

# Pricing Ordered Items

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## ABSTRACT

We study the revenue guarantees and approximability of item pricing. Recent work shows that with  $n$  heterogeneous items, item-pricing guarantees an  $O(\log n)$  approximation to the optimal revenue achievable by any (buy-many) mechanism, even when buyers have arbitrarily combinatorial valuations. However, finding good item prices is challenging – it is known that even under unit-demand valuations, it is NP-hard to find item prices that approximate the revenue of the optimal item pricing better than  $O(\sqrt{n})$ .

Our work provides a more fine-grained analysis of the revenue guarantees and computational complexity in terms of the number of item “categories” which may be significantly fewer than  $n$ . We assume the items are partitioned in  $k$  categories so that items within a category are totally-ordered and a buyer’s value for a bundle depends only on the best item contained from every category.

We show that item-pricing guarantees an  $O(\log k)$  approximation to the optimal (buy-many) revenue and provide a PTAS for computing the optimal item-pricing when  $k$  is constant. We also provide a matching lower bound showing that the problem is (strongly) NP-hard even when  $k = 1$ . Our results naturally extend to the case where items are only partially ordered, in which case the revenue guarantees and computational complexity depend on the width of the partial ordering, i.e. the largest set for which no two items are comparable.

## CCS CONCEPTS

• **Theory of computation** → **Algorithmic mechanism design; Computational pricing and auctions.**

## KEYWORDS

buy-many mechanisms, item pricing, revenue maximization, ordered item values

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## 1 INTRODUCTION

A dominant theme within algorithmic mechanism design is simplicity versus optimality – establishing that simple mechanisms can approximate optimal ones within many settings. The simple mechanism in most of these results is an *item pricing*, where the seller determines a fixed price for each item and buyers can purchase any set of items at the sum of the corresponding prices. Item pricings are also an important class of mechanisms from a practical viewpoint – most real world mechanisms are indeed item pricing mechanisms. However, despite their simplicity and popularity, finding good item prices for multi-item settings is a notoriously challenging problem and it is known to be inapproximable within a factor better than  $\sqrt{n}$  even for unit-demand buyers [10].

In this paper, we focus on structured mechanism design instances and perform a fine grained analysis of the approximability of item pricing as well as its approximate optimality. We consider a standard multi-parameter mechanism design setting where a revenue maximizing seller offers multiple items for sale to a buyer whose value for the items is drawn from a known distribution. We define a new parameterization over value distributions wherein items can be partitioned into a few categories and items within each category can be ordered by desirability. We show that the number of categories governs both the approximability and approximate optimality of item pricings.

*Ordered items and the approximability of item pricing.* At the heart of our parameterization is the so-called FedEx Problem that was first studied by Fiat et al [23]. In the FedEx Problem, the items offered by the seller correspond to shipping times for a package; each buyer has a deadline for shipping their package and obtains a fixed value if the shipping time meets their deadline. The FedEx Problem occupies a sweet-spot between single-parameter mechanism design settings where a buyer’s preferences can be fully described through a scalar value; and multi-parameter settings where different (sets of) items bring the buyer different values. Accordingly it exhibits some but not all of the complexity of multi-parameter settings. Indeed, as we show, in contrast to the general case, the optimal item pricing for FedEx instances can be computed in polynomial time.

The FedEx Problem is a special case of “totally ordered” settings where items can be ranked by quality and every buyer type weakly

prefers a higher ranked item to a lower ranked one.<sup>1</sup> More generally, we consider settings where items can be partitioned into  $k$  categories such that within each category items are totally ordered by quality:

Consider, for example, a car dealership that sells  $k$  different models of cars. Each model comes in a variety of different trims – the most basic trim along with a sequence of upgrades. For any particular model or category, every buyer has the same ordering of values over different trims although values differ arbitrarily across buyers and across categories. One buyer may value the luxury trim \$5,000 higher than the standard trim and another may value them the same, but no buyer values the standard trim more than the luxury trim.

For another example, consider an internet service provider such as Comcast, AT&T or Spectrum that offers multiple products such as TV, internet, and phone service. Each individual product has quality or service levels that are ordered. In particular, every buyer weakly prefers higher internet speeds to lower speeds and unlimited talk time to limited talk time. However, buyers may assign different values to different combinations of the three services.

Yet another example is of a streaming media company such as Netflix or Amazon pricing TV shows. The seasons of each show form an ordered list – viewers typically would not watch season 2 without watching season 1, for example.  $k$  shows, accordingly, correspond to a  $k$  category setting. Each category contains items corresponding to the first several seasons of the same show.

We emphasize that both the totally ordered setting and the  $k$ -category setting are multi-parameter settings where the buyer's values are combinatorial and described as functions over the set of items allocated to the buyer. Beyond the ordering over items within each category, we make no assumptions on the buyer's values over sets of items.

Our main computational finding is that the approximability of item pricing is governed by the parameter  $k$ . For the totally ordered ( $k = 1$ ) and  $k$ -category settings, we provide a polynomial time approximation scheme with a running time that depends exponentially on  $k$ . For any given  $\epsilon > 0$ , our algorithm returns an item pricing that approximates the revenue of the optimal item pricing within a factor of  $(1 + \epsilon)$  and runs in time  $\text{poly}(m, n^{\text{poly}(k/\epsilon)}, b)$  where  $n$  is the number of items,  $m$  is the support of the distribution and  $b$  is the bit complexity of buyer's value distribution. Our approximation scheme is almost the best possible<sup>2</sup>, as we show that finding the optimal item pricing is strongly NP-hard even for  $k = 1$ . Our algorithm is particularly relevant and useful when  $k$  is a small constant such as in the examples described above.

<sup>1</sup>In the FedEx setting, for example, every buyer weakly prefers earlier shipping times to later ones.

<sup>2</sup>There dependency on  $k$  and  $\epsilon$  may be improved.

**THEOREM 1.1.** *For any distribution of support size  $m$  over  $k$ -category valuations  $v : 2^{[n]} \rightarrow [1, 2^b]$ , we can compute an  $(1 + \epsilon)$ -approximate item pricing in  $\text{poly}(m, n^{\text{poly}(k, 1/\epsilon)}, b)$  time.*

*The approximate optimality of item pricing.* As aforementioned, a central problem in multi-item mechanism design is approximating the revenue of the optimal mechanism in multi-parameter settings by simple mechanisms like item pricing. In fact, in the kinds of settings we study in this paper (with no assumptions on the value distributions), it is known that no simple mechanisms can provide any finite approximation to the optimal revenue in the worst case. This general case impossibility of simple-versus-optimal results has led to two complementary lines of work in recent years.

The first looks at structured settings for which approximately-optimal mechanisms can be characterized. The FedEx problem [23, 35] and its extensions to so-called “interdimensional” settings [19, 20] belong to this line of work; in these settings, the optimal mechanism can have an exponential or even unbounded description complexity but under appropriate assumptions, mechanisms with polynomial menu size provide an approximation. Another series of works [1, 11, 34] bounds the revenue gap between item pricings and optimal mechanisms assuming that the buyer's values are subadditive and independent across different items.

The second line of work places an extra incentive constraint on the revenue maximization problem. Instead of viewing a mechanism as a one time interaction between the seller and a buyer, it is assumed that the buyer can visit the mechanism multiple times purchasing different bundles of items. In this “buy-many” setting, complicated mechanisms that extracted arbitrarily higher revenue than simpler ones are no longer incentive compatible as the buyer can buy multiple cheaper options instead of a single expensive one. In fact, recent work [15] shows that item-pricing achieves a  $\Theta(\log n)$  approximation to the optimal buy-many mechanism and this is tight in a strong sense as no simple mechanism, i.e. one with polynomial description complexity, can approximate the optimal revenue better than a logarithmic factor.

Our work unifies the two approaches and considers the revenue approximation of item pricing in more structured buy-many settings. Our first finding is that item pricing is the optimal buy-many mechanism in the FedEx setting. More generally, we find that the revenue guarantees of item pricing are again governed by the parameter  $k$  of our parameterization. In the totally ordered setting where  $k = 1$ , we show that item pricing is no longer optimal but achieves a constant factor approximation to the optimal buy-many revenue. For  $k$  categories, we show that the approximation is  $\Theta(\log k)$ . This gives a smooth degradation of the revenue guarantee as the instances become less and less structured.

**THEOREM 1.2.** *For any distribution over  $k$ -category valuation functions, the optimal item pricing guarantees a  $1/\Theta(\log k)$  fraction of the revenue achievable by the optimal buy-many mechanism.*

*Implications for Buy-Many Mechanism Design.* Even though our focus in this work is on item pricing and its revenue guarantees, our result gives the first computationally efficient algorithm for computing approximately optimal buy-many mechanisms in structured settings. In contrast to the setting of buy-one mechanisms

where the optimal mechanism can typically be computed via a linear program of polynomial size in the support of the distribution of values, no such algorithm is known for buy-many settings. In fact, we observe that the  $\sqrt{n}$  inapproximability of item-pricing even for unit-demand settings, directly implies a  $\sqrt{n}/\log(n)$  inapproximability for buy-many mechanisms as one can efficiently convert any buy-many mechanism into an item pricing one with a logarithmic loss in approximation<sup>3</sup>. Our results show that for structured settings, the optimal buy-many mechanism is efficiently approximable and that such an approximation can be achieved via item pricing.

We remark that being able to obtain approximate item pricing or buy-many mechanisms is important even in cases where the optimal buy-one mechanism might be easier to compute. This is because buy-one mechanisms may be inherently complex and difficult for the buyers to understand and participate in. More significantly, in many settings, it may be unrealistic to expect that the revenue promised by a buy-one mechanism is achievable in practice. For cases like shopping from a retail store, it may not be feasible to implement a buy-one mechanism as buyers faced with superadditive prices would break their desired bundle into smaller ones visiting the store multiple times. This would result in significantly lower revenue than expected by the buy-one model.

*Extensions.* We further consider settings where there is a partial ordering over items. Consider, for example, an electronics company that manufactures both cameras and cell phones. Some cell phones capture all of the features of certain cameras, and therefore all buyers weakly prefer the former to the latter. But not all cameras and cell phones are comparable. We say that an item  $i$  dominates another item  $j$  if for every set  $S$  of items containing both  $i$  and  $j$ , every buyer is indifferent between getting  $S$  or  $S \setminus \{j\}$ .

We use the parameter  $k$  to denote the “width” of the partial ordering over items—the size of the largest set of incomparable items or the longest anti-chain in the partial ordering. Note, that the  $k$ -category setting is a special case of this more general width- $k$  setting. Our PTAS for item pricing of Theorem 1.1 as well as the buy-many revenue approximation result of Theorem 1.2 naturally extend to this more general setting with the same guarantees.

A more relaxed condition for partial ordering across items specifies that item  $i$  dominates another item  $j$  if all sets of items  $S$  that do not contain items  $i$  or  $j$ , adding  $i$  to  $S$  is always preferable to adding  $j$ . Unfortunately, we show that under such a weak condition, pricing cannot guarantee a constant fraction of the optimal buy-many revenue even in simple settings. In fact, even with additive buyers over totally ordered items, we show that no buy-many mechanism with polynomial description complexity can achieve better than  $1/o(\log \log n)$  fraction of the optimal buy-many revenue (see Section 4.4). It is an interesting open question left by our work to show that this bound is indeed achievable by item pricing.

*Our techniques.* Our techniques are easiest to understand in the context of a unit-demand buyer with totally ordered items. Our analysis of the gap between item pricings and optimal buy-many mechanisms in this setting hinges on a characterization of the

buyer’s optimal buy-many strategy. Faced with a menu of randomized options, the buyer essentially behaves like a Pandora’s box algorithm which at every step opens a box (i.e. purchases a lottery) and obtains a random reward. Because the same lotteries can be purchased any number of times, the buyer’s optimal strategy is to pick a single lottery repeatedly until an item of a certain minimum value is instantiated. This characterization allows us to relate the buyer’s utility to the value of the item(s) bought by the buyer. We can then apply a lemma from [15] that relates the revenue obtained by an item pricing to the change in the buyer’s utility at different scalings of that item pricing.

In order to approximate the optimal item pricing for a unit demand buyer with totally ordered items, we view the buyer as additive over item *upgrades*: the purchase of an item  $i$  can be viewed equivalently as the purchase of the base item 1 along with a series of upgrades, 1 to 2, 2 to 3, and so on till  $i$ . The benefit in doing so is that with some slight loss in approximation, we can group upgrades into different pricing scales, and price each scale independently. This permits a dynamic programming based algorithm for optimizing the prices of the upgrades. The pricing found in this manner can be easily converted into an item pricing with the same revenue.

