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Regularity of Stationary Boltzmann Equation in Convex Domains

Hongxu Chen & Chanwoo Kim

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Abstract

The higher regularity estimate has been a challenging question regarding the Boltzmann equation in bounded domains. Indeed, it is well-known to have "*the non-existence of a second order derivative at the boundary*" in Guo et al. (Invent Math 207:115–290, 2017) even for symmetric convex domains such as a disk or sphere. In this paper, we answer this question in the affirmative by constructing the $C^{1,\beta}$ solutions away from the grazing boundary, for any $\beta < 1$, to the stationary Boltzmann equation with the non-isothermal diffuse boundary condition in strictly convex domains, as long as a smooth wall temperature has small fluctuation pointwisely.

1. Introduction

An interesting physical application of the kinetic theory is its mesoscopic description of the heat transfer of rarefied gas. The *quantitative description of the stationary state* and a derivation of macroscopic models (as the Knudsen number $\mathcal{K}n \to \infty$) can be achieved through the famous steady Boltzmann equation

$$v \cdot \nabla_x F = \frac{1}{\mathcal{K}n} Q(F, F), \quad (x, v) \in \Omega \times \mathbb{R}^3, \tag{1.1}$$

where the hard sphere collision operator $Q(F, F)$ takes the form

$$\begin{aligned} Q(F_1, F_2) &:= Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |(v-u)\cdot\omega| \big[F_1(u')F_2(v') - F_1(u)F_2(v)\big] \mathrm{d}\omega \mathrm{d}u, \end{aligned} \tag{1.2}$$

with $u' = u + [(v-u)\cdot\omega]\omega$, $v' = v - [(v-u)\cdot\omega]\omega$ with $\omega \in \mathbb{S}^2$.

In particular, when the gas interacts with a non-isothermal boundary it is well-known that the non-equilibrium steady states can be constructed by the Boltzmann

equation (1.1). The kinetic description of the boundary interaction with the gas particles has been extensively investigated in various aspects (see [3–7,16,17,21, 22] and the references therein). In this paper we are interested in one of the basic and physical conditions, the so-called *diffuse reflection boundary condition*, which takes into account an instantaneous thermal equilibrating with the non-constant wall temperature of a reflecting gas particle:

$$F(x,v)|_{n(x)\cdot v<0} = M_W(x,v)\int_{n(x)\cdot u>0} F(x,u)\{n(x)\cdot u\}\mathrm{d}u, \quad x\in\partial\Omega. \tag{1.3}$$

Here the outward normal at the boundary $\partial\Omega$ is denoted by $n(x)$, and we define the wall Maxwellian associated with the described wall temperature $T_W(x)$ at $x\in\partial\Omega$:

$$M_W(x,v) = \sqrt{\frac{2\pi}{T_W}}M_{1,0,T_W} := \frac{1}{2\pi[T_W(x)]^2}e^{-\frac{|v|^2}{2T_W(x)}}. \tag{1.4}$$

Recently, a unique stationary solution of (1.1) with (1.3) in general bounded domains has been constructed in an L^∞-space when the non-constant wall temperature is a small fluctuation around any constant temperature T_0 in [9] (see [15] for the construction in convex domains). Moreover, the authors prove that such non-equilibrium solutions are dynamically and asymptotically stable. We also refer to relevant literatures [11] and the references therein for the PDE aspects of non-equilibrium steady states. As an important application of such construction the authors further derive the Fourier law (Navier-Stokes-Fourier system, more precisely) rigorously as $\mathcal{K}n\to\infty$ in [10]. On the other hand, for each fixed finite Knudsen number $\mathcal{K}n$, they formulate a criterion of the Fourier law in mesoscopic level in [9]. Utilizing the available numeric results, they illustrate the violation of such a criterion, which demonstrates a deviation from the Fourier law for each fixed finite Knudsen number $\mathcal{K}n$.

Qualitatively the kinetic and macroscopic descriptions of heat transfer are remarkably different in the presence of boundaries in particular. In the absence of fluid velocity flow, a macroscopic description via the Fourier law is given by the Laplace equation with suitable boundary condition, which enjoys analytic smoothness of the solutions. On the other hand, the kinetic description from the Boltzmann equation (1.1) possesses a boundary singularity intrinsically ([18]). Such a drastic discrepancy comes from the convection effect $\mathcal{K}nv\cdot\nabla_x F$, which has small factor but non-zero for any finite Knudsen number $\mathcal{K}n>0$. Indeed, it is very interesting to study the *quantitative* effect of such a convection term $\mathcal{K}nv\cdot\nabla_x F$ in the interaction of the boundary and collision process in the limiting process $\mathcal{K}n\to\infty$. Our work in this paper originates from this motivation.

As the first step toward this goal, in this paper we are looking for the smoothness of the stationary Boltzmann equation for fixed $\mathcal{K}n\sim 1$, comparable to the regularity of the corresponding (in a sense of $\mathcal{K}n\to\infty$) elliptic equation, for which *the Schauder estimates* are available. More precisely the main purpose of this paper is to develop a robust and unified *higher regularity* estimate in $C_x^{1,\beta}$ with the aid of weights for the stationary Boltzmann solutions to (1.1) with the diffuse reflection boundary condition (1.3) in the *convex* domains. For this purpose we focus on the

convex domain as a discontinuous singularity appears in the non-convex domain in general [13,18].

In general convex domains, regularity estimates at most up to the first derivatives away from the so-called *grazing set*

$$\gamma_0 := \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v = 0\} \tag{1.5}$$

has been established in [1,2,14] for the nonlinear dynamical Boltzmann equation. The key idea of the approach is based on the so-called kinetic distance, which is almost invariant along the characteristics. With the aid of such weight a generic singularity $\frac{1}{n(x)\cdot v}$ of the first order derivatives can be controlled. We refer to [8] for the regularity of the stationary linear equation up to the first derivatives. *However,* any higher regularity beyond the first order derivatives away from the boundary has been a challenging question. Apparently any second order derivatives estimate seems impossible due to the well-known *"non-existence of second order spatial normal derivative at the boundary"* in [14] even in the convex domain, or even in symmetric domains. We note that the mechanism of such phenomenon is against the conventional effect of the collision in some sense, which will be described in Section 1.2. Throughout this paper we will use the following notations:

Notations: $f \lesssim g \Leftrightarrow$ there exists $0 < C < \infty$ such that $0 \leqq f \leqq Cg$; $f \sim g \Leftrightarrow$ there exists $0 < C < \infty$ such that $0 \leqq \frac{f}{C} \leqq g \leqq Cf$; $f \ll g \Leftrightarrow$ there exists a small constant $c > 0$ such that $0 \leqq f \leqq cg$; $f = O(g) \Leftrightarrow |f| \lesssim g$; $f = o(g) \Leftrightarrow |f| \ll g$.

1.1. Main theorem

Throughout this paper we assume the domain is defined as $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$ via a C^3 function $\xi : \mathbb{R}^3 \to \mathbb{R}$. Equivalently we assume that for all $q \in \partial\Omega$, there exists a C^3 function η_q and $0 < \delta_1 \ll 1$, such that

$$\eta_q : B_+(0; \delta_1) \ni \mathbf{x}_q := (\mathbf{x}_{q,1}, \mathbf{x}_{q,2}, \mathbf{x}_{q,3}) \to \mathbb{R}^3, \tag{1.6}$$

where the map is one-to-one and onto to the image $\mathcal{O}_q := \eta_q(B_+(0; \delta_1))$ when δ_1 is sufficiently small. Moreover, $\eta_q(\mathbf{x}_q) \in \partial\Omega$ if and only if $\mathbf{x}_{q,3} = 0$ within the image of η_q. We refer to [10] for the construction of such ξ and η_q. We further assume that the domain is strictly convex in the following sense:

$$\sum_{i,j=1}^{3} \zeta_i \zeta_j \partial_i \partial_j \xi(x) \gtrsim |\zeta|^2 \quad \text{for all } x \in \bar{\Omega} \text{ and } \zeta \in \mathbb{R}^3. \tag{1.7}$$

Without loss of generality we may assume that $\nabla\xi \neq 0$ near $\partial\Omega$.

In order to control the generic singularity at the boundary we adopt the following weight of [14]:

Definition 1. For sufficiently small $0 < \varepsilon \ll \|\xi\|_{C^2}$, we define a kinetic distance

$$\begin{aligned} &\alpha(x, v) := \chi_\varepsilon(\tilde{\alpha}(x, v)), \quad (x, v) \in \bar{\Omega} \times \mathbb{R}^3 \\ &\tilde{\alpha}(x, v) := \sqrt{|v \cdot \nabla_x \xi(x)|^2 - 2\xi(x)(v \cdot \nabla_x^2 \xi(x) \cdot v)}, \quad (x, v) \in \bar{\Omega} \times \mathbb{R}^3, \end{aligned} \tag{1.8}$$

where $\chi_a : [0,\infty) \to [0,\infty)$ stands for a non-decreasing smooth function such that

$$\begin{aligned} &\chi_a(s) = s \text{ for } s \in [0,a],\ \ \chi_a(s) = 2a \text{ for } s \in [4a,\infty], \\ &\text{and } |\chi_a'(\tau)| \leqq 1 \text{ for } \tau \in [0,\infty). \end{aligned} \tag{1.9}$$

We note that $\alpha \equiv 0$ on the grazing set γ_0. From a computation, we have $|v \cdot \nabla_x \alpha(x,v)| \leqq |v|\alpha(x,v)$, together with $\tau\chi_\varepsilon'(\tau) \leqq \chi_\varepsilon(\tau)$, this implies

$$e^{-|v|s}\alpha(x-sv,v) \leqq \alpha(x,v) \leqq e^{|v|s}\alpha(x-sv,v) \text{ as long as } x - sv \in \bar{\Omega}. \tag{1.10}$$

The definition of $\alpha, \tilde{\alpha}$ in (1.8) implies

$$\tilde{\alpha}(x,v) \gtrsim \alpha(x,v). \tag{1.11}$$

We extend the outward normal in the domain:

$$n(x) := \chi_{\varepsilon/2}'(\text{dist}(x,\partial\Omega))\nabla\xi(x)/|\nabla\xi(x)| \ \text{ for all } x \in \bar{\Omega}. \tag{1.12}$$

In particular, we note that $n(x) \equiv 0$ when $\text{dist}(x,\partial\Omega) \geqq 2\varepsilon$. In order to explore the "better" behavior of the tangential derivative versus the normal derivative we define a G-derivative (which is a matrix)

$$\nabla_\| f(x) = G(x)\nabla_x f(x), \tag{1.13}$$

where

$$G(x) := \big(I - n(x) \otimes n(x)\big). \tag{1.14}$$

Note that near the boundary, from (1.12) we have

$$G(x)n(x) = 0 \text{ for } \text{dist}(x,\partial\Omega) \leqq \varepsilon/2. \tag{1.15}$$

From the definition of n in (1.12), the G-derivative is actually a full derivative away from the boundary: if $\text{dist}(x,\partial\Omega) \geqq 2\varepsilon$, then $G(x)\nabla_x = \nabla_x$.

Main Theorem. Fix $\mathcal{K}n > 0$. Assume the domain is convex (1.7) and the boundary is C^3. Suppose $\sup_{x\in\partial\Omega}|T_W(x) - T_0| \ll 1$ for some constant $T_0 > 0$ and $T_W(x) \in C^1(\partial\Omega)$. For given $\mathfrak{m} > 0$ we construct a unique solution

$$F(x,v) = \mathfrak{m}M_{1,0,T_0}(v) + \sqrt{M_{1,0,T_0}(v)}f(x,v) \geqq 0, \tag{1.16}$$

to the stationary Boltzmann equation (1.1) and the diffuse reflection boundary condition (1.3) such that $\iint_{\Omega\times\mathbb{R}^3} f\sqrt{M_{1,0,T_0}(v)} = 0$, and

$$\|wf\|_\infty \lesssim \|T_W - T_0\|_{L^\infty(\partial\Omega)}, \quad w(v) := e^{\varrho|v|^2} \text{ for some } 0 < \varrho < 1/4. \tag{1.17}$$

Moreover, f (and F) belongs to $C^1(\bar{\Omega}\times\mathbb{R}^3\backslash\gamma_0)$ locally and satisfies

$$\|w_{\tilde{\theta}}(v)\alpha(x,v)\nabla_x f(x,v)\|_{L^\infty(\Omega\times\mathbb{R}^3)} \lesssim \|T_W - T_0\|_{C^1(\partial\Omega)}, \tag{1.18}$$

$$\|w_{\tilde{\theta}/2}(v)|v|\nabla_\| f(x,v)\|_{L^\infty(\Omega\times\mathbb{R}^3)} \lesssim \|T_W - T_0\|_{C^1(\partial\Omega)}, \tag{1.19}$$

$$\||v|^2\nabla_v f(x,v)\|_{L^\infty(\Omega\times\mathbb{R}^3)} \lesssim \|T_W - T_0\|_{C^1(\partial\Omega)}, \tag{1.20}$$

where $w_{\tilde{\theta}}(v) = e^{\tilde{\theta}|v|^2}$ with $0 < \tilde{\theta} \ll \varrho$.

If we further assume $T_W(x) \in C^2(\partial\Omega)$, then for any $0 \leqq \beta < 1$, the solution $F(x, v)$ belongs to $C^{1,\beta}(\bar{\Omega} \times \mathbb{R}^3 \backslash \gamma_0)$ locally. Moreover,

$$\sup_{x,y\in\Omega} \left\| w_{\tilde{\theta}}(v)|v|^2 \min\left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{2+\beta} \frac{\nabla_x f(x,v) - \nabla_x f(y,v)}{|x-y|^\beta} \right\|_{L^\infty(\mathbb{R}^3_v)}$$
$$\lesssim \|T_W - T_0\|_{C^2(\partial\Omega)}, \tag{1.21}$$

$$\sup_{x,y\in\Omega} \left\| w_{\tilde{\theta}/2}(v)|v|^2 \min\left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta} \frac{|\nabla_\parallel f(x,v) - \nabla_\parallel f(y,v)|}{|x-y|^\beta} \right\|_{L^\infty(\mathbb{R}^3_v)}$$
$$\lesssim \|T_W - T_0\|_{C^2(\partial\Omega)}. \tag{1.22}$$

$$\sup_{x,y\in\Omega} \left\| w_{\tilde{\theta}/2}(v)|v|^3 \min\left\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \right\}^{1+\beta} \frac{\nabla_v f(x,v) - \nabla_v f(y,v)}{|x-y|^\beta} \right\|_{L^\infty(\mathbb{R}^3_v)}$$
$$\lesssim \|T_W - T_0\|_{C^2(\partial\Omega)}. \tag{1.23}$$

Remark 1. The unique solvability and the pointwise estimate has been established in [9]. We record the statement of the theorem in Section 2 for the sake of readers' convenience.

Remark 2. The second estimate (1.19) implies that any tangential spatial derivatives of $f(x, v)$ does not blow up near the grazing set. Also comparing the $C^{1,\beta}$ estimates (1.21) and (1.22), the weight in the semi-norm of the tangential spatial derivative has a lower power in terms of α than the one for the normal derivative.

Remark 3. Estimating differential quotient with respect to v has some subtle (probably technical) issue, since the trajectory is not stable at $v = 0$ near the boundary. Since our motivation of the paper is investigating the regularity in space we omit to discuss them. This issue (instability of the trajectory at $v = 0$ in the Hölder norm estimate) will be discussed in a forthcoming paper, [20].

1.2. Major difficulties

In this section we illustrate the major difficulties, and in the next sections we will explain the key ideas and analytical development to overcome such obstacles. A generic feature of the boundary problem of the Boltzmann equation is a singularity of solutions, which originates mainly from 1) characteristics feature of the phase boundary $\partial\Omega \times \mathbb{R}^3$ with respect to the transport operator (i.e. the phase boundary is always characteristic but not uniformly characteristic at the grazing set γ_0 of (1.5)), and 2) the mixing effect of the collision operator.

The effect of characteristics phase boundary can appear in several ways. Depending on the shape of the domain, the generic boundary singularity at γ_0 can propagate inside the domain and affect the global dynamics. Indeed, it has been proved in [18] that any general non-convex domains admit smooth initial datum which will produce the discontinuity for the Boltzmann solution in a stable manner, which propagates along the trajectory. Although such discontinuity-type singularities may stay near the boundary for the convex domains, its derivatives blow up near the grazing set. Actually it is not merely the effect of characteristics phase boundary but also the mixing effect of the collision operator: the mixing immediately produces a singular source term for the normal derivative at the boundary. In

[14], the authors quantify the rate of the blow-up with respect to the kinetic distance of (1.8) and study the mixing effect by the collision operator in terms of the kinetic distance. As a result they establish the first order derivatives estimate for the dynamical Boltzmann equation. On the other hand, the kinetic distance produces *a loss of moment* and they utilize a fast decay weight $e^{-C\langle v\rangle t}$ to recover such a loss. In other words, the success of the approach in [14] to the dynamic problem can be achieved in the space losing its exponential moment quickly (exponentially). Evidently utilizing such functional spaces is not possible in the stationary problem, which is one of the major difficulties to establish the main theorem.

The effect of the nonlinear collision operator is complex, in particular, within the interaction of the transport operator, which eventually restricts our regularity strictly below two derivatives in any L^p-space: the boundary singularity of Boltzmann solutions appears as $\frac{\partial F}{\partial n} \sim \frac{Q(F,F)}{n(x)\cdot v} \notin L^1_{loc}$, while the leading order term of any second order derivatives $\nabla_{x,v}\partial_n$ contains a factor of $Q(\nabla_x F, F)(x_{\mathbf{b}}(x,v), v)$ at a backward exit position $x_{\mathbf{b}}(x,v)$ which is defined through a backward exit time $t_{\mathbf{b}}$:

$$t_{\mathbf{b}}(x,v) := \sup\{s > 0 : x - sv \in \Omega\}, \quad x_{\mathbf{b}}(x,v) := x - t_{\mathbf{b}}(x,v)v. \quad (1.24)$$

Due to a lack of symmetry of $\frac{\partial F}{\partial n}$, in particular for the diffuse reflection boundary condition, any possibility of cancellation in the integration formula $Q(\frac{\partial F}{\partial n}, F)$ can be expelled generically in [14]. Then it follows that $|\frac{\partial^2 F}{\partial n^2}(x,v)| = \infty$ for all v for some $x \in \partial\Omega$. This singularity likely appears at all boundary points with all velocities then propagates along the trajectory inside the domain, and masses up all directional derivatives. Even strictly below the second derivatives estimate, at first glance it is not obvious that the similar failure is avoidable in our weighted $C^{1,\beta}$. Moreover, we encounter similar type of, but much more geometrically involved, terms associated with the diffuse reflection boundary condition intertwined with the transport operator. Such non-integrable singularities could barge in the higher order estimates, which is the other major difficulty of the proof.

1.3. Regularizing via the mixing of the binary collision, transport, and diffuse reflection

To overcome such difficulties described in Section 1.2., we establish a novel and robust *quantitative* estimate of regularization effect (in space and velocity) of the *velocity mixing* via the diffuse reflection boundary condition (1.3) or/and the binary collision (1.1) intertwined with the *transport operator*.

We demonstrate the scheme first for $\nabla_x F$, of which the most singular term comes from the boundary contribution such as

$$\nabla_x x_{\mathbf{b}}(x,v) \int_{n(x_{\mathbf{b}})\cdot v^1 > 0} \nabla_{x_{\mathbf{b}}} F(x_{\mathbf{b}}(x,v), v^1) |n(x_{\mathbf{b}}(x,v)) \cdot v^1| \mathrm{d}v^1. \quad (1.25)$$

Upon using the transport operator once again, the contribution of the collision operator (ignoring the singularity of Q for simplicity) can be viewed as

$$\nabla_x x_{\mathbf{b}} \int_{n(x_{\mathbf{b}})\cdot v^1 > 0} \int_{t_{\mathbf{b}}(x_{\mathbf{b}}, v^1)}^{0} \int_{\mathbb{R}^3} \nabla_x F(x_{\mathbf{b}} - sv^1, u) |n(x_{\mathbf{b}}) \cdot v^1| \mathrm{d}u \mathrm{d}s \mathrm{d}v^1. \quad (1.26)$$

A key observation is that the x-derivative has a natural relation with the v^1-derivative as

$$\nabla_x F(x_{\mathbf{b}} - (t^1 - s)v^1, u) = \frac{\nabla_{v^1}[F(x_{\mathbf{b}} - (t^1 - s)v^1, u)]}{-(t^1 - s)}. \tag{1.27}$$

When $t^1 - s$ has a positive lower bound, thanks to the v^1-integral from the diffuse reflection boundary condition, we are able to remove such a v^1-derivative completely from F. As a result of the integration by parts, the singularity of $\nabla_{v^1} t_{\mathbf{b}}(x_{\mathbf{b}}, v^1)$ occurs, which will be compensated by the boundary measure and thus we obtain a bound like $\nabla_x x_{\mathbf{b}}(x, v) \times \|F\|_\infty$. When $t^1 - s$ is small we use the so-called the nonlocal-to-local estimate and derive $O(|t^1 - s|)\alpha(x, v)^{-1}\|\alpha \nabla_x F\|_\infty$. We will describe the nonlocal-to-local estimate and its application in detail at the next subsection.

On the other hand, the boundary contribution of (1.25) upon applying the transport operator appears as

$$\nabla_x x_{\mathbf{b}}(x, v) \int_{n(x_{\mathbf{b}})\cdot v^1 > 0} \nabla_{x_{\mathbf{b}}} F(x_{\mathbf{b}}(x^1, v^1), v^1)|n(x^1) \cdot v^1| \mathrm{d}v^1, \tag{1.28}$$

where $x^1 = x_{\mathbf{b}}(x, v)$. The key idea is to convert v^1-integration to the integration in $(x^2, t_{\mathbf{b}}(x^1, v^1)) = (x_{\mathbf{b}}(x^1, v^1), t_{\mathbf{b}}(x^1, v^1))$, while the change of variables produces a factor of the Jacobian as $\frac{|n(x^2)\cdot v^1|}{t_{\mathbf{b}}(x^1, v^1)^3}$. Then we are able to move $\nabla_{x_{\mathbf{b}}}$-derivative from F via the integration by parts, while the derivative to the geometric components arise. Using the convexity and boundary measure crucially we are able to bound this amount by $\nabla_x x_{\mathbf{b}}(x, v) \times \|F\|_\infty$.

1.4. Higher regularity

For the higher regularity estimate in the weighted $C^{1,\beta}$-space, we 1) adopt the idea of Section 1.3 with stronger weight in α, 2) crucially establish a "better" estimate for the tangential derivatives, 3) use the full range of the nonlocal-to-local estimate, and 4) carefully study the possibly harmful (which has been explained in the last paragraph of Section 1.2.) term $\frac{1}{|x-y|^\beta} \int_{t_{\mathbf{b}}(y,v)}^{t_{\mathbf{b}}(x,v)} Q(\nabla_x F, F)(x - sv, v)\mathrm{d}s$.

By expressing $\frac{\nabla_x F(x,v) - \nabla_x F(y,v)}{|x-y|^\beta}$ along the trajectories (see (7.36)–(7.48) for the details), we notice that the difference is singular at least as

$$\frac{\nabla_x x_{\mathbf{b}}(x, v) - \nabla_x x_{\mathbf{b}}(y, v)}{|x - y|^\beta} \int_{n(x_{\mathbf{b}})\cdot v^1 > 0} \nabla_{x_{\mathbf{b}}} F(x_{\mathbf{b}}, v^1)|n(x_{\mathbf{b}}) \cdot v^1| \mathrm{d}v^1, \tag{1.29}$$

where the integration is bounded using the weighted C^1-estimate. By the mean value type estimate and the computation of $\nabla_x^2 x_{\mathbf{b}}$, for $x(\tau) = \tau x + (1 - \tau)y$, we derive that the difference quotient of $\nabla_x x_{\mathbf{b}}$ is bounded by

$$|x - y|^{1-\beta} \int_0^1 \frac{|v|^3}{\alpha^3(x(\tau), v)} \mathrm{d}\tau. \tag{1.30}$$

We prove that $\alpha(x(\tau), v) \gtrsim \min\{\alpha(x, v), \alpha(y, v)\}$ for $|x-y| < \min\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\}$ in the convex domains, for which we use the weight of $\min\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\}^{2+\beta}$ for $\frac{\nabla_x F(x,v)-\nabla_x F(y,v)}{|x-y|^\beta}$. The convexity of the domain is crucial since any similar type of the bound is false for the non-convex domains in general.

Unfortunately this estimate with the weight of the power $2+\beta$ is too singular! In particular the difference quotient of $\nabla_x F$ contains

$$\nabla_x x_{\mathbf{b}}(x, v) \int \frac{\nabla_{x_{\mathbf{b}}} F(x_{\mathbf{b}}(x, v), v^1) - \nabla_{x_{\mathbf{b}}} F(x_{\mathbf{b}}(y, v), v^1)}{|x - y|^\beta} |n(x_{\mathbf{b}}) \cdot v^1| \mathrm{d}v^1,$$

in which the control of the possible singularity of $|n(x_{\mathbf{b}}) \cdot v^1|^{-(1+\beta)}$ would be non-integrable. To overcome it, realizing that $\nabla_{x_{\mathbf{b}}} F$ is $\nabla_{\|} F$, we establish an estimate of the difference quotient for the tangential derivatives $\frac{\nabla_{\|} F(x,v)-\nabla_{\|} F(y,v)}{|x-y|^\beta}$ with the weight $\min\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\}$ for a lower power than $2+\beta$. The optimal power is examined through (1.29), which turns out to be $1+\beta$.

To estimate the difference quotient with different weights, we first employ delicate splitting for the boundary integral and the time integral depending on how the trajectories from two different points hit the boundary. Then we adopt the idea of the scheme of Section 1.3 when $t^1 - s$ has a positive lower bound. On the other hand, when $t^1 - s$ is small we use the weight and derive

$$\begin{aligned} &\left\| \min\left\{ \frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|} \right\}^\beta \nabla F \right\|_\infty \\ &\times \int_{\text{small interval}} \int \frac{1}{\min\left\{ \frac{\alpha(x-sv,u)}{|u|}, \frac{\alpha(y-sv,u)}{|u|} \right\}^\beta} \mathrm{d}u\mathrm{d}s. \end{aligned} \tag{1.31}$$

The second author and collaborators studied a similar estimate of (1.31) in [14]. In this paper we elaborate the so-called nonlocal-to-local estimate, which consists of analytical and geometrical arguments: first we study the u-integration of the integrand and derive a gain of power such as, for $1 < \beta < 3$

$$\frac{1}{\min\{\xi(x - sv, u), \xi(y - sv, u)\}^{\frac{\beta-1}{2}}}, \tag{1.32}$$

where $\xi(x)$ can be understood as the distance from x to the boundary. Second we employ $s \mapsto \xi(x-sv, u)$ with the Jacobian $\mathrm{d}s = \frac{1}{|u \cdot \nabla \xi|} \mathrm{d}\xi(x-sv, u)$ and recover a power of α as in the bound of ξ through the geometric velocity lemma. We crucially utilize such a gain of α to extract a smallness in (1.31).

Lastly we discuss the possible harmful term $\frac{1}{|x-y|^\beta} \int_{t_{\mathbf{b}}(y,v)}^{t_{\mathbf{b}}(x,v)} Q(\nabla_x F, F)(x - sv, v)\mathrm{d}s$. First we apply the α-weighted bound for $\nabla_x F$ and then establish $Q(\nabla_x F, F)(x - sv, v) \sim \ln|\xi(x - sv)|$. Upon the time integration on $[t_{\mathbf{b}}(y, v), t_{\mathbf{b}}(x, v)]$ we derive a bound $\min\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\} \ln\left(\min\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\}\right)$. For $|x - y| < \min\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\}$, we realize the difference quotient is bounded. Of course such bound blows up if $\beta = 1$.

Now we state the outline for our paper. In Section 2 we prove several lemmas which serve as preliminary. Section 3 and Section 4 are devoted to establish the ideas in Section 1.3 as well as the nonlocal-to-local estimate and (1.18). In Section 5 and Section 6 we establish the rest of weighted C^1 estimates. Finally, in Section 7 we prove the weighted $C^{1,\beta}$ estimate.

2. Preliminaries

2.1. Basic notions

We first record the unique existence theorem of [9].

Existence Theorem of [9] Assume the domain is open bounded and the boundary is smooth. For $\mathfrak{m} > 0$ and $0 < \varrho < 1/4$, if $\sup_{x\in\partial\Omega}|T_W(x) - T_0| \ll 1$, then there exists a unique mild solution

$$F(x, v) = \mathfrak{m}M_{1,0,T_0}(v) + \sqrt{M_{1,0,T_0}(v)}f(x, v) \geqq 0, \tag{2.1}$$

with $\iint_{\Omega\times\mathbb{R}^3} f\sqrt{M_{1,0,T_0}(v)} = 0$ to (1.1) and (1.3) such that

$$\begin{aligned} \|wf\|_{L^\infty(\bar{\Omega}\times\mathbb{R}^3)} &:= \|wf\|_\infty \lesssim \|T_W - T_0\|_{L^\infty(\partial\Omega)}, \\ w(v) &:= e^{\varrho|v|^2} \text{ with } 0 < \varrho < 1/4. \end{aligned} \tag{2.2}$$

Without loss of generality, we assume $\mathfrak{m} = 1$, $T_0 = 1$ in (1.16). Then we define the reference global Maxwellian and its perturbation:

$$\mu := M_{1,0,1}, \quad F(x, v) = \mu(v) + \sqrt{\mu(v)}f(x, v).$$

Plugging (1.16) into (1.1) and (1.3), we obtain the equation and boundary condition for f:

$$v \cdot \nabla_x f + \nu(v)f = K(f) + \Gamma(f, f), \tag{2.3}$$

$$f(x, v)|_{n(x)\cdot v<0} = \frac{M_W(x, v)}{\sqrt{\mu(v)}} \int_{n(x)\cdot u>0} f(x, u)\sqrt{\mu(u)}\{n(x) \cdot u\}\mathrm{d}u + r(x, v). \tag{2.4}$$

Here $\nu(v)$, $K(f)$, $\Gamma(f, f)$ are the linear Boltzmann operator(see [12]) given by

$$\begin{aligned} \nu(v)f &= -\frac{Q(\mu, \sqrt{\mu}f)}{\sqrt{\mu}}, \quad K(f) = \frac{Q(\sqrt{\mu}f, \mu)}{\sqrt{\mu}}, \\ \Gamma(f, f) &= \frac{Q(\sqrt{\mu}f, \sqrt{\mu}f)}{\sqrt{\mu}}. \end{aligned} \tag{2.5}$$

The $r(x, v)$ is the remainder term. By $\sqrt{2\pi}\int_{n(x)\cdot u>0}\sqrt{\mu(u)}\{n(x) \cdot u\}\mathrm{d}u = 1$, this term is given by

$$r(x, v) := \frac{M_W(x, v)/\sqrt{2\pi} - \mu(v)}{\sqrt{\mu(v)}}. \tag{2.6}$$

Consider a linear transport equation with the inflow boundary condition

$$v \cdot \nabla_x f + \nu(v) f = h(x, v), \quad (x, v) \in \Omega \times \mathbb{R}^3, \tag{2.7}$$

$$f(x, v) = g(x, v), \quad (x, v) \in \gamma_-. \tag{2.8}$$

As we can not rely on the Gronwall-type estimate, we will use the Duhamel's formula to express the equation along the trajectory:

$$\begin{aligned} f(x, v) = & \mathbf{1}_{t \geqq t_{\mathbf{b}}} e^{-\nu(v) t_{\mathbf{b}}(x, v)} f(x_{\mathbf{b}}(x, v), v) \\ & + \mathbf{1}_{t < t_{\mathbf{b}}} e^{-\nu(v) t} f(x - tv, v) \\ & + \int_{\max\{0, t - t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} h(x - (t-s)v, v) \mathrm{d}s. \end{aligned} \tag{2.9}$$

Here we fix $t \gg 1$.

In order to obtain C^1 estimate we take the spatial derivative to (2.9) to get

$$\partial_{x_j} f(x, v) = \mathbf{1}_{t \geqq t_{\mathbf{b}}} e^{-\nu(v) t_{\mathbf{b}}(x, v)} \partial_{x_j} [f(x_{\mathbf{b}}(x, v), v)] \tag{2.10}$$

$$- \mathbf{1}_{t \geqq t_{\mathbf{b}}} \nu(v) \partial_{x_j} t_{\mathbf{b}}(x, v) e^{-\nu(v) t_{\mathbf{b}}(x, v)} f(x_{\mathbf{b}}(x, v), v) \tag{2.11}$$

$$+ \mathbf{1}_{t < t_{\mathbf{b}}} e^{-\nu(v) t} \partial_{x_j} [f(x - tv, v)] \tag{2.12}$$

$$+ \int_{\max\{0, t - t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \partial_{x_j} [h(x - (t-s)v, v)] \mathrm{d}s \tag{2.13}$$

$$- \mathbf{1}_{t \geqq t_{\mathbf{b}}} \partial_{x_j} t_{\mathbf{b}} e^{-\nu(v) t_{\mathbf{b}}} h(x - t_{\mathbf{b}} v, v), \tag{2.14}$$

where $x_{\mathbf{b}}(x, v)$ and $t_{\mathbf{b}}(x, v)$ represent the backward exit position and time which are defined in (1.24). The derivative of $t_{\mathbf{b}}(x, v)$ and $x_{\mathbf{b}}(x, v)$ has singular behavior as stated in (2.32), such singularity will be cancelled by our weight α defined in (1.8). With a compatibility condition it is standard to check the piecewise formula (2.10)–(2.14) is actually a weak derivative of f and continuous across $\{t = t_{\mathbf{b}}(x, v)\}$ (see [13]) for the details.

Definition 2. Recall the backward exit position $x_{\mathbf{b}}$ and backward exit time $t_{\mathbf{b}}$ in (1.24), we define a stochastic cycles as $(x^0, v^0) = (x, v) \in \bar{\Omega} \times \mathbb{R}^3$ and inductively

$$x^1 := x_{\mathbf{b}}(x, v), \ v^1 \in \{v^1 \in \mathbb{R}^3 : n(x^1) \cdot v^1 > 0\}, \tag{2.15}$$

$$v^k \in \{v^k \in \mathbb{R}^3 : n(x^k) \cdot v^k > 0\}, \quad \text{for } k \geqq 1, \tag{2.16}$$

$$x^{k+1} := x_{\mathbf{b}}(x^k, v^k), \ t_{\mathbf{b}}^k := t_{\mathbf{b}}(x^k, v^k) \ \text{ for } n(x^k) \cdot v^k \geqq 0. \tag{2.17}$$

Choose $t \geqq 0$. We define $t^0 = t$ and

$$t^k = t - \{t_{\mathbf{b}} + t_{\mathbf{b}}^1 + \cdots + t_{\mathbf{b}}^{k-1}\}, \quad \text{for } k \geqq 1. \tag{2.18}$$

Remark 4. Here x^{k+1} depends on $(x, v, x^1, v^1, \cdots, x^k, v^k)$, while v^k is a free parameter whose domain (2.16) only depends on x^k.

Recall (1.6). Since the boundary is compact and C^3, for fixed $0 < \delta_1 \ll 1$ we may choose a finite number of $p \in \tilde{\mathcal{P}} \subset \partial\Omega$ and $0 < \delta_2 \ll 1$ such that $\mathcal{O}_p = \eta_p(B_+(0;\delta_1)) \subset B(p;\delta_2) \cap \bar{\Omega}$ and $\{\mathcal{O}_p\}$ forms a finite covering of $\partial\Omega$. We further choose an interior covering $\mathcal{O}_0 \subset \Omega$ such that $\{\mathcal{O}_p\}_{p\in\mathcal{P}}$ with $\mathcal{P} = \tilde{\mathcal{P}} \cup \{0\}$ forms an open covering of $\bar{\Omega}$. We define a partition of unity

$$\mathbf{1}_{\bar{\Omega}}(x) = \sum_{p\in\mathcal{P}} \iota_p(x) \text{ such that } 0 \leqq \iota_p(x) \leqq 1, \quad \iota_p(x) \equiv 0 \text{ for } x \notin \mathcal{O}_p. \tag{2.19}$$

Without loss of generality (see [19]) we can always reparametrize η_p such that $\partial_{\mathbf{x}_{p,i}}\eta_p \neq 0$ for $i = 1, 2, 3$ at $\mathbf{x}_{p,3} = 0$, and an *orthogonality* holds as

$$\partial_{\mathbf{x}_{p,i}}\eta_p \cdot \partial_{\mathbf{x}_{p,j}}\eta_p = 0 \text{ at } \mathbf{x}_{p,3} = 0 \text{ for } i \neq j \text{ and } i, j \in \{1, 2, 3\}. \tag{2.20}$$

At $\mathbf{x}_{p,3} = 0$, the $\mathbf{x}_{p,3}$ derivative gives the outward normal

$$n_p(\mathbf{x}_p) = \frac{\partial_{\mathbf{x}_{p,3}}\eta_p}{\langle \partial_{\mathbf{x}_{p,3}}\eta_p, \partial_{\mathbf{x}_{p,3}}\eta_p\rangle}. \tag{2.21}$$

For simplicity, we denote

$$\partial_i\eta_p(\mathbf{x}_p) := \partial_{\mathbf{x}_{p,i}}\eta_p. \tag{2.22}$$

Definition 3. For $x \in \bar{\Omega}$, we choose $p \in \mathcal{P}$ as in (1.6). We define

$$T_{\mathbf{x}_p} = \begin{pmatrix} \frac{\partial_1\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,11}(\mathbf{x}_p)}} & \frac{\partial_2\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,22}(\mathbf{x}_p)}} & \frac{\partial_3\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,33}(\mathbf{x}_p)}} \end{pmatrix}^t, \tag{2.23}$$

with $g_{p,ij}(\mathbf{x}_p) = \langle \partial_i\eta_p(\mathbf{x}_p), \partial_j\eta_p(\mathbf{x}_p)\rangle$ for $i, j \in \{1, 2, 3\}$. Here A^t stands the transpose of a matrix A. Note that when $\mathbf{x}_{p,3} = 0$, $T_{\mathbf{x}_p}\frac{\partial_i\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,ii}(\mathbf{x}_p)}} = e_i$ for $i = 1, 2, 3$ where $\{e_i\}$ is a standard basis of $\mathbb{R}^3$.

We define

$$\mathbf{v}_j(\mathbf{x}_p) = \frac{\partial_j\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,jj}(\mathbf{x}_p)}} \cdot v. \tag{2.24}$$

We note that from (2.20), the map $T_{\mathbf{x}_p}$ is an orthonormal matrix when $\mathbf{x}_{p,3} = 0$. Therefore both maps $v \to \mathbf{v}(\mathbf{x}_p)$ and $\mathbf{v}(\mathbf{x}_p) \to v$ have a unit Jacobian at $\mathbf{x}_{p,3} = 0$. This fact induces a new representation of boundary integration of diffuse boundary condition in (2.4): For $x \in \partial\Omega$ and $p \in \mathcal{P}$ as in (1.6),

$$\begin{aligned} &\int_{n(x)\cdot v>0} f(x, v)\sqrt{\mu(v)}\{n(x) \cdot v\}\mathrm{d}v \\ &= \int_{\mathbf{v}_{p,3}>0} f(\eta_p(\mathbf{x}_p), T^t_{\mathbf{x}_p}\mathbf{v}(\mathbf{x}_p))\sqrt{\mu(\mathbf{v}(\mathbf{x}_p))}\mathbf{v}_3(\mathbf{x}_p)\mathrm{d}\mathbf{v}(\mathbf{x}_p). \end{aligned} \tag{2.25}$$

We have used the fact of $\mu(v) = \mu(|v|) = \mu(|T^t_{\mathbf{x}_p}\mathbf{v}(\mathbf{x}_p)|) = \mu(|\mathbf{v}(\mathbf{x}_p)|) = \mu(\mathbf{v}(\mathbf{x}_p))$ and $\mathbf{x}_{p,3} = 0$.

Now we reparametrize the stochastic cycle using the local chart defined in Definition 2.

Definition 4. Recall the stochastic cycles (2.16). For each cycle x^k let us choose $p^k \in \mathcal{P}$ in (1.6). Then we denote

$$\begin{aligned} \mathbf{x}^k_{p^k} &:= (\mathbf{x}^k_{p^k,1}, \mathbf{x}^k_{p^k,2}, 0) \text{ such that } \eta_{p^k}(\mathbf{x}^k_{p^k}) = x^k, \quad \text{for } k \geqq 1, \\ \mathbf{v}^k_{p^k,i} &:= \frac{\partial_j \eta_{p^k}(\mathbf{x}^k_{p^k})}{\sqrt{g_{p^k,jj}(\mathbf{x}^k_{p^k})}} \cdot v^k, \quad \text{for } k \geqq 1. \end{aligned} \tag{2.26}$$

From (2.21) we denote the outward normal at x^k as

$$n(x^k) = n_{p^k}(\mathbf{x}^k_{p^k}). \tag{2.27}$$

Conventionally, we denote

$$\mathbf{x}^0_{p^0} := x^0 = x, \quad \mathbf{v}^0_{p^0} := v^0 = v. \tag{2.28}$$

We define

$$\partial_{\mathbf{x}^k_{p^k,i}}[a(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k)] := \frac{\partial \eta_{p^k}(\mathbf{x}^k_{p^k,i})}{\partial \mathbf{x}^k_{p^k,i}} \cdot \nabla_x a(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k), \quad i = 1, 2. \tag{2.29}$$

Conventionally we denote

$$\underline{\nabla}_{x^k} a(x^k, v^k) = \big(\partial_{\mathbf{x}^k_{p^k,1}}[a(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k)], \partial_{\mathbf{x}^k_{p^k,2}}[a(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k)]\big).$$

2.2. Properties of stochastic cycle

In this subsection we list useful properties of the stochastic cycle defined in Definitions 2 and 4.

Lemma 2.1. *For the $t_{\mathbf{b}}$ and $x_{\mathbf{b}}$ defined in (2.16) and (2.17), the derivative reads*

$$\frac{\partial t^{k+1}_{\mathbf{b}}}{\partial x^{k+1}_j} = \frac{1}{\mathbf{v}_3(\mathbf{x}_{p^{k+2}})} \frac{\partial_3 \eta_{p^{k+2}}(x^{k+2})}{\sqrt{g_{p^{k+2},33}(x^{k+2})}} \cdot e_j, \tag{2.30}$$

$$\frac{\partial t^{k+1}_{\mathbf{b}}}{\partial v^{k+1}_j} = -\frac{t^{k+1}_{\mathbf{b}} e_j}{\mathbf{v}^{k+1}_{p^{k+2},3}} \cdot \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}. \tag{2.31}$$

And thus

$$\begin{aligned} \nabla_x t_{\mathbf{b}} &= \frac{n(x_{\mathbf{b}})}{n(x_{\mathbf{b}}) \cdot v}, \quad \nabla_v t_{\mathbf{b}} = -\frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{n(x_{\mathbf{b}}) \cdot v}, \\ \nabla_x x_{\mathbf{b}} &= Id_{3\times 3} - \frac{n(x_{\mathbf{b}}) \otimes v}{n(x_{\mathbf{b}}) \cdot v}, \quad \nabla_v x_{\mathbf{b}} = -t_{\mathbf{b}} Id + \frac{t_{\mathbf{b}} n(x_{\mathbf{b}}) \otimes v}{n(x_{\mathbf{b}}) \cdot v}. \end{aligned} \tag{2.32}$$

For $i = 1, 2$,

$$\frac{\partial \mathbf{x}^{k+2}_{p^{k+2},i}}{\partial x^{k+1}_j} = \frac{1}{\sqrt{g_{p^{k+2},ii}(\mathbf{x}^{k+2}_{p^{k+2}})}} \times \left[\frac{\partial_i \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})}{\sqrt{g_{p^{k+2},ii}(\mathbf{x}^{k+2}_{p^{k+2}})}} - \frac{\mathbf{v}^{k+2}_{p^{k+2},i}}{\mathbf{v}^{k+2}_{p^{k+2},3}} \frac{\partial_3 \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})}{\sqrt{g_{p^{k+2},33}(\mathbf{x}^{k+2}_{p^{k+2}})}} \right] \cdot e_j, \tag{2.33}$$

$$\frac{\partial \mathbf{x}^{k+2}_{p^{k+2},i}}{\partial \mathbf{x}^{k+1}_{p^{k+1},j}} = \frac{1}{\sqrt{g_{p^{k+2},ii}(\mathbf{x}^{k+2}_{p^{k+2}})}} \times \left[\frac{\partial_i \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})}{\sqrt{g_{p^{k+2},ii}(\mathbf{x}^{k+2}_{p^{k+2}})}} - \frac{\mathbf{v}^{k+2}_{p^{k+2},i}}{\mathbf{v}^{k+2}_{p^{k+2},3}} \frac{\partial_3 \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})}{\sqrt{g_{p^{k+2},33}(\mathbf{x}^{k+2}_{p^{k+2}})}} \right] \times \cdot \partial_j \eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), \tag{2.34}$$

$$\frac{\partial \mathbf{x}^{k+2}_{p^{k+2},i}}{\partial v^{k+1}_j} = -t^{k+1}_{\mathbf{b}} e_j \cdot \frac{1}{\sqrt{g_{p^{k+2},ii}(\mathbf{x}^{k+2}_{p^{k+2}}))}} \left[\frac{\partial_i \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}}\Big|_{x^{k+2}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},i}}{\mathbf{v}^{k+1}_{p^{k+2},3}} \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}} \right]. \tag{2.35}$$

Proof. First of all we have

$$\begin{aligned} x^{k+2} = \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}) &= x^{k+1} - t^{k+1}_{\mathbf{b}} v^{k+1} \\ &= \eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}) - t^{k+1}_{\mathbf{b}} v^{k+1}. \end{aligned} \tag{2.36}$$

Proof of (2.30) We take $\frac{\partial}{\partial x^{k+1}_j}$ to (2.36) to get

$$\begin{aligned} \sum_{l=1,2} \frac{\partial \mathbf{x}^{k+2}_{p^{k+2},l}}{\partial x^{k+1}_j} \frac{\partial \eta_{p^{k+2}}}{\partial \mathbf{x}^{k+2}_{p^{k+2},l}}\Big|_{x^{k+2}} &= -t^{k+1}_{\mathbf{b}} \frac{\partial v^{k+1}}{\partial x^{k+1}_j} - \frac{\partial t^{k+1}_{\mathbf{b}}}{\partial x^{k+1}_j} v^{k+1} + e_j \\ &= -\frac{\partial t^{k+1}_{\mathbf{b}}}{\partial x^{k+1}_j} v^{k+1} + e_j. \end{aligned} \tag{2.37}$$

Then we take an inner product with $\frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}$ to (2.37) to have

$$\begin{aligned} &\sum_{l=1,2} \frac{\partial \mathbf{x}^{k+2}_{p^{k+2},l}}{\partial x^{k+1}_j} \frac{\partial \eta_{p^{k+2}}}{\partial \mathbf{x}^{k+2}_{p^{k+2},l}}\Big|_{x^{k+2}} \cdot \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}} \\ &= -\frac{\partial t^{k+1}_{\mathbf{b}}}{\partial x^{k+1}_j} v^{k+1} \cdot \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}} + e_j \cdot \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}. \end{aligned} \tag{2.38}$$

Due to (2.20) the LHS equals zero. Now we consider the RHS. From (2.24)

$$v^{k+1}\cdot\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}=\mathbf{v}_3(\mathbf{x}_{p^{k+2}}).$$

From (2.38), we conclude (2.30). □

Proof of (2.31) We apply ∂v_j^{k+1} to (2.36) and take $\cdot\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}$ to have

$$\begin{aligned}\frac{\partial x^{k+2}}{\partial v_j^{k+1}}\cdot\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}&=\sum_{l=1}^{2}\frac{\partial\mathbf{x}^{k+2}_{p^{k+2},l}}{\partial v_j^{k+1}}\frac{\partial\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})}{\partial\mathbf{x}^{k+2}_{p^{k+2},l}}\cdot\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}\\&=-\Big\{t_{\mathbf{b}}^{k+1}e_j+v^{k+1}\frac{\partial t_{\mathbf{b}}^{k+1}}{\partial v_j^{k+1}}\Big\}\cdot\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}.\end{aligned}$$

Thus we apply (2.20) and (2.26) and use (2.24) to obtain (2.31). □

Proof of (2.32) The first line of (2.32) follows directly from (2.30) and (2.31). For the second line we take ∂x_j^{k+1} and ∂v_j^{k+1} to (2.36). Again using (2.30) and (2.31) we conclude (2.32). □

Proof of (2.33) We take inner product with $\frac{\partial_i\eta_{p^{k+2}}}{g_{p^{k+2},ii}}\Big|_{x^{k+2}}$ to (2.37) to have

$$\begin{aligned}&\sum_{l=1,2}\frac{\partial\mathbf{x}^{k+2}_{p^{k+2},l}}{\partial x_j^{k+1}}\frac{\partial\eta_{p^{k+2}}}{\partial\mathbf{x}^{k+2}_{p^{k+2},l}}\Big|_{x^{k+2}}\cdot\frac{\partial_i\eta_{p^{k+2}}}{g_{p^{k+2},ii}}\Big|_{x^{k+2}}\\&=\frac{\partial\mathbf{x}^{k+2}_{p^{k+2},i}}{\partial x_j^{k+1}}=-\frac{\partial t_{\mathbf{b}}^{k+1}}{\partial x_j^{k+1}}v^{k+1}\cdot\frac{\partial_i\eta_{p^{k+2}}}{g_{p^{k+2},ii}}\Big|_{x^{k+2}}+e_j\cdot\frac{\partial_i\eta_{p^{k+2}}}{g_{p^{k+2},ii}}\Big|_{x^{k+2}}.\end{aligned}$$

By (2.24),

$$v^{k+1}\cdot\frac{\partial_i\eta_{p^{k+2}}}{g_{p^{k+2},ii}}\Big|_{x^{k+2}}=\frac{\mathbf{v}_i(\mathbf{x}_{p^{k+2}})}{\sqrt{g_{p^{k+2},ii}}}.$$

Then, from (2.30), we conclude (2.33). □

Proof of (2.34) Since

$$\frac{\partial\mathbf{x}^{k+2}_{p^{k+2},i}}{\partial\mathbf{x}^{k+1}_{p^{k+1},j}}=\nabla_{x^{k+1}}\mathbf{x}^{k+2}_{p^{k+2},i}\cdot\partial_{\mathbf{x}^{k+1}_{p^{k+1},j}}\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}),$$

by (2.33) we conclude (2.34). □

Proof of (2.35) For $i=1,2$, $j=1,2,3$, we apply ∂v_j^{k+1} to (2.36) and take $\cdot\frac{\partial_i\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}}\Big|_{x^{k+2}}$ to obtain

$$
\begin{aligned}
&\frac{\partial x^{k+2}}{\partial v_j^{k+1}} \cdot \frac{\partial_i \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}}\Big|_{x^{k+2}} \\
&= \sum_{l=1}^{2} \frac{\partial \mathbf{x}^{k+2}_{p^{k+2},l}}{\partial v_j^{k+1}} \frac{\partial \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})}{\partial \mathbf{x}^{k+2}_{p^{k+2},l}} \cdot \frac{\partial_i \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}}\Big|_{x^{k+2}} \\
&= \frac{\partial \mathbf{x}^{k+2}_{p^{k+2},i}}{\partial v_j^{k+1}} \sqrt{g_{p^{k+2},ii}(\mathbf{x}^{k+2}_{p^{k+2}})} \\
&= -\Big\{ \frac{\partial t_{\mathbf{b}}^{k+1}}{\partial v_j^{k+1}} v^{k+1} + t_{\mathbf{b}}^{k+1} \frac{\partial v^{k+1}}{\partial v_j^{k+1}} \Big\} \cdot \frac{\partial_i \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}}\Big|_{x^{k+2}} \\
= \quad & -\Big\{ t_{\mathbf{b}}^{k+1} e_j - \frac{t_{\mathbf{b}}^{k+1} e_j}{\mathbf{v}^{k+1}_{p^{k+2},3}} \cdot \frac{\partial_3 \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}} v^{k+1} \Big\} \cdot \frac{\partial_i \eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},ii}}}\Big|_{x^{k+2}}.
\end{aligned}
$$

Then we apply (2.31) to obtain (2.35). □

The following two lemmas are immediate consequences of Lemma 2.1.

Lemma 2.2.

$$
t_{\mathbf{b}}(x, v) \lesssim \frac{|n(x_{\mathbf{b}}(x, v)) \cdot v|}{|v|^2}, \tag{2.39}
$$

and thus

$$
|\nabla_v t_{\mathbf{b}}| \lesssim \frac{1}{|v|^2}, \quad |\nabla_v x_{\mathbf{b}}| \lesssim \frac{1}{|v|}. \tag{2.40}
$$

$$
|\nabla_v T^t_{\mathbf{x}^1_p}| \lesssim \frac{\|\eta\|_{C^2}}{|v|}. \tag{2.41}
$$

Proof. Clearly, (2.39) follows from $\frac{n(x_{\mathbf{b}})\cdot v}{|v|} \gtrsim x - x_{\mathbf{b}} = t_{\mathbf{b}}|v|$.

By (2.32) and (2.39) we have

$$
\begin{aligned}
|\nabla_v t_{\mathbf{b}}| &\lesssim \frac{|n(x_{\mathbf{b}}) \cdot v|}{|n(x_{\mathbf{b}}) \cdot v|} \frac{1}{|v|^2} \lesssim \frac{1}{|v|^2}, \\
|\nabla_v x_{\mathbf{b}}| &\lesssim \frac{|n(x_{\mathbf{b}}) \cdot v|}{|v|^2} + \frac{|n(x_{\mathbf{b}}) \cdot v||v|}{|n(x_{\mathbf{b}}) \cdot v||v|^2} \lesssim \frac{1}{|v|}.
\end{aligned}
$$

For (2.41) by the definition of $T_{\mathbf{x}_p}$ in (2.23), and using (2.35), we have

$$
|\nabla_v T^t_{\mathbf{x}^1_p}| \lesssim \|\eta\|_{C^2} \times |\nabla_v [\mathbf{x}^1_{p^1,1} + \mathbf{x}^1_{p^2,1}]| \lesssim \|\eta\|_{C^2} \frac{|v| t_{\mathbf{b}}}{|n(\eta_{p^1}(\mathbf{x}^1_{p^1})) \cdot v|} \lesssim \frac{\|\eta\|_{C^2}}{|v|},
$$

where we have used (2.39) in the last inequality.

