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FINITE DIMENSIONAL MODELS FOR RANDOM MICROSTRUCTURES

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Abstract. Finite dimensional (FD) models, i.e., deterministic functions of space depending on finite sets of random variables, are used extensively in applications to generate samples of random fields $Z(x)$ and construct approximations of solutions $U(x)$ of ordinary or partial differential equations whose random coefficients depend on $Z(x)$. FD models of $Z(x)$ and $U(x)$ constitute surrogates of these random fields which target various properties, e.g., mean/correlation functions or sample properties. We establish conditions under which samples of FD models can be used as substitutes for samples of $Z(x)$ and $U(x)$ for two types of random fields $Z(x)$ and a simple stochastic equation. Some of these conditions are illustrated by numerical examples.

1. Introduction

Material properties exhibit random spatial fluctuations which can be represented by scalar-/vector-/matrix-valued random fields $\{Z(x),\ x \in D\}$, where $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded subset specifying the domain of a material specimen. Stresses, strains and other material responses $\{U(x),\ x \in D\}$ to boundary conditions and other actions satisfy ordinary or partial differential equations with random coefficients which depend on $Z(x)$. Analytical solutions of these stochastic equations are possible in simple cases of limited practical interest. Generally, numerical methods have to be employed for solution.

The implementation of numerical methods requires to discretize both the physical and the probability spaces. The finite element/difference methods are the standard tools for discretizing the physical space. *Finite dimensional* (FD) models are commonly used to discretize the probability space. They are deterministic functions of finitely many arguments of which some are random variables. For example, the material random field $Z(x)$, which is an uncountable family of random elements indexed by $x \in D$, can be represented by FD random fields $\{Z_n(x),\ x \in D\}$, i.e., deterministic functions of $x \in D$ which depend on n random variables $(Z_1, \ldots, Z_n)$, see [10] for the construction of these models. We say that $Z(x)$ has *infinite stochastic dimension* while $Z_n(x)$ has *finite stochastic dimension* equal to the number n of random variables in its definition. Denote by $U_n(x)$ the solution of the defining equation of $U(x)$ with $Z_n(x)$ in place of $Z(x)$. The random fields $U(x)$ and $U_n(x)$, which are complex functionals of $Z(x)$ and $Z_n(x)$, are referred to as *analytical/target* and *numerical/approximate* solutions of material responses. They differ since $Z(x)$ and $Z_n(x)$ differ. The size of the discrepancy between $U(x)$ and $U_n(x)$ depends on that between $Z(x)$ and $Z_n(x)$ and the structure of the defining equation for material responses.

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The discussion is limited to material properties which vary continuously over the specimen domain, so that the functions describing these properties are elements of the space of scalar/vector-valued continuous functions $C(D)$ defined on bounded subsets D of $\mathbb{R}^d$, $d = 1, 2, 3$. To satisfy this requirement, $Z(x)$ is assumed to be homogeneous with continuous samples, i.e., its samples are elements of $C(D)$ and its statistics are invariant to space shift. To satisfy physics constraints, $Z(x)$ is assumed to be a non-Gaussian translation field, i.e., $Z(x) = h\big(G(x)\big)$, $x \in D$, where $h : C(D) \to C(D)$ is continuous in the topology induced by the 'sup' metric of $C(D)$ and $G(x)$ is a zero-mean homogeneous Gaussian field with continuous samples defined on a probability space $\big(\Omega, \mathcal{F}, P\big)$. Since h is continuous, it is measurable from $\big(C(D), \mathcal{C}\big)$ to $\big(C(D), \mathcal{C}\big)$, where $\mathcal{C}$ is the Borel σ-field generated by the open sets $\mathcal{D}$ of $C(D)$. The probability measures induced by the random fields $G(x)$ and $Z(x)$ on $\big(C(D), \mathcal{C}\big)$ are $P\big(G^{-1}(A)\big)$ and $P\big(G^{-1}(Z^{-1}(A))\big)$, $A \in \mathcal{C}$.