## 1.1 Other Related Work

The computational complexity of item pricing for a single buyer has been studied previously for a variety of valuation functions. One widely studied setting is the  $k$ -hypergraph pricing problem, where each possible realization of the buyer is unit-demand over a set of at most  $k$  items. It has been shown that there exists an algorithm with competitive ratio  $O(\min(k, \sqrt{n \log n}))$  [10] (also see [2, 5, 27]), and is hard to approximate within  $\Omega(\min(k^{1-\epsilon}, n^{1/2-\epsilon}))$  under the Exponential Time Hypothesis [10] (also see [3, 8, 9]). Such results also extend to a single-minded buyer that wants an entire set of at most  $k$  items. The specific case where  $k = 2$  is called the graph vertex pricing, for which there is an efficient algorithm with competitive ratio 4 [2]. No efficient algorithm can give an approximation ratio better than 4 assuming the Unique Games Conjecture [31] (also see [27, 29]). Another special case is the tollbooth problem, where the buyer demands a path on a path graph. This problem is strongly NP-hard [22], and a PTAS is known [26] (also see [2, 24]).

Another line of work studies the problem of selling to a unit-demand buyer with item values drawn from independent distributions. For general distributions, computing the optimal item pricing is NP-hard [18]. The optimal item pricing revenue can be approximated to within a factor of 2 (providing a 4-approximation to the optimal revenue overall) [11, 12], and a PTAS (or QPTAS) exists if the item values are drawn from monotone hazard rate (or regular) distributions [6]. The problem of finding the revenue from the optimal mechanism for a unit-demand buyer with independent item values has been further studied: it is known that no efficient exact algorithm exists unless the polynomial-time hierarchy collapses [17], and a QPTAS exists [30].

The recent decade has seen much work on approximating the optimal revenue via simple mechanisms such as item pricing and grand bundle pricing: for a single unit-demand buyer [11, 13]; an additive buyer [1, 28, 32]; a subadditive buyer [14, 34]; as well as for multi-buyer settings [7, 12, 14, 21, 36]. All of these results require

<sup>3</sup>The efficient algorithm comes from the constructive proof of the item pricing mechanism that  $O(\log n)$ -approximates the optimal buy-many mechanism in [15]

independence across individual item values. For correlated item values, simple mechanisms cannot provide finite approximations to the optimal revenue and bounded gaps are only known in comparison to the optimal *buy-many* revenue. [4] shows that item pricing gives  $O(\log n)$ -approximation to the optimal buy-many revenue for a unit-demand buyer. [15] further generalizes the result to a general-valued buyer, and [16] characterizes the tight menu-size complexity of the mechanism needed for  $(1 + \epsilon)$ -approximation in revenue.

## 1.2 Organization

We present our results by iteratively building from the simplest case of the FedEx-Problem in Section 3, to the case of totally-ordered items in Section 4 and finally the general case with partially ordered items in Section 5.

## 2 DEFINITIONS

We study the multidimensional mechanism design problem where the seller has  $n$  heterogeneous items to sell to a single buyer, and aims to maximize the revenue. The buyer's value type is specified by a valuation function  $v : 2^{[n]} \rightarrow \mathbb{R}_{\geq 0}$  that assigns a non-negative value to every set of items. The valuation functions are monotone: for any  $S, T \subseteq [n]$  with  $S \subseteq T$  and any valuation function  $v$ , we have  $v(S) \leq v(T)$ . We study the *Bayesian setting*, where the buyer's valuation function  $v$  is drawn from a publicly known distribution  $\mathcal{D}$  over the set of all monotone valuation functions.

*Unit-demand Buyers.* We say that a buyer is *unit-demand* over all items, if the buyer is only interested in purchasing one item, and her value for any set of items is solely determined by the item that is most valuable to her. In other words, for any set  $S \subseteq [n]$ ,  $v(S) = \max_{i \in S} v(\{i\})$ . When there is no ambiguity, we use  $v_i$  to denote  $v(\{i\})$  for a unit-demand buyer of type  $v$ .

*Totally-ordered Items.* We say that a unit-demand buyer has *totally-ordered* values, if for every possible value realization  $v$  of the buyer,  $v_1 \leq v_2 \leq \dots \leq v_n$ .

*Partially-ordered Items.* Let  $\leq$  denote a partial ordering over the  $n$  items. We say that the buyer has *partially-ordered* values with respect to the relation  $\leq$  if for every realizable valuation function  $v$ , every pair of items  $i$  and  $j$  with  $i \leq j$ , and every set  $S \subseteq [n]$ , we have  $v(S \cup \{i, j\}) = v(S \cup \{j\})$ . We say that the item  $j$  dominates  $i$ . In other words, the buyer may discard from his allocated set any item that is dominated by another item in his allocation with no loss in value. As a consequence, the only “interesting” allocations over partially-ordered items are sets that form *antichains*, i.e. where no two items are comparable. An important parameter of a partially ordered set is its *width* that is defined to be the size of the largest antichain. We use  $k$  to denote the width of the partial ordering  $\leq$ . An important special case of partially-ordered items is the  $k$ -category setting where items are partitioned in  $k$ -categories. In this setting, items within a category are totally ordered and every buyer's value for a bundle depends only on the best item of each category it contains.

*Input Model for the Computational Problem.* When we study computational problems, we assume that the input distribution  $\mathcal{D}$  is

provided explicitly over a support of size  $m$ . Each buyer type  $v$  in the support is a vector of size  $O(n^k)$  that specifies the buyer's value  $v(T)$  for all possible sets  $T$  of size at most  $k$ ,<sup>4</sup> and is accompanied with a probability of realization  $\Pr[v]$ . We further assume that the value  $v(T)$  of each set of items is either 0, or in range  $[1, \mathcal{R}]$ . Without loss of generality we assume that each buyer type  $v$  in the support  $\mathcal{D}$  is non-trivial:  $v([n]) \geq 1$ .

*Single-buyer Mechanisms.* By the Taxation Principle [33], any single-buyer mechanism can be described as a menu of possible outcomes, and the buyer can select one menu option. Each outcome  $\lambda = (x, p) \in \Delta(2^{[n]}) \times \mathbb{R}_{\geq 0}$  is a lottery that is specified by a randomized allocation  $x$  over the sets of items, and a price  $p$  that is the payment of the buyer if she wants to get such an allocation. For any set  $S \subseteq [n]$ ,  $x_S$  denotes the probability that only items in set  $S$  are allocated to the buyer, and we have  $\|x\|_1 = 1$ .<sup>5</sup> We will use  $x(\lambda)$  and  $p(\lambda)$  to denote the allocation and the payment of any lottery  $\lambda$ . For any buyer of valuation function  $v$ , her *value* for lottery  $\lambda$  is defined by  $v(\lambda) \equiv \mathbb{E}_{S \sim x} v(S)$ ; her *utility* for purchasing  $\lambda$  is defined by  $u_v(\lambda) \equiv v(\lambda) - p(\lambda)$ . We will also use  $S \sim \lambda$  to denote a set of items drawn from set distribution  $x(\lambda)$ .

Given a mechanism  $\mathcal{M}$  with a menu of lotteries  $\Lambda$ , the buyer selects the menu option  $\lambda$  that maximizes her utility  $u_v(\lambda)$ . When there are multiple lotteries with the same highest utility for the buyer, the seller can choose the most expensive lottery to sell to the buyer. Without loss of generality, we assume that for any allocation  $x \in \Delta(2^{[n]})$  over the sets of items, there is a corresponding price  $p(x)$  such that  $(x, p(x)) \in \Lambda$ . We also use the pricing function  $p$  as an alternative definition of the mechanism  $\mathcal{M}$ . The buyer's utility is defined as  $u_p(v) = v(x) - p(x)$ . The buyer's payment is  $\text{REV}_p(v) = p(x)$ , and we write the revenue of mechanism  $p$  as  $\text{REV}_p = \mathbb{E}_{v \sim \mathcal{D}} \text{REV}_p(v)$ . Since the mechanism only allows the buyer to interact with the mechanism for once, it is also called *buy-one mechanism*.

*Buy-many Mechanisms.* In an (adaptively) buy-many mechanism, the buyer is allowed to interact with the mechanism for multiple times. To be more precise, a buy-many mechanism  $\mathcal{M}$  generated by a set  $\Lambda$  of lotteries can be defined as follows. The buyer can adaptively purchase a (random) sequence of lotteries in  $\Lambda$ , which means that in each step, the buyer can decide which lottery to purchase given the instantiation of the previous lotteries in the sequence. The buyer gets the union of all items allocated in each step and pays the sum of the prices of all purchased lotteries. For any adaptive algorithm  $\mathcal{A}$ , define  $\Lambda_{\mathcal{A}} = (\lambda_{\mathcal{A},1}, \lambda_{\mathcal{A},2}, \dots)$  to be the random sequence of lotteries purchased by the buyer of type  $v$ . The expected value of the buyer is

$$v(\Lambda_{\mathcal{A}}) \equiv \mathbb{E}_{(S_1, S_2, \dots) \sim (\lambda_{\mathcal{A},1}, \lambda_{\mathcal{A},2}, \dots)} v\left(\bigcup_{i \geq 1} S_i\right),$$

<sup>4</sup>Note that it suffices to specify the buyer's value over sets of size at most  $k$ , where  $k$  is the width of the partial ordering over items, because the buyer only desires sets that form antichains.

<sup>5</sup>Throughout the paper we assume  $\|x\|_1 = 1$  since in an adaptively buy-many mechanism, a buyer will only purchase a lottery with total allocation 1, otherwise she can repeatedly purchase the lottery and get a larger utility.

and the payment of the buyer is

$$p(\Lambda_{\mathcal{A}}) \equiv \mathbb{E}_{\mathcal{A}} \sum_{i \geq 1} p(\lambda_{\mathcal{A},i}).$$

Any buy-many mechanism can be described by a *buy-one* menu, where the buyer is only allowed to purchase a single lottery. This is because the expected outcome of any adaptive algorithm  $\mathcal{A}$  can be described by the allocation  $\cup_i (S_i \sim \lambda_{\mathcal{A},i})$ , and an expected payment  $p(\Lambda_{\mathcal{A}})$ . We say that a buy-one menu  $\Lambda$  satisfies the *buy-many constraint*, if for every adaptive algorithm  $\mathcal{A}$ , there exists a cheaper single lottery  $\lambda \in \Lambda$  dominating it. Rigorously speaking, there exists  $\lambda \in \Lambda$  with  $p(\lambda) \leq p(\Lambda_{\mathcal{A}})$  such that there exists a coupling between a random draw  $S$  from  $\lambda$ , and the union of random draws  $S'$  from  $\Lambda_{\mathcal{A}}$ , satisfying  $S \supseteq S'$ . Intuitively, a buy-one menu satisfies the buy-many constraint, if the buyer always prefers to purchase a single option from the menu, even if she has the option to adaptively interact with the mechanism for multiple times. In later sections, when we refer to a “buy-many mechanism” with menu  $\Lambda$ , we are always referring to a buy-one mechanism with menu  $\Lambda$  that satisfies the buy-many constraint.

### 3 WARM-UP: ITEM PRICING IN THE FEDEX SETTING

In this section, we study the item pricing in the FedEx setting [23]. The buyer’s value distribution in the FedEx problem has the following structure. Any buyer type  $v$  is defined by the pair of parameters  $(i_v, v_H)$  with  $v_i = 0$  for  $i < i_v$  and  $v_i = v_H$  otherwise. In other words, the buyer is totally-ordered and has at most two distinct values for all items, with the lower value being 0.

#### 3.1 The Optimality of Item Pricing

Our first observation is that item pricing achieves the optimal revenue obtained by any buy-many mechanism.

**THEOREM 3.1.** *For any value distribution in the FedEx setting, there exists an item pricing that achieves the optimal buy-many revenue.*

**PROOF.** Consider a buyer with value function  $v$  in the FedEx setting. Recall that the buyer only values items with index  $\geq i_v$  and values all of them equally. Therefore the buyer obtains the same value from an allocation  $x = (x_1, x_2, \dots, x_n)$  as from an allocation  $x'$  where  $x'_{i_v} = \sum_{i \geq i_v} x_i$  and  $x'_i = 0$  for  $i \neq i_v$ .

Given any buy-many menu  $\{(x, p)\}$ , consider replacing every lottery  $(x, p)$  with  $n$  different options:

$$(x^{(1)}, p^{(1)}) = ((\sum_{j \geq 1} x_j, 0, \dots, 0), p),$$

$$(x^{(2)}, p^{(2)}) = ((0, \sum_{j \geq 2} x_j, 0, \dots, 0), p), \dots,$$

$(x^{(n)}, p^{(n)}) = ((0, \dots, 0, x_n), p)$ . By our observation above, for every buyer type  $v$ , one of the  $n$  new options bring the same utility to the buyer as  $(x, p)$  and all other options bring lower utility. As a result, the new mechanism is identical in its allocations and revenue to the original one.

Observe that the new mechanism sells each item separately (but with different probabilities of allocation). We have the following observation:

**OBSERVATION 3.2.** *In a mechanism that sells each item separately, an adaptively buy-many buyer always purchases an item with allocation 1.*

The observation is true since if the buyer of type  $v$  purchases a lottery  $\lambda$  that sells item  $i_v$  with probability  $x_{i_v}$ , the buyer can repeatedly purchase the same lottery until she gets the item, which increases her utility. We may therefore drop any options that allocate items with probability less than 1 from the menu without changing the allocations or revenue of the mechanism. This final mechanism is an item pricing, and Theorem 3.1 follows.  $\square$

#### 3.2 A Poly-Time Algorithm for Finding Optimal Item Pricings

In this section, we show that the optimal item pricing in the FedEx setting can be computed efficiently. We actually prove a stronger result: for each realized buyer type  $v$ , if the buyer has at most two distinct item values, the optimal item pricing can be computed in polynomial time via dynamic programming. For each buyer type  $v$ , let  $v_L$  and  $v_H$  denote the two different item values in  $v$ , and let  $i_v$  be the smallest item type with item value  $v_H$ . In other words,  $v_1 = v_2 = \dots = v_{i_v-1} = v_L$ , and  $v_{i_v} = v_{i_v+1} = \dots = v_n = v_H$ . If  $v_1 = v_n$ , we define  $i_v = 1$  and  $v_L = v_H = v_1$ . The FedEx Problem is a special case with  $v_L = 0$ .