Then the lemma follows. □

Lemma 2.3. *The following map is one-to-one:*

$$
\begin{aligned}
&v^{k+1} \in \{ n(x^{k+1}) \cdot v^{k+1} > 0 : x_{\mathbf{b}}(x^{k+1}, v^{k+1}) \in B(p^{k+2}, \delta_2) \} \\
&\mapsto (\mathbf{x}^{k+2}_{p^{k+2},1}, \mathbf{x}^{k+2}_{p^{k+2},2}, t_{\mathbf{b}}^{k+1}),
\end{aligned} \tag{2.42}
$$

with

$$\det\left(\frac{\partial(\mathbf{x}^{k+2}_{p^{k+2},1},\mathbf{x}^{k+2}_{p^{k+2},2},t^{k+1}_{\mathbf{b}})}{\partial v^{k+1}}\right) = \frac{1}{\sqrt{g_{p^{k+2},11}(\mathbf{x}^{k+2}_{p^{k+2}})g_{p^{k+2},22}(\mathbf{x}^{k+2}_{p^{k+2}})}}\frac{|t^{k+1}_{\mathbf{b}}|^3}{|n(x^{k+2})\cdot v^{k+1}|}. \tag{2.43}$$

Proof. Combining (2.31) and (2.35) we conclude

$$\begin{aligned}
&\det\left(\frac{\partial(\mathbf{x}^{k+2}_{p^{k+2},1},\mathbf{x}^{k+2}_{p^{k+2},2},t^{k+1}_{\mathbf{b}})}{\partial v^{k+1}}\right)\\
&= |t^{k+1}_{\mathbf{b}}|^3\det\begin{pmatrix}
-\frac{1}{\mathbf{v}^{k+1}_{p^{k+2},3}}\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}} \\
\frac{1}{\sqrt{g_{p^{k+2},11}(\mathbf{x}^{k+2}_{p^{k+2}})}}\left[\frac{\partial_1\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},11}}}\Big|_{x^{k+2}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\mathbf{v}^{k+1}_{p^{k+2},3}}\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}\right] \\
\frac{1}{\sqrt{g_{p^{k+2},22}(\mathbf{x}^{k+2}_{p^{k+2}})}}\left[\frac{\partial_2\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},22}}}\Big|_{x^{k+2}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},2}}{\mathbf{v}^{k+1}_{p^{k+2},3}}\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}\right]
\end{pmatrix}\\
&= -|t^{k+1}_{\mathbf{b}}|^3\frac{1}{\mathbf{v}^{k+1}_{p^{k+2},3}}\frac{1}{\sqrt{g_{p^{k+2},11}(\mathbf{x}^{k+2}_{p^{k+2}})g_{p^{k+2},22}(\mathbf{x}^{k+2}_{p^{k+2}})}}\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}\\
&\quad\cdot\left(\left[\frac{\partial_1\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},11}}}\Big|_{x^{k+2}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},1}}{\mathbf{v}^{k+1}_{p^{k+2},3}}\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}\right]\right.\\
&\quad\left.\times\left[\frac{\partial_2\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},22}}}\Big|_{x^{k+2}} - \frac{\mathbf{v}^{k+1}_{p^{k+1},2}}{\mathbf{v}^{k+1}_{p^{k+2},3}}\frac{\partial_3\eta_{p^{k+2}}}{\sqrt{g_{p^{k+2},33}}}\Big|_{x^{k+2}}\right]\right)\\
&= \frac{1}{\sqrt{g_{p^{k+2},11}(\mathbf{x}^{k+2}_{p^{k+2}})g_{p^{k+2},22}(\mathbf{x}^{k+2}_{p^{k+2}})}}\frac{|t^{k+1}_{\mathbf{b}}|^3}{\mathbf{v}^{k+1}_{p^{k+2},3}} = (2.43),
\end{aligned}$$

where we have used (2.20).

Now we prove the map (2.42) is one to one. Assume that there exists v and $\tilde{v}$ satisfy $x_{\mathbf{b}}(x^{k+1},v) = x_{\mathbf{b}}(x^{k+1},\tilde{v})$ and $t_{\mathbf{b}}(x^{k+1},v) = t_{\mathbf{b}}(x^{k+1},\tilde{v})$. We choose $p\in\partial\Omega$ near $x_{\mathbf{b}}(x^{k+1},v)$ and use the same parametrization. Then, by an expansion, for some $\bar{v}\in\{a\tilde{v}+(1-a)v : a\in[0,1]\}$,

$$0 = \begin{pmatrix}\mathbf{x}_{p,1}(x^{k+1},\tilde{v})\\ \mathbf{x}_{p,2}(x^{k+1},\tilde{v})\\ t_{\mathbf{b}}(x^{k+1},\tilde{v})\end{pmatrix} - \begin{pmatrix}\mathbf{x}_{p,1}(x^{k+1},v)\\ \mathbf{x}_{p,2}(x^{k+1},v)\\ t_{\mathbf{b}}(x^{k+1},v)\end{pmatrix} = \begin{pmatrix}\nabla_v\mathbf{x}_{p,1}(x^{k+1},\bar{v})\\ \nabla_v\mathbf{x}_{p,2}(x^{k+1},\bar{v})\\ \nabla_v t_{\mathbf{b}}(x^{k+1},\bar{v})\end{pmatrix}(\tilde{v}-v).$$

This equality can be true only if the determinant of the Jacobian matrix equals zero. Then (2.43) implies that $t_{\mathbf{b}}(x^{k+1},\bar{v}) = 0$. But this implies $x^{k+1} = x_{\mathbf{b}}(x^{k+1},\bar{v})$ and hence $n(x^{k+1})\cdot\bar{v} = 0$ which is out of our domain. □

The next lemma describe the properties of a convex domain.

Lemma 2.4. *Given a C^2 convex domain defined in (1.7),*

$$|n_{p^{k+j}}(\mathbf{x}_{p^{k+j}}^{k+j})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))|\sim|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^2,\quad j=1,2,$$
$$|\mathbf{v}_{p^{k+1},3}^{k+1}|/|\mathbf{v}_{p^{k+1}}^{k+1}|\sim|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|. \tag{2.44}$$

For $j'=1,2$,

$$\left|\frac{\partial[n_{p^{k+j}}(\mathbf{x}_{p^{k+j}}^{k+j})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))]}{\partial\mathbf{x}_{p^{k+2},j'}^{k+2}}\right|$$
$$\lesssim\|\eta\|_{C^2}|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|,\quad j=1,2. \tag{2.45}$$

Proof. First we prove (2.44). By Taylor's expansion, for $x,y\in\partial\Omega$ and some $0\leqq t\leqq 1$,

$$\xi(y)-\xi(x)=0-0=\nabla\xi(x)\cdot(y-x)+\frac{1}{2}(y-x)^T\nabla^2\xi(x+t(y-x))(y-x).$$

Thus, from (1.12),

$$|n(x)\cdot(x-y)|\sim(y-x)^T\nabla^2\xi(x+t(y-x))(y-x).$$

From the convexity (1.7), we have

$$|n_{p^{k+j}}(\mathbf{x}_{p^{k+j}}^{k+j})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))|\geqq C_\Omega|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^2.$$

Since ξ is C^2 at least,

$$|\{x^{k+1}-y\}\cdot n(x^{k+1})|\leqq\|\xi\|_{C^2}|x^{k+1}-y|^2.$$

Also notice that

$$|n_{p^{k+1}}(\mathbf{x}_{p^{k+1}}^{k+1})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))|=|\mathbf{v}_{p^{k+1},3}^{k+1}|(t^{k+2}-t^{k+1}),$$

and thus

$$\frac{|\mathbf{v}_{p^{k+1},3}^{k+1}|}{|\mathbf{v}_{p^{k+1}}^{k+1}|}\geqq\frac{1}{|\mathbf{v}_{p^{k+1}}^{k+1}|}\frac{C_\Omega}{|t^{k+1}-t^{k+2}|}\left|x^{k+1}-x^{k+2}\right|^2$$
$$=C_\Omega|x^{k+1}-x^{k+2}|=C_\Omega|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|.$$

By the same computation we can easily conclude

$$\frac{|\mathbf{v}_{p^{k+1},3}^{k+1}|}{|\mathbf{v}_{p^{k+j}}^{k+j}|}\leqq C_\xi|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|.$$

Then we prove (2.45). For $j = 1$, $j' = 1, 2$ we have

$$\begin{aligned}
&\left|\frac{\partial[n_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}))]}{\partial \mathbf{x}^{k+2}_{p^{k+2},j'}}\right| \\
&\leqq |n_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}})\cdot\partial_{j'}\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})| \\
&= \Big|n_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}})\cdot\partial_{j'}\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}) \\
&\quad + n_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}})\cdot\big[\partial_{j'}\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})-\partial_{j'}\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}})\big]\Big| \\
&\leqq 0+\|\eta\|_{C^2}|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})|, \qquad (2.46)
\end{aligned}$$

where we applied (2.20) and (2.27).

For $j = 2$, we have

$$\begin{aligned}
&\left|\frac{\partial[n_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}))]}{\partial \mathbf{x}^{k+2}_{p^{k+2},j'}}\right| \\
&\lesssim |n_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})\cdot\partial_{j'}\eta_{p^{k+2}}|+\|\eta\|_{C^2}|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})| \\
&= \|\eta\|_{C^2}|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})|,
\end{aligned}$$

where we applied (2.20) and (2.27). □

2.3. Properties of tangential derivative

Aiming the regularity estimate of (1.19) without the $\alpha-$weight, we establish several properties of the tangential derivative. We summarize them in Lemma 2.5–2.7.

Lemma 2.5. *For $x = \eta_p(\mathbf{x}_p) \in \partial\Omega$, we have the following equivalence:*

$$|G(x)\nabla_x f(x,v)| \sim \Big|\sum_{j=1,2}\partial_{\mathbf{x}_{p,j}} f(\eta_p(\mathbf{x}_p),v)\Big|. \qquad (2.47)$$

Proof. By (2.23) we have

$$\partial_i\eta_p(\mathbf{x}_p) = \sqrt{g_{p,ii}(\mathbf{x}_p)}\,T^t_{\mathbf{x}_p}e_i.$$

Denote $\mathfrak{F}(x) = \nabla_x f(x,v)T^t_{\mathbf{x}_p}$, we have

$$\begin{aligned}
&\sum_{j=1,2}\partial_{\mathbf{x}_{p,j}} f(\eta_p(\mathbf{x}_p),v) \\
&= \sqrt{g_{p,11}(\mathbf{x}_p)}\nabla_x f(x,v)T^t_{\mathbf{x}_p}e_1+\sqrt{g_{p,22}(\mathbf{x}_p)}\nabla_x f(x,v)T^t_{\mathbf{x}_p}e_2 \\
&= \sqrt{g_{p,11}(\mathbf{x}_p)}\mathfrak{F}e_1+\sqrt{g_{p,22}(\mathbf{x}_p)}\mathfrak{F}e_2.
\end{aligned}$$

We also have

$$\begin{aligned} G(x)\nabla_x f(x,v) &= \nabla_x f\Big(T^t_{\mathbf{x}_p} T_{\mathbf{x}_p} - T^t_{\mathbf{x}_p} e_3 e_3^t T_{\mathbf{x}_p}\Big) \\ &= \mathfrak{F}\Big(T_{\mathbf{x}_p} - e_3 e_3^t T_{\mathbf{x}_p}\Big) = \mathfrak{F}\Big(I - e_3\otimes e_3\Big) T_{\mathbf{x}_p} \\ &= \big(\mathfrak{F}e_1\ \mathfrak{F}e_2\ 0\big)\, T_{\mathbf{x}_p} = \mathfrak{F}e_1 \frac{\partial_1\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,11}(\mathbf{x}_p)}} + \mathfrak{F}e_2 \frac{\partial_2\eta_p(\mathbf{x}_p)}{\sqrt{g_{p,22}(\mathbf{x}_p)}}. \end{aligned}$$

Since $\partial_1\eta_p(\mathbf{x}_p) \perp \partial_2\eta_p(\mathbf{x}_p)$,

$$|G(x)\nabla_x f(x,v)| \sim \Big|\sqrt{g_{p,11}(\mathbf{x}_p)}\mathfrak{F}e_1 + \sqrt{g_{p,22}(\mathbf{x}_p)}\mathfrak{F}e_2\Big| \sim \Big|\sum_{j=1,2} \partial_{\mathbf{x}_{p,j}} f(\eta_p(\mathbf{x}_p), v)\Big|.$$

Lemma 2.6. *For any $s \in [0, t_{\mathbf{b}}]$, we have*

$$|G(x) - G(x - sv)| \lesssim \frac{\tilde{\alpha}(x,v)}{|v|}. \tag{2.48}$$

Thus

$$|G(x)\nabla_x f(x - sv)| \lesssim \frac{\|w_{\tilde{\theta}/2}|v|\nabla_\parallel f\| + \|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{|v| w_{\tilde{\theta}/2}(v)}. \tag{2.49}$$

Proof. By the definition (1.14), we have

$$\begin{aligned} |G(x) - G(x - sv)| \ &\leqq \big|n(x - sv)\otimes(n(x - sv) - n(x))\big| \\ &\quad + \big|(n(x - sv) - n(x))\otimes n(x)\big|. \end{aligned}$$

Then by (1.12) we have

$$\begin{aligned} |\nabla_x n(x)| &\lesssim \Big|\nabla_x\Big[\chi'_{\varepsilon/2}(dist(x,\partial\Omega))\Big]\Big| + \Big|\chi'_{\varepsilon/2}(dist(x,\partial\Omega))\nabla_x \frac{\nabla_x\xi(x)}{|\nabla\xi(x)|}\Big| \\ &\lesssim |\chi''\times\nabla_x dist(x,\partial\Omega)| \\ &\quad + \Big|\chi'_{\varepsilon/2}(dist(x,\partial\Omega))\frac{|\nabla\xi(x)|\nabla^2\xi(x) - \nabla\xi(x)\otimes\frac{\nabla^2\xi(x)\nabla\xi(x)}{|\nabla\xi(x)|}}{|\nabla\xi(x)|^2}\Big| \\ &\lesssim 1 + \chi'_{\varepsilon/2}(dist(x,\partial\Omega))\frac{|\nabla^2\xi(x)|}{|\nabla\xi(x)|}. \end{aligned}$$

From (1.9) and (1.12) we have $|\nabla\xi(x)| \gtrsim 1$ when $dist(x,\partial\Omega) \ll 1$. When $dist(x,\partial\Omega) \gtrsim 1$ we take ε to be small enough such that $\chi'_\varepsilon(dist(x,\partial\Omega)) = 0$. Hence

$$|\nabla_x n(x)| \lesssim \|\xi\|_{C^2}. \tag{2.50}$$

Then we use (2.39) to have

$$|n(x - sv) - n(x)| \lesssim t_{\mathbf{b}}|v|\|\xi\|_{C^2} \lesssim \frac{\tilde{\alpha}(x,v)}{|v|}.$$

Thus we conclude (2.48).

Last we prove (2.49). We rewrite

$$
\begin{aligned}
&|G(x)\nabla_x f(x-sv,v)| \\
&\quad = |G(x-sv)\nabla_x f(x-sv,v) + [G(x)-G(x-sv)]\nabla_x f(x-sv,v)| \\
&\quad \lesssim \frac{\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty}{|v| w_{\tilde\theta/2}(v)} + \left|\frac{\tilde\alpha(x,v)}{|v|}\nabla_x f(x-sv,v)\right| \\
&\quad \lesssim \frac{\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty}{|v| w_{\tilde\theta/2}(v)} + \frac{\|w_{\tilde\theta/2}\tilde\alpha\nabla_x f\|_\infty}{|v| w_{\tilde\theta/2}(v)} \\
&\quad \lesssim \frac{\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{|v| w_{\tilde\theta/2}(v)},
\end{aligned}
$$

where we applied (1.10), and we used (1.8) to have $w_{\tilde\theta/2}(v)\tilde\alpha(x,v) \lesssim w_{\tilde\theta}(v)\alpha(x,v)$. Then we conclude the lemma. □

Lemma 2.7. *For $x_{\mathbf{b}}(x,v) = \eta_{p^1}(\mathbf{x}^1_{p^1})$ and $i = 1, 2$, we have*

$$
\left|G(x)\nabla_x \mathbf{x}^1_{p^1,i}\right| \lesssim 1, \tag{2.51}
$$

$$
\left|G(x)\nabla_x t_{\mathbf{b}}(x,v)\right| \lesssim \frac{1}{|v|}. \tag{2.52}
$$

Proof. By (2.48) in Lemma 2.6 we have

$$
\left|G(x)\nabla_x \mathbf{x}^1_{p^1,i}\right| \lesssim \underbrace{\left|G(x_{\mathbf{b}})\nabla_x \mathbf{x}^1_{p^1,i}\right|}_{(2.53)_1} + \underbrace{\frac{\tilde\alpha(x,v)}{|v|}|\nabla_x \mathbf{x}^1_{p^1,i}|}_{(2.53)_2}. \tag{2.53}
$$

By (2.33), the definition of $\mathbf{v}^1_{p^1,3}$, $n(x_{\mathbf{b}})$ in (2.26) and (1.15), we have

$$
(2.53)_1 \lesssim 1 + \left||v|G(x_{\mathbf{b}})\frac{n(x_{\mathbf{b}})}{|n(x_{\mathbf{b}})\cdot v|}\right| = 1.
$$

Again by (2.33) and using definition of $\tilde\alpha$ in (1.8), we have

$$
(2.53)_2 \lesssim \frac{\tilde\alpha(x,v)}{|v|} + \frac{\tilde\alpha(x,v)}{|v|}\frac{|v|}{|n(x_{\mathbf{b}})\cdot v|} \lesssim 1.
$$

We conclude (2.51).

For (2.52) by (2.30) we have

$$
(2.53) \lesssim 0 + \frac{\tilde\alpha}{|v|}\frac{1}{|n(x_{\mathbf{b}})\cdot v|} \lesssim \frac{1}{|v|}.
$$

We conclude (2.52). □

2.4. Properties of Hölder's estimate

To prove the $C^{1,\beta}$ estimate (1.21)–(1.23) we need several $C^{1,\beta}$ estimate for $x_{\mathbf{b}}$ and $t_{\mathbf{b}}$. We summarize them in Lemma 2.8 and Lemma 2.10. Lemma 2.9 serves as a key ingredient to prove Lemma 2.8.

Lemma 2.8. *We have the following estimates:*

$$\Big|\frac{x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{\beta}}, \tag{2.54}$$

$$\Big|\frac{t_{\mathbf{b}}(x,v)-t_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{|v|\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{\beta}}, \tag{2.55}$$

$$\begin{aligned}\frac{|e^{-C\nu t_{\mathbf{b}}(x,v)}-e^{-C\nu t_{\mathbf{b}}(y,v)}|^{\beta}}{|x-y|^{\beta}} &\lesssim \Big|\frac{e^{-C\nu t_{\mathbf{b}}(x,v)}-e^{-C\nu t_{\mathbf{b}}(y,v)}}{|x-y|^{\beta}}\Big| \\ &\lesssim \frac{1}{|v|\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{\beta}},\end{aligned} \tag{2.56}$$

$$\Big|\frac{n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))}{|x-y|^{\beta}}\Big| \lesssim \|\xi\|_{C^2}\frac{1}{\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{\beta}}, \tag{2.57}$$

$$\Big|\frac{\nabla_x x_{\mathbf{b}}(x,v)-\nabla_x x_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{2+\beta}}, \tag{2.58}$$

$$\Big|\frac{\nabla_x t_{\mathbf{b}}(x,v)-\nabla_x t_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{|v|\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{2+\beta}}, \tag{2.59}$$

$$\Big|\frac{\nabla_v x_{\mathbf{b}}(x,v)-\nabla_v x_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{|v|\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}, \tag{2.60}$$

$$\Big|\frac{\nabla_v t_{\mathbf{b}}(x,v)-\nabla_v t_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{|v|^2\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}, \tag{2.61}$$

$$\Big|G(y)\frac{\nabla_x x_{\mathbf{b}}(x,v)-\nabla_x x_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}, \tag{2.62}$$

$$\Big|G(y)\frac{\nabla_x t_{\mathbf{b}}(x,v)-\nabla_x t_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{1}{|v|\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}, \tag{2.63}$$

$$\Big|\frac{f_s(x,v)-f_s(y,v)}{|x-y|^{\beta}}\Big| \lesssim \frac{\|wf_s\|_{\infty}^{1-\beta}\|w_{\tilde{\theta}}\alpha\nabla_x f_s\|_{\infty}^{\beta}}{w_{\tilde{\theta}}(v)\min\{\alpha(x,v),\alpha(y,v)\}^{\beta}}. \tag{2.64}$$

For $x_{\mathbf{b}}(x,v)=\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)})$, $x_{\mathbf{b}}(y,v)=\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})$*(see the definition* (7.4) *in Section 7),* $i\in\{1,2\}$*, we have*

$$\frac{|\mathbf{x}^1_{p^1(x)}-\mathbf{x}^1_{p^1(y)}|}{|x-y|^\beta}\lesssim\frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta},\tag{2.65}$$

$$\frac{|T_{\mathbf{x}^1_{p^1(x)}}-T_{\mathbf{x}^1_{p^1(y)}}|}{|x-y|^\beta}\lesssim\frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta},\tag{2.66}$$

$$\frac{|\nabla_x\mathbf{x}^1_{p^1(x)}-\nabla_x\mathbf{x}^1_{p^1(y)}|}{|x-y|^\beta}\lesssim\frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta},\tag{2.67}$$

$$\frac{|\nabla_v\mathbf{x}^1_{p^1(x)}-\nabla_v\mathbf{x}^1_{p^1(y)}|}{|x-y|^\beta}\lesssim\frac{1}{|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},\tag{2.68}$$

$$\frac{|\nabla_v T_{\mathbf{x}^1_{p^1(x)}}-\nabla_v T_{\mathbf{x}^1_{p^1(y)}}|}{|x-y|^\beta}\lesssim\frac{1}{|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.\tag{2.69}$$

When $x,y\in\partial\Omega$*,*

$$\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|}{|x-y|}\lesssim 1,\tag{2.70}$$

for $\tilde\theta\ll 1$*,*

$$\frac{|M_w(x,v)-M_w(y,v)|}{\sqrt{\mu(v)}|x-y|^\beta}\lesssim w^{-1}_{\tilde\theta}(v)\|T_w-T_0\|_{C^1}.\tag{2.71}$$

For $x=\eta_{p(x)}(\mathbf{x}_{p(x)})$, $y=\eta_{p(y)}(\mathbf{x}_{p(y)})\in\partial\Omega$*, and* $x_{\mathbf{b}}(x,v)=\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)})$, $x_{\mathbf{b}}(y,v)=\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})$*. For* $i,j\in\{1,2\}$ *we have*

$$\frac{\left|\partial_{\mathbf{x}_{p(x),j}}\mathbf{x}^1_{p^1(x),i}-\partial_{\mathbf{x}_{p(y),j}}\mathbf{x}^1_{p^1(y),i}\right|}{|x-y|^\beta}\lesssim\frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^3}.\tag{2.72}$$

We need the next lemma to prove this

Lemma 2.9. *Define*

$$x(\tau):=(1-\tau)x+\tau y,\quad |\dot x(\tau)|=|x-y|.\tag{2.73}$$

If $|x-y|\leqq\varepsilon\min\{\frac{\tilde\alpha(x,v)}{|v|},\frac{\tilde\alpha(y,v)}{|v|}\}\ll 1$*, then*

$$\tilde\alpha(x(\tau),v)\gtrsim\min\left\{\tilde\alpha(x,v),\tilde\alpha(y,v)\right\}.\tag{2.74}$$

Proof. By the definition (1.8) we have

$$\tilde{\alpha}^2(x(\tau), v) = |\nabla\xi(x(\tau)) \cdot v|^2 - 2\xi(x(\tau))(v \cdot \nabla^2\xi(x(\tau)) \cdot v). \tag{2.75}$$

We expand $|\nabla\xi(x(\tau)) \cdot v|^2$ and $-2\xi(x(\tau))(v \cdot \nabla^2\xi(x(\tau)) \cdot v)$ separately: we expand in τ as

$$|\nabla\xi(x(\tau)) \cdot v|^2 = |\nabla\xi(x(0)) \cdot v|^2 + \underbrace{\int_0^\tau d\tau' 2(\nabla\xi(x(\tau')) \cdot v)\dot{x}(\tau) \cdot \nabla^2\xi(x(\tau')) \cdot v}_{(2.76)_*}, \tag{2.76}$$

$$-2\xi(x(\tau))(v \cdot \nabla^2\xi(x(\tau)) \cdot v) = -2\xi(x(\tau))\{v \cdot \nabla^2\xi(x(0)) \cdot v + O(|x - y|)\|\xi\|_{C^3}|v|^2\}, \tag{2.77}$$

where we have used (2.73).

For $(2.76)_*$ we further expand in τ' and obtain

$$(2.75) = |\nabla\xi(x) \cdot v|^2 + 2(\nabla\xi(x) \cdot v)O(|x - y|)|v|\|\xi\|_{C^2} + \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \dot{x}(\tau'') \cdot \nabla^2\xi(x(\tau'')) \cdot v\dot{x}(\tau'') \cdot \nabla^2\xi(x(\tau'')) \cdot v \tag{2.78}$$

$$+ \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' 2(\nabla\xi(x(\tau'')) \cdot v)\dot{x}(\tau'')\dot{x}(\tau'')\nabla^3\xi(x(\tau'')) \cdot v - 2\xi(x(\tau))O(|v|^2). \tag{2.79}$$

From the convexity (1.7) we have

$$(2.78) + (2.79) = O(1)|\dot{x}|^2\|\xi\|_{C^3}|v|^2 = O(\varepsilon^2)\min\left\{\frac{\alpha(x, v)}{|v|}, \frac{\alpha(y, v)}{|v|}\right\}^2 |v|^2. \tag{2.80}$$

From (2.80) we have

$$(2.75) + 2\xi(x(\tau))O(|v|^2) = |\nabla\xi(x) \cdot v|^2 + O(\varepsilon)\alpha(x, v)\min\{\alpha(x, v), \alpha(y, v)\} + O(\varepsilon^2)\min\{\alpha(x, v), \alpha(y, v)\}^2. \tag{2.81}$$

Now we claim

$$-\xi(x(\tau)) \geqq \min\{-\xi(x), -\xi(y)\}. \tag{2.82}$$

From $\frac{d}{d\tau}(-\xi(x(\tau))) = -\dot{x}(\tau) \cdot \nabla_x\xi(x(\tau))$ and convexity (1.7),

$$\frac{d^2}{d\tau^2}(-\xi(x(\tau))) = -\dot{x}(\tau) \cdot \nabla_x^2\xi(x(\tau)) \cdot \dot{x(\tau)} \lesssim -|\dot{x}(\tau)|^2 \leqq 0.$$

Thus $-\xi(x(\tau))$ is a concave function of τ. From $0 \leqq \tau \leqq 1$, we prove our claim (2.82) as

$$-\xi(x(\tau)) = -\xi(x((1-\tau)\cdot 0 + \tau\cdot 1)) \geqq -(1-\tau)\xi(x(0)) - \tau\xi(x(1))$$
$$= -(1-\tau)\xi(x) - \tau\xi(y) \geqq \min\{-\xi(x), -\xi(y)\}.$$

Now combining (2.81) and (2.82) we conclude that

$$(2.75) \gtrsim |\nabla\xi(x)\cdot v|^2 + \min\{-\xi(x), -\xi(y)\}|v|^2 + O(\varepsilon)\alpha(x,v)\min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\} + O(\varepsilon^2)\min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\}^2. \tag{2.83}$$

Similarly we can set $x(\tau) = (1-\tau)y + \tau x$. From $x(0) = y$, following the same argument we derive

$$(2.75) \geqq |\nabla\xi(y)\cdot v|^2 + \min\{-\xi(x), -\xi(y)\}|v|^2 + O(\varepsilon)\tilde{\alpha}(y,v)\min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\} + O(\varepsilon^2)\min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\}^2. \tag{2.84}$$

From the definition of (1.8) using (2.83) and (2.84) we have

$$(2.75) \geqq \min\left\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\right\}^2 - O(\varepsilon)\min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\}\Big\}^2.$$

Hence from $\varepsilon \ll 1$ we conclude (2.74). □

Now we start the proof of Lemma 2.8.

Proof of Lemma 2.8. For all estimates we assume $|x-y| \leqq \varepsilon \min\{\frac{\tilde{\alpha}(x,v)}{|v|}, \frac{\tilde{\alpha}(y,v)}{|v|}\}$, otherwise the lemma follows immediately by (2.32). Thus we can apply (2.74) during the whole proof. We will use the $x(\tau)$ defined in (2.73).

Proof of (2.54) We have

$$\frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|}{|x-y|^\beta} = \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_x x_{\mathbf{b}}(x(\tau),v)$$
$$= \frac{1}{|x-y|^\beta}\int_0^1 |\dot{x}(\tau)||\nabla_x x_{\mathbf{b}}(x(\tau),v)|\mathrm{d}\tau$$
$$\lesssim \frac{1}{|x-y|^{\beta-1}}\int_0^1 \frac{|v|}{\tilde{\alpha}(x(\tau),v)} \lesssim \frac{|v|^\beta}{\min\{\alpha(x,v),\alpha(y,v)\}^\beta},$$

where we have used Lemma 2.9, (1.11), (2.32) and $\beta < 1$ in the last line. □

Proof of (2.55) We have

$$\frac{|t_{\mathbf{b}}(x,v) - t_{\mathbf{b}}(y,v)|}{|x-y|^\beta} = \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_x t_{\mathbf{b}}(x(\tau),v)$$
$$= \frac{1}{|x-y|^\beta}\int_0^1 |\dot{x}(\tau)||\nabla_x t_{\mathbf{b}}(x(\tau),v)|\mathrm{d}\tau$$
$$= |x-y|^{1-\beta}\int_0^1 \frac{1}{\tilde{\alpha}(x(\tau),v)} \lesssim \frac{|v|^\beta}{|v|\min\{\alpha(x,v),\alpha(y,v)\}^\beta},$$

where we have used Lemma 2.9, (1.11), (2.32) and $\beta < 1$ in the last line.

Proof of (2.56) The first inequality is clear since $|e^{-C\nu t_{\mathbf{b}}(x,v)} - e^{-C\nu t_{\mathbf{b}}(y,v)}| \lesssim 1$.

To prove the second inequality we have

$$\begin{aligned}
&\frac{|e^{-C\nu t_{\mathbf{b}}(x,v)} - e^{-C\nu t_{\mathbf{b}}(y,v)}|}{|x-y|^\beta} \\
&= \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau} e^{-C\nu t_{\mathbf{b}}(x(\tau),v)} \\
&\lesssim \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau (\nu t_{\mathbf{b}}(x(\tau),v)) e^{-C\nu t_{\mathbf{b}}(x(\tau),v)} |x-y| \frac{1}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|} \\
&\lesssim |x-y|^{1-\beta} \frac{1}{\min\{\tilde{\alpha}(x,v), \tilde{\alpha}(y,v)\}} \lesssim \frac{1}{|v| \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^\beta},
\end{aligned}$$

where we have used (2.32) in the second line, Lemma 2.9 and (1.11) in the last line. □

Proof of (2.57) Since

$$\begin{aligned}
\frac{|n(x_{\mathbf{b}}(x,v)) - n(x_{\mathbf{b}}(y,v))|}{|x-y|^\beta} &= \frac{|n(x_{\mathbf{b}}(x,v)) - n(x_{\mathbf{b}}(y,v))|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|} \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|}{|x-y|^\beta} \\
&\lesssim \|\xi\|_{C^2} \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|}{|x-y|^\beta}.
\end{aligned}$$

By (2.54) we derive (2.57). □

Proof of (2.58) We have

$$\begin{aligned}
\frac{|\nabla_x x_{\mathbf{b}}(x,v) - \nabla_x x_{\mathbf{b}}(y,v)|}{|x-y|^\beta} &= \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \nabla_x x_{\mathbf{b}}(x(\tau),v) \\
&= \frac{1}{|x-y|^\beta}\int_0^1 |\dot{x}(\tau)||\nabla_x \nabla_x x_{\mathbf{b}}(x(\tau),v)|\mathrm{d}\tau \\
&\lesssim |x-y|^{1-\beta}\int_0^1 \frac{|v|^3}{|\tilde{\alpha}(x(\tau),v)|^3}. \qquad (2.85)
\end{aligned}$$

Here we have used (2.32) to have

$$\begin{aligned}
&|\nabla_x(\nabla_x x_{\mathbf{b}}(x(\tau),v))| \\
&\lesssim \left[\frac{\|\eta\|_{C^2}|v||n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2} + \|\eta\|_{C^2}\frac{|v||n(x_{\mathbf{b}}(x(\tau),v))\otimes v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}\right] \\
&\quad \times |\nabla_x x_{\mathbf{b}}(x(\tau),v)| \qquad (2.86) \\
&\lesssim \frac{|v|^2}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2} + \frac{|v|^3}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^3} \lesssim \frac{|v|^3}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^3}, \qquad (2.87)
\end{aligned}$$

where we have used $|n(x_{\mathbf{b}}(x(\tau),v))\cdot v| \leqq |v|$ in the last inequality. Then by Lemma 2.9 and (1.11) we obtain (2.58). □

Proof of (2.59) We have

$$\begin{aligned}\frac{|\nabla_x t_{\mathbf{b}}(x,v)-\nabla_x t_{\mathbf{b}}(y,v)|}{|x-y|^\beta} &= \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_x t_{\mathbf{b}}(x(\tau),v)\\ &= \frac{1}{|x-y|^\beta}\int_0^1 |\dot{x}(\tau)||\nabla_x\nabla_x t_{\mathbf{b}}(x(\tau),v)|\mathrm{d}\tau\\ &\lesssim |x-y|^{1-\beta}\int_0^1 \frac{|v|^2}{|\tilde{\alpha}(x(\tau),v)|^3},\end{aligned}$$

where we have used (2.32) to conclude

$$\begin{aligned}|\nabla_x(\nabla_x t_{\mathbf{b}}(x(\tau),v))| &\lesssim \frac{\|\eta\|_{C^2}(|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|)}{|n(x_{\mathbf{b}}(x(\tau),v))|^2}\nabla_x x_{\mathbf{b}}(x(\tau),v)\\ &\quad+\frac{n(x_{\mathbf{b}}(x(\tau)),v)\|\eta\|_{C^2}|v|}{|n(x_{\mathbf{b}}(x(\tau),v))|^2}\nabla_x x_{\mathbf{b}}(x(\tau),v)\\ &\lesssim \frac{|v|^2}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^3}.\end{aligned}\tag{2.88}$$

Thus, by Lemma 2.9 and (1.11), we obtain (2.59). □

Proof of (2.60) We have

$$\begin{aligned}\frac{|\nabla_v x_{\mathbf{b}}(x,v)-\nabla_v x_{\mathbf{b}}(y,v)|}{|x-y|^\beta} &= \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_v t_{\mathbf{b}}(x(\tau),v)\\ &= \frac{1}{|x-y|^\beta}\int_0^1 |\dot{x}(\tau)||\nabla_x\nabla_v x_{\mathbf{b}}(x(\tau),v)|\mathrm{d}\tau\\ &\lesssim |x-y|^{1-\beta}\int_0^1 \frac{|v|}{|\tilde{\alpha}(x(\tau),v)|^2},\end{aligned}$$

where we have used (2.32) and (2.39) to conclude that

$$\begin{aligned}|\nabla_x\nabla_v x_{\mathbf{b}}(x(\tau),v)| &\lesssim |\nabla_x t_{\mathbf{b}}(x(\tau),v)|+|\nabla_x t_{\mathbf{b}}(x(\tau),v)|\frac{|v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}\\ &\quad+|t_{\mathbf{b}}(x(\tau),v)|\frac{|\nabla_x x_{\mathbf{b}}(x(\tau),v)|[\|\eta\|_{C^2}|v|^2+|v|^2]}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}\\ &\lesssim \frac{|v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}+\frac{|n(x_{\mathbf{b}}(x(\tau),v)\cdot v)|}{|v|^2}\frac{|v|^3}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^3}\\ &\lesssim \frac{|v|}{|n(x_{\mathbf{b}}(x(\tau)),v)\cdot v|^2}.\end{aligned}$$

Thus, by Lemma 2.9 and (1.11), we conclude (2.60). □

Proof of (2.61) We have

$$\begin{aligned}\frac{|\nabla_v t_{\mathbf{b}}(x,v)-\nabla_v t_{\mathbf{b}}(y,v)|}{|x-y|^\beta} &= \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_v t_{\mathbf{b}}(x(\tau),v)\\ &= \frac{1}{|x-y|^\beta}\int_0^1 |\dot{x}(\tau)||\nabla_x\nabla_v t_{\mathbf{b}}(x(\tau),v)|\mathrm{d}\tau\\ &\lesssim |x-y|^{1-\beta}\int_0^1 \frac{1}{|\tilde{\alpha}(x(\tau),v)|^2},\end{aligned}$$

where we have used (2.32) and (2.39) to conclude that

$$\begin{aligned}
&|\nabla_x\nabla_v t_{\mathbf{b}}(x(\tau),v)|\\
&\lesssim |\nabla_x t_{\mathbf{b}}(x(\tau),v)|\frac{1}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}\\
&+|t_{\mathbf{b}}(x(\tau),v)|\frac{|\nabla_x x_{\mathbf{b}}(x(\tau),v)|[\|\eta\|_{C^2}|v|+|v|]}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}\\
&\lesssim \frac{1}{|v||n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}+\frac{|n(x_{\mathbf{b}}(x(\tau),v)\cdot v)|}{|v|^2}\frac{|v|^2}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^3}\\
&\lesssim \frac{1}{|n(x_{\mathbf{b}}(x(\tau)),v)\cdot v|^2},
\end{aligned}$$

Thus by Lemma 2.9 and (1.11) we conclude (2.61). □

Proof of (2.62) From (2.85) and (2.86), we bound

$$\begin{aligned}
&|G(y)\nabla_x(\nabla_x x_{\mathbf{b}}(x(\tau),v))|\\
&\lesssim\Big[\frac{\|\eta\|_{C^2}|v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}+\frac{\|\eta\|_{C^2}|v||G(y)n(x_{\mathbf{b}}(x(\tau),v))\otimes v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}\Big]|\nabla_x x_{\mathbf{b}}(\tau,v)|\\
&\lesssim\frac{|v|^2}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2},
\end{aligned}$$

where we have used

$$\begin{aligned}
|G(y)n(x_{\mathbf{b}}(x(\tau),v))|&\lesssim |G(y)n(y)|+|n(x_{\mathbf{b}}(x(\tau,v)))-n(y)|\\
&\lesssim |x_{\mathbf{b}}(x(\tau),v)-y|\lesssim |x_{\mathbf{b}}(x(\tau),v)-x(\tau)|+|x(\tau)-y|\\
&\lesssim \min\left\{\frac{\tilde\alpha(x,v)}{|v|},\frac{\tilde\alpha(y,v)}{|v|}\right\}.
\end{aligned}\tag{2.89}$$

Thus

$$\Big|G(y)\frac{\nabla_x x_{\mathbf{b}}(x,v)-\nabla_x x_{\mathbf{b}}(y,v)}{|x-y|^\beta}\Big|\lesssim |x-y|^{1-\beta}\int_0^1\frac{|v|^2}{|\tilde\alpha(x(\tau),v)|^2},$$

and we conclude (2.62) from (1.11). □

Proof of (2.63) From (2.88) we have

$$\begin{aligned}
&\frac{|G(y)[\nabla_x t_{\mathbf{b}}(x,v)-\nabla_x t_{\mathbf{b}}(y,v)]|}{|x-y|^\beta}\\
&\lesssim\frac{1}{|x-y|^\beta}\times\int_0^1 d\tau\Big|\frac{\|\eta\|_{C^2}}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}\nabla_x x_{\mathbf{b}}(x(\tau),v)\\
&+\|\eta\|_{C^2}|v|\frac{G(y)n(x_{\mathbf{b}}(x(\tau),v))}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}\nabla_x x_{\mathbf{b}}(x(\tau),v)\Big|\\
&\lesssim\frac{1}{|x-y|^\beta}\int_0^1 d\tau\frac{|v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2}\lesssim\frac{1}{|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},
\end{aligned}$$

where we have used (2.89) to $G(y)n(x_{\mathbf{b}}(x(\tau),v))$ and (1.11). □

Proof of (2.64) By (2.74) in Lemma 2.9 we have

$$\begin{aligned}\frac{|f_s(x,v)-f_s(y,v)|}{|x-y|^\beta} &\lesssim w^{\beta-1}(v)\|wf_s\|_\infty^{1-\beta}\frac{|f_s(x,v)-f_s(y,v)|^\beta}{|x-y|^\beta}\\ &\lesssim w^{\beta-1}(v)\|wf_s\|_\infty^{1-\beta}\frac{1}{|x-y|^\beta}\Big|\int_0^1 \mathrm{d}\tau \dot{x}(\tau)\cdot\nabla_x f_s(x(\tau),v)\Big|^\beta\\ &\lesssim w^{\beta-1}(v)w_{\tilde\theta}^{-\beta}(v)\|wf_s\|_\infty^{1-\beta}\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^\beta\Big|\int_0^1 \mathrm{d}\tau\frac{1}{\tilde\alpha(x(\tau),v)}\Big|^\beta\\ &\lesssim w_{\tilde\theta}^{-1}(v)\frac{\|wf_s\|_\infty^{1-\beta}\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^\beta}{\min\{\alpha(x,v),\alpha(y,v)\}^\beta},\end{aligned}$$

where we have used $\tilde\theta \ll \varrho$ to have

$$w_{\tilde\theta}^{-\beta}(v)w^{\beta-1}(v) = e^{|v|^2[(\beta-1)\varrho-\beta\tilde\theta]} \leqq e^{-\tilde\theta|v|^2}. \qquad \square$$

Proof of (2.65) From (2.33), (2.26) and (2.32),

$$\nabla_x \mathbf{x}^1_{p^1(x),i} \sim \frac{v\cdot\partial_i\eta_{p^1(x)}n(x_{\mathbf{b}}(x,v))}{|n(x_{\mathbf{b}}(x,v))\cdot v|}.$$

Then we apply the same computation as the proof of (2.54) to conclude (2.65). $\square$

Proof of (2.66) For this estimate we can assume $|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)| \ll 1$, otherwise, for $|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)| \gtrsim \delta$ we use (2.54) and (2.23) to have

$$\begin{aligned}\frac{|T_{\mathbf{x}^1_{p^1(x)}}-T_{\mathbf{x}^1_{p^1(y)}}|}{|x-y|^\beta} &= \frac{|T_{\mathbf{x}^1_{p^1(x)}}-T_{\mathbf{x}^1_{p^1(y)}}|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta}\\ &\lesssim \frac{\|\eta\|_{C^1}}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta}.\end{aligned}$$

With $|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)| \ll 1$, we can assume that $x_{\mathbf{b}}(x,v)$ and $x_{\mathbf{b}}(y,v)$ correspond to the same p in (2.19). Then we drop the dependency on p and write $p^1(x)=p^1(y)=p$. The variable depend on x are $\mathbf{x}^1_{p^1(x)}$ and $\mathbf{x}^1_{p^1(y)}$. Thus

$$\frac{|T_{\mathbf{x}^1_{p^1(x)}}-T_{\mathbf{x}^1_{p^1(y)}}|}{|x-y|^\beta} = \frac{|T_{\mathbf{x}^1_{p^1(x)}}-T_{\mathbf{x}^1_{p^1(y)}}|}{|\mathbf{x}^1_{p^1(x)}-\mathbf{x}^1_{p^1(y)}|^\beta}\frac{|\mathbf{x}^1_{p^1(x)}-\mathbf{x}^1_{p^1(y)}|^\beta}{|x-y|^\beta} \lesssim \frac{\|\eta\|_{C^2}}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta},$$

where we have used intermediate value theorem to T in (2.23) and (2.65). $\square$

Proof of (2.67) Following the proof of (2.65), it is straightforward to verify that

$$\nabla_x[\nabla_x\mathbf{x}^1_{p^1(x),i}] \lesssim \frac{|v|^3}{|n(x_{\mathbf{b}}(x,v))\cdot v|^3}.$$

Then we follow the proof of (2.58) to conclude (2.67). $\square$

Proof of (2.68) Following the proof of (2.65), it is straightforward to verify that

$$\nabla_v[\nabla_x\mathbf{x}^1_{p^1(x),i}] \lesssim \frac{|v|}{|n(x_{\mathbf{b}}(x,v))\cdot v|^2}.$$

Then we follow the proof of (2.60) to conclude (2.68). □

Proof of (2.69) The v-derivative reads as

$$\nabla_v \frac{\partial_i \eta_p(\mathbf{x}^1_{p^1(x)})}{\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(x)})}} = \sum_{j=1,2} \nabla_v \mathbf{x}^1_{p^1(x),j} \frac{\partial_{ij}\eta\sqrt{g_{p,ii}} - \frac{\partial_{ij}\eta\partial_i\eta}{\sqrt{g_{p,ii}}}}{g_{p,ii}}.$$

Similar to the proof of (2.66), we can assume $x_{\mathbf{b}}(x,v)$ and $x_{\mathbf{b}}(y,v)$ correspond to the same p. Then we apply (2.40), (2.65) and (2.68) to have

$$\begin{aligned}
&\frac{|\nabla_v \frac{\partial_i \eta_p(\mathbf{x}^1_{p^1(x)})}{\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(x)})}} - \nabla_v \frac{\partial_i \eta_p(\mathbf{x}^1_{p^1(y)})}{\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(y)})}}|}{|x-y|^\beta} \lesssim \sum_{j=1,2} \frac{|\nabla_v \mathbf{x}^1_{p^1(x),j} - \nabla_v \mathbf{x}^1_{p^1(y),j}|}{|x-y|^\beta} \\
&+ \frac{1}{|v|} \frac{|\mathbf{x}^1_{p^1(x)} - \mathbf{x}^1_{p^1(y)}|}{|x-y|^\beta} \frac{1}{|\mathbf{x}^1_{p^1(x)} - \mathbf{x}^1_{p^1(y)}|^\beta} \\
&\times \Bigg[\frac{\partial_{ij}\eta_p(\mathbf{x}^1_{p^1(x)})\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(x)})} - \frac{\partial_{ij}\eta_p(\mathbf{x}^1_{p^1(x)})\partial_i\eta_p(\mathbf{x}^1_{p^1(x)})}{\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(x)})}}}{g_{p,ii}(\mathbf{x}^1_{p^1(x)})} \\
&- \frac{\partial_{ij}\eta_p(\mathbf{x}^1_{p^1(y)})\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(y)})} - \frac{\partial_{ij}\eta_p(\mathbf{x}^1_{p^1(y)})\partial_i\eta_p(\mathbf{x}^1_{p^1(y)})}{\sqrt{g_{p,ii}(\mathbf{x}^1_{p^1(y)})}}}{g_{p,ii}(\mathbf{x}^1_{p^1(y)})} \Bigg] \\
&\lesssim \frac{\|\eta\|_{C^3}}{|v| \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.
\end{aligned}$$

In the last line we used intermediate value theorem for $\partial_{ij}\eta$. By definition of T in (2.23) we conclude (2.69). □

Proof of (2.70) Since $|x-y| \ll 1$, we can assume that $x, y \in B(p;\delta_2)$, where $B(p;\delta_2)$ is defined in (2.19). Then both x, y correspond to the same p. Only for proof of this estimate we denote

$$\begin{aligned}
&x = \eta_p(\mathbf{x}_p(x)), \quad y = \eta_p(\mathbf{x}_p(y)), \\
&\eta_p(\mathbf{x}_p(\tau)) := \tau\eta_p(\mathbf{x}_p(x)) + (1-\tau)\eta_p(\mathbf{x}_p(y)).
\end{aligned}$$

By mean value theorem, there exists $c \in [0,1]$ such that

$$x - y = \eta_p(\mathbf{x}_p(x)) - \eta_p(\mathbf{x}_p(y)) = \nabla\eta_p\big(c\mathbf{x}_p(x) + (1-c)\mathbf{x}_p(y)\big)\big(\mathbf{x}_p(x) - \mathbf{x}_p(y)\big).$$

Thus

$$\begin{aligned}
&\frac{|x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(x)), v) - x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(y)), v)|}{|x-y|} \\
&= \frac{1}{|x-y|} \Big| \int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau} x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)), v) \Big|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|x-y|}\Big|\int_0^1 \mathrm{d}\tau \nabla_x x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v)\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_p(\mathbf{x}_p(\tau))\Big| \\
&= \frac{1}{|x-y|}\int_0^1 \mathrm{d}\tau \nabla_x x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v)\nabla\eta_p\big(c\mathbf{x}_p(x)+(1-c)\mathbf{x}_p(y)\big)\big(\mathbf{x}_p(x)-\mathbf{x}_p(y)\big) \\
&\lesssim \frac{|\mathbf{x}_p(x)-\mathbf{x}_p(y)|}{|x-y|}\Big|\int_0^1 \mathrm{d}\tau \nabla_x \eta_p\big(c\mathbf{x}_p(x)+(1-c)\mathbf{x}_p(y)\big) \\
&\qquad\qquad - \frac{n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\otimes v}{|n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\cdot v|}\nabla\eta_p\big(c\mathbf{x}_p(x)+(1-c)\mathbf{x}_p(y)\big)\Big| \\
&\lesssim \|\eta\|_{C^1} + \frac{n(\eta_p\big(c\mathbf{x}_p(x)+(1-c)\mathbf{x}_p(y)\big))\nabla\eta_p\big(c\mathbf{x}_p(x)+(1-c)\mathbf{x}_p(y)\big)|v|}{|n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\cdot v|} \\
&+ \frac{|v|[n(\eta_p\big(c\mathbf{x}_p(x)(1-c)\mathbf{x}_p(y)\big)) - n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))]}{|n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\cdot v|} \\
&\lesssim 1 + \frac{\|\xi\|_{C^2}|x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v)-\eta_p(\mathbf{x}_p(\tau))|}{|n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\cdot v|} \\
&\quad + \frac{|\eta_p(\mathbf{x}_p(\tau))-\eta_p\big(c\mathbf{x}_p(x)+(1-c)\mathbf{x}_p(y)\big)|}{|n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\cdot v|} \\
&\lesssim 1 + \frac{\alpha(n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau))),v))+|v||x-y|}{|n(x_{\mathbf{b}}(\eta_p(\mathbf{x}_p(\tau)),v))\cdot v|} \lesssim 1.
\end{aligned}
$$

In the fourth line we have used (2.32). In the last three lines we have used (2.48) and $|x-y| \leqq O(\varepsilon)\min\{\frac{\tilde{\alpha}(x,v)}{|v|}, \frac{\tilde{\alpha}(y,v)}{|v|}\}$. □

Proof of (2.71) Since $\|T_w - T_0\|_\infty \ll 1$ from **Existence Theorem**, by the definition of M_w in (1.4) we apply the mean value theorem to have

$$
\begin{aligned}
\frac{|M_w(x,v)-M_w(y,v)|}{\sqrt{\mu(v)}|x-y|^\beta} &\lesssim \frac{|M_w(x,v)-M_w(y,v)|}{\sqrt{\mu(v)}|x-y|} \leqq \Big\|\frac{\nabla_x M_w(x,v)}{\sqrt{\mu(v)}}\Big\|_{L^\infty_x} \\
&\lesssim_{T_0} \Big\|\nabla_x T_w|v|^2\frac{M_w(x)}{\sqrt{\mu(v)}}\Big\|_{L^\infty_x} \lesssim w_{\tilde{\theta}}^{-1}(v)\|T_w - T_0\|_{C^1}.
\end{aligned}
$$

□

Proof of (2.72) From (1.6) it is equivalent to compute

$$
\begin{aligned}
&\frac{|\partial_j\eta_{p(x)}(\mathbf{x}_{p(x)})\nabla_x\mathbf{x}^1_{p^1(x),i} - \partial_j\eta_{p(y)}(\mathbf{x}_{p(y)})\nabla_x\mathbf{x}^1_{p^1(y),i}|}{|x-y|^\beta} \\
&\lesssim \frac{|\partial_j\eta_{p(x)}(\mathbf{x}_{p(x)}) - \partial_j\eta_{p(y)}(\mathbf{x}_{p(y)})||\nabla_x\mathbf{x}^1_{p^1(x),i}|}{|x-y|^\beta} \\
&\quad + \frac{\big|\partial_j\eta_{p(x)}(\mathbf{x}_{p(x)})[\nabla_x\mathbf{x}^1_{p^1(x),i} - \nabla_x\mathbf{x}^1_{p^1(y),i}]\big|}{|x-y|^\beta} \\
&\lesssim \frac{\|\eta\|_{C^2}|v|}{\alpha(x,v)} + \|\eta\|_{C^1}\frac{|\nabla_x\mathbf{x}^1_{p^1(x),i} - \nabla_x\mathbf{x}^1_{p^1(y),i}|}{|x-y|^\beta},
\end{aligned}
$$

where we have used (2.34). Denote $x_{\mathbf{b}}(x(\tau),v) = \eta_{p^1(x(\tau))}(\mathbf{x}^1_{p^1(x(\tau))})$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\nabla_x \mathbf{x}^1_{p^1(x(\tau)),i}\big) &= \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\nabla_{x(\tau)}\nabla_x \mathbf{x}^1_{p^1(x(\tau)),i}\\ &= \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\nabla_{x(\tau)}\mathbf{x}^1_{p^1(x(\tau))}\nabla_{\mathbf{x}^1_{p^1(x(\tau))}}\nabla_x \mathbf{x}^1_{p^1(x(\tau)),i}.\end{aligned}$$

Applying (2.33), we further bound

$$\begin{aligned}&\frac{|\nabla_x \mathbf{x}^1_{p^1(x),i} - \nabla_x \mathbf{x}^1_{p^1(y),i}|}{|x-y|^\beta}\\ &= \frac{1}{|x-y|^\beta}\Big|\int_0^1 \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\nabla_x \mathbf{x}^1_{p^1(x(\tau)),i}\big)\Big|\\ &= \frac{1}{|x-y|^\beta}\int_0^1 \mathrm{d}\tau |\dot{x}(\tau)||\nabla_{x(\tau)}\mathbf{x}^1_{p^1(x(\tau))}\nabla_{\mathbf{x}^1_{p^1(x(\tau))}}\nabla_x \mathbf{x}^1_{p^1(x(\tau)),i}|\\ &\lesssim \frac{|\nabla_x \mathbf{x}^1_{p^1(x(\tau))}|}{g_{p^1(x(\tau)),ii}(\mathbf{x}^1_{p^1(x(\tau))})}\|\eta\|_{C^2}\Big[\frac{1}{g_{p^1(x(\tau)),ii}(\mathbf{x}^1_{p^1(x(\tau))})}\\ &\quad + \frac{|v|^2}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|^2 g_{p^1(x(\tau)),33}(\mathbf{x}^1_{p^1(x(\tau))})}\Big]\\ &\lesssim_\eta \frac{|v|^3}{\alpha^3(x(\tau),v)} \lesssim \frac{1}{\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^3}.\end{aligned}$$

In the third line we have applied the derivative to (2.33). In the fourth line we have used (1.11) and $|\nabla_x \mathbf{x}^1_{p^1}| \lesssim \frac{|v|}{|n(x_{\mathbf{b}}(x(\tau),v))\cdot v|}$ from (2.33). □

Lemma 2.10. *For any $s \in [0, \min(t_{\mathbf{b}}(x,v), t_{\mathbf{b}}(y,v))]$, we have*

$$\begin{aligned}&\Big|\frac{G(x)\nabla_x f_s(x-sv,v) - G(y)\nabla_x f_s(y-sv,v)}{|x-y|^\beta}\Big|\\ &\lesssim \Big|\frac{\nabla_\parallel f_s(x-sv,v) - \nabla_\parallel f_s(y-sv,v)}{|x-y|^\beta}\Big|\\ &\quad + \frac{\tilde{\alpha}(x,v)}{|v|}\Big|\frac{\nabla_x f_s(x-sv,v) - \nabla_x f_s(y-sv,v)}{|x-y|^\beta}\Big| + \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f_s\|_\infty}{w_{\tilde{\theta}}(v)\alpha(y-sv,v)}.\end{aligned} \tag{2.90}$$

Proof. First we rewrite

$$\begin{aligned}&G(x)\nabla_x f_s(x-sv,v) - G(y)\nabla_x f_s(y-sv,v)\\ &\quad = G(x-sv)\nabla_x f_s(x-sv,v) - G(y-sv)\nabla_x f_s(y-sv,v) &(2.91)\\ &\quad + \big(G(x)-G(x-sv)\big)\nabla_x f_s(x-sv,v) &(2.92)\\ &\quad + \big(G(y-sv)-G(y)\big)\nabla_x f_s(y-sv,v). &(2.93)\end{aligned}$$

Note that from (1.13) the contribution of (2.91) appears in (2.90) .