We establish conditions under which samples of the FD random fields $Z_n(x)$ and $U_n(x)$, which can be constructed numerically, can be used as approximations for samples of the target random fields $Z(x)$ and $U(x)$. To establish these conditions, we examine the weak and almost sure (a.s) convergences of $Z_n(x)$ to $Z(x)$ in the space of continuous functions $C(D)$. The weak convergence, denoted by $Z_n \Longrightarrow Z$, means for real-valued random fields that $P\big(\sup_{x\in D} |Z(x) - Z_n(x)| > \varepsilon\big) \to 0$ as $n \to \infty$ for any $\varepsilon > 0$ so that the probability measure of the "bad subset" $\Omega_n(\varepsilon) = \{\omega \in \Omega : \sup_{x\in D} |Z(x,\omega) - Z_n(x,\omega)| > \varepsilon\}$ of the sample space Ω which contains pairs of target and FD samples which differ by more than ε in the metric of $C(D)$, is small for sufficiently large n. If $Z_n \Longrightarrow Z$, the subsets $\Omega_n(\varepsilon)$ and $\Omega_m(\varepsilon)$ for large $m \neq n$ have small measures but, generally, differ. The almost sure (a.s.) convergence, denoted by $Z_n \overset{\text{a.s.}}{\to} Z$, implies that once the measure of $\Omega_n(\varepsilon)$ gets small for some n_0 it remains small and its measure decreases as $n \geq n_0$ increases. This type of convergence is desirable since it guarantees that the accuracy of the FD models improves with n. We also examine the convergence of the FD solution $U_n(x)$ to $U(x)$ for a simple stochastic problem, the one-dimensional transport equation.

The paper is organized as follows. Section 2 deals with scalar-valued material random fields $Z(x)$. Mean square periodic and bounded frequency range fields are discussed in Sects. 2.1 and 2.2. Finite dimensional (FD) models $\{Z_n(x)\}$ of $Z(x)$ and theorems for their weak and a.s. convergence to $Z(x)$ are in Sects. 2.1.1-2.1.2 and Sects. 2.2.1-2.2.2, respectively. Section 2.3 illustrates numerically some of the theoretical results of Sects. 2.1 and 2.2. Section 3 deals with vector-valued material random fields $Z(x)$ and follows the structure of Sect. 2. Mean square periodic and bounded frequency range fields are in Sects. 3.1 and 3.2. Finite dimensional (FD) models $\{Z_n(x)\}$ of $Z(x)$ and theorems for their weak and a.s. convergence to $Z(x)$ are in Sects. 3.1.1-3.1.2 and Sects. 3.2.1-3.2.2, respectively. Section 3.3 applies results of Sects. 3.1-3.2 to construct models for compliance tensors and present numerical results in Sects. 3.3.1 and 3.3.2. Section 4 examines the response of a simple stochastic differential equation, the one-dimensional transport equation. Some final comments are in Sect. 5.

2. Scalar-valued microstructure models

Most of the results in this section related on properties of random fields with finite variance are available in the literature, e.g., [1, 12, 21]. The presentation of these properties follows closely these references.

Let $Z(x)$, $x \in \mathbb{R}^d$, be a real-valued random field defined on a probability space $\big(\Omega, \mathcal{F}, P\big)$ with zero mean and finite variance. The assumption $E[Z(x)] = 0$ is not restrictive since, if $E[Z(x)] \neq 0$, it can be added to the samples of $Z(x)$. If D is a closed and bounded subset of $\mathbb{R}^d$ and the correlation function $c(x,y) = E[Z(x)\,Z(y)]$ is continuous on $D \times D$ and, therefore, square integrable on this subset, then $Z(x) = \sum_{k=1}^{\infty} \lambda_k^{1/2}\, \xi_k\, v_k(x)$

and $c(x,y) = \sum_{k=1}^{\infty} \lambda_k \, v_k(x) \, v_k(y)$, where $\{\lambda_k\}$ and $\{v_k(x)\}$ are the eigenvalues and eigenfunctions of $c(x,y)$ in $D \times D$ and $\{\xi_k\}$ are uncorrelated random variables with zero means and unit variances. The series representation of $Z(x)$ converges in mean square (m.s.) and that of $c(x,y)$ converges absolutely and uniformly by Mercer's theorem [6] (Sects. IV.1 and IV.3). It is assumed that the real-valued random field $Z(x)$ has the following properties 1 and 2 and property 3 or 4.