**THEOREM 3.3.** *In the totally-ordered setting, if each realized buyer type has at most two distinct item values, then the optimal item pricing can be computed in polynomial time.*

**PROOF.** Let  $\Pr[v]$  be the realization probability of  $v$  under input value distribution  $\mathcal{D}$ . Without loss of generality, we only study item pricings with monotone item prices  $p_1 \leq p_2 \leq \dots \leq p_n$ . For a buyer type with  $v_H > v_L$ , the buyer would either purchase item 1, or item  $i_v$ , or nothing. For a buyer type with  $v_H = v_L$ , the buyer would either purchase item 1, or nothing.

To compute the optimal item pricing, we first find a set of feasible prices for each item, then use a dynamic program to find the optimal item pricing. Define  $\Pi_L = \{v_1 | v \sim \mathcal{D}, i \in [n]\} \cup \{0\}$  be the set of all possible values for item 1, including 0. Let  $\Pi^* = \{z | z = v_H - v_L + y, y \in \Pi_L, v \sim \mathcal{D}\} \cup \Pi_L$ . We first observe that we may restrict prices to lie in a set of polynomial size without loss in revenue. The proof of this lemma is deferred to Section A.

**LEMMA 3.4.** *There exists an optimal item pricing such that  $p_1 \in \Pi_L$ , and  $p_i \in \Pi^*$  for each  $i \geq 2$ .*

Now we are ready to find the optimal item pricing. Let  $F[y, i, z]$  denote the total revenue from buyer types  $v$  with  $i_v \leq i$ , under a monotone item pricing that has already priced the first  $i$  items, with  $p_1 = y$  and  $p_i = z$ . Then we have the following recursive formula:

$$F[y, i, z] = \max_{z' \leq z, z' \in \Pi^*} \left\{ F[y, i-1, z'] + \sum_{v: i_v=i} \Pr[v] G[v, y, z] \right\},$$

where  $G[v, y, z]$  is the payment of buyer type  $v$  with item price  $y$  for item 1, and price  $z$  for item  $i_v$ . In other words,  $G[v, y, z] = z$  if  $v_H - z \geq v_1 - y$  and  $v_H \geq z$ ;  $G[v, y, z] = y$  if  $v_1 - y > v_H - z$  and  $v_1 \geq y$ ;  $G[v, y, z] = 0$  otherwise. The recursive formula is based on the following fact: if  $p_1$  is fixed, the revenue contribution of buyer types with  $i_v = i$  only depends on  $p_i$ . The optimal item pricing revenue we want to compute is  $\max_{y \in \Pi_L, z \in \Pi^*, z \geq y} F[y, n, z]$ . Since the table has a polynomial number of entries, and the inductive steps can be computed in polynomial time, the total running time

is also polynomial in the number of items and the support size of the distribution.  $\square$

## 4 ITEM PRICING IN THE TOTALLY-ORDERED SETTING

In the previous section we observed that for the FedEx setting, item pricing is not only optimal but also polynomial-time computable. When considering the general totally-ordered setting, both properties no longer hold. We show that for a general-valued buyer, item pricings may achieve strictly less revenue than the optimal buy-many mechanism. We complement this result by showing that item pricing gives a constant approximation in revenue to the optimal buy-many mechanism. Next, we show that computing the optimal item pricing in the totally-ordered setting is strongly NP-hard, thus there is no FPTAS algorithm finding the revenue obtained by the optimal item pricing. We complement the hardness result by providing a PTAS computing an approximately optimal item pricing, thus giving a tight characterization of the computational complexity of the problem.

### 4.1 Item Pricing Is a Constant Approximation to the Optimal Buy-Many Revenue

We first provide an example which shows that in the totally-ordered setting, the optimal item pricing and the optimal buy-many mechanism may have a constant factor revenue gap. Consider the following example: Let there be 2 items and 3 unit-demand buyers, with the following values for items 1 and 2 respectively, each realized with probability  $\frac{1}{3}$ :

$$v^{(1)} = (0, 5), v^{(2)} = (1, 3), v^{(3)} = (1, 2).$$

The optimal buy-one mechanism has the following menu:

$$\lambda_1 = ((0, 1), 5), \lambda_2 = \left( \left( \frac{2}{3}, \frac{1}{3} \right), \frac{5}{3} \right), \lambda_3 = ((1, 0), 1).$$

The lotteries are written in the form of  $((x_1, x_2), p)$  where  $x_1$  and  $x_2$  are the probabilities of the buyer getting items 1 or 2 respectively, and  $p$  is the price for this lottery. In the mechanism, each buyer  $v^{(i)}$  prefers to purchase lottery  $\lambda_i$ . Observe that the menu of lotteries satisfies the buy-many constraint. This is because to achieve the allocation of any  $\lambda_i$  using the other two lotteries in the menu, one always needs to pay more than  $p_i$ . Thus the mechanism is also the optimal buy-many mechanism, with revenue  $\frac{23}{9}$ . Observe that the optimal item pricing for this instance is  $p_1 = 1, p_2 = 3$ , which yields a revenue of  $\frac{7}{3} < \frac{23}{9}$ . Thus there can be a constant gap between the optimal item pricing revenue and the optimal revenue obtained by any (buy-many) mechanism.

Then we show that item pricings can actually achieve a constant fraction of the revenue obtained by the optimal buy-many mechanism.

**THEOREM 4.1.** *For any unit-demand buyer with totally-ordered value for all items, item pricing gives a 5.4 approximation in revenue to the optimal buy-many mechanism.*

**PROOF.** Let  $p$  be the optimal buy-many mechanism. Define  $q$  to be the following item pricing:  $q_i$ , the price of item  $i$ , is the cheapest price at which an adaptive buyer can obtain an item with index

at least  $i$  with probability 1 from repeatedly purchasing a single lottery. In other words,

$$q_i = \min_{x \in \Delta([n]): \sum_{j \geq i} x_j = 1} p(x).$$

We will show that a scaling of  $q$  gives a constant fraction of the optimal revenue obtained by buy-many mechanisms. We use the following lemma from [15] that relates the revenue of an appropriate scaling of  $q$  to the change in the buyer's utility as the pricing function changes from a low scaling factor,  $\ell$ , to a high one,  $h$ .

**LEMMA 4.2.** *(Lemma 3.1 of [15]) For any pricing  $q$  and any  $0 < \ell \leq h$ , let  $\alpha$  be drawn from  $[\ell, h]$  with density function  $\frac{1}{\alpha \log(h/\ell)}$ . Then for any valuation function  $v$ ,*

$$\mathbb{E}_\alpha [REV_{\alpha q}(v)] \geq \frac{u_{\ell q}(v) - u_{h q}(v)}{\ln(h/\ell)}.$$

In order to utilize this lemma, choosing  $h = 1$ , we show that the buyer obtains a low utility under pricing  $q$  and high enough utility at an appropriate scaling  $\ell q$ . We begin with two observations. The first shows that in any buy-many mechanism, the value of any set obtained by the buyer with non-zero probability is at least the expected utility of the buyer.

**LEMMA 4.3.** *For any buyer type  $v$  and any buy-many mechanism  $p$ , the buyer purchases  $\lambda = (x, p(x))$  in  $p$ , then for any set  $T$  in the support of  $x$ ,  $v(T) \geq u_p(v)$ .*

**PROOF.** Since  $\lambda$  is the optimal lottery purchased by the buyer, purchasing it is also the optimal adaptive strategy of the buyer. Thus, if the buyer buys  $\lambda$  and gets any set  $T$  in the support of  $x$  allocated, she would not purchase another lottery on the menu, in particular,  $\lambda$ . Since the value gain of purchasing  $\lambda$  with set  $T$  at hand is at most  $v(\lambda) - v(T)$ , therefore  $v(\lambda) - v(T) \leq p(\lambda)$  which is the price of purchasing  $\lambda$ . Thus  $v(T) \geq v(\lambda) - p(\lambda) = u_p(v)$ .  $\square$

We emphasize that the lemma holds for arbitrary buyer types and not just unit-demand valuations. The second observation shows that for any lottery, its price in  $p$  is lower bounded by the price of some item in its support in item pricing  $q$ .

**LEMMA 4.4.** *For any allocation  $x \in \Delta([n])$ , there exists an item  $i$  in the support of  $x$ , such that  $q_i \leq p(x)$ .*

**PROOF.** Let  $i$  be the item with the lowest type in the support of  $x$ . Then we have  $\sum_{j \geq i} x_j = 1$ , thus  $q_i \leq p(x)$  by definition of  $q$ .  $\square$

Now we come back to the proof of the theorem. Fix any buyer type  $v$ . We will consider four different pricing mechanisms: the optimal buy-many pricing  $p$ , the item pricing  $q$  constructed above, and their scalings  $\beta p$  and  $\ell q$  with  $\beta, \ell \in [0, 1]$ .

Let  $\lambda'$  denote the lottery the buyer purchases under pricing  $\beta p$ . By Lemma 4.4, there exists an item  $j$  in the support of  $\lambda'$ , such that  $p(\lambda') \geq q_j$ . Then

$$REV_{\beta p}(v) = \beta p(\lambda') \geq \beta q_j. \quad (1)$$

Next, let  $\lambda$  denote the lottery the buyer purchases under pricing  $p$ . Then, by Lemma 4.3, we have

$$\begin{aligned} v_j &\geq u_{\beta p}(v) \geq v(\lambda) - \beta p(\lambda) = v(\lambda) - p(\lambda) + (1 - \beta)p(\lambda) \\ &= u_p(v) + (1 - \beta)REV_p(v). \end{aligned} \quad (2)$$

where the second inequality follows by noting that the buyer has the option of purchasing  $\lambda$  under  $\beta p$ . Next, we note that since the buyer has the option of purchasing item  $j$  under pricing  $\ell q$ , we have

$$u_{\ell q}(v) \geq v_j - \ell q_j. \quad (3)$$

Finally, by the definition of the pricing  $q$ , it has identical individual item prices as in  $p$ , so it offers a strictly subset of options in  $p$ . Thus the buyer can purchase any set of items more cheaply under  $p$  than under  $q$ , which means

$$u_p(v) \geq u_q(v). \quad (4)$$

By  $\frac{\ell}{\beta}(1) + (2) + (3) + (4)$ ,

$$u_{\ell q}(v) - u_q(v) \geq (1 - \beta)\text{REV}_p(v) - \frac{\ell}{\beta}\text{REV}_{\beta p}(v). \quad (5)$$

By applying Lemma 4.2 to (5), there exists a random scaling factor  $\alpha$ , such that

$$\begin{aligned} \text{REV}_{\alpha q}(v) &\geq \frac{u_{\ell q}(v) - u_q(v)}{\ln(1/\ell)} \\ &\geq \frac{1}{\ln(1/\ell)} \left( (1 - \beta)\text{REV}_p(v) - \frac{\ell}{\beta}\text{REV}_{\beta p}(v) \right). \end{aligned} \quad (6)$$

Since  $p$  is the optimal buy-many mechanism, it achieves higher revenue than  $\ell p$ , which means  $\mathbb{E}_{v \sim D}[\text{REV}_p(v)] \geq \mathbb{E}_{v \sim D}[\text{REV}_{\ell p}(v)]$ . Taking the expectation over  $v$  on both sides of (6), we have

$$\begin{aligned} \text{REV}_{\alpha q} &= \mathbb{E}_{v \sim D}[\text{REV}_{\alpha q}(v)] \\ &\geq \frac{1}{\ln(1/\ell)} \left( (1 - \beta)\text{REV}_p - \frac{\ell}{\beta}\text{REV}_{\beta p} \right) \\ &\geq \frac{1}{\ln(1/\ell)} \left( 1 - \beta - \frac{\ell}{\beta} \right) \text{REV}_p. \end{aligned}$$

Take  $\ell = 0.03485$  and  $\beta = 0.18668$ , we have  $\text{REV}_{\alpha q} \geq 0.18668\text{REV}_p$ . Since  $\alpha q$  is a (randomized) item pricing, thus there exists an item pricing that gives a constant  $1/0.18668 < 5.4$ -approximation to the revenue obtained by the optimal buy-many mechanism.  $\square$

## 4.2 Hardness of Computing the Optimal Item Pricing

In this section, we show that it's strongly NP-hard to compute the optimal revenue that can be obtained by item pricing mechanisms, which means that there exists no FPTAS for the problem unless  $P=NP$ . Let ORDEREDITEMPRICING denote the following problem: For a unit-demand buyer with ordered valuation over  $n$  items, let  $D$  be the value distribution with support size  $m$ . Compute the optimal revenue obtained by item pricing.