For (2.92) and (2.93) we apply (2.48) and rearrange terms to derive that

$$
\begin{aligned}
&|[G(x)-G(x-sv)]\nabla_x f_s(x-sv,v)-[G(y)-G(y-sv)]\nabla_x f_s(y-sv,v)|\\
&\quad=|[G(x)-G(x-sv)][\nabla_x f_s(x-sv,v)-\nabla_x f_s(y-sv,v)]\\
&\qquad+[G(x)-G(y)+G(y-sv)-G(x-sv)]\nabla_x f_s(y-sv,v)|\\
&\quad\lesssim \frac{\tilde{\alpha}(x,v)}{|v|}|\nabla_x f_s(x-sv,v)-\nabla_x f_s(y-sv,v)|\\
&\qquad+|n(x)-n(y)+n(y-sv)-n(x-sv)|\frac{\|\alpha\nabla_x f_s\|_\infty}{\alpha(y-sv,u)}.
\end{aligned}
\tag{2.94}
$$

Applying mean value theorem to $n(x)-n(y)$, $n(x-sv)-n(x-sv)$ with (2.50), we conclude the lemma. □

2.5. *Properties of boundary condition and collision operators*

In this subsection we list some properties of the boundary condition and collision operators. We summarize the property of diffuse boundary condition in Lemma 2.11. The property of the collision operator is summarized in Lemma 2.12 and Lemma 2.13.

Lemma 2.11. *For the diffuse boundary condition of f in (2.4), let $x_{\mathbf{b}}(x,v)=\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)})\in\partial\Omega$ (see (7.4)), we have*

$$
\|r\|_\infty<\infty,\quad |\partial_{\mathbf{x}^1_{p^1,i}} r(\eta_{p^1}(\mathbf{x}^1_{p^1}),v)|\lesssim \|T_W-T_0\|_{C^1},
\tag{2.95}
$$

$$
\begin{aligned}
&||v|^2\nabla_v[r(x_{\mathbf{b}}(x,v),v)]|\lesssim \|T_W-T_0\|_{C^1},\\
&\left|w_{\tilde{\theta}}(v)|v|^2\frac{\partial_{\mathbf{x}^1_{p^1(x),i}} r(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v)-\partial_{\mathbf{x}^1_{p^1(y),i}} r(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}),v)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\right|
\end{aligned}
\tag{2.96}
$$

$$
\begin{aligned}
&\lesssim \|T_W-T_0\|_{C^2},\\
&\left|w_{\tilde{\theta}}(v)|v|^2\frac{\partial_{\mathbf{x}^1_{p^1(x),i}} M_W(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v)-\partial_{\mathbf{x}^1_{p^1(y),i}} M_W(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}),v)}{\sqrt{\mu(v)}|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\right|\\
&\lesssim \|T_W-T_0\|_{C^2},
\end{aligned}
\tag{2.97}
$$

$$
\begin{aligned}
&\frac{1}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\
&\quad\times\left|\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0} f(x_{\mathbf{b}}(x,v),v^1)\sqrt{\mu(v^1)}\{n(x_{\mathbf{b}}(x,v))\cdot v^1\}\mathrm{d}v^1\right.\\
&\qquad\left.-\int_{n(x_{\mathbf{b}}(y,v))\cdot v^1>0} f(x_{\mathbf{b}}(y,v),v^1)\sqrt{\mu(v^1)}\{n(x_{\mathbf{b}}(y,v))\cdot v^1\}\mathrm{d}v^1\right|\\
&\quad\lesssim \|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty.
\end{aligned}
\tag{2.98}
$$

Proof. From (2.6), it is straightforward to derive the estimate for $\|r\|_\infty$. We take derivative to r to obtain

$$
|\partial_{\mathbf{x}^1_{p^1,i}} r(\eta_{p^1}(\mathbf{x}^1_{p^1}),v)|
$$

$$
\begin{aligned}
&= \Big| \frac{\partial_i \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{2\pi\mu(v)}} \nabla_x \Big(\frac{1}{2\pi[T_W(x_{\mathbf{b}}(x,v))]^2} e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \Big) \Big| \\
&= \Big| \frac{\partial_i \eta_{p^1}(\mathbf{x}^1_{p^1}) \nabla_x T_W(x_{\mathbf{b}}(x,v))}{\sqrt{2\pi\mu}} \Big(\frac{-1}{\pi[T_W(x_{\mathbf{b}}(x,v))]^3} + \frac{|v|^2}{4\pi[T_W(x_{\mathbf{b}}(x,v))]^4} \Big) \\
&\quad \times e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \Big| \\
&\lesssim_{T_0} \|T_W - T_0\|_{C^1}, \qquad (2.99)
\end{aligned}
$$

where we have used $\|T_W - T_0\|_\infty \ll 1$ from **Existence Theorem**. Here we note that the ∇_x above represents the partial derivative.

Then we take v derivative to have

$$
\begin{aligned}
&|\nabla_v r(x_{\mathbf{b}}(x,v),v)| \\
&= |\nabla_v \frac{M_W}{\sqrt{2\pi\mu}} - \nabla_v \sqrt{\mu}| \lesssim \Big| \nabla_v \frac{e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}}}{\sqrt{\mu(v)}[T_W(x_{\mathbf{b}}(x,v))]^2} \Big| \\
&\lesssim \|T_W - T_0\|_{C^1} \frac{1}{\mu(v)} \times \Big| \nabla_v e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \sqrt{\mu(v)} \\
&\quad + \nabla_v [\sqrt{\mu(v)} T_W^2(x_{\mathbf{b}}(x,v))] e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \Big| \\
&\lesssim \|T_W - T_0\|_{C^1} \frac{1}{\mu(v)} \times e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \sqrt{\mu(v)} |v|^2 |\nabla_v x_{\mathbf{b}}(x,v)| \\
&\lesssim \|T_W - T_0\|_{C^1} \frac{e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} |v|}{\sqrt{\mu(v)}}, \qquad (2.100)
\end{aligned}
$$

where we have used (2.40) in the last line. Since the coefficient for $|v|^2$ in exponent is negative, we conclude (2.95).

For (2.96) from (2.99) we apply the mean value theorem to bound

$$
\begin{aligned}
&\frac{|\partial_i \eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) - \partial_i \eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \lesssim \|\eta\|_{C^2}, \\
&\frac{|\nabla_x T_W(x_{\mathbf{b}}(x,v)) - \nabla_x T_W(x_{\mathbf{b}}(y,v))|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \lesssim \|\nabla_x^2 T_W\|_\infty, \\
&w_{\tilde\theta}(v)|v|^2 \frac{\big| e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} - e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(y,v))}} \big|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \lesssim \|\nabla_x T_W\|_\infty, \\
&w_{\tilde\theta}(v)|v|^2 \Big| \frac{\frac{-1}{\pi[T_W(x_{\mathbf{b}}(x,v))]^3} + \frac{1}{\pi[T_W(x_{\mathbf{b}}(y,v))]^3} + \frac{|v|^2}{4\pi[T_W(x_{\mathbf{b}}(x,v))]^4} - \frac{|v|^2}{4\pi[T_W(x_{\mathbf{b}}(x,v))]^4}}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \Big| \\
&\times e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \\
&\lesssim \|T_W\|_{C^1} |v|^4 e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}} \lesssim \|\nabla_x^2 T_W\|_\infty,
\end{aligned}
$$

and thus (2.96) follows from $\tilde{\theta} \ll \frac{1}{T_W(x)}$.

Since $\partial_{\mathbf{x}^1_{p^1,i}} r(\eta_{p^1}(\mathbf{x}^1_{p^1}), v) = \frac{1}{\sqrt{2\pi}} \partial_{\mathbf{x}^1_{p^1,i}} M_W(\eta_{p^1}(\mathbf{x}^1_{p^1}), v)$, (2.97) also follows.

Last we prove (2.98). We rewrite the LHS of (2.98) as

$$\frac{1}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \times \Big[\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1 > 0, n(x_{\mathbf{b}}(y,v))\cdot v^1 > 0} \mathrm{d}v^1 \\ f(x_{\mathbf{b}}(x,v), v^1)\sqrt{\mu(v^1)} |n(x_{\mathbf{b}}(x,v))\cdot v^1| - f(x_{\mathbf{b}}(y,v), v^1)\sqrt{\mu(v^1)} |n(x_{\mathbf{b}}(y,v))\cdot v^1| \Big] \tag{2.101}$$

$$+ \frac{\int_{|n(x_{\mathbf{b}}(x,v)) - n(x_{\mathbf{b}}(y,v))| \geqq \frac{n(x_{\mathbf{b}}(x,v))\cdot v^1}{|v^1|} > 0} f(x_{\mathbf{b}}(x,v), v^1)\sqrt{\mu(v^1)} |n(x_{\mathbf{b}}(x,v))\cdot v^1|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\ + \frac{\int_{|n(x_{\mathbf{b}}(x,v)) - n(x_{\mathbf{b}}(y,v))| \geqq \frac{n(x_{\mathbf{b}}(x,v))\cdot v^1}{|v^1|} > 0} f(x_{\mathbf{b}}(y,v), v^1)\sqrt{\mu(v^1)} |n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}. \tag{2.102}$$

Clearly from (2.64) and (2.57), we have

$$|(2.101)| \lesssim \int_{n(x_{\mathbf{b}}(x,v))\cdot v^1 > 0, n(x_{\mathbf{b}}(y,v))\cdot v^1 > 0} \mathrm{d}v^1 \\ \Big[\frac{\|wf\|_\infty^{1-\beta} \|w_{\tilde{\theta}} \alpha \nabla_x f\|_\infty^\beta \sqrt{\mu(v^1)}}{\min\big\{\alpha(x_{\mathbf{b}}(x,v), v^1), \alpha(x_{\mathbf{b}}(y,v), v^1)\big\}^\beta} + \|\eta\|_{C^2} \|wf\|_\infty \sqrt{\mu(v^1)} \Big] \\ \lesssim \|w_{\tilde{\theta}} \alpha \nabla_x f\|_\infty + \|wf\|_\infty,$$

where we have used Young's inequality with $1 - \beta + \beta = 1$ and definition of α in (1.8) with $\beta < 1$.

For (2.102), from (2.57) we bound

$$\frac{\int_{|n(x_{\mathbf{b}}(x,v)) - n(x_{\mathbf{b}}(y,v))| \geqq \frac{n(x_{\mathbf{b}}(x,v))\cdot v^1}{|v^1|} > 0} f(x_{\mathbf{b}}(x,v), v^1)\sqrt{\mu(v^1)} |n(x)\cdot v^1|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\ \lesssim \frac{|n(x_{\mathbf{b}}(x,v)) - n(x_{\mathbf{b}}(y,v))|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \int f(x_{\mathbf{b}}(x,v), v^1)\sqrt{\mu(v^1)} |v^1| \lesssim \|wf\|_\infty.$$

Then we conclude the lemma. □

Aside from the boundary condition, we also need to estimate the collision operator. The next two lemmas describe the properties of the collision operator K and Γ.

Lemma 2.12. *The linear Boltzmann operator $K(f)$ in (2.5) is given by*

$$Kf(x,v) = \int_{\mathbb{R}^3} \mathbf{k}(v,u) f(x,u) \mathrm{d}u.$$

The kernel $\mathbf{k}(v,u)$ satisfies:

$$|\mathbf{k}(v,u)| \lesssim \mathbf{k}_{\varrho}(v,u), \ \ |\nabla_u \mathbf{k}(v,u)|$$

$$\lesssim \langle u\rangle \mathbf{k}_\varrho(v,u)/|v-u|,\ \mathbf{k}_\varrho(v,u) := e^{-\varrho|v-u|^2}/|v-u|. \tag{2.103}$$

And for $3 > c \geqq 0$,

$$\int_{\mathbb{R}^3} \mathbf{k}_\varrho(v,u)\frac{1}{|u|^c}\mathrm{d}u \lesssim \frac{1}{|v|^c}. \tag{2.104}$$

Moreover, for the operator ν *and* Γ *in* (2.5), *we have*

$$|K(f)+\Gamma(f,f)| = O(1)\|wf\|_\infty, \tag{2.105}$$

$$\nu = O(\sqrt{|v|^2+1}) \gtrsim 1, \quad |\nabla_v \nu| \lesssim 1, \tag{2.106}$$

$$|\nabla_v \Gamma(f,f)| \lesssim \frac{\|wf\|_\infty^2}{|v|^2} + \frac{\|wf\|_\infty \||v|^2\nabla_v f\|_\infty}{|v|^2}, \tag{2.107}$$

$$|\nabla_x \Gamma(f,f)(v)| = O(\|wf\|_\infty)\Big\{|\nabla_x f(v)| + \int_{\mathbb{R}^3} \mathbf{k}_\varrho(v,u)|\nabla_x f(u)|\mathrm{d}u\Big\}, \tag{2.108}$$

$$|G(x)\nabla_x \Gamma(f,f)(x,v)| = O(\|wf\|_\infty)\{|G(x)\nabla_x f(x,v)| + \int_{\mathbb{R}^3} \mathbf{k}_\varrho(v,u)|G(x)\nabla_x f(x,u)|\}. \tag{2.109}$$

Proof of (2.103) We define

$$\Gamma_{\text{gain}}(f_1,f_2) = \frac{Q_{\text{gain}}(\sqrt{\mu}f_1,\sqrt{\mu}f_2)}{\sqrt{\mu}}, \quad \Gamma_{\text{loss}}(f_1,f_2) = \frac{Q_{\text{loss}}(\sqrt{\mu}f_1,\sqrt{\mu}f_2)}{\sqrt{\mu}}, \tag{2.110}$$

where Q_{gain}, Q_{loss} are defined in (1.2).

By the Grad estimate in [12],

$$\Gamma_{\text{gain}}(\sqrt{\mu},f) + \Gamma_{\text{gain}}(f,\sqrt{\mu}) = \int_{\mathbb{R}^3} \mathbf{k}_2(v,u)f(u)\mathrm{d}u, \tag{2.111}$$

$$\nu(\sqrt{\mu}f) = \int_{\mathbb{R}^3} \mathbf{k}_1(v,u)f(u)\mathrm{d}u,$$

where

$$\mathbf{k}_1(v,u) = C_{\mathbf{k}_1}|u-v|e^{-\frac{|v|^2+|u|^2}{2}}, \tag{2.112}$$

$$\mathbf{k}_2(v,u) = C_{\mathbf{k}_2}\frac{1}{|u-v|}e^{-\frac{1}{4}|u-v|^2-\frac{1}{4}\frac{(|u|^2-|v|^2)^2}{|u-v|^2}}. \tag{2.113}$$

We compute the derivative:

$$|\nabla_u \mathbf{k}_1(v,u)| \lesssim e^{-\frac{|u|^2+|v|^2}{2}} + |u||u-v|e^{-\frac{|u|^2+|v|^2}{2}} \lesssim e^{-\frac{|v-u|^2}{4}} \lesssim \frac{e^{-\varrho|v-u|^2}}{|v-u|^2}.$$

And

$$|\nabla_u \mathbf{k}_2(v,u)| \lesssim \frac{1}{|v-u|^2}e^{-\frac{1}{4}|v-u|^2} + \frac{1}{|v-u|}e^{-\frac{1}{4}|v-u|^2-\frac{1}{4}\frac{(|u|^2-|v|^2)^2}{|v-u|^2}}$$

$$\times\left[|v-u|+\frac{|u|\Big||u|^2-|v|^2\Big||v-u|^2-(|u|^2-|v|^2)^2|v-u|}{|v-u|^4}\right]$$
$$\lesssim \frac{e^{-\varrho|v-u|^2}\langle u\rangle}{|v-u|^2},$$

where we have used

$$e^{-\frac{1}{4}\frac{(|u|^2-|v|^2)^2}{|v-u|^2}}\frac{|u|\Big||u|^2-|v|^2\Big||v-u|^2}{|v-u|^4}\lesssim\frac{|u|}{|v-u|},$$
$$e^{-\frac{1}{4}\frac{(|u|^2-|v|^2)^2}{|v-u|^2}}\frac{(|u|^2-|v|^2)^2|v-u|}{|v-u|^4}\lesssim\frac{1}{|v-u|}.$$

□

Proof of (2.104) We consider two cases. When $|u|>\frac{|v|}{2}$, we have

$$\int_{|u|>\frac{|v|}{2}}\mathbf{k}_\varrho(v,u)\frac{1}{|u|^c}\mathrm{d}u\lesssim\int_{|u|>\frac{|v|}{2}}\frac{e^{-\varrho|v-u|^2}}{|v-u|}\frac{1}{|u|^c}\mathrm{d}u$$
$$\lesssim\frac{1}{|v|^c}\int_{\mathbb{R}^3}\frac{e^{-\varrho|v-u|^2}}{|v-u|}\mathrm{d}u\lesssim\frac{1}{|v|^c}.$$

When $|u|\leqq\frac{|v|}{2}$ we bound $|v-u|\geqq\frac{|v|}{2}$, and thus

$$\int_{|u|\leqq\frac{|v|}{2}}\mathbf{k}(v,u)\frac{1}{|u|^c}\mathrm{d}u\lesssim\frac{e^{-\varrho|v|^2/2}}{|v|}\int_{|u|\leqq\frac{|v|}{2}}\frac{1}{|u|^c}\mathrm{d}u$$
$$\lesssim\frac{e^{-\varrho|v|^2/2}}{|v|}\int_{0\leqq r\leqq\frac{|v|}{2}}\int_{\partial B(0,r)}\mathrm{d}S\frac{1}{|r|^c}\mathrm{d}r$$
$$\lesssim\frac{e^{-\varrho|v|^2/2}}{|v|}|v|^{3-c}\lesssim\frac{e^{-\varrho|v|^2/2}|v|^2}{|v|^c}\lesssim\frac{1}{|v|^c}.$$

In the second line we used the polar coordinate with $|u|=r$. In the third line we used $c<3$ to compute the r integral.

Then we conclude (2.104). □

Proof of (2.105) For $K(f)$ we bound

$$K(f)\lesssim\|f\|_\infty\int_{\mathbb{R}^3}|\mathbf{k}(v,u)|\mathrm{d}u\lesssim\|f\|_\infty\lesssim\|wf\|_\infty,$$

where we have used $|\mathbf{k}(v,u)|\lesssim\mathbf{k}_\varrho(v,u)\in L^1_u$.

For Γ, clearly

$$|\Gamma_{\text{gain}}(f,f)|\lesssim|\Gamma_{\text{gain}}(e^{-\varrho|v|^2},|f|)|\times\|wf\|_\infty. \tag{2.114}$$

By (2.111) we bound $|\Gamma_{\text{gain}}(e^{-\varrho|v|^2},|f|)|$ using different exponent of $\mathbf{k}_2(v,u)$, we conclude that

$$\Gamma_{\text{gain}}(f,f)\lesssim\|wf\|_\infty^2\lesssim\|wf\|_\infty.$$

For the other term we bound

$$|\nu(\sqrt{\mu}f)f(v)| \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} |v-u| e^{-\varrho|v|^2} \sqrt{\mu(u)}|f(u)| \\ \lesssim \|wf\|_\infty^2 \int_{\mathbb{R}^3} |v-u| e^{-C|v-u|^2} \lesssim \|wf\|_\infty, \tag{2.115}$$

where we have used

$$e^{-\varrho|v|^2} e^{-\varrho|u|^2} \lesssim e^{-C(|v|^2+|u|^2)} \lesssim e^{-\frac{C}{2}|v-u|^2}. \qquad \square$$

The proof for (2.106) is standard (see Chapter 3 in [12]).

Proof of (2.107) The velocity derivative for the nonlinear Boltzmann operator reads

$$\nabla_v \Gamma(f,f) = \nabla_v \left(\Gamma_{\text{gain}}(f,f) - \Gamma_{\text{loss}}(f,f)\right) \\ = \Gamma_{\text{gain}}(\nabla_v f, f) + \Gamma_{\text{gain}}(f, \nabla_v f) - \Gamma_{\text{loss}}(\nabla_v f, f) - \Gamma_{\text{loss}}(f, \nabla_v f) \tag{2.116} \\ + \Gamma_{v,\text{gain}}(f,f) - \Gamma_{v,\text{loss}}(f,f). \tag{2.117}$$

Here we have defined

$$\Gamma_{v,\text{gain}}(f,f) - \Gamma_{v,\text{loss}}(f,f) \\ := \int_{\mathbb{R}^3}\int_{\S^2} |u\cdot\omega| f(v+u_\perp) f(v+u_\parallel) \nabla_v \sqrt{\mu(v+u)} \mathrm{d}\omega \mathrm{d}u \\ - \int_{\mathbb{R}^3}\int_{\S^2} |u\cdot\omega| f(v+u) f(v) \nabla_v \sqrt{\mu(v+u)} \mathrm{d}\omega \mathrm{d}u. \tag{2.118}$$

From (2.114) we have

$$|\Gamma_{\text{gain}}(f, \partial_v f) + \Gamma_{\text{gain}}(\partial_v f, f)| \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} |\partial_v f| \mathrm{d}u. \tag{2.119}$$

For $|\nu(\sqrt{\mu}\partial_v f)f(v)|$ we have

$$|\nu(\sqrt{\mu}\partial_v f)f(v)| \lesssim \|wf\|_\infty e^{-\varrho|v|^2} \nu(\sqrt{\mu}\partial_v f)(v) \\ \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} |v-u| e^{-\varrho|v|^2} \sqrt{\mu(u)} |\partial_v f(u)| \\ \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} |\partial_v f(u)| \mathrm{d}u, \tag{2.120}$$

where we have used $e^{-\varrho|v|^2}|v-u|\sqrt{\mu(u)} \lesssim \frac{e^{-\varrho|v-u|^2}}{|v-u|}$.

Then we combine (2.119) and (2.120), and use (2.104) with $c = 2$ to conclude

$$(2.116) \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} |\nabla_v f| \\ \lesssim \|wf\|_\infty \||v|^2 \nabla_v f\|_\infty \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} \frac{1}{|u|^2}$$

$$\lesssim \|wf\|_\infty \frac{\||v|^2\nabla_v f\|_\infty}{|v|^2}.$$

Then we further compute

$$\begin{aligned}(2.117) &\lesssim \|wf\|_\infty^2 \int_{\mathbb{R}^3} |u|[e^{-\varrho|v+u_\perp|^2} e^{-\varrho|v+u_\parallel|^2} + e^{-\varrho|v+u|^2} e^{-\varrho|v|^2}] e^{-\frac{|v+u|^2}{2}} \\ &\lesssim \|wf\|_\infty^2 \int_{\mathbb{R}^3} |u| e^{-c|v|^2} e^{-c|u|^2} \mathrm{d}u \\ &\lesssim \|wf\|_\infty^2 \frac{|v|^2 e^{-c|v|^2}}{|v|^2} \lesssim \frac{\|wf\|_\infty^2}{|v|^2},\end{aligned}$$

where we have used

$$\begin{aligned}e^{-\varrho|v+u_\perp|^2} e^{-\varrho|v+u_\parallel|^2} &= e^{-(\varrho|v|^2+2v\cdot(u_\perp+u_\parallel)+\varrho|u|^2)} e^{-\varrho|v|^2} \\ &= e^{-\varrho|v+u|^2} e^{-\varrho|v|^2},\end{aligned}$$

and

$$\begin{aligned}e^{-\varrho|v+u|^2} e^{-\varrho|v|^2} &= e^{-\varrho|v|^2/2} e^{-\varrho(3|v|^2/2+2v\cdot u+|u|^2)} \\ &= e^{-\varrho|v|^2/2} e^{-\varrho(\sqrt{3/2}v+\sqrt{2/3}u)^2} e^{-u^2/3}.\end{aligned}$$

□

Proof of (2.108) Replacing the ∇_v by ∇_x in (2.119) and (2.120), we have

$$|\Gamma_{\text{gain}}(f, \partial_x f) + \Gamma_{\text{gain}}(\partial_x f, f)| \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} |\partial_x f| \mathrm{d}u,$$

$$|\nu(\sqrt{\mu}\partial_x f) f(v)| \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} |\partial_x f(u)| \mathrm{d}u.$$

□

Proof of (2.109) Since

$$\begin{aligned}G(x)\nabla_x \Gamma(f, f)(x, v) &= G(x)\Gamma(\nabla_x f, f) + G(x)\Gamma(f, \nabla_x f) \\ &= \Gamma(G(x)\nabla_x f, f) + \Gamma(f, G(x)\nabla_x f),\end{aligned} \tag{2.121}$$

from (2.108), we conclude (2.109). □

Lemma 2.13. *If* $0 < \frac{\tilde\theta}{4} < \varrho$ *and if* $0 < \tilde\varrho < \varrho - \frac{\tilde\theta}{4}$,

$$\mathbf{k} \frac{e^{\tilde\theta|v|^2}}{e^{\tilde\theta|u|^2}} \lesssim \mathbf{k}_{\tilde\varrho}(v, u), \tag{2.122}$$

where $\mathbf{k}$ *is defined in* (2.103). *Also,*

$$|\nabla_v \mathbf{k}(v, u)| \frac{e^{\tilde\theta|v|^2}}{e^{\tilde\theta|u|^2}} \lesssim \frac{\langle v\rangle \mathbf{k}_{\tilde\varrho}(v, u)}{|v-u|}. \tag{2.123}$$

As a consequence, when $\bar\theta \ll \varrho$, *we have*

$$w_{\bar\theta}(v)|K(f) + \Gamma(f, f)| \lesssim \|w_{\bar\theta} f\|_\infty \leqq \|wf\|_\infty. \tag{2.124}$$

Proof of (2.122) Note that the $\mathbf{k}$ in (2.103) equals to $\mathbf{k}_1(v,u)+\mathbf{k}_2(v,u)$ in (2.112), (2.113), then

$$\mathbf{k}(v,u)\frac{e^{\tilde{\theta}|v|^2}}{e^{\tilde{\theta}|u|^2}} \lesssim \frac{1}{|v-u|}\exp\left\{-\varrho|v-u|^2-\varrho\frac{||v|^2-|u|^2|^2}{|v-u|^2}+\tilde{\theta}|v|^2-\tilde{\theta}|u|^2\right\}.$$

Let $v-u=\eta$ and $u=v-\eta$. Then the exponent equals

$$\begin{aligned}
&-\varrho|\eta|^2-\varrho\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\tilde{\theta}\{|v-\eta|^2-|v|^2\}\\
&=-2\varrho|\eta|^2+4\varrho v\cdot\eta-4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}-\tilde{\theta}\{|\eta|^2-2v\cdot\eta\}\\
&=(-2\varrho-\tilde{\theta})|\eta|^2+(4\varrho+\bar{\theta})v\cdot\eta-4\varrho\frac{\{v\cdot\eta\}^2}{|\eta|^2}.
\end{aligned}$$

If $0<\tilde{\theta}<4\varrho$ then the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is

$$(4\varrho+\bar{\theta})^2-4(-2\varrho-\tilde{\theta})(-4\varrho)=4\tilde{\theta}^2-16\varrho\tilde{\theta}<0.$$

Hence, the quadratic form is negative definite. We thus have, for $0<\tilde{\varrho}<\varrho-\frac{\tilde{\theta}}{4}$, the following perturbed quadratic form is still negative definite: $-(\varrho-\tilde{\varrho})|\eta|^2-(\varrho-\tilde{\varrho})\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\tilde{\theta}\{|\eta|^2-2v\cdot\eta\}\leqq 0$. □

Proof of (2.123) Taking the derivative to (2.112) and (2.113) we have

$$\begin{aligned}
\nabla_v\mathbf{k}_1(v,u)&=\frac{v-u}{|v-u|}\mathbf{k}_1(v,u)-v\mathbf{k}_1(v,u),\\
\nabla_v\mathbf{k}_2(v,u)&=\frac{v-u}{|v-u|^2}\mathbf{k}_2(v,u)\\
&\quad-\mathbf{k}_2(v,u)\Big[\frac{v-u}{2}+\frac{v(|u|^2-|v|^2)|v-u|^2-(|u|-|v|^2)^2|v-u|}{2|v-u|^4}\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
|\nabla_v\mathbf{k}(v,u)\frac{e^{\tilde{\theta}|v|^2}}{e^{\tilde{\theta}|u|^2}}| &\lesssim \Big[\frac{1}{|v-u|}+|v-u|+\langle v\rangle\Big][\mathbf{k}_1(v,u)+\mathbf{k}_2(v,u)]\frac{e^{\tilde{\theta}|v|^2}}{e^{\tilde{\theta}|u|^2}}\\
&\lesssim \Big[\frac{1}{|v-u|}+|v-u|+\langle v\rangle\Big]\mathbf{k}_{\tilde{\varrho}}(v,u)\lesssim\frac{\langle v\rangle\mathbf{k}_c(v,u)}{|v-u|}
\end{aligned}$$

for $c<\tilde{\varrho}$. In the last line we have applied (2.122). □

Proof of (2.124) By Lemma 2.12, we have

$$\begin{aligned}
|w_{\bar{\theta}}(v)Kf| &\lesssim \int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)\frac{w_{\bar{\theta}}(v)}{w_{\bar{\theta}}(u)}w_{\bar{\theta}}(u)|f(u)|\mathrm{d}u\\
&\lesssim \|w_{\bar{\theta}}f\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\tilde{\varrho}}(v,u)\mathrm{d}u\lesssim\|w_{\bar{\theta}}f\|_\infty,
\end{aligned}\tag{2.125}$$

where we have used $\mathbf{k}_{\tilde{\varrho}}(v,u)\in L^1_u$.

For Γ, we follow (2.114) and have

$$w_{\bar\theta}(v)|\Gamma_{\text{gain}}(f,f)| \lesssim \|wf\|_\infty w_{\bar\theta}(v)|\Gamma_{\text{gain}}(e^{-\theta|v|^2},f)| \lesssim \|w_{\bar\theta}f\|_\infty,$$

here we bound $|\Gamma_{\text{gain}}(e^{-\theta|v|^2},f)|$ using different exponent of $\mathbf{k}_2(v,u)$ in (2.111), and we apply the same computation as (2.125) to have

$$w_{\bar\theta}(v)|\Gamma_{\text{gain}}(e^{-\theta|v|^2},f)| \lesssim w_{\bar\theta}(v)\int_{\mathbb{R}^3}\mathbf{k}_\varrho(v,u)|f(u)| \lesssim \|w_{\bar\theta}f\|_\infty.$$

For the other term we follow (2.115) to have

$$w_{\bar\theta}(v)|\nu(\sqrt{\mu}f)f(v)| \lesssim \|wf\|_\infty \int_{\mathbb{R}^3}|v-u|e^{-(\varrho-\bar\theta)|v|^2}\sqrt{\mu(u)}|f(u)| \lesssim \|w_{\bar\theta}f\|_\infty.$$

□

3. Differentiation Along the Stochastic Cycles: Mixing via Diffuse Reflection and Transport

The main purpose of this section is to provide crucial differentiation form of the transport equation with the diffuse reflection boundary condition, which will be stated in Proposition 1. Several geometric integration by parts will be employed as being described in Section 1.3.

Consider a sequence of linear transport equation for $\ell \geqq 1$ with the inflow boundary condition

$$v\cdot\nabla_x f^\ell + \nu(v)f^\ell = h^\ell(x,v), \quad (x,v)\in\Omega\times\mathbb{R}^3, \tag{3.1}$$

$$f^\ell(x,v) = g^\ell(x,v), \quad (x,v)\in\gamma_-. \tag{3.2}$$

Here we set $f^0 = 0$.

Later we will substitute the h^ℓ by the sequence of collision operator defined as

$$h^\ell(x,v) := K(f^{\ell-1}) + \Gamma(f^{\ell-1},f^{\ell-1}), \tag{3.3}$$

and g^ℓ by the sequence of boundary condition:

$$f^\ell(x,v)|_{\gamma_-} = \frac{M_W(x,v)}{\sqrt{\mu(v)}}\int_{n(x)\cdot v^1>0} f^{\ell-1}(x,v^1)\sqrt{\mu(v^1)}\{n(x)\cdot v^1\}\mathrm{d}v^1 + r(x,v), \tag{3.4}$$

where $r(x,v)$ is defined in (2.6).

Note that from the collision operator (3.3) and boundary condition (3.4) and $f^0 = 0$,

$$h^1(x,v) = 0, \quad g^1(x,v) = r(x,v).$$

We have the following expansion:

Proposition 1. *Suppose f solves inhomogeneous steady transport equation (3.1) with the diffuse BC (3.4). Then*

$$
\begin{aligned}
& w_{\tilde{\theta}}(v)\partial_{x_i} f^{\ell}(x,v) \\
&= O(1) w_{\tilde{\theta}}(v) \frac{n_i(x^1)}{\alpha(x,v)} \\
&\quad \times \left\{ \frac{\nu(v)}{w_{\bar{\theta}}(v)} \|w_{\bar{\theta}} f^{\ell}\|_{L^\infty(\partial\Omega)} + |v| \frac{M_W(x^1,v)}{\sqrt{\mu(v)}} \|w_{\bar{\theta}} f^{\ell-1}\|_{L^\infty(\partial\Omega)} \right. \qquad (3.5) \\
&\quad \left. + |v| |\underline{\nabla}_{x^1} r(x^1)| (1 + \|w_{\bar{\theta}} f^{\ell-1}\|_{L^\infty(\partial\Omega)}) \right\} \\
&\quad + \frac{O(1)}{\alpha(x,v)} e^{-\nu(v)t} (w_{\tilde{\theta}} \alpha \partial_{x_i} f^{\ell})(x - tv, v) \qquad (3.6) \\
&\quad + \int_{\max\{0, t - t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} w_{\tilde{\theta}}(v) \partial_{x_i} \underbrace{h^{\ell}(x - (t-s)v, v)}_{(3.7)_*} \, \mathrm{d}s \qquad (3.7) \\
&\quad + O(1) e^{-\nu(v) t_{\mathbf{b}}} \frac{n_i(x^1)|v|}{\alpha(x,v)} \frac{w_{\tilde{\theta}}(v) M_W(x^1,v)}{\sqrt{\mu(v)}} \int_{n(x^1)\cdot v^1 > 0} \mathrm{d}v^1 \{n^1 \cdot v^1\} \sqrt{\mu(v^1)} \qquad (3.8) \\
&\quad \times \left\{ e^{-\nu(v^1) t^1} \frac{1}{\alpha(x^1, v^1)} (\alpha \underline{\nabla}_{x^1} f^{\ell-1})(x^1 - t^1 v^1, v^1) \right. \qquad (3.9) \\
&\quad \left. + \int_{\max\{0, t^1 - t^1_{\mathbf{b}}\}}^{t^1} e^{-\nu(v^1)(t^1 - s^1)} \underline{\nabla}_{x^1} \underbrace{h^{\ell-1}(x^1 - (t^1 - s^1)v^1, v^1)}_{(3.10)_*} \, \mathrm{d}s^1 \right\}. \qquad (3.10)
\end{aligned}
$$

Here $\bar{\theta} > \tilde{\theta}$, $\underline{\nabla}_{x^1} a(x^1 + \cdot)$ stands the tangential derivative $\nabla_{\mathbf{x}^1_{p^1}} [a(\eta_{p^1}(\mathbf{x}^1_{p^1}) + \cdot)]$ in a local coordinate of (1.6) as in (2.29).

To estimate the trace of f^{ℓ} in (3.5), we need the following lemma:

Lemma 3.1. (Theorem (9.2.1) in [5]) *For f^{ℓ} satisfying (3.1), we have the following property for the trace of f^{ℓ}:*

$$
\|f^{\ell}\|_{L^\infty(\partial\Omega)} \lesssim \|f^{\ell}\|_{\infty} + \|h^{\ell}\|_{\infty}. \qquad (3.11)
$$

The following lemma is a direct consequence of Lemma 3.1

Lemma 3.2. *Let $h^{\ell}(x,v) := K(f^{\ell-1}) + \Gamma(f^{\ell-1}, f^{\ell-1})$, then the trace of f^{ℓ} is well-defined. Moreover, for some $\bar{\theta} > \tilde{\theta}$, we have*

$$
\|f^{\ell}\|_{L^\infty(\partial\Omega)} \leqq \|w_{\bar{\theta}} f^{\ell}\|_{L^\infty(\partial\Omega)} \lesssim \|w f^{\ell}\|_{\infty} + \|w f^{\ell-1}\|_{\infty}. \qquad (3.12)
$$

Proof. $w_{\bar{\theta}} f^{\ell}$ satisfies

$$
v \cdot \nabla_x (w_{\bar{\theta}} f^{\ell}) + \nu(v)(w_{\bar{\theta}} f^{\ell}) = w_{\bar{\theta}} h^{\ell}(x,v).
$$

Applying Lemma 3.1 to the above equation, we have

$$\begin{aligned}\|w_{\tilde\theta} f^\ell\|_{L^\infty(\partial\Omega)} &\lesssim \|w_{\tilde\theta} f^\ell\|_\infty + \|w_{\tilde\theta} h^\ell\|_\infty \\ &\lesssim \|w f^\ell\|_\infty + \|w f^{\ell-1}\|_\infty,\end{aligned}$$

where we have applied (2.124) to h^ℓ. □

Proof of Proposition 1. Consider f^ℓ solves (3.1) and (3.4). Choose $t \gg 1$. Recall (2.29). Same as (2.10)–(2.13), for $k \geqq 1$, $n(\mathbf{x}^k_{p^k}) \cdot v^k > 0$, and $i = 1, 2$, or $k = 0$ with $i = 1, 2, 3$,

$$\begin{aligned}
&w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^k_{p^k,i}}[f^\ell(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k)] \\
&= \mathbf{1}_{t^k \geqq t^k_{\mathbf{b}}} e^{-\nu^k t^k_{\mathbf{b}}} w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^k_{p^k,i}}[f^\ell(x_{\mathbf{b}}(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k), v^k)] && (3.13)\\
&\quad - \mathbf{1}_{t^k \geqq t^k_{\mathbf{b}}} \nu^k \partial_{\mathbf{x}^k_{p^k,i}} t^k_{\mathbf{b}} e^{-\nu^k t^k_{\mathbf{b}}} w_{\tilde\theta}(v^k) f^\ell(x_{\mathbf{b}}(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k), v^k) && (3.14)\\
&\quad + \mathbf{1}_{t^k < t^k_{\mathbf{b}}} e^{-\nu^k t^k} w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^k_{p^k,i}}[f^\ell(\eta_{p^k}(\mathbf{x}^k_{p^k}) - t^k v^k, v^k)] && (3.15)\\
&\quad + \int_{\max\{0, t^k - t^k_{\mathbf{b}}\}}^{t^k} e^{-\nu^k (t^k - s^k)} w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^k_{p^k,i}}[h^\ell(\eta_{p^k}(\mathbf{x}^k_{p^k}) - (t^k - s^k)v^k, v^k)]\mathrm{d}s^k && (3.16)\\
&\quad + \partial_{\mathbf{x}^k_{p^k,i}} t^k_{\mathbf{b}} e^{-\nu^k t^k_{\mathbf{b}}} w_{\tilde\theta}(v^k) h^\ell(x_{\mathbf{b}}(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k), v^k), && (3.17)
\end{aligned}$$

where we denoted $\nu^k = \nu(v^k)$.

Estimate of (3.13). From (3.4) and (2.25) with replacing f by f^ℓ, for $k \geqq 1$ with $i = 1, 2$, or $k = 0$ with $i = 1, 2, 3$, if $x_{\mathbf{b}}(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k) \in \mathcal{O}_{p^{k+1}}$ then

$$\begin{aligned}
&w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^k_{p^k,i}}[f^\ell(x_{\mathbf{b}}(\eta_{p^k}(\mathbf{x}^k_{p^k}), v^k), v^k)] \\
&= \sum_{j=1,2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},j}}{\partial \mathbf{x}^k_{p^k,i}} w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^{k+1}_{p^{k+1},j}}[f^\ell(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), v^k)] && (3.18)\\
&= \sum_{j=1,2} \frac{\partial \mathbf{x}^{k+1}_{p^{k+1},j}}{\partial \mathbf{x}^k_{p^k,i}} \Big[\frac{w_{\tilde\theta}(v^k) M_W(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), v^k)}{\sqrt{\mu(v^k)}} \\
&\quad \times \int_{\mathbf{v}^{k+1}_{p^{k+1},3} > 0} \sqrt{\mu(\mathbf{v}^{k+1}_{p^{k+1}})}\, \mathbf{v}^{k+1}_{p^{k+1},3} \mathrm{d}\mathbf{v}^{k+1}_{p^{k+1}} && (3.19)\\
&\quad \times \underbrace{\partial_{\mathbf{x}^{k+1}_{p^{k+1},j}}[f^{\ell-1}(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}})]}_{(3.19)_*} \\
&\quad + O(1) \sum_{j=1,2} w_{\tilde\theta}(v^k)\partial_{\mathbf{x}^{k+1}_{p^{k+1},j}} r(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), v^k)\{1 + \|f^{\ell-1}\|_{L^\infty(\partial\Omega)}\} \Big]. && (3.20)
\end{aligned}$$

Note that the above equalities for $k = 0$ gives an identity of $\partial_{x_i}[f^\ell(x_{\mathbf{b}}(x, v), v)]$.

It is relatively simple to derive that, from (2.34),

$$(3.20) = O(1)|w_{\tilde{\theta}} r(x^{k+1})|\{1 + \|w_{\tilde{\theta}} f^{\ell-1}\|_{L^\infty(\partial\Omega)}\}. \tag{3.21}$$

Now we consider (3.19). We compute $(3.19)_* = (2.29)_{(k,i,a)\to(k+1,j,f^{\ell-1})} +$ (3.22). Here (3.22) is given by

$$\begin{aligned}
&\left(\partial_{\mathbf{x}^{k+1}_{p^{k+1},j}} T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}}\right) \cdot \nabla_v f^{\ell-1}(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}}) \\
&= \sum_{l,m} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},j}} \left(\frac{\partial_m \eta_{p^{k+1},l}(\mathbf{x}^{k+1}_{p^{k+1}})}{\sqrt{g_{p^{k+1},mm}(\mathbf{x}^{k+1}_{p^{k+1}})}} \right) \mathbf{v}^{k+1}_{p^{k+1},m} \\
&\quad \times \partial_{v_l} f^{\ell-1}(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}}) \\
&= \sum_{m,n} (3.23)_{mn} \mathbf{v}^{k+1}_{p^{k+1},m} \partial_{\mathbf{v}^{k+1}_{p^{k+1},n}} [f^{\ell-1}(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}})],
\end{aligned} \tag{3.22}$$

where

$$(3.23)_{mn} := \sum_{l} \frac{\partial}{\partial \mathbf{x}^{k+1}_{p^{k+1},j}} \left(\frac{\partial_m \eta_{p^{k+1},l}(\mathbf{x}^{k+1}_{p^{k+1}})}{\sqrt{g_{p^{k+1},mm}(\mathbf{x}^{k+1}_{p^{k+1}})}} \right) \frac{\partial_n \eta_{p^{k+1},l}(\mathbf{x}^{k+1}_{p^{k+1}})}{\sqrt{g_{p^{k+1},nn}(\mathbf{x}^{k+1}_{p^{k+1}})}}. \tag{3.23}$$

Here we have used (2.23) and (2.24).

First we consider the contribution of (3.22) in (3.19). We substitute (3.22)–(3.23) for $(3.19)_*$ and then apply the integration by parts with respect to $\partial_{\mathbf{v}^{k+1}_{p^{k+1}}}$ to derive that

$$\begin{aligned}
&\int_{\mathbf{v}^{k+1}_{p^{k+1},3}>0} f^{\ell-1}(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}}) \\
&\qquad \times \sum_{m,n} (3.23)_{mn} \partial_{\mathbf{v}^{k+1}_{p^{k+1},n}} \left[\mathbf{v}^{k+1}_{p^{k+1},m} \mathbf{v}^{k+1}_{p^{k+1},3} \sqrt{\mu(\mathbf{v}^{k+1}_{p^{k+1}})}\right] \mathrm{d}\mathbf{v}^{k+1}_{p^{k+1}} \\
&= O(1)\|\eta\|_{C^2} \|w_{\tilde{\theta}} f^{\ell-1}\|_{L^\infty(\partial\Omega)}.
\end{aligned} \tag{3.24}$$

Here we have used

$$\begin{aligned}
&f^{\ell-1}(\eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}), T^t_{\mathbf{x}^{k+1}_{p^{k+1}}} \mathbf{v}^{k+1}_{p^{k+1}}) \sum_{m,n} (3.23)_{mn} \mathbf{v}^{k+1}_{p^{k+1},m} \mathbf{v}^{k+1}_{p^{k+1},3} \sqrt{\mu(\mathbf{v}^{k+1}_{p^{k+1}})} \\
&\equiv 0 \text{ when } \mathbf{v}^{k+1}_{p^{k+1},3} = 0
\end{aligned}$$

for $\|f^{\ell-1}\|_{L^\infty(\partial\Omega)} < \infty$ by Lemma 3.2.

Estimate of the contribution of $(2.29)_{(k,i,a)\to(k+1,j,f^{\ell-1})}$ *in* (3.19). Since the velocity variables of

$(2.29)_{(k,i,a)\to(k+1,j,f^{\ell-1})}$ is written in Cartesian coordinate as v^{k+1} (not $\mathbf{v}^{k+1}_{p^{k+1}}$) we rewrite the $\mathbf{v}^{k+1}_{p^{k+1}}$-integration of (3.19) in v^{k+1}-integration. Then, along the trajectory, $(2.29)_{(k,i,a)\to(k+1,j,f^{\ell-1})}$ can be represented by (3.13)–(3.16) with $(k, i, \ell) \to$

$(k+1, j, \ell-1)$. Here we further replace $(3.13)_{(k,i,\ell)\to(k+1,j,\ell-1)}$ by $\mathbf{1}_{t^{k+1}\geqq t_{\mathbf{b}}^{k+1}}\times e^{-\nu^{k+1}t_{\mathbf{b}}^{k+1}}(3.18)_{(k,i,j,\ell)\to(k+1,j,j',\ell-1)}$. We note that we do not use a further expansion of (3.19)–(3.20). Throughout the process, we derive an identity

$$
\begin{aligned}
&(3.19)\text{ with }[(3.19)_* = (2.29)_{(k,i,a)\to(k+1,j,f^{\ell-1})}]\\
&=\int_{n^{k+1}\cdot v^{k+1}>0}\mathbf{1}_{t^{k+1}\geqq t_{\mathbf{b}}^{k+1}}e^{-\nu^{k+1}t_{\mathbf{b}}^{k+1}}\sum_{p^{k+2}\in\mathcal{P}}\iota_{p^{k+2}}(x_{\mathbf{b}}(x^{k+1},v^{k+1}))\\
&\quad\times\sum_{j'=1,2}\frac{\partial\mathbf{x}_{p^{k+2},j'}^{k+2}}{\partial\mathbf{x}_{p^{k+1},j}^{k+1}}\partial_{\mathbf{x}_{p^{k+2},j'}^{k+2}}[f^{\ell-1}(\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}),v^{k+1})]\\
&\quad\times\sqrt{\mu(v^{k+1})}\{n^{k+1}\cdot v^{k+1}\}\mathrm{d}v^{k+1} \qquad (3.25)\\
&\quad+\int_{n^{k+1}\cdot v^{k+1}>0}\sum_{p^{k+2}\in\mathcal{P}}\iota_{p^{k+2}}(x_{\mathbf{b}}(x^{k+1},v^{k+1}))\\
&\quad\times[(3.14)+(3.15)+(3.16)]_{(k,i,\ell)\to(k+1,j,\ell-1)}\\
&\quad\times\sqrt{\mu(v^{k+1})}\{n^{k+1}\cdot v^{k+1}\}\mathrm{d}v^{k+1}. \qquad (3.26)
\end{aligned}
$$

Here we have denoted $n^{k+1}=n(x^{k+1})$. It is relatively easy to derive

$$
\begin{aligned}
(3.26)&=O(1)\|w_{\bar\theta}f^{\ell-1}\|_{L^\infty(\partial\Omega)}\\
&\quad+O(1)[(3.9)+(3.10)]_{(t^1,x^1,v^1,f)\to(t^{k+1},x^{k+1},v^{k+1},f^{\ell-1})},
\end{aligned}
\qquad (3.27)
$$

where we have used $|\partial_{\mathbf{x}_{p^{k+1},j}^{k+1}}t_{\mathbf{b}}^{k+1}|\leqq\frac{1}{n(x^{k+2})\cdot v^{k+1}}$ from (2.32).

In order to take off $\partial_{\mathbf{x}_{p^{k+2},j'}^{k+2}}$ from $f^{\ell-1}$ in (3.25) we use the change of variables of (2.42). Note that

$$
v^{k+1}=(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))/t_{\mathbf{b}}^{k+1}. \qquad (3.28)
$$

Now we apply the change of variables of (2.42) and derive that

$$
\begin{aligned}
&=\sum_{p^{k+2}\in\mathcal{P}}\iint_{|\mathbf{x}_{p^{k+2}}^{k+2}|<\delta_1}\int_0^{t^{k+1}}e^{-\nu(v^{k+1})t_{\mathbf{b}}^{k+1}}\iota_{p^{k+2}}(\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))\\
&\quad\times\sum_{j'=1,2}\frac{\partial\mathbf{x}_{p^{k+2},j'}^{k+2}}{\partial\mathbf{x}_{p^{k+1},j}^{k+1}}\partial_{\mathbf{x}_{p^{k+2},j'}^{k+2}}[f^{\ell-1}(\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}),v^{k+1})]\\
&\quad\times\frac{n_{p^{k+2}}(\mathbf{x}_{p^{k+1}}^{k+1})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))}{t_{\mathbf{b}}^{k+1}}\\
&\quad\times\frac{n_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})\cdot(x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2}))}{|t_{\mathbf{b}}^{k+1}|^4}\\
&\quad\times\sqrt{\mu(v^{k+1})}\mathrm{d}t_{\mathbf{b}}^{k+1}\sqrt{g_{p^{k+2},11}g_{p^{k+2},22}}\mathrm{d}\mathbf{x}_{p^{k+2},1}^{k+2}\mathrm{d}\mathbf{x}_{p^{k+2},2}^{k+2}. \qquad (3.29)
\end{aligned}
$$

Here we read $g_{p^{k+2},ii}$ at $\mathbf{x}^{k+2}_{p^{k+2}}$.

We apply the integration by parts with respect to $\partial_{\mathbf{x}^{k+2}_{p^{k+2},j'}}$ for $j'=1,2$. For $\iota_{p^{k+2}}(\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}))=0$ when $|\mathbf{x}^{k+2}_{p^{k+2}}|=\delta_1$ from (2.19), such contribution of $|\mathbf{x}^{k+2}_{p^{k+2}}|=\delta_1$ vanishes. Then we derive

$$
\begin{aligned}
(3.25) = & \sum_{p^{k+2}\in\mathcal{P}} \iint \int_0^{t^{k+1}} \partial_{\mathbf{x}^{k+2}_{p^{k+2},j'}} \\
& \times \Big[e^{-\nu(v^{k+1})t^{k+1}_{\mathbf{b}}} \sqrt{\mu(v^{k+1})}\, \iota_{p^{k+2}}(\eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})) \Big] \cdots \qquad (3.30) \\
& + \sum_{p^{k+2}\in\mathcal{P}} \iint \int_0^{t^{k+1}} \partial_{\mathbf{x}^{k+2}_{p^{k+2},j'}} \Big[\sum_{j'=1,2} \frac{\partial \mathbf{x}^{k+2}_{p^{k+2},j'}}{\partial \mathbf{x}^{k+1}_{p^{k+1},j}} \sqrt{g_{p^{k+2},11} g_{p^{k+2},22}} \Big] \cdots \qquad (3.31) \\
& + \sum_{p^{k+2}\in\mathcal{P}} \iint \int_0^{t^{k+1}} \partial_{\mathbf{x}^{k+2}_{p^{k+2},j'}} \Big[\frac{n_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}}) \cdot (x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}))}{t^{k+1}_{\mathbf{b}}} \\
& \qquad \cdot \frac{n_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}) \cdot (x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}))}{|t^{k+1}_{\mathbf{b}}|^4} \Big] \cdots . \qquad (3.32)
\end{aligned}
$$

From (2.34) and (2.44), (2.45), we derive that

$$
\begin{aligned}
& \Big| \frac{\partial}{\partial \mathbf{x}^{k+2}_{p^{k+2},j'}} \Big(\sum_{j'=1,2} \frac{\partial \mathbf{x}^{k+2}_{p^{k+2},j'}}{\partial \mathbf{x}^{k+1}_{p^{k+1},j}} \sqrt{g_{p^{k+2},11} g_{p^{k+2},22}} \Big) \Big| \\
& \lesssim \|\eta\|_{C^2} \Big\{ 1 + \frac{|\mathbf{v}^{k+2}_{p^{k+2},\|}|}{|\mathbf{v}^{k+2}_{p^{k+2},3}|^2} |\partial_3 \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}) \cdot \partial_j \eta_{p^{k+1}}(\mathbf{x}^{k+1}_{p^{k+1}})| \Big\} \\
& \leqq O(\|\eta\|_{C^2}) \Big\{ 1 + \frac{|\mathbf{v}^{k+2}_{p^{k+2}}|}{|\mathbf{v}^{k+2}_{p^{k+2},3}|^2} |x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})| \Big\} \\
& \leqq O(\|\eta\|_{C^2}) \frac{1}{|\mathbf{v}^{k+2}_{p^{k+2},3}|} = O(\|\eta\|_{C^2}) \frac{|t^{k+1}_{\mathbf{b}}|}{|n_{p^{k+2}}(\mathbf{x}^{k+1}_{p^{k+1}}) \cdot (x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}}))|} .
\end{aligned} \qquad (3.33)
$$

Now using (3.28) for (3.30), (3.33) for (3.31), and (2.45) for (3.32), we derive that

$$
\begin{aligned}
& |(3.25)| \\
& \lesssim \|\eta\|_{C^2} \|f^{\ell-1}\|_{L^\infty(\partial\Omega)} \iint \int_0^{t^{k+1}} e^{-\nu_0 t^{k+1}_{\mathbf{b}}} \\
& \times \Big[\frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})|^3}{|t^{k+1}_{\mathbf{b}}|^5} + \frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})|^2}{|t^{k+1}_{\mathbf{b}}|^4} \Big] e^{-\frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}^{k+2}_{p^{k+2}})|^2}{4|t^{k+1}_{\mathbf{b}}|^2}}
\end{aligned}
$$

$$\lesssim \|\eta\|_{C^2}\|w_{\tilde\theta} f^{\ell-1}\|_{L^\infty(\partial\Omega)} \int_0^\infty \frac{e^{-\nu_0 t_{\mathbf{b}}^{k+1}}}{|t_{\mathbf{b}}^{k+1}|^{1/2}} \iint \frac{1}{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^{3/2}}$$
$$\lesssim \|\eta\|_{C^2}\|w_{\tilde\theta} f^{\ell-1}\|_{L^\infty(\partial\Omega)}, \tag{3.34}$$

where we have used Lemma 3.2 and

$$\Big[\frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^3}{|t_{\mathbf{b}}^{k+1}|^5} + \frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^2}{|t_{\mathbf{b}}^{k+1}|^4}\Big] e^{-\frac{|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^2}{4|t_{\mathbf{b}}^{k+1}|^2}}$$
$$\leqq \frac{1}{|t_{\mathbf{b}}^{k+1}|^{1/2}} \frac{1}{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^{3/2}}$$
$$\times \Big[\frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^{9/2}}{|t_{\mathbf{b}}^{k+1}|^{9/2}} + \frac{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^{7/2}}{|t_{\mathbf{b}}^{k+1}|^{7/2}}\Big] e^{-\frac{|x^{k+1}-\eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^2}{4|t_{\mathbf{b}}^{k+1}|^2}}$$
$$\lesssim \frac{1}{|t_{\mathbf{b}}^{k+1}|^{1/2}} \frac{1}{|x^{k+1} - \eta_{p^{k+2}}(\mathbf{x}_{p^{k+2}}^{k+2})|^{3/2}}.$$

Finally collecting terms (3.13)–(3.16), and (3.21), (3.34) and setting $k = 0$, we prove the Proposition 1. □

4. Mixing via the Binary Collision and Transport

In this section we mainly establish the integration by parts technique mentioned in Section 1.3 using the mixing of the binary collision and the transport operator. In particular, we will prove Proposition 2. As direct consequence of Proposition 1 and Proposition 2 we will give a proof of the (1.18) in **Main Theorem**.

