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Additive Schwarz preconditioners for C^0 interior penalty methods for the obstacle problem of clamped Kirchhoff plates

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Abstract

The discrete variational inequalities resulting from C^0 interior penalty methods for the obstacle problem of clamped Kirchhoff plates can be solved by the primal-dual active set algorithm. We develop and analyze additive Schwarz preconditioners for the auxiliary systems that appear in each iteration of the primal-dual active set algorithm. Numerical results corroborate our theoretical estimates.

KEYWORDS

additive Schwarz preconditioners, C^0 interior penalty method, fourth-order variational inequality, obstacle problem for clamped Kirchhoff plates

1 | INTRODUCTION

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$, $f \in L_2(\Omega)$, and $\psi \in H^3(\Omega) \cap W^{2,\infty}(\Omega)$ such that $\psi < 0$ on $\partial\Omega$. The obstacle problem of clamped Kirchhoff plates on Ω is to find

$$u = \operatorname{argmin}_{v \in K} \left[\frac{1}{2} a(v, v) - (f, v) \right], \quad (1)$$

where $(\cdot, \cdot)$ is the inner product for $L_2(\Omega)$,

$$a(v, w) = \int_{\Omega} D^2 v : D^2 w \, dx = \int_{\Omega} \sum_{i,j=1}^2 \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right) \left(\frac{\partial^2 w}{\partial x_i \partial x_j} \right) dx, \quad (2)$$

and K is the subset of $H_0^2(\Omega)$ defined by

$$K = \{v \in H_0^2(\Omega) : v \geq \psi \text{ on } \Omega\}. \quad (3)$$

Here and throughout this paper, we follow the standard notation for L_2 -based Sobolev spaces [1–3].

Since K is a nonempty closed convex subset of $H_0^2(\Omega)$, it follows from the standard theory of calculus of variations [4, 5] that the obstacle problem (1)–(3) has a unique solution $u \in K$ characterized by the following fourth-order variational inequality

$$a(u, v - u) - (f, v - u) \geq 0, \quad \forall v \in K.$$

A quadratic C^0 interior penalty method for the obstacle problem of clamped Kirchhoff plates was investigated in Brenner et al. [6]. The discrete variational inequalities resulting from the C^0 interior penalty method can be solved by a primal-dual active set (PDAS) algorithm [7–10]. In each iteration of the PDAS algorithm, an auxiliary system of equations involving the inactive nodes has to be solved. Due to the nature of the fourth-order problem itself, such auxiliary systems are very ill-conditioned. The goal of this paper is to develop one-level and two-level additive Schwarz preconditioners for these auxiliary systems.

Domain decomposition methods for second-order variational inequalities can be found in References [11–22]. However, the literature on domain decomposition methods for fourth-order variational inequalities is very limited. In Scarpini [23], Scarpini discussed an alternating Schwarz algorithm for the plate obstacle problem discretized by a mixed finite element methods. Most recently, additive Schwarz preconditioners for a partition of unity method for the obstacle problem of clamped Kirchhoff plates were investigated in Brenner et al. [24].

The rest of the paper is organized as follows. We recall the C^0 interior penalty method and the PDAS algorithm in Sections 2 and 3. Then we define and analyze one-level and two-level additive Schwarz preconditioners for the auxiliary systems in Sections 4 and 5, respectively. Numerical results are presented in Section 6, and we conclude the paper in Section 7.

2 | C^0 INTERIOR PENALTY METHODS

For solving fourth order problems, the C^0 interior penalty method has certain advantages [6, 25, 26]. It combines ideas of continuous and discontinuous Galerkin methods, and the stabilization technique.

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω consisting of convex quadrilaterals, and let V_h be the standard Q_k finite element space associated with $\mathcal{T}_h$,

$$V_h = \{v \in H_0^1(\Omega) : v|_D \in Q_k(D), \quad \forall D \in \mathcal{T}_h\}.$$

We denote by $\mathcal{N}_h$ the set of nodes in Ω associated with V_h .

The discrete problem of the obstacle problem (1)–(3) resulting from the C^0 interior penalty method is to find

$$u_h = \operatorname{argmin}_{v \in K_h} \left[\frac{1}{2} a_h(v, v) - (f, v) \right], \quad (4)$$

where

$$\begin{aligned} a_h(v, w) = & \sum_{D \in \mathcal{T}_h} \sum_{i,j=1}^2 \int_D \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j} dx + \sum_{e \in \mathcal{E}_h} \frac{\eta}{|e|} \int_e \left[\left[\frac{\partial v}{\partial n} \right] \right] \left[\left[\frac{\partial w}{\partial n} \right] \right] ds \\ & + \sum_{e \in \mathcal{E}_h} \int_e \left(\left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} \left[\left[\frac{\partial w}{\partial n} \right] \right] + \left\{ \left\{ \frac{\partial^2 w}{\partial n^2} \right\} \right\} \left[\left[\frac{\partial v}{\partial n} \right] \right] \right) ds, \end{aligned} \quad (5)$$

$$K_h = \{v \in V_h : v(p) \geq \psi(p), \quad \forall p \in \mathcal{N}_h\}, \quad (6)$$

and η in $a_h(\cdot, \cdot)$ is a sufficiently large penalty parameter. We denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and the jump $[[\cdot]]$ and the average $\{\{\cdot\}\}$ are defined as follows. On an edge shared by two elements, that is, $e \in \partial D_- \cap \partial D_+$, we pick n_e to be the unit normal vector pointing from D_- to D_+ , and define

$$\left[\left[\frac{\partial v}{\partial n} \right] \right] = \frac{\partial v_+}{\partial n_e} - \frac{\partial v_-}{\partial n_e} \quad \text{and} \quad \left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} = \frac{1}{2} \left(\frac{\partial^2 v_+}{\partial n_e^2} + \frac{\partial^2 v_-}{\partial n_e^2} \right).$$

For an edge on the boundary of Ω , we take n_e to be the outward pointing unit normal vector, and define

$$\left[\left[\frac{\partial v}{\partial n} \right] \right] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

Then it follows from the standard theory that the unique solution of (4) can be characterized by the discrete variational inequality

$$a_h(u_h, v - u_h) - (f, v - u_h) \geq 0, \quad \forall v \in K_h. \quad (7)$$

Moreover, we have ([6], theorem 3.4)

$$\|u - u_h\|_h \leq Ch^\alpha,$$

where $\|\cdot\|_h$ is the mesh-dependent energy norm defined by

$$\|v\|_h^2 = \sum_{D \in \mathcal{T}_h} |v|_{H^2(D)}^2 + \sum_{e \in \mathcal{E}_h} |e| \|\{\{\partial^2 v / \partial n^2\}\}\|_{L_2(e)}^2 + \sum_{e \in \mathcal{E}_h} |e|^{-1} \|[[\partial v / \partial n]]\|_{L_2(e)}^2,$$

h is the mesh size of the triangulation, the positive constant C is independent of h , and the index of elliptic regularity $\alpha \in \left(\frac{1}{2}, 1\right]$ is determined by the interior angles of Ω .