**THEOREM 4.5.** *ORDEREDITEMPRICING is strongly NP-hard, even when each realized buyer has at most three distinct item values.*

**PROOF.** We prove the theorem via a reduction from MAX-CUT.

For any MAX-CUT instance with graph  $G(V, E)$ , let  $n = |V| > 180$  be large enough. Consider an instance of ORDEREDITEMPRICING with  $n + 1$  items. For convenience, we assume that each node in  $V$  also has an index in  $[n]$ . We want the following properties of the optimal item pricing for the instance:

- (1) The optimal item pricing has integral item prices for each item;
- (2)  $p_{n+1} = 6n$ , and there is a set of buyer types purchasing item  $n + 1$  with realization probability  $q_1 = 0.9$  that do not depend on the structure of the graph and contribute  $R_1(n)$  to the total revenue;
- (3)  $p_i = 3i - 1$  or  $3i - 2$ , and there is a set of buyer types purchasing items in  $[n]$  with realization probability  $q_2 < \frac{1}{12n}q_1$  that do not depend on the structure of the graph and contribute  $R_2(n)$  to the total revenue;
- (4) In addition to all previous buyers, for each  $(i, j) \in E$  with  $i < j$ , there exists a set  $T_{ij}$  of buyer types, and real number  $R_{ij}(n) > 0$  that is irrelevant to the graph structure such that: if  $p_j - p_i = 3(j - i)$ , then the revenue contribution from  $T_{ij}$  is  $R_{ij}(n)$ ; if  $p_j - p_i \neq 3(j - i)$ , then the revenue contribution from  $T_{ij}$  is  $R_{ij}(n) + \frac{1}{n^{10}}$ . The realization probability of any buyer type is polynomially bounded by  $n$  (at least  $\text{poly}(n^{-1})$ ).

Before going to the construction, we first show the strongly NP-hardness of ORDEREDITEMPRICING for an instance with above properties. This proves the claim of the Theorem.

Given an instance with such properties, we can calculate the revenue of the optimal item pricing for the instance. The total revenue contributed from buyer types from Property 2 and 3 is  $R_1(n) + R_2(n)$ . For any cut  $C = (V_1, V \setminus V_1)$ , if each item  $i$  in  $V_1$  is priced  $3i - 1$ , while each item  $j$  in  $V \setminus V_1$  is priced  $3j - 2$ , the total revenue contributed from buyer types from Property 4 is  $\sum_{(i,j) \in E} R_{ij}(n) + \frac{1}{n^{10}}|C|$ . Thus, for a graph  $G(V, E)$  with maximum cut  $c_{\max}$ , the corresponding instance of ORDEREDITEMPRICING has maximum revenue

$$h(G) = R_1(n) + R_2(n) + \sum_{(i,j) \in E} R_{ij}(n) + \frac{1}{n^{10}}c_{\max}.$$

This builds a bijection between the maximum cut of  $G$ , and the optimal item pricing revenue of the ORDEREDITEMPRICING instance constructed from  $G$ . Since all inputs for the ORDEREDITEMPRICING instance are polynomially bounded, we know that problem ORDEREDITEMPRICING is strongly NP-hard from the APX-hardness of MAX-CUT.

Now let's go back to show how to construct the ORDEREDITEMPRICING instance satisfying every property.

**Property 1.** To make sure that the optimal item pricing has all integral prices, we only need to construct the distribution such that each buyer type has integral value for every item. For such an instance, if the optimal item pricing does not have integral price for every item, we can round up the price for each item to the closet integer without reducing the revenue. The reason is that such a round up procedure does not change the incentive of any buyer type. If a buyer  $v$  prefers to purchase item  $i$  to  $j$  under pricing  $p$ , which means  $v_i - p_i \geq v_j - p_j$ , then  $v_i - \lceil p_i \rceil \geq v_j - \lceil p_j \rceil$  since  $v_i$  and  $v_j$  are both integer. Thus we can only focus on the class of item pricing with integral item prices.

**Property 2.** Construct a buyer type with value  $v_{n+1} = 6n$  for item  $n + 1$ , and  $v_i = 0$  for all  $i \leq n$ . In other words, the buyer only wants to purchase item  $n + 1$  with value  $6n$ , and is not interested in any other item. The buyer type appears with probability  $q_1$ , where  $q_1$  is

very close to 1. We will make sure that the rest of the buyer types will contribute less than  $q_1$  revenue, so the optimal pricing will not set a price less than  $6n$  for item  $n+1$ . This will be done by letting the rest of the buyer types have maximum item value  $\leq 6n$  for each item and appear with probability  $< \frac{1}{6n}q_1 = \frac{3}{20}n$  in total. The revenue contribution of the buyer is  $R_1(n) = 6nq_1$  by setting  $p_{n+1} = 6n$ .

**Property 3.** For each  $i \in [n]$ , construct a buyer type  $v$  with value  $v_1 = v_2 = \dots = v_{i-1} = 0$ ,  $v_i = v_{i+1} = \dots = v_{n+1} = 3i - 1$ , which appears with probability  $q'_2 = \frac{1}{36n^2}q_1 = \frac{1}{40n^2}$ ; a buyer type  $v'$  with value  $v'_1 = \dots = v'_{i-1} = 0$ ,  $v'_i = \dots = v'_{n+1} = 3i - 2$ , which appears with probability  $\frac{1}{3i-2}q'_2$ ; and a buyer  $v''$  with value  $v''_1 = \dots = v''_{i-1} = 0$ ,  $v''_i = \dots = v''_{n+1} = 3i - 2$ ,  $v''_{n+1} = 6n$ , which appears with probability  $q'_2$ . Under any item pricing, all buyers would purchase item  $i$ , item  $n+1$ , or nothing. Note that the price for item  $n+1$  has been fixed to  $6n$  by Property 2. We have the following cases:

- If  $p_i \leq 3i - 3$ , then all three buyer types prefer to purchase item  $i$ , which lead to total revenue  $(3i-3)(q'_2 + \frac{1}{3i-2}q'_2 + q'_2) = \frac{(3i-3)(6i-3)}{3i-2}q'_2 < 6nq'_2$  for the three buyer types.
- If  $p_i \geq 3i$ , then buyer  $v$  and buyer  $v'$  cannot afford to purchase any item, while buyer  $v''$  prefers to purchase item  $n+1$ , which lead to revenue  $6nq'_2$ .
- If  $p_i = 3i - 1$ , then buyer  $v$  purchases item  $i$ ; buyer  $v'$  purchases nothing; buyer  $v''$  purchases item  $n+1$ , which leads to total revenue  $(3i-1)q'_2 + 6nq'_2$ .
- If  $p_i = 3i - 2$ , then the first buyer purchase item  $i$ , the second buyer purchases item  $i$ , the third buyer purchases item  $n+1$ , which leads to revenue  $(3i-2)(q'_2 + \frac{1}{3i-2}q'_2) + 6nq'_2 = (3i-1)q'_2 + 6nq'_2$ .

Thus setting  $p_i = 3i - 1$  or  $3i - 2$  gives the same optimal revenue for the three buyers, while setting any other price leads to a revenue loss of at least  $(3i-1)q'_2$ . We set  $q'_2$  to be large enough such that the rest of the buyers (to be defined in Property 4) cannot contribute  $q'_2$  revenue, which can be done by letting the rest of the buyer types have maximum item value  $\leq 6n$  and appear with probability  $< \frac{1}{6n}q'_2 = \frac{1}{240}n^3$  in total. So the optimal pricing only sets  $p_i = 3i - 1$  or  $3i - 2$  for item  $i$ .

The total revenue contribution of the buyers is  $R_2(n) = \sum_{i=1}^n ((3i-1)q'_2 + 6nq'_2)$ . The total realization probability of the buyers added in this property is at most  $3q'_2$  for each  $i$ , thus at most  $3nq'_2 = \frac{3}{40n}$ . The set of the buyers and the revenue contribution only depend on  $n$ , and does not depend on the graph structure.

**Property 4.** For each edge  $(i, j) \in E$  with  $i < j$ , let  $x = 3i - 2$ , and  $y = 3j - 2$ . Consider set  $T_{ij}$  of buyer types formed by the following 4 types of buyers  $v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}$ :

- $v_1^{(1)} = \dots = v_{i-1}^{(1)} = 0$ ,  $v_i^{(1)} = \dots = v_{j-1}^{(1)} = x$ ,  
 $v_j^{(1)} = \dots = v_{n+1}^{(1)} = y$ ;
- $v_1^{(2)} = \dots = v_{i-1}^{(2)} = 0$ ,  $v_i^{(2)} = \dots = v_{j-1}^{(2)} = x + 1$ ,  
 $v_j^{(2)} = \dots = v_{n+1}^{(2)} = y + 1$ ;
- $v_1^{(3)} = \dots = v_{i-1}^{(3)} = 0$ ,  $v_i^{(3)} = \dots = v_{j-1}^{(3)} = x$ ,  
 $v_j^{(3)} = \dots = v_{n+1}^{(3)} = y + 1$ ;

$$\bullet v_1^{(4)} = \dots = v_{i-1}^{(4)} = 0, v_i^{(4)} = \dots = v_{j-1}^{(4)} = x + 1, \\ v_j^{(4)} = \dots = v_{n+1}^{(4)} = y.$$

In other words, the four type of buyers would purchase item  $i$ , item  $j$  or nothing, and has slightly different values for item  $i$  and  $j$ . Our goal here is to determine the appearance probability of each buyer type in the value distribution, such that if  $p_i \equiv p_j \pmod 3$ , then these buyer types contribute  $R_{ij}(n)$  to the total revenue; if  $p_i \not\equiv p_j \pmod 3$ , then these buyer types contribute  $R_{ij}(n) + n^{-10}$  to the total revenue.

Let  $A$  be the following  $4 \times 4$  outcome matrix, such that each element of the matrix corresponds to the payment of a (row) buyer under a specific (item) item pricing:

$$\text{pricing } (p_i, p_j) \begin{matrix} & \begin{matrix} (v_i, v_j) \text{ of the buyer} \\ (x, y) & (x+1, y+1) & (x, y+1) & (x+1, y) \end{matrix} \\ \begin{matrix} (x, y) \\ (x+1, y+1) \\ (x, y+1) \\ (x+1, y) \end{matrix} & \begin{bmatrix} y & y & y & x \\ 0 & y+1 & y+1 & x+1 \\ x & x & y+1 & x \\ y & y & y & y \end{bmatrix} \end{matrix}$$

For any vector  $\mathbf{z} \in \mathbb{R}_{\geq 0}^4$ ,  $A\mathbf{z}$  correspond to the vector of the revenues of the 4 pricings  $(p_i, p_j) = (x, y), (x+1, y+1), (x, y+1), (x+1, y)$ , given that  $z_\ell$  buyers of type  $\ell$  appear. To satisfy Property 4, we need to find a vector  $\mathbf{z}$  such that  $A\mathbf{z} = (R_{ij}, R_{ij}, R_{ij} + n^{-10}, R_{ij} + n^{-10})^T$  for some  $R_{ij} \geq 0$ .

By solving  $\mathbf{a}$  for  $A\mathbf{a} = (1, 1, 1, 1)^T$  and  $\mathbf{b}$  for  $A\mathbf{b} = (0, 0, 1, 1)^T$ , we get

$$\mathbf{a} = \left( \frac{1}{y^2 + y}, \frac{x}{y(y+1)(y+1-x)}, \frac{y-x}{y(y+1-x)}, 0 \right)^T, \\ \mathbf{b} = \left( \frac{1}{y^2 + y}, -\frac{x^2 + y(y+1)^2 - xy(y+2)}{y(y+1)(y+1-x)(y-x)}, \frac{y-x}{y(y+1-x)}, \frac{1}{y-x} \right)^T.$$

Taking

$$\mathbf{z} = 2n^{-10}y^3\mathbf{a} + n^{-10}\mathbf{b} \\ = n^{-10} \left( \frac{2y^3 + 1}{y^2 + y}, \frac{2x(y-x)y^3 + xy^2 + 2x - y^3 - 2y^2 - y}{y(y+1)(y+1-x)(y-x)}, \right. \\ \left. \frac{(2y^3 + 1)(y-x)}{y(y+1-x)}, \frac{1}{y-x} \right),$$

we have  $\mathbf{z} \geq 0$ , and

$$A\mathbf{z} = (2n^{-10}y^3, 2n^{-10}y^3, 2n^{-10}y^3 + n^{-10}, 2n^{-10}y^3 + n^{-10}).$$

Thus if with probability  $z_\ell$  the buyer has type  $v^{(\ell)}$ , the four buyer types contribute  $R_{ij}(n) = 2n^{-10}y^3$  revenue if  $p_i \equiv p_j \pmod 3$ , and  $R_{ij}(n) + n^{-10} = 2n^{-10}y^3 + n^{-10}$  otherwise.

