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On the Hypergraph Connectivity of Skeleta of Polytopes

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Abstract
We show that for every d-dimensional polytope, the hypergraph whose nodes are k-faces and whose hyperedges are $(k+1)$-faces of the polytope is strongly $(d-k)$-vertex connected, for each $0 \leq k \leq d-1$.

Keywords Polytopes · Connectivity · Skeleta · Hypergraphs

Mathematics Subject Classification 52B05 · 05C40

1 Introduction

Balinski proved that the edge graph of any d-dimensional polytope is d-vertex connected [2]. That is, removing fewer than d of the vertices leaves the remaining vertices connected via edges. A number of natural generalizations of this result have since been investigated. Sallee found bounds for several different notions of connectivity of incidence graphs between r-faces and s-faces of a polytope [4]. More recently, Athanasiadis considered the graphs $\mathcal{G}_k(P)$ for a convex polytope P, whose nodes are the k-faces of P, and with two nodes adjacent if the corresponding k-faces are both contained in the same $(k+1)$-face. Vertex connectivity of $\mathcal{G}_k(P)$ is equivalent to one of the connectivity notions on the incidence graphs considered by Sallee. Athanasiadis

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described exactly the minimum vertex connectivity of $\mathcal{G}_k(P)$ over all d-polytopes for every k and d [1].

Let P be a convex d-dimensional polytope. We denote by $\mathcal{H}_k(P)$ the hypergraph whose nodes are the k-faces of polytope P, and whose hyperedges correspond naturally to the $(k+1)$-faces of P. We say a hypergraph is *strongly* α*-vertex connected* if removing fewer than α nodes along with all hyperedges incident to each removed node leaves the remaining nodes connected. Using tropical geometry, Maclagan and the second author showed that for every *rational* d-polytope, $\mathcal{H}_k(P)$ is strongly $(d-k)$-vertex connected [3]. Our main result is generalizing this statement to all polytopes:

Theorem 1.1 *For every d-polytope P, the hypergraph $\mathcal{H}_k(P)$ is strongly $(d-k)$-vertex connected, for each* $0 \leq k \leq d-1$.

The result is tight. For simple polytopes, each k-face is contained in exactly $d-k$ of the $(k+1)$-faces, so the hypergraph $\mathcal{H}_k(P)$ cannot have higher connectivity.

2 Proof of the Result

We say that a pure k-dimensional polyhedral complex is *c-connected through codimension one* if after removing fewer than c *closed* maximal faces, the remaining maximal faces are connected via paths through faces of dimension $k-1$. That is, for any two remaining maximal faces F, F', there remains a sequence $F = G_1, \ldots, G_\ell = F'$ of maximal faces such that for each i, $G_i \cap G_{i+1}$ is a face of dimension $k-1$ not belonging to a removed face. The *m-skeleton* of a polytope Q is the polyhedral complex whose maximal faces are the m-dimensional faces of Q. Then Theorem 1.1 can be rephrased as the following equivalent form on the polar dual $Q = P^{\Delta}$.

Theorem 2.1 *For every d-polytope Q, the $(d-k-1)$-skeleton is $(d-k)$-connected through codimension one, for each* $0 \leq k \leq d-1$. *Equivalently, the k-skeleton of Q is $(k+1)$-connected through codimension one for each* $0 \leq k \leq d-1$.

We will need some lemmas before proceeding with the proof by induction on dimension.

Lemma 2.2 *Let F, G, R be three distinct k-faces of a d-polytope Q, for some* $1 \leq k \leq d-1$. *Then there is a hyperplane intersecting F and G and avoiding R. Moreover, the hyperplane can be chosen to avoid all vertices of Q.*

Proof Let $f \in F$ and $g \in G$ be relative interior points, and let L be the line through f and g. Let Q' be the smallest face of Q containing $F \cup G$. By convexity, $L \cap Q \subset Q'$ and L meets the boundary of Q' only at the two points f and g. In particular L does not meet R or any other face of dimension $\leq k$.