3 | A PDAS ALGORITHM

Let $\lambda_h : \mathcal{N}_h \rightarrow \mathbb{R}$ be defined by

$$a_h(u_h, v) - (f, v) = \sum_{p \in \mathcal{N}_h} \lambda_h(p) v(p), \quad \forall v \in V_h. \quad (8)$$

The discrete variational inequality Equation (7) is then equivalent to Equation (8) together with the optimality conditions

$$u_h(p) - \psi(p) \geq 0, \quad \lambda_h(p) \geq 0 \quad \text{and} \quad (u_h(p) - \psi(p)) \lambda_h(p) = 0, \quad \forall p \in \mathcal{N}_h,$$

which can be written concisely as

$$\lambda_h(p) = \max(0, \lambda_h(p) + c(\psi(p) - u_h(p))), \quad \forall p \in \mathcal{N}_h, \quad (9)$$

where c is a positive number.

The system defined by Equations (8) and (9) can be solved by a PDAS method.

Given any approximation (u_k, λ_k) of (u_h, λ_h) , the next approximation is obtained by solving the following system of equations

$$a_h(u_{k+1}, v) - (f, v) = \sum_{p \in \mathcal{N}_h} \lambda_{k+1}(p) v(p), \quad \forall v \in V_h, \quad (10a)$$

$$u_{k+1}(p) = \psi(p), \quad \forall p \in \mathcal{A}_k, \quad (10b)$$

$$\lambda_{k+1}(p) = 0, \quad \forall p \in \mathcal{I}_k, \quad (10c)$$

where $\mathcal{A}_k = \{p \in \mathcal{N}_h : \lambda_k(p) + c(\psi(p) - u_k(p)) > 0\}$ is the active set determined by (u_k, λ_k) , and $\mathcal{I}_k = \mathcal{N}_h \setminus \mathcal{A}_k$. The iteration terminates when $\mathcal{A}_{k+1} = \mathcal{A}_k$. Given a sufficiently accurate initial guess, the PDAS algorithm converges superlinearly to the unique solution of (7) (cf. Hintermüller et al. [9]).

From Equations (10b) and (10c), we can reduce (10a) to an auxiliary system that only involves the unknowns of $u_{k+1}(p)$ for $p \in \mathcal{I}_k$. But even so, the reduced auxiliary system is still a large, sparse, and ill-conditioned system for small h . Such system can be solved efficiently by a preconditioned Krylov method such as the preconditioned conjugate gradient method.

Let $\widetilde{\mathcal{N}}_h$ be a subset of $\mathcal{N}_h$. We define $\widetilde{T}_h : V_h \rightarrow V_h$, the truncation operator, by

$$(\widetilde{T}_h v)(p) = \begin{cases} v(p), & p \in \widetilde{\mathcal{N}}_h, \\ 0, & p \in \mathcal{N}_h \setminus \widetilde{\mathcal{N}}_h. \end{cases} \quad (11)$$

Then $\widetilde{T}_h$ is a projection from V_h onto $\widetilde{V}_h = \widetilde{T}_h V_h$. Moreover, from the standard estimates for bivariate polynomials, we have for all $v \in V_h$,

$$\|v\|_{L_2(\Omega)}^2 \approx h^2 \sum_{p \in \mathcal{N}_h} v^2(p),$$

which implies

$$\|\widetilde{T}_h v\|_{L_2(\Omega)} \leq C_* \|v\|_{L_2(\Omega)}, \quad \forall v \in V_h. \quad (12)$$

Let $\widetilde{A}_h : \widetilde{V}_h \rightarrow \widetilde{V}_h'$ be defined by

$$\langle \widetilde{A}_h v, w \rangle = a_h(v, w) \quad \forall v, w \in \widetilde{V}_h, \quad (13)$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $\widetilde{V}_h' \times \widetilde{V}_h$. From Equations (5) and (11), it holds that for η sufficiently large [25],

$$C_1 |v|_{H^2(\Omega, \mathcal{T}_h)}^2 \leq \langle \widetilde{A}_h v, v \rangle \leq C_2 |v|_{H^2(\Omega, \mathcal{T}_h)}^2 \quad \forall v \in \widetilde{V}_h, \quad (14)$$

where

$$|v|_{H^2(\Omega, \mathcal{T}_h)}^2 = \sum_{D \in \mathcal{T}_h} |v|_{H^2(D)}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|[[[\partial v / \partial n]]]\|_{L_2(e)}^2, \quad (15)$$

and the constants C_1 and C_2 depend only on the shape regularity of $\mathcal{T}_h$.

Remark 3.1 On the finite element space, $|\cdot|_{H^2(\Omega, \mathcal{T}_h)}$ and $\|\cdot\|_h$ are equivalent [25].

In the context of solving the discrete variational inequality (7), the set $\widetilde{\mathcal{N}}_h$ represents the inactive set that appears in an iteration of the PDAS algorithm and $\widetilde{A}_h$ represents the stiffness matrix for the corresponding auxiliary system. Our goal is to construct and analyze additive Schwarz preconditioners for $\widetilde{A}_h$ whose performance is independent of $\widetilde{\mathcal{N}}_h$.

4 | A ONE-LEVEL ADDITIVE SCHWARZ PRECONDITIONER

In this section, we develop a one-level additive Schwarz preconditioner for the auxiliary system defined by $\widetilde{A}_h$.

Let Ω_j , $1 \leq j \leq J$, be overlapping subdomains of Ω such that $\Omega = \bigcup_{j=1}^J \Omega_j$, $\text{diam } \Omega_j \approx H$, and the boundaries of Ω_j are aligned with $\mathcal{T}_h$. We assume that there exist nonnegative $\theta_j \in C^\infty(\bar{\Omega})$ for $1 \leq j \leq J$ such that

$$\theta_j = 0, \quad \text{on } \Omega \setminus \Omega_j, \quad (16a)$$

$$\sum_{j=1}^J \theta_j = 1, \quad \text{on } \bar{\Omega}, \quad (16b)$$

$$\|\nabla \theta_j\|_{L_\infty(\Omega)} \leq \frac{C_\dagger}{\delta}, \quad \|\nabla^2 \theta_j\|_{L_\infty(\Omega)} \leq \frac{C_\dagger}{\delta^2}, \quad (16c)$$

where $\nabla^2 \theta_j$ is the Hessian of θ_j , $\delta > 0$ measures the overlap among subdomains, and $C_\dagger$ is a positive constant independent of h, H , and J . Moreover, we assume that

$$\text{any point in } \Omega \text{ can belong to at most } N_c \text{ many subdomains}, \quad (17)$$

where the positive integer N_c is independent of h, H, J , and δ .