Now we compute the total realization probability of the four buyer types in the distribution:  $\|\mathbf{z}\|_1 < 10y^3n^{-10} < 270n^{-7}$  by the definition of  $\mathbf{z}$  and  $y < 3n$ . Since there are less than  $\frac{1}{2}n^2$  edges, the total realization probability of all buyer types added in this property is less than  $\frac{1}{2}n^2 \cdot 270n^{-7} = 135n^{-5} < \frac{1}{240n^3}$  for  $n > 180$  as required in Property 3. The total realization property of all buyer types added in Property 3 and 4 is less than  $\frac{3}{40n} + \frac{1}{240n^3} < \frac{3}{20n}$  as required in Property 2. This completes the proof of the correctness of the construction.  $\square$

### 4.3 A PTAS for Computing a Near-Optimal Item Pricing

We complement the strongly NP-hardness result by developing a PTAS algorithm for computing the optimal item pricing for a unit-demand buyer with totally-ordered values.

**THEOREM 4.6.** *For a unit-demand buyer with totally ordered item values, there exists an algorithm running in  $\text{poly}(m, n^{\text{poly}(1/\epsilon)}, \log \mathcal{R})$  time that computes an item pricing that is  $(1 + \epsilon)$ -approximation in revenue to the optimal item pricing.<sup>6</sup>*

**PROOF.** We prove the theorem in several steps.

- (1) There exists a near optimal item pricing where all prices are non-decreasing powers of  $(1 + \epsilon^2)$ . Let  $\Pi = \{(1 + \epsilon^2)^r \mid r \in \mathbb{Z}\} \cup \{0\}$ . Then for all item pricings  $p$ , there exists  $q^{(1)} \in \Pi^n$ , such that for all value functions  $v$ ,

$$\text{REV}_{q^{(1)}}(v) \geq (1 - O(\epsilon))\text{REV}_p(v).$$

- (2) At a small loss in revenue, we can restrict prices to lie in a small set. In particular, for all value distributions  $\mathcal{D}$  with value range  $\mathcal{R}$ , there exists an efficiently computable set  $\Pi^* \subset \Pi$  with  $|\Pi^*| = \text{poly}(1/\epsilon, \log \mathcal{R})$  such that for all item pricings  $q^{(1)} \in \Pi^n$ , there exists an item pricing  $q^{(2)} \in \Pi^{*n}$  satisfying

$$\text{REV}_{q^{(2)}}(\mathcal{D}) \geq (1 - O(\epsilon))\text{REV}_{q^{(1)}}(\mathcal{D}).$$

- (3) Next, we define for each unit demand buyer an additive value function that closely mimics it. For a unit-demand value function  $v$ , define  $v_i^\oplus = v_i - v_{i-1}$  and let  $v^\oplus$  be the value function that assigns to a set  $S \subseteq [n]$  the value  $\sum_{i \in S} v_i^\oplus$ . We also define a new kind of pricing that we will call an *interval prefix pricing*. Given a partition of the  $n$  items into  $t$  intervals,  $I_{i_0, i_1}, I_{i_1, i_2}, \dots, I_{i_{t-1}, i_t}$  with  $i_0 = 0$  and  $i_t = n$ , an interval prefix pricing  $q$  is a menu with  $n$  options; The  $j$ th option allocates the set  $I_{i_\ell, j} = \{i_\ell + 1, i_\ell + 2, \dots, j\}$  at price  $q_j$  where  $i_\ell < j \leq i_{\ell+1}$ .

We furthermore say that an interval prefix pricing  $q$  satisfies *price gap*  $(\gamma, \delta)$  if (1) menu options corresponding to different intervals are priced multiplicatively apart: for all  $i, j$ , and  $\ell$  with  $i \leq i_\ell < j$ ,  $q_j \geq (1 + \epsilon^2)^\gamma q_i$ , (2) and, menu options corresponding to any single interval are priced multiplicatively close to each other: for all  $i, j$ , and  $\ell$  with  $i_\ell < i < j \leq i_{\ell+1}$ ,  $q_j \leq (1 + \epsilon^2)^\delta q_i$ .

We show that for value distribution  $\mathcal{D}$  and its corresponding additive value distribution  $\mathcal{D}^\oplus$  and item pricing  $q^{(2)} \in \Pi^{*n}$ , there exists an efficiently computable set  $\Pi'$  with  $|\Pi'| = |\Pi^*|$  and an interval prefix pricing  $q^{(3)}$  with  $q_i^{(3)} \in \Pi'$  for all  $i \in [n]$  and price gap  $(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$ , such that

$$\text{REV}_{q^{(3)}}(\mathcal{D}^\oplus) \geq (1 - O(\epsilon))\text{REV}_{q^{(2)}}(\mathcal{D}).$$

The converse also holds: for every unit demand value function  $v$  and interval prefix pricing  $q$  with price gap

$(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$ , we can efficiently compute an item pricing  $q^{(4)}$  such that

$$\text{REV}_{q^{(4)}}(v) \geq (1 - O(\epsilon))\text{REV}_q(v^\oplus).$$

- (4) Finally, we show that for any distribution over additive values  $v^\oplus$  and any set  $\Pi'$  of values, an optimal interval prefix pricing  $q$ , with  $q_i \in \Pi'$  for all  $i \in [n]$  and price gap  $(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$ , can be found in time polynomial in  $|\Pi'|$ ,  $n^{\text{poly}(1/\epsilon)}$ , and  $m$ .

The algorithm can be described as follows. By the last step, we can efficiently compute the optimal interval prefix pricing  $q$  with price gap  $(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$  for the distribution  $\mathcal{D}^\oplus$  over additive buyer  $v^\oplus$  that corresponds to the unit-demand distribution  $\mathcal{D}$ , such that all item prices are in  $\Pi'$ . By Step 3, we can efficiently compute an item pricing  $q^{(4)}$  with  $\text{REV}_{q^{(4)}}(v) \geq (1 - O(\epsilon))\text{REV}_q(v^\oplus)$ . Also by the first three steps,

$$\begin{aligned} & \text{REV}_{q^{(4)}}(\mathcal{D}) \\ & \geq (1 - O(\epsilon))\text{REV}_q(\mathcal{D}^\oplus) \geq (1 - O(\epsilon))\text{REV}_{q^{(3)}}(\mathcal{D}^\oplus) \\ & \geq (1 - O(\epsilon))\text{REV}_{q^{(2)}}(\mathcal{D}) \geq (1 - O(\epsilon))\text{REV}_{q^{(1)}}(\mathcal{D}) \\ & \geq (1 - O(\epsilon))\text{REV}_p(\mathcal{D}) \end{aligned}$$

for the optimal item pricing  $p$ .  $q^{(4)}$  can be found in  $\text{poly}(m, n^{\text{poly}(1/\epsilon)}, |\Pi'|) = \text{poly}(m, \log \mathcal{R}, n^{\text{poly}(1/\epsilon)})$  time.

Now we elaborate on each step in more detail. For simplicity, we assume  $\frac{1}{\epsilon}$  is an integer.

*Step 1.* We first introduce a useful lemma for approximating the revenue of a mechanism  $p$  via another pricing function  $q$  that approximates  $p$  closely *pointwise*. Slightly different forms of the lemma appears in multiple papers in the simple-versus-optimal mechanism design and the menu-size complexity literature, and is attributed to Nisan<sup>7</sup>.

**LEMMA 4.7.** *For any  $\epsilon > 0$ , let  $p$  and  $q$  be two pricing functions satisfying  $q(\lambda) \leq p(\lambda) \leq (1 + \epsilon)q(\lambda)$  for all random allocations  $\lambda \in \Delta(2^{[n]})$ . Then for scaling factor  $\alpha = (1 + \epsilon)^{-1/\sqrt{\epsilon}}$  and any valuation function  $v$ ,*

$$\text{REV}_{\alpha q}(v) \geq (1 - 3\sqrt{\epsilon})\text{REV}_p(v).$$

Now come back to Step 1 of the proof. Consider the optimal item pricing  $p$ . Let  $q$  be the item pricing that rounds the price of each item down to the closest integral power of  $(1 + \epsilon^2)$ . Then for any lottery  $\lambda$ ,  $q(\lambda) \leq p(\lambda) < (1 + \epsilon^2)q(\lambda)$ . By Lemma 4.7,  $q^{(1)} = (1 + \epsilon^2)^{-1/\epsilon} q$  is an item pricing with power-of- $(1 + \epsilon^2)$  prices for each item, and achieves  $(1 - O(\epsilon))$  fraction of the revenue of  $p$ . This means that focusing on finding the optimal item pricing with discretized item prices only loses a  $(1 - O(\epsilon))$ -factor in revenue.

Any item pricing is equivalent to an item pricing with non-decreasing item prices in the totally ordered setting, since for any pricing  $p$ , if there exist two items  $i < j$  with  $p_i > p_j$ , then no buyer ever prefers  $i$  to  $j$  since item  $i$  is worse in quality with higher prices. Thus setting  $p_i = p_j$  keeps the incentive of all buyer types unchanged, while after finite such operations, we can get an item pricing with non-decreasing item prices and the same revenue.

<sup>6</sup>The dependency on  $\log \mathcal{R}$  in the running time can actually be removed due to Lemma 5.4.

<sup>7</sup>For a more detailed discussion of the history of the idea of the lemma, we would point the readers to footnote 24 of [25].

*Step 2.* It suffices to show that there exists an item pricing with all item prices being either 0 or bounded in range  $[\Omega(\epsilon^2), \mathcal{R}]$ , that achieves an  $(1 - O(\epsilon))$  fraction of the revenue of  $q^{(1)}$ .

The proof has the same nudging idea as Lemma 4.7. Let  $q^{(2)}$  be defined as follows. For each item  $i$  with  $q_i^{(1)} \leq \epsilon^2$ ,  $q_i^{(2)} = 0$ ; otherwise,  $q_i^{(2)} = (1 - \epsilon)q_i^{(1)}$ . Then for each buyer type  $v$ , assume that she purchases item  $i$  under  $q^{(1)}$  and  $j$  under  $q^{(2)}$ . Since the buyer prefers  $i$  over  $j$  under  $q^{(1)}$ , we have

$$v_i - q_i^{(1)} \geq v_j - q_j^{(1)}; \quad (7)$$

Since the buyer prefers  $j$  over  $i$  under  $q^{(2)}$ , and  $(1 - \epsilon)(q^{(1)} - \epsilon^2) \leq q^{(2)} \leq (1 - \epsilon)q^{(1)}$ , we have

$$v_j - (1 - \epsilon)(q_j^{(1)} - \epsilon^2) \geq v_j - q_j^{(2)} \geq v_i - q_i^{(2)} \geq v_i - (1 - \epsilon)q_i^{(1)}. \quad (8)$$

Add inequalities (7) and (8), we have  $q_j^{(1)} \geq q_i^{(1)} - \epsilon + \epsilon^2$ . Thus

$$q_j^{(2)} \geq (1 - \epsilon)(q_j^{(1)} - \epsilon^2) \geq (1 - \epsilon)q_i^{(1)} - \epsilon + \epsilon^2.$$

Taking the expectation over  $v \sim \mathcal{D}$ . Observe that  $\text{REV}_{q^{(1)}}(\mathcal{D}) \geq 1$  by selling only item  $n$  at price 1, since in the input model we assume that each valuation function has at least value 1 for some item. We have

$$\text{REV}_{q^{(2)}}(\mathcal{D}) \geq (1 - \epsilon)\text{REV}_{q^{(1)}}(\mathcal{D}) - \epsilon + \epsilon^2 = (1 - O(\epsilon))\text{REV}_{q^{(1)}}(\mathcal{D}).$$

The item prices of  $q^{(2)}$  are in some set  $\Pi^*$  with  $|\Pi^*| = O(\frac{1}{\epsilon} \log \frac{\mathcal{R}}{\epsilon})$  since all item prices in  $q^{(2)}$  are in range  $[(1 - \epsilon)\epsilon^2, \mathcal{R}]$  and are powers of  $(1 + \epsilon^2)$  in  $q^{(1)}$  multiplied by  $(1 - \epsilon)$ .

*Step 3.* For item pricing  $q^{(2)}$ , consider the following  $\frac{1}{\epsilon}$  sets of prices: for each  $\ell \in [\frac{1}{\epsilon}]$ , let

$$V_\ell = \left\{ (1 + \epsilon^2)^s \mid s \in \mathbb{Z}, s \equiv a \bmod \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2} \text{ for a s.t. } \frac{\ell}{\epsilon^2} \ln \frac{1}{\epsilon^2} \leq a < \frac{\ell+1}{\epsilon^2} \ln \frac{1}{\epsilon^2} \right\}.$$

Since  $V_\ell$  are disjoint sets with the union being the set of all power-of- $(1 + \epsilon^2)$  prices, there exists a set of prices  $V_\ell$  such that in item pricing  $q^{(2)}$ , the revenue contribution from items with price in  $V_\ell$  is at most  $\epsilon$  fraction of the total revenue. Consider the following item pricing  $q'$ : for any item with price  $q_i^{(2)} \notin V_\ell$ ,  $q'_i = q_i^{(2)}$ ; for any item with  $q_i^{(2)} \in V_\ell$ ,  $q'_i$  is set to the smallest  $q_j^{(2)} \notin V_\ell$  with  $j > i$ . In other words, the prices of all items in  $V_\ell$  are raised while the incentives of all buyer types that do not purchase an item with price in  $V_\ell$  in  $q^{(2)}$  stay unchanged. Thus the revenue of  $q'$  is at least  $(1 - O(\epsilon))$  fraction of the revenue of  $q^{(2)}$ ,  $\text{REV}_{q'}(\mathcal{D}) \geq (1 - O(\epsilon))\text{REV}_{q^{(2)}}(\mathcal{D})$ .