We consider a solution of the Boltzmann equation (3.1) with $h^\ell(x, v)$ given by (3.3), and the diffuse BC (3.4). The main result is an estimate of (3.3)-contribution in (3.7) and (3.10).

Proposition 2. *We bound* $(3.7)_{(3.7)_*=Kf^{\ell-1}}$, $(3.7)_{(3.7)_*=\Gamma(f^{\ell-1},f^{\ell-1})}$, *and* $(3.8)\cdot(3.10)_{(3.10)_*=Kf^{\ell-2}+\Gamma(f^{\ell-2},f^{\ell-2})}$ *respectively as*

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t e^{-\nu(v)(t-s)} w_{\tilde\theta}(v)\partial_{x_i} K f^{\ell-1}(x-(t-s)v, v)\mathrm{d}s$$
$$\leqq \frac{O(1)}{\alpha(x,v)}\Big\{\Big(\varepsilon + \sup_{0\leqq i\leqq \ell-1}\|w f^{\ell-1-i}\|_\infty\Big) \sup_{0\leqq i\leqq \ell-1}\|\alpha\nabla_x f^{\ell-1-i}\|_\infty$$
$$+ \varepsilon^{-1}\sup_{0\leqq i\leqq \ell-1}\|w f^{\ell-1-i}\|_\infty\Big\}, \tag{4.1}$$
$$e^{-\nu(v)t_{\mathbf{b}}}\frac{n_i(x^1)|v|}{\alpha(x,v)}\frac{w_{\tilde\theta}(v)M_W(x^1,v)}{\sqrt{\mu(v)}}\int_{n(x^1)\cdot v^1>0}\mathrm{d}v^1\{n^1\cdot v^1\}\sqrt{\mu(v^1)}$$

$$\times \int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} e^{-\nu(v^1)(t^1-s^1)} \underline{\nabla}_{x^1} K f^{\ell-1}(x^1-(t^1-s^1)v^1, v^1)\mathrm{d}s^1$$

$$\leqq \frac{O(1)}{\alpha(x,v)} \times \Big\{ \varepsilon \sup_{0\leqq i\leqq \ell-1} \|w_{\tilde{\theta}} \alpha \nabla_x f^{\ell-1-i}\|_\infty + \varepsilon^{-1} \sup_{0\leqq i\leqq \ell-1} \|w f^{\ell-1-i}\|_\infty \Big\}, \tag{4.2}$$

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} w_{\tilde{\theta}} \partial_{x_i} \Gamma(f^{\ell-1}, f^{\ell-1})(x-(t-s)v, v)\mathrm{d}s$$

$$+ e^{-\nu(v)t_{\mathbf{b}}} \frac{n_i(x^1)|v|}{\alpha(x,v)} \frac{w_{\tilde{\theta}}(v) M_W(x^1, v)}{\sqrt{\mu(v)}} \int_{n(x^1)\cdot v^1>0} \mathrm{d}v^1 \{n^1\cdot v^1\}\sqrt{\mu(v^1)}$$

$$\times \int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} e^{-\nu(v^1)(t^1-s^1)} \underline{\nabla}_{x^1} \Gamma(f^{\ell-1}, f^{\ell-1})(x^1-(t^1-s^1)v^1, v^1)\mathrm{d}s^1$$

$$\leqq \frac{O(1)}{\alpha(x,v)} \Big(\varepsilon + \sup_{0\leqq i\leqq \ell-1} \|w f^{\ell-1-i}\|_\infty \Big) \sup_{0\leqq i\leqq \ell-1} \|w_{\tilde{\theta}} \alpha \nabla_x f^{\ell-1-i}\|_\infty. \tag{4.3}$$

Proof of (1.18) *in Main Theorem* By Lemma 3.2, we have

$$\sup_{i\geqq 0} \|w_{\tilde{\theta}} f^i\|_{L^\infty(\partial\Omega)} \lesssim \sup_{i\geqq 0} \|w f^i\|_\infty + \sup_{i\geqq 1} \|w f^{i-1}\|_\infty.$$

Combining Proposition 1 and Proposition 2 we obtain that for $t \gg 1$ and $\varepsilon \ll 1$,

$$\|w_{\tilde{\theta}} \alpha \nabla_x f^\ell\|_\infty \leqq o(1) \sup_{i\leqq \ell-1} \|w_{\tilde{\theta}} \alpha \nabla_x f^i\|_\infty + C(\varepsilon)\|T_W - T_0\|_{C^1} \sup_{i\geqq 0} \|w f^i\|_\infty,$$

where the $\|T_W - T_0\|_{C^1}$ comes from $|\underline{\nabla}_{x^1} r(x^1)|$ in (3.5).

By a standard argument we pass the limit and conclude that the unique solution in **Existence Theorem** satisfies the weighted C^1 estimate (1.18). □

In Section 4.2 we give a proof of the proposition. In Section 4.1 we give a key lemma for the proof.

4.1. Nonlocal-to-local estimate and small time contributions

The key lemma to prove Proposition 2 is the following *Nonlocal-to-Local estimate*:

Lemma 4.1. *Denote* $x' = x-(t-s)v, \quad y' = y-(t-s)v$. *Assume* $(t,x,v) \in [0,\infty)\times\bar{\Omega}\times\mathbb{R}^3$ *and* $t - t_{\mathbf{b}}(x,v) \leqq t-t_1 \leqq t-t_2 \leqq t$. *Then for* $0<\beta<1$ *and some* $C_1>0$,

$$\int_{t-t_2}^{t-t_1} \int_{\mathbb{R}^3} \frac{e^{-C\langle v\rangle(t-s)} e^{-\varrho|v-u|^2}}{|v-u|\alpha(x',u)} \mathrm{d}u\mathrm{d}s$$

$$\lesssim |e^{-C_1\nu t_1} - e^{-C_1\nu t_2}|^\beta \Big[\langle v\rangle^{-1/2} \big(1 + |\ln|v|| + |\ln\alpha(x,v)|\big) + \frac{1}{|v|^{1-\beta}} \Big]$$

$$\lesssim \frac{|e^{-C_1\nu t_1} - e^{-C_1\nu t_2}|^\beta}{\alpha(x,v)}. \tag{4.4}$$

Thus

$$\int_{t-t_{\mathbf{b}}(x,v)}^{t}\int_{\mathbb{R}^3} \frac{e^{-C\langle v\rangle(t-s)}e^{-\varrho|v-u|^2}}{|v-u|\alpha(x',u)}\mathrm{d}u\mathrm{d}s \lesssim \frac{1}{\alpha(x,v)}, \tag{4.5}$$

and for $\varepsilon \ll 1$,

$$\int_{t-t_{\mathbf{b}}(x,v)}^{t}\int_{\mathbb{R}^3} \mathbf{1}_{s\geqq t-\varepsilon}\frac{e^{-C\langle v\rangle(t-s)}e^{-\varrho|v-u|^2}}{|v-u|\alpha(x',u)}\mathrm{d}u\mathrm{d}s \leqq \frac{O(\varepsilon)}{\alpha(x,v)}. \tag{4.6}$$

For $1 < p < 3$, *we have*

$$\begin{aligned} &int_{t-t_{\mathbf{b}}}^{t} e^{-\nu(t-s)}\mathrm{d}s\int_{\mathbb{R}^3}\mathrm{d}u\frac{\mathbf{k}(v,u)}{|u|^2\min\left\{\frac{\alpha(x',u)}{|u|},\frac{\alpha(y',u)}{|u|}\right\}^p} \\ &\lesssim \frac{\min\{1,O(t)\}}{|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^p}, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\int_{t-t_{\mathbf{b}}}^{t} e^{-\nu(t-s)}\mathrm{d}s\int_{\mathbb{R}^3}\mathrm{d}u\mathbf{1}_{s\geqq t-\varepsilon}\frac{\mathbf{k}(v,u)}{|u|^2\min\left\{\frac{\alpha(x',u)}{|u|},\frac{\alpha(y',u)}{|u|}\right\}^p} \\ &\lesssim \frac{\varepsilon}{|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^p}. \end{aligned} \tag{4.8}$$

For $\beta < 1$,

$$\int_{t-t_{\mathbf{b}}(x,v)}^{t}\frac{e^{-C\langle v\rangle(t-s)}}{|v|\min\{\xi(x'),\xi(y')\}^{\beta/2}} \lesssim \frac{1}{|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta}. \tag{4.9}$$

$$\int_{\mathbb{R}^3}\mathbf{k}(v,u)\frac{1}{[\alpha(x,u)]^\beta}\mathrm{d}u \lesssim 1. \tag{4.10}$$

$$\begin{aligned} &\int_{t-t_{\mathbf{b}}(x,v)}^{t}\int_{\mathbb{R}^3}\frac{e^{-C\langle v\rangle(t-s)}e^{-\varrho|v-u|^2}}{|v-u|^2}\frac{1}{\min\{\alpha(x',u),\alpha(y',u)\}^\beta}\mathrm{d}u\mathrm{d}s \\ &\lesssim \frac{1}{\min\{\alpha(x,v),\alpha(y,v)\}^\beta}. \end{aligned} \tag{4.11}$$

Remark 5. We note that (4.6) can be considered as a boarderline case of Lemma 10 in [14] in which the integral $1/\alpha^\beta$ is considered for $\beta \gneqq 1$.

The proof of Proposition 2 only require (4.5) and (4.6). But in the Section 7, in the proof of the weighted $C^{1,\beta}$ estimate (1.21), the nonlocal-to-local type estimate will be involved with different power of α. We summarize all these estimates in this single lemma.

Proof. During the whole proof we assume $\alpha = \tilde{\alpha}$. For the other case, when $\alpha \gtrsim 1$, the lemma follows from $\mathbf{k}(v,u) \in L^1_u$. □

Proof of (4.4) (4.5) *and* (4.6) We only prove (4.4). (4.5) and (4.6) follow directly from (4.4).

Step 1. We claim that, for $y \in \bar{\Omega}$ and $\varrho > 0$,

$$\int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|} \frac{1}{\alpha(y,u)} \mathrm{d}u \lesssim 1 + |\ln|\xi(y)|| + |\ln|v||. \tag{4.12}$$

Recall (2.24) and set $\mathbf{v} = \mathbf{v}(y)$ and $\mathbf{u} = \mathbf{u}(y)$. For $|u| \geqq O(1)|v|$,

$$\left[|\mathbf{u}_3(y)|^2 + |\xi(y)||u|^2\right]^{1/2} \gtrsim \left[|\mathbf{u}_3(y)|^2 + |\xi(y)||v|^2\right]^{1/2}. \tag{4.13}$$

Thus

$$\begin{aligned}
\int_{\frac{|v|}{4} \leqq |u| \leqq 4|v|} &\lesssim \iint \frac{e^{-\varrho|\mathbf{v}_\parallel - \mathbf{u}_\parallel|^2}}{|\mathbf{v}_\parallel - \mathbf{u}_\parallel|} \mathrm{d}\mathbf{u}_\parallel \int_0^{4|v|} \frac{\mathrm{d}\mathbf{u}_3}{\left[|\mathbf{u}_3|^2 + |\xi(y)||v|^2\right]^{1/2}} \\
&\leqq \int_0^{4|v|} \frac{\mathrm{d}\mathbf{u}_3}{\left[|\mathbf{u}_3|^2 + |\xi(y)||v|^2\right]^{1/2}} \\
&= \ln\left(\sqrt{|\mathbf{u}_3|^2 + |\xi(y)||v|^2} + |\mathbf{u}_3|\right)\Big|_0^{4|v|} \\
&= \ln(\sqrt{16|v|^2 + |\xi(y)||v|^2} + 16|v|^2) - \ln(\sqrt{|\xi(y)||v|^2}) \\
&\leqq \ln|v| + \ln|\xi(y)|.
\end{aligned} \tag{4.14}$$

If $|u| \geqq 4|v|$ then $|u-v|^2 \geqq \frac{|v|^2}{4} + \frac{|u|^2}{4}$ and hence $e^{-\varrho|v-u|^2} \leqq e^{-\frac{\varrho}{8}|v|^2} e^{-\frac{\varrho}{8}|u|^2} e^{-\frac{\varrho}{2}|v-u|^2}$. This, together with (4.13), implies

$$\begin{aligned}
\int_{|u| \geqq 4|v|} &\lesssim e^{-\frac{\varrho}{8}|v|^2} \iint \frac{e^{-\frac{\varrho}{2}|\mathbf{v}_\parallel - \mathbf{u}_\parallel|^2}}{|\mathbf{v}_\parallel - \mathbf{u}_\parallel|} \mathrm{d}\mathbf{u}_\parallel \int_0^{\infty} \frac{e^{-\frac{\varrho}{8}|\mathbf{u}_3|^2}}{\left[|\mathbf{u}_3|^2 + |\xi(y)||v|^2\right]^{1/2}} \mathrm{d}\mathbf{u}_3 \\
&\lesssim e^{-\frac{\varrho}{8}|v|^2} \int_0^{\infty} \frac{e^{-\frac{\varrho}{8}|\mathbf{u}_3|^2}}{\left[|\mathbf{u}_3|^2 + |\xi(y)||v|^2\right]^{1/2}} \mathrm{d}\mathbf{u}_3 \\
&\lesssim e^{-\frac{\varrho}{8}|v|^2} + e^{-\frac{\varrho}{8}|v|^2} \int_0^{1} \frac{\mathrm{d}\mathbf{u}_3}{\left[|\mathbf{u}_3|^2 + |\xi(y)||v|^2\right]^{1/2}} \\
&= e^{-\frac{\varrho}{8}|v|^2} + e^{-\frac{\varrho}{8}|v|^2} \ln\left(\sqrt{|\mathbf{u}_3|^2 + |\xi(y)||v|^2} + |\mathbf{u}_3|\right)\Big|_0^1 \\
&= e^{-\frac{\varrho}{8}|v|^2} \left\{1 + \ln(\sqrt{1 + |\xi(y)||v|^2} + 1) - \ln(\sqrt{|\xi(y)||v|^2})\right\} \\
&\leqq e^{-\frac{\varrho}{8}|v|^2} \{\ln|v| + \ln|\xi(y)|\}.
\end{aligned} \tag{4.15}$$

For $|u| \leqq \frac{|v|}{4}$, we have $|v-u| \geqq \big||v| - |u|\big| \geqq |v| - \frac{|v|}{4} \geqq \frac{|v|}{2}$. We have

$$\int_{|u| \leqq \frac{|v|}{4}} \lesssim \frac{e^{-\frac{\varrho}{4}|v|^2}}{|v|} \int_{|\mathbf{u}_3| + |\mathbf{u}_\parallel| \leqq \frac{|v|}{2}} \frac{\mathrm{d}\mathbf{u}_3 \mathrm{d}\mathbf{u}_\parallel}{\left[|\mathbf{u}_3|^2 + |\xi(y)||\mathbf{u}_\parallel|^2\right]^{1/2}}$$

$$\lesssim |v|e^{-\frac{\varrho}{4}|v|^2}\int_{|\tilde{\mathbf{u}}_3|\leqq\frac{1}{2}}\int_{|\tilde{\mathbf{u}}_\parallel|\leqq\frac{1}{2}}\frac{\mathrm{d}\tilde{\mathbf{u}}_\parallel\mathrm{d}\tilde{\mathbf{u}}_3}{\left[|\tilde{\mathbf{u}}_3|^2+|\xi(y)||\tilde{\mathbf{u}}_\parallel|^2\right]^{1/2}},$$

where we have used $|v|\tilde{u}=u$. Using the polar coordinate $\tilde{\mathbf{u}}_1=|\tilde{\mathbf{u}}_\parallel|\cos\rho$, $\tilde{\mathbf{u}}_2=|\tilde{\mathbf{u}}_\parallel|\sin\rho$, we have

$$\begin{aligned}
&\int_{|u|\leqq\frac{|v|}{4}}\lesssim |v|e^{-\frac{\varrho}{4}|v|^2}\int_0^{\frac{1}{2}}\mathrm{d}\tilde{\mathbf{u}}_3\int_0^{2\pi}\int_0^{\sqrt{1/2}}\frac{|\tilde{\mathbf{u}}_\parallel|\mathrm{d}|\tilde{\mathbf{u}}_\parallel|\mathrm{d}\rho}{\left[|\tilde{\mathbf{u}}_3|^2+|\xi(y)||\tilde{\mathbf{u}}_\parallel|^2\right]^{1/2}}\\
&\lesssim |v|e^{-\frac{\varrho}{4}|v|^2}\int_0^{\frac{1}{2}}\mathrm{d}\tilde{\mathbf{u}}_3\int_0^{1/2}\frac{\mathrm{d}|\tilde{\mathbf{u}}_\parallel|^2}{\left[|\tilde{\mathbf{u}}_3|^2+|\xi(y)||\tilde{\mathbf{u}}_\parallel|^2\right]^{1/2}}\\
&=|v|e^{-\frac{\varrho}{4}|v|^2}\int_0^{\frac{1}{2}}\mathrm{d}\tilde{\mathbf{u}}_3\frac{1}{|\xi(y)|}\Big(\sqrt{|\tilde{\mathbf{u}}_3|^2+\frac{|\xi(y)|}{2}}-|\tilde{\mathbf{u}}_3|\Big)\\
&=\frac{|v|e^{-\frac{\varrho}{4}|v|^2}}{|\xi(y)|}\\
&\quad\times\left\{\frac{1}{2}|\tilde{\mathbf{u}}_3|\sqrt{|\tilde{\mathbf{u}}_3|^2+\frac{|\xi|}{2}}+\frac{|\xi|}{4}\log\Big(\sqrt{|\tilde{\mathbf{u}}_3|^2+\frac{|\xi|}{2}}+|\tilde{\mathbf{u}}_3|\Big)-\frac{1}{2}|\tilde{\mathbf{u}}_3|^{1/2}\right\}\Big|_{|\tilde{\mathbf{u}}_3|=0}^{|\tilde{\mathbf{u}}_3|=1/2}\\
&=\frac{|v|e^{-C|v|^2}}{|\xi|}\left\{\frac{1}{4}\sqrt{\frac{1}{4}+\frac{|\xi|}{2}}+\frac{\xi}{4}\log\Big(\sqrt{\frac{1}{4}+\frac{|\xi|}{2}}+\frac{1}{2}\Big)-\frac{|\xi|}{4}\log\Big(\sqrt{\frac{|\xi|}{2}}\Big)-\frac{1}{8}\right\}\\
&\lesssim\frac{|v|e^{-C|v|^2}}{|\xi|}\Big[|\xi|\log(|\xi|)+|\xi|\log\Big(1+\sqrt{1+|\xi|}\Big)\Big]\\
&\lesssim 1+\log(|\xi(y)|).
\end{aligned}\tag{4.16}$$

Collecting terms from (4.14), (4.15), and (4.16), we prove (4.12).

Step 2. We prove the following statement: for $x\in\partial\Omega$, we can choose $0<\tilde{\delta}\ll_\Omega 1$ such that

$$\begin{aligned}
\tilde{\delta}^{1/2}|v\cdot\nabla\xi(x-(t-s)v)|\gtrsim_\Omega\ & |v|\sqrt{-\xi(x-(t-s)v)},\\
&\text{for}\quad s\in\Big[t-t_{\mathbf{b}}(x,v),t-t_{\mathbf{b}}(x,v)+\tilde{t}\Big]\cup\Big[t-\tilde{t},t\Big],
\end{aligned}\tag{4.17}$$

$$\begin{aligned}
\tilde{\delta}^{1/2}\times\alpha(x,v)\lesssim_\Omega\ & |v|\sqrt{-\xi(x-(t-s)v)},\\
&\text{for } s\in\Big[t-t_{\mathbf{b}}(x,v)+\tilde{t},t-\tilde{t}\Big],
\end{aligned}\tag{4.18}$$

here $\tilde{t}=\min\left\{\frac{t_{\mathbf{b}}(x,v)}{2},\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}\right\}$. We note that when $\tilde{t}<\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}$, (4.18) vanishes.

If $v=0$ or $v\cdot\nabla\xi(x)\leqq 0$, since $x\in\partial\Omega$, then (4.17) and (4.18) hold clearly with $t_{\mathbf{b}}(x,v)=\tilde{t}=0$. We may assume $v\neq 0$ and $v\cdot\nabla\xi(x)>0$. Since $v\cdot\nabla\xi(x)>0$, from (1.10), we have $|v\cdot\nabla\xi(x_{\mathbf{b}}(x,v))|>0$. Then we must have $v\cdot\nabla\xi(x_{\mathbf{b}}(x,v))<0$, otherwise $v\cdot\nabla\xi(x_{\mathbf{b}})>0$ implies $x_{\mathbf{b}}$ is not the backward exit position defined in (1.24). By the mean value theorem there exists at least one $t^*\in(t-t_{\mathbf{b}}(x,v),t)$ such that $v\cdot\nabla\xi(x-t^*v)=0$. Moreover due to the convexity

in (1.7) we have $\frac{d}{ds}\big(v\cdot\nabla\xi(x-(t-s)v)\big)=v\cdot\nabla^2\xi(x-(t-s)v)\cdot v\geqq C_\xi|v|^2$, and therefore $t^*\in(t-t_{\mathbf{b}}(x,v),t)$ is unique.

Let $s\in\big[t-\tilde{t},t\big]$ for $0<\tilde{\delta}\ll 1$ and $\tilde{t}\leqq\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}$. Then from the fact that $v\cdot\nabla_x\xi(x-(t-\tau)v)$ is non-decreasing function in $\tau\in[t-t^*,t]$ and $x\in\partial\Omega$,

$$\begin{aligned}|v|^2(-1)\xi(x-(t-s)v)\ &=\int_s^t|v|^2v\cdot\nabla_x\xi(x-(t-\tau)v)\mathrm{d}\tau\\&\leqq\tilde{\delta}\alpha(x,v)|v\cdot\nabla_x\xi(x)|.\end{aligned}\tag{4.19}$$

Since $|v\cdot\nabla_x\xi(x)|\leqq\alpha(x,v)\leqq C_\Omega\alpha(x-(t-s)v,v)\leqq C_\Omega\{|v\cdot\nabla\xi(x-(t-s)v)|+\|\nabla_x^2\xi\|_\infty|v|\sqrt{-\xi(x-(t-s)v)}\}$, we choose $\tilde{\delta}\ll(C_\Omega\|\nabla_x^2\xi\|_\infty^{-2}$ and absorb

$$\begin{aligned}&\tilde{\delta}\alpha(x,v)\times C_\Omega\{|v\cdot\nabla\xi(x-(t-s)v)|+\|\nabla_x^2\xi\|_\infty|v|\sqrt{-\xi(x-(t-s)v)}\}\\&\quad\leqq\tilde{\delta}\times\{C_\Omega\|\nabla_x^2\xi\|_\infty|v|\sqrt{-\xi(x-(t-s)v)}\}^2\end{aligned}$$

by the left hand side of (4.19). This gives (4.17) for $s\in\big[t-\tilde{t},t\big]$. The proof for $s\in\big[t-t_{\mathbf{b}}(x,v),t-t_{\mathbf{b}}(x,v)+\tilde{t}\big]$ is same.

For (4.18), we assume $\tilde{t}=\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}$, otherwise (4.18) vanishes. For $\xi(x-(t-s)v)$ is non-increasing in $s\in[t-t^*,t]$, we have $|v|^2(-1)\xi(x-(t-s)v)\geqq|v|^2(-1)\xi\Big(x-\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}v\Big)$ for $s\in\big[t-t^*,t-\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}\big]$. By an expansion, for $s^*:=t-\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}$,

$$\begin{aligned}&|v|^2(-1)\xi\Big(x-\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}v\Big)\\&=|v|^2(v\cdot\nabla_x\xi(x))\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}+\int_{s^*}^t\int_{s^*}^\tau|v|^2v\cdot\nabla_x^2\xi(x-(t-\tau')v)\cdot v\mathrm{d}\tau'\mathrm{d}\tau.\end{aligned}\tag{4.20}$$

The last term of (4.20) is bounded by $\|\nabla_x^2\xi\|_\infty\tilde{\delta}^2\big(\frac{\alpha(x,v)}{|v|^2}\big)^2|v|^4\leqq\|\nabla_x^2\xi\|_\infty\tilde{\delta}^2\alpha(x,v)^2$. Since $v\cdot\nabla_x\xi(x)\leqq\alpha(x,v)$, for $\tilde{\delta}\ll\|\nabla_x^2\xi\|_\infty^{-1/2}$, the right hand side of (4.20) is bounded below by $\frac{\tilde{\delta}}{2}\alpha(x,v)^2$. This completes the proof of (4.18) when $s\in\big[t-t^*,t-\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}\big]$. The proof for the case of $s\in\big[t-t_{\mathbf{b}}(x,v)+\tilde{\delta}\frac{\alpha(x,v)}{|v|^2},t-t^*\big]$ is same.

Step 3. From (4.12), for the proof of (4.4), it suffices to estimate

$$\begin{aligned}&\int_{t-t_{\mathbf{b}}(x,v)}^t\mathbf{1}_{t-t_2\geqq s\geqq t-t_1}e^{-C\langle v\rangle(t-s)}\big|\ln|\xi(x-(t-s)v)|\big|\mathrm{d}s\\&+\int_{t-t_{\mathbf{b}}(x,v)}^t\mathbf{1}_{t-t_2\geqq s\geqq t-t_1}e^{-C\langle v\rangle(t-s)}\big(1+\big|\ln|v|\big|\big)\mathrm{d}s.\end{aligned}\tag{4.21}$$

We simply bound the second term of (4.21) as

$$(1+|\ln|v||)\int_{t-t_1}^{t-t_2}e^{-C\langle v\rangle(t-s)}\lesssim(1+|\ln|v||)\langle v\rangle^{-1}|e^{-C\langle v\rangle t_2}-e^{-C\langle v\rangle t_1}|.\tag{4.22}$$

For the first term of (4.21), we first assume $x \in \partial\Omega$. For utilizing (4.17) and (4.18), we split the first term of (4.21) as

$$\underbrace{\int_{t-\tilde{t}}^{t} + \int_{t-t_{\mathbf{b}}(x,v)}^{t-t_{\mathbf{b}}(x,v)+\tilde{t}}}_{(4.23)_1} + \underbrace{\int_{t-t_{\mathbf{b}}(x,v)+\tilde{t}}^{t-\tilde{t}}}_{(4.23)_2}. \tag{4.23}$$

Without loss of generality, we assume $t-t_2 \in [t-\tilde{t}, t], t-t_1 \in [t-t_{\mathbf{b}}(x,v)+\tilde{t}, t-\tilde{t}]$. For the first term $(4.23)_1$ we use a change of variables $s \mapsto -\xi(x-(t-s)v)$ in $s \in [t-t_{\mathbf{b}}(x,v), t-t^*]$ and $s \in [t-t^*, t]$ separately with $\mathrm{d}s = |v \cdot \nabla_x \xi(x-(t-s)v)|^{-1}\mathrm{d}|\xi|$. From (4.19) we have $|\xi(x-(t-s)v)| \leqq \tilde{\delta}\frac{\alpha^2(x,v)}{|v|^2}$. Then applying Hölder inequality with $\beta + (1-\beta) = 1$ and using (4.17), we get

$$\begin{aligned}
&(4.23)_1 \mathbf{1}_{\{t-t_2 \in [t-\tilde{t},t]\}, t-t_1 \in [t-t_{\mathbf{b}}(x,v)+\tilde{t}, t-\tilde{t}]} \\
&\lesssim \Big(\Big[\int_{t-\tilde{t}}^{t-t_2} e^{-C\langle v\rangle(t-s)/\beta} \mathrm{d}s \Big]^{\beta} + \Big[\int_{t-t_1}^{t-t_{\mathbf{b}}(x,v)+\tilde{t}} e^{-C\langle v\rangle(t-s)/\beta} \mathrm{d}s \Big]^{\beta} \Big) \\
&\quad \times \Big[\int_0^{\tilde{\delta}\frac{\alpha^2(x,v)}{|v|^2}} |\ln|\xi||^{1/(1-\beta)} \frac{\mathrm{d}|\xi|}{\tilde{\delta}^{-1/2}|v|\sqrt{|\xi|}} \Big]^{1-\beta} \\
&\lesssim |e^{-C\langle v\rangle t_2/\beta} - e^{-C\langle v\rangle t_1/\beta}|^{\beta} \frac{1}{|v|^{1-\beta}},
\end{aligned} \tag{4.24}$$

where we have used $t-\tilde{t} > t-t_1$, $\quad t-t_2 > t-t_{\mathbf{b}}(x,v)+\tilde{t}$ and $\frac{|\ln|\xi||^{1/(1-\beta)}}{\sqrt{\xi}} \in L^1_{loc}(0,\infty)$ for $\beta < 1$ in the last line.

On the other hand, from (4.18),

$$\begin{aligned}
(4.23)_2 &\leqq \int_{t-t_1}^{t-t_2} e^{-C\langle v\rangle(t-s)} \Big| \ln\Big(\tilde{\delta}\frac{\alpha(x,v)^2}{|v|^2}\Big)\Big| \mathrm{d}s \\
&\leqq 2\int_{t-t_1}^{t-t_2} e^{-C\langle v\rangle(t-s)} \{|\ln\tilde{\delta}| + |\ln\alpha(x,v)| + |\ln|v||\} \mathrm{d}s \\
&\leqq 2|e^{-C\langle v\rangle t_2} - e^{-C\langle v\rangle t_1}| \langle v\rangle^{-1/2} \{|\ln\tilde{\delta}| + |\ln\alpha(x,v)| + |\ln|v||\},
\end{aligned} \tag{4.25}$$

where we have used a similar estimate of (4.22).

Now as assume $x \notin \partial\Omega$. We find $\bar{x} \in \partial\Omega$ and $\bar{t}$ so that

$$x = \bar{x} - (\bar{t}-t)v \text{ and } \bar{t} > t.$$

Then clearly, $x-(t-s)v = \bar{x} - (\bar{t}-s)v$. Since $\bar{x} \in \partial\Omega$, applying the same computation as (4.24) and (4.25), the first term of (4.21) is bounded by

$$\begin{aligned}
&\int_{\bar{t}-t_{\mathbf{b}}(\bar{x},v)}^{\bar{t}} \mathbf{1}_{t-t_2 \geqq s \geqq t-t_1} e^{-C\langle v\rangle(\bar{t}-s)} |\ln|\xi(\bar{x}-(\bar{t}-s)v)|| \mathrm{d}s \\
&\lesssim (4.24) + |e^{-C\langle v\rangle t_2} - e^{-C\langle v\rangle t_1}| \langle v\rangle^{-1/2} \{|\ln\tilde{\delta}| + |\ln\alpha(\bar{x},v)| + |\ln|v||\}.
\end{aligned} \tag{4.26}$$

Here we used $\alpha(\bar{x}, v) \sim \alpha(x, v)$ from Lemma 1.10 to obtain the same upper bound as (4.24). Again using $\alpha(\bar{x}, v) \sim \alpha(x, v)$, we conclude

$$(4.26) \lesssim (4.24) + (4.25).$$

Then we conclude the first inequality in (4.4) using $\beta < 1$.

For the second inequality, from (1.8) and (1.9) we bound a term of the upper bound of second line of (4.4) as

$$\begin{aligned}
&\{\langle v\rangle^{-1/2}\big(1 + |\ln|v|| + |\ln\alpha(x, v)|\big) + \frac{1}{|v|^{1-\beta}}\} \times \frac{\alpha(x, v)}{\alpha(x, v)} \\
\leqq &\{1 + \langle v\rangle^{-1/2} \min\{1, |v|\}\,|\ln|v|| + \alpha|\ln|\alpha|| + \frac{\alpha(x, v)}{|v|^{1-\beta}}\} \times \frac{1}{\alpha(x, v)} \lesssim \frac{1}{\alpha(x, v)},
\end{aligned}$$

where we have used $\alpha(x, v) \leqq \min\{1, |v|\}$ and $1 - \beta < 1$. □

Proof of (4.7) *and* (4.8) Again we only prove (4.7). Clearly, we have

$$\begin{aligned}
&\int_0^t e^{-\nu(t-s)}\mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v, u)}{|u|^2 \min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|}, \frac{\alpha(y-(t-s)v,u)}{|u|}\right\}^p} \\
\leqq &\int_0^t e^{-\nu(t-s)}\mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v, u)}{|u|^2\big(\frac{\alpha(x-(t-s)v,u)}{|u|}\big)^p} \\
&+ \int_0^t e^{-\nu(t-s)}\mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v, u)}{|u|^2\big(\frac{\alpha(y-(t-s)v,u)}{|u|}\big)^p}.
\end{aligned}$$

By Lemma 1 in [14], we have

$$\begin{aligned}
&\int_0^t e^{-\nu(t-s)}\mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v, u)}{\alpha^p(x - (t - s)v, u)} \frac{|u|^{p-2}}{|v|^{p-2}} |v|^{p-2} \\
\lesssim &|v|^{p-2}\Big[\min\{1, t\} \times \min\{\frac{1}{|v|^2\alpha^{p-2}(x, v)}, \frac{\alpha^{1/2-p/2}(x, v)}{|v|^{p-1}}\} \\
&+ \frac{1}{\alpha^{p-1}(x, v)} \int_0^t e^{-\frac{C}{2}\langle v\rangle(t-s)}\mathrm{d}s\Big].
\end{aligned}$$

We bound

$$\begin{aligned}
&\frac{1}{|v|^2\alpha^{p-2}(x, v)} \leqq \frac{\alpha^2(x, v)}{|v|^2} \frac{1}{\alpha^p(x, v)} \leqq \frac{1}{\alpha^p(x, v)}, \\
&\frac{\alpha^{1/2-p/2}(x, v)}{|v|^{p-1}} \leqq \frac{\alpha^{p-1}(x, v)}{|v|^{p-1}} \frac{1}{\alpha^{p/2-1/2+p-1}(x, v)} \\
&\leqq \frac{1}{\alpha^{3p/2-3/2}(x, v)} \lesssim \frac{1}{\alpha^p(x, v)}, \\
&\frac{1}{\alpha^{p-1}(x, v)} \int_0^t e^{-\frac{C}{2}\langle v\rangle(t-s)}\mathrm{d}s \lesssim \min\{1, O(t)\} \frac{1}{\alpha^p(x, v)},
\end{aligned}$$

where we have used $\alpha \leqq 1$ and $p < 3$.

Then we conclude (4.7). □

Proof of (4.9) Since $\frac{1}{\min\{\xi(x'),\xi(y')\}^{\beta/2}} \leqq \frac{1}{\xi^{\beta/2}(x')} + \frac{1}{\xi^{\beta/2}(y')}$, we only need to prove

$$\int_{t-t_{\mathbf{b}}(x,v)}^{t} \frac{e^{-C\langle v\rangle(t-s)}}{|v|\xi(x')^{\beta/2}} \lesssim \frac{1}{|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{\beta}}.$$

We split the integral as

$$\int_{t-\tilde{\delta}\frac{\tilde{\alpha}(x,v)}{|v|^2}}^{t} + \int_{t-t_{\mathbf{b}}(x,v)}^{t-t_{\mathbf{b}}(x,v)+\tilde{\delta}\frac{\tilde{\alpha}(x,v)}{|v|^2}} + \int_{t-t_{\mathbf{b}}(x,v)+\tilde{\delta}\frac{\tilde{\alpha}(x,v)}{|v|^2}}^{t-\tilde{\delta}\frac{\tilde{\alpha}(x,v)}{|v|^2}}.$$

Similarly to (4.24), the first two terms are bounded by

$$2\int_0^{\tilde{\delta}\frac{\alpha^2(x,v)}{|v|}} \frac{1}{|v||\xi|^{\beta/2}} \frac{d|\xi|}{\tilde{\delta}^{-1/2}|v|\sqrt{|\xi|}} \lesssim \frac{1}{|v|^2}.$$

Similarly to (4.25), the third term is bounded by

$$\int_0^t \frac{e^{-C\langle v\rangle(t-s)}}{|v|} \frac{1}{|\tilde{\delta}|^{\beta/2}\frac{\alpha^{\beta}(x,v)}{|v|^{\beta}}} \mathrm{d}s \lesssim \frac{1}{|v|\langle v\rangle \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{\beta}}.$$

□

Proof of (4.10) Similarly to the proof of (4.6) we consider three cases: $\frac{|v|}{4} \leqq |u| \leqq 4|v|$, $|u| \geqq 4|v|$, $|u| \leqq \frac{|v|}{4}$.

By a similar computation as to that for (4.14),(4.15) and (4.16), we obtain

$$\int_{\frac{|v|}{4}\leqq|u|\leqq 4|v|} \lesssim \iint \frac{e^{-\varrho|\mathbf{v}_{\parallel}-\mathbf{u}_{\parallel}|^2}}{|\mathbf{v}_{\parallel}-\mathbf{u}_{\parallel}|}\mathrm{d}\mathbf{u}_{\parallel} \int_0^{4|v|} \frac{\mathrm{d}\mathbf{u}_3}{\left[|\mathbf{u}_3|^2+|\xi(y)||v|^2\right]^{\beta/2}} \lesssim 1,$$

$$\int_{|u|\geqq 4|v|} \lesssim e^{-\frac{\varrho}{8}|v|^2} \iint \frac{e^{-\frac{\varrho}{2}|\mathbf{v}_{\parallel}-\mathbf{u}_{\parallel}|^2}}{|\mathbf{v}_{\parallel}-\mathbf{u}_{\parallel}|}\mathrm{d}\mathbf{u}_{\parallel} \int_0^{\infty} \frac{e^{-\frac{\varrho}{8}|\mathbf{u}_3|^2}}{\left[|\mathbf{u}_3|^2+|\xi(y)||v|^2\right]^{\beta/2}}\mathrm{d}\mathbf{u}_3 \lesssim 1,$$

$$\int_{|u|\leqq\frac{|v|}{4}} \lesssim |v|e^{-\frac{\varrho}{2}|v|^2} \int_{|\tilde{\mathbf{u}}_3|\leqq\frac{1}{2}} \int_{|\tilde{\mathbf{u}}_{\parallel}|\leqq\frac{1}{2}} \frac{\mathrm{d}\tilde{\mathbf{u}}_{\parallel}\mathrm{d}\tilde{\mathbf{u}}_3}{|\tilde{\mathbf{u}}_3|^{\beta}} \lesssim 1.$$

Here we used $\frac{1}{|u|^{\beta}} \in L^1_u$. □

Proof of (4.11) By Hölder inequality with $\frac{1}{3}+\frac{1}{3/2}=1$ and split $|v-u|^2 = |v-u|^{4/3+\varepsilon}|v-u|^{2/3-\varepsilon}$ we have

$$\begin{aligned}
&\int_{t-t_{\mathbf{b}}(x,v)}^{t} \mathrm{d}s e^{-C\langle v\rangle(t-s)} \\
&\quad\times \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|^2} \frac{1}{\min\{\alpha(x-(t-s)v,u),\alpha(y-(t-s)v,u)\}^{\beta}}\mathrm{d}u \\
&\lesssim \int_{t-t_{\mathbf{b}}(x,v)}^{t} e^{-C\langle v\rangle(t-s)}\mathrm{d}s\Big(\int_{\mathbb{R}^3} \frac{e^{\varrho|v-u|^2}}{|v-u|^{2+\varepsilon}}\mathrm{d}u\Big)^{2/3}
\end{aligned}$$

$$\times \Big(\int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|^{2-\varepsilon}} \frac{1}{\min\{\alpha(x-(t-s)v,u), \alpha(y-(t-s)v,u)\}^{3\beta}} \mathrm{d}u \Big)^{1/3}$$

$$\lesssim \Big(\int_{t-t_{\mathbf{b}}(x,v)}^{t} e^{-C\langle v\rangle(t-s)} \times \int_{\mathbb{R}^3} \frac{e^{-\varrho|v-u|^2}}{|v-u|^{2-\varepsilon}} \frac{1}{\min\{\alpha(x-(t-s)v,u), \alpha(y-(t-s)v,u)\}^{3\beta}} \mathrm{d}u \Big)^{1/3}$$

$$\times \Big(\int_{t-t_{\mathbf{b}}(x,v)}^{t} e^{-C\langle v\rangle(t-s)} \Big)^{2/3}. \tag{4.27}$$

Since $3\beta < 3$ by (4.7) we have

$$(4.27) \lesssim \Big(\frac{1}{\min\{\alpha(x,v), \alpha(y,v)\}^{3\beta}} \Big)^{1/3},$$

then we finish the proof. □

4.2. *Proof of Proposition 2*

Step 1. Convert ∇_x for ∇_v along the trajectory using binary collision

Expansion of $(3.7)_{(3.7)_*=Kf^{\ell-1}(x-(t-s)v,v)}$. First we consider (3.7) with $(3.7)_* = Kf^{\ell-1}(x-(t-s)v,v) = \int_{\mathbb{R}^3} \mathbf{k}(v,u) f^{\ell-1}(x-(t-s)v,u)\mathrm{d}u$. Temporarily denote $y = x-(t-s)v$. Proposition 1 gives a formula of $w_{\tilde{\theta}}(u)\partial_{x_i} f^{\ell-1}(y,u)$ by (3.5)–(3.10) with $h^\ell = (3.3)$ and $(x,v,\ell) \to (y,u,\ell-1)$. We split the contribution of (3.7) with $(3.7)_* = Kf^{\ell-2}(y-(s-s^0)u,u)$, which is (4.34), and the rest. The rest is given as

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \mathrm{d}s\, e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u\, \mathbf{k}(v,u) \frac{w_{\tilde{\theta}}(v)}{w_{\tilde{\theta}}(u)} \Big\{ O(1) \frac{n_i(x^1) w_{\tilde{\theta}}(u)}{\alpha(y,u)}$$

$$\times \Big(\frac{\nu(u)}{w_{\tilde{\theta}}(u)} \|w_{\tilde{\theta}} f^{\ell-1}\|_{L^\infty(\partial\Omega)} + |u| \frac{M_W(x^1,u)}{\sqrt{\mu(u)}} \|w_{\tilde{\theta}} f^{\ell-2}\|_{L^\infty(\partial\Omega)} \tag{4.28}$$

$$+ |u||\underline{\nabla}_{x^1} r(x^1)|(1 + \|w_{\tilde{\theta}} f^{\ell-2}\|_{L^\infty(\partial\Omega)}) \Big)$$

$$+ \frac{O(1)}{\alpha(y,u)} e^{-\nu(u)s} (w_{\tilde{\theta}} \alpha \partial_{x_i} f^{\ell-1})(y-su,u) \tag{4.29}$$

$$+ \int_{\max\{0,s-t_{\mathbf{b}}\}}^{s} e^{-\nu(u)(t-s)} w_{\tilde{\theta}}(u) \partial_{x_i} \Gamma(f^{\ell-2}, f^{\ell-2})(y-(s-s')v,v)\mathrm{d}s' \tag{4.30}$$

$$+ O(1) e^{-\nu(u)t_{\mathbf{b}}} \frac{n_i(x^1)|u|}{\alpha(y,u)} \frac{w_{\tilde{\theta}}(u) M_W(x^1,u)}{\sqrt{\mu(u)}} \int_{n(x^1)\cdot v^1 > 0} \mathrm{d}v^1 \{n^1 \cdot v^1\} \sqrt{\mu(v^1)} \tag{4.31}$$

$$\times \Big(e^{-\nu(v^1)t^1} \frac{1}{\alpha(x^1,v^1)} (\alpha \underline{\nabla}_{x^1} f^{\ell-2})(x^1 - t^1 v^1, v^1) \tag{4.32}$$

$$+\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} e^{-\nu(v^1)(t^1-s^1)}\nabla_{x^1}\underbrace{h^{\ell-2}(x^1-(t^1-s^1)v^1,v^1)}_{h^{\ell-2}\text{ in }(3.3)}\,\mathrm{d}s^1\Big)\Big\}, \tag{4.33}$$

where we intentionally have abused the notations as $x^1 = x^1(y,u), t^1 = s - t_{\mathbf{b}}(y,u)$ for the sake of simplicity (see (2.16) and (2.17)). We will estimate (4.28)–(4.33) later together with the other expansions.

Now we focus on the contribution of (3.7) of $(3.7)_* = Kf^{\ell-2}(y-(s-s^0)u,u)$. We split the time integration in s^0 as

$$\begin{aligned}
&\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\,\mathbf{k}(v,u)w_{\tilde\theta}(v)\\
&\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s}\mathrm{d}s^0\, e^{-\nu(u)(s-s^0)}\{\mathbf{1}_{s^0\leqq s-\varepsilon}+\mathbf{1}_{s^0>s-\varepsilon}\}\\
&\times\int_{\mathbb{R}^3}\mathrm{d}u'\,\mathbf{k}(u,u')\partial_{x_j}f^{\ell-2}(x-(t-s)v-(s-s^0)u,u').
\end{aligned}\tag{4.34}$$

Note that $\partial_{x_i}f^{\ell-2}(x-(t-s)v-(s-s^0)u,u') = \frac{-1}{s-s^0}\partial_{u_i}[f^{\ell-2}(x-(t-s)v-(s-s^0)u,u')]$. Applying an integration by parts with respect to ∂_{u_i}, we derive an identity of the contribution of $\{s^0\leqq s-\varepsilon\}$ in (4.34) as

$$\begin{aligned}
&\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}w_{\tilde\theta}(v)\int_{\mathbb{R}^3}\mathrm{d}u\\
&\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s}\mathrm{d}s^0\, e^{-\nu(u)(s-s^0)}\frac{\mathbf{1}_{s^0\leqq s-\varepsilon}}{s-s^0}\\
&\times\int_{\mathbb{R}^3}\mathrm{d}u'\,\partial_{u_i}[\mathbf{k}(v,u)\mathbf{k}(u,u')]f^{\ell-2}(x-(t-s)v-(s-s^0)u,u')\\
&-\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}w_{\tilde\theta}(v)\int_{\mathbb{R}^3}\mathrm{d}u\\
&\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s}\mathrm{d}s^0\,\partial_{u_i}\nu(u)e^{-\nu(u)(s-s^0)}\mathbf{1}_{s^0\leqq s-\varepsilon}\\
&\times\int_{\mathbb{R}^3}\mathrm{d}u'\,\mathbf{k}(v,u)\mathbf{k}(u,u')f^{\ell-2}(x-(t-s)v-(s-s^0)u,u')\\
&+\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}w_{\tilde\theta}(v)\int_{\mathbb{R}^3}\mathrm{d}u\\
&\times\mathbf{1}_{s\geqq t_{\mathbf{b}}(y,u)}e^{-\nu(u)t_{\mathbf{b}}(y,u)}\frac{\mathbf{1}_{t_{\mathbf{b}}(y,u)\geqq\varepsilon}}{t_{\mathbf{b}}(y,u)}\frac{\partial t_{\mathbf{b}}(y,u)}{\partial u_i}\\
&\times\int_{\mathbb{R}^3}\mathrm{d}u'\,\mathbf{k}(v,u)\mathbf{k}(u,u')f^{\ell-2}(x-(t-s)v-t_{\mathbf{b}}(y,u)u,u').
\end{aligned}\tag{4.35}$$

From (2.103) and Lemma 2.13, for the first term in (4.35) we have

$$w_{\tilde\theta}(v)\partial_{u_i}[\mathbf{k}(v,u)\mathbf{k}(u,u')]f^{\ell-2}(x-(t-s)v-(s-s^0)u,u')$$

$$\lesssim \frac{w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)}\frac{w_{\tilde\theta}(u)}{w_{\tilde\theta}(u')}\Big(\partial_{u_i}\mathbf{k}(v,u)\mathbf{k}(u,u')+\partial_{u_i}\mathbf{k}(u,u')\mathbf{k}(v,u)\Big)\frac{w_{\tilde\theta}(u')}{w(u')}\|wf^{\ell-2}\|_\infty$$

$$\lesssim \mathbf{k}_{\tilde\varrho}(v,u)\mathbf{k}_{\tilde\varrho}(u,u')\Big(\frac{1}{|v-u|}+\frac{1}{|u-u'|}\Big)\frac{\langle u\rangle}{\langle u'\rangle}\frac{w_{\tilde\theta}(u')\langle u'\rangle}{w(u')}\|wf^{\ell-2}\|_\infty$$

$$\lesssim \mathbf{k}_{\tilde\varrho}(v,u)\mathbf{k}_{\tilde\varrho/2}(u,u')\Big(\frac{1}{|v-u|}+\frac{1}{|u-u'|}\Big)\|wf^{\ell-2}\|_\infty. \tag{4.36}$$

Since $\frac{\mathbf{k}_\varrho(v,u)}{|v-u|}, \frac{\mathbf{k}_\varrho(u,u')}{|u-u'|}\in L^1_u$, the first term of (4.35) is bounded by

$$O(\varepsilon^{-1})\|wf^{\ell-2}\|_\infty. \tag{4.37}$$

For the second term in (4.35), similarly to (4.36) we have

$$w_{\tilde\theta}(v)\mathbf{k}(v,u)\mathbf{k}(u,u')f^{\ell-2}(x-(t-s)v-(s-s^0)u,u')$$
$$\lesssim \mathbf{k}_{\tilde\varrho}(v,u)\mathbf{k}_{\tilde\varrho}(u,u')\|wf^{\ell-2}\|_\infty.$$

Thus the second term is bounded by

$$O(\varepsilon^{-1})\|wf^{\ell-2}\|_\infty. \tag{4.38}$$

From (2.103) and (2.32), we conclude the third term in (4.35) is bounded by

$$\|f^{\ell-2}\|_{L^\infty(\partial\Omega)}\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)}$$
$$\times\int_{\mathbb{R}^3}\mathrm{d}u\mathbf{k}_\varrho(v,u)\frac{\mathbf{1}_{t_{\mathbf{b}}(y,u)\geqq\varepsilon}}{t_{\mathbf{b}}(y,u)}\frac{\partial t_{\mathbf{b}}(y,u)}{\partial u_i}\int_{\mathbb{R}^3}\mathrm{d}u'\mathbf{k}_\varrho(u,u')$$
$$\lesssim \|w_{\tilde\theta}f^{\ell-2}\|_{L^\infty(\partial\Omega)}\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\mathbf{k}_\varrho(v,u)}{\alpha(y,u)}. \tag{4.39}$$

This term will be estimated later using Lemma 4.1.

On the other hand by Lemma 2.13 the contribution of $\{s^0>s-\varepsilon\}$ in (4.34) is controlled by

$$\|w_{\tilde\theta}\alpha\nabla_x f^{\ell-2}\|_\infty\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\,\mathbf{k}(v,u)\frac{w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)}$$
$$\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^s\int_{\mathbb{R}^3}\mathbf{1}_{s^0>s-\varepsilon}$$
$$\times e^{-\nu_0(u)(s-s^0)}\mathbf{k}(u,u')\frac{w_{\tilde\theta}(u)}{w_{\tilde\theta}(u')\alpha(y-(s-s^0)u,u')}\mathrm{d}u'\mathrm{d}s^0$$
$$\lesssim \|w_{\tilde\theta}\alpha\nabla_x f^{\ell-2}\|_\infty\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\,\mathbf{k}_{\tilde\varrho}(v,u)$$
$$\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^s\int_{\mathbb{R}^3}\mathbf{1}_{s^0>s-\varepsilon}$$
$$\times e^{-\nu_0\langle u\rangle(s-s^0)}\mathbf{k}_{\tilde\varrho}(u,u')\frac{1}{\alpha(y-(s-s^0)u,u')}\mathrm{d}u'\mathrm{d}s^0. \tag{4.40}$$

This term will be estimate later using Lemma 4.1.

Expansion of $(3.10)_{(3.10)_*=Kf^{\ell-2}(x^1-(t^1-s^1)v^1,v^1)}$ *and* $(4.33)_{h^{\ell-2}=Kf^{\ell-3}(x^1-(t^1-s^1),v^1)}$. We split

$$
\begin{aligned}
&(3.10)_{(3.10)_*=Kf^{\ell-2}(x^1-(t^1-s^1)v^1,v^1)}\\
&= \underbrace{\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} \mathbf{1}_{s^1\leqq t^1-\varepsilon}\cdots}_{(4.41)_1} + \underbrace{\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} \mathbf{1}_{s^1\geqq t^1-\varepsilon}\cdots}_{(4.41)_2}, \qquad (4.41)\\
&(4.33)_{h^{\ell-2}=Kf^{\ell-3}(x^1-(t^1-s^1),v^1)}\\
&= \underbrace{\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} \mathbf{1}_{s^1\leqq t^1-\varepsilon}\cdots}_{(4.42)_1} + \underbrace{\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1} \mathbf{1}_{s^1\geqq t^1-\varepsilon}\cdots}_{(4.42)_2}. \qquad (4.42)
\end{aligned}
$$

We simply derive an intermediate estimate (see (2.29))

$$
\begin{aligned}
&|(4.41)_2|+|(4.42)_2|\\
&\leqq \sum_{i=0,1}\int_{\max\{t^2,t^1-\varepsilon\}}^{t^1} e^{-\nu_0\langle v^1\rangle(t^1-s^1)}\\
&\quad\times\int_{\mathbb{R}^3}\mathbf{k}_{\mathcal{Q}}(v^1,u')|\nabla_{\mathbf{x}^1_{p^1}}\eta_{p^1}|\frac{\alpha|\nabla_x f^{\ell-2-i}(x^1-(t^1-s^1)v^1,u')|}{\alpha(x^1-(t^1-s^1)v^1,u')}\mathrm{d}u'\mathrm{d}s^1\\
&\leqq \|\eta\|_{C^1}\sum_{i=0,1}\|\alpha\nabla_x f^{\ell-2-i}\|_\infty \sup_{t^1,x^1,v^1}\int_{\max\{t^2,t^1-\varepsilon\}}^{t^1} e^{-\nu_0\langle v^1\rangle(t^1-s^1)}\\
&\quad\times\int_{\mathbb{R}^3}\mathbf{k}_{\mathcal{Q}}(v^1,u')\frac{1}{\alpha(x^1-(t^1-s^1)v^1,u')}\mathrm{d}u'\mathrm{d}s^1. \qquad (4.43)
\end{aligned}
$$

This term, together with (4.40), will be estimate later using Lemma 4.1.