Remark 4.1 Given a coarse triangulation $\mathcal{T}_H$ of Ω consisting of convex quadrilaterals, suppose $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, we can then construct Ω_j by enlarging the elements of $\mathcal{T}_H$ by the amount of δ so that each Ω_j is the union of the elements in $\mathcal{T}_h$. The construction of θ_j that satisfy (16) is standard [27].

Now, let $\tilde{V}_j$, $1 \leq j \leq J$, be the subspace of $\tilde{V}_h$ whose members vanish at all nodes outside Ω_j , and let $\tilde{A}_j : \tilde{V}_j \rightarrow \tilde{V}_j'$ be defined by

$$\langle \tilde{A}_j v, w \rangle = a_h(v, w), \quad \forall v, w \in \tilde{V}_j.$$

We have an analog of (14) that

$$C_3 |v|_{H^2(\Omega_j, \mathcal{T}_h)}^2 \leq \langle \tilde{A}_j v, v \rangle \leq C_4 |v|_{H^2(\Omega_j, \mathcal{T}_h)}^2, \quad \forall v \in \tilde{V}_j, \quad (18)$$

where

$$|v|_{H^2(\Omega_j, \mathcal{T}_h)}^2 = \sum_{\substack{D \in \mathcal{T}_h \\ D \subset \Omega_j}} |v|_{H^2(D)}^2 + \sum_{\substack{e \in \mathcal{E}_h \\ e \subset \bar{\Omega}_j}} \frac{1}{|e|} \|[[\partial v / \partial n]]\|_{L_2(e)}^2.$$

The one-level additive Schwarz preconditioner $B_{OL} : \tilde{V}_h' \rightarrow \tilde{V}_h$ is defined by

$$B_{OL} = \sum_{j=1}^J \tilde{I}_j \tilde{A}_j^{-1} \tilde{I}_j',$$

where $\tilde{I}_j : \tilde{V}_j \rightarrow \tilde{V}_h$, $1 \leq j \leq J$, is the natural injection operator, and $\tilde{I}_j' : \tilde{V}_h' \rightarrow \tilde{V}_j'$ is the transpose of $\tilde{I}_j$.

Next, we derive an estimate for the condition number of $B_{OL} \tilde{A}_h$.

From now on, to avoid the proliferation of constants, we use the notation $A \lesssim B$ to represent the statement $A \leq \text{constant} \times B$, where the positive constant depends only on the shape regularity of the meshes and not h, H, J, δ , nor $\tilde{\mathcal{N}}_h$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Let $\Pi_h : C^0(\bar{\Omega}) \rightarrow V_h$ be the nodal interpolation operator. We have the following interpolation error estimate ([28], lemma 3.2).

Lemma 4.1 *It holds that, for all $\xi \in H_0^2(\Omega)$,*

$$\|\xi - \Pi_h \xi\|_{L_2(\Omega)} + h\|\xi - \Pi_h \xi\|_{H^1(\Omega)} + h^2\|\Pi_h \xi\|_{H^2(\Omega, \mathcal{T}_h)} \lesssim h^2\|\xi\|_{H^2(\Omega)}. \quad (20)$$

We begin with an upper bound for the eigenvalues of $B_{OL}\tilde{A}_h$.

Lemma 4.2 *The following upper bound for the eigenvalues of $B_{OL}\tilde{A}_h$ holds,*

$$\lambda_{\max}(B_{OL}\tilde{A}_h) \lesssim 1. \quad (21)$$

Proof. Let $v \in \tilde{V}_h$ be arbitrary. For any $v_j \in \tilde{V}_j$ such that $v = \sum_{j=1}^J \tilde{I}_j v_j$, we have, by Equations (14), (17), the Cauchy–Schwarz inequality, and Equation (18) that

$$\langle \tilde{A}_h v, v \rangle \approx |v|_{H^2(\Omega, \mathcal{T}_h)}^2 \lesssim \sum_{j=1}^J |\tilde{I}_j v_j|_{H^2(\Omega, \mathcal{T}_h)}^2 = \sum_{j=1}^J |v_j|_{H^2(\Omega_j, \mathcal{T}_h)}^2 \lesssim \sum_{j=1}^J \langle \tilde{A}_j v_j, v_j \rangle,$$

which implies

$$\langle \tilde{A}_h v, v \rangle \lesssim \min_{\substack{v = \sum_{j=1}^J \tilde{I}_j v_j \\ v_j \in \tilde{V}_j}} \sum_{j=1}^J \langle \tilde{A}_j v_j, v_j \rangle.$$

Hence Equation (21) follows from (22) and the standard additive Schwarz theory [2, 17, 29–31]. ■

Next, we derive a lower bound for the eigenvalues of $B_{OL}\tilde{A}_h$.

Lemma 4.3 *It holds that*

$$\lambda_{\min}(B_{OL}\tilde{A}_h) \gtrsim \delta^4, \quad (23)$$

where δ measures the overlap among the subdomains.

Proof. Let $v \in \tilde{V}_h$ be arbitrary. We define

$$v_j = \Pi_h(\theta_j v), \quad 1 \leq j \leq J. \quad (24)$$

From Equation (16a), we know $v_j \in \tilde{V}_j$, and furthermore from Equation (16b), it holds that

$$\sum_{j=1}^J v_j = \sum_{j=1}^J \Pi_h(\theta_j v) = \Pi_h\left(\sum_{j=1}^J \theta_j\right)v = \Pi_h v = v.$$

It then follows from Equations (24), (20), and (16c) that

$$\begin{aligned} |v_j|_{H^2(D)}^2 &= |\Pi_h(\theta_j v)|_{H^2(D)}^2 \lesssim |\theta_j v|_{H^2(D)}^2 \\ &\lesssim \|\theta_j\|_{L_\infty(D)}^2 |v|_{H^2(D)}^2 + \|\nabla \theta_j\|_{L_\infty(D)}^2 |v|_{H^1(D)}^2 + \|\nabla^2 \theta_j\|_{L_\infty(D)}^2 \|v\|_{L_2(D)}^2 \\ &\lesssim |v|_{H^2(D)}^2 + \frac{1}{\delta^2} |v|_{H^1(D)}^2 + \frac{1}{\delta^4} \|v\|_{L_2(D)}^2. \end{aligned} \quad (25)$$