Now we construct an interval prefix pricing  $q$  as follows. The price of each item  $q_i$  is set to  $q'_i$ , with all items naturally grouped to intervals as follows: for any integer  $r$ , all items with  $\log_{1+\epsilon^2} q_i$  in range  $[\frac{r}{\epsilon^3} \ln \frac{1}{\epsilon^2} + \frac{\ell+1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{r+1}{\epsilon^3} \ln \frac{1}{\epsilon^2} + \frac{\ell}{\epsilon^2} \ln \frac{1}{\epsilon^2}]$  are grouped to an interval. Then for any two items  $i, j$  in the same interval, the logarithm (with base  $1 + \epsilon^2$ ) of the two prices differs by at most  $\frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2}$ ; for any two items  $i < j$  in different intervals, the prices differ by a factor at least  $(1 + \epsilon^2)^{\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}} > \frac{1}{\epsilon^2}$ . Thus  $q$  has price gap  $(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$ .

Let  $I_{i_0, i_1}, I_{i_1, i_2}, \dots, I_{i_{t-1}, i_t}$  be the interval partitioning of the entire set of items. Now we study the price of any set  $T = \{j_1, j_2, \dots, j_b\}$  of items under such an interval prefix pricing  $q$ . Suppose that the item with the largest index  $j_b$  is in interval  $I_{i_{d-1}, i_d}$ . Then the price  $q(T)$  of the set is lower-bounded by  $q'_{j_b}$ , and upper-bounded by  $q'_{j_b}$  plus the prices of all items in intervals prior to  $I_{i_{d-1}, i_d}$ . The payment for interval  $I_{i_{d-2}, i_{d-1}}$  is at most

$$q'_{i_{d-1}} < \epsilon^2 q'_{i_{d-1}+1} \leq \epsilon^2 q'_{j_b}.$$

Similarly the payment for intervals with smaller indexes are geometrically smaller, and sum up to at most  $\epsilon^2 q'_{j_b}$ . This means that  $q'_{j_b} \leq q(T) \leq (1 + 2\epsilon^2)q'_{j_b} < (1 + 4\epsilon^2)q'_{j_b}$ .

On the other hand, we introduce a new mechanism  $q''$  called *prefix pricing*. In such a mechanism, the buyer can pay  $q'_i$  to obtain a prefix set of items  $\{1, 2, \dots, i\}$ . For any unit-demand buyer  $v$  with totally-ordered item values, she buys set  $\{1, 2, \dots, i\}$  under  $q''$  if and only if she buys  $i$  under  $q'$ . Also under  $q''$ , buyer  $v$  and  $v^\oplus$  purchase the same set. Thus  $\text{REV}_{q''}(v^\oplus) = \text{REV}_{q''}(v) = \text{REV}_{q'}(v)$ . In prefix pricing  $q''$ , the price of set  $T$  is  $q''(T) = q'_{j_b}$ , which is the price of the smallest prefix that contains  $j_b$ . Thus

$$(1 + 2\epsilon^2)^{-1} q(T) \leq q''(T) \leq q(T) \leq (1 + 2\epsilon^2) q''(T). \quad (9)$$

By Lemma 4.7, there exists a scaling  $q^{(3)} = (1 + 4\epsilon^2)^{-1/(2\epsilon)} q$  of interval prefix pricing  $q$  which gives  $(1 - O(\epsilon))$ -fraction of the revenue of  $q''$  for any buyer  $v^\oplus$ . Thus

$$\begin{aligned} & \text{REV}_{q^{(3)}}(\mathcal{D}^\oplus) \\ & \geq (1 - O(\epsilon))\text{REV}_{q''}(\mathcal{D}^\oplus) = (1 - O(\epsilon))\text{REV}_{q'}(\mathcal{D}) \\ & \geq (1 - O(\epsilon))\text{REV}_{q^{(2)}}(\mathcal{D}). \end{aligned}$$

Since all item prices in  $q$  are in  $\Pi^*$ , we have that all item prices in  $q^{(3)}$  are in  $\Pi' = \{(1 + 4\epsilon^2)^{-1/(2\epsilon)} y \mid y \in \Pi^*\}$ .

The converse also holds: given any interval prefix pricing  $q$  with price gap  $(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$ , we can define prefix pricing  $q''$  such that  $q''(\{1, 2, \dots, i\}) = q_i$ . Then (9) still holds, which means that by Lemma 4.7, there exists a scaling  $q^o = (1 + 4\epsilon^2)^{-1/(2\epsilon)} q''$  of prefix pricing  $q''$  which gives  $(1 - O(\epsilon))$ -fraction of the revenue of  $q$  for any additive buyer  $v^\oplus$ . Let  $q^{(4)}$  be the item pricing with  $q_i^{(4)} = q^o(\{1, 2, \dots, i\})$ . Then

$$\text{REV}_{q^{(4)}}(v) = \text{REV}_{q^o}(v) = \text{REV}_{q^o}(v^\oplus) \geq (1 - O(\epsilon))\text{REV}_q(v).$$

*Step 4.* We aim to solve the following problem: find the optimal interval prefix pricing  $q$  with price gap  $(\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$  for an additive buyer, such that the item prices  $q_i$  that define  $q$  are in set  $\Pi'$ .

This can be solved via the following dynamic program. Since the buyer is additive, the revenue contribution from each interval can be calculated separately without worrying about the incentive of the buyer. Let  $F[i, z]$  be the optimal revenue of the interval prefix pricing (with price gap  $(\gamma, \delta) = (\frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2})$ ) and without prefix buying constraints) from only items in prefix set  $\{1, 2, \dots, i\}$ , with the last item price being  $q_i = z$ . Then we can write the following recursive formula:

$$F[i, z] = \max_{j < i, y \geq z(1+\epsilon^2)^{-\delta}, w \leq y(1+\epsilon^2)^{-\gamma}, y, z \in \Pi'} \{F[j, w] + G[j, i, y, z]\},$$

where  $G[j, i, y, z]$  denotes the optimal revenue of a pricing  $q$  which sells a prefix of interval  $I_{j,i} = \{j+1, j+2, \dots, i\}$  with price  $q_{j+1} = y$  and  $q_i = q(\{j+1, j+2, \dots, i\}) = z$  to an additive buyer with distribution  $\mathcal{D}^\oplus$ . In the recursive formula, we want all items in  $I_{j,i}$  to be priced in range  $[y, z]$  with  $y \geq z(1 + \epsilon^2)^{-\delta}$ , with all items in  $\{1, 2, \dots, j\}$  to be priced at most  $w \leq y(1 + \epsilon^2)^{-\gamma}$ . The objective we want to solve is  $\max_{z \in \Pi'} F[n, z]$ .

Now we analyze the running time of the dynamic program. Each  $i, j, y, z$  in the recursive formula has  $\text{poly}(m, n, \frac{1}{\epsilon})$  possibilities. The running time of calculating each  $G[j, i, y, z]$  is  $\text{poly}(m, n^\delta) = \text{poly}(m, n^{\frac{1}{\epsilon^3} \ln \frac{1}{\epsilon^2}})$ , since there are only at most  $\delta$  different prices for the items in  $I_{j,i}$ , thus at most  $\text{poly}(n^\delta)$  different non-decreasing pricings. Therefore the dynamic program can be solved in  $\text{poly}(m, n^{\text{poly}(1/\epsilon)}, |\Pi'|)$  time.  $\square$

#### 4.4 Discussion on Additive Buyers

In Section 4.1, we showed that for a unit-demand buyer with totally ordered item values, item pricing gives a constant approximation to the optimal buy-many revenue. We want to investigate whether such nice properties generalize to other class of valuation functions. For example, for an additive buyer with totally-ordered item values, can item pricings achieve a constant fraction of the optimal buy-many revenue? Unfortunately, we give a negative answer to the question through the following theorem.

**THEOREM 4.8.** *For any additive buyer with totally ordered value for all items, item pricing cannot achieve an approximation ratio  $O(\log \log n)$  to the optimal (deterministic) buy-many mechanism.*

The proof uses a reduction from an additive buyer with unordered item values, which is known to have  $\Omega(\log n)$  revenue gap between the optimal item pricing and the optimal deterministic buy-many mechanism [15]. The proof is omitted due to page limit.

### 5 ITEM PRICING IN THE PARTIALLY-ORDERED SETTING

In this section, we generalize the results for a unit-demand buyer in the totally-ordered setting to a general-valued buyer in the partially-ordered setting. We first show that item pricing gives an  $O(\log k)$ -approximation in revenue to the optimal buy-many mechanism, where  $k$  is the width of the ordered set of items. Then we provide a PTAS algorithm for finding a near-optimal item pricing when  $k$  is a constant. This way, we show that the width  $k$  of the ordered set is the key parameter in both the performance and the computational complexity of the item pricing mechanisms.

#### 5.1 Item Pricing Gives an $O(\log k)$ Approximation in Revenue to the Optimal Buy-Many Mechanism

For a general-valued buyer, [15] shows that item pricing gives an  $O(\log n)$  approximation in revenue to the optimal buy-many mechanism. Here we improve the approximation ratio to  $O(\log k)$ , which is only related to the width of the partially ordered item set.

**THEOREM 5.1.** *For any buyer with partially ordered values for all items with width  $k$ , item pricing gives an  $O(\log k)$  approximation in revenue to the optimal buy-many mechanism.*

Similar to the totally-ordered setting, given an optimal mechanism  $p$ , we define item pricing  $q$  such that each  $q_i$  is the cheapest way to get an item that dominates  $i$  in  $p$ . We show that a scaling of  $q$  gives  $O(\log k)$ -approximation to the optimal revenue. Compared to the totally-ordered setting, the major difference is that Lemma 4.4 no longer holds. Instead, we show that for any allocation  $x$  of items, there exists a set  $T$  in the support of  $x$ , such that  $q(T) \leq k^2 p(x)$ . The rest of the proof mostly still goes through.

**PROOF.** Let  $S$  be the collection of sets that can be demanded by the buyer, i.e. the set of antichains in the partial ordering. Given the optimal buy-many mechanism  $p$ , define item pricing  $q$  as follows:  $q_i$ , the price of item  $i$ , is defined by the cheapest way of getting an item that dominates it. In other words, if  $S_i = \{j \in [n] \mid j \geq i\}$  is the set of items that dominate  $i$ ,

$$q_i = \min_{x \in \Delta(S): \sum_{T \cap S_i \neq \emptyset} x_T = 1} p(x) = \min_{x \in \Delta(S)} \frac{p(x)}{\sum_{T \cap S_i \neq \emptyset} x_T}.$$

We show that a scaling of  $q$  gives a  $1/O(\log k)$  fraction of the optimal revenue obtained by buy-many mechanisms. Lemma 4.3 still holds, while Lemma 4.4 needs to be modified as follows.

**LEMMA 5.2.** *For any allocation  $x \in \Delta(S)$ , there exists a set  $T$  in the support of  $x$ , such that  $q(T) \leq k^2 p(x)$ .*

**PROOF.** Let  $S$  be the set of items, such that for any  $i \in S$ , the probability of getting an item that dominates  $i$  is less than  $\frac{1}{k}$ . In other words,

$$S = \left\{ i \in [n] \mid \sum_{T \cap S_i \neq \emptyset} x_T < \frac{1}{k} \right\}.$$

We first observe that there exists a set  $T$  in the support of  $x$ , such that  $T$  does not contain any item in  $S$ . We reason this as follows. Let  $S' \subseteq S$  be an antichain such that for any element  $i \in S$ , there is an element  $j \in S'$  such that  $i \supseteq j$ . Intuitively,  $S'$  is the “bottom” of set  $S$  in the preference graph such that all elements in  $S$  dominate some element in  $S'$ . Then for any  $i \in S'$ , the probability that a set drawn from  $x$  contains an element that dominates  $i$  is less than  $\frac{1}{k}$ . Since  $S'$  is an antichain and there are at most  $k$  elements in  $S'$ , by union bound, the probability that a set drawn from  $x$  contains an element that dominates some element in  $S'$  is less than 1. Thus there exists a set  $T$  in the support of  $x$ , such that  $T$  does not contain any item that dominates some element in  $S'$ , thus does not contain any item in  $S$ .

For any item  $i \in T$ , by the definition of  $q_i$ ,

$$q_i \leq \frac{p(x)}{\sum_{T \cap S_i \neq \emptyset} x_T} \leq \frac{p(x)}{1/k} = kp(x).$$

Then

$$q(T) = \sum_{i \in T} q_i \leq \sum_{i \in T} kp(x) = |T|kp(x) \leq k^2 p(x).$$

$\square$

Now we are ready to prove the theorem. The same as in the proof of Theorem 4.1, we fix any buyer type  $v$ , and define the four mechanisms as follows. Under the optimal pricing  $p$ , the buyer purchases lottery  $\lambda$ ; under pricing  $\beta p$  the buyer purchases lottery  $\lambda'$ ; under item pricing  $q$ , the buyer purchases set  $S$ ; under pricing  $\ell q$  the buyer purchases set  $S'$ . Here we define  $\beta = \frac{1}{2}$  and  $\ell = \frac{1}{8k^2}$ .