1. *Weakly homogeneous*, i.e., the correlation function $c(\xi) = E[Z(x+\xi)\,Z(x)]$, $\xi = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$, depends only on the lag ξ. The correlation function $c(\xi)$ and the spectral density $s(\nu)$, $\nu = (\nu_1,\ldots,\nu_d)$, provided it exists, are related by [1](Chap. 2)

$$c(\xi) = \int_{\mathbb{R}^d} e^{i\,\xi\cdot\nu}\, dS(\nu) = \int_{\mathbb{R}^d} e^{i\,\xi\cdot\nu}\, s(\nu)\, d\nu, \quad \text{and } s(\nu) = \frac{1}{(2\,\pi)^d}\int_{\mathbb{R}^d} e^{-i\,\xi\cdot\nu}\, c(\xi)\, d\xi, \tag{2.1}$$

where $S(\nu)$ is a bounded real-valued measure such that $\int_\Lambda dS(\nu) \geq 0$ for all Borel measurable $\Lambda \subset \mathbb{R}^d$. If $S(\nu)$ is absolutely continuous with respect to the Lebesgue measure, then the spectral density $s(\nu)$ exists and $dS(\nu) = s(\nu)\,d\nu$ [1] (Theorem 2.1.2). It is assumed that $S(\nu)$ has this property so that the spectral density exists. Since $Z(x)$ is real-valued, its correlation function and spectral density are real-valued even functions, so that Eq. 2.1 can be given in the form $c(\xi) = \int_{\mathbb{R}^d} \cos(\xi\cdot\nu)\, s(\nu)\, d\nu$ [1] (Sect. 2.4)

The field $Z(x)$ admits the spectral representation

$$Z(x) = \int_{\mathbb{R}^d} e^{i\,\nu\cdot x}\, dW(\nu), \tag{2.2}$$

where $W(\nu)$ is a zero-mean, complex-valued process with orthogonal increments whose first two moments are $E[dW(\nu)] = 0$ and $E[|dW(\nu)|^2] = E[dW(\nu)\,dW(\nu)^*] = dS(\nu) = s(\nu)\,d\nu$. Since $Z(x)$ is real, the above spectral representation takes the form [1] (Sect. 2.3)

$$Z(x) = \int_{\mathbb{R}^d} \big(\cos(\nu\cdot x)\, dU(\nu) - \sin(\nu\cdot x)\, dV(\nu)\big), \tag{2.3}$$

as the imaginary part $\int_{\mathbb{R}^d} \big(\cos(\nu\cdot x)\, dV(\nu) + \sin(\nu\cdot x)\, dU(\nu)\big)$ of the right side of Eq. 2.2 must vanish, where $U(\nu)$ and $V(\nu)$ are zero-mean, real-valued random fields with orthogonal increments of moments $E[dU(\nu)] = E[dV(\nu)] = 0$, $E[dU(\nu)\,dV(\nu')] = 0$ for all ν, ν' and $E[|dU(\nu)|^2] = E[|dV(\nu)|^2] = dS(\nu) = s(\nu)\,d\nu$, see [11] (Theorem 13).

2. *Continuous*, i.e., the samples $Z(x,\omega)$ of $Z(x)$ are real-valued continuous functions for almost all $\omega \in \Omega$ so that almost all samples of $Z(x)$ are members of $C(D)$.

3. *Mean square (m.s.) periodic*, i.e., the statistics of $Z(x)$ repeat over bounded rectangles. A precise definition is in the subsequent subsection. Periodic material properties are commonly used in mechanics to characterize large or infinite material specimens by their properties over finite subsets, referred to as unit cells [15] (Chap. 3).

4. *Bounded frequency range*, i.e., the support of the spectral density $s(\nu)$ in Eq. 2.1 is a bounded rectangle D_ν of $\mathbb{R}^d$, a common assumption in applications which seems to be consistent with physics.

2.1. **Mean square periodic fields.** For simplicity, we limit our discussion to two-dimensional material specimens in bounded rectangular domains $D = [-a_1, a_1] \times [-a_2, a_2]$ or $D = [0, T_1] \times [0, T_2]$, where $T_i = 2\,a_i$, $i = 1, 2$, whose properties are described by real-valued, weakly homogeneous random fields $Z(x)$, $x \in \mathbb{R}^2$, which are periodic in the sense of the following definition.

Definition 2.1. The random field $Z(x)$ is said to be D- or (T_1, T_2)-*m.s. periodic* if

$$E\big[\big(Z(x_1+T_1, x_2+T_2) - Z(x_1,x_2)\big)^2\big] = 0, \quad x = (x_1,x_2) \in \mathbb{R}^2. \tag{2.4}$$

This definition extends directly to real-valued random fields defined on $\mathbb{R}^d$, $d > 2$.

Theorem 2.2. *If $Z(x)$ is D-m.s. periodic, then*

$$E\big[\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1, x_2)\big)^2\big] = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.5}$$

and the correlation function is (T_1, T_2)-periodic, i.e.,

$$c(\xi_1, \xi_2) = E\big[Z(x_1 + \xi_1 + k_1\,T_1, x_2 + \xi_2 + k_2\,T_2)\,Z(x_1, x_2)\big], \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{2.6}$$

for any integers k_1 and k_2.