Moreover, for $e \in \overline{\Omega}_j$, an arbitrary edge from $\mathcal{E}_h$, we have, from Equation (24), the trace theorem with scaling, Equations (20), and (16c),

$$\begin{aligned} \frac{1}{|e|} \|[[[\partial v_j / \partial n]]]\|_{L_2(e)}^2 &= \frac{1}{|e|} \|[[[\partial(\Pi_h(\theta_j v)) / \partial n]]]\|_{L_2(e)}^2 \\ &\lesssim \frac{1}{|e|} \|[[[\partial(\Pi_h(\theta_j v) - \theta_j v) / \partial n]]]\|_{L_2(e)}^2 + \frac{1}{|e|} \|[[[\partial(\theta_j v) / \partial n]]]\|_{L_2(e)}^2 \end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{|e|} \sum_{D \in \mathcal{T}_e} \|\partial(\Pi_h(\theta_j v) - \theta_j v)_D / \partial n\|_{L_2(e)}^2 + \frac{1}{|e|} \|[[\theta_j(\partial v / \partial n)]]\|_{L_2(e)}^2 \\
&\lesssim \sum_{D \in \mathcal{T}_e} [(\text{diam } D)^{-2} \|\Pi_h(\theta_j v) - \theta_j v\|_{H^1(D)}^2 + \|\Pi_h(\theta_j v) - \theta_j v\|_{H^2(D)}^2] + \frac{1}{|e|} \|[[\partial v / \partial n]]\|_{L_2(e)}^2 \\
&\lesssim \sum_{D \in \mathcal{T}_e} \|\theta_j v\|_{H^2(D)}^2 + \frac{1}{|e|} \|[[\partial v / \partial n]]\|_{L_2(e)}^2 \\
&\lesssim \sum_{D \in \mathcal{T}_e} \left(\|v\|_{H^2(D)}^2 + \frac{1}{\delta^2} \|v\|_{H^1(D)}^2 + \frac{1}{\delta^4} \|v\|_{L_2(D)}^2 \right) + \frac{1}{|e|} \|[[\partial v / \partial n]]\|_{L_2(e)}^2, \tag{26}
\end{aligned}$$

where $\mathcal{T}_e$ is the set of elements in $\mathcal{T}_h$ sharing e as a common edge.

We can now conclude from Equations (19), (17), (25), (26), and Poincaré–Friedrichs inequalities [32] that

$$\begin{aligned}
\sum_{j=1}^J |v_j|_{H^2(\Omega_j, \mathcal{T}_h)}^2 &= \sum_{j=1}^J \left(\sum_{\substack{D \in \mathcal{T}_h \\ D \subset \Omega_j}} |v_j|_{H^2(D)}^2 + \sum_{\substack{e \in \mathcal{E}_h \\ e \subset \Omega_j}} \frac{1}{|e|} \|[[\partial v_j / \partial n]]\|_{L_2(e)}^2 \right) \\
&\lesssim \sum_{D \in \mathcal{T}_h} \left(\|v\|_{H^2(D)}^2 + \frac{1}{\delta^2} \|v\|_{H^1(D)}^2 + \frac{1}{\delta^4} \|v\|_{L_2(D)}^2 \right) + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|[[\partial v / \partial n]]\|_{L_2(e)}^2 \\
&\lesssim \frac{1}{\delta^4} |v|_{H^2(\Omega, \mathcal{T}_h)}^2. \tag{27}
\end{aligned}$$

Therefore, Equation (23) follows from Equations (14), (18), (27) and the standard additive Schwarz theory. ■

Combining Lemma 4.2 and Lemma 4.3, we obtain the following estimate on the condition number of the one-level additive Schwarz preconditioned auxiliary system.

Theorem 4.4 *The condition number of $B_{\text{OL}} \tilde{A}_h$ satisfies*

$$\kappa(B_{\text{OL}} \tilde{A}_h) = \frac{\lambda_{\max}(B_{\text{OL}} \tilde{A}_h)}{\lambda_{\min}(B_{\text{OL}} \tilde{A}_h)} \lesssim \delta^{-4}, \tag{28}$$

where δ measures the overlap among the subdomains.

Remark 4.2 Under the assumption that the subdomains Ω_j , $1 \leq j \leq J$, are shape regular, we can improve the estimate Equation (28) to

$$\kappa(B_{\text{OL}} \tilde{A}_h) \lesssim H^{-1} \delta^{-3}, \tag{29}$$

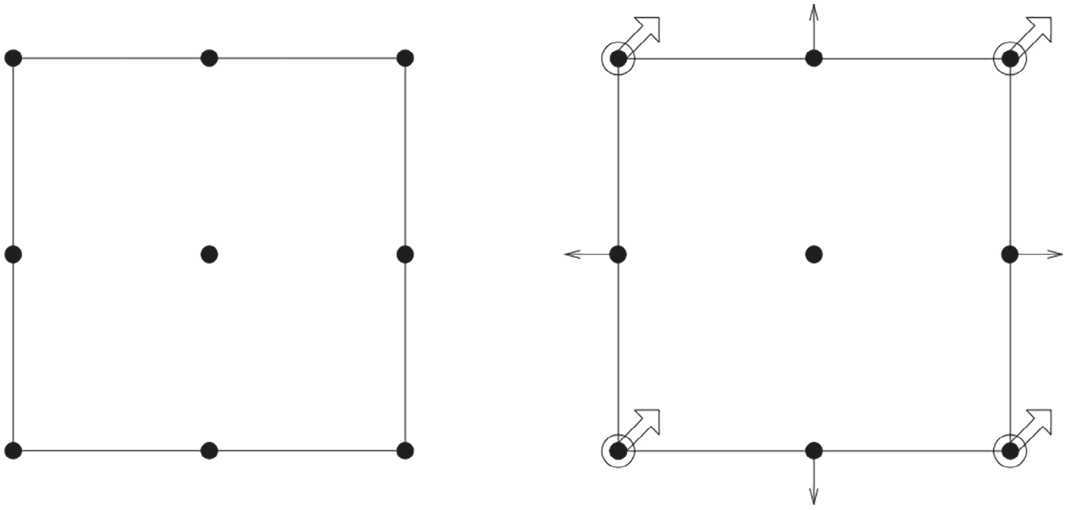
by similar arguments in Brenner and Wang [28]. In the case of small overlap among the subdomains $\delta \approx h$, we have

$$\kappa(B_{\text{OL}} \tilde{A}_h) \lesssim H^{-1} h^{-3}, \tag{30}$$

which implies that for a fixed h the condition number will increase as H decreases (or J increases). In the case of generous overlap among the subdomains $\delta \approx H$, we have

$$\kappa(B_{\text{OL}} \tilde{A}_h) \lesssim H^{-4}, \tag{31}$$

which indicates that $\kappa(B_{\text{OL}} \tilde{A}_h)$ remains constant as h decreases provided H (or J) is fixed.

