.7469 \times 10^{-2}$).

The optimal state, optimal control and active set computed on a mesh with $h = 9.7469 \times 10^{-2}$ are displayed in Fig. 5. The exact optimal state, optimal control and the exact active set are clearly captured.

7. Concluding Remarks

We have studied in this paper a C^1 virtual element method for an elliptic distributed optimal control problem with pointwise state constraints using the simplest virtual element for fourth-order problems. Our approach can also be applied to other C^1 virtual elements[27,33] and nonconforming virtual elements[3,52] for fourth-order problems. It can also be extended to three-dimensional optimal control problems by using the C^1 virtual elements in Ref. 5.

It is also possible to use this approach to design and analyze new virtual element methods for optimal control problems with both state and control constraints. In the case of classical nonconforming finite element methods and C^0 interior penalty methods, this has been carried out in Refs. 17 and 23.

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