Now we consider $(4.41)_1$ and $(4.42)_1$. Recall (2.29). The key observation is the following interchange of spatial derivatives and velocity derivatives: for $t^1\neq s^1$ and $i=0,1$,

$$
\begin{aligned}
&\partial_{\mathbf{x}^1_{p^1,j}}\Big[\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')\mathrm{d}u'\Big]\\
&=\sum_{\ell=1}^3\frac{\partial\eta_{p^1,\ell}(\mathbf{x}^1_{p^1})}{\partial\mathbf{x}^1_{p^1,j}}\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')\partial_{x_\ell}f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')\mathrm{d}u'\\
&=-\frac{1}{t^1-s^1}\sum_{\ell=1}^3\frac{\partial\eta_{p^1,\ell}(\mathbf{x}^1_{p^1})}{\partial\mathbf{x}^1_{p^1,j}}\\
&\quad\times\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')\partial_{v^1_\ell}[f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')]\mathrm{d}u'\\
&=-\frac{1}{t^1-s^1}\sum_{\ell=1}^3\frac{\partial\eta_{p^1,\ell}(\mathbf{x}^1_{p^1})}{\partial\mathbf{x}^1_{p^1,j}}
\end{aligned}
$$

$$\times \partial_{v_\ell^1}\Big[\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')\mathrm{d}u'\Big] \tag{4.44}$$

$$+\frac{1}{t^1-s^1}\sum_{\ell=1}^{3}\frac{\partial\eta_{p^1,\ell}(\mathbf{x}^1_{p^1})}{\partial\mathbf{x}^1_{p^1,j}}$$

$$\times\int_{\mathbb{R}^3}\partial_{v_\ell^1}\mathbf{k}(v^1,u')f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')\mathrm{d}u'. \tag{4.45}$$

Now we consider the contribution of (4.44) in $(4.41)_1$ and $(4.42.)_1$ inside v^1-integration in (3.8) and (4.31). From the integration by parts with respect to $\partial_{v_\ell^1}$, for $i=0,1$,

$$\begin{aligned}
&\int_{n^1\cdot v^1>0}[\text{the contribution of (4.44) in } (4.41)_1 \text{ and } (4.42)_1]\\
&\quad\times\sqrt{\mu(v^1)}\{n^1\cdot v^1\}\mathrm{d}v^1\\
&=\int_{n^1\cdot v^1>0}\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1}\mathbf{1}_{s^1\leqq t^1-\varepsilon}e^{-\nu(v^1)(t^1-s^1)}\frac{1}{t^1-s^1}\sum_{\ell=1}^{3}\frac{\partial\eta_{p^1,\ell}(\mathbf{x}^1_{p^1})}{\partial\mathbf{x}^1_{p^1,j}}\\
&\quad\times\partial_{v_\ell^1}\Big[\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')\mathrm{d}u'\Big]\mathrm{d}s^1\\
&\quad\times\sqrt{\mu(v^1)}\{n^1\cdot v^1\}\mathrm{d}v^1\\
&=-\int_{n^1\cdot v^1>0}\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1}\frac{\mathbf{1}_{\varepsilon\leqq t^1-s^1}}{t^1-s^1}\sum_{\ell=1}^{3}\frac{\partial\eta_{p^1,\ell}(\mathbf{x}^1_{p^1})}{\partial\mathbf{x}^1_{p^1,j}}\\
&\quad\times\partial_{v_\ell^1}\big[\sqrt{\mu(v^1)}\{n^1\cdot v^1\}e^{-\nu(v^1)(t^1-s^1)}\big]\\
&\quad\times\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')f^{\ell-2-i}(\eta_{p^1}(\mathbf{x}^1_{p^1})-(t^1-s^1)v^1,u')\mathrm{d}u'\mathrm{d}s^1\mathrm{d}v^1\\
&\quad+\int_{n^1\cdot v^1=0}\int_{\max\{0,t^1-t^1_{\mathbf{b}}\}}^{t^1}\cdots\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')f^{\ell-2-i}(\cdot,u')\mathrm{d}u'\mathrm{d}s^1\\
&\quad\times\sqrt{\mu(v^1)}\{n^1\cdot v^1\}\mathrm{d}v^1\\
&\quad+\int_{n^1\cdot v^1>0}\mathbf{1}_{t^1\geqq t^1_{\mathbf{b}}\geqq\varepsilon}\frac{e^{-\nu(v^1)t^1_{\mathbf{b}}}}{t^1_{\mathbf{b}}}\partial_{\mathbf{x}^1_{p^1,j}}\eta\cdot\nabla_{v^1}t^1_{\mathbf{b}}\\
&\quad\times\int_{\mathbb{R}^3}\mathbf{k}(v^1,u')f^{\ell-2-i}(x^2,u')\mathrm{d}u'\sqrt{\mu(v^1)}\{n^1\cdot v^1\}\mathrm{d}v^1\\
&=O(\varepsilon^{-1})\|\eta\|_{C^1}[\|wf^{\ell-2-i}\|_\infty+\|f^{\ell-2-i}\|_{L^\infty(\partial\Omega)}]\\
&\quad\times\Big\{1+\int_{n^1\cdot v^1>0}\frac{|n^1\cdot v^1|}{|n^2\cdot v^1|}\sqrt{\mu(v^1)}\mathrm{d}v^1\Big\}\\
&=O(\varepsilon^{-1})\|\eta\|_{C^1}[\|wf^{\ell-2-i}\|_\infty+\|w_{\bar\theta}f^{\ell-2-i}\|_{L^\infty(\partial\Omega)}].
\end{aligned}\tag{4.46}$$

Here we have used (2.32) and (1.10). Also we have used $\|wf^{\ell-2-i}\|_\infty<\infty$ and $\|f^{\ell-2-i}\|_{L^\infty(\partial\Omega)}<\infty$ from Lemma 3.2 to derived $\int_{n^1\cdot v^1=0}\cdots=0$.

From (2.103), (1.6) and $t^1 - s^1 \geqq \varepsilon$, we bound the contribution of (4.45) in $(4.41)_1$ and $(4.42)_1$ by

$$\begin{aligned}&\int_{n(x^1)\cdot v^1>0} [\text{the contribution of (4.45) of } (4.41)_1 \text{ and } (4.42)_1] \\ &\quad \times \sqrt{\mu(v^1)}\{n^1\cdot v^1\}\mathrm{d}v^1 = O(\varepsilon^{-1})\|\eta\|_{C^1}\sup_{i=0,1}\|wf^{\ell-2-i}\|_\infty.\end{aligned} \tag{4.47}$$

Now we consider contributions of Γ in (3.7), (3.10), (4.30), and (4.33). From (2.108)

$$\begin{aligned}&|(3.7)_{(3.7)_*=\Gamma(f^{\ell-1},f^{\ell-1})}| + |(3.10)_{(3.10)_*=\Gamma(f^{\ell-2},f^{\ell-2})}| \\ &\quad + |(4.30)| + |(4.33)_{h=\Gamma(f^{\ell-3},f^{\ell-3})}| \\ &\leqq \sup_i O(\|wf^{\ell-1-i}\|_\infty) \\ &\quad \times \sup_i \sum_{j=0,1}\Big\{\int_{\max\{0,t^j-t^j_{\mathbf{b}}\}}^{t^j} e^{-\nu_0\langle v^j\rangle(t^j-s^j)} w_{\tilde\theta}(v^j) \\ &\quad \times |\nabla_x f^{\ell-1-i}(x^j-(t^j-s^j)v^j, v^j)|\mathrm{d}u\mathrm{d}s^j \\ &\quad + \int_{\max\{0,t^j-t^j_{\mathbf{b}}\}}^{t^j} e^{-\nu_0\langle v^j\rangle(t^j-s^j)} \\ &\quad \times \int_{\mathbb{R}^3}\mathbf{k}(v^j,u)\frac{w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)}|\nabla_x f^{\ell-1-i}(x^j-(t^j-s^j)v^j,u)|\mathrm{d}u\mathrm{d}s^j\Big\} \\ &\leqq \sup_i O(\|wf^{\ell-1-i}\|_\infty)\sup_i\|w_{\tilde\theta}\alpha\nabla_x f^{\ell-1-i}\|_\infty \\ &\quad \times \Big\{\sum_{j=0,1}\int_{\max\{0,t^j-t^j_{\mathbf{b}}\}}^{t^j} e^{-\nu_0\langle v^j\rangle(t^j-s^j)}\frac{1}{\alpha(x^j-(t^j-s^j)v^j,v^j)}\mathrm{d}s \qquad (4.48) \\ &\quad + \sum_{j=0,1}\int_{\max\{0,t^j-t^j_{\mathbf{b}}\}}^{t^j} e^{-\nu_0\langle v^j\rangle(t^j-s^j)} \\ &\quad \times \int_{\mathbb{R}^3}\mathbf{k}_{\tilde Q}(v^j,u)\frac{1}{\alpha(x^j-(t^j-s^j)v^j,u)}\mathrm{d}u\mathrm{d}s^j\Big\}, \qquad (4.49)\end{aligned}$$

where we have used Lemma 2.13 in the last line.

From (1.10)

$$(4.48) \leqq e^{C_\Omega}\sum_{j=0,1}\int_0^t e^{-\nu_0\langle v^j\rangle(t^j-s^j)}\mathrm{d}s^j \times \alpha(x^j,v^j)^{-1} \leqq \frac{C_\Omega}{\nu_0\langle v^j\rangle}\alpha(x^j,v^j)^{-1}. \tag{4.50}$$

Step 2. Estimate of (4.49), (4.43), (4.40) and (4.39) using Lemma 4.1.

Clearly $\|\alpha\nabla_x f^\ell\|_\infty \leqq \|w_{\tilde\theta}\alpha\nabla_x f^\ell\|_\infty$, applying Lemma 4.1 we bound

$$|(4.39)| \leqq \frac{O(\|f^{\ell-2}\|_{L^\infty(\partial\Omega)})}{\alpha(x,v)},$$

$$|(4.40)| + |(4.43)| \leqq \frac{O(\varepsilon)}{\alpha(x,v)} \sup_{0\leqq i\leqq \ell-1} \|w_{\tilde\theta}\alpha\nabla_x f^{\ell-1-i}\|_\infty, \tag{4.51}$$

$$|(4.49)| \leqq \frac{O(\sup_{0\leqq i\leqq \ell-1}\|wf^{\ell-1-i}\|_\infty)}{\alpha(x,v)} \sup_{0\leqq i\leqq \ell-1} \|w_{\tilde\theta}\alpha\nabla_x f^{\ell-1-i}\|_\infty.$$

First we prove (4.1). From (4.6)

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u\, \mathbf{k}(v,u)(4.28)$$
$$\lesssim \frac{1+\|\theta\|_{C^1}}{\alpha(x,v)}(1+\sup_{i=0,1}\|w_{\tilde\theta} f^{\ell-1-i}\|_{L^\infty(\partial\Omega)}),$$
$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u\, \mathbf{k}(v,u)\{(4.29)+(4.31)(4.32)\}$$
$$\lesssim \frac{e^{-\nu_0 t}}{\alpha(x,v)}\|w_{\tilde\theta}\alpha\nabla_x f^{\ell-1}\|_\infty.$$

From (4.48)–(4.50) and (4.51)

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)}$$
$$\times \int_{\mathbb{R}^3} \mathrm{d}u\, \mathbf{k}(v,u)\{(4.30)+(4.31)(4.33)_{h^{\ell-2}=\Gamma(f^{\ell-3},f^{\ell-3})}\}$$
$$\leqq \frac{O(\sup_{0\leqq i\leqq \ell-1}\|wf^{\ell-1-i}\|_\infty)}{\alpha(x,v)} \sup_{0\leqq i\leqq \ell-1}\|w_{\tilde\theta}\alpha\nabla_x f^{\ell-1-i}\|_\infty.$$

From (4.42) $\leqq$ $(4.42)_1$ + (4.43) and (4.46), (4.47), (4.51)

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s\, e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u\, \mathbf{k}(v,u)(4.33)_{h^{\ell-2}=Kf^{\ell-3}(x^1-(t^1-s^1),v^1)}$$
$$\leqq \frac{1}{\alpha(x,v)} \times \Big\{ O(\varepsilon) \sup_{0\leqq i\leqq \ell-1}\|\alpha\nabla_x f^{\ell-1-i}\|_\infty$$
$$+ O(\varepsilon^{-1})[\sup_{0\leqq i\leqq \ell-1}\|wf^{\ell-1-i}\|_\infty + \sup_{0\leqq i\leqq \ell-1}\|w_{\tilde\theta} f^{\ell-1-i}\|_{L^\infty(\partial\Omega)}]\Big\}.$$

From $|(4.34)| \leqq |(4.35)| + |(4.40)|$ and (4.51),

$$|(4.34)| \leqq \frac{1}{\alpha(x,v)} \times \Big\{ O(\varepsilon^{-1})[\|wf^{\ell-2}\|_\infty + \sup_{0\leqq i\leqq \ell-1}\|w_{\tilde\theta} f^{\ell-1-i}\|_{L^\infty(\partial\Omega)}]$$
$$+ O(\varepsilon) \sup_{0\leqq i\leqq \ell-1}\|w_{\tilde\theta}\alpha\nabla_x f^{\ell-1-i}\|_\infty\Big\}.$$

By Lemma 3.2, we have

$$\sup_{i\geqq 0}\|w_{\tilde\theta} f^i\|_{L^\infty(\partial\Omega)} \lesssim \sup_{i\geqq 0}\|wf^i\|_\infty + \sup_{i\geqq 1}\|wf^{i-1}\|_\infty. \tag{4.52}$$

Collecting the terms and applying (4.52), we complete the proof of (4.1).

The proof of (4.2) comes from (4.41), (4.43), (4.46), (4.47) and (4.52). We prove (4.3) from (4.48)–(4.50) and (4.2).

5. C^1 Estimate of Tangential Derivative and Continuity of C^1 Solution

5.1. C^1 estimate of tangential derivative

In this subsection we prove (1.19) in the **Main Theorem**. Section 3 and 4 already conclude the estimate (1.18), from now on we will drop the super index in f^ℓ and only analyze the property of $\nabla_x f$.

Proof of (1.19) For $x \in \Omega$ we use (2.10)–(2.14) to have

$$G(x)\nabla_x f(x,v) = \mathbf{1}_{t\geqq t_{\mathbf{b}}} e^{-\nu(v)t_{\mathbf{b}}} G(x) \sum_{i=1,2} \nabla_x \mathbf{x}^1_{p^1,i} \partial_{\mathbf{x}^1_{p^1,i}} f(\eta_{p^1}(\mathbf{x}^1_{p^1}), v) \tag{5.1}$$

$$- \mathbf{1}_{t\geqq t_{\mathbf{b}}} \nu(v) \nabla_x t_{\mathbf{b}} e^{-\nu(v)t_{\mathbf{b}}} G(x) f(x_{\mathbf{b}}(x,v), v) \tag{5.2}$$

$$+ \mathbf{1}_{t<t_{\mathbf{b}}} e^{-\nu(v)t} G(x) \nabla_x f(x - tv, v) \tag{5.3}$$

$$- \mathbf{1}_{t\geqq t_{\mathbf{b}}} G(x) \nabla_x t_{\mathbf{b}} e^{-\nu(v)t_{\mathbf{b}}} h(x - t_{\mathbf{b}} v, v) \tag{5.4}$$

$$+ \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} G(x) e^{-\nu(v)(t-s)} \nabla_x h(x - (t-s)v, v) \mathrm{d}s, \tag{5.5}$$

where $h = K(f) + \Gamma(f,f)$.

We focus on the estimate of (5.5). (5.1)-(5.4) will be estimated with (5.5) together.

Estimate of (5.5) **with** $h = K(f)$. Let $y = x - (t-s)v$. Rewriting $G(x) = G(x) - G(y) + G(y)$ and applying (2.48) in Lemma 2.6 to $G(x) - G(y)$ we have

$$|(5.5)\mathbf{1}_{h=K(f)}| \lesssim \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} G(y) \int_{\mathbb{R}^3} \mathbf{k}(v,u) \nabla_x f(y,u) \mathrm{d}u\mathrm{d}s \Big| \tag{5.6}$$

$$+ \Big| \frac{\tilde{\alpha}(x,v)}{|v|} \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_{\mathbb{R}^3} \mathbf{k}(v,u) \nabla_x f(y,u) \mathrm{d}u\mathrm{d}s \Big|. \tag{5.7}$$

Then applying (4.5) in Lemma 4.1 with $y = x - (t-s)v$ and (2.122) we obtain

$$(5.7) \lesssim \frac{\tilde{\alpha}(x,v)}{|v| w_{\tilde{\theta}}(v)} \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_{\mathbb{R}^3} \frac{\mathbf{k}(v,u)}{\alpha(y,u)} \frac{w_{\tilde{\theta}}(v)}{w_{\tilde{\theta}}(u)} w_{\tilde{\theta}}(u) \alpha(y,u) \nabla_x f(y,u) \mathrm{d}u\mathrm{d}s$$

$$\lesssim \|w_{\tilde{\theta}} \alpha \nabla_x f\|_\infty \frac{\alpha(x,v)}{w_{\tilde{\theta}/2}(v)|v|} \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_{\mathbb{R}^3} \frac{\mathbf{k}_{\tilde{\varrho}}(v,u)}{\alpha(y,u)} \mathrm{d}u\mathrm{d}s \lesssim \frac{\|w_{\tilde{\theta}} \alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|}.$$

We focus on (5.6). We further expand $G(y)\nabla_x f(y,u)$ along u:

(5.6)

$$\lesssim \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)}$$

$$\times \int_{\mathbb{R}^3} \mathbf{k}(v,u) G(y) \sum_{i=1,2} \nabla_x \mathbf{x}^1_{p^1(u),i} \partial_{\mathbf{x}^1_{p^1(u),i}} f(\eta_{p^1(u)}(\mathbf{x}^1_{p^1(u)}),u) \mathrm{d}u \mathrm{d}s \Big| \tag{5.8}$$

$$+ \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathbf{k}(v,u) G(y) \nabla_x t_{\mathbf{b}}(y,u) f(y - t_{\mathbf{b}}(y,u)u, u) \mathrm{d}u \mathrm{d}s \Big| \tag{5.9}$$

$$+ \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} e^{-\nu(u)s} \int_{\mathbb{R}^3} \mathbf{k}(v,u) G(y) \nabla_x f(y - su, u) \mathrm{d}u \mathrm{d}s \Big| \tag{5.10}$$

$$+ \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)}$$

$$\times \int_{\mathbb{R}^3} \mathbf{k}(v,u) G(y) \nabla_x t_{\mathbf{b}}(y,u) e^{-\nu(u) t_{\mathbf{b}}(y,u)} h(y - t_{\mathbf{b}}(y,u)u, u) \mathrm{d}u \mathrm{d}s \Big| \tag{5.11}$$

$$+ \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \mathrm{d}s$$

$$\times \int_{\mathbb{R}^3} \mathbf{k}(v,u) \mathrm{d}u G(y) \int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s} \mathrm{d}s' \int_{\mathbb{R}^3} \mathbf{k}(u,u') \nabla_x f(y - (s-s')u, u') \Big| \tag{5.12}$$

$$+ \Big| \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \mathrm{d}s$$

$$\times \int_{\mathbb{R}^3} \mathbf{k}(v,u) \mathrm{d}u G(y) \int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s} \mathrm{d}s' \nabla_x \Gamma(f,f)(y - (s-s')u, u') \Big|.$$

In (5.8) we denoted $x_{\mathbf{b}}(x-(t-s)v,u) = \eta_{p^1(u)}(\mathbf{x}^1_{p^1(u)})$. (5.13)

Then we estimate (5.1)–(5.4) together with (5.8)–(5.11).

First we estimate (5.1) and (5.8). We start from (5.1). From Section 3 we have

$$|\partial_{\mathbf{x}^1_{p^1,i}} f(\eta_{p^1}(\mathbf{x}^1_{p^1}),v)| = (3.19) + (3.20) \lesssim (3.21) + (3.24) + (3.27) + (3.34)$$
$$\lesssim \frac{M_W(x_{\mathbf{b}},v)}{\sqrt{\mu(v)}} \|T_W - T_0\|_{C^1} [\|wf\|_\infty + \|w_{\tilde\theta} f\|_{L^\infty(\partial\Omega)}].$$

Thus by (2.51)

$$|(5.1)| \lesssim \frac{\|T_W - T_0\|_{C^1} [\|wf\|_\infty + \|w_{\tilde\theta} f\|_{L^\infty(\partial\Omega)}]}{w_{\tilde\theta/2}(v)|v|}. \tag{5.14}$$

For (5.8) similarly by (2.51) we apply (2.122) and (2.104) with $c=1$ to have

$$(5.8) \lesssim \frac{\|T_W - T_0\|_{C^1} [\|wf\|_\infty + \|w_{\tilde\theta} f\|_{L^\infty(\partial\Omega)}]}{w_{\tilde\theta/2}(v)}$$
$$\times \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)} \frac{1}{|u|} \mathrm{d}u \mathrm{d}s$$
$$\lesssim \frac{[\|wf\|_\infty + \|w_{\tilde\theta} f\|_{L^\infty(\partial\Omega)}]}{w_{\tilde\theta/2}(v)|v|}. \tag{5.15}$$

Then we estimate (5.2) and (5.9). We start from (5.2). By (2.52) we conclude

$$|(5.2)| \lesssim \frac{\|w_{\tilde{\theta}/2} f\|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|} \lesssim \frac{\|w_{\tilde{\theta}} f\|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}. \tag{5.16}$$

For (5.9) similarly by (2.52) we apply (2.104) with $c = 1$ to have

$$(5.9) \lesssim \frac{\|w_{\tilde{\theta}/2} f\|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)} \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} \frac{1}{|u|} \mathrm{d}u\mathrm{d}s \lesssim \frac{\|w_{\tilde{\theta}} f\|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}. \tag{5.17}$$

Then we estimate (5.3) and (5.10). For (5.3) we apply (2.49) to have

$$|(5.3)| \lesssim e^{-t}\Big[\frac{\|w_{\tilde{\theta}/2}|v|\nabla_\parallel f(x,v)\|_\infty}{w_{\tilde{\theta}/2}(v)|v|} + \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|}\Big]. \tag{5.18}$$

Similarly for (5.10) applying (2.49) and (2.104) with $c = 1$ we have

$$\begin{aligned}(5.10) &\lesssim e^{-t} \frac{\|w_{\tilde{\theta}/2}|v|\nabla_\parallel f\|_\infty + \|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)} \\ &\quad \times \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} \frac{1}{|u|} \mathrm{d}u\mathrm{d}s \\ &\lesssim e^{-t}\Big[\frac{\|w_{\tilde{\theta}/2}|v|\nabla_\parallel f(x,v)\|_\infty}{|v|} + \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|}\Big].\end{aligned} \tag{5.19}$$

Then we estimate (5.4) and (5.11). For (5.4), by (2.52) and (3.3) we have

$$\begin{aligned}|(5.4)| &\lesssim \frac{K(f)+\Gamma(f,f)}{|v|} \lesssim \frac{(\|f\|_{L^\infty(\partial\Omega)}+1)}{|v|} \int_{\mathbb{R}^3} |\mathbf{k}(v,u) f(x-t_{\mathbf{b}}v,u)| \mathrm{d}u \\ &\lesssim \frac{(\|f\|_{L^\infty(\partial\Omega)}+1)\|w_{\tilde{\theta}/2} f\|_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|} \int_{\mathbb{R}^3} \mathbf{k}_{\varrho}(v,u) \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} \mathrm{d}u \\ &\lesssim \frac{\|w_{\tilde{\theta}} f\|_{L^\infty(\partial\Omega)} + \|w_{\tilde{\theta}} f\|^2_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}.\end{aligned} \tag{5.20}$$

For (5.11) similarly by (2.52) and (2.104) with $c = 1$ we have

$$\begin{aligned}(5.11) &\lesssim \frac{\|w_{\tilde{\theta}} f\|_{L^\infty(\partial\Omega)} + \|w_{\tilde{\theta}} f\|^2_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)} \\ &\quad \times \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{w_{\tilde{\theta}/2}(v)}{w_{\tilde{\theta}/2}(u)} \frac{1}{|u|} \\ &\lesssim \frac{\|w_{\tilde{\theta}} f\|_{L^\infty(\partial\Omega)} + \|w_{\tilde{\theta}} f\|^2_{L^\infty(\partial\Omega)}}{w_{\tilde{\theta}/2}(v)|v|}.\end{aligned} \tag{5.21}$$

Last we estimate (5.12), this estimate is the most delicate one. We apply the decomposition (4.34) to $\mathrm{d}s'$.

When $s' > s - \varepsilon$, we apply (2.49) in Lemma 2.6 to have

$$
\begin{aligned}
&(5.12)\mathbf{1}_{s'>s-\varepsilon} \\
&\lesssim \frac{\||v|\nabla_{\|} f\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)} \qquad (5.22)\\
&\quad\times \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\int_{\mathbb{R}^3}\mathbf{k}(v,u)\frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)}\mathrm{d}u\mathrm{d}s\int_{s-\varepsilon}^{s}\mathrm{d}s'\int_{\mathbb{R}^3}\mathbf{k}(u,u')\frac{w_{\tilde\theta/2}(u)}{w_{\tilde\theta/2}(u')}\frac{|u|}{|u'|}\frac{|v|}{|u|}\\
&\lesssim o(1)\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty + \|w_{\tilde\theta/2}|v|\nabla_{\|} f\|_\infty}{w_{\tilde\theta/2}(v)|v|}, \qquad (5.23)
\end{aligned}
$$

where we have applied (2.104) twice with $c = 1$.

On the other hand when $s' \leqq s - \varepsilon$, we exchange ∇_x for ∇_u:

$$
\nabla_x f(x-(t-s)v-(s-s')u,u') = \frac{-1}{s-s'}\nabla_u[f(x-(t-s)v-(s-s')u,u')].
$$

Then we perform an integration by parts with respect to $\mathrm{d}u$ and obtain

$$
\begin{aligned}
&(5.12)\mathbf{1}_{s'\leqq s-\varepsilon}\\
&= \Big|\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\\
&\quad\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s}\mathrm{d}s'\, e^{-\nu(u)(s-s')}\frac{\mathbf{1}_{s'\leqq s-\varepsilon}}{s-s'}\\
&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u' G(y)\,\nabla_u[\mathbf{k}(v,u)\mathbf{k}(u,u')]f(y-(s-s')u,u')\\
&\quad-\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\\
&\quad\times\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s}\mathrm{d}s'\,\nabla_u\nu(u)e^{-\nu(u)(s-s')}\mathbf{1}_{s'\leqq s-\varepsilon}\\
&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u' G(y)\,\mathbf{k}(v,u)\mathbf{k}(u,u')f(y-(s-s')u,u')\\
&\quad+\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s\, e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\\
&\quad\times\mathbf{1}_{s\geqq t_{\mathbf{b}}(y,u)}e^{-\nu(u)t_{\mathbf{b}}(y,u)}\frac{\mathbf{1}_{t_{\mathbf{b}}(y,u)\geqq\varepsilon}}{t_{\mathbf{b}}(y,u)}\nabla_u t_{\mathbf{b}}(y,u)\\
&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u' G(y)\,\mathbf{k}(v,u)\mathbf{k}(u,u')f(y-t_{\mathbf{b}}(y,u)u,u')\Big|. \qquad (5.24)
\end{aligned}
$$

We bound $|G(y)| \leqq 1$. Then applying (2.103) and (2.106) the first and second terms of (5.24) are bounded by

$$
\frac{O(\varepsilon^{-1})\|w_{\tilde\theta}f\|_\infty}{w_{\tilde\theta/2}(v)}\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}\mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^{s}\mathrm{d}s' e^{-\nu(u)(s-s')}
$$

$$\times \int_{\mathbb{R}^3} \frac{\mathbf{k}(v,u)\langle u\rangle^2}{|v-u|}\frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta}(u)}\frac{|u|}{|u|}\frac{\mathbf{k}(u,u')}{|u-u'|}\frac{w_{\tilde\theta}(u)}{w_{\tilde\theta}(u')}$$
$$\lesssim \frac{O(\varepsilon^{-1})\|wf\|_\infty}{w_{\tilde\theta/2}(v)}\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\int_{\max\{0,s-t_{\mathbf{b}}(y,u)\}}^s \mathrm{d}s' e^{-\nu(u)(s-s')}$$
$$\times \int_{\mathbb{R}^3}\frac{\mathbf{k}(v,u)}{|v-u|}\frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)}\frac{1}{|u|}\frac{\mathbf{k}_{\tilde\varrho}(u,u')}{|u-u'|}\lesssim \frac{O(\varepsilon^{-1})\|wf\|_\infty}{w_{\tilde\theta/2}(v)|v|}.$$

where we have used (2.122), $\langle u\rangle^2|u|w_{\tilde\theta}^{-1}(u)\lesssim w_{\tilde\theta/2}^{-1}(u)$ and (2.104) with $c=1$.

For the third term we apply (2.32) and (2.48) to have

$$\left|G(y)\frac{\nabla_u t_{\mathbf{b}}(y,u)}{t_{\mathbf{b}}(y,u)}\right| = \left|\frac{G(y)n(x_{\mathbf{b}}(y,u))}{n(x_{\mathbf{b}}(y,u))\cdot u}\right|$$
$$= \left|\frac{[G(y)-G(x_{\mathbf{b}}(y,u))]n(x_{\mathbf{b}}(y,u))}{n(x_{\mathbf{b}}(y,u))\cdot u}+\frac{G(x_{\mathbf{b}}(y,u))n(x_{\mathbf{b}}(y,u))}{n(x_{\mathbf{b}}(y,u))\cdot u}\right|$$
$$\lesssim \frac{\tilde\alpha(y,u)}{|n(x_{\mathbf{b}}(y,u))\cdot u||u|}\lesssim\frac{1}{|u|}. \tag{5.25}$$

Thus applying (2.104) with $c=1$ the third term is bounded by

$$\frac{\|w_{\tilde\theta/2}f\|_{L^\infty(\partial\Omega)}}{w_{\tilde\theta/2}(v)}\int_{\max\{0,t-t_{\mathbf{b}}\}}^t \mathrm{d}s e^{-\nu(v)(t-s)}$$
$$\times\int_{\mathbb{R}^3}\mathrm{d}u\mathbf{k}(v,u)\frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)}\frac{1}{|u|}e^{-\nu t_{\mathbf{b}}(y,u)}\int_{\mathbb{R}^3}\mathbf{k}(u,u')\lesssim\frac{\|w_{\tilde\theta}f\|_{L^\infty(\partial\Omega)}}{w_{\tilde\theta/2}(v)|v|}.$$

Therefore, we conclude

$$(5.12)\mathbf{1}_{s'<s-\varepsilon}\lesssim O(\varepsilon^{-1})\frac{\|wf\|_\infty+\|w_{\tilde\theta}f\|_{L^\infty(\partial\Omega)}}{w_{\tilde\theta/2}(v)|v|}. \tag{5.26}$$

We estimate (5.13) together with (5.5)$\mathbf{1}_{h=\Gamma}$. We apply (2.109), (2.49) and (4.11) with $c=1$ to have

$$|(5.5)\mathbf{1}_{h=\Gamma}|$$
$$\lesssim\int_{\max\{t-t_{\mathbf{b}}\}}^t e^{-\nu(v)(t-s)}\|wf\|\|w_{\tilde\theta}f\|_\infty$$
$$\times\left|G(x)\nabla_x f(x-(t-s)v)+\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(v,u)|G(x)\nabla_x f(x-(t-s)v,u)|\right|$$
$$\lesssim\frac{\|wf\|_\infty[\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty+\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty]}{w_{\tilde\theta/2}(v)}$$
$$\times\left[\frac{1}{|v|}+\int_{\max\{0,t-t_{\mathbf{b}}\}}^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(v,u)\frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)}\frac{1}{|u|}\mathrm{d}u\mathrm{d}s\right]$$
$$\lesssim\|wf\|_\infty\frac{\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty}{|v|}+\frac{\|wf\|_\infty\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{|v|}$$

$$\lesssim \frac{\|wf\|_\infty[\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty]}{w_{\tilde\theta/2}(v)|v|}. \tag{5.27}$$

Then, similarly, we have

(5.13)

$$\begin{aligned}
&\lesssim \int_{\max\{t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_\varrho(v,u) \frac{\|wf\|_\infty[\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty]}{w_{\tilde\theta/2}(u)|u|} \\
&\lesssim \frac{\|wf\|_\infty[\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty]}{w_{\tilde\theta/2}(v)} \\
&\quad\times \int_{\max\{t-t_{\mathbf{b}}\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathbf{k}_\varrho(v,u) \frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)} \frac{1}{|u|} \\
&\lesssim \frac{\|wf\|_\infty[\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty]}{w_{\tilde\theta/2}(v)|v|}.
\end{aligned} \tag{5.28}$$

Now we estimate $\|w_{\tilde\theta} f\|_{L^\infty(\partial\Omega)}$. Similar to Lemma 3.2, we let $h = K(f) + \Gamma(f, f)$, following the same proof as to that of Lemma 3.2 and **Existence Theorem**, we have

$$\|w_{\tilde\theta} f\|_{L^\infty(\partial\Omega)} \lesssim \|wf\|_\infty = o(1). \tag{5.29}$$

With the above estimate, we combine (5.15), (5.17),(5.19),(5.21),(5.23), (5.26), (5.27) and (5.28), then we conclude

$$(5.5) \lesssim \big[o(1) + e^{-t}\big]\Big[\frac{\|wf\|_\infty}{w_{\tilde\theta/2}(v)|v|} + \frac{\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty}{w_{\tilde\theta/2}(v)|v|}\Big] + \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)|v|}. \tag{5.30}$$

Combining (5.18),(5.14), (5.30),(5.16) and (5.29) we conclude

$$\begin{aligned}
|\nabla_\parallel f| \lesssim\ & O(\varepsilon^{-1})\Big[\frac{\|wf\|_\infty}{w_{\tilde\theta/2}|v|} + \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}|v|}\Big] \\
&+\big[\|wf\|_\infty + e^{-t}\big]\frac{\|w_{\tilde\theta/2}|v|\nabla_\parallel f\|_\infty}{w_{\tilde\theta/2}|v|}.
\end{aligned} \tag{5.31}$$

Then from $t \gg 1$ and $\|wf\|_\infty \ll 1$ in the **Existence Theorem** we conclude (1.19). □

5.2. *Continuity of C^1 solution*

In this section we prove the continuity of $\nabla_x f$. The continuity of $G(x)\nabla_x f$ will follow directly from the continuity of $\nabla_x f$. We only need to prove $\nabla_x f$ is continuous at $t = t_{\mathbf{b}}(x, v)$. We consider (5.1) - (5.5) without $G(x)$. When $t = t_{\mathbf{b}}(x, v)$, (5.3) reads

$$e^{-\nu t_{\mathbf{b}}} \nabla_x f(x - t_{\mathbf{b}} v, v).$$

At the boundary $x - t_{\mathbf{b}} v = x_{\mathbf{b}}(x, v) = \eta_{p^1}(\mathbf{x}^1_{p^1})$, we use the notation (2.28),(2.29) and decompose the spatial derivative as

$$v \cdot \nabla_x f = \sum_{i=1}^{2} \mathbf{v}^1_{p^1,i} \frac{\partial_{\mathbf{x}^1_{p^1,i}} f}{\sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} + \mathbf{v}^1_{p^1,3} \frac{\partial_{\mathbf{x}^1_{p^1,3}} f}{\sqrt{g_{p^1,33}(\mathbf{x}_{p^1})}}.$$

Then from the equation (2.3), we derive

$$\frac{\partial_{\mathbf{x}^1_{p^1,3}} f}{\sqrt{g_{p^1,33}(\mathbf{x}_{p^1})}} = \underbrace{-\sum_{i=1}^{2} \frac{\mathbf{v}^1_{p^1,i} \partial_{\mathbf{x}^1_{p^1,i}} f}{\mathbf{v}^1_{p^1,3} \sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}}}_{(5.32)_1} - \underbrace{\frac{\nu(v) f}{\mathbf{v}^1_{p^1,3}}}_{(5.32)_2} + \underbrace{\frac{K(f) + \Gamma(f,f)}{\mathbf{v}^1_{p^1,3}}}_{(5.32)_3} \tag{5.32}$$

Plugging these terms back to the spatial derivative $\nabla_x f$, we derive that

$$\begin{aligned} \nabla_x f &= \nabla_x f T T^t \\ &= \left(\frac{\partial_{\mathbf{x}^1_{p^1,1}} f}{\sqrt{g_{p^1,11}(\mathbf{x}_{p^1})}}, \frac{\partial_{\mathbf{x}^1_{p^1,2}} f}{\sqrt{g_{p^1,22}(\mathbf{x}_{p^1})}}, (5.32) \right) \begin{pmatrix} \frac{\partial_1 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,11}(\mathbf{x}^1_{p^1})}} \\ \frac{\partial_2 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,22}(\mathbf{x}^1_{p^1})}} \\ \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \end{pmatrix}. \end{aligned} \tag{5.33}$$

Then the contribution of $(5.32)_1$ in (5.33) is

$$\begin{aligned} &\left(\frac{\partial_{\mathbf{x}^1_{p^1,1}} f}{\sqrt{g_{p^1,11}(\mathbf{x}_{p^1})}}, \frac{\partial_{\mathbf{x}^1_{p^1,2}} f}{\sqrt{g_{p^1,22}(\mathbf{x}_{p^1})}}, (5.32)_1 \right) \begin{pmatrix} \frac{\partial_1 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,11}(\mathbf{x}^1_{p^1})}} \\ \frac{\partial_2 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,22}(\mathbf{x}^1_{p^1})}} \\ \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{2} \left[\frac{\partial_{\mathbf{x}^1_{p^1,i}} f}{\sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} \frac{\partial_i \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,ii}(\mathbf{x}^1_{p^1})}} - \frac{\mathbf{v}^1_{p^1,i} \partial_{\mathbf{x}^1_{p^1,i}} f}{\mathbf{v}^1_{p^1,3} \sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \right] \cdot e_1 \\ \sum_{i=1}^{2} \left[\frac{\partial_{\mathbf{x}^1_{p^1,i}} f}{\sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} \frac{\partial_i \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,ii}(\mathbf{x}^1_{p^1})}} - \frac{\mathbf{v}^1_{p^1,i} \partial_{\mathbf{x}^1_{p^1,i}} f}{\mathbf{v}^1_{p^1,3} \sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \right] \cdot e_2 \\ \sum_{i=1}^{2} \left[\frac{\partial_{\mathbf{x}^1_{p^1,i}} f}{\sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} \frac{\partial_i \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,ii}(\mathbf{x}^1_{p^1})}} - \frac{\mathbf{v}^1_{p^1,i} \partial_{\mathbf{x}^1_{p^1,i}} f}{\mathbf{v}^1_{p^1,3} \sqrt{g_{p^1,ii}(\mathbf{x}_{p^1})}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \right] \cdot e_3 \end{pmatrix}^t, \end{aligned}$$

which is exactly the same as $\sum_{i=1,2} \nabla_x \mathbf{x}^1_{p^1,i} \partial_{\mathbf{x}^1_{p^1,i}} f(\eta_{p^1}(\mathbf{x}^1_{p^1}), v)$ in (5.1) by applying (2.33).

The contribution of $(5.32)_2$ and $(5.32)_3$ in (5.33) are

$$((5.32)_2 + (5.32)_3) \cdot \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}}$$

$$= -\begin{pmatrix} \frac{\nu(v)f}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \cdot e_1 \\ \frac{\nu(v)f}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \cdot e_2 \\ \frac{\nu(v)f}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \cdot e_3 \end{pmatrix}^t + \begin{pmatrix} \frac{K(f)+\Gamma(f,f)}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \cdot e_1 \\ \frac{K(f)+\Gamma(f,f)}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \cdot e_2 \\ \frac{K(f)+\Gamma(f,f)}{\mathbf{v}^1_{p^1,3}} \frac{\partial_3 \eta_{p^1}(\mathbf{x}^1_{p^1})}{\sqrt{g_{p^1,33}(\mathbf{x}^1_{p^1})}} \cdot e_3 \end{pmatrix}^t,$$

which is exactly the same as $-\nu(v)\nabla_x t_{\mathbf{b}} f$ in (5.3) and $\nabla_x t_{\mathbf{b}} h$ in (5.4) by applying (2.32).

Thus $\nabla_x f$ is continuous at $t = t_{\mathbf{b}}$.

6. C^1_v Estimate

In this section we prove the C^1_v estimate, which is (1.20) in **Main Theorem**.

Proof of (1.20) We take the v derivative to (2.9) and have

$$\begin{aligned}
\nabla_v f(x,v) =& \mathbf{1}_{t \geqq t_{\mathbf{b}}} e^{-\nu t_{\mathbf{b}}} \nabla_v [f(x_{\mathbf{b}}, v)] && (6.1)\\
&- \mathbf{1}_{t \geqq t_{\mathbf{b}}} \nu \nabla_v t_{\mathbf{b}}(x,v) e^{-\nu t_{\mathbf{b}}} f(x_{\mathbf{b}}, v) && (6.2)\\
&- \mathbf{1}_{t \geqq t_{\mathbf{b}}} \nabla_v \nu(v) t_{\mathbf{b}} e^{-\nu t_{\mathbf{b}}} f(x_{\mathbf{b}}, v) && (6.3)\\
&+ \mathbf{1}_{t \leqq t_{\mathbf{b}}} e^{-\nu t} \nabla_v [f(x - tv, v)] && (6.4)\\
&- \mathbf{1}_{t \leqq t_{\mathbf{b}}} \nabla_v \nu(v) t e^{-\nu t} f(x - tv, v) && (6.5)\\
&- \int_{\max\{0, t - t_{\mathbf{b}}\}}^{t} \nabla_v \nu(v) e^{-\nu(t-s)} (t-s) h(x - (t-s)v, v) \mathrm{d}s && (6.6)\\
&+ \int_{\max\{0, t - t_{\mathbf{b}}\}}^{t} e^{-\nu(t-s)} \nabla_v [h(x - (t-s)v, v)] \mathrm{d}s && (6.7)\\
&- \mathbf{1}_{t \geqq t_{\mathbf{b}}} \nabla_v t_{\mathbf{b}}(x,v) e^{-\nu t_{\mathbf{b}}} h(x - t_{\mathbf{b}} v, v). && (6.8)
\end{aligned}$$

Note that from Section 5.2, we have $\|wf\|_{L^\infty(\partial\Omega)} \leqq \|wf\|_\infty$, then in the following estimate we will not specify the norm on $\partial\Omega$.

First we estimate (6.1) and (6.7), which are the most delicate.

For (6.1), we apply the boundary condition (2.25) to obtain

$$\begin{aligned}
|(6.1)| &\lesssim \nabla_v \Big[\frac{M_W(x_{\mathbf{b}}, v)}{\sqrt{\mu(v)}} \int_{\mathbf{v}^1_3 > 0} f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p} \mathbf{v}^1) \sqrt{\mu(\mathbf{v}^1)} \mathbf{v}^1_3 \mathrm{d}\mathbf{v}^1 + r(x_{\mathbf{b}}, v) \Big] \\
&\lesssim \frac{\|wf\|_\infty \|T_W - T_0\|_{C^1} + \||v|^2 \nabla_v [r(x_{\mathbf{b}}, v)]\|_\infty}{|v|^2} && (6.9)\\
&\quad + \frac{|v|^2 M_W(x_{\mathbf{b}}, v)}{\sqrt{\mu(v)}|v|^2} \Big| \int_{\mathbf{v}^1_3 > 0} \underbrace{\nabla_v [f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p} \mathbf{v}^1)]}_{(6.10)_*} \sqrt{\mu(\mathbf{v}^1)} \mathbf{v}^1_3 \mathrm{d}\mathbf{v}^1 \Big|. && (6.10)
\end{aligned}$$

In the second line we used the same computation as (2.100) to have

$$\Big|\nabla_v \frac{M_W(x_{\mathbf{b}}, v)}{\sqrt{\mu(v)}}\Big| \lesssim_{T_W} \frac{e^{-\frac{|v|^2}{2T_W(x_{\mathbf{b}}(x,v))}}|v|}{\sqrt{\mu(v)}} \lesssim \frac{1}{|v|^2}.$$

Applying (2.95) we have

$$(6.9) \lesssim \frac{\|wf\|_\infty + 1}{|v|^2}. \tag{6.11}$$

Using chain rule we have

$$(6.10)_* = \underbrace{\nabla_v x_{\mathbf{b}} \nabla_x f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p}\mathbf{v}^1)}_{(6.10)_1} + \underbrace{\nabla_v T^t_{\mathbf{x}^1_p}\mathbf{v}^1 \nabla_v f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p}\mathbf{v}^1)}_{(6.10)_2}.$$

Then we estimate the contribution of both terms above in (6.10). Applying (2.40), the contribution of $(6.10)_1$ is bounded by

$$\begin{aligned}&\frac{1}{|v|^2}\int_{n(x_{\mathbf{b}})\cdot v^1>0} \frac{\alpha(x_{\mathbf{b}}, v^1)}{|v^1|}|\nabla_x f(x_{\mathbf{b}}, v^1)|\mu^{1/4}(v^1)\mathrm{d}v^1 \\ &\lesssim \frac{1}{|v|^2}\int_{n(x_{\mathbf{b}})\cdot v^1>0} \frac{\|\alpha\nabla_x f\|_\infty}{|v^1|}\mu^{1/4}(v^1)\mathrm{d}v^1 \lesssim \frac{\|\alpha\nabla_x f\|_\infty}{|v|^2},\end{aligned} \tag{6.12}$$

where we have used (1.8) to have $|n(x_{\mathbf{b}})\cdot v^1|\sqrt{\mu(v^1)} \lesssim \alpha(x_{\mathbf{b}}, v^1)\mu^{1/4}(v^1)$.

For the contribution of $(6.10)_2$, we exchange the v derivative into $\mathbf{v}^1$ derivative:

$$\nabla_v f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p}\mathbf{v}^1) = \nabla_{\mathbf{v}^1}[f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p}\mathbf{v}^1)]T_{\mathbf{x}^1_p}.$$

Then the contribution of $(6.10)_2$ in (6.12) can be written as

$$\begin{aligned}&\Big|\frac{O(1)}{|v|}\int_{\mathbf{v}^1_3>0} \nabla_v T^t_{\mathbf{x}^1_p}\mathbf{v}^1 \nabla_{\mathbf{v}^1}[f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p}\mathbf{v}^1)]T_{\mathbf{x}^1_p}\mathbf{v}^1_3\sqrt{\mu(\mathbf{v}^1)}\mathrm{d}\mathbf{v}^1\Big| \\ &= \Big|\frac{O(1)}{|v|}\int_{\mathbf{v}^1_3>0} \nabla_v T^t_{\mathbf{x}^1_p} f(x_{\mathbf{b}}, T^t_{\mathbf{x}^1_p}\mathbf{v}^1)T_{\mathbf{x}^1_p}\nabla_{\mathbf{v}^1}\big[\mathbf{v}^1\mathbf{v}^1_3\sqrt{\mu(\mathbf{v}^1)}\big]\mathrm{d}\mathbf{v}^1\Big| \\ &\lesssim \frac{O(1)}{|v|}\int_{\mathbf{v}^1_3>0}\frac{1}{|v|}\|\eta\|_{C^2}\|wf\|_\infty\mu^{1/4}(\mathbf{v}^1)\mathrm{d}\mathbf{v}^1 \lesssim \frac{\|wf\|_\infty}{|v|^2},\end{aligned} \tag{6.13}$$

where we applied an integration by parts to $\mathrm{d}v^1$ in the second line, and used (2.41) in the third line.

Combining (6.12), (6.13) and (6.11) we conclude

$$|(6.1)| \lesssim \frac{\|wf\|_\infty + \|\alpha\nabla_x f\|_\infty}{|v|^2}. \tag{6.14}$$

Then we estimate (6.7). For $h = K(f)$, we compute

$$\int_{\max\{0,t-t_{\mathbf{b}}\}}^t e^{-\nu(t-s)}\int_{\mathbb{R}^3}|\nabla_v\big[\mathbf{k}(v,u)f(x-(t-s)v,u)\big]|\mathrm{d}s$$

$$
\begin{aligned}
&= \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(t-s)} \\
&\quad \times \int_{\mathbb{R}^3} \left| \nabla_v \mathbf{k}(v,u) f(x-(t-s)v,u) + \mathbf{k}(v,u) \nabla_v [f(x-(t-s)v,u)] \right| \\
&\lesssim \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(t-s)} \\
&\quad \times \int_{\mathbb{R}^3} \left[\|wf\|_\infty \frac{w^{-1}(u)\langle v\rangle \mathbf{k}_\varrho(v,u)}{|v-u|} + |\mathbf{k}(v,u)(t-s) \nabla_x f(x-(t-s)v,u)| \right] \\
&\lesssim \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} \Big[\|wf\|_\infty \frac{e^{-\varrho|v-u|^2/2}}{|v-u|^2} \frac{1}{|v|^2} \\
&\qquad + t_{\mathbf{b}} \|w_{\tilde\theta} \alpha \nabla_x f\|_\infty \frac{w_{\tilde\theta}(v)\mathbf{k}(v,u)}{w_{\tilde\theta}(u)\alpha(x-(t-s)v,u)} \frac{1}{w_{\tilde\theta}(v)} \Big] \\
&\lesssim \frac{\|wf\|_\infty}{|v|^2} + \frac{\tilde\alpha(x,v)}{w_{\tilde\theta}(v)|v|^2} \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{\alpha(x-(t-s)v,v)} \lesssim \frac{\|wf\|_\infty + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{|v|^2}.
\end{aligned} \tag{6.15}
$$

In the sixth line we have used

$$
\begin{aligned}
w^{-1}(u) e^{-\varrho|v-u|^2} \langle v\rangle &= e^{-\varrho|v-u|^2/2} e^{-\varrho|v-u|^2/2} e^{-\varrho|u|^2} \langle v\rangle \\
&\lesssim e^{-\varrho|v-u|^2/2} \frac{e^{-C|v|^2}\langle v\rangle |v|^2}{|v|^2} \lesssim \frac{e^{-\varrho|v-u|^2/2}}{|v|^2}.
\end{aligned}
$$

In the last line we have applied $\frac{e^{-\varrho|v-u|^2/2}}{|v-u|^2} \in L^1_u$, (4.5) in Lemma 4.1 and (2.39).

For $h = \Gamma(f,f)$ we apply (2.107) to have

$$
\begin{aligned}
|(6.7)\mathbf{1}_{h=\Gamma(f,f)}| &\lesssim \frac{1}{|v|^2} \int_{\max\{0,t-t_{\mathbf{b}}\}}^{t} e^{-\nu(t-s)} [\|wf\|_\infty^2 + \|wf\|_\infty \||v|^2 \nabla_v f\|_\infty] \\
&\lesssim \frac{\|wf\|_\infty^2 + \|wf\|_\infty \||v|^2\nabla_v f\|_\infty}{|v|^2}.
\end{aligned} \tag{6.16}
$$

Then we estimate all the other terms, which follow from more direct computation. For (6.2) we apply (2.40) and (2.106); for (6.3) we apply (2.106); for (6.4) we apply (2.40) and $t \leqq t_{\mathbf{b}} \leqq \frac{\tilde\alpha(x,v)}{|v|^2}$; for (6.5) we apply (2.106); for (6.6) we apply (2.106) and (2.124); for (6.8) we apply (2.40) and (2.105), then we obtain the following bound:

$$
|(6.2)| \lesssim \frac{\|\nu f\|_\infty}{|v|^2} \lesssim \frac{\|wf\|_\infty}{|v|^2}, \tag{6.17}
$$

$$
|(6.3)| \lesssim \frac{\||v|^2 f\|_\infty}{|v|^2} \lesssim \frac{\|wf\|_\infty}{|v|^2}, \tag{6.18}
$$

$$
\begin{aligned}
|(6.4)| &\lesssim e^{-t} t \nabla_x f(x-tv,v) + e^{-t} \nabla_v f(x-tv,v) \\
&\lesssim e^{-t} \frac{\|\alpha\nabla_x f\|_\infty}{|v^2|} + e^{-t} \frac{\||v|^2\nabla_v f\|_\infty}{|v|^2},
\end{aligned} \tag{6.19}
$$

$$|(6.5)| \lesssim te^{-t}\frac{\||v|^2 f\|_\infty}{|v|^2} \lesssim \frac{\|wf\|_\infty}{|v|^2}, \tag{6.20}$$

$$|(6.6)| \lesssim \frac{\||v|^2 h\|_\infty}{|v|^2}\int_{\max\{0,t-t_{\mathbf{b}}\}}^{t}(t-s)e^{-\nu(t-s)}\mathrm{d}s \lesssim \frac{\|w_{\tilde{\theta}}h\|_\infty}{|v|^2}$$
$$\lesssim \frac{\|wf\|_\infty}{|v|^2}, \tag{6.21}$$

$$|(6.8)| \lesssim \frac{\|wf\|_\infty}{|v|^2}. \tag{6.22}$$

Combining (6.14)–(6.22) we conclude

$$|\nabla_v f| \lesssim \|T_W - T_0\|_{C^1}\frac{(e^{-t}+\|wf\|_\infty)\||v|^2\nabla_v f\|_\infty + \|wf\|_\infty + \|\alpha\nabla_x f\|_\infty}{|v|^2}.$$

Since $e^{-t} \ll 1$ from $t \gg 1$ and $\|wf\|_\infty \ll 1$ from **Existence Theorem**, we conclude the proof. □

7. $C^{1,\beta}$ Solutions in Convex Domains

7.1. Proof of (1.21) and (1.22)

Given the continuity of the C^1 solution in Section 5.2, in this section we prove the Hölder regularity, which are (1.21) and (1.22) in the **Main Theorem**.

For simplicity we denote

$$[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}$$
$$:= \sup_{x,y\in\Omega}\Big\|w_{\tilde{\theta}}(v)|v|^2\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{2+\beta}\frac{\nabla_x f(x,v)-\nabla_x f(y,v)}{|x-y|^\beta}\Big\|_{L^\infty_v}, \tag{7.1}$$

$$[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}$$
$$:= \sup_{x,y\in\Omega}\Big\|w_{\tilde{\theta}/2}(v)|v|^2\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}\frac{|\nabla_\parallel f(x,v)-\nabla_\parallel f(y,v)|}{|x-y|^\beta}\Big\|_{L^\infty_v}. \tag{7.2}$$

Here $\nabla_\parallel = G(x)\nabla_x$ and G is defined in (1.14). We note that the weight in (7.1) and (7.2) are different in terms of the power.

To prove the weighted $C^{1,\beta}$ we will estimate the characteristics starting from two different positions x and y. In result we define the backward exit time and position that correspond to x and y.

The first backward exit position and time are denoted using the same notation as

$$x_{\mathbf{b}}(x,v), \quad x_{\mathbf{b}}(y,v), \quad t_{\mathbf{b}}(x,v), \quad t_{\mathbf{b}}(y,v).$$

For simplicity we denote the second backward exit position and time as

$$\begin{aligned} x^2_{\mathbf{b}}(x) &= x_{\mathbf{b}}(x_{\mathbf{b}}(x,v),v^1), \quad x^2_{\mathbf{b}}(y) = x_{\mathbf{b}}(x_{\mathbf{b}}(y,v),v^1),\\ t^2_{\mathbf{b}}(x) &= t_{\mathbf{b}}(x_{\mathbf{b}}(x,v),v^1), \quad t^2_{\mathbf{b}}(y) = t_{\mathbf{b}}(x_{\mathbf{b}}(y,v),v^1). \end{aligned} \tag{7.3}$$

Similarly to Definition 4, we choose $p^1(x)$, $p^2(x)$, $p^1(y)$, $p^2(y) \in \mathcal{P}$ such that

$$\mathbf{x}^i_{p^i(x)} := (\mathbf{x}^i_{p^i(x),1}, \mathbf{x}^i_{p^i(x),2}, 0) \text{ such that } \eta_{p^i(x)}(\mathbf{x}^i_{p^i(x)}) = \begin{cases} x_{\mathbf{b}}(x,v), & i=1; \\ x^2_{\mathbf{b}}(x), & i=2. \end{cases} \tag{7.4}$$

$$\mathbf{x}^i_{p^i(y)} := (\mathbf{x}^i_{p^i(y),1}, \mathbf{x}^i_{p^i(y),2}, 0) \text{ such that } \eta_{p^i(y)}(\mathbf{x}^i_{p^i(y)}) = \begin{cases} x_{\mathbf{b}}(y,v), & i=1; \\ x^2_{\mathbf{b}}(y), & i=2. \end{cases} \tag{7.5}$$

Since $\|wf\|_\infty$, $\||v|\nabla_\parallel f\|_\infty$, $\||v|^2\nabla_v f\|_\infty$, $\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty$ are all bounded from previous sections, for simplicity, throughout this section we assume

$$\begin{aligned} &\max\{\|wf\|_\infty, \||v|\nabla_\parallel f\|_\infty, \||v|^2\nabla_v f\|_\infty\} \\ &\leqq \|\alpha\nabla_x f\|_\infty \leqq \|\alpha\nabla_x f\|^2_\infty \leqq \|w_{\tilde\theta}\alpha\nabla_x f\|^2_\infty. \end{aligned} \tag{7.6}$$

Note that the above assumption implies $\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty \geqq 1$. Then the bound of (7.1) and (7.2) are given by the following proposition.

Proposition 3. *Suppose $F = \mu + \sqrt{\mu} f$ solves the steady Boltzmann equation (1.1) with boundary condition (1.3), then*

$$[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}} \lesssim o(1)[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}} + C_\varepsilon \|T_W - T_0\|_{C^2} \|w_{\tilde\theta}\alpha\nabla_x f\|^2_\infty, \tag{7.7}$$

and

$$[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}} \lesssim o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}} + C_\varepsilon \|T_W - T_0\|_{C^2} \|w_{\tilde\theta}\alpha\nabla_x f\|^2_\infty, \tag{7.8}$$

where $C_\varepsilon \gg 1$.