FIGURE 1 Q_2 element and Q_4 Bogner–Fox–Schmit element

5 | A TWO-LEVEL ADDITIVE SCHWARZ PRECONDITIONER

In this section, we develop a two-level additive Schwarz preconditioner for the auxiliary system defined by $\tilde{A}_h$. In addition to subdomain solves, the two-level additive Schwarz preconditioner contains a coarse grid solve.

Let $\mathcal{T}_H$ be a coarse quasi-uniform triangulation for Ω whose mesh size is comparable to the diameters of the subdomains Ω_j , $1 \leq j \leq J$, and $V_H \subset H_0^1(\Omega)$ be the Q_k finite element space associated with $\mathcal{T}_H$. For example, we can take $\mathcal{T}_H$ to be the triangulation mentioned in Remark 4.1.

Since the Q_{k+2} Bogner–Fox–Schmit tensor product element is a C^1 relative of the Q_k tensor product element [25], we define $W_H \subset H_0^2(\Omega)$ to be the Q_{k+2} Bogner–Fox–Schmit finite element space associated with $\mathcal{T}_H$. The two spaces V_H and W_H can be connected by an enriching operator E_H which is constructed by the averaging technique as follows.

For simplicity, we take V_H to be the Q_2 finite element space and W_H to be the Q_4 Bogner–Fox–Schmit finite element space. The Q_2 element and the Q_4 Bogner–Fox–Schmit element are depicted in Figure 1, where we use the solid dot $\bullet$ to denote pointwise evaluation of the shape functions, the circle $\circ$ to denote pointwise evaluation of all the first order derivatives of the shape functions, the arrow $\uparrow$ to denote pointwise evaluation of the normal derivative of the shape functions, and the arrow $\nearrow$ to denote pointwise evaluation of the mixed second order derivative of the shape functions.

The operator $E_H : V_H \rightarrow W_H$ is defined by

$$(E_H v)(p) = v(p) \quad (31a)$$

$$\nabla(E_H v)(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D \in \mathcal{T}_p} \nabla v_D(p) \quad (31b)$$

$$\frac{\partial(E_H v)}{\partial n_e}(m_e) = \frac{1}{|\mathcal{T}_e|} \sum_{D \in \mathcal{T}_e} \frac{\partial v_D}{\partial n_e}(m_e) \quad (31c)$$

$$\frac{\partial^2(E_H v)}{\partial x_1 \partial x_2}(p) = \frac{1}{|\mathcal{T}_p|} \sum_{D \in \mathcal{T}_p} \frac{\partial^2 v_D}{\partial x_1 \partial x_2}(p) \quad (31d)$$

where Equation (31a) is defined for any interior node p associated with V_H , $\mathcal{T}_p$ is the set of elements in $\mathcal{T}_H$ sharing p as a common vertex, $|\mathcal{T}_p|$ is the number of elements in $\mathcal{T}_p$, Equations (31b) and (31d) are defined for any interior vertex p of $\mathcal{T}_H$, $\mathcal{T}_e$ is the set of elements in $\mathcal{T}_H$ sharing e as a common edge, $|\mathcal{T}_e|$ is the number of elements in $\mathcal{T}_e$, and Equation (31c) is defined for any interior edge e of $\mathcal{T}_H$ with midpoint m_e . The construction of E_H can be extended to other Q_k finite element spaces.

Now we define $I_0 : V_H \rightarrow V_h$ by

$$I_0 v = \Pi_h E_H v, \quad \forall v \in V_H. \quad (32)$$

The following property of the operator I_0 holds ([28], lemma 3.3).

Lemma 5.1 *For all $v \in V_H$, we have*

$$\|v - I_0 v\|_{L_2(\Omega)} + H|v - I_0 v|_{H^1(\Omega)} + H^2|I_0 v|_{H^2(\Omega, \mathcal{T}_h)} \lesssim H^2|v|_{H^2(\Omega, \mathcal{T}_H)}, \quad (33)$$

where $|\cdot|_{H^2(\Omega, \mathcal{T}_H)}$ is the analog of $|\cdot|_{H^2(\Omega, \mathcal{T}_h)}$ for $v \in V_H$.

We define $\tilde{V}_0 \subset \tilde{V}_h$ by

$$\tilde{V}_0 = \tilde{T}_h I_0 V_H,$$

and the operator $\tilde{A}_0 : \tilde{V}_0 \rightarrow \tilde{V}_0'$ by

$$\langle \tilde{A}_0 v, w \rangle = a_h(v, w), \quad \forall v, w \in \tilde{V}_0. \quad (34)$$

Similarly, we have

$$\langle \tilde{A}_0 v, v \rangle \approx |v|_{H^2(\Omega, \mathcal{T}_h)}^2, \quad \forall v \in \tilde{V}_0. \quad (35)$$

Then the two-level additive Schwarz preconditioner $B_{TL} : \tilde{V}_h' \rightarrow \tilde{V}_h$ is given by

$$B_{TL} = \sum_{j=0}^J \tilde{I}_j \tilde{A}_j^{-1} \tilde{I}_j^t,$$

where $\tilde{I}_j : \tilde{V}_j \rightarrow \tilde{V}_h$, $0 \leq j \leq J$, is the natural injection operator, and $\tilde{I}_j^t$ is the transpose of $\tilde{I}_j$.

The proof of the following Lemma is similar to Lemma 4.2.

Lemma 5.2 *It holds that*

$$\lambda_{\max}(B_{TL} \tilde{A}_h) \lesssim 1. \quad (36)$$

We now turn our attention to a lower bound for the eigenvalues of $B_{TL} \tilde{A}_h$.

Let $\Pi_H : C^0(\bar{\Omega}) \rightarrow V_H$ be the nodal interpolation operator and $J_h^H : V_h \rightarrow V_H$ be defined by

$$J_h^H v = \Pi_H E_h v, \quad \forall v \in V_h,$$

where $E_h : V_h \rightarrow W_h$ is an enriching operator defined by the averaging technique, and W_h is the Q_{k+2} Bogner–Fox–Schmit finite element space associated with $\mathcal{T}_h$.

The following estimate holds for J_h^H ([28], lemma 3.5).

Lemma 5.3 *For all $v \in V_h$, it is valid that*

$$\|v - J_h^H v\|_{L_2(\Omega)} + H|v - J_h^H v|_{H^1(\Omega)} + H^2|J_h^H v|_{H^2(\Omega, \mathcal{T}_H)} \lesssim H^2|v|_{H^2(\Omega, \mathcal{T}_h)}. \quad (37)$$

Moreover, let $R_0 : V_h \rightarrow \tilde{V}_0$ be defined by

$$R_0 v = \tilde{T}_h I_0 J_h^H v, \quad \forall v \in V_h. \quad (38)$$

Then we have the following estimate for the operator R_0 .