By Lemma 5.2, there exists a set  $T$  in the support of  $\lambda'$ , such that  $p(\lambda') \geq \frac{1}{k^2} q(T)$ . Then

$$\text{REV}_{\beta p}(v) = \beta p(\lambda') \geq \frac{\beta}{k^2} q(T). \quad (10)$$

Inequality (2) still holds (by replacing item  $j$  in (2) with set  $T$ ):

$$v(T) \geq u_p(v) + (1 - \beta) \text{REV}_{\beta p}(v). \quad (11)$$

Since the buyer has a larger utility purchasing item  $S'$  than  $T$  under pricing  $\ell q$ , we have

$$u_{\ell q}(v) \geq v(T) - \ell q(T). \quad (12)$$

Since the buyer has a larger utility purchasing lottery  $\lambda$  than buying the collection of lottery  $\lambda_i$  for every  $i \in S$  that defines  $q_i$  in  $p$  (which has price  $q_i$ ), and  $\lambda_i$  allocates the buyer an item that dominates  $i$ , we have

$$u_p(v) \geq v(\cup \lambda_i) - \sum_{i \in S} p(\lambda_i) \geq v(S) - \sum_{i \in S} q_i = v(S) - q(S) = u_q(v). \quad (13)$$

By  $\frac{\ell k^2}{\beta}$  (10) + (11) + (12) + (13), and apply  $\beta = \frac{1}{2}$  and  $\ell = \frac{1}{8k^2}$ ,

$$\begin{aligned} u_{\ell q}(v) - u_q(v) &\geq (1 - \beta) \text{REV}_{\beta p}(v) - \frac{\ell k^2}{\beta} \text{REV}_{\beta p}(v) \\ &= \frac{1}{2} \text{REV}_{\beta p}(v) - \frac{1}{4} \text{REV}_{\beta p}(v). \end{aligned} \quad (14)$$

By applying Lemma 4.2 to (14), there exists a random scaling factor  $\alpha$ , such that

$$\begin{aligned} \text{REV}_{\alpha q}(v) &\geq \frac{u_{\ell q}(v) - u_q(v)}{\ln(1/\ell)} \\ &\geq \frac{1}{\ln(8k^2)} \left( \frac{1}{2} \text{REV}_{\beta p}(v) - \frac{1}{4} \text{REV}_{\beta p}(v) \right). \end{aligned} \quad (15)$$

Since  $p$  is the optimal buy-many mechanism, it achieves higher revenue than  $\ell p$ , which means  $\mathbb{E}_{v \sim D} [\text{REV}_p(v)] \geq \mathbb{E}_{v \sim D} [\text{REV}_{\ell p}(v)]$ . Taking the expectation over  $v$  on both sides of (15), the same as (6) we have

$$\text{REV}_{\alpha q} \geq \frac{1}{\ln(8k^2)} \left( \frac{1}{2} \text{REV}_p - \frac{1}{4} \text{REV}_{\beta p} \right) = \frac{1}{4 \ln(8k^2)} \text{REV}_p.$$

Thus there exists an item pricing which gives a  $\frac{1}{O(\log k)}$  fraction of the revenue obtained by the optimal buy-many mechanism.  $\square$

## 5.2 A PTAS Algorithm for Computing a Near-Optimal Item Pricing in Partially-Ordered Setting

In this section, we generalize the approximation algorithm for the totally-ordered setting to the partially ordered setting, where the width of the entire set of items is  $k$ . When  $k$  is a constant, the algorithm is a PTAS.

**THEOREM 5.3.** *For a general-valued buyer with partially ordered values, if the partially ordered set containing all items has width  $k$ , then there exists an algorithm running in  $\text{poly}(m, n^{\text{poly}(k, 1/\epsilon)}, \log \mathcal{R})$  time that computes an item pricing that is  $(1 + \epsilon)$ -approximation in revenue to the optimal item pricing.*

**PROOF SKETCH.** The algorithm is similar to the totally-ordered setting, but we need to define the generalized multi-dimensional prefixes and intervals. We also define slightly different interval prefix pricing and additive buyer.

- **Prefix:** For any set  $T \in \mathcal{S}$ , the prefix parameterized by  $T$  is a set of items  $P_T \supseteq T$  such that item  $j \in [n]$  is in  $P_T$  if and only if there exists  $i \in T$  such that  $j \leq i$ . Such a definition ensures that for  $T_1, T_2 \in \mathcal{S}$ ,  $T_1 \leq T_2$ ,  $P_{T_1} \subseteq P_{T_2}$ .
- **Interval:** For two sets  $T, T' \in \mathcal{S}$  with  $T \subseteq T'$ , interval  $I_{T, T'} = P_{T'} \setminus P_T$  is a set of items with contiguous item types between  $T$  and  $T'$  in the ordering graph.
- **Interval prefix pricing:** Let  $I = (I_{T_0, T_1}, I_{T_1, T_2}, \dots, I_{T_{\ell-1}, T_\ell})$  be a partition of the  $n$  items into  $\ell$  intervals, with  $T_0 = \emptyset$ , and  $T_\ell$  be a set of items that dominates all other items (with  $P_{T_\ell} = [n]$ ). An interval prefix pricing  $q$  is a mechanism defined by a vector of item prices  $(q_1, q_2, \dots, q_n)$ : For any set  $S_\ell \subseteq I_\ell$  of items, there is a menu allocating a set of items  $P_{S_\ell}^* = \cup_{1 \leq j \leq \ell-1} T_j \cup S_\ell$ , with price  $q(P_{S_\ell}^*) = \sum_{i \in P_{S_\ell}^*} q_i$ . In other words, to purchase any set of items in  $S_\ell$ , the buyer also needs to purchase all sets of items  $T_1, \dots, T_{\ell-1}$  that define the previous intervals.
- **Additive-over-intervals buyer:** Given an interval partition  $I = (I_{T_0, T_1}, I_{T_1, T_2}, \dots, I_{T_{\ell-1}, T_\ell})$ , for an arbitrary value function  $v$  and any set  $S = S_1 \cup S_2 \cup \dots \cup S_\ell$  of items with  $S_\ell \subseteq I_\ell$  for every  $\ell \in [\ell]$ , define

$$v_I^\oplus(S) = \sum_{\ell=1}^{\ell} (v(S_\ell \cup T_{\ell-1}) - v(T_{\ell-1})).$$

In other words,  $v_I^\oplus(S_\ell)$  is the value gain of getting set  $S_\ell$ , when the buyer  $v$  has a set of items  $T_{\ell-1}$  at hand.  $v$  and  $v_I^\oplus$  has the same behavior under an interval prefix pricing defined by interval partition  $I$ .

The proof for the totally-ordered setting can be generalized to the partially ordered setting, if we use the above generalized definitions of the terms. The key steps are shown as follows.

- (1) There exists a near optimal item pricing where all prices are powers of  $(1 + \epsilon^2)$ . Let  $\Pi = \{(1 + \epsilon^2)^r | r \in \mathbb{Z}\} \cup \{0\}$ . Then for all item pricings  $p$ , there exists  $q^{(1)} \in \Pi^n$ , such that for all value functions  $v$ ,

$$\text{REV}_{q^{(1)}}(v) \geq (1 - O(\epsilon)) \text{REV}_p(v).$$

Furthermore, without loss of generality, we can assume that for the item pricing  $q^{(1)}$  we consider, set  $\{i | q_i^{(1)} \leq y\}$  is a prefix for any  $y \in \mathbb{R}$ .

- (2) At a small loss in revenue, we can restrict prices to lie in a small set. In particular, for all value distributions  $\mathcal{D}$  with value range  $\mathcal{R}$ , there exists an efficiently computable set  $\Pi^* \subset \Pi$  with  $|\Pi^*| = \text{poly}(1/\epsilon, k, \log n, \log \mathcal{R})$  such that for all item pricings  $q^{(1)} \in \Pi^n$ , there exists an item pricing

$q^{(2)} \in \Pi^{*n}$  satisfying

$$\text{REV}_{q^{(2)}}(\mathcal{D}) \geq (1 - O(\epsilon))\text{REV}_{q^{(1)}}(\mathcal{D}).$$

- (3) Given a partition of the  $n$  items into  $t$  intervals,  $I = (I_{T_0, T_1}, I_{T_1, T_2}, \dots, I_{T_{t-1}, T_t})$  with  $T_0 = \emptyset$ , and  $T_t$  be a set of items that dominates all other items (with  $P_{T_t} = [n]$ ). We furthermore say that for any interval partition  $I$ , an item pricing  $q$  satisfies *price gap*  $(\gamma, \delta)$  if (1) items corresponding to different intervals are priced multiplicatively apart: for all  $i, j$ , and  $\ell$  with  $i \in P_{T_\ell}$  and  $j \notin P_{T_\ell}$ ,  $q_j \geq (1 + \epsilon^2)^\gamma q_i$ , (2) and, menu options corresponding to any single interval are priced multiplicatively close to each other: for all  $i, j$ , and  $\ell$  with  $i, j \in I_\ell$ ,  $q_j \leq (1 + \epsilon^2)^\delta q_i$ . We show that for value distribution  $\mathcal{D}$  and item pricing  $q^{(2)} \in \Pi^{*n}$ , there exists an item pricing  $q^{(3)}$  with  $q_i^{(3)} \in \Pi^*$  for all  $i \in [n]$  and price gap  $(\gamma, \delta) = (\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , such that

$$\text{REV}_{q^{(3)}}(\mathcal{D}) \geq (1 - O(\epsilon))\text{REV}_{q^{(2)}}(\mathcal{D}).$$

- (4) We define a new kind of pricing that we will call an *interval prefix pricing*. Given a partition of the  $n$  items into  $t$  intervals  $I = (I_{T_0, T_1}, I_{T_1, T_2}, \dots, I_{T_{t-1}, T_t})$ , an interval prefix pricing  $q$  is a mechanism defined by a vector of item prices  $(q_1, q_2, \dots, q_n)$ : For any set  $S_\ell \subseteq I_\ell$  of items, there is a menu allocating a set of items  $P_{S_\ell}^* = \cup_{1 \leq j \leq \ell-1} T_j \cup S_\ell$ , with price  $q(P_{S_\ell}^*) = \sum_{i \in P_{S_\ell}^*} q_i$ . In other words, to purchase any set of items in  $S_\ell$ , the buyer also needs to purchase all sets of items  $T_1, \dots, T_{\ell-1}$  that define the previous intervals. We show that for every value function  $v$  and item pricing  $q^{(3)} \in \Pi^{*n}$  with price gap  $(\gamma, \delta) = (\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , there exists an efficiently computable set  $\Pi'$  with  $|\Pi'| = |\Pi^*|$  and an interval prefix pricing  $q^{(4)}$  with  $q_i^{(4)} \in \Pi'$  for all  $i \in [n]$  and price gap  $(\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , such that

$$\text{REV}_{q^{(4)}}(v) \geq (1 - O(\epsilon))\text{REV}_{q^{(3)}}(v).$$

The converse is also true: for every value function  $v$  and interval prefix pricing  $q$  with price gap  $(\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , we can efficiently compute an item pricing  $q^{(5)}$  such that

$$\text{REV}_{q^{(5)}}(v) \geq (1 - O(\epsilon))\text{REV}_q(v).$$

- (5) We define for each arbitrary-valued buyer  $v$  and interval partitioning  $I$  an *additive-over-intervals* value function that closely mimics it. For an arbitrary value function  $v$ , for any set  $S = S_1 \cup S_2 \cup \dots \cup S_t$  of items with  $S_\ell \subseteq I_\ell$  for every  $\ell \in [t]$ , define

$$v_I^\oplus(S) = \sum_{\ell=1}^t (v(S_\ell \cup T_{\ell-1}) - v(T_{\ell-1})).$$

In other words,  $v_I^\oplus(S_\ell)$  is the value gain of getting set  $S_\ell$ , when the buyer  $v$  has set of items  $T_{\ell-1}$  at hand. We write  $\mathcal{D}_I^\oplus$  as the distribution of  $v_I^\oplus$  corresponding to  $v \sim \mathcal{D}$ .  $v$  and  $v_I^\oplus$  have the same behavior under an interval prefix pricing defined by interval partition  $I$ . When the interval partition is clear from the context, we will omit  $I$  and write  $v^\oplus = v_I^\oplus$ .