These two estimates together conclude (1.21) and (1.22).

Below we present two lemmas regarding the collision operators. These two lemmas will be used to estimate the difference of the collision operators.

Lemma 7.1. *For $h(x,v) = Kf(x,v) + \Gamma(f,f)(x,v)$, we have*

$$\frac{|h(x,v) - h(y,v)|}{|x-y|^\beta} \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}, \tag{7.9}$$

$$|\nabla_x h(x,v)| \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)} \int_{\mathbb{R}^3} \frac{\mathbf{k}_{\tilde\varrho}(v,u)}{\alpha(x,u)} \mathrm{d}u + \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)\alpha(x,v)}. \tag{7.10}$$

Proof. Since

$$\begin{aligned} &|\Gamma(f,f)(x,v) - \Gamma(f,f)(y,v)| \\ &\quad = \Gamma(f(x) - f(y), f(x))(v) + \Gamma(f(y), f(x) - f(y))(v) \\ &\quad \lesssim \|wf\|_\infty \Big(\int_{\mathbb{R}^3} \mathbf{k}(v,u)|f(x,u) - f(y,u)|\mathrm{d}u + |f(x,v) - f(y,v)| \Big), \end{aligned} \tag{7.11}$$

we have

$$\begin{aligned}
&\frac{|h(x,v)-h(y,v)|}{|x-y|^\beta}\\
&=\frac{|K[f(x,v)-f(y,v)]+\Gamma(f,f)(x,v)-\Gamma(f,f)(y,v)|}{|x-y|^\beta}\\
&\lesssim \int_{\mathbb{R}^3}\mathbf{k}(v,u)\frac{f(x,u)-f(y,u)}{|x-y|^\beta}\mathrm{d}u\\
&\quad+\|wf\|_\infty\int_{\mathbb{R}^3}\mathbf{k}(v,u)\frac{f(x,u)-f(y,u)}{|x-y|^\beta}\mathrm{d}u+\frac{|f(x,v)-f(y,v)|}{|x-y|^\beta}\\
&\lesssim \frac{\|wf\|_\infty\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\\
&\quad\times\Big[\int_{\mathbb{R}^3}\frac{w_{\tilde\theta}(u)\mathbf{k}_\varrho(v,u)}{w_{\tilde\theta}(u)\min\{\alpha(x,u),\alpha(y,u)\}^\beta}+\frac{1}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}\Big]\\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta},
\end{aligned}$$

where we have applied (2.64) and (4.10) in Lemma 4.1.

Then we prove (7.10). Clearly from Lemma 2.13,

$$\begin{aligned}
|\nabla_x Kf(x,v)|&=|\int_{\mathbb{R}^3}\mathbf{k}(v,u)\nabla_x f(x,u)|\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_{\mathbb{R}^3}\frac{\mathbf{k}(v,u)}{\alpha(x,u)}\frac{w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)}\\
&\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_{\mathbb{R}^3}\frac{\mathbf{k}_{\tilde\varrho}(v,u)}{\alpha(x,u)}.
\end{aligned}$$

For Γ, we bound

$$\begin{aligned}
|\nabla_x\Gamma(f,f)(x,v)|&=|\Gamma(\nabla_x f,f)+\Gamma(f,\nabla_x f)|\\
&\lesssim\|wf\|_\infty\Big[|\nabla_x f(x,v)|+\int_{\mathbb{R}^3}|\mathbf{k}(v,u)\nabla_x f(x,u)|\Big]\\
&\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)\alpha(x,v)}+\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_{\mathbb{R}^3}\frac{\mathbf{k}(v,u)}{\alpha(x,u)}\frac{w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)}\mathrm{d}u,
\end{aligned}$$

again by Lemma 2.13 we conclude the lemma. □

Lemma 7.2. *Denote $x'=x-(t-s)v$, $y'=y-(t-s)v$. For the difference of $\nabla_x\Gamma$, we have*

$$\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}\Big|\frac{\nabla_x\Gamma(f,f)(x',v)-\nabla_x\Gamma(f,f)(y',v)}{|x-y|^\beta}\Big|\\
&\lesssim\frac{o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}+\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta}(v)|v|^2\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{2+\beta}},
\end{aligned}\tag{7.12}$$

and

$$\int_0^t e^{-\nu(v)(t-s)}\Big|\frac{G(x')\nabla_x\Gamma(f,f)(x',v)-G(y')\nabla_x\Gamma(f,f)(y',v)}{|x-y|^\beta}\Big| \lesssim \frac{o(1)[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}+\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.13}$$

Proof. We rewrite the difference of $\nabla_x\Gamma$ as

$$\begin{aligned}&\nabla_x\Gamma(f,f)(x,v)-\nabla_x\Gamma(f,f)(y,v)\\&=\Gamma(\nabla_x f(x)-\nabla_x f(y),f(x))(v)+\Gamma(f(x)-f(y),\nabla_x f(x))(v)\\&\quad+\Gamma(\nabla_x f(y),f(x)-f(y))(v)+\Gamma(f(y),\nabla_x f(x)-\nabla_x f(y))(v).\end{aligned}$$

Then by (2.114) we bound

$$|\Gamma(\nabla_x f(x)-\nabla_x f(y),f(x))(v)|\lesssim\|wf\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\mathcal{Q}}(v,u)|\nabla_x f(x,u)-\nabla_x f(y,u)|\mathrm{d}u,$$

$$\begin{aligned}|\Gamma(f(y),\nabla_x f(x)-\nabla_x f(y))(v)|&\lesssim\|wf\|_\infty\big(\nabla_x f(x,v)-\nabla_x f(y,v)\big)\\&\quad+\|wf\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\mathcal{Q}}(v,u)|\nabla_x f(x,u)-\nabla_x f(y,u)|.\end{aligned}$$

For $\Gamma(f(x)-f(y),\nabla_x f(x))(v)$, we bound Γ_{loss} in (2.110) as

$$\begin{aligned}|\Gamma_{\mathrm{loss}}(f(x)-f(y),\nabla_x f(x))(x)|&\lesssim|\nabla_x f(x,v)|\nu(\sqrt{\mu}(f(x)-f(y)))\\&\lesssim|\nabla_x f(x,v)|\int_{\mathbb{R}^3}\mathbf{k}_{\mathcal{Q}}(v,u)|f(x,u)-f(y,u)|.\end{aligned}$$

Then we use Carleman's representation (see [14]) to write the Γ_{gain} in (2.110) as

$$\begin{aligned}&\frac{|\Gamma_{\mathrm{gain}}(f(x)-f(y),\nabla_x f(x))(v)|}{|x-y|^\beta}\\&\lesssim\int_{\mathbb{R}^3}\mathrm{d}u\frac{|\nabla_x f(x,u)|}{|v-u|}\int_{(u-v)\cdot\omega=0}\frac{|f(x,v+\omega)-f(y,v+\omega)|}{|x-y|^\beta}\\&\lesssim\int_{\mathbb{R}^3}\mathrm{d}u\mathbf{k}(v,u)|\nabla_x f(x,u)|\int_{\mathbb{S}^2}\frac{w_{\tilde\theta}^{-1}(v+\omega)\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{\min\{\alpha(x,v+\omega),\alpha(y,v+\omega)\}^\beta}\\&\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta}(v)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{w_{\tilde\theta}(v)\mathbf{k}(v,u)}{w_{\tilde\theta}(u)\alpha(x,u)\min\{\xi(x),\xi(y)\}^{\beta/2}}\\&\quad\times\int_{\mathbb{S}^2}\mathrm{d}\omega e^{-\tilde\theta|v|^2/2}\frac{w_{\tilde\theta}^{-1}(v+\omega)}{|v+\omega|^\beta}\\&\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta}^{3/2}(v)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\mathbf{k}_{\tilde{\mathcal{Q}}}(v,u)}{\alpha(x,u)\min\{\xi(x),\xi(y)\}^{\beta/2}}\\&\lesssim\frac{w_{\tilde\theta}^{-3/2}(v)}{\min\{\xi(x),\xi(y)\}^{\beta/2}}[1+\log|\xi(y)|+|\log|v||].\end{aligned} \tag{7.14}$$

Here we applied (2.64) in the third line. In the fourth line we have used that for $|\omega| \leqq 1$,

$$w_{\tilde{\theta}}^{-1}(v+\omega) = w_{\tilde{\theta}}^{-1}(v+\omega) w_{\tilde{\theta}}^{-1}(v+\omega) \lesssim w_{\tilde{\theta}}^{-1}(v+\omega) e^{-\tilde{\theta}|v|^2} e^{\tilde{\theta} v\cdot\omega}$$
$$\lesssim w_{\tilde{\theta}}^{-1}(v+\omega) e^{-\tilde{\theta}|v|^2} e^{\frac{\tilde{\theta}|v|^2}{2}} e^{\tilde{\theta}|w|^2} \lesssim w_{\tilde{\theta}}^{-1}(v+\omega) e^{-\tilde{\theta}|v|^2/2}.$$

and $\alpha(x,v) \geqq \sqrt{-\xi(x)}|v|$. In the sixth line we use Lemma 2.13 and in the last line we use (4.12).

Thus by (2.64) with (7.6) we obtain

$$\frac{|\nabla_x \Gamma(f,f)(x',v) - \nabla_x \Gamma(f,f)(y',v)|}{|x-y|^\beta}$$
$$\leqq \Big[O(\|wf\|_\infty) \frac{|\nabla_x f(x',v) - \nabla_x f(y',v)|}{|x-y|^\beta}$$
$$+ O(\|wf\|_\infty) \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{|\nabla_x f(x',u) - \nabla_x f(y',u)|}{|x-y|^\beta} \Big] \tag{7.15}$$
$$+ \Big[|\nabla_x f(x',v)| \int_{\mathbb{R}^3} \mathbf{k}(v,u) \frac{|f(x',u) - f(y',u)|}{|x-y|^\beta} \mathrm{d}u$$
$$+ \frac{|f(x',v) - f(y',v)|}{|x-y|^\beta} \int_{\mathbb{R}^3} \mathbf{k}(v,u) |\nabla_x f(x',u)| \mathrm{d}u \Big] \tag{7.16}$$
$$+ \frac{\|w_{\tilde{\theta}} \alpha \nabla_x f\|_\infty^2}{w_{\tilde{\theta}}^{3/2}(v)} \Big[\frac{|\log|\xi(x')||}{\min\{\xi(x'),\xi(y')\}^{\beta/2}} + \frac{|\log|v||}{\min\{\xi(x'),\xi(y')\}^{\beta/2}} \Big]. \tag{7.17}$$

We bound $w_{\tilde{\theta}}^{-3/2}(v)|v| \lesssim w_{\tilde{\theta}}^{-1}(v)$. By (4.7) and (4.10) in Lemma 4.1 we have

$$\int_0^t e^{-\nu(v)(t-s)} (7.15)$$
$$\lesssim \frac{\|wf_s\|_\infty [\nabla_x f_s(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}}{w_{\tilde{\theta}}(v)|v|^2 \min\Big\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\Big\}^{2+\beta}}$$
$$+ \frac{\|wf_s\|_\infty [\nabla_x f_s(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}}{w_{\tilde{\theta}}(v)} \int_0^t e^{-\nu(v)(t-s)}$$
$$\times \int_{\mathbb{R}^3} \frac{w_{\tilde{\theta}}(v)\mathbf{k}(v,u)}{w_{\tilde{\theta}}(u)|u|^2 \min\Big\{\frac{\alpha(x',u)}{|u|}, \frac{\alpha(y',u)}{|u|}\Big\}^{2+\beta}}$$
$$\lesssim \frac{\|wf_s\|_\infty [\nabla_x f_s(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}}{w_{\tilde{\theta}}(v)|v|^2 \min\Big\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\Big\}^{2+\beta}}. \tag{7.18}$$

By (2.64),(4.7) and (4.10) in Lemma 4.1 we have

$$\int_0^t e^{-\nu(v)(t-s)} (7.16)$$

$$
\begin{aligned}
&\lesssim \frac{w_{\tilde\theta}^{-2}(v)|v|\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{|v|\alpha(x,v)}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\frac{w_{\tilde\theta}(v)\mathbf{k}_\varrho(v,u)}{w_{\tilde\theta}(u)\min\{\alpha(x',u),\alpha(y',u)\}^\beta}\\
&\quad+\frac{w_{\tilde\theta}^{-2}(v)|v|\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{|v|\min\{\alpha(x,v),\alpha(y,v)\}^\beta}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\frac{w_{\tilde\theta}(v)\mathbf{k}_\varrho(v,u)}{w_{\tilde\theta}(u)\alpha(x',u)}\\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}\\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}.
\end{aligned}
\tag{7.19}
$$

For (7.17), we bound $|\log\xi(x')|\lesssim\frac{1}{\min\{\xi(x'),\xi(y')\}^\varepsilon}$ and $w_{\tilde\theta}^{-3/2}(v)|v|\lesssim w_{\tilde\theta}^{-1}(v)$. Then we have

$$
\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}(7.17)\\
&\lesssim \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2\Big[\int_0^t\frac{w_{\tilde\theta}^{-3/2}(v)e^{-\nu(v)(t-s)}}{\min\{\xi(x'),\xi(y')\}^{\beta/2+\varepsilon}}+\int_0^t\frac{w_{\tilde\theta}^{-3/2}(v)e^{-\nu(v)(t-s)}|\log|v||}{\min\{\xi(x'),\xi(y')\}^{\beta/2}}\Big]\\
&\lesssim \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2\Big[\int_0^t\frac{w_{\tilde\theta}^{-3/2}(v)|v|e^{-\nu(v)(t-s)}}{|v|\min\{\xi(x'),\xi(y')\}^{\beta/2+\varepsilon}}\\
&\quad+\int_0^t\frac{w_{\tilde\theta}^{-3/2}(v)|v||\log|v||e^{-\nu(v)(t-s)}}{|v|\min\{\xi(x'),\xi(y')\}^{\beta/2}}\Big]\\
&\lesssim \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2\Big[\int_0^t\frac{w_{\tilde\theta}^{-1}(v)e^{-\nu(v)(t-s)}}{|v|\min\{\xi(x'),\xi(y')\}^{\beta/2+\varepsilon}}+\int_0^t\frac{w_{\tilde\theta}^{-1}(v)e^{-\nu(v)(t-s)}}{|v|\min\{\xi(x'),\xi(y')\}^{\beta/2}}\Big]\\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{\beta}},
\end{aligned}
\tag{7.20}
$$

where we have applied Lemma 4.1 in the last line with small enough ε such that $\beta/2+\varepsilon\ll 1$.

Using $\|wf\|_\infty\ll 1$ from **Existence Theorem** we conclude (7.12).

Then we prove (7.13). From (7.15)–(7.17) we can rewrite

$$
\begin{aligned}
&\frac{|G(x')\nabla_x\Gamma(f,f)(x',v)-G(y')\nabla_x\Gamma(f,f)(y',v)|}{|x-y|^\beta}\\
&=\Big|\frac{G(y')-G(x')}{|x-y|^\beta}\nabla_x\Gamma(f,f)(y',v)+G(x')\frac{\nabla_x\Gamma(f,f)(x',v)-\nabla_x\Gamma(f,f)(y',v)}{|x-y|^\beta}\Big|\\
&\lesssim\frac{|G(y')-G(x')|}{|x'-y'|^\beta}\|wf\|_\infty\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty\\
&\quad\times\Big[\frac{w_{\tilde\theta}^{-1}(v)|v|}{|v||\alpha(y,v)|}+w_{\tilde\theta/2}^{-1}(v)\int_{\mathbb{R}^3}\frac{w_{\tilde\theta/2}(v)|u|\mathbf{k}(v,u)}{w_{\tilde\theta}(u)|u|\alpha(y-(t-s)v,u)}\Big] \qquad (7.21)\\
&\quad+|G(x')\times\big[(7.15)+(7.16)+(7.17)\big]|. \qquad (7.22)
\end{aligned}
$$

We bound $w_{\tilde{\theta}}^{-1}(v)|v| \lesssim w_{\tilde{\theta}/2}^{-1}(v)$. Then we have

$$\begin{aligned}
&\int_0^t e^{-\nu(v)(t-s)}(7.21)\\
&\lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}}\\
&\quad+\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)}\int_0^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\frac{\mathbf{k}_{\tilde{\varrho}}(v,u)}{|u|^2\min\left\{\frac{\alpha(x',u)}{|u|},\frac{\alpha(y',u)}{|u|}\right\}}\mathrm{d}u\\
&\lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},
\end{aligned}$$

where we have used (2.57) in the first line, (2.122) in the second line and (4.5) in Lemma 4.1 with $\frac{\alpha(x,v)}{|v|}\lesssim 1$ in the last line.

For (7.22), since $|G|\lesssim 1$, from (7.20) and (7.19) the contribution of (7.16)(7.17) is already included in (7.13). Then we consider the contribution of (7.15), which reads

$$\begin{aligned}
&\|wf\|_\infty\frac{|G(x')[\nabla_x f(x',v)-\nabla_x f(y',v)]|}{|x-y|^\beta}\\
&\quad+\|wf\|_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\varrho}(v,u)\frac{|G(x')[\nabla_x f(x',u)-\nabla_x f(y',u)]|}{|x-y|^\beta}.
\end{aligned}$$

Since $w_{\tilde{\theta}}^{-1}(v)|v|\lesssim w_{\tilde{\theta}/2}^{-1}(v)$, we rewrite

$$\begin{aligned}
&\left|G(x')\frac{\nabla_x f(x',v)-\nabla_x f(y',v)}{|x-y|^\beta}\right|\\
&=\left|\frac{\nabla_\parallel f(x',v)-\nabla_\parallel f(y',v)}{|x-y|^\beta}+\frac{\left[G(x')-G(y')\right]\nabla_x f(y',v)}{|x-y|^\beta}\right|\\
&\lesssim\left|\frac{\nabla_\parallel f(x',v)-\nabla_\parallel f(y',v)}{|x-y|^\beta}\right|+\frac{w_{\tilde{\theta}}^{-1}(v)|v|\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{|v|\alpha(y',v)}\\
&\lesssim\frac{[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x',v)}{|v|},\frac{\alpha(y',v)}{|v|}\right\}^{1+\beta}}\\
&\quad+\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x',v)}{|v|},\frac{\alpha(y',v)}{|v|}\right\}^{1+\beta}}.
\end{aligned}$$

Thus applying (2.122) and (4.7) in Lemma 4.1 with $p=1+\beta$ we obtain

$$\int_0^t e^{-\nu(v)(t-s)}|G(x')(7.15)|$$

$$
\begin{aligned}
&\lesssim \|wf\|_\infty \frac{[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}} + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}} \\
&\quad + \frac{\|wf\|_\infty}{w_{\tilde\theta/2}(v)} \int_0^t e^{-\nu(v)(t-s)} \\
&\quad \times \int_{\mathbb{R}^3} w_{\tilde\theta/2}(v)\mathbf{k}(v,u) \frac{[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}} + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(u)|u|^2 \min\left\{\frac{\alpha(x',u)}{|u|}, \frac{\alpha(y',u)}{|u|}\right\}^{1+\beta}} \\
&\lesssim \|wf\|_\infty \frac{[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}} + \|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.
\end{aligned}
$$

Finally by $\|wf\|_\infty \ll 1$ from the **Existence Theorem** we conclude the lemma. □

We need one more lemma before the proof. In the following lemma we express the difference quotient in (7.1),(7.2) along the characteristics. This lemma will significantly simplify the proof of Proposition 3.

Lemma 7.3. *Suppose f solves inhomogenenous steady transport equation with the diffuse BC. Then*

$$
\begin{aligned}
&\frac{\partial_{x_i} f(x,v) - \partial_{x_i} f(y,v)}{|x-y|^\beta} \\
&= \frac{O(1)\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty + o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}} \qquad (7.23) \\
&\quad + \frac{O(1)}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}} \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta} \\
&\quad \times w_{\tilde\theta}(v)|v|^2 \frac{|\partial^1_{\mathbf{x}_{p^1(x),i}} f(x_{\mathbf{b}}(x,v),v) - \partial^1_{\mathbf{x}_{p^1(y),i}} f(x_{\mathbf{b}}(y,v),v)|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \qquad (7.24)
\end{aligned}
$$

$$+\int_{\max\{0,t-\min\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}\}}^{t}\frac{e^{-\nu(t-s)}[\partial_{x_i}h(x-(t-s)v,v)-\partial_{x_i}h(y-(t-s)v,v)]}{|x-y|^{\beta}}. \tag{7.25}$$

We denote $[G(x)\nabla_x f(x,v)]_i$ *as the* i*-th component of* $G(x)\nabla_x f$*, then*

$$\frac{[G(x)\nabla_x f(x,v)]_i-[G(y)\nabla_x f(y,v)]_i}{|x-y|^{\beta}}$$

$$=\frac{\tilde{\alpha}(x,v)\times(7.25)}{|v|} \tag{7.26}$$

$$+\frac{O(1)\|w_{\tilde{\theta}}\alpha\nabla_x f\|_{\infty}+o(1)[\nabla_{x_\parallel}f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}+o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}} \tag{7.27}$$

$$+\int_{\max\{0,t-\min\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}\}}^{t}\Big[\frac{e^{-\nu(t-s)}[G(x-(t-s)v)\nabla_x h(x-(t-s)v,v)]_i}{|x-y|^{\beta}} \tag{7.28}$$

$$-\frac{e^{-\nu(t-s)}[G(y-(t-s)v)\nabla_x h(y-(t-s)v,v)]_i}{|x-y|^{\beta}}\Big]$$

$$+O(1)\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^{\beta}}{|x-y|^{\beta}}\frac{|\partial_{\mathbf{x}^1_{p^1(x),i}}f(x_{\mathbf{b}}(x,v),v)-\partial_{\mathbf{x}^1_{p^1(y),i}}f(x_{\mathbf{b}}(y,v),v)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^{\beta}}. \tag{7.29}$$

Proof. For f satisfying (2.7), different to (2.10)–(2.14), we use $\min\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}$ to split the cases. For simplicity, denote $t_m(v)=\min\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}$. We express $\nabla_x f(x,v)$ along the trajectory as:

$$\partial_{x_i}f(x,v)=\mathbf{1}_{t\geqq t_m(v)}\sum_{i=1,2}e^{-\nu t_{\mathbf{b}}}\partial_x\mathbf{x}^1_{p^1(x),i}\partial_{\mathbf{x}^1_{p^1(x),i}}f(x_{\mathbf{b}}(x,v),v) \tag{7.30}$$

$$-\mathbf{1}_{t\geqq t_m(v)}\nu\nabla_x t_{\mathbf{b}}(x,v)e^{-\nu t_{\mathbf{b}}(x,v)}f(x_{\mathbf{b}}(x,v),v) \tag{7.31}$$

$$+\mathbf{1}_{t<t_m(v)}e^{-\nu t}\partial_{x_i}f(x-tv,v) \tag{7.32}$$

$$+\mathbf{1}_{t<t_m(v)}\int_0^t e^{-\nu(t-s)}\partial_{x_i}[h(x-(t-s)v,v)]\mathrm{d}s \tag{7.33}$$

$$+\mathbf{1}_{t\geqq t_m(v)}\int_{t-t_{\mathbf{b}}(x,v)}^{t}e^{-\nu(t-s)}\partial_{x_i}[h(x-(t-s)v,v)]\mathrm{d}s \tag{7.34}$$

$$-\mathbf{1}_{t\geqq t_m(v)}\partial_{x_i}t_m(v)e^{-\nu t_m(v)}h(x-t_m(v)v,v)\mathrm{d}s. \tag{7.35}$$

Here we note that when $t\geqq t_m(v)$, we evaluate $\partial_{x_i}f(x,v)$ along the characteristics at $x_{\mathbf{b}}(x,v)$ regardless the relationship between $t_{\mathbf{b}}(x,v)$ and $t_{\mathbf{b}}(y,v)$.

Taking the difference of $\partial_{x_i} f(x,v)$ and $\partial_{y_i} f(y,v)$ using (7.30)–(7.35) we obtain

$$\begin{aligned}
&\frac{\partial_{x_i} f(x,v) - \partial_{x_i} f(y,v)}{|x-y|^\beta} \\
&= \mathbf{1}_{t \geqq t_m(v)} \frac{O(1)\nu |\partial_{x_i} t_{\mathbf{b}}(x,v) - \partial_{x_i} t_{\mathbf{b}}(y,v)| e^{-\nu t_{\mathbf{b}}(x,v)} f(x_{\mathbf{b}}(x,v),v)}{|x-y|^\beta} \qquad (7.36)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \frac{\partial_{x_i} t_{\mathbf{b}}(y,v) \nu |x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|}{|x-y|^\beta} \\
&\quad \times \frac{e^{-\nu t_{\mathbf{b}}} f(x_{\mathbf{b}}(x,v),v) - e^{-\nu t_{\mathbf{b}}} f(x_{\mathbf{b}}(y,v),v)}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|} \qquad (7.37)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \frac{\partial_{x_i} t_{\mathbf{b}}(y,v) \nu \big[e^{-\nu t_{\mathbf{b}}} f(x_{\mathbf{b}}(y,v),v) - e^{-\nu t_{\mathbf{b}}(y,v)} f(x_{\mathbf{b}}(y,v),v) \big]}{|x-y|^\beta} \qquad (7.38)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \sum_{i=1,2} \frac{\partial_{x_i} \mathbf{x}^1_{p^1(x),i} - \partial_{x_i} \mathbf{x}^1_{p^1(y),i}}{|x-y|^\beta} e^{-\nu(v) t_{\mathbf{b}}(x,v)} \partial_{\mathbf{x}^1_{p^1(x),i}} f(x_{\mathbf{b}}(x,v),v) \qquad (7.39)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \sum_{i=1,2} \partial_{x_i} \mathbf{x}^1_{p^1(y),i} \frac{e^{-\nu t_{\mathbf{b}}(x,v)} - e^{-\nu t_{\mathbf{b}}(y,v)}}{|x-y|^\beta} \partial_{\mathbf{x}^1_{p^1(y),i}} f(x_{\mathbf{b}}(y,v),v) \qquad (7.40)\\
&\quad + \mathbf{1}_{t < t_m(v)} \frac{e^{-\nu t} [\partial_{x_i} f(x - tv, v) - \partial_{x_i} f(y - tv, v)]}{|x - tv - (y - tv)|^\beta} \qquad (7.41)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \sum_{i=1,2} e^{-\nu(v) t_{\mathbf{b}}(x,v)} |\partial_{x_i} \mathbf{x}^1_{p^1(x),i}| \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta} \\
&\quad \times \frac{|\partial_{\mathbf{x}^1_{p^1,i}} f(x_{\mathbf{b}}(x,v),v) - \partial_{\mathbf{x}^1_{p^1,i}} f(x_{\mathbf{b}}(y,v),v)|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \qquad (7.42)\\
&\quad + \mathbf{1}_{t < t_m(v)} \int_0^t \frac{e^{-\nu(t-s)} \big[\partial_{x_i} h(x - (t-s)v, v) - \partial_{x_i} h(y - (t-s)v, v) \big]}{|x-y|^\beta} \qquad (7.43)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \int_{t - t_m(v)}^t \frac{e^{-\nu(t-s)} \big[\partial_{x_i} h(x - (t-s)v, v) - \partial_{x_i} h(y - (t-s)v, v) \big]}{|x-y|^\beta} \qquad (7.44)\\
&\quad + \mathbf{1}_{t \geqq t_m(v)} \int_{\min\{t - t_{\mathbf{b}}(x,v), t - t_{\mathbf{b}}(y,v)\}}^{t - t_m(v)} \\
&\Big[\mathbf{1}_{t_{\mathbf{b}}(x,v) > t_{\mathbf{b}}(y,v)} \frac{e^{-\nu(t-s)} \partial_{x_i} h(x - (t-s)v, v)}{|x-y|^\beta} \qquad (7.45)\\
&\quad + \mathbf{1}_{t_{\mathbf{b}}(y,v) > t_{\mathbf{b}}(x,v)} \frac{e^{-\nu(t-s)} \partial_{x_i} h(y - (t-s)v, v)}{|x-y|^\beta} \Big]
\end{aligned}$$

$$+\mathbf{1}_{t\geqq t_m(v)}\partial_{x_i}t_{\mathbf{b}}(x,v)e^{-\nu(v)t_{\mathbf{b}}(x,v)}\frac{h(x-t_{\mathbf{b}}(x,v)v,v)-h(y-t_{\mathbf{b}}(y,v)v,v)}{|x-y|^{\beta}} \tag{7.46}$$

$$-\mathbf{1}_{t\geqq t_m(v)}\frac{\partial_{x_i}t_{\mathbf{b}}(x,v)-\partial_{x_i}t_{\mathbf{b}}(y,v)}{|x-y|^{\beta}}e^{-\nu(v)t_{\mathbf{b}}(x,v)}h(y-t_{\mathbf{b}}(y,v)v,v) \tag{7.47}$$

$$-\mathbf{1}_{t\geqq t_m(v)}\frac{e^{-\nu(v)t_{\mathbf{b}}(x,v)}-e^{-\nu(v)t_{\mathbf{b}}(y,v)}}{|x-y|^{\beta}}\partial_{x_i}t_{\mathbf{b}}(y,v)h(y-t_{\mathbf{b}}(y,v),v). \tag{7.48}$$

First we estimate (7.36)–(7.41). For (7.36) we apply (2.59); for (7.37) we apply (2.32),(2.64),(2.54) and (7.6); for (7.38) we apply (2.32) and (2.56), then we obtain

$$(7.36)=\frac{O(1)w_{\tilde\theta}(v)|v|f(x_{\mathbf{b}}(x,v),v)}{w_{\tilde\theta}|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}=\frac{O(1)\|wf\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}, \tag{7.49}$$

$$\begin{aligned}(7.37)&=O(1)\partial_{x_i}t_{\mathbf{b}}(x,v)\frac{f(x_{\mathbf{b}}(x,v),v)-f(x_{\mathbf{b}}(y,v),v)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|}\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|}{|x-y|^{\beta}}\\&=\frac{O(1)}{\min\{\alpha(x,v),\alpha(y,v)\}}\frac{\|w_{\tilde\theta}(v)\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}}\frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{\beta}}\\&=\frac{O(1)\|\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}},\end{aligned} \tag{7.50}$$

$$\begin{aligned}(7.38)&=\frac{O(1)\|w_{\tilde\theta}f\|_\infty}{w_{\tilde\theta}(v)\alpha(x,v)}\frac{1}{|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{\beta}}\\&=\frac{O(1)\|wf\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}\end{aligned} \tag{7.51}$$

$$=\frac{O(1)\|wf\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}. \tag{7.52}$$

From Section 4,

$$\partial_{\mathbf{x}^1_{p^1(x),i}}f(\eta_{p^1}(\mathbf{x}^1_{p^1(x)}),v)=(3.19)+(3.20)=O(1)\frac{M_W(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v)}{\sqrt{\mu(v)}}\|\alpha\nabla_x f\|_\infty,$$

thus with $\tilde\theta\ll 1$,

$$\max\{\left||v|\partial_{\mathbf{x}^1_{p^1(x),i}}f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v)\right|,\left||v|^2\partial_{\mathbf{x}^1_{p^1(x),i}}f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v)\right|\}$$
$$=O(1)\frac{\|\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}.$$

Then for (7.39) we apply (2.58); for (7.40) we apply (2.32) and (2.56), then we obtain

$$(7.39) = \frac{O(1)|v|^2 \partial_{\mathbf{x}^1_{p^1,i}} f(x_{\mathbf{b}}(x,v),v)}{|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}} = \frac{O(1)\|\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}, \tag{7.53}$$

$$(7.40) = \frac{O(1)|v| \partial_{\mathbf{x}^1_{p^1,i}} f(x_{\mathbf{b}}(x,v),v)}{|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(x,v)}{|v|}\right\}^{1+\beta}} = \frac{O(1)\|\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(x,v)}{|v|}\right\}^{1+\beta}} \tag{7.54}$$

$$= \frac{O(1)\|\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(x,v)}{|v|}\right\}^{2+\beta}}, \tag{7.55}$$

$$(7.41) = O(1)\frac{e^{-\nu t}}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}} [\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}. \tag{7.56}$$

Therefore, from (7.6) and $t \gg 1$, we conclude

$$(7.36) + \cdots + (7.41) = (7.23).$$

For (7.42), from (2.33), such contribution is included in (7.24).
The contribution of (7.43) and (7.44) are already included in (7.25).
Then we estimate (7.45)–(7.48). We apply (7.10) to (7.45) to have

$$\begin{aligned}(7.45) &= O(1)\frac{\|w_{\tilde{\theta}}\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)} \int_{\min\{t-t_{\mathbf{b}}(x,v), t-t_{\mathbf{b}}(y,v)\}}^{\max\{t-t_{\mathbf{b}}(x,v), t-t_{\mathbf{b}}(y,v)\}} \int_{\mathbb{R}^3} \frac{e^{-\nu(t-s)}\mathbf{k}_{\tilde{Q}}(v,u)}{\alpha(x-(t-s)v,u)|x-y|^\beta} \\ &= \frac{O(1)\|w_{\tilde{\theta}}\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)\alpha(x,v)} \frac{|e^{-C_1 \nu t_{\mathbf{b}}(x,v)} - e^{-C_1 \nu t_{\mathbf{b}}(y,v)}|^\beta}{|x-y|^\beta} \\ &= \frac{O(1)\|w_{\tilde{\theta}}\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}} \qquad (7.57) \\ &= \frac{O(1)\|w_{\tilde{\theta}}\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}, \qquad (7.58)\end{aligned}$$

where we have applied (4.4) of Lemma 4.1 in second line, (2.56) in the third line.

Then for (7.46) we apply (2.32),(2.54) and (7.9); for (7.47) we apply (2.59) and (2.105); for (7.48) we apply (2.56) and (2.105), then we obtain

$$\begin{aligned}(7.46) &= O(1)\frac{1}{\alpha(x,v)} \frac{h(x_{\mathbf{b}}(x,v),v) - h(x_{\mathbf{b}}(y,v),v)}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta} \\ &= \frac{O(1)\|w_{\tilde{\theta}}\alpha \nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)\min\{\alpha(x,v),\alpha(y,v)\}^2} \frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^\beta}\end{aligned}$$

$$= \frac{O(1)\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\{\alpha(x,v),\alpha(y,v)\}^{2+\beta}}, \tag{7.59}$$

$$(7.47) = O(1)\frac{\|w_{\tilde{\theta}}|v|h\|_\infty}{w_{\tilde{\theta}}(v)|v|}\frac{\nabla_x t_{\mathbf{b}}(x,v)-\nabla_x t_{\mathbf{b}}(y,v)}{|x-y|^\beta}$$

$$= \frac{O(1)\|wf\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\{\alpha(x,v),\alpha(y,v)\}^{2+\beta}}, \tag{7.60}$$

$$(7.48) = \frac{O(1)\|w_{\tilde{\theta}}h\|_\infty}{w_{\tilde{\theta}}(v)\alpha(y,v)}\frac{e^{-\nu t_{\mathbf{b}}(x,v)}-e^{-\nu t_{\mathbf{b}}(y,v)}}{|x-y|^\beta} = \frac{O(1)\|wf\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\{\alpha(x,v),\alpha(y,v)\}^{1+\beta}} \tag{7.61}$$

$$= \frac{O(1)\|wf\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\{\alpha(x,v),\alpha(y,v)\}^{2+\beta}}. \tag{7.62}$$

Therefore, by (7.6), collecting (7.58),(7.59), (7.60),(7.62) we conclude

$$(7.45)+\cdots+(7.48) = (7.23).$$

Then we prove the estimate (7.26)-(7.29).
We rewrite

$$\frac{G(x)\nabla_x f(x,v)-G(y)\nabla_x f(y,v)}{|x-y|^\beta} = \frac{G(y)-G(x)}{|x-y|^\beta}\nabla_x f(x,v) \tag{7.63}$$

$$+ G(y)\frac{\nabla_x f(x,v)-\nabla_x f(y,v)}{|x-y|^\beta}. \tag{7.64}$$

By (2.57) we conclude that

$$(7.63)_i = O(1)\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)\alpha(x,v)}. \tag{7.65}$$

Then we consider (7.64). Note that (7.36-(7.48)) represent the i-th component of $\nabla_x f(x,v)-\nabla_x f(y,v)$, for convenience we define a notation that represents the vector consists of the element (7.36):

$$[(7.36)] = [(7.36)_{i=1}, (7.36)_{i=2}, (7.36)_{i=3}].$$

Similarly we can define the same notation for (7.37)–(7.48). We can use this representation to express $G(y)\frac{\nabla_x f(x,v)-\nabla_x f(y,v)}{|x-y|^\beta}$.

Then for $G(y)[(7.36)]$ we apply (2.63); for $G(y)[(7.37)]$ we apply (2.52),(2.54),(2.64) and (7.6); for $G(y)[(7.38)]$ we apply $|G| = O(1)$; for $G(y)[(7.39)]$ we apply (2.62) and (2.47); for $G(y)[(7.40)]$ we apply $|G| = O(1)$; for $G(y)[(7.42)]$ we apply (2.51), then we obtain

$$\begin{aligned}[G(y)[(7.36)]]_i &= O(1)\frac{\|w_{\tilde{\theta}}|v|f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}} \\ &= \frac{O(1)\|wf\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},\end{aligned} \tag{7.66}$$

$$[G(y)[(7.37)]]_i = O(1)\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}, \tag{7.67}$$

$$[G(y)[(7.38)]]_i = O(1)(7.51) = O(1)\frac{\|wf\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}, \tag{7.68}$$

$$\begin{aligned}[G(y)[(7.39)]]_i &= O(1)\frac{\|w_{\tilde\theta}(v)|v|^2\partial^1_{\mathbf{x}^1_{p^1,i}}f\|_\infty}{|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}\\ &= O(1)\frac{\|wf\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},\end{aligned} \tag{7.69}$$

$$[G(y)[(7.40)]]_i = O(1)(7.54) = O(1)\frac{\|wf\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|^2}\right\}^{1+\beta}},$$

$$[G(y)[(7.42)]]_i = (7.28). \tag{7.70}$$

For $[G(y)[(7.41)]]_i$, we rewrite

$$\begin{aligned}&G(y)\frac{\nabla_x f(x-tv,v)-\nabla_x f(y-tv,v)}{|x-y|^\beta}\\ &=\underbrace{\frac{G(x)\nabla_x f(x-tv,v)-G(y)\nabla_x f(y-tv,v)}{|x-y|^\beta}}_{(7.71)_1}+\underbrace{\frac{[G(y)-G(x)]\nabla_x f(x-tv,v)}{|x-y|^\beta}}_{(7.71)_2}.\end{aligned} \tag{7.71}$$

We apply (2.57) to have

$$[(7.71)_2]_i = O(1)\frac{\||v|^2 w_{\tilde\theta/2}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)|v|^2} = \frac{O(1)\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|^2},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.$$

For $(7.71)_1$ we apply (2.90) and conclude

$$\begin{aligned}|(7.71)_1| &= O(1)\frac{[\nabla_{x_\parallel}f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}}{w_{\tilde\theta/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}\\ &\quad+\frac{\frac{\tilde\alpha(x,v)}{|v|}[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}\\ &\quad+\frac{|v|O(1)\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}\\ &= O(1)\frac{[\nabla_{x_\parallel}f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}+[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}+\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},\end{aligned}$$

where we have used $\tilde{\alpha}(v)w_{\tilde{\theta}}^{-1}(v) \lesssim \alpha(v)w_{\tilde{\theta}/2}^{-1}(v)$, $|v|w_{\tilde{\theta}}^{-1}(v) \lesssim w_{\tilde{\theta}/2}^{-1}(v)$. Thus with $e^{-\nu t} \ll 1$ when $t \gg 1$,

$$[G(y)[(7.41)]]_i = o(1)\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty + [\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}} + [\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde{\theta}/2}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}, \tag{7.72}$$

which is already included in (7.27).

Then we estimate $[G(y)[(7.43)]]_i$. We rewrite

$$\begin{aligned}
&\frac{G(y)[\nabla_x h(x-(t-s)v) - \nabla_x h(y-(t-s)v)]}{|x-y|^\beta} \\
&= \underbrace{\frac{G(x)\nabla_x h(x-(t-s)v) - G(y)\nabla_x h(y-(t-s)v)}{|x-y|^\beta}}_{(7.73)_1} \\
&\quad + \underbrace{\frac{G(y)-G(x)}{|x-y|^\beta}\nabla_x h(x-(t-s)v)}_{(7.73)_2}.
\end{aligned} \tag{7.73}$$

We bound $w_{\tilde{\theta}}^{-1}(v)|v| \lesssim w_{\tilde{\theta}/2}^{-1}(v)$. From (2.57) and (7.10) the contribution of $(7.73)_2$ in $[G(y)[(7.43)]]_i$ is

$$\begin{aligned}
&O(1)\|\xi\|_{C^2}\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)}\int_0^t\int_{\mathbb{R}^3}\frac{e^{-\nu(t-s)}\mathbf{k}_{\tilde{\varrho}}(v,u)}{\alpha(x-(t-s)v,u)} \\
&= O(1)\|\xi\|_{C^2}\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty |v| w_{\tilde{\theta}}^{-1}(v)}{|v|\alpha(x,v)} \\
&= O(1)\|\xi\|_{C^2}\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},
\end{aligned}$$

which is included in (7.27).

For $(7.73)_1$ we apply (2.90) in Lemma 2.10. Then such contribution in $[G(y)[(7.43)]]_i$ equals to

$$\begin{aligned}
&\frac{[\nabla_\parallel h(x-(t-s)v,u)]_i - [\nabla_\parallel h(y-(t-s)v,u)]_i}{|x-y|^\beta} \\
&+ \frac{\tilde{\alpha}(x,v)}{|v|}\frac{\partial_{x_i}h(x-(t-s)v,v) - \partial_{x_i}h(y-(t-s)v,v)}{|x-y|^\beta} \\
&+ \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},
\end{aligned}$$

which are included in (7.26),(7.27) and (7.29) respectively.

Last we estimate (7.45)–(7.48). For (7.45) we apply $|G(y)| = O(1)$; for (7.46) we apply (2.54),(7.9) and (2.63); for (7.47) we apply (2.105),(2.32) and (2.63) ; for (7.48) we apply $|G| = O(1)$, then we obtain

$$[G(y)[(7.45)]]_i = O(1)(7.57) = \frac{O(1)\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}, \tag{7.74}$$

$$\begin{aligned}[G(y)[(7.46)]]_i &= \frac{O(1)}{|v|}\frac{h(x_{\mathbf{b}}(x,v),v) - h(x_{\mathbf{b}}(y,v),v)}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}\frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta} \\ &\lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|\min\{\alpha(x,v),\alpha(y,v)\}}\frac{1}{\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^\beta} \\ &\lesssim \frac{1}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}, \end{aligned} \tag{7.75}$$

$$\begin{aligned}[G(y)[(7.47)]]_i &= \frac{O(1)|v|\|h\|_\infty}{w_{\tilde{\theta}}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}} \\ &= \frac{O(1)\|wf\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}, \end{aligned} \tag{7.76}$$

$$[G(y)[(7.48)]]_i = O(1)(7.61) = O(1)\frac{\|wf\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.77}$$

These four estimates are all included in (7.27).
We conclude the lemma. □

Now are ready to prove Proposition 3.

Proof of Proposition 3. We will use 3 steps to prove this proposition. Since we already expand the difference quotient in Lemma 7.3, we mainly estimate (7.24),(7.25) and (7.26),(7.28),(7.29). The estimate of (7.24) is put in Step 1 and the estimate of (7.25) is put in Step 2. Thus Step 1 and Step 2 together conclude (7.7). In Step 3 we estimate (7.26),(7.28),(7.29) and conclude (7.8).

Before going into these steps we first list some estimates for the α-weight. We will heavily rely on these estimates to make the computation more precise. For $0 \leqq s \leqq t_{\mathbf{b}}(x_{\mathbf{b}}(x,v), v^1)$, we have

$$\alpha(x_{\mathbf{b}}(x,v),v^1) \sim \alpha(x_{\mathbf{b}}(x,v) - sv^1, v^1) \sim \alpha(x_{\mathbf{b}}(x_{\mathbf{b}}(x,v),v^1),v^1), \tag{7.78}$$

and

$$\begin{aligned}&|n(x_{\mathbf{b}}(x,v))\cdot v^1|e^{-C|v^1|^2} \\ &= [\mathbf{1}_{|n(x_{\mathbf{b}}(x,v))\cdot v^1|\geqq 1} + \mathbf{1}_{|n(x_{\mathbf{b}}(x,v))\cdot v^1|<1}]|n(x_{\mathbf{b}}(x,v))\cdot v^1|e^{-C|v^1|^2}\end{aligned}$$

$$\lesssim \alpha(x_{\mathbf{b}}(x,v),v^1)\frac{|n(x_{\mathbf{b}}(x,v))\cdot v^1|}{\alpha(x_{\mathbf{b}}(x,v),v^1)}e^{-C|v^1|^2}+\alpha(x_{\mathbf{b}}(x,v),v^1)e^{-C|v^1|^2}$$
$$\lesssim \alpha(x_{\mathbf{b}}(x,v),v^1)e^{-C|v^1|^2/2}, \tag{7.79}$$

where we have used (1.8) and (1.10) in the derivation.

Suppose $\alpha(x_{\mathbf{b}}(x,v)) \leqq \alpha(x_{\mathbf{b}}(y,v))$. We let $\varepsilon \ll 1$ such that $\beta+\varepsilon<1$ and $\frac{1-\beta}{1-\beta-\varepsilon}<2$. Then we apply the Hölder's inequality with $(\beta+\varepsilon)+(1-\beta-\varepsilon)=1$ to have

$$\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\frac{e^{-C|v^1|^2}|n(x_{\mathbf{b}}(x,v))\cdot v^1|}{|v^1|^2\min\left\{\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|},\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|}\right\}^{1+\beta}}\mathrm{d}v^1$$
$$=\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\frac{e^{-C|v^1|^2}}{|v^1|^{1-\beta}\alpha^{\beta}(x_{\mathbf{b}}(x,v),v^1)}\mathrm{d}v^1$$
$$\lesssim\Big(\int\frac{e^{-C|v^1|^2}}{|v^1|^{\frac{1-\beta}{1-\beta-\varepsilon}}}\Big)^{1-\beta-\varepsilon}\Big(\int\frac{e^{-C|v^1|^2}}{\alpha^{\frac{\beta}{\beta+\varepsilon}}(x_{\mathbf{b}}(x,v),v^1)}\Big)^{\beta+\varepsilon}\lesssim 1, \tag{7.80}$$

where we have used

$$\int\frac{e^{-C|v^1|^2}}{\alpha^{\frac{\beta}{\beta+\varepsilon}}(x_{\mathbf{b}}(x,v),v^1)}$$
$$\lesssim\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\mathbf{1}_{\alpha(x_{\mathbf{b}}(x,v),v^1)\geqq 1}\frac{e^{-C|v^1|^2}}{\alpha^{\frac{\beta}{\beta+\varepsilon}}(x_{\mathbf{b}}(x,v),v^1)}\mathrm{d}v^1$$
$$+\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\mathbf{1}_{\alpha(x_{\mathbf{b}}(x,v),v^1)\leqq 1}\frac{e^{-C|v^1|^2}}{|n(x_{\mathbf{b}}(x,v))\cdot v^1|^{\frac{\beta}{\beta+\varepsilon}}}\mathrm{d}v^1\lesssim 1.$$

Then we start the proof. □

Step 1: estimate of (7.24).
We focus on

$$w_{\tilde{\theta}}(v)|v|^2\frac{|\partial_{\mathbf{x}^1_{p^1(x),i}}f(x_{\mathbf{b}}(x,v),v)-\partial_{\mathbf{x}^1_{p^1(y),i}}f(x_{\mathbf{b}}(y,v),v)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^{\beta}}. \tag{7.81}$$

Applying the diffuse boundary condition (2.4) we get

$$\partial_{\mathbf{x}^1_{p^1(x),i}}f(x_{\mathbf{b}}(x,v),v)$$
$$=\frac{M_W(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu(v)}}\int_{\mathbf{v}^1_{p^1(x),3}>0}$$
$$\Big\{\underbrace{\Big[\Big(\partial_{\mathbf{x}^1_{p^1(x),i}}T^t_{\mathbf{x}^1_{p^1(x)}\mathbf{v}^1_{p^1}}\Big)\cdot\nabla_v f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1(x)})}_{(7.82)_1}$$

$$
\begin{aligned}
&+ \underbrace{\partial_{\mathbf{x}^1_{p^1(x),j}} f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}} \mathbf{v}^1_{p^1(x)})}_{(7.82)_2} \Big] \sqrt{\mu(\mathbf{v}^1_{p^1(x)})} \mathbf{v}^1_{p^1(x),3} \mathrm{d}\mathbf{v}^1_{p^1(x)} \\
&+ \frac{\partial_{\mathbf{x}^1_{p^1(x),i}} M_W(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu(v)}} \\
&\times \int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0} f(x_{\mathbf{b}}(x,v),v^1)\sqrt{\mu(v^1)}\{n(x_{\mathbf{b}}(x,v))\cdot v^1\} \Big\} \mathrm{d}v^1 \\
&+ \partial_{\mathbf{x}^1_{p^1(x),i}} r(x_{\mathbf{b}}(x,v),v). \qquad (7.82)
\end{aligned}
$$

From Lemma 2.11 the contribution of the last two terms of (7.82) in (7.81) is bounded by

$$\|T_W - T_0\|_{C^2}[\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty + 1].$$

Velocity derivative: first we consider the contribution of $(7.82)_1$ in (7.24). From (3.22), we rewrite $(7.82)_1$ as

$$
\begin{aligned}
&\int_{\mathbf{v}^1_{p^1,3}>0} \sum_{m,n} (3.23)_{mn,k+1\to 1}(x) \mathbf{v}^1_{p^1,m} \\
&\quad \times \partial_{\mathbf{v}^1_{p^1,n}} [f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}} \mathbf{v}^1_{p^1})] \sqrt{\mu(\mathbf{v}^1_{p^1})} \mathbf{v}^1_{p^1,3} \mathrm{d}\mathbf{v}^1_{p^1} \\
&= \int_{\mathbf{v}^1_{p^1,3}>0} \sum_{m,n} (3.23)_{mn,k+1\to 1}(x) \\
&\quad \times f(\eta_{p^1}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}} \mathbf{v}^1_{p^1}) \partial_{\mathbf{v}^1_{p^1,n}} \Big[\sqrt{\mu(\mathbf{v}^1_{p^1})} \mathbf{v}^1_{p^1,3} \mathbf{v}^1_{p^1,m}\Big] \mathrm{d}\mathbf{v}^1_{p^1}. \qquad (7.83)
\end{aligned}
$$

Here we dropped the x dependence in $\mathbf{v}^1_{p^1(x)}$ since it becomes a dummy variable.

From Section 5.2, $\|wf\|_{L^\infty(\partial\Omega)} \leqq \|wf\|_\infty$, for the rest estimate we will not specify the norm on $\partial\Omega$. We apply (3.24) to have

$$(7.83) = O(1)\|\eta\|_{C^2}\|wf\|_\infty. \qquad (7.84)$$

Then the contribution of $(7.82)_1$ in (7.24) can be written as

$$
\begin{aligned}
&\frac{w_{\tilde{\theta}}(v)|v|^2}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta} \Big[\frac{M_W(x_{\mathbf{b}}(x,v),v) - M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}} \times (7.83) \qquad (7.85) \\
&\quad + \frac{M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}} \\
&\quad \times \int_{\mathbf{v}^1_{p^1,3}>0} \big[f(\eta_{p^1}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}} \mathbf{v}^1_{p^1}) - f(\eta_{p^1}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}} \mathbf{v}^1_{p^1})\big] \\
&\quad \times \sum_{m,n} (3.23)_{mn,k+1\to 1}(x) \partial_{\mathbf{v}^1_{p^1,n}} \Big[\sqrt{\mu(\mathbf{v}^1_{p^1})} \mathbf{v}^1_{p^1,3} \mathbf{v}^1_{p^1,m}\Big] \mathrm{d}\mathbf{v}^1_{p^1} \qquad (7.86) \\
&\quad + \frac{M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}} \int_{\mathbf{v}^1_{p^1,3}>0} \sum_{m,n} \partial_{\mathbf{v}^1_{p^1,n}} \Big[\sqrt{\mu(\mathbf{v}^1_{p^1})} \mathbf{v}^1_{p^1,3} \mathbf{v}^1_{p^1,m}\Big]
\end{aligned}
$$

$$\times f(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1})\Big[(3.23)_{mn,k+1\to 1}(x) - -(3.23)_{mn,k+1\to 1}(y)\Big]\mathrm{d}v^1\Big]. \tag{7.87}$$

For (7.85), since $\tilde{\theta} \ll 1$, from (7.84) we derive that

$$\begin{aligned}(7.85) &\lesssim \|\eta\|_{C^2}\|wf\|_\infty \frac{w_{\tilde{\theta}}(v)|v|^2[M_W(x_{\mathbf{b}}(x,v),v) - M_W(x_{\mathbf{b}}(y,v),v)]}{\sqrt{\mu(v)}|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\ &\lesssim \|\eta\|_{C^2}\|wf\|_\infty \|T_W\|_{C^1}. \end{aligned} \tag{7.88}$$

For (7.86), from (2.64) with (7.6) and (2.54) we compute

$$\begin{aligned} &\frac{f_s(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1}) - f_s(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1})}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\ &\lesssim \frac{f_s(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1}) - f_s(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1})}{|\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) - \eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})|^\beta} \\ &\quad + \frac{f_s(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1}) - f_s(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1})}{|T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1} - T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1}|} \\ &\quad \times \frac{|T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1} - T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1}|}{|\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) - \eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})|} \\ &\lesssim \frac{\|\alpha\nabla_x f_s\|_\infty}{\min\left\{\alpha(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1}), \alpha(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1})\right\}^\beta} \\ &\quad + \frac{\||v|^2\nabla_v f\|_\infty \|\eta\|_{C^2}|\mathbf{v}^1_{p^1}|}{|\mathbf{v}^1_{p^1}|^2}, \end{aligned} \tag{7.89}$$

where we have used the definition of $T_{\mathbf{x}_p}$ in (2.23) and the mean value theorem regarding $\nabla_v f$ in the last line.

Since $\sum_{m,n}(3.23)_{mn,k+1\to 1}(x)\partial_{\mathbf{v}^1_{p^1,n}}\Big[\sqrt{\mu(\mathbf{v}^1_{p^1})}\mathbf{v}^1_{p^1,3}\mathbf{v}^1_{p^1,m}\Big] \lesssim 1$, we have

$$\begin{aligned}(7.86) &\lesssim \|\alpha\nabla_x f\|_\infty \frac{w_{\tilde{\theta}}(v)|v|^2 M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}} \\ &\quad \times \int_{\mathbf{v}^1_{p^1,3}>0} \frac{\mu^{1/4}(\mathbf{v}^1_{p^1})}{\min\left\{\alpha(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1_{p^1}), \alpha(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1_{p^1})\right\}^\beta}\mathrm{d}\mathbf{v}^1_{p^1} \\ &\quad + \|\eta\|_{C^2}\||v|^2\nabla_v f\|_\infty \frac{w_{\tilde{\theta}}(v)|v|^2 M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}} \int_{\mathbf{v}^1_{p^1,3}>0} \frac{\mu^{1/4}(\mathbf{v}^1_{p^1})}{|\mathbf{v}^1_{p^1}|}\mathrm{d}\mathbf{v}^1_{p^1} \\ &\lesssim \big(\|\alpha\nabla_x f\|_\infty + \||v|^2\nabla_v f\|_\infty\big)\frac{|v|^2 M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}} \end{aligned}$$

$$\times \Big[\int_{\mathbb{R}^3} \Big(\frac{\mu^{1/4}(\mathbf{v}^1_{p^1})}{|n(x_{\mathbf{b}}(x,v)) \cdot T^t_{\mathbf{x}^1_{p^1(x)}} \mathbf{v}^1_{p^1}|^\beta} + \frac{\mu^{1/4}(\mathbf{v}^1_{p^1})}{|n(x_{\mathbf{b}}(y,v)) \cdot T^t_{\mathbf{x}^1_{p^1(y)}} \mathbf{v}^1_{p^1}|^\beta} \mathrm{d}\mathbf{v}^1_{p^1} \Big)$$
$$+ \int_{\mathbb{R}^3} \frac{\mu^{1/4}(\mathbf{v}^1_{p^1})}{|\mathbf{v}^1_{p^1}|} \mathbf{v}^1_{p^1} \Big]$$
$$\lesssim \|\alpha \nabla_x f\|_\infty + \||v|^2 \nabla_v f\|_\infty, \tag{7.90}$$

where we have used $\beta < 1$ in the last line.