Lemma 5.4 *It holds that*

$$\|v - R_0 v\|_{L_2(\Omega)} + h|v - R_0 v|_{H^1(\Omega)} + h^2|v - R_0 v|_{H^2(\Omega, \mathcal{T}_h)} \lesssim H^2|v|_{H^2(\Omega, \mathcal{T}_h)}, \quad \forall v \in \tilde{V}_h. \quad (39)$$

Proof. For all $v \in \tilde{V}_h$, it follows from Equations (38), (12), (33), and (37) that

$$\begin{aligned} \|v - R_0 v\|_{L_2(\Omega)} &= \|v - \tilde{T}_h I_0 J_h^H v\|_{L_2(\Omega)} \\ &= \|\tilde{T}_h(v - I_0 J_h^H v)\|_{L_2(\Omega)} \\ &\lesssim \|v - I_0 J_h^H v\|_{L_2(\Omega)} \\ &\lesssim \|v - J_h^H v\|_{L_2(\Omega)} + \|I_0 J_h^H v - J_h^H v\|_{L_2(\Omega)} \\ &\lesssim H^2|v|_{H^2(\Omega, \mathcal{T}_h)} + H^2|J_h^H v|_{H^2(\Omega, \mathcal{T}_h)} \\ &\lesssim H^2|v|_{H^2(\Omega, \mathcal{T}_h)}. \end{aligned} \quad (40)$$

From standard inverse estimates, we know that for all $D \in \mathcal{T}_h$,

$$|v - R_0 v|_{H^1(D)} \lesssim h^{-1} \|v - R_0 v\|_{L_2(D)} \quad \text{and} \quad |v - R_0 v|_{H^2(D)} \lesssim h^{-2} \|v - R_0 v\|_{L_2(D)}. \quad (41)$$

Moreover, for any $e \in \mathcal{E}_h$, by the trace theorem with scaling and Equation (41), we have

$$\begin{aligned} \frac{1}{|e|} \|[(\partial(v - R_0 v)/\partial n)]\|_{L_2(e)}^2 &\lesssim \frac{1}{|e|} \sum_{D \in \mathcal{T}_e} \|\partial(v - R_0 v)_D / \partial n\|_{L_2(e)}^2 \\ &\lesssim \sum_{D \in \mathcal{T}_e} [(\text{diam } D)^{-2} |v - R_0 v|_{H^1(D)}^2 + |v - R_0 v|_{H^2(D)}^2] \\ &\lesssim \sum_{D \in \mathcal{T}_e} (\text{diam } D)^{-4} \|v - R_0 v\|_{L_2(D)}^2, \end{aligned} \quad (42)$$

where $\mathcal{T}_e$ is the set of the elements in $\mathcal{T}_h$ sharing e as a common edge.

By summing up Equations (41) and (42) over all the elements in $\mathcal{T}_h$ and all the edges in $\mathcal{E}_h$, we obtain

$$|v - R_0 v|_{H^1(\Omega)} \lesssim h^{-1} \|v - R_0 v\|_{L_2(\Omega)} \quad \text{and} \quad |v - R_0 v|_{H^2(\Omega, \mathcal{T}_h)} \lesssim h^{-2} \|v - R_0 v\|_{L_2(\Omega)},$$

which together with Equation (40) implies (39). $\blacksquare$

The lower bound for the eigenvalues of $B_{TL} \tilde{A}_h$ is then given by the following lemma.

Lemma 5.5 *It holds that*

$$\lambda_{\min}(B_{TL} \tilde{A}_h) \gtrsim \frac{1}{\min((H/h)^4, \delta^{-4})}. \quad (43)$$

Proof. Let $v \in \tilde{V}_h$ be arbitrary, $v_0 = R_0 v \in \tilde{V}_0$, and $v_j = \Pi_h(\theta_j(v - v_0)) \in \tilde{V}_j$, $1 \leq j \leq J$. It is clear that

$$\sum_{j=0}^J v_j = v_0 + \sum_{j=1}^J \Pi_h(\theta_j(v - v_0)) = v_0 + \Pi_h \left[\left(\sum_{j=1}^J \theta_j \right) (v - v_0) \right] = v_0 + v - v_0 = v.$$

First of all, from Equations (35), (39), and (14), we have

$$\langle \tilde{A}_0 v_0, v_0 \rangle \approx |R_0 v|_{H^2(\Omega, \mathcal{T}_h)}^2 \lesssim \left(\frac{H}{h} \right)^4 |v|_{H^2(\Omega, \mathcal{T}_h)}^2 \approx \left(\frac{H}{h} \right)^4 \langle \tilde{A}_h v, v \rangle. \quad (44)$$

Next, we consider v_j for $1 \leq j \leq J$.

Let $D \subset \Omega_j$ be an arbitrary element in $\mathcal{T}_h$ and $e \subset \bar{\Omega}_j$ be an edge from $\mathcal{E}_h$. Since $v - v_0 \in \tilde{V}_h$, by using similar discussions as we did in Equations (25) and (26), we have

$$|v_j|_{H^2(D)}^2 \lesssim |v - v_0|_{H^2(D)}^2 + \frac{1}{\delta^2} |v - v_0|_{H^1(D)}^2 + \frac{1}{\delta^4} \|v - v_0\|_{L_2(D)}^2,$$

and

$$\begin{aligned} \frac{1}{|e|} \|[[\partial v_j / \partial n]]\|_{L_2(e)}^2 &\lesssim \sum_{D \in \mathcal{T}_e} \left(|v - v_0|_{H^2(D)}^2 + \frac{1}{\delta^2} |v - v_0|_{H^1(D)}^2 + \frac{1}{\delta^4} \|v - v_0\|_{L_2(D)}^2 \right) \\ &\quad + \frac{1}{|e|} \|[[\partial(v - v_0) / \partial n]]\|_{L_2(e)}^2, \end{aligned}$$