We show that for every additive-over-intervals value function  $v^\oplus$  and interval prefix pricing  $q^{(4)}$  with  $q_i^{(4)} \in \Pi'$ , and with price gap  $(\gamma, \delta) = (\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , there exists an efficiently computable set  $\Pi^o$  with  $|\Pi^o| = |\Pi^*|$  and an item  $q^{(6)}$  with  $q_i^{(6)} \in \Pi^o$  for all  $i \in [n]$  and price gap  $(\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , such that

$$\text{REV}_{q^{(6)}}(v^\oplus) \geq (1 - O(\epsilon))\text{REV}_{q^{(4)}}(v).$$

The converse also holds: for every value function  $v$  and item pricing  $q$  with price gap  $(\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , we can efficiently compute an interval prefix pricing  $q^{(7)}$  with price gap  $(\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$  such that

$$\text{REV}_{q^{(7)}}(v) \geq (1 - O(\epsilon))\text{REV}_q(v^\oplus).$$

- (6) Finally, we show that for any distribution  $\mathcal{D}$  over arbitrary values and any set  $\Pi^o$  of values, an optimal item pricing  $q$  for value distribution  $\mathcal{D}_I^\oplus$  and the corresponding interval partition  $I$ , with  $q_i \in \Pi^o$  for all  $i \in [n]$  and price gap  $(\frac{1}{\epsilon^2} \ln \frac{k}{\epsilon^2}, \frac{k}{\epsilon^3} \ln \frac{k}{\epsilon^2})$ , can be found in time polynomial in  $|\Pi^o|$ ,  $n^{k \cdot \text{poly}(1/\epsilon)}$ , and  $m$ .

The complete reasoning is similar to the proof of Theorem 4.6 and is omitted.  $\square$

We further notice that for a unit-demand buyer, the dependency on  $\log \mathcal{R}$  can be removed. This is enabled via the following lemma.

**LEMMA 5.4.** *Let  $V$  be a set of non-negative real numbers. Then we can efficiently find a set of power-of- $(1 + \epsilon^2)$  prices  $V'$  with  $|V'| = O(|V|^2 \frac{1}{\epsilon^2} \ln \frac{1}{\epsilon^2})$  satisfying the following: For any buyer that is unit-demand over  $n$  items such that for any buyer type  $v$  and item  $i$  the buyer has value  $v_i \in V$ , the optimal power-of- $(1 + \epsilon^2)$  item pricing  $p$  satisfy  $p_i \in V'$  for any  $i \in [n]$ .*

Given the lemma, since there are at most  $mn$  different item values in the input, we have  $|V| = mn$  in the lemma. The lemma can replace Step 2 in the proof of Theorem 4.6 to remove the running time's dependency on  $\log \mathcal{R}$ . It can also be applied to Theorem 5.3 for a unit-demand buyer. The proof of Lemma 5.4 is omitted.

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## A OMITTED PROOFS FOR THE FEDEX SETTING

**LEMMA 3.4.** *There exists an optimal item pricing such that  $p_1 \in \Pi_L$ , and  $p_i \in \Pi^*$  for each  $i \geq 2$ .*

**PROOF.** We start with an arbitrary optimal item pricing  $p$  with monotone item prices. Notice that a buyer  $v$  will either purchase item 1 or item  $i_v$  due to the monotonicity of item prices.

If  $p_1 \notin \Pi_L$ , and  $p_1 = p_2 = \dots = p_\ell < p_{\ell+1}$ , suppose that we raise the price of item 1, 2,  $\dots$ ,  $\ell$  by a small enough  $\epsilon$ . For a buyer of type  $v$ , if the buyer prefers to purchase item 1 previously, then

$v_1 \geq p_1$ . Since  $p_1 \notin \Pi_L$ , and  $v_1 \in \Pi_L$ , we know that  $v_1 > p_1$ , thus  $v_1 \geq p_1 + \epsilon$  for a small enough  $\epsilon$ . Thus after the perturbation, she either still prefers to purchase item 1 and pay  $\epsilon$  more, or she will switch to purchase a more expensive item, which means that her payment increases. For any buyer that does not prefer to purchase item 1 before perturbation, her preferred item does not change after the price changes. Therefore, after the operation, the total revenue does not decrease. The value  $\epsilon$  can be chosen so as to enforce either  $p_1 + \epsilon \in \Pi_L$ , or  $p_1 + \epsilon = p_{\ell+1}$ . By repeating this operation, we will have  $p_1 = p_2 = \dots = p_j \in \Pi_L$  for some  $j \in [n]$ , while maintaining the optimality of  $p$ .

If  $\ell$  is the smallest index such that  $p_\ell \notin \Pi^*$ , and  $p_\ell = p_{\ell+1} = \dots = p_j < p_{j+1}$ , suppose that we raise the price of items  $\ell, \ell+1, \dots, j$  by a small enough  $\epsilon$ . For a buyer of type  $v$ , if  $i_v \notin [\ell, j]$ , the buyer's incentive does not change after the perturbation. Otherwise when  $\ell \leq i_v \leq j$ , if the buyer prefers to purchase item  $i_v$  before the perturbation, then  $v_{i_v} - p_{i_v} \geq v_1 - p_1$ , and  $v_{i_v} \geq p_{i_v}$ . Since  $p_1 \in \Pi_L$ ,  $v_{i_v} - v_1 + p_1$  and  $v_{i_v}$  are both in  $\Pi^*$ . By  $p_{i_v} = p_\ell \notin \Pi^*$ ,  $v_{i_v} - p_{i_v} > v_1 - p_1$  and  $v_{i_v} > p_{i_v}$ . Therefore after the perturbation, for small enough  $\epsilon$ , she still prefers to purchase item  $i_v$  and pay  $\epsilon$  more. Therefore, after the operation, the total revenue does not decrease. The value  $\epsilon$  can be as large as making  $p_\ell + \epsilon \in \Pi^*$ , or making  $p_\ell + \epsilon = p_{j+1}$ . By repeating this operation, we will have  $p_i \in \Pi^*$  for each  $i \geq 2$ , while maintaining the optimality of  $p$ . This finishes the proof of the lemma.  $\square$

## REFERENCES

- [1] Moshe Babaioff, Nicole Immorlica, Brendan Lucier, and S Matthew Weinberg. 2014. A simple and approximately optimal mechanism for an additive buyer. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*. IEEE, 21–30.
- [2] Maria-Florina Balcan and Avrim Blum. 2006. Approximation algorithms and online mechanisms for item pricing. In *Proceedings of the 7th ACM Conference on Electronic Commerce*. 29–35.
- [3] Patrick Briest. 2008. Uniform budgets and the envy-free pricing problem. In *International Colloquium on Automata, Languages, and Programming*. Springer, 808–819.
- [4] Patrick Briest, Shuchi Chawla, Robert Kleinberg, and S Matthew Weinberg. 2015. Pricing lotteries. *Journal of Economic Theory* 156 (2015), 144–174.
- [5] Patrick Briest and Piotr Krysta. 2006. Single-minded unlimited supply pricing on sparse instances. In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*. 1093–1102.
- [6] Yang Cai and Constantinos Daskalakis. 2011. Extreme-value theorems for optimal multidimensional pricing. In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*. IEEE, 522–531.
- [7] Yang Cai and Mingfei Zhao. 2017. Simple mechanisms for subadditive buyers via duality. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*. 170–183.
- [8] Parinya Chalermsook, Julia Chuzhoy, Sampath Kannan, and Sanjeev Khanna. 2012. Improved hardness results for profit maximization pricing problems with unlimited supply. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*. Springer, 73–84.
- [9] Parinya Chalermsook, Bundit Laekhanukit, and Danupon Nanongkai. 2013. Graph products revisited: Tight approximation hardness of induced matching, poset dimension and more. In *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms*. SIAM, 1557–1576.
- [10] Parinya Chalermsook, Bundit Laekhanukit, and Danupon Nanongkai. 2013. Independent set, induced matching, and pricing: Connections and tight (subexponential time) approximation hardnesses. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*. IEEE, 370–379.
- [11] Shuchi Chawla, Jason D Hartline, and Robert Kleinberg. 2007. Algorithmic pricing via virtual valuations. In *Proceedings of the 8th ACM Conference on Electronic Commerce*. 243–251.
- [12] Shuchi Chawla, Jason D Hartline, David L Malec, and Balasubramanian Sivan. 2010. Multi-parameter mechanism design and sequential posted pricing. In *Proceedings of the forty-second ACM symposium on Theory of computing*. 311–320.
- [13] Shuchi Chawla, David Malec, and Balasubramanian Sivan. 2015. The power of randomness in bayesian optimal mechanism design. *Games and Economic Behavior* 91 (2015), 297–317.
- [14] Shuchi Chawla and J Benjamin Miller. 2016. Mechanism design for subadditive agents via an ex ante relaxation. In *Proceedings of the 2016 ACM Conference on Economics and Computation*. 579–596.
- [15] Shuchi Chawla, Yifeng Teng, and Christos Tzamos. 2019. Buy-Many Mechanisms are Not Much Better than Item Pricing. In *Proceedings of the 2019 ACM Conference on Economics and Computation*. 237–238.
- [16] Shuchi Chawla, Yifeng Teng, and Christos Tzamos. 2020. Menu-size complexity and revenue continuity of buy-many mechanisms. In *Proceedings of the 21st ACM Conference on Economics and Computation*. 475–476.
- [17] Xi Chen, Ilias Diakonikolas, Anthi Orfanou, Dimitris Paparas, Xiaorui Sun, and Mihalis Yannakakis. 2015. On the complexity of optimal lottery pricing and randomized mechanisms. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*. IEEE, 1464–1479.
- [18] Xi Chen, Ilias Diakonikolas, Dimitris Paparas, Xiaorui Sun, and Mihalis Yannakakis. 2014. The complexity of optimal multidimensional pricing. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms*. SIAM, 1319–1328.
- [19] Nikhil R Devanur, Kira Goldner, Raghuvansh R Saxena, Ariel Schwartzman, and S Matthew Weinberg. 2020. Optimal mechanism design for single-minded agents. In *Proceedings of the 21st ACM Conference on Economics and Computation*. 193–256.
- [20] Nikhil R Devanur and S Matthew Weinberg. 2017. The optimal mechanism for selling to a budget constrained buyer: The general case. In *Proceedings of the 2017 ACM Conference on Economics and Computation*. 39–40.
- [21] Paul Dütting, Thomas Kesselheim, and Brendan Lucier. 2020. An  $O(\log \log m)$  Prophet Inequality for Subadditive Combinatorial Auctions. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. 306–317. <https://doi.org/10.1109/FOCS46700.2020.00037>
- [22] Khaled Elbassioni, Rajiv Raman, Saurabh Ray, and René Sitters. 2009. On profit-maximizing pricing for the highway and tollbooth problems. In *International Symposium on Algorithmic Game Theory*. Springer, 275–286.
- [23] Amos Fiat, Kira Goldner, Anna R Karlin, and Elias Koutsoupias. 2016. The FedEx Problem. In *Proceedings of the 2016 ACM Conference on Economics and Computation*. 21–22.
- [24] Iftah Gamzu and Danny Segev. 2010. A sublogarithmic approximation for highway and tollbooth pricing. In *International Colloquium on Automata, Languages, and Programming*. Springer, 582–593.
- [25] Yannai A Gonczarowski and S Matthew Weinberg. 2021. The sample complexity of up-to- $\epsilon$  multi-dimensional revenue maximization. *Journal of the ACM (JACM)* 68, 3 (2021), 1–28.
- [26] Fabrizio Grandoni and Thomas Rothvoß. 2016. Pricing on paths: A ptas for the highway problem. *SIAM J. Comput.* 45, 2 (2016), 216–231.
- [27] Venkatesan Guruswami, Jason D Hartline, Anna R Karlin, David Kempe, Claire Kenyon, and Frank McSherry. 2005. On profit-maximizing envy-free pricing.. In *SODA*, Vol. 5. Citeseer, 1164–1173.
- [28] Sergiu Hart, Noam Nisan, et al. 2013. *The menu-size complexity of auctions*. Center for the Study of Rationality.
- [29] Rohit Khandekar, Tracy Kimbrel, Konstantin Makarychev, and Maxim Sviridenko. 2009. On hardness of pricing items for single-minded bidders. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*. Springer, 202–216.
- [30] Praveesh Kothari, Sahil Singla, Divyarthi Mohan, Ariel Schwartzman, and S Matthew Weinberg. 2019. Approximation Schemes for a Unit-Demand Buyer with Independent Items via Symmetries. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE, 220–232.
- [31] Euiwoong Lee. 2015. Hardness of graph pricing through generalized max-dicut. In *Proceedings of the forty-seventh annual ACM symposium on Theory of Computing*. 391–399.
- [32] Xinye Li and Andrew Chi-Chih Yao. 2013. On revenue maximization for selling multiple independently distributed items. *Proceedings of the National Academy of Sciences* 110, 28 (2013), 11232–11237.
- [33] Jean-Charles Rochet. 1985. The taxation principle and multi-time Hamilton-Jacobi equations. *Journal of Mathematical Economics* 14, 2 (1985), 113–128.
- [34] Aviad Rubinstein and S Matthew Weinberg. 2018. Simple mechanisms for a subadditive buyer and applications to revenue monotonicity. *ACM Transactions on Economics and Computation (TEAC)* 6, 3-4 (2018), 1–25.
- [35] Raghuvansh R Saxena, Ariel Schwartzman, and S Matthew Weinberg. 2018. The menu complexity of “one-and-a-half-dimensional” mechanism design. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 2026–2035.
- [36] Andrew Chi-Chih Yao. 2014. An n-to-1 bidder reduction for multi-item auctions and its applications. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*. SIAM, 92–109.