Proof. We have

$$\begin{aligned}
&E\big[\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1, x_2)\big)^2\big] \\
&= E\bigg[\bigg(\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2)\big) \\
&+ \big(Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2) - Z(x_1, x_2)\big)\bigg)^2\bigg] \\
&= 2\,E\big[\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2) \\
&\times\ \big(Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2) - Z(x_1, x_2)\big)\big] \\
&+ E\big[\big(Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2) - Z(x_1, x_2)\big)^2\big],
\end{aligned}$$

since $E\big[\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2)\big)^2\big] = 0$ by the definition of m.s. periodicity. Also, $\big|E\big[\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2)\,\big(Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2) - Z(x_1, x_2)\big)\big]\big| = 0$ by the properties of $Z(x)$ and the Cauchy-Schwarz inequality. Accordingly, the above equality yields the recursive formula $E\big[\big(Z(x_1 + k_1\,T_1, x_2 + k_2\,T_2) - Z(x_1, x_2)\big)^2\big] = E\big[\big(Z(x_1 + (k_1 - 1)\,T_1, x_2 + (k_2 - 1)\,T_2) - Z(x_1, x_2)\big)^2\big]$, which implies the stated property. Similar arguments yield the periodicity of the correlation function of $Z(x)$. □

The properties in Eqs. 2.5 and 2.6 extend directly to vector-valued random fields with m.s. periodic components of the same periods T_1 and T_2, as shown in a subsequent subsection. They also hold for matrix-valued random fields since they can be reset as vector-valued random fields.

Theorem 2.3. *If the partial derivatives $\partial c(\xi)/\partial \xi_1$, $\partial c(\xi)/\partial \xi_2$ and $\partial^2 c(\xi)/\partial \xi_1\,\partial \xi_2$ of the correlation function $c(\xi) = E[Z(x + \xi)\,Z(x)]$ of a weakly homogeneous, D-m.s. periodic random field $Z(x)$ are continuous in D, then its Fourier series*

$$c(\xi) = \sum_{k,l} s_{kl}\,e^{i\,\nu_{kl}\cdot\xi} \quad \text{with } s_{kl} = \frac{1}{4\,a_1\,a_2}\int_D c(\xi)\,e^{-i\,\nu_{kl}\cdot\xi}\,d\xi, \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{2.7}$$

converges absolutely and uniformly, where $\sum_{k,l} := \sum_{k,l=\pm1,\pm2,\ldots}$, $\nu_{i,1} = 2\,\pi/T_i = \pi/a_i$, $\nu_{i,k} = k\,\nu_{i,1}$ for $i = 1, 2$ and $k = \pm1, \pm2, \ldots$ and $\nu_{kl} = (\nu_{1,k}, \nu_{2,l})$. The Fourier coefficients $\{s_{kl}\}$ are real-valued and $s_{kl} = s_{-k,-l}$.

Proof. Under the stated conditions, the Fourier series of the correlation function of $Z(x)$ converges to $c(\xi)$ absolutely and uniformly and has the form in Eq. 2.7 [19] (Sects. 7.1 to 7.3). That s_{kl} is real-valued follows by calculating the integral in Eq. 2.7 as the sums of the integrals over $D_1 \cup D_1'$ and $D_2 \cup D_2'$, where $D_1 = (0, a_1) \times (0, a_2)$, $D_1' = (-a_1, 0) \times (-a_2, 0)$, $D_2 = (0, a_1) \times (-a_2, 0)$, $D_2' = (-a_1, 0) \times (0, a_2)$. For example, $\int_{D_1 \cup D_1'} c(\xi)\,\exp(-i\,\nu_{kl}\cdot\xi)\,d\xi = 2\int_{D_1} c(\xi)\,\cos(\nu_{kl}\cdot\xi)\,d\xi$, which is real since the correlation function of $Z(x)$ is real-valued, see also Eq. 2.1. The Fourier coefficient $s_{-k,-l}$ corresponding to the frequency $\nu_{-k,-l} = (-\nu_{1,k}, -\nu_{2,l})$ coincides with s_{kl}, as it results by using the change of variables $\xi = -\eta$ in the integral of Eq. 2.7. □

This section defines FD models $Z_{m.n}(x)$ of $Z(x)$, establishes conditions under which the finite dimensional distributions of $Z_{m.n}(x)$ converge to those of $Z(x)$, develops criteria for the tightness of the sequence $Z_{m.n}(x)$ of FD models and establishes conditions for the weak convergence of the sequence of fields $Z_{m.n}(x)$ to $Z(x)$ in the space $C(D)$ of real-valued continuous functions endowed with the uniform metric.