which together with Equations (18), (19), (17), (39), and (14) imply

$$\begin{aligned} \sum_{j=1}^J \langle \tilde{A}_j v_j, v_j \rangle &\approx \sum_{j=1}^J |v_j|_{H^2(\Omega_j, \mathcal{T}_h)}^2 \\ &= \sum_{j=1}^J \left(\sum_{\substack{D \in \mathcal{T}_h \\ D \subset \Omega_j}} |v_j|_{H^2(D)}^2 + \sum_{\substack{e \in \mathcal{E}_h \\ e \subset \bar{\Omega}_j}} \frac{1}{|e|} \|[[\partial v_j / \partial n]]\|_{L_2(e)}^2 \right) \\ &\lesssim \sum_{D \in \mathcal{T}_h} \left(|v - v_0|_{H^2(D)}^2 + \frac{1}{\delta^2} |v - v_0|_{H^1(D)}^2 + \frac{1}{\delta^4} \|v - v_0\|_{L_2(D)}^2 \right) + \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \|[[\partial(v - v_0) / \partial n]]\|_{L_2(e)}^2 \\ &\lesssim |v - v_0|_{H^2(\Omega, \mathcal{T}_h)}^2 + \frac{1}{\delta^2} |v - v_0|_{H^1(\Omega)}^2 + \frac{1}{\delta^4} \|v - v_0\|_{L_2(\Omega)}^2 \\ &= |v - R_0 v|_{H^2(\Omega, \mathcal{T}_h)}^2 + \frac{1}{\delta^2} |v - R_0 v|_{H^1(\Omega)}^2 + \frac{1}{\delta^4} \|v - R_0 v\|_{L_2(\Omega)}^2 \\ &\lesssim \left(\frac{H^4}{h^4} + \frac{H^4}{h^2 \delta^2} + \frac{H^4}{\delta^4} \right) |v|_{H^2(\Omega, \mathcal{T}_h)}^2 \\ &\lesssim \left(\frac{H}{h} \right)^4 \langle \tilde{A}_h v, v \rangle. \end{aligned} \tag{45}$$

Hence by combining Equations (44) and (45), we obtain

$$\sum_{j=0}^J \langle \tilde{A}_j v_j, v_j \rangle \lesssim (H/h)^4 \langle \tilde{A}_h v, v \rangle,$$

which implies

$$\lambda_{\min}(B_{\text{TL}} \tilde{A}_h) \gtrsim (H/h)^{-4}$$

by the standard additive Schwarz theory.

On the other hand, we can also use the decomposition of v in the one-level case to conclude that

$$\lambda_{\min}(B_{\text{TL}} \tilde{A}_h) \gtrsim \delta^4.$$

Therefore, the estimate (43) holds. ■

Combining Lemma 5.2 and Lemma 5.5, we have the following estimate on the condition number of $B_{\text{TL}} \tilde{A}_h$.

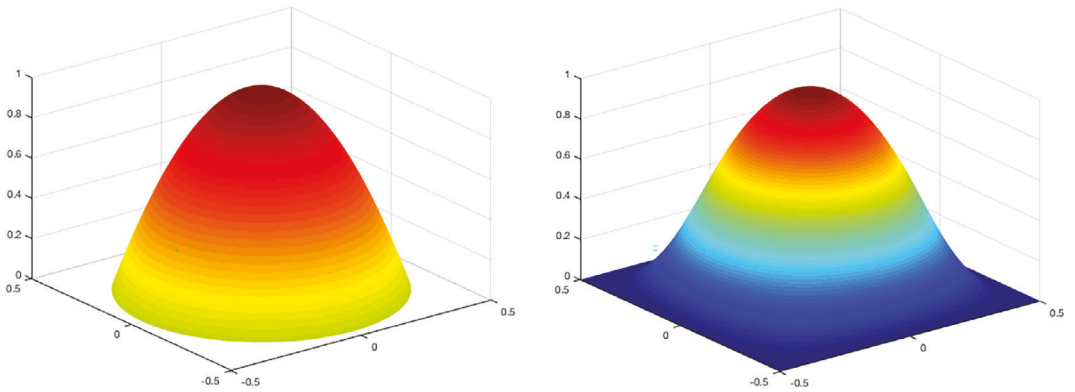


FIGURE 2 Left: graph of the obstacle function ψ ; right: graph of the numerical solution u_h at refinement level 8

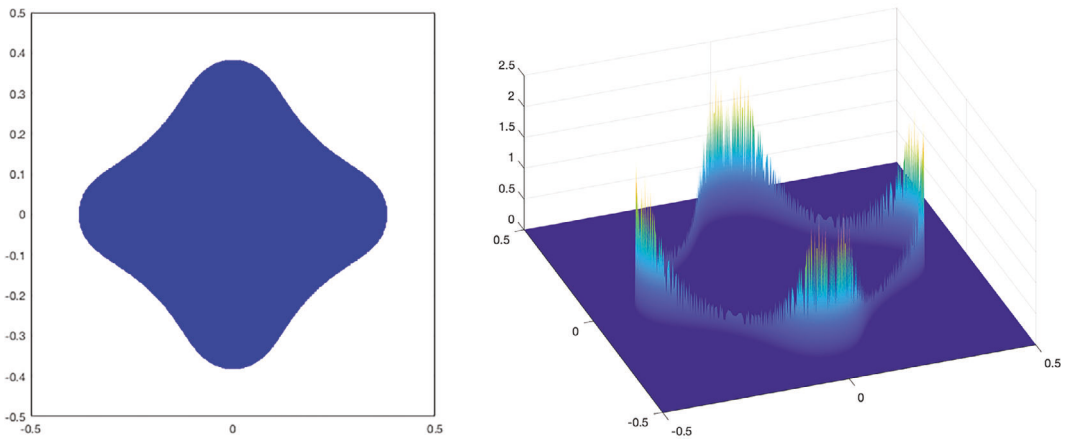


FIGURE 3 Left: discrete active set $\mathcal{A}_h$ at refinement level 8; right: graph of the discrete Lagrange multiplier λ_h at refinement level 8

Theorem 5.6 *It holds that*

$$\kappa(B_{\text{TL}}\tilde{A}_h) \lesssim \min((H/h)^4, \delta^{-4}). \quad (46)$$

Remark 5.1 Because of the necessity of a truncation operator in the construction of $\tilde{V}_0$ when the obstacle is present, the estimate Equation (46) is different from the one for the plate bending problem without obstacles (cf. Brenner and Wang [28]) that takes the form

$$\kappa(B_{\text{TL}}A_h) \lesssim (1 + (H/\delta)^4).$$

Remark 5.2 Under the assumption that the subdomains $\Omega_1, \dots, \Omega_J$ are shape regular, the condition number estimate Equation (46) can be improved to (cf. Brenner and Wang [28])

$$\kappa(B_{\text{TL}}\tilde{A}_h) \lesssim \min((H/h)^4, H^{-1}\delta^{-3}), \quad (47)$$

which indicates that the two-level additive Schwarz algorithm is scalable as long as H/h is bounded, and the condition number for the two-level algorithm is (up to a constant) at least as good as the one-level algorithm.