Last we estimate (7.87). From (3.23) and (2.32) we compute

$$\frac{(3.23)_{mn}(x) - -(3.23)_{mn}(y)}{|\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) - \eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})|^\beta}$$
$$\lesssim \Big[\frac{\partial}{\partial \mathbf{x}^1_{p^1(x),j}} \Big(\frac{\partial_m \eta_{p^1(x),l}(\mathbf{x}^1_{p^1(x)})}{\sqrt{g_{p^1(x),mm}(\mathbf{x}^1_{p^1(x)})}} \Big) \frac{\frac{\partial_n \eta_{p^1(x),l}(\mathbf{x}^1_{p^1(x)})}{\sqrt{g_{p^1(x),nn}(\mathbf{x}^1_{p^1(x)})}} - \frac{\partial_n \eta_{p^1(y),l}(\mathbf{x}^1_{p^1(y)})}{\sqrt{g_{p^1(y),nn}(\mathbf{x}^1_{p^1(y)})}}}{|\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) - \eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})|^\beta}$$
$$+ \frac{\frac{\partial}{\partial \mathbf{x}^1_{p^1(x),j}} \Big(\frac{\partial_m \eta_{p^1(x),l}(\mathbf{x}^1_{p^1(x)})}{\sqrt{g_{p^1(x),mm}(\mathbf{x}^1_{p^1(x)})}} \Big) - \frac{\partial}{\partial \mathbf{x}^1_{p^1(y),j}} \Big(\frac{\partial_m \eta_{p^1(y),l}(\mathbf{x}^1_{p^1(y)})}{\sqrt{g_{p^1(y),mm}(\mathbf{x}^1_{p^1(y)})}} \Big)}{|\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) - \eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)})|^\beta}$$
$$\times \frac{\partial_n \eta_{p^1(y),l}(\mathbf{x}^1_{p^1(y)})}{\sqrt{g_{p^1(y),nn}(\mathbf{x}^1_{p^1(y)})}} \Big] \lesssim \|\eta\|_{C^3},$$

where we have used $\eta \in C^3$ and mean value theorem in the last line. Thus we conclude

$$(7.87) \lesssim \|\eta\|_{C^3} \|wf\|_\infty. \tag{7.91}$$

Combining (7.88),(7.90) and (7.91), we conclude that the contribution of the velocity derivative $(7.82)_1$ in (7.81) has an upper bound

$$|(7.81)_{(7.82)_1}| \lesssim \|\eta\|_{C^3} \|T_W\|_{C^1} \|\alpha \nabla_x f\|_\infty, \tag{7.92}$$

where we used (7.6).

Spatial derivative: we consider the contribution of the spatial derivative $(7.82)_2$ in (7.24). We rewrite the $\mathbf{v}^1_{p^1}$-integration using v^1 integration and get

$$(7.82)_2 = e^{-\nu t_{\mathbf{b}}(x,v)} \frac{M_W(x_{\mathbf{b}}(x,v), v)}{\sqrt{\mu(v)}}$$
$$\times \int_{n(x_{\mathbf{b}}(x,v)) \cdot v^1 > 0} \underbrace{\partial_{\mathbf{x}^1_{p^1(x),i}} \Big[f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), v^1) \Big] \sqrt{\mu(v^1)} |n(x_{\mathbf{b}}(x,v)) \cdot v^1| \, \mathrm{d}v^1}_{(7.93)_*}. \tag{7.93}$$

Then the contribution of $(7.82)_2$ in (7.24) can be written as

$$\Big[\frac{w_{\tilde\theta}(v)|v|^2[M_W(x_{\mathbf{b}}(x,v),v)-M_W(x_{\mathbf{b}}(y,v),v)]}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta\sqrt{\mu(v)}}\times(7.93) \tag{7.94}$$

$$+\frac{|v|^2M_W(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu(v)}|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}$$
$$\times\Big(\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\frac{\partial}{\partial\mathbf{x}^1_{p^1(x),i}}f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v^1)\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(x,v))\cdot v^1|\mathrm{d}v^1$$
$$-\int_{n(x_{\mathbf{b}}(y,v))\cdot v^1>0}\frac{\partial}{\partial\mathbf{x}^1_{p^1(y),i}}f(\eta_{p^1}(\mathbf{x}^1_{p^1(y)}),v^1)\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(y,v))\cdot v^1|\mathrm{d}v^1\Big)\Big]. \tag{7.95}$$

From (2.47) in Lemma 2.5,

$$(7.93)\lesssim\|\,|v|\nabla_\parallel f\|_\infty\int\sqrt{\mu(v^1)}\frac{|n(x_{\mathbf{b}}(x,v))\cdot v^1|}{|v^1|}\mathrm{d}v^1\lesssim\|\,|v|\nabla_\parallel f\|_\infty.$$

Thus applying (2.54) and (2.71) we derive that

$$(7.94)\lesssim\|\eta\|_{C^2}\|\,|v|\nabla_\parallel f\|_\infty. \tag{7.96}$$

For (7.95), we express $\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v^1)$ and $\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(y),i}}f(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}),v^1)$ using (7.30)–(7.35) with the notation (7.3):

$$\mathbf{1}_{t^1\geqq\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}e^{-\nu t^2_{\mathbf{b}}(x)}\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}\big[f(x_{\mathbf{b}}(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v^1),v^1)\big] \tag{7.97}$$

$$-\mathbf{1}_{t^1\geqq\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}\nu\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}t^2_{\mathbf{b}}(x)e^{-\nu t^2_{\mathbf{b}}(x)}f(x^2_{\mathbf{b}}(x),v^1) \tag{7.98}$$

$$+\mathbf{1}_{t^1\leqq\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}e^{-\nu t^1}\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}\big[f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)})-t^1v^1,v^1)\big] \tag{7.99}$$

$$+\mathbf{1}_{t^1\leqq\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}$$
$$\times\int_0^{t^1}e^{-\nu(t^1-s^1)}\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}\big[h(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)})-(t^1-s^1)v^1,v^1)\big]\mathrm{d}s^1 \tag{7.100}$$

$$+\mathbf{1}_{t^1\geqq\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}$$
$$\times\int_{t^1-t^2_{\mathbf{b}}(x)}^{t^1}e^{-\nu(t^1-s^1)}\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}\big[h(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)})-(t^1-s^1)v^1,v^1)\big]\mathrm{d}s^1 \tag{7.101}$$

$$+\mathbf{1}_{t^1\geqq\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}}t^2_{\mathbf{b}}(x)e^{-\nu t^2_{\mathbf{b}}(x)}h(x^2_{\mathbf{b}}(x),v^1). \tag{7.102}$$

We first estimate the boundary term (7.97). We split (7.97) into two cases using $\min\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\}$. We put the discussion for $\min\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\}\leqq\varepsilon$, or $|v|\geqq\varepsilon$ together with the estimate of (7.98)–(7.102). Here we discuss the case that $\min\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\}\geqq\varepsilon$ and $|v|\leqq\varepsilon^{-1}$.

For this case the difference quotient of (7.97) reads as

$$\begin{aligned}&\frac{\mathbf{1}_{\{\min\{n(x_{\mathbf{b}}(x,v))\cdot v^1,n(x_{\mathbf{b}}(y,v))\cdot v^1\}\geqq\varepsilon,|v|\leqq\varepsilon^{-1}\}}}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^{\beta}}\\&\quad\times\Big(\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}(7.97)(x)\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(x,v))\cdot v^1|\mathrm{d}v^1\\&\quad-\int_{n(x_{\mathbf{b}}(y,v))\cdot v^1>0}(7.97)(y)\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(y,v))\cdot v^1|\mathrm{d}v^1\Big).\end{aligned}\tag{7.103}$$

We perform the change of variable (2.42) and use (3.29) to rewrite thing's as

$$\begin{aligned}&\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\frac{\partial}{\partial\mathbf{x}^1_{p^1,i}}[f(\eta_{p^2}(\mathbf{x}^2_{p^2}),v^1)]\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(x,v))\cdot v^1|\mathrm{d}v^1\\&=\sum_{p^2\in\mathcal{P}}\iint_{|\mathbf{x}^2_{p^2}|<\delta_1}\int_0^{t-t_{\mathbf{b}}(x,v)}e^{-\nu(v^1)t^1_{\mathbf{b}}}\iota_{p^2}(\eta_{p^2}(\mathbf{x}^2_{p^2}))\\&\quad\times\sum_{j'=1,2}\frac{\partial\mathbf{x}^2_{p^2,j'}}{\partial\mathbf{x}^1_{p^1(x),j}}\partial_{\mathbf{x}^2_{p^2,j'}}[f(\eta_{p^2}(\mathbf{x}^2_{p^2}),v^1)]\\&\quad\times\frac{n_{p^1(x)}(\mathbf{x}^1_{p^1(x)})\cdot(x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))}{t^2_{\mathbf{b}}}\frac{n_{p^2}(\mathbf{x}^2_{p^2})\cdot(x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))}{|t^2_{\mathbf{b}}|^4}\\&\quad\times e^{-\frac{|x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}}\mathrm{d}t^2_{\mathbf{b}}\sqrt{g_{p^2,11}g_{p^2,22}}\mathrm{d}\mathbf{x}^2_{p^2,1}\mathrm{d}\mathbf{x}^2_{p^2,2}.\end{aligned}\tag{7.104}$$

Here we dropped the x dependence on $p^2(x)$ since $\mathbf{x}^2_{p^2}$ becomes dummy variable after the change of variable.

In (7.104) the variables that depend on x are $t_{\mathbf{b}}(x,v)$, $x_{\mathbf{b}}(x,v)$, $\mathbf{x}^1_{p^1(x)}$. Thus we have

$$\begin{aligned}(7.103)&=\frac{(7.104)(x)-(7.104)(y)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^{\beta}}\\&=\frac{1}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^{\beta}}\Big[\sum_{p^2\in\mathcal{P}}\iint_{|\mathbf{x}^2_{p^2}|<\delta_1}\int_{t-\max\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}}^{t-\min\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}}\cdots\qquad(7.105)\\&\quad+\sum_{p^2\in\mathcal{P}}\iint_{|\mathbf{x}^2_{p^2}|<\delta_1}\int_0^{t-\max\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}}\Big[\frac{\partial\mathbf{x}^2_{p^2,j'}}{\partial\mathbf{x}^1_{p^1(x),j}}-\frac{\partial\mathbf{x}^2_{p^2,j'}}{\partial\mathbf{x}^1_{p^1(y),j}}\Big]\cdots\qquad(7.106)\\&\quad+\Big[\frac{n_{p^1(x)}(\mathbf{x}^1_{p^1(x)})\cdot(x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))}{t^2_{\mathbf{b}}}\end{aligned}$$

$$- \frac{n_{p^1(y)}(\mathbf{x}^1_{p^1(y)}) \cdot (x_{\mathbf{b}}(y,v) - \eta_{p^2}(\mathbf{x}^2_{p^2}))}{t^2_{\mathbf{b}}}\Big] \cdots \tag{7.107}$$

$$+ \Big[\frac{n_{p^2}(\mathbf{x}^2_{p^2}) \cdot (x_{\mathbf{b}}(x,v) - \eta_{p^2}(\mathbf{x}^2_{p^2}))}{|t^2_{\mathbf{b}}|^4} - \frac{n_{p^2}(\mathbf{x}^2_{p^2}) \cdot (x_{\mathbf{b}}(y,v) - \eta_{p^2}(\mathbf{x}^2_{p^2}))}{|t^2_{\mathbf{b}}|^4}\Big] \cdots \tag{7.108}$$

$$+ \Big[e^{-\frac{|x_{\mathbf{b}}(x,v) - \eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}} - e^{-\frac{|x_{\mathbf{b}}(y,v) - \eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}}\Big]\Big] \cdots . \tag{7.109}$$

Since $\min\{\alpha(x_{\mathbf{b}}(x,v), v^1), \alpha(x_{\mathbf{b}}(y,v), v^1)\} \geqq \varepsilon$, from (1.11), clearly we have $|n(x_{\mathbf{b}}(x,v)) \cdot v^1| \gtrsim \varepsilon$. Moreover, due to $|v^1| \leqq \varepsilon^{-1}$, we have a lower bound for $t^2_{\mathbf{b}}$ from (2.39):

$$t^2_{\mathbf{b}} \gtrsim \frac{1}{|v^1|} \frac{\min\big\{|n(x_{\mathbf{b}}(x,v)) \cdot v^1|, |n(x_{\mathbf{b}}(y,v)) \cdot v^1|\big\}}{|v^1|} \gtrsim \varepsilon^3.$$

From (2.47) in Lemma 2.5 and Lemma 2.8 we obtain the following estimate:

$$\Big|\partial_{\mathbf{x}^2_{p^2,j'}}[f(\eta_{p^2}(\mathbf{x}^2_{p^2}), v^1)]\Big| \leqq \||v|\nabla_{\|} f\|_\infty < \infty, \quad \Big|\frac{\partial \mathbf{x}^2_{p^2,j'}}{\partial \mathbf{x}^1_{p^1,j}}\Big| \lesssim 1,$$

$$\Big|\frac{n_{p^1(x)}(\mathbf{x}^1_{p^1(x)}) \cdot (x_{\mathbf{b}}(x,v) - \eta_{p^2}(\mathbf{x}^2_{p^2}))}{t^2_{\mathbf{b}}}\Big| \lesssim_\Omega O(\varepsilon^{-3}),$$

$$\Big|\frac{n_{p^2}(\mathbf{x}^2_{p^2}) \cdot (x_{\mathbf{b}}(x,v) - \eta_{p^2}(\mathbf{x}^2_{p^2}))}{|t^2_{\mathbf{b}}|^4}\Big| \lesssim_\Omega O(\varepsilon^{-12}).$$

Now we estimate (7.105)–(7.109). By (2.56) we compute

$$\begin{aligned}
&\frac{|(7.105)|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\
&\lesssim \frac{O(\varepsilon^{-15})}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \iint \int_{t-\max\{t_{\mathbf{b}}(x,v), t_{\mathbf{b}}(y,v)\}}^{t-\min\{t_{\mathbf{b}}(x,v), t_{\mathbf{b}}(y,v)\}} e^{-\nu t^2_{\mathbf{b}}} \\
&\lesssim O(\varepsilon^{-15}) \frac{\big|e^{-\nu t_{\mathbf{b}}(x,v)} - e^{-\nu t_{\mathbf{b}}(y,v)}\big|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\
&= O(\varepsilon^{-15}) \frac{\big|e^{-\nu t_{\mathbf{b}}(x,v)} - e^{-\nu t_{\mathbf{b}}(y,v)}\big|}{|x-y|^\beta} \frac{|x-y|^\beta}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\
&\lesssim \frac{O(\varepsilon^{-15})}{\min\big\{\alpha(x_{\mathbf{b}}(x,v), v^1), \alpha(x_{\mathbf{b}}(y,v), v^1)\big\}} \frac{|x-y|^\beta}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\
&\lesssim O(\varepsilon^{-16}) \frac{|x-y|^\beta}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}.
\end{aligned} \tag{7.110}$$

The extra term $\frac{|x-y|^\beta}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}$ will be cancelled by $\frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta}$ in (7.24).

Then we estimate (7.106). By (2.72) we have

$$\begin{aligned}&\frac{\left|\frac{\partial \mathbf{x}^2_{p^2,j'}}{\partial \mathbf{x}^1_{p^1(x),j}}-\frac{\partial \mathbf{x}^2_{p^2,j'}}{\partial \mathbf{x}^1_{p^1(y),j}}\right|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\mathbf{1}_{\min\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\}\geqq\varepsilon}\\ &\lesssim \frac{|v^1|^3}{\min\left\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\right\}^3}\lesssim O(\varepsilon^{-6}).\end{aligned}$$

Thus

$$\frac{|(7.106)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\lesssim O(\varepsilon^{-15})\int_0^\infty e^{-\nu_0 t^2_{\mathbf{b}}}\frac{\left|\frac{\partial \mathbf{x}^2_{p^2,j'}}{\partial \mathbf{x}^1_{p^1(x),j}}-\frac{\partial \mathbf{x}^2_{p^2,j'}}{\partial \mathbf{x}^1_{p^1(y),j}}\right|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\lesssim O(\varepsilon^{-21}). \tag{7.111}$$

Then we estimate (7.107). By (2.57) we compute

$$\begin{aligned}&\frac{|n_{p^1(x)}(\mathbf{x}^1_{p^1(x)})\cdot(x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))-n_{p^1(y)}(\mathbf{x}^1_{p^1(y)})\cdot(x_{\mathbf{b}}(y,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))|}{t^2_{\mathbf{b}}|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\ &\lesssim O(\varepsilon^{-3})\Big[\frac{|n_{p^1(x)}(\mathbf{x}^1_{p^1(x)})-n_{p^1(y)}(\mathbf{x}^1_{p^1(y)})||x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\ &\quad+\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\Big]\lesssim O(\varepsilon^{-3}).\end{aligned}$$

Thus

$$\frac{|(7.107)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\lesssim O(\varepsilon^{-15}). \tag{7.112}$$

For (7.108) we compute the difference as

$$\begin{aligned}&\frac{|n_{p^2}(\mathbf{x}^2_{p^2})\cdot(x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))-n_{p^2}(\mathbf{x}^2_{p^2})\cdot(x_{\mathbf{b}}(y,v)-\eta_{p^2}(\mathbf{x}^2_{p^2}))|}{|t^2_{\mathbf{b}}|^4|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\ &\lesssim O(\varepsilon^{-12})\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\lesssim O(\varepsilon^{-12}).\end{aligned}$$

Thus

$$\frac{|(7.108)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\lesssim O(\varepsilon^{-12}). \tag{7.113}$$

Last we estimate (7.109). By mean value theorem,

$$\frac{\left|e^{-\frac{|x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}}-e^{-\frac{|x_{\mathbf{b}}(y,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}}\right|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}$$

$$\lesssim \frac{\left| e^{-\frac{|x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}} - e^{-\frac{|x_{\mathbf{b}}(y,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})|}{4|t^2_{\mathbf{b}}|}} \right|}{|x_{\mathbf{b}}(x,v)-\eta_{p^2}(\mathbf{x}^2_{p^2})-x_{\mathbf{b}}(y,v)+\eta_{p^2}(\mathbf{x}^2_{p^2})|^\beta/|t^2_{\mathbf{b}}|^\beta} \frac{1}{|t^2_{\mathbf{b}}|^\beta} \lesssim O(\varepsilon^{3\beta}).$$

Thus

$$\frac{|(7.109)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta} \lesssim O(\varepsilon^{-15-3\beta}). \tag{7.114}$$

Therefore, from (7.6), we collect (7.110),(7.111),(7.112),(7.113) and (7.114) to conclude

$$\begin{aligned} &|(7.103)| \\ &\lesssim \Big[O(\varepsilon^{-21}) + O(\varepsilon^{-16})\frac{|x-y|^\beta}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\Big]\|w_{\tilde\theta}\alpha\nabla_x f\|^2_\infty. \end{aligned} \tag{7.115}$$

Then we estimate the rest terms in (7.97)–(7.102). First we rewrite the contribution of these terms in (7.95) into

$$\begin{aligned} &\frac{\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}(7.97)\mathbf{1}_{\dots}(x)+\cdots+(7.102)(x)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta} \\ &\quad - \frac{\int_{n(x_{\mathbf{b}}(y,v))\cdot v^1>0}(7.97)\mathbf{1}_{\dots}(y)+\cdots+(7.102)(y)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta} \\ &= \frac{\int_{|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))|\geqq \frac{n(x_{\mathbf{b}}(x,v)\cdot v^1)}{|v^1|}>0}\cdots}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta} \\ &\quad - \frac{\int_{|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))|\geqq \frac{n(x_{\mathbf{b}}(y,v)\cdot v^1)}{|v^1|}>0}\cdots}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta} \qquad (7.116) \\ &\quad + \int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0, n(x_{\mathbf{b}}(y,v))\cdot v^1>0} \frac{|n(x_{\mathbf{b}}(x,v))\cdot v^1|-|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\sqrt{\mu(v^1)} \\ &\quad \times \big[(7.97)\mathbf{1}_{\dots}(x)+\cdots+(7.102)(x)\big] \qquad (7.117) \\ &\quad + \int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0, n(x_{\mathbf{b}}(y,v))\cdot v^1>0} |n(x_{\mathbf{b}}(y,v))\cdot v^1|\sqrt{\mu(v^1)} \\ &\quad \times \frac{(7.97)\mathbf{1}_{\dots}(x)+\cdots+(7.102)(x)-(7.97)\mathbf{1}_{\dots}(y)-\cdots-(7.102)(y)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}. \qquad (7.118) \end{aligned}$$

By (2.47) in Lemma 2.5,

$$\begin{aligned} &|(7.97)+\cdots+(7.102)| \lesssim (5.1)+\cdots(5.5) \\ &\lesssim (5.31) \lesssim \frac{\||v|\nabla_\parallel f\|_\infty + \|\alpha\nabla_x f\|_\infty}{|v^1|}. \end{aligned}$$

For (7.116), from (2.57) and (7.6) we have

$$\frac{1}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\int_{|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))|\geqq\frac{n(x_{\mathbf{b}}(x,v))\cdot v^1}{|v^1|}>0}|(7.97)+\cdots+(7.102)|$$

$$
\begin{aligned}
&\lesssim \frac{\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\
&\quad\times\int_{|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))|\geqq\frac{n(x_{\mathbf{b}}(x,v))\cdot v^1}{|v^1|}>0}\frac{|n(x_{\mathbf{b}}(x,v))\cdot v^1|}{|v|^1}\sqrt{\mu(v^1)}\\
&\lesssim \frac{\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\lesssim\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty.
\end{aligned}
$$

Similarly

$$
\frac{1}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\int_{|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))|\geqq\frac{n(x_{\mathbf{b}}(y,v))\cdot v^1}{|v^1|}>0}\cdots\lesssim\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty.
$$

Thus

$$
|(7.116)|\lesssim\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty. \tag{7.119}
$$

For (7.117), applying (2.57) we have

$$
\begin{aligned}
|(7.117)|&\lesssim\int\frac{|n(x_{\mathbf{b}}(x,v))\cdot v^1|-|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\sqrt{\mu(v^1)}\cdots\\
&\lesssim\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty.
\end{aligned} \tag{7.120}
$$

Then we focus (7.118), this estimate is the most delicate one. First of all we bound

$$
\begin{aligned}
|n(x_{\mathbf{b}}(y,v)\cdot v^1)|\leqq&\underbrace{|n(x_{\mathbf{b}}(x,v))-n(x_{\mathbf{b}}(y,v))||v^1|}_{(7.121)_1}\\
&+\underbrace{\min\left\{n(x_{\mathbf{b}}(x,v))\cdot v^1,n(x_{\mathbf{b}}(y,v))\cdot v^1\right\}}_{(7.121)_2}.
\end{aligned} \tag{7.121}
$$

By (2.57) the contribution of $(7.121)_1$ in (7.118) is bounded by

$$
\begin{aligned}
&\frac{|(7.118)_{(7.121)_1}|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\
&\lesssim\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0,n(x_{\mathbf{b}}(y,v))\cdot v^1>0}\\
&\frac{|n(x_{\mathbf{b}}(x,v))\cdot v^1|-|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\sqrt{\mu(v^1)}\cdots\mathrm{d}v^1\\
&\lesssim\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0,n(x_{\mathbf{b}}(y,v))\cdot v^1>0}\sqrt{\mu(v^1)}\lesssim\|\eta\|_{C^2}\|\alpha\nabla_x f\|_\infty.
\end{aligned} \tag{7.122}
$$

We focus on the contribution of $(7.121)_2$ in (7.118). Then without loss generality, we can assume

$$
|n(x_{\mathbf{b}}(y,v))\cdot v^1|=\min\left\{|n(x_{\mathbf{b}}(x,v))\cdot v^1|,|n(x_{\mathbf{b}}(y,v))\cdot v^1|\right\}. \tag{7.123}
$$

In result we can replace $|n(x_{\mathbf{b}}(y,v))\cdot v^1|$ or $|n(x_{\mathbf{b}}(y,v))\cdot v^1|$ by $\min\{|n(x_{\mathbf{b}}(x,v))\cdot v^1|, |n(x_{\mathbf{b}}(y,v))\cdot v^1|\}$.

Note that from (2.47) in Lemma 2.5,

$$\sum_{i=1,2}\partial_{\mathbf{x}^1_{p^1(x),i}} f(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),v^1)\sim G(x_{\mathbf{b}}(x,v))\nabla_x f(x_{\mathbf{b}}(x,v),v^1),$$

and we have an expression of

$$G(x_{\mathbf{b}}(x,v))\nabla_x f(x_{\mathbf{b}}(x,v),v^1)-G(x_{\mathbf{b}}(y,v))\nabla_x f(x_{\mathbf{b}}(y,v),v^1)$$

from Lemma 7.3. Thus the contribution of (7.97)–(7.102) in (7.118) can be expressed using (7.26)-(7.29), with replacing $x\to x_{\mathbf{b}}(x,v)$, $y\to x_{\mathbf{b}}(y,v)$, $v\to v^1$, $\mathbf{x}^1_{p^1,i}\to\mathbf{x}^i_{p^2,i}$, $x_{\mathbf{b}}(x,v)\to\eta_{p^2(x)}(\mathbf{x}^2_{p^2(x)})$, $x_{\mathbf{b}}(y,v)\to\eta_{p^2(y)}(\mathbf{x}^2_{p^2(y)})$.

From (7.80) we derive that the contribution of (7.27) is bounded by

$$\begin{aligned}
&\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0, n(x_{\mathbf{b}}(y,v))\cdot v^1>0}\\
&\quad\frac{|n(x_{\mathbf{b}}(y,v))\cdot v^1|\sqrt{\mu(v^1)}}{|v^1|^2\min\left\{\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|},\frac{\alpha(x_{\mathbf{b}}(y,v),v^1)}{|v^1|}\right\}^{1+\beta}}\\
&\quad\times\left[\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty+o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}+o(1)[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}\right]\\
&\quad\lesssim\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty+o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}+o(1)[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}},
\end{aligned}\tag{7.124}$$

where we have used (7.80).

Then we estimate the contribution of (7.28), (7.29) and (7.26).

We begin with (7.29). Note that we only need to consider the case of $\min\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\}\leqq\varepsilon$, or $|v^1|\geqq\varepsilon^{-1}$. We derive

$$\begin{aligned}
&\int\mathbf{1}_{\min\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\}\leqq\varepsilon,\text{ or }|v^1|\geqq\varepsilon^{-1}}\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(y,v)\cdot v^1)|\\
&\quad\times\sum_{j=1,2}\left[e^{-\nu t^2_{\mathbf{b}}(x)}\frac{|x^2_{\mathbf{b}}(x)-x^2_{\mathbf{b}}(y)|^\beta}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\right.\\
&\quad\left.\times\frac{\partial_{\mathbf{x}^2_{p^2(x),j}}f(\eta_{p^2(x)}(\mathbf{x}^2_{p^2(x)}),v^1)-\partial_{\mathbf{x}^2_{p^2(y),j}}f(\eta_{p^2(y)}(\mathbf{x}^2_{p^2(y)}),v^1)}{|\eta_{p^2(x)}(\mathbf{x}^2_{p^2(x)})-\eta_{p^2(y)}(\mathbf{x}^2_{p^2(y)})|^\beta}\right]\\
&\quad\lesssim\int_{\varepsilon>n(x_{\mathbf{b}}(x,v))\cdot v^1>0,\varepsilon>n(x_{\mathbf{b}}(y,v))\cdot v^1>0,\text{ or }|v^1|\geqq\varepsilon^{-1}}\sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(x,v))\cdot v^1|\\
&\quad\times\frac{[\nabla_{x_\parallel}f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}}{|v^1|^2\min\left\{\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|},\frac{\alpha(x_{\mathbf{b}}(y,v),v^1)}{|v^1|}\right\}^{1+\beta}}\\
&\quad\lesssim O(\varepsilon)[\nabla_{x_\parallel}f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}},
\end{aligned}\tag{7.125}$$

where we have applied (2.70) and (2.47) in Lemma 2.5 to the fourth line, (7.123) and (7.80) to the integral in the last line.

Then we focus on the contribution of (7.25). First we consider $h = K(f)$. Denote

$$x^s = x_{\mathbf{b}}(x,v) - (t^1 - s^1)v^1, \quad y^s = x_{\mathbf{b}}(y,v) - (t^1 - s^1)v^1,$$

we need to compute

$$\begin{aligned} &\int \sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(y,v))\cdot v^1| \int_0^{t^1} \mathrm{d}s^1 e^{-\nu(v^1)(t^1-s^1)} \\ &\times \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v^1,u) \underbrace{\frac{G(x^s)\nabla_x f(x^s,u) - G(y^s)\nabla_x f(y^s,u)}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}}_{(7.126)_*}. \end{aligned} \tag{7.126}$$

We use the decomposition (4.41) for the $\mathrm{d}s^1$ integral. When $t^1 - s^1 \leqq \varepsilon$, we apply (4.8) in Lemma 4.1 with $p = 1+\beta$ and (7.123) to conclude that

$$\begin{aligned} &|(7.126)\mathbf{1}_{t^1-s^1\leqq\varepsilon}| \\ &\lesssim [\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}} \int \sqrt{\mu(v^1)}|n(x_{\mathbf{b}}(y,v))\cdot v^1|\mathrm{d}v^1 \\ &\quad\times \int_{t^1-\varepsilon}^{t^1} \mathrm{d}s^1 e^{-\nu(v^1)(t^1-s^1)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v^1,u) \frac{1}{|u|^2 \min\left\{\frac{\alpha(x^s,u)}{|u|}, \frac{\alpha(y^s,u)}{|u|}\right\}^{1+\beta}} \\ &\lesssim O(\varepsilon)[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}} \int \sqrt{\mu(v^1)} \frac{|n(x_{\mathbf{b}}(y,v),v^1)|}{|v^1|^2 \min\left\{\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v|^1}, \frac{\alpha(x_{\mathbf{b}}(y,v),v^1)}{|v^1|}\right\}^{1+\beta}} \mathrm{d}v^1 \\ &\lesssim O(\varepsilon)[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}. \end{aligned}$$

When $t^1 - s^1 \geqq \varepsilon$. We rewrite things as

$$(7.126)_* = \underbrace{\frac{\Big[G(x^s) - G(y^s)\Big]\nabla_x f(y^s,u)}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}}_{(7.127)_1} + \underbrace{\frac{G(y^s)\Big[\nabla_x f(x^s,u) - \nabla_x f(y^s,u)\Big]}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}}_{(7.127)_2}. \tag{7.127}$$

By (2.57) we have $(7.127)_1 \lesssim \frac{\|\alpha\nabla_x f\|_\infty}{\alpha(y^s,u)}$. Thus, the contribution in (7.126) is bounded by

$$\begin{aligned} &\|\alpha\nabla_x f\|_\infty \int \sqrt{\mu(v)^1}|n(x_{\mathbf{b}}(y,v))\cdot v^1|\mathrm{d}v^1 \int_0^{t^1} \mathrm{d}s^1 e^{-\nu(v^1)}(t^1 - s^1) \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v^1,u)}{\alpha(y^s,u)} \\ &\lesssim \|\alpha\nabla_x f\|_\infty \int \sqrt{\mu(v^1)} \frac{|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{\min\left\{\alpha(x_{\mathbf{b}}(x,v),v^1), \alpha(x_{\mathbf{b}}(y,v),v^1)\right\}} \mathrm{d}v^1 \lesssim \|\alpha\nabla_x f\|_\infty, \end{aligned}$$

where we have used (4.5) in Lemma 4.1 and (7.121).

Then we focus on the contribution of $(7.127)_2$. We exchange ∇_x for ∇_{v^1}:

$$\nabla_x f(x^s,u) = \nabla_x f(x_{\mathbf{b}}(x,v) - (t^1 - s^1)v^1, u) = \frac{\nabla_{v^1} f(x_{\mathbf{b}}(x,v) - (t^1 - s^1)v^1, u)}{-(t^1 - s^1)}.$$

Then we perform an integration by parts for $\mathrm{d}v^1$. The $\mathrm{d}v^1$ integral in (7.126)$\mathbf{1}_{t^1-s^1\geqq\varepsilon}$ becomes

$$\Big|\int \nabla_{v^1}\Big[|n(x_{\mathbf{b}}(y,v))\cdot v^1|\sqrt{\mu(v^1)}\int_{\max\{t^1-t^2_{\mathbf{b}}(x),t^1-t^2_{\mathbf{b}}(y)\}}^{t^1} \times\frac{e^{-\nu(v^1)(t^1-s^1)}}{-(t^1-s^1)}\mathbf{k}_{\mathcal{Q}}(v^1,u)G(y^s)\Big]\frac{f(x^s,u)-f(y^s,u)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\Big|$$

$$\lesssim \Big|\int\int_{\max\{t^1-t^2_{\mathbf{b}}(x),t^1-t^2_{\mathbf{b}}(y)\}}^{t^1}\nabla_{v^1}\Big[|n(x_{\mathbf{b}}(y,v))\cdot v^1|\sqrt{\mu(v^1)}e^{-\nu(v^1)(t^1-s^1)}\Big]\cdots\Big| \tag{7.128}$$

$$+\Big|\int\int_{\max\{t^1-t^2_{\mathbf{b}}(x),t^1-t^2_{\mathbf{b}}(y)\}}^{t^1}\nabla_{v^1}\mathbf{k}_{\mathcal{Q}}(v^1,u)\cdots\Big| \tag{7.129}$$

$$+\Big|\int\int_{\max\{t^1-t^2_{\mathbf{b}}(x),t^1-t^2_{\mathbf{b}}(y)\}}^{t^1}\nabla_{v^1}G(x_{\mathbf{b}}(y,v)-(t^1-s^1)v^1)\cdots\Big| \tag{7.130}$$

$$+\Big|\int\nabla_{v^1}\min\Big\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\Big\}\frac{e^{-\nu(v^1)\min\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\}}}{\min\Big\{t^2_{\mathbf{b}}(x),t^2_{\mathbf{b}}(y)\Big\}}\cdots\Big|. \tag{7.131}$$

For (7.128), since

$$|\nabla_{v^1}[n(x_{\mathbf{b}}(y,v)\cdot v^1)\sqrt{\mu(v^1)}e^{-\nu(v^1)(t^1-s^1)}]|\lesssim\mu^{1/4}(v^1)e^{-\nu(v^1)(t^1-s^1)/2},$$

by (2.64) with (7.6) and (4.10) we have

$$\begin{aligned}(7.128)&\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty\int\mu^{1/4}(v^1)\int e^{-\nu(v^1)(t^1-s^1)/2}\mathrm{d}s^1\\&\quad\times\int_{\mathbb{R}^3}\frac{\mathbf{k}_{\mathcal{Q}}(v^1,u)}{\min\Big\{\alpha(x^s,v^1),\alpha(y^s,v^1)\Big\}^\beta}\mathrm{d}u\\&\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty\int\mu^{1/4}(v^1)\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty.\end{aligned}$$

For (7.129) from (2.103), we have $\nabla_{v^1}\mathbf{k}(v^1,u)\lesssim\frac{\langle v^1\rangle\mathbf{k}(v^1,u)}{|v^1-u|}$. Then by (4.11) in Lemma 4.1 we have

$$\begin{aligned}(7.129)&\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty\int|n(x_{\mathbf{b}}(y,v))\cdot v^1|\sqrt{\mu(v^1)}\langle v^1\rangle\mathrm{d}v^1\\&\quad\times\int e^{-\nu(v^1)(t^1-s^1)}\mathrm{d}s^1\int_{\mathbb{R}^3}\frac{\mathbf{k}_{\mathcal{Q}}(v^1,u)}{|v^1-u|\min\Big\{\alpha(x^s,v^1),\alpha(y^s,v^1)\Big\}^\beta}\mathrm{d}u\\&\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty\int\frac{|n(x_{\mathbf{b}}(y,v))\cdot v^1|\mu^{1/4}(v^1)}{\min\Big\{\alpha(x_{\mathbf{b}}(x,v),v^1),\alpha(x_{\mathbf{b}}(y,v),v^1)\Big\}^\beta}\\&\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty.\end{aligned}$$

For (7.130), since $|\nabla_{v^1} G(x_{\mathbf{b}}(y,v) - (t^1 - s^1)v^1)| \lesssim \|\xi\|_{C^2}(t^1 - s^1)$, and $(t^1 - s^1)e^{-\nu(v^1)(t^1-s^1)} \lesssim e^{-\nu(v^1)(t^1-s^1)/2}$, we have

$$(7.130) \lesssim O(\varepsilon^{-1})\|\alpha \nabla_x f\|_\infty \int \mu^{1/4}(v^1)\mathrm{d}v^1 \lesssim O(\varepsilon^{-1})\|\xi\|_{C^2}\|\alpha\nabla_x f\|_\infty.$$

For (7.131), since we consider $t^1 - s^1 \geqq \varepsilon$, $\min\{t^2_{\mathbf{b}}(x), t^2_{\mathbf{b}}(y)\} \geqq \varepsilon$. From (2.32) we have

$$\nabla_{v^1} \min\left\{t^2_{\mathbf{b}}(x), t^2_{\mathbf{b}}(y)\right\} \frac{e^{-\nu(v^1)\min\{t^2_{\mathbf{b}}(x), t^2_{\mathbf{b}}(y)\}}}{\min\left\{t^2_{\mathbf{b}}(x), t^2_{\mathbf{b}}(y)\right\}} \lesssim O(\varepsilon^{-1}).$$

Denote

$$x^b = x_{\mathbf{b}}(x,v) - \min\left\{t^2_{\mathbf{b}}(x), t^2_{\mathbf{b}}(y)\right\} v^1, \quad y^b = x_{\mathbf{b}}(y,v) - \min\left\{t^2_{\mathbf{b}}(x), t^2_{\mathbf{b}}(y)\right\} v^1.$$

Using (2.32) and from (4.10) in Lemma 4.1 we have

$$\begin{aligned}(7.131) &\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty \int \frac{|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|n(x_{\mathbf{b}}(y,v))\cdot v^1|}\sqrt{\mu(v^1)} \\ &\quad \times \int_{\mathbb{R}^3} \frac{\mathbf{k}_{\varrho}(v^1,u)}{\min\left\{\alpha(x^b,u), \alpha(y^b,u)\right\}^\beta} \\ &\lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty \int \sqrt{\mu(v^1)} \lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty.\end{aligned}$$

Thus the contribution of $(7.127)_2$ in (7.126) is bounded by

$$O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty. \tag{7.132}$$

Then we obtain

$$|(7.126)| \lesssim O(\varepsilon^{-1})\|\alpha\nabla_x f\|_\infty + O(\varepsilon)[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}. \tag{7.133}$$

Then we consider $h = \Gamma(f,f)$. We use (7.13) in Lemma 7.2 and (7.123) and (7.78) to obtain

$$\begin{aligned}&\int |n(x_{\mathbf{b}}(y,v))\cdot v^1|\sqrt{\mu(v^1)}\int_{\max\{t^x,t^y\}}^{t^1} \mathrm{d}s^1 e^{-\nu(v^1)(t^1-s^1)} \\ &\quad \times \frac{|G(x_{\mathbf{b}}(x,v))\nabla_x\Gamma(f,f)(x^s,v^1) - G(x_{\mathbf{b}}(y,v))\nabla_x\Gamma(f,f)(y^s,v^1)|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\ &\lesssim \left(\|\alpha\nabla_x f\|^2_\infty + \|wf\|_\infty[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}\right) \\ &\quad \times \int \sqrt{\mu(v^1)}\frac{|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|v^1|^2\min\left\{\alpha(x_{\mathbf{b}}(x,v),v^1), \alpha(x_{\mathbf{b}}(y,v),v^1)\right\}^{1+\beta}}\mathrm{d}v^1 \\ &\lesssim \|\alpha\nabla_x f\|^2_\infty + o(1)[\nabla_{x_\parallel} f(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}.\end{aligned} \tag{7.134}$$

The last term is (7.26). This estimate is similar to the contribution of (7.28). Note that $\sqrt{\mu(v^1)}\tilde{\alpha}(x_{\mathbf{b}}(x,v),v^1) \lesssim \mu^{1/4}(v^1)\alpha(x_{\mathbf{b}}(x,v),v^1)$, we need to compute

$$\begin{aligned}&\int \mu^{1/4}(v^1)|n(x_{\mathbf{b}}(y,v))\cdot v^1| \int_0^{t^1} \mathrm{d}s^1 e^{-\nu(v^1)(t^1-s^1)} \\ &\times \int_{\mathbb{R}^3} \mathrm{d}u \frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|}\mathbf{k}(v^1,u)\frac{\nabla_x f(x^s,u)-\nabla_x f(y^s,u)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}.\end{aligned} \tag{7.135}$$

Again we first consider $t^1-s^1 \leqq \varepsilon$. We apply (4.8) in Lemma 4.1 with $p=2+\beta$ and (7.123) to obtain

$$\begin{aligned}&|(7.135)\mathbf{1}_{t^1-s^1\leqq\varepsilon}| \\ &\lesssim [\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}} \int \mu^{1/4}(v^1)|n(x_{\mathbf{b}}(y,v),v^1)|\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|} \\ &\quad\times \int_{t^1-\varepsilon}^{t^1} \mathrm{d}s^1 e^{-\nu(v^1)(t^1-s^1)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v^1,u)\frac{1}{|u|^2\min\left\{\frac{\alpha(x^s,u)}{|u|},\frac{\alpha(y^s,u)}{|u|}\right\}^{2+\beta}} \\ &\lesssim O(\varepsilon)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}} \int \mu^{1/4}(v^1)\frac{|n(x_{\mathbf{b}}(y,v),v^1)|}{|v^1|^2\min\left\{\frac{\alpha(x_{\mathbf{b}}(x,v),v^1)}{|v^1|},\frac{\alpha(x_{\mathbf{b}}(y,v),v^1)}{|v^1|}\right\}^{1+\beta}} \\ &\lesssim O(\varepsilon)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}.\end{aligned}$$

For $t^1-s^1 \geqq \varepsilon$, we apply the same integration by parts technique as in (7.128)–(7.131). The only difference is, here we do not have an extra term $G(y^s)$. But this term does not apply a role in the estimates of (7.128),(7.129) and (7.131). Thus for this case, we have the same upper bound as (7.132).

Combining (7.115),(7.119),(7.120),(7.122),(7.124), (7.133),(7.125), (7.134) and (7.135), we conclude that

$$\begin{aligned}(7.95) \lesssim\ & o(1)\Big[[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}} + [\nabla_{\|} f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}\Big] \\ &+\Big[O(\varepsilon^{-21})+O(\varepsilon^{-16})\frac{|x-y|^\beta}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\Big]\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty^2.\end{aligned} \tag{7.136}$$

This, together with (7.96) and (7.92), leads to the conclusion:

$$\begin{aligned}(7.24) \lesssim\ & \frac{\|T_W - T_0\|_{C^2}}{w_{\tilde{\theta}}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}} \\ &\times\Big[o(1)\Big[[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}} + [\nabla_{\|} f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}\Big] \\ &+ O(\varepsilon^{-21})\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty^2\Big],\end{aligned} \tag{7.137}$$

where we have applied (2.54) to $\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta}$.

Step 2: estimate of (7.25).

Now we estimate the contribution of the collision operator. First we consider $h = \Gamma(f, f)$. Applying (7.12) in Lemma 7.2 we have

$$(7.25)\mathbf{1}_{h=\Gamma} \lesssim \frac{o(1)[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}} + \|w_{\tilde{\theta}}\alpha\nabla_x f\|^2_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}. \tag{7.138}$$

Now we focus on the estimate for $h = K(f)$, which is

$$\int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v,u) \underbrace{\frac{\partial_{x_i} f(x-(t-s)v, u) - \partial_{x_i} f(y-(t-s)v, u)}{|x-y|^\beta}}_{(7.139)_*}. \tag{7.139}$$

Since $|x-y| = |x-(t-s)v - [y-(t-s)v]|$, we express $(7.139)_*$ by (7.23)-(7.25). The contribution of (7.23) in (7.139) is bounded by

$$\begin{aligned}
&\frac{\left[o(1)[\nabla_{x_\parallel} f(\cdot, v)]_{C^{0,\beta}_{x;1+\beta}} + \|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty\right]}{w_{\tilde{\theta}}(v)} \\
&\quad\times \int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \frac{w_{\tilde{\theta}}(v)\mathbf{k}(v,u)}{w_{\tilde{\theta}}(u)|u|^2 \min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|}, \frac{\alpha(x-(t-s)v,u)}{|u|}\right\}^{2+\beta}} \\
&\lesssim \frac{o(1)[\nabla_{x_\parallel} f(\cdot, v)]_{C^{0,\beta}_{x;1+\beta}} + \|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}},
\end{aligned} \tag{7.140}$$

where we have applied Lemma 4.1.

Then we consider the contribution of (7.24) in (7.139). By (7.137), (2.122) and (4.7), such contribution is bounded by

$$\begin{aligned}
&\|T_W - T_0\|_{C^2} \frac{o(1)\left[[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}} + [\nabla_\parallel f(\cdot, v)]_{C^{0,\beta}_{x,1+\beta}}\right] + O(\varepsilon^{-21})\|\alpha\nabla_x f\|^2_\infty}{w_{\tilde{\theta}}(v)} \\
&\quad\times \int_0^t \mathrm{d}s e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \frac{w_{\tilde{\theta}}(v)\mathbf{k}(v,u)}{w_{\tilde{\theta}}(u)|u|^2 \min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|}, \frac{\alpha(y-(t-s)v,u)}{|u|}\right\}^{1+\beta}} \\
&\lesssim \|T_W - T_0\|_{C^2} \frac{o(1)\left[[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}} + [\nabla_\parallel f(\cdot, v)]_{C^{0,\beta}_{x,1+\beta}}\right] + O(\varepsilon^{-21})\|\alpha\nabla_x f\|^2_\infty}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.
\end{aligned} \tag{7.141}$$

Then we focus the contribution of the collision operator, the (7.25) in (7.139). We first estimate $h = \Gamma(f, f)$. By Lemma 7.2, such contribution in (7.139) is bounded by

$$\frac{1}{w_{\tilde{\theta}}(v)} \int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u w_{\tilde{\theta}}(v)\mathbf{k}(v,u) \frac{o(1)[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}} + \|w_{\tilde{\theta}}\alpha\nabla_x f\|^2_\infty}{w_{\tilde{\theta}}(u)|u|^2 \min\left\{\frac{\alpha(x,u)}{|u|}, \frac{\alpha(y,u)}{|u|}\right\}^{2+\beta}}$$

$$\lesssim \frac{o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}} + \|w_{\tilde\theta}\alpha\nabla_x f\|^2_\infty}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}, \tag{7.142}$$

where we have used Lemma 4.1 and Lemma 2.13.

Then we estimate $h = K(f)$, which is the most delicate one. We denote $t^s_m(u) = \min\{t_{\mathbf{b}}(x-(t-s)v,u), t_{\mathbf{b}}(y-s(t-s)v,u)\}$. We need to compute

$$\int_{t-t_m(v)}^t \mathrm{d}s e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v,u) \int_{s-t^s_m(u)}^s \mathrm{d}s' e^{-\nu(s-s')} \int_{\mathbb{R}^3} \mathrm{d}u' \\ \times \mathbf{k}(u,u') \frac{\nabla_x f(x-(t-s)v-(s-s')u,u') - \nabla_x f(y-(t-s)v-(s-s')u,u')}{|x-y|^\beta}. \tag{7.143}$$

We first decompose the s' integration as

$$\underbrace{\int_{s-\varepsilon}^s \mathrm{d}s sps}_{(7.144)_1} + \underbrace{\int_0^{s-\varepsilon} \mathrm{d}s'}_{(7.144)_2}. \tag{7.144}$$

Applying (4.8) in Lemma 4.1 with $p = 2+\beta$ we conclude that the contribution of $(7.144)_1$ in (7.143) is bounded by

$$\frac{[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}}{w_{\tilde\theta}(v)} \int_{t_m(s)}^t \mathrm{d}s e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v,u) w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)} \int_{s-\varepsilon}^s \mathrm{d}s' e^{-\nu(u)(s-s')} \\ \times \int_{\mathbb{R}^3} \mathrm{d}u' \frac{\mathbf{k}(u,u') w_{\tilde\theta}(u)}{w_{\tilde\theta}(u')|u'|^2 \min\left\{\frac{\alpha(x-(t-s)v-(s-s')u,u')}{|u'|}, \frac{\alpha(y-(t-s)v-(s-s')u,u')}{|u'|}\right\}^{2+\beta}} \\ \lesssim \frac{O(\varepsilon)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}.$$

Then we consider contribution of $(7.144)_2$. For simplicity we denote

$$x'' = x-(t-s)v-(s-s')u, \quad y'' = y-(t-s)v-(s-s')u, \\ x''-y'' = x-y. \tag{7.145}$$

We exchange ∇_x for ∇_u:

$$\nabla_x f(x-(t-s)v-(s-s')u,u') - \nabla_x f(y-(t-s)v-(s-s')u,u') \\ = \nabla_u[f(x'',u') - f(y'',u')] \frac{-1}{s-s'}.$$

Since $s-s' \geq \varepsilon$ the contribution of $(7.144)_2$ in (7.143) is

$$\int_{t-t_m(v)}^t \mathrm{d}s e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v,u) \int_{s-t^s_m(u)}^{s-\varepsilon} e^{-\nu(u)(s-s')} \mathrm{d}s' \mathbf{1}_{s-s'\geq\varepsilon} \\ \times \int_{\mathbb{R}^3} \mathrm{d}u' \mathbf{k}(u,u') \frac{\nabla_u[f(x'',u') - f(y'',u')]}{|x-y|^\beta} \frac{-1}{s-s'}. \tag{7.146}$$

Then we integrate by part for $\mathrm{d}u$ to have

(7.146)

$$= \int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{1}_{s-s'\geqq\varepsilon}$$

$$\times \Big[\nabla_u[\mathbf{k}(v,u)\mathbf{k}(u,u')] \int_{s-t_m^s(u)}^{s-\varepsilon} \frac{e^{-\nu(u)(s-s')}}{s-s'}\mathrm{d}s' \int_{\mathbb{R}^3} \mathrm{d}u' \frac{f(x'',u')-f(y'',u')}{|x-y|^\beta} \tag{7.147}$$

$$+ \mathbf{k}(v,u) \int_{s-t_m^s(u)}^{s-\varepsilon} \frac{\nabla_u e^{-\nu(u)(s-s')}}{s-s'}\mathrm{d}s' \int_{\mathbb{R}^3} \mathrm{d}u' \mathbf{k}(u,u') \frac{f(x'',u')-f(y'',u')}{|x-y|^\beta} \tag{7.148}$$

$$+ \mathbf{k}(v,u)\nabla_u t_m^s(u) \frac{e^{-\nu t_m^s(u)}}{t_m^s(u)} \int_{\mathbb{R}^3} \mathrm{d}u' \mathbf{k}(u,u') \frac{f(x^b,u')-f(y^b,u')}{|x-y|^\beta}\Big]. \tag{7.149}$$

Here we denoted

$$x^b = x-(t-s)v-t_m^s(u)u, \quad y^b = y-(t-s)v-t_m^s(u)u. \tag{7.150}$$

First we estimate (7.147). We begin with $\nabla_u \mathbf{k}(u,u')$. Since $w_{\tilde\theta}^{-1}(u')\langle u'\rangle |u'|^2 \lesssim 1$, from (2.64) with (7.6) and (2.103) we have

$$\begin{aligned}
&|(7.147)| \\
&\lesssim O(\varepsilon^{-1}) \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)} \int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \frac{\mathbf{k}(v,u)w_{\tilde\theta}(v)}{w_{\tilde\theta}(u)} \\
&\times \int_{s-t_m^s(u)}^{s-\varepsilon} e^{-\nu(u)(s-s')}\mathrm{d}s' \int_{\mathbb{R}^3} \mathrm{d}u' |u'|^2 w_{\tilde\theta}^{-1}(u') \\
&\times \frac{\nabla_u \mathbf{k}(u,u')w_{\tilde\theta}(u)}{w_{\tilde\theta}(u')} \frac{1}{|u'|^2 \min\{\alpha(x'',u'),\alpha(y'',u')\}^\beta} \\
&\lesssim \frac{O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)} \int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(v)(t-s)} \\
&\times \int_{\mathbb{R}^3} \frac{\mathrm{d}u \mathbf{k}_{\tilde Q}(v,u)}{|u|^2 \min\{\alpha(x-(t-s)v,u),\alpha(y-(t-s)v,u)\}^\beta} \\
&\lesssim \frac{O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2} \lesssim \frac{O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}},
\end{aligned} \tag{7.151}$$

where we have used (4.11), (2.122) and (2.123) in the fifth line, (4.10) in the last line.

The term with $\nabla_u \mathbf{k}(v,u)$ can be similarly bounded by

$$\frac{O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}. \tag{7.152}$$

Then we estimate (7.148). From (2.64) with (7.6) we have

$$
\begin{aligned}
&|(7.148)| \\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_0^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{w_{\tilde\theta}(u)\mathbf{k}(v,u)}{w_{\tilde\theta}(v)}\int_0^{s-\varepsilon} e^{-\nu(u)(s-s')}\mathrm{d}s' \\
&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u'|u'|^2 w_{\tilde\theta}^{-1}(u')\frac{w_{\tilde\theta}(u)\mathbf{k}(u,u')}{w_{\tilde\theta}(u')|u'|^2\min\{\alpha(x'',u'),\alpha(y'',u')\}^\beta} \\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_0^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{1}{|u|^2}\mathbf{k}_{\tilde\varrho}(v,u) \\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}},
\end{aligned}
\tag{7.153}
$$

where we have used Lemma 2.13 in the second line and (4.10) in Lemma 4.1 in the fourth line.

Last we estimate (7.149). Since we are considering $s-s'\geqq\varepsilon$, we have $t_{\mathbf{b}}(x-(t-s)v,u)\geqq\varepsilon$. From (2.64) with (7.6) and (2.32), we have

$$
\begin{aligned}
&|(7.149)| \\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_{t-t_m(v)}^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\frac{w_{\tilde\theta}(u)\mathbf{k}(v,u)}{w_{\tilde\theta}(v)}\mathrm{d}u\frac{e^{-\nu(u)t_m^s(u)}}{t_m^s(u)} \\
&\quad\times\nabla_u t_m^s(u)\int_{\mathbb{R}^3}\mathrm{d}u'|u'|w_{\tilde\theta}^{-1}(u')\frac{w_{\tilde\theta}(u)\mathbf{k}(u,u')}{|u'|w_{\tilde\theta}(u')\min\left\{\alpha(x^b,u'),\alpha(y^b,u')\right\}^\beta} \\
&\lesssim O(\varepsilon^{-1})\frac{\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)}\int_{t-t_m(v)}^t \mathrm{d}s e^{-\nu(v)(t-s)} \\
&\quad\times\int_{\mathbb{R}^3}\frac{\mathbf{k}_{\tilde\varrho}(v,u)}{|u|^2\min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|},\frac{\alpha(y-(t-s)v,u)}{|u|}\right\}}\mathrm{d}u \\
&\lesssim \frac{O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}}\lesssim\frac{O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}},
\end{aligned}
\tag{7.154}
$$

where we have used Lemma 2.13 in the fourth line and (4.10) in Lemma 4.1 in the last line.