We may assume that Q is a d-dimensional polytope in $\mathbb{R}^d$. Let π be a corank one linear map from $\mathbb{R}^d$ to $\mathbb{R}^{d-1}$ such that the image of L is a point. Then the image $R' = \pi(R)$ does not contain $\pi(L)$, and each vertex $v_1, \ldots, v_n$ of Q has $v_i' = \pi(v_i) \neq \pi(L)$ since L does not contain any of the vertices.

Since R' is convex and does not contain $\pi(L)$, there is a hyperplane through $\pi(L)$ which does not meet R'. Since R' is compact, the set of normal vectors of such hyperplanes form a full dimensional open set in $\mathbb{RP}^{d-1}$. (More precisely, it is the interior of the dual cone, and its negative, of the pointed cone generated by R' after a translation that sends $\pi(L)$ to the origin.) On the other hand, the condition that such a hyperplane contains each v_i' is a codimension one closed condition. Thus, as there are finitely many v_i', the cone of such normal vectors restricted to those whose hyperplane *does not* contain any v_i' is non-empty. In particular, there is a hyperplane H' through $\pi(L)$ which does not meet R' or any of the v_i'. Its preimage $\pi^{-1}(H)$ is a desired hyperplane. ◻

Lemma 2.3 *Let Q be a polytope and H a hyperplane intersecting Q but not containing any vertices of Q. The map ϕ mapping a face F to $F \cap H$ is a poset isomorphism from the poset of faces of Q that meet H to the face poset of $Q \cap H$.*

Proof For any face F of Q which meets H, since H does not contain any vertices of F, F is not contained in H and H meets the relative interior of F, so $\dim(F \cap H) = \dim F - 1$. Moreover, $F \cap H$ is indeed a face of $Q \cap H$: any supporting hyperplane for F in Q is also a supporting hyperplane for $F \cap H$ in $Q \cap H$. On the other hand, for any face F' of $Q \cap H$, let $x \in F'$ be a relative interior point in F', and let F be the unique face of Q for which x is a relative interior point. Then x is also in the relative interior of $F \cap H$. Since F' and $F \cap H$ are two faces of $Q \cap H$ that meet in their relative interiors, we have $F \cap H = F'$. So ϕ is a surjective map between the desired sets. If $F \cap H = G \cap H$ for k-faces F, G meeting H, then F and G would have a common relative interior point, which implies $F = G$. Thus ϕ is injective. It is clear that ϕ preserves the inclusion relation. ◻

Proof of Theorem 2.1 We will use induction on k. The statement is trivial for $k = 0$, as we are not removing any faces, and the vertices of a polytope are connected through the empty face. The case when $k = 1$ is clear, as removing a single edge does not disconnect the vertex-edge graph of any polytope.

Suppose $2 \le k \le d - 1$. Let Q be a d-polytope and $\mathcal{B}$ be any set of k k-faces of Q to remove. We need to find a path between any two k-faces $F, G \notin \mathcal{B}$, through codimension-one faces, which we will call *ridge paths*. Arbitrarily choose any $R \in \mathcal{B}$. Lemma 2.2 gives a hyperplane H intersecting F and G, and avoiding R and vertices of Q. Let $Q' = Q \cap H$. Since H intersects F and G, $F' = F \cap H$ and $G' = G \cap H$ are two $(k-1)$-faces of Q' by Lemma 2.3. Moreover, each face in $\mathcal{B} \setminus \{R\}$ corresponds to at most one $(k-1)$-dimensional face in Q'. Call these faces $\mathcal{B}'$. As $|\mathcal{B}'| \le k - 1$, by induction there is a ridge path in Q' connecting F' to G' and avoiding each face in $\mathcal{B}'$. Using Lemma 2.3, we can lift this path back up to a ridge path connecting F to G in Q avoiding $\mathcal{B}$. ◻

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Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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