TABLE 1 Average condition number of auxiliary system without a preconditioner, and number of iterations of the PDAS algorithm

	$\kappa(\tilde{A}_h)$	PDAS Iterations
$\ell = 1$	1.7200×10^0	2
$\ell = 2$	7.2687×10^1	5
$\ell = 3$	1.7912×10^2	10
$\ell = 4$	1.8327×10^3	12
$\ell = 5$	3.1546×10^4	17
$\ell = 6$	5.2298×10^5	32
$\ell = 7$	8.4722×10^6	49

Abbreviation: PDAS, primal-dual active set.

TABLE 2 Average condition number for one-level additive Schwarz preconditioned auxiliary system with small overlap

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	4.4972×10^0	—	—	—
$\ell = 3$	5.5042×10^0	6.4417×10^0	—	—
$\ell = 4$	4.2146×10^0	1.2558×10^1	2.4693×10^1	—
$\ell = 5$	5.7727×10^0	6.6345×10^1	1.1999×10^2	3.1176×10^2
$\ell = 6$	1.4470×10^1	4.6800×10^2	8.9931×10^2	1.6871×10^3

TABLE 3 Average condition number for two-level additive Schwarz preconditioned auxiliary system with small overlap

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	4.6253×10^0	—	—	—
$\ell = 3$	5.4757×10^0	5.5599×10^0	—	—
$\ell = 4$	4.8622×10^0	7.3983×10^0	6.2173×10^0	—
$\ell = 5$	6.4056×10^0	2.8187×10^1	8.6746×10^0	6.6549×10^0
$\ell = 6$	1.4899×10^1	1.7420×10^2	2.7963×10^1	9.5560×10^0

Moreover, we can deduce from Equation (47) that

$$\kappa(B_{\text{TL}}\tilde{A}_h) < \kappa(B_{\text{OL}}\tilde{A}_h),$$

if $H^5 \ll h$ in the case of small overlap $\delta \approx h$, and if $H^2 \ll h$ in the case of generous overlap $\delta \approx H$.

Furthermore, in the case of generous overlap $\delta \approx H$, $\kappa(B_{\text{TL}}\tilde{A}_h)$ remains constant as h decreases provided that H is kept fixed.

6 | NUMERICAL RESULTS

We consider the obstacle problem (cf. Brenner et al. [33]) with $\Omega = (-0.5, 0.5)^2$, $f = 0$, and $\psi = 1 - 5|x|^2 + |x|^4$. We discretize the model problem by the C^0 interior penalty method that is based

TABLE 4 Average condition number for one-level additive Schwarz preconditioned auxiliary system with generous overlap

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	1.0000×10^0	—	—	—
$\ell = 3$	1.0000×10^0	2.8958×10^0	—	—
$\ell = 4$	1.0000×10^0	2.6102×10^0	5.9018×10^0	—
$\ell = 5$	1.0000×10^0	2.6319×10^0	6.4099×10^0	5.2953×10^1
$\ell = 6$	1.0000×10^0	2.5289×10^0	6.6582×10^0	5.5126×10^1

TABLE 5 Average condition number for two-level additive Schwarz preconditioned auxiliary system with generous overlap

	$J = 4$	$J = 16$	$J = 64$	$J = 256$
$\ell = 2$	1.2083×10^0	—	—	—
$\ell = 3$	1.2500×10^0	2.9011×10^0	—	—
$\ell = 4$	1.2500×10^0	2.6830×10^0	4.2245×10^0	—
$\ell = 5$	1.2500×10^0	2.6803×10^0	4.5906×10^0	1.0738×10^1
$\ell = 6$	1.2501×10^0	2.5504×10^0	4.9739×10^0	1.0999×10^1

on a rectangular mesh, and choose V_h to be the standard Q_2 finite element space with the mesh size $h = 2^{-\ell}$, where ℓ is the refinement level. The resulting discrete variational inequalities are solved by the PDAS algorithm in Section 3, where the constant c in Equation (9) is chosen to be 10^8 . The initial guess for the PDAS algorithm is taken to be the solution at the previous level or zero when $\ell = 1$.

Graphs of the obstacle function ψ and the numerical solution u_h (at refinement level 8) are given in Figure 2. Moreover, the discrete active set $\mathcal{A}_h$ and the discrete Lagrange multiplier λ_h (at refinement level 8) are displayed in Figure 3, from which we observe that λ_h is positive along the boundary of the active set $\mathcal{A}_h$. These figures match the ones obtained by a partition of unity method in Brenner et al. [24].

For comparison, we first calculate the condition number of the un-preconditioned auxiliary system $\tilde{A}_h$ in each iteration of the PDAS algorithm and then average them. The average condition numbers and numbers of iterations of the PDAS algorithm for various ℓ levels are presented in Table 1.

We then apply the one-level and two-level additive Schwarz preconditioners on the auxiliary system in each iteration of the PDAS algorithm. The average condition numbers of both preconditioned auxiliary systems for 4, 16, 64, and 256 subdomains with small overlap, $\delta = h$, are reported in Tables 2 and 3 respectively. Comparing to average condition numbers of the unpreconditioned auxiliary system in Table 1, both the one-level and two-level algorithms show significant improvement.

Moreover, we observe that the condition numbers for the one-level algorithm in Table 2 agree with Equation (30) in Remark 4.2. In addition, a comparison of Tables 2 and 3 shows that $\kappa(B_{TL}\tilde{A}_h)$ is smaller than $\kappa(B_{OL}\tilde{A}_h)$ for J large.

Similar simulations for generous overlap $\delta = H$ are also performed. The average condition numbers of the one-level and two-level additive Schwarz preconditioned auxiliary systems for various number of subdomains are presented in Tables 4 and 5. From the behavior of the condition numbers, we notice that if H or J is fixed, both $\kappa(B_{OL}\tilde{A}_h)$ and $\kappa(B_{TL}\tilde{A}_h)$ remain constant as h decreases, which coincide with our theoretical results in Remark 4.2 and Remark 5.2, respectively. In addition, as in the case of small overlap, we can see that $\kappa(B_{TL}\tilde{A}_h)$ is smaller than $\kappa(B_{OL}\tilde{A}_h)$ for J large.

7 | CONCLUDING REMARKS

We investigate the one-level and two-level additive Schwarz domain decomposition preconditioners for the auxiliary systems that appear in a PDAS algorithm for solving the obstacle problem of clamped Kirchhoff plates, where the discrete problem is discretized by C^0 interior penalty methods. We demonstrate that both preconditioners improve the condition numbers of the auxiliary systems dramatically. The condition number estimates for the one-level additive Schwarz preconditioner are identical to the ones for the plate bending problem without obstacles, whereas the condition number estimates for the two-level additive Schwarz preconditioner are different because of the necessity of the a truncation operator in the construction of the coarse problem. All theoretical results are confirmed by numerical results.

The results in this paper can also be applied to the general Kirchhoff plate model described in (1.2.45) of Ciarlet [3].

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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