Then combining (7.151), (7.152), (7.153) and (7.154) we conclude

$$
|(7.143)|\lesssim\frac{o(1)[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}+O(\varepsilon^{-1})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}}.
\tag{7.155}
$$

Combining (7.155), (7.142), (7.141), (7.140) and (7.138) we conclude that

$$\begin{aligned}(7.25) \lesssim & \frac{\|T_W - T_0\|_{C^2}}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}} \\ & \times \Big[o(1)\Big[[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}} \\ & + [\nabla_{\|} f(\cdot, v)]_{C^{0,\beta}_{x,1+\beta}}\Big] + O(\varepsilon^{-21})\|w_{\tilde{\theta}}\alpha\nabla_x f\|^2_\infty \Big].\end{aligned} \tag{7.156}$$

Finally from (7.23)–(7.25) and the estimate (7.156), (7.137), we conclude the proof of (7.7).

Step 3: proof of (7.8).

Now we prove (7.8). From Lemma 7.3, (7.24) is already bounded from (7.137). For (7.26), since $w^{-1}_{\tilde{\theta}}(v)\tilde{\alpha}(x,v) \lesssim w^{-1}_{\tilde{\theta}/2}(v)\alpha(x,v)$, by (7.156) we conclude

$$\begin{aligned}(7.26) &\lesssim \frac{\tilde{\alpha}(x,v)}{|v|} \times (7.156) \\ &= \frac{o(1)\Big[[\nabla_x f(\cdot, v)]_{C^{0,\beta}_{x,2+\beta}} + [\nabla_{\|} f(\cdot, v)]_{C^{0,\beta}_{x,1+\beta}}\Big] + O(\varepsilon^{-21})\|w_{\tilde{\theta}}\alpha\nabla_x f\|^2_\infty}{w_{\tilde{\theta}/2}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.\end{aligned} \tag{7.157}$$

Then we only need to estimate (7.28). First we consider $h = \Gamma(f, f)$. Such contribution is directly bounded using (7.13) in Lemma 7.2, thus

$$(7.28)_{h=\Gamma} \lesssim \frac{o(1)[\nabla_{\|} f(\cdot, v)]_{C^{0,\beta}_{x,1+\beta}} + \|w_{\tilde{\theta}}\alpha\nabla_x f\|^2_\infty}{w_{\tilde{\theta}/2}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.158}$$

Then we consider $h = K(f)$, which reads

$$\begin{aligned}&\int_{t-t_m(v)}^{t} \mathrm{d}s e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \mathbf{k}(v,u) \\ &\times \frac{G(x-(t-s)v)\nabla_x f(x-(t-s)v, u) - G(y-(t-s)v)\nabla_x f(y-(t-s)v, u)}{|x-y|^\beta}.\end{aligned} \tag{7.159}$$

We express (7.159) by (7.26)-(7.29) along u.

Note that

$$(7.26) \lesssim (7.157), \quad (7.29) \lesssim (7.137),$$

we conclude that the contribution of (7.26),(7.27) and (7.29) in (7.159) are bounded by

$$\begin{aligned}
&\frac{1}{w_{\tilde\theta/2}(v)}\int_{t-t_m(v)}^{t}\mathrm{d}s e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u w_{\tilde\theta/2}(v)\mathbf{k}(v,u)\\
&\quad\times\frac{o(1)\Big[[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}+[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}\Big]+O(\varepsilon^{-21})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta/2}(u)|u|^2\min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|},\frac{\alpha(y-(t-s)v,u)}{|u|}\right\}^{1+\beta}}\\
&\lesssim\frac{o(1)\Big[[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}+[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}\Big]+O(\varepsilon^{-21})\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},
\end{aligned}\tag{7.160}$$

where we have used (4.7) in Lemma 4.1 with $p=1+\beta$.

Then we focus on the contribution of the double collision operator (7.28). By Lemma 7.2 the contribution of $h=\Gamma$ is bounded by

$$\begin{aligned}
&\frac{1}{w_{\tilde\theta/2}(v)}\int_{t-t_m(v)}^{t}\mathrm{d}s e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u w_{\tilde\theta/2}(v)\mathbf{k}(v,u)\\
&\quad\times\frac{o(1)[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}+\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta/2}(u)|u|^2\min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|},\frac{\alpha(y-(t-s)v,u)}{|u|}\right\}^{1+\beta}}\\
&\lesssim\frac{o(1)[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}+\|w_{\tilde\theta}\alpha\nabla_x f\|_\infty^2}{w_{\tilde\theta/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.
\end{aligned}\tag{7.161}$$

Last we focus on the contribution of $h=K(f)$. Recall the notation x'',y'' in (7.145). We need to compute

$$\begin{aligned}
&\int_{t-t_m(v)}^{t}\mathrm{d}s e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\mathbf{k}(v,u)\int_{s-t_m^s(u)}^{s}\mathrm{d}s' e^{-\nu(s-s')}\int_{\mathbb{R}^3}\mathrm{d}u'\mathbf{k}(u,u')\\
&\quad\times\underbrace{\frac{G(x'')\nabla_x f(x',u')-G(y'')\nabla_x f(y'',u')}{|x-y|^\beta}}_{(7.162)_*}.
\end{aligned}\tag{7.162}$$

We apply the decomposition (7.144) for $\mathrm{d}s'$.

When $s-s'\leqq\varepsilon$, by (4.8) in Lemma 4.1 with $p=1+\beta$ we have

$$\begin{aligned}
&(7.162)\mathbf{1}_{s-s'\leqq\varepsilon}\frac{1}{w_{\tilde\theta/2}(v)}\\
&\lesssim\int_{t-t_m(v)}^{t}\mathrm{d}s e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{w_{\tilde\theta/2}(v)\mathbf{k}(v,u)}{w_{\tilde\theta/2}(u)}\int_{s-\varepsilon}^{s}\mathrm{d}s' e^{-\nu(s-s')}\\
&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u' w_{\tilde\theta/2}(u)\mathbf{k}(u,u')\frac{[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}}{w_{\tilde\theta/2}(u')|u'|^2\min\{\alpha(x'',u'),\alpha(y'',u')\}^{1+\beta}}
\end{aligned}$$

$$\lesssim O(\varepsilon)\frac{[\nabla_{\parallel} f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.163}$$

When $s-s'\geqq\varepsilon$, we rewrite

$$(7.162)_*=\frac{[G(x'')-G(y'')]\nabla_x f(x'',u')}{|x-y|^\beta} \tag{7.164}$$

$$+\frac{G(y'')[\nabla_x f(x'',u')-\nabla_x f(y'',u')]}{|x-y|^\beta}. \tag{7.165}$$

For (7.164), since $|u'|w^{-1}_{\tilde{\theta}}(u')\lesssim w^{-1}_{\tilde{\theta}/2}(u')$, we apply (2.57) to conclude that the contribution of (7.164) in (7.162) is bounded by

$$\begin{aligned}&\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)}\int_{t-t_m(v)}^t \mathrm{d}s e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{w_{\tilde{\theta}/2}(v)\mathbf{k}(v,u)}{w_{\tilde{\theta}/2}(u)}\\&\quad\times\int_{s-t^s_m(u)}^s \mathrm{d}s' e^{-\nu(s-s')}\int_{\mathbb{R}^3}\mathrm{d}u'|u'|\frac{w_{\tilde{\theta}/2}(u)\mathbf{k}(u,u')}{w_{\tilde{\theta}/2}(u')|u'|\alpha(x'',u')}\\&\lesssim\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}},\end{aligned} \tag{7.166}$$

where we have applied Lemma 2.13 and (4.5) in Lemma 4.1.

For (7.165), we exchange $\nabla_x f(x'',u')=\frac{\nabla_u f(x'',u')}{-(s-s')}$ and perform an integration by parts to $\mathrm{d}u$. Since $|G(y'')|\lesssim 1$, the contribution of (7.165) in (7.162) is bounded by (7.147),(7.148),(7.149) and with an extra term that corresponds to the derivative of $G(y'')$:

$$\begin{aligned}&\Big|\int_{t-t_m(v)}^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\mathbf{1}_{s-s'\geqq\varepsilon}\\&\quad\times\mathbf{k}(v,u)\int_{s-t^s_m(u)}^{s-\varepsilon}\nabla_u G(y'')e^{-\nu(u)(s-s')}\mathrm{d}s'\\&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u'\mathbf{k}(u,u')\frac{f(x'',u')-f(y'',u')}{|x-y|^\beta}\Big|\\&\lesssim\frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)}\int_0^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{w_{\tilde{\theta}/2}(u)\mathbf{k}(v,u)}{w_{\tilde{\theta}/2}(v)}\\&\quad\times\int_0^{s-\varepsilon} e^{-\nu(u)(s-s')}(s-s')\|\xi\|_{C^2}\mathrm{d}s'\\&\quad\times\int_{\mathbb{R}^3}\mathrm{d}u' w_{\tilde{\theta}/2}(u)\mathbf{k}(u,u')\frac{|u'|^2}{w_{\tilde{\theta}}(u')|u'|^2\min\{\alpha(x'',u'),\alpha(y'',u')\}^\beta}\end{aligned}$$

$$\lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)}\int_0^t \mathrm{d}s e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\mathbf{k}_{\tilde{\varrho}}(v,u)}{|u|^2}\int_{s-t_m^s(u)}^{s-\varepsilon} e^{-\nu(u)(s-s')/2}\mathrm{d}s'$$
$$\lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.167}$$

Here we have used (2.64) with (7.6) in the fourth line and Lemma 2.13 in the second last line.

Thus the contribution of (7.165) in (7.162) is bounded by

$$\begin{aligned}&(7.147)+(7.148)+(7.149)+(7.167)\\ &\lesssim (7.151)+(7.153)+(7.154)+(7.167)\\ &\lesssim \frac{O(\varepsilon^{-1})\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.\end{aligned} \tag{7.168}$$

This, together with (7.166) and (7.163), leads to the conclusion:

$$(7.162)\lesssim \frac{o(1)\Big[[\nabla_x f(\cdot,v)]_{C^{0,\beta}_{x,2+\beta}}+[\nabla_\parallel f(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}\Big]+O(\varepsilon^{-21})\|w_{\tilde{\theta}}\alpha\nabla_x f\|_\infty^2}{w_{\tilde{\theta}/2}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.169}$$

Finally collecting (7.169),(7.161),(7.160),(7.158),(7.157) and Lemma 7.3 we conclude the proof of (7.8). □

7.2. *Proof of (1.23)*

In this section we prove the Hölde regularity (1.23). For simplicity, we denote

$$\begin{aligned}&[\nabla_v f_s(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}\\ &:=\sup_{x,y\in\Omega}\left\|w_{\tilde{\theta}/2}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}\frac{|\nabla_v f_s(x,v)-\nabla_v f_s(y,v)|}{|x-y|^\beta}\right\|_{L^\infty_v}.\end{aligned} \tag{7.170}$$

We will use (7.6) for simplifying the proof.

Proof of (1.23) Similaly to (7.30) - (7.35) and (6.1) - (6.8), we use $t_m(v)=\min\{t_{\mathbf{b}}(x,v),t_{\mathbf{b}}(y,v)\}$ to express $\nabla_v f_s(x,v)$ as

$$\begin{aligned}\nabla_v f_s(x,v)=&\mathbf{1}_{t\geqq t_m}e^{-\nu t_{\mathbf{b}}(x,v)}\nabla_v[f_s(x_{\mathbf{b}}(x,v),v)]\\ &-\mathbf{1}_{t\geqq t_m}\nu\nabla_v t_{\mathbf{b}}(x,v)e^{-\nu t_{\mathbf{b}}(x,v)}f_s(x_{\mathbf{b}}(x,v),v)\\ &-\mathbf{1}_{t\geqq t_m}\nabla_v\nu(v)t_{\mathbf{b}}(x,v)e^{-\nu t_{\mathbf{b}}(x,v)}f_s(x_{\mathbf{b}}(x,v),v)\\ &+\mathbf{1}_{t\leqq t_m}e^{-\nu t}\nabla_v[f_s(x-tv,v)]\\ &-\mathbf{1}_{t\leqq t_m}\nabla_v\nu(v)te^{-\nu t}f_s(x-tv,v)\end{aligned}$$

$$
\begin{aligned}
&- \mathbf{1}_{t \geqq t_m} \int_{t - t_{\mathbf{b}}(x,v)}^{t} \nabla_v \nu(v) e^{-\nu(t-s)} (t-s) h(x - (t-s)v, v) \mathrm{d}s \\
&+ \mathbf{1}_{t \geqq t_m} \int_{t - t_{\mathbf{b}}(x,v)}^{t} e^{-\nu(t-s)} \nabla_v [h(x - (t-s)v, v)] \mathrm{d}s \\
&- \mathbf{1}_{t \leqq t_m} \int_{0}^{t} \nabla_v \nu(v) e^{-\nu(t-s)} (t-s) h(x - (t-s)v, v) \mathrm{d}s \\
&+ \mathbf{1}_{t \leqq t_m} \int_{0}^{t} e^{-\nu(t-s)} \nabla_v [h(x - (t-s)v, v)] \mathrm{d}s \\
&- \mathbf{1}_{t \leqq t_m} \nabla_v t_{\mathbf{b}}(x, v) e^{-\nu t_{\mathbf{b}}(x,v)} h(x - t_{\mathbf{b}}(x, v)v, v).
\end{aligned}
$$

Taking the difference of $\nabla_v f_s(x, v)$ and $\nabla_v f_s(y, v)$ using the above equation we have

$$
\begin{aligned}
&\frac{\nabla_v f_s(x, v) - \nabla_v f_s(y, v)}{|x - y|^\beta} \\
&= \mathbf{1}_{t \geqq t_m} \frac{e^{-\nu t_{\mathbf{b}}(x,v)} - e^{-\nu t_{\mathbf{b}}(y,v)}}{|x - y|^\beta} \nabla_v [f_s(x_{\mathbf{b}}(x, v), v)] && (7.171) \\
&\quad + \mathbf{1}_{t \geqq t_m} e^{-\nu t_{\mathbf{b}}(y,v)} \frac{\nabla_v [f_s(x_{\mathbf{b}}(y, v), v)] - \nabla_v [f_s(x_{\mathbf{b}}(x, v), v)]}{|x - y|^\beta} && (7.172) \\
&\quad - \mathbf{1}_{t \geqq t_m} \frac{\nu [\nabla_v t_{\mathbf{b}}(x, v) - \nabla_v t_{\mathbf{b}}(y, v)] + \nabla_v \nu(v) [t_{\mathbf{b}}(x, v) - t_{\mathbf{b}}(y, v)]}{|x - y|^\beta} \\
&\quad \times e^{-\nu t_{\mathbf{b}}(x,v)} f_s(x_{\mathbf{b}}(x, v), v) && (7.173) \\
&\quad - \mathbf{1}_{t \geqq t_m} [\nu \nabla_v t_{\mathbf{b}}(y, v) + \nabla_v \nu t_{\mathbf{b}}(y, v)] \frac{e^{-\nu t_{\mathbf{b}}(y,v)} - e^{-\nu t_{\mathbf{b}}(x,v)}}{|x - y|^\beta} f_s(x_{\mathbf{b}}(x, v), v) && (7.174) \\
&\quad - \mathbf{1}_{t \geqq t_m} [\nu \nabla_v t_{\mathbf{b}}(y, v) + \nabla_v \nu t_{\mathbf{b}}(y, v)] e^{-\nu t_{\mathbf{b}}(y,v)} \frac{f_s(x_{\mathbf{b}}(y, v), v) - f_s(x_{\mathbf{b}}(x, v), v)}{|x - y|^\beta} && (7.175) \\
&\quad + \mathbf{1}_{t \leqq t_m} e^{-\nu t} \frac{\nabla_v [f_s(x - tv, v)] - \nabla_v [f_s(y - tv, v)]}{|x - y|^\beta} && (7.176) \\
&\quad - \mathbf{1}_{t \leqq t_m} \nabla_v \nu t e^{-\nu t} \frac{f_s(x - tv, v) - f_s(y - tv, v)}{|x - y|^\beta} && (7.177) \\
&\quad - \mathbf{1}_{t \geqq t_m} \int_{t - t_m}^{t} \nabla_v \nu e^{-\nu(t-s)} (t-s) \frac{h(x - (t-s)v, v) - h(y - (t-s)v, v)}{|x - y|^\beta} \mathrm{d}s && (7.178) \\
&\quad - \mathbf{1}_{t \geqq t_m} \frac{1}{|x - y|^\beta} \int_{\min\{t - t_{\mathbf{b}}(x,v), t - t_{\mathbf{b}}(y,v)\}}^{t - t_m} \\
&\quad \times \Big[\mathbf{1}_{t_{\mathbf{b}}(x,v) \geqq t_{\mathbf{b}}(y,v)} \nabla_v \nu e^{-\nu(t-s)} h(x - (t-s)v, v) \\
&\quad + \mathbf{1}_{t_{\mathbf{b}}(x,v) \leqq t_{\mathbf{b}}(y,v)} \nabla_v \nu e^{-\nu(t-s)} h(y - (t-s)v, v) \Big] \mathrm{d}s && (7.179) \\
&\quad + \mathbf{1}_{t \geqq t_m} \int_{t - t_m}^{t} e^{-\nu(t-s)} \frac{\nabla_v [h(x - (t-s)v, v)] - \nabla_v [h(y - (t-s)v, v)]}{|x - y|^\beta} \mathrm{d}s && (7.180)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{1}_{t\geqq t_m}\frac{1}{|x-y|^\beta}\int_{\min\{t-t_{\mathbf{b}}(x,v),t-t_{\mathbf{b}}(y,v)\}}^{t-t_m}\\
&\times\Big[\mathbf{1}_{t_{\mathbf{b}}(x,v)\geqq t_{\mathbf{b}}(y,v)}e^{-\nu(t-s)}\nabla_v[h(x-(t-s)v,v)]\\
&+\mathbf{1}_{t_{\mathbf{b}}(x,v)\leqq t_{\mathbf{b}}(y,v)}e^{-\nu(t-s)}\nabla_v[h(y-(t-s)v,v)]\Big]\mathrm{d}s \qquad (7.181)\\
&-\mathbf{1}_{t\leqq t_m}\int_0^t\nabla_v\nu e^{-\nu(t-s)}(t-s)\frac{h(x-(t-s)v,v)-h(y-(t-s)v,v)}{|x-y|^\beta}\mathrm{d}s \qquad (7.182)\\
&+\mathbf{1}_{t\leqq t_m}\int_0^t e^{-\nu(t-s)}\frac{\nabla_v[h(x-(t-s)v,v)]-\nabla_v[h(y-(t-s)v,v)]}{|x-y|^\beta}\mathrm{d}s \qquad (7.183)\\
&-\mathbf{1}_{t\leqq t_m}\frac{\nabla_v t_{\mathbf{b}}(x,v)e^{-\nu t_{\mathbf{b}}(x,v)}-\nabla_v t_{\mathbf{b}}(y,v)e^{-\nu t_{\mathbf{b}}(y,v)}}{|x-y|^\beta}h(x-t_{\mathbf{b}}(x,v)v,v) \qquad (7.184)\\
&-\mathbf{1}_{t\leqq t_m}\nabla_v t_{\mathbf{b}}(y,v)e^{-\nu t_{\mathbf{b}}(y,v)}\frac{h(y-t_{\mathbf{b}}(y,v),v)-h(x-t_{\mathbf{b}}(x,v),v)}{|x-y|^\beta}. \qquad (7.185)
\end{aligned}
$$

The estimate of (7.172), (7.180) and (7.183) are the most delicate, we will estimate them in Step 2 and Step 3. Now we estimate the rest terms in Step 1.

Step 1: Estimate of the rest of the terms.

We will use the estimate

$$
\frac{|v|^c}{w_{\tilde\theta}(v)}\lesssim\frac{1}{w_{\tilde\theta/2}(v)|v|^3},\ \text{for } c\geqq -2. \qquad (7.186)
$$

For (7.171) we apply (2.56), (2.40) and (7.186) to have

$$
\begin{aligned}
&|(7.171)|\\
&\lesssim\frac{1}{|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta}[|\nabla_v x_{\mathbf{b}}(x,v)||\nabla_x f_s(x_{\mathbf{b}}(x,v),v)|\\
&\quad+|\nabla_v f_s(x_{\mathbf{b}}(x,v),v)|]\\
&\lesssim\frac{1}{w_{\tilde\theta/2}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}[\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty\\
&\quad+\|w_{\tilde\theta/2}(v)|v|^2\nabla_v f_s\|_\infty].
\end{aligned}
$$

For (7.173) we apply (2.61) and (2.56) to have

$$
|(7.173)|\lesssim\|wf_s\|_\infty\frac{1}{w_{\tilde\theta}(v)|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.
$$

For (7.174) we apply (2.39), (2.40) and (2.56) to have

$$
|(7.174)|\lesssim\|wf_s\|_\infty\frac{1}{w_{\tilde\theta}(v)|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{\beta}}.
$$

For (7.175) we apply (2.39), (2.40), (2.54) and (2.64) to have

$$|(7.175)| \lesssim \frac{1}{|v|}\|wf_s\|_\infty \frac{1}{w_{\tilde\theta}(v)\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{2\beta}}.$$

For (7.176) we use $t \leqq t_m$ and (2.39) to have

$$\begin{aligned}
&|(7.176)|\\
&\lesssim e^{-\nu t}\Big[\frac{t[\nabla_x f_s(x-tv,v)-\nabla_x f_s(y-tv,v)]+\nabla_v f_s(x-tv,v)-\nabla_v f_s(y-tv,v)}{|x-y|^\beta}\Big]\\
&\lesssim \frac{[\nabla_x f_s(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde\theta}(v)|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}+\frac{o(1)[\nabla_v f_s(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}}{w_{\tilde\theta/2}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.
\end{aligned}$$

For (7.177) we apply (2.64) and (7.6) to have

$$|(7.177)| \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}.$$

For (7.178) we apply (7.9) to have

$$|(7.178)| \lesssim \frac{\|wf_s\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}.$$

For (7.179) we apply (2.56) and (2.124) to have

$$|(7.179)| \lesssim \frac{\|wf_s\|_\infty}{w_{\tilde\theta}(v)}\frac{|t_{\mathbf{b}}(x,v)-t_{\mathbf{b}}(y,v)|}{|x-y|^\beta} \lesssim \frac{\|wf_s\|_\infty}{w_{\tilde\theta}(v)|v|\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta}.$$

For (7.181) we apply the same computation in (6.16) and (6.15), and use (7.6), (2.56) to have

$$|(7.181)| \lesssim \frac{|t_{\mathbf{b}}(x,v)-t_{\mathbf{b}}(y,v)|}{|x-y|^\beta}\frac{\|wf_s\|_\infty}{w_{\tilde\theta/2}(v)|v|^2} \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta/2}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^\beta}.$$

For (7.182) we apply (7.9) to have

$$|(7.182)| \lesssim \frac{\|wf_s\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}.$$

For (7.184), we apply (2.61), (2.56), (2.40) and (2.124) to have

$$\begin{aligned}
&|(7.184)|\\
&\lesssim \frac{\|wf_s\|_\infty}{w_{\tilde\theta}(v)}\times\Big[\frac{|\nabla_v t_{\mathbf{b}}(x,v)-\nabla_v t_{\mathbf{b}}(y,v)|}{|x-y|^\beta}+\frac{|\nabla_v t_{\mathbf{b}}(y,v)||e^{-\nu t_{\mathbf{b}}(x,v)}-e^{-\nu t_{\mathbf{b}}(y,v)}|}{|x-y|^\beta}\Big]\\
&\lesssim \frac{\|wf_s\|_\infty}{w_{\tilde\theta}(v)}\times\Big[\frac{1}{|v|^2\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}+\frac{1}{|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{\beta}}\Big].
\end{aligned}$$

For (7.185) we apply (2.40), (7.9) and (7.6) to have

$$|(7.185)| \lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde{\theta}}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^\beta}.$$

Combining all the estimates above we use (7.6), (7.186) and Proposition 3 to conclude

$$\begin{aligned}&(7.171) - - (7.185) \quad \text{except} \quad (7.172), (7.180), (7.183)\\ &\lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f_s\|_\infty^2 + o(1)[\nabla_v f_s(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}}{w_{\tilde{\theta}/2}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}.\end{aligned} \tag{7.187}$$

Step 2: estimate of (7.172). We apply the boundary condition (2.25) with the notation (7.4) to have

$$\begin{aligned}&\nabla_v f_s(x_{\mathbf{b}}(x,v),v)\\ &= \nabla_v\Big[\frac{M_w(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu_0(v)}}\int_{\mathbf{v}_3^1>0} f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1\\ &\quad + r(x_{\mathbf{b}}(x,v),v)\Big]\\ &= \nabla_v\Big[\frac{M_w(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu_0(v)}}\Big]\int_{\mathbf{v}_3^1>0} f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1 \qquad (7.188)\\ &\quad + \nabla_v[r(x_{\mathbf{b}}(x,v),v)] \qquad (7.189)\\ &\quad + \frac{M_w(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu_0(v)}}\int_{\mathbf{v}_3^1>0}\nabla_v[f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)]\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1. \qquad (7.190)\end{aligned}$$

First we consider the contribution of (7.188) in (7.172), which equals

$$\begin{aligned}&\frac{\nabla_v\Big[\frac{M_w(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu_0(v)}}\Big] - \nabla_v\Big[\frac{M_w(x_{\mathbf{b}}(y,v),v)}{\sqrt{\mu_0(v)}}\Big]}{|x-y|^\beta}\\ &\quad\times\int_{\mathbf{v}_3^1>0} f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1 \qquad (7.191)\\ &\quad + \nabla_v\Big[\frac{M_w(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu_0(v)}}\Big]\\ &\quad\times\int_{\mathbf{v}_3^1>0}\frac{f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1) - f_s(x_{\mathbf{b}}(y,v), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x-y|^\beta}\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1. \qquad (7.192)\end{aligned}$$

For (7.191) we apply the definition in (1.4) to have

$$\nabla_v\Big[\frac{M_w(x_{\mathbf{b}}(x,v),v)}{\sqrt{\mu_0(v)}}\Big] = \nabla_v\frac{e^{-\frac{|v|^2}{2T_w(x_{\mathbf{b}}(x,v))}}e^{\frac{|v|^2}{4T_0}}T_0}{\sqrt{2\pi}[T_w(x_{\mathbf{b}}(x,v))]^2}$$

$$= \frac{T_0}{\sqrt{2\pi}} \frac{1}{[T_w(x_{\mathbf{b}}(x,v))]^2} \Big(e^{-\frac{|v|^2}{2T_w(x_{\mathbf{b}}(x,v))}} e^{\frac{|v|^2}{4T_0}} [-\nabla_v (\frac{|v|^2}{2T_w(x_{\mathbf{b}}(x,v))}) + \frac{v}{2}] \Big)$$
$$+ \frac{T_0}{\sqrt{2\pi}} e^{-\frac{|v|^2}{2T_w(x_{\mathbf{b}}(x,v))}} e^{\frac{|v|^2}{4T_0}} \nabla_v \Big(\frac{1}{[T_w(x_{\mathbf{b}}(x,v))]^2} \Big).$$

Then we apply (2.60), (2.54) and (2.40) to bound

$$\begin{aligned}
&\frac{|\nabla_v (\frac{|v|^2}{2T_w(x_{\mathbf{b}}(x,v))}) - \nabla_v (\frac{|v|^2}{2T_w(x_{\mathbf{b}}(y,v))})|}{|x-y|^\beta} \\
&\lesssim \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta} \times \Big[\frac{|\frac{|v|}{T_w(x_{\mathbf{b}}(x,v))} - \frac{|v|}{T_w(x_{\mathbf{b}}(y,v))}|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \\
&\quad + \frac{|v|^2 |\frac{\nabla_v x_{\mathbf{b}}(x,v)}{[T_w(x_{\mathbf{b}}(x,v))]^2} - \frac{\nabla_v x_{\mathbf{b}}(y,v)}{[T_w(x_{\mathbf{b}}(x,v))]^2} + \frac{\nabla_v x_{\mathbf{b}}(y,v)}{[T_w(x_{\mathbf{b}}(y,v))]^2} - \frac{\nabla_v x_{\mathbf{b}}(y,v)}{[T_w(x_{\mathbf{b}}(x,v))]^2}|}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \Big] \\
&\lesssim \frac{\|T_w\|_{C^1} |v|}{\min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^\beta} + \frac{|v|}{\min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^{1+\beta}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{|\nabla_v (\frac{1}{2T_w(x_{\mathbf{b}}(x,v))}) - \nabla_v (\frac{1}{2T_w(x_{\mathbf{b}}(y,v))})|}{|x-y|^\beta} \\
&\lesssim \frac{|\frac{1}{T_w(x_{\mathbf{b}}(x,v))} - \frac{1}{T_w(x_{\mathbf{b}}(y,v))}|}{|x-y|^\beta} \\
&\quad + \frac{|\frac{\nabla_v x_{\mathbf{b}}(x,v)}{[T_w(x_{\mathbf{b}}(x,v))]^2} - \frac{\nabla_v x_{\mathbf{b}}(y,v)}{[T_w(x_{\mathbf{b}}(x,v))]^2} + \frac{\nabla_v x_{\mathbf{b}}(y,v)}{[T_w(x_{\mathbf{b}}(y,v))]^2} - \frac{\nabla_v x_{\mathbf{b}}(y,v)}{[T_w(x_{\mathbf{b}}(x,v))]^2}|}{|x-y|^\beta} \\
&\lesssim \frac{\|T_w\|_{C^1}}{\min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^\beta} + \frac{1}{|v| \min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^{1+\beta}} \\
&\quad + \frac{\|T_w\|_{C^1}}{|v| \min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^\beta}.
\end{aligned}$$

We combine the estimate above to bound (7.191) as

$$\begin{aligned}
(7.191) &\lesssim \frac{\|w f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3 \min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^{1+\beta}} \int_{\mathbf{v}_3^1 > 0} \sqrt{\mu_0(\mathbf{v}^1)} \mathbf{v}_3^1 d\mathbf{v}^1 \\
&\lesssim \frac{\|w f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3 \min\Big\{ \frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|} \Big\}^{1+\beta}}, \qquad (7.193)
\end{aligned}$$

where we have used that for $c \in \{1, 0, -1\}$ and $\|T_w - T_0\|_\infty \ll 1$,

$$e^{-\frac{|v|^2}{2T_w(x_{\mathbf{b}}(x,v))}} e^{\frac{|v|^2}{4T_0}} |v|^c \lesssim \frac{1}{w_{\tilde\theta}(v)|v|^3}.$$

For (7.192) we apply the above estimate, the same computation as (2.100) and (7.89), (7.90), then use (2.54) and (7.6) to obtain

$$(7.192) \lesssim \frac{\|w_{\tilde{\theta}}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde{\theta}}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.194}$$

Then we consider the contribution of (7.189) in (7.172). By the definition of r in (2.6), we apply the same bound in (7.193) to bound such contribution as

$$\frac{1}{w_{\tilde{\theta}}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.195}$$

Then we consider the contribution of (7.190) in (7.172), which equals

$$\frac{1}{\sqrt{\mu_0(v)}}\frac{M_w(x_{\mathbf{b}}(x,v)) - M_w(x_{\mathbf{b}}(y,v))}{|x-y|^\beta}$$
$$\times \int_{\mathbf{v}_3^1>0} \nabla_v[f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)]\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1 \tag{7.196}$$
$$+ \frac{M_w(x_{\mathbf{b}}(y,v))}{\sqrt{\mu_0(v)}} \int_{\mathbf{v}_3^1>0} \frac{\nabla_v[f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)] - \nabla_v[f_s(x_{\mathbf{b}}(y,v), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)]}{|x-y|^\beta}$$
$$\times \sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 d\mathbf{v}^1. \tag{7.197}$$

Applying (2.71) and (2.54) we have

$$(7.196) \lesssim \frac{\|wf_s\|_\infty + \|\alpha\nabla_x f_s\|_\infty}{\sqrt{\mu_0(v)}} \frac{M_w(x_{\mathbf{b}}(x,v)) - M_w(x_{\mathbf{b}}(y,v))}{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta} \frac{|x_{\mathbf{b}}(x,v) - x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta}$$
$$\lesssim \frac{\|wf_s\|_\infty + \|\alpha\nabla_x f_s\|_\infty}{w_{\tilde{\theta}}(v)\min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^\beta}. \tag{7.198}$$

Here for the $\mathbf{v}^1$ integral we apply the bound (6.12), (6.13) for (6.10).

Then we focus on (7.197). We use the notation (7.4) and (7.5) the partial x derivative, then we have

$$(7.197) = \frac{M_w(x_{\mathbf{b}}(y,v))}{\sqrt{\mu_0(v)}} \int_{\mathbf{v}_3^1>0} d\mathbf{v}^1 \sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1$$
$$\times \Big[\sum_{i=1,2} \Big(\frac{\nabla_v \mathbf{x}^1_{p^1(x),i} \partial_{\mathbf{x}^1_{p^1(x),i}} f_s(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)}{|x-y|^\beta}$$
$$- \frac{\nabla_v \mathbf{x}^1_{p^1(y),i} \partial_{\mathbf{x}^1_{p^1(y),i}} f_s(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x-y|^\beta} \Big) \tag{7.199}$$

$$+\frac{\nabla_v T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1\nabla_v f_s(x_{\mathbf{b}}(x,v),T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)-\nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1\nabla_v f_s(x_{\mathbf{b}}(y,v),T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x-y|^\beta}\Big]. \tag{7.200}$$

For (7.199) we have

$$\begin{aligned}(7.199)=&\frac{M_w(x_{\mathbf{b}}(y,v))w_{\tilde\theta}(v)|v|^2}{\sqrt{\mu_0(v)}w_{\tilde\theta}(v)|v|^2}\int_{\mathbf{v}^1_3>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}^1_3\\&\times\sum_{i=1,2}\Big[\frac{|\nabla_v\mathbf{x}^1_{p^1(x),i}-\nabla_v\mathbf{x}^1_{p^1(y),i}|}{|x-y|^\beta}\partial_{\mathbf{x}^1_{p^1(x),i}}f_s(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)\\&+\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta}\nabla_v\mathbf{x}^1_{p^1(y),i}\\&\times\frac{\partial_{\mathbf{x}^1_{p^1(x),i}}f_s(\eta_{p^1(x)}(\mathbf{x}^1_{p^1(x)}),T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)-\partial_{\mathbf{x}^1_{p^1(y),i}}f_s(\eta_{p^1(y)}(\mathbf{x}^1_{p^1(y)}),T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\Big].\end{aligned}$$

By changing the $\mathrm{d}\mathbf{v}^1$ integral back to $\mathrm{d}v^1$ integral in (2.25) and applying (2.60) and (2.40), we have

$$\begin{aligned}&(7.199)\\&\lesssim\frac{1}{w_{\tilde\theta}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}\\&\times\Big[\int_{n(x_{\mathbf{b}}(y,v))\cdot v^1>0}\|\alpha\nabla_x f_s\|_\infty\sqrt{\mu_0(v^1)}\frac{|n(x_{\mathbf{b}}(y,v))\cdot v^1|}{|n(x_{\mathbf{b}}(y,v))\cdot v^1|}\mathrm{d}v^1 \qquad (7.201)\\&+\frac{1}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\\&\times\Big|\int_{n(x_{\mathbf{b}}(x,v))\cdot v^1>0}\sqrt{\mu_0(v^1)}|n(x_{\mathbf{b}}(x,v))\cdot v^1|\partial_{\mathbf{x}^1_{p^1(x),i}}f_s(x_{\mathbf{b}}(x,v),v^1)\mathrm{d}v^1\\&-\int_{n(x_{\mathbf{b}}(y,v))\cdot v^1>0}\sqrt{\mu_0(v^1)}|n(x_{\mathbf{b}}(y,v))\cdot v^1|\partial_{\mathbf{x}^1_{p^1(y),i}}f_s(x_{\mathbf{b}}(y,v),v^1)\mathrm{d}v^1\Big|\Big]. \qquad (7.202)\end{aligned}$$

Clearly

$$(7.201)\lesssim\frac{\|\alpha\nabla_x f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.203}$$

The other term is the same as (7.95), which is bounded by (7.136). Thus by Proposition 3,

$$(7.202)\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|^2_\infty}{w_{\tilde\theta}(v)|v|^3\min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.204}$$

For (7.200) we have

$$
\begin{aligned}
(7.200) \lesssim\ & \frac{M_w(x_{\mathbf{b}}(y,v))w_{\tilde\theta}(v)|v|^2}{\sqrt{\mu_0(v)}w_{\tilde\theta}(v)|v|^2}\Big[\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1 \\
& \times \frac{\Big|\nabla_v T^t_{\mathbf{x}^1_{p^1(x)}} - \nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}\Big|}{|x-y|^\beta}\frac{|\mathbf{v}^1|\|\,|v|^2\nabla_v f_s\|_\infty}{|\mathbf{v}^1|^2} \qquad (7.205)\\
& + \Big|\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1\nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1 \\
& \times \frac{\nabla_v f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1) - \nabla_v f_s(x_{\mathbf{b}}(y,v), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x-y|^\beta}\Big|\Big]. \qquad (7.206)
\end{aligned}
$$

Applying (2.69) we have

$$
(7.205) \lesssim \frac{\||v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}. \qquad (7.207)
$$

For the other term (7.206), we exchange the v-derivative to $\mathbf{v}^1$-derivative, then

$$
\begin{aligned}
(7.206) &= \frac{1}{w_{\tilde\theta}(v)|v|^3}\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1\nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1 \\
&\quad\times\frac{T_{\mathbf{x}^1_{p^1(x)}}\nabla_{\mathbf{v}^1} f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1) - T_{\mathbf{x}^1_{p^1(y)}}\nabla_{\mathbf{v}^1} f_s(x_{\mathbf{b}}(y,v), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x-y|^\beta} \\
&\lesssim \frac{1}{w_{\tilde\theta}(v)|v|^3}\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1|\nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}|\mathbf{v}^1 \\
&\quad\times\frac{\big|T_{\mathbf{x}^1_{p^1(x)}} - T_{\mathbf{x}^1_{p^1(y)}}\big|}{|x-y|^\beta}|T^t_{\mathbf{x}^1_{p^1(x)}}|\frac{\||v|^2\nabla_v f_s\|}{|\mathbf{v}^1|^2} \\
&\quad+\frac{1}{w_{\tilde\theta}(v)|v|^3}\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1\nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1 T_{\mathbf{x}^1_{p^1(y)}} \\
&\quad\times\frac{\nabla_{\mathbf{v}^1}\big[f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1) - f_s(x_{\mathbf{b}}(y,v), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)\big]}{|x-y|^\beta} \\
&\lesssim \frac{1}{w_{\tilde\theta}(v)|v|^3\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1\frac{\||v|^2\nabla_v f_s\|_\infty}{|\mathbf{v}^1|} \\
&\quad+\frac{1}{w_{\tilde\theta}(v)|v|^3}\Big|\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\nabla_{\mathbf{v}^1}[\sqrt{\mu_0(\mathbf{v}^1)}\mathbf{v}_3^1\mathbf{v}^1]\nabla_v T^t_{\mathbf{x}^1_{p^1(y)}}T_{\mathbf{x}^1_{p^1(y)}} \\
&\quad\times\frac{f_s(x_{\mathbf{b}}(x,v), T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1) - f_s(x_{\mathbf{b}}(y,v), T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)}{|x-y|^\beta}\Big|
\end{aligned}
$$

$$\lesssim \frac{1}{w_{\tilde\theta}(v)|v|^3}\times\Big[\frac{1}{\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}\||v|^2\nabla_v f_s\|_\infty+\frac{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}{|x-y|^\beta}$$
$$\times\int_{\mathbf{v}_3^1>0}\mathrm{d}\mathbf{v}^1\mu_0^{1/4}(\mathbf{v}^1)\frac{|f_s(x_{\mathbf{b}}(x,v),T^t_{\mathbf{x}^1_{p^1(x)}}\mathbf{v}^1)-f_s(x_{\mathbf{b}}(y,v),T^t_{\mathbf{x}^1_{p^1(y)}}\mathbf{v}^1)|}{|x_{\mathbf{b}}(x,v)-x_{\mathbf{b}}(y,v)|^\beta}\Big]$$
$$\lesssim\frac{\||v|^2\nabla_v f_s\|_\infty+\|\alpha\nabla_x f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}.\tag{7.208}$$

In the sixth last line we have used (2.66), (2.54), (2.41) and applied an integration by part for $\mathrm{d}\mathbf{v}^1$. In the third last line, we have used (2.41). In the last line we have used (2.54) and applied the same computation as (7.89), (7.90).

Combining (7.193), (7.194), (7.195), (7.198), (7.203), (7.204), (7.207) and (7.208), we use (7.6) and (7.186) to conclude

$$(7.172)\lesssim\frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta/2}(v)|v|^3\min\Big\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\Big\}^{1+\beta}}.\tag{7.209}$$

Step 3: estimate of (7.180) **and** (7.183). We focus on (7.180), the estimate of (7.183) is the same. First we consider $h=K(f_s)$, we compute

$$\int_{t-t_m}^t e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\nabla_v[\mathbf{k}(v,u)[f(x-(t-s)v,u)-f(y-(t-s)v,u)]]}{|x-y|^\beta}\mathrm{d}s$$
$$=\int_{t-t_m}^t e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\nabla_v[\mathbf{k}(v,u)][f(x-(t-s)v,u)-f(y-(t-s)v,u)]}{|x-y|^\beta}\mathrm{d}s\tag{7.210}$$
$$+\int_{t-t_m}^t e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\mathbf{k}(v,u)\nabla_v[f(x-(t-s)v,u)-f(y-(t-s)v,u)]}{|x-y|^\beta}\mathrm{d}s.\tag{7.211}$$

For (7.210) we have

$$(7.210)\lesssim\int_{t-t_m}^t e^{-\nu(t-s)}\int_{\mathbb{R}^3}\mathrm{d}u\frac{\mathbf{k}_{\tilde\varrho}(v,u)}{|v-u|}\frac{\langle v\rangle\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)\min\{\alpha(x-(t-s)v,u),\alpha(y-(t-s)v,u)\}^\beta}\mathrm{d}s$$
$$\lesssim\int_{t-t_m}^t e^{-\nu(t-s)}\frac{\langle v\rangle\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}\mathrm{d}s$$
$$\lesssim\frac{\langle v\rangle\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}.\tag{7.212}$$

In the second line we have used (2.64), Lemma 2.13, (7.6) and (2.103). In the third line we have used (4.11).

For (7.211) we have

(7.211)

$$
\begin{aligned}
&= \int_{t-t_m}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u \\
&\quad \times \frac{(t-s)\mathbf{k}(v,u)[\nabla_x f(x-(t-s)v,u) - \nabla_x f(y-(t-s)v,u)]}{|x-y|^\beta} \mathrm{d}s \\
&= \int_{t-t_m}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} \mathrm{d}u t_m \mathbf{k}(v,u) \\
&\quad \times \frac{w_{\tilde\theta}(v)[\nabla_x f_s(\cdot,v)]_{C^{0,\beta}_{x;2+\beta}}}{w_{\tilde\theta}(v)w_{\tilde\theta}(u)|u|^2 \min\left\{\frac{\alpha(x-(t-s)v,u)}{|u|}, \frac{\alpha(y-(t-s)v,u)}{|u|}\right\}^{2+\beta}} \mathrm{d}s \\
&\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)|v|^2 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{2+\beta}} \frac{\min\{\alpha(x,v), \alpha(y,v)\}}{|v|^2} \\
&= \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \qquad (7.213)
\end{aligned}
$$

In the second line we have used $(t-s) \leqq t_m$. In the last two lines we have used (4.7), (2.39) and Lemma 2.13.

Next we consider $h = \Gamma(f_s, f_s)$. We use (2.117) and (2.116) to compute

$$
\begin{aligned}
&\int_{t-t_m}^{t} e^{-\nu(t-s)} \frac{\nabla_v[\Gamma(f_s,f_s)(x-(t-s)v,v) - \Gamma(f_s,f_s)(x-(t-s)v,v)]}{|x-y|^\beta} \mathrm{d}s \\
&= \int_{t-t_m}^{t} e^{-\nu(t-s)} \frac{\Gamma(\nabla_v f_s, f_s)(x-(t-s)v,v) - \Gamma(\nabla_v f_s, f_s)(y-(t-s)v,v)}{|x-y|^\beta} \mathrm{d}s \qquad (7.214) \\
&\quad + \int_{t-t_m}^{t} e^{-\nu(t-s)} \frac{\Gamma(f_s, \nabla_v f_s)(x-(t-s)v,v) - \Gamma(f_s, \nabla_v f_s)(y-(t-s)v,v)}{|x-y|^\beta} \mathrm{d}s \qquad (7.215) \\
&\quad + \int_{t-t_m}^{t} e^{-\nu(t-s)}(t-s) \frac{\Gamma(\nabla_x f_s, f_s)(x-(t-s)v,v) - \Gamma(\nabla_x f_s, f_s)(y-(t-s)v,v)}{|x-y|^\beta} \mathrm{d}s \qquad (7.216) \\
&\quad + \int_{t-t_m}^{t} e^{-\nu(t-s)}(t-s) \frac{\Gamma(f_s, \nabla_x f_s)(x-(t-s)v,v) - \Gamma(f_s, \nabla_x f_s)(y-(t-s)v,v)}{|x-y|^\beta} \mathrm{d}s \qquad (7.217) \\
&\quad + \int_{t-t_m}^{t} e^{-\nu(t-s)} \frac{\Gamma_{v,\mathrm{gain}}(f_s,f_s)(x-(t-s)v,v) - \Gamma_{v,\mathrm{gain}}(f_s,f_s)(y-(t-s)v,v)}{|x-y|^\beta} \mathrm{d}s \qquad (7.218) \\
&\quad - \int_{t-t_m}^{t} e^{-\nu(t-s)} \frac{\Gamma_{v,\mathrm{loss}}(f_s,f_s)(x-(t-s)v,v) - \Gamma_{v,\mathrm{loss}}(f_s,f_s)(y-(t-s)v,v)}{|x-y|^\beta} \mathrm{d}s. \qquad (7.219)
\end{aligned}
$$

Applying (2.39) for $t-s \leqq t_m$, we use Proposition 3 to have

$$|(7.216)+(7.217)| \lesssim \frac{\min\{\alpha(x,v),\alpha(y,v)\}}{|v|^2} \times (7.12)$$
$$\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2}{w_{\tilde\theta}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.220}$$

For (7.214) and (7.215), we apply the same estimate as (7.15) and (7.16) with replacing ∇_x derivative by ∇_v derivative. By the same computation as (7.14), (7.17) change to

$$\int_{\mathbb{R}^3} du \mathbf{k}(v,u)|\nabla_v f_s(x,u)| \int_{\mathbb{S}^2} \frac{w_{\tilde\theta}^{-1}(v+\omega)\|w_{\tilde\theta}\alpha\nabla_x f_s\|}{\min\{\alpha(x,v+\omega),\alpha(y,v+\omega)\}^\beta}$$
$$\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta/2}(v)} \int_{\mathbb{R}^3} du \frac{w_{\tilde\theta/2}(v)}{w_{\tilde\theta/2}(u)|u|^2 \min\{\xi(x),\xi(y)\}^{\beta/2}}$$
$$\times \int_{\mathbb{S}^2} d\omega e^{-\tilde\theta|v|^2/2} \frac{w_{\tilde\theta}^{-1}(v+\omega)}{|v+\omega|^\beta}$$
$$\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta}(v)} \int_{\mathbb{R}^3} du \frac{\mathbf{k}_{\tilde\varrho}(v,u)}{|u|^2 \min\{\xi(x),\xi(y)\}^{\beta/2}}$$
$$\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty w_{\tilde\theta}^{-1}(v)}{|v|^2 \min\{\xi(x),\xi(y)\}^{\beta/2}}.$$

Applying the same computation as (7.20), the contribution of the above term is bounded by

$$\frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.221}$$

For the contribution of (7.15) with replacing ∇_x by ∇_v, we apply the same computation as (7.18) to bound such contribution by

$$\frac{\|wf_s\|_\infty [\nabla_v f_s(\cdot,v)]_{C^{0,\beta}_{x,1+\beta}}}{w_{\tilde\theta/2}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|},\frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.222}$$

For the contribution of (7.16) with replacing ∇_x by ∇_v, we apply the same computation as (7.19) to bound such contribution by

$$\frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta}^{3/2}(v)|v|^2} \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \frac{w_{\tilde\theta}(v)\mathbf{k}(v,u)}{w_{\tilde\theta}(u)\min\{\alpha(x',u),\alpha(y',u)\}^\beta}$$
$$+ \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta}^{3/2}(v)|v|^2 \min\{\alpha(x,v),\alpha(y,v)\}^\beta} \int_0^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} \frac{w_{\tilde\theta/2}(v)\mathbf{k}(v,u)|v|^2}{w_{\tilde\theta/2}(u)|u|^2}$$

$$\lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty \|w_{\tilde\theta/2}|v|^2\nabla_v f_s\|_\infty}{w_{\tilde\theta}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.223}$$

Combining (7.221), (7.222), (7.223) and using (7.6), (7.186), $\|wf_s\|_\infty \ll 1$ from the **Existence Theorem**, we conclude that

$$|(7.214)+(7.215)| \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2 + o(1)[\nabla_v f_s(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}}{w_{\tilde\theta/2}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.224}$$

For (7.218) and (7.219), we use (2.118) to have

$$\begin{aligned}\Gamma_{v,\text{gain}}(f_s,f_s)(x,v)-\Gamma_{v,\text{gain}}(f_s,f_s)(y,v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathrm{d}\omega\mathrm{d}u|u\cdot\omega|\frac{|v+u|}{2T_w}\sqrt{\mu(v+u)}\\ &\times\Big([f_s(x,v+u_\perp)-f_s(y,v+u_\perp)]f_s(x,v+u_\parallel)\\ &+[f_s(x,v+u_\parallel)-f_s(y,v+u_\parallel)f_s(y,v+u_\perp)]\Big).\end{aligned}$$

Clearly $\frac{|v+u|}{2T_w}\sqrt{\mu(v+u)} \lesssim \mu^{1/4}(v+u)$, using such bound for $\frac{|v+u|}{2T_w}\sqrt{\mu(v+u)}$ the above term has the same form as $\Gamma_{\text{gain}}(f_s(x)-f_s(y),\, f_s(x))+\Gamma_{\text{gain}}(f_s(y),\, f_s(x)-f_s(y))$ in (2.110). Thus we can use (7.9) to bound (7.218) as

$$(7.218) \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}. \tag{7.225}$$

Similarly for (7.219) we also have

$$(7.219) \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty}{w_{\tilde\theta}(v)\min\{\alpha(x,v),\alpha(y,v)\}^\beta}. \tag{7.226}$$

Combining (7.212), (7.213), (7.220), (7.224), (7.225) and (7.226) we use (7.186) and (7.6) to conclude

$$(7.180) \lesssim \frac{\|w_{\tilde\theta}\alpha\nabla_x f_s\|_\infty^2 + o(1)[\nabla_v f_s(\cdot,v)]_{C^{0,\beta}_{x;1+\beta}}}{w_{\tilde\theta/2}(v)|v|^3 \min\left\{\frac{\alpha(x,v)}{|v|}, \frac{\alpha(y,v)}{|v|}\right\}^{1+\beta}}. \tag{7.227}$$

Finally, we combine (7.187), (7.209) and (7.227), and conclude (1.23). □

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References

1. Cao, Y.: Regularity of Boltzmann equation with external fields in convex domains of diffuse reflection. *SIAM J. Math. Anal.* **51**(4), 3195–3275, 2019
2. Cao, Y.: Rarefied gas dynamics with external fields under specular reflection boundary condition, *preprint*
3. Cao, Y., Kim, C., Lee, D.: Global strong solutions of the Vlasov–Poisson–Boltzmann system in bounded domains. *Arch. Rational . Mech. Anal.* **233**(3), 1027–1130, 2019
4. Cao, Y., Kim, C.: Lipschitz continuous solutions of the Vlasov-Maxwell systems with a conductor boundary condition. https://doi.org/10.48550/arXiv.2203.01615
5. Cercignani, C., Illner, R., Pulvirenti, M.: *The Mathematical Theory of Dilute Gases*. Applied Mathematical Sciences, 106. Springer, New York, 1994
6. Chen, H., Kim, C., Li, Q.: Local Well-posedness of Vlasov–Possion–Boltzmann system with generalized diffuse boundary condition. *J. Stat. Phys.* **179**, 535–631, 2020
7. Chen, H.: Cercignani-Lampis boundary in the Boltzmann theory. *Kinet. Relat. Models* **13**(3), 549–597, 2020
8. Chen, I., Hsia, C., Kawagoe, D.: Regularity for diffuse reflection boundary problem to the stationary linearized Boltzmann equation in a convex domain. *Ann. Inst. H. Poincare Anal. Non Lineaire* **36**(3), 745–782, 2019
9. Esposito, R., Guo, Y., Kim, C., Marra, R.: Non-isothermal boundary in the Boltzmann theory and Fourier law. *Commun. Math. Phys* **323**(1), 177–239, 2013
10. Esposito, R., Guo, Y., Kim, C., Marra, R.: Stationary solutions to the Boltzmann equation in the Hydrodynamic limit. *Ann. PDE* **4**(1), 1, 2018
11. Esposito, R., Marra, R.: Stationary Non equilibrium States in Kinetic Theory. J. Stat. Phys, 2020
12. Glassey, R.: *The Cauchy Problem in Kinetic Theory.* Vol. 52. SIAM, 1996
13. Guo, Y., Kim, C., Tonon, D., Trescases, A.: BV-regularity of the Boltzmann equation in non-convex domains. *Arch. Ration. Mech. Anal.* **220**, 1045–1093, 2016
14. Guo, Y., Kim, C., Tonon, D., Trescases, A.: Regularity of the Boltzmann Equation in Convex Domains. *Invent. Math* **207**, 115–290, 2017
15. Guiraud, J.P.: Probleme aux limites interieur pour l'equation de Boltzmann en regime stationnaire, faiblement non lineaire. *J. de Méc.* **11**(2), 443–490, 1972
16. Jin, J., Kim, C.: Damping of kinetic transport equation with diffuse boundary condition. https://doi.org/10.48550/arXiv.2011.11582
17. Jang, J., Kim, C.: Incompressible Euler limit from Boltzmann equation with Diffuse Boundary Condition for Analytic data. *Ann. PDE* **7**(2), 1–103, 2021
18. Kim, C.: Formation and Propagation of Discontinuity for Boltzmann Equation in Non-Convex Domains. *Commun. Math. Phys.* **308**, 641–701, 2011
19. Kim, C., Lee, D.: The Boltzmann equation with specular boundary condition in convex domains. *Commun. Pure Appl. Math.* **71**(3), 411–504, 2018
20. Kim, C., Lee, D.: Holder Regularity of the Boltzmann equation Past an Obstacle. https://doi.org/10.48550/arXiv.2111.07558
21. Maxwell, J.C.: On the dynamical theory of gases. *Philos. Trans. R. Soc. Lond.* **157**, 49–88, 1866
22. Sone, Y.: Molecular Gas Dynamics. Theory, Techniques, and Applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Inc., Boston (2007)

HONGXU CHEN & CHANWOO KIM
Department of Mathematics,
The University of Wisconsin-Madison,
Madison
USA.
e-mail: hchen463@wisc.edu
e-mail: chanwoo.kim@wisc.edu

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