2.1.1. *Finite dimensional (FD) models.* The spectral representation of $Z(x)$ in Eq. 2.3 for m.s. periodic random fields takes the form

$$Z(x) = \sum_{k,l} \big[U_{kl}\,\cos(\nu_{kl}\cdot x) - V_{kl}\,\sin(\nu_{kl}\cdot x)\big], \quad x \in \mathbb{R}^2, \tag{2.8}$$

where $\{U_{kl}\}$ and $\{V_{kl}\}$ are zero-mean uncorrelated random variables with variances $\{s_{kl}\}$, see Eq. 2.3. Consider the family of finite dimensional (FD) random fields,

$$Z_{m,n}(x) = \sum_{|k|\le m, |l|\le n} \big[U_{kl}\,\cos(\nu_{kl}\cdot x) - V_{kl}\,\sin(\nu_{kl}\cdot x)\big], \quad x \in \mathbb{R}^2, \tag{2.9}$$

obtained by truncation of the infinite series representation of $Z(x)$ in Eq. 2.8. These FD models are deterministic functions of $x \in D$ which depend on finite sets of random variables. The random fields $Z_{m,n}(x)$ are weakly homogeneous since $E[Z_{m,n}(x)] = 0$ and

$$\begin{aligned} c_{m,n}(x,y) = E\big[Z_{m,n}(x)\,Z_{m,n}(y)\big] &= \sum_{|k|\le m, |l|\le n} s_{kl}\,\big(\cos(\nu_{kl}\cdot x)\,\cos(\nu_{kl}\cdot y) + \\ &+ \sin(\nu_{kl}\cdot x)\,\sin(\nu_{kl}\cdot y)\big) = \sum_{|k|\le m, |l|\le n} s_{kl}\,\cos\big(\nu_{kl}\cdot(x-y)\big). \end{aligned} \tag{2.10}$$

This correlation function converges absolutely and uniformly to the correlation function $c(\xi)$ as $m, n \to \infty$ by Mercer's theorem [12] (Sect. 6.2) since the spectral representation of $Z(x)$ in Eq. 2.9 constitutes a truncated Karhunen-Loève expansion of $Z(x)$ as the trigonometric functions of this representation are the eigenfunctions of the correlation function of $Z(x)$.

The random fields $Z(x)$ and $Z_{m,n}(x)$ in Eqs. 2.8 and 2.9 are defined in the second moment sense, i.e., only their mean and correlation functions are known, unless $Z(x)$ is Gaussian in which case $\{U_{kl}\}$ and $\{V_{kl}\}$ are independent Gaussian variables. If $Z(x)$ is not Gaussian, $\{U_{kl}\}$ and $\{V_{kl}\}$ are uncorrelated but dependent non-Gaussian variables with unknown distributions. For these fields, we construct samples of the random variables $\{U_{kl}(\omega)\}$ and $\{V_{kl}(\omega)\}$ from samples $Z(x,\omega)$, $\omega \in \Omega$, of $Z(x)$ by projection, i.e., $\langle Z(\cdot,\omega), \cos(\nu_{kl}\cdot\cdot)\rangle = U_{kl}(\omega)\,\int_D \cos^2(\nu_{kl}\cdot x)\,dx$, so that

$$U_{kl}(\omega) = \frac{1}{2\,a_1\,a_2}\int_D Z(x,\omega)\,\cos(\nu_{kl}\cdot x)\,dx \quad \text{and } V_{kl}(\omega) = \frac{1}{2\,a_1\,a_2}\int_D Z(x,\omega)\,\sin(\nu_{kl}\cdot x)\,dx. \tag{2.11}$$

This construction pairs target and FD samples, i.e., the samples of $Z(x,\omega)$ of $Z(x)$ with the samples $Z_{m.n}(x,\omega)$ of $Z_{m,n}(x)$, via the samples $U_{kl}(\omega)$ and $V_{kl}(\omega)$ of the random coefficients of FD fields. It can be used for Gaussian and non-Gaussian fields.

Note also that the random fields $Z(x)$ and $Z_{m,n}(x)$ depend on countable sets of random variables, the random coefficients $\{U_{kl}\}$ and $\{V_{kl}\}$. These sets are infinite for $Z(x)$ and finite for $Z_{m,n}(x)$. Accordingly, $Z_{m,n}(x)$ can be used in numerical calculations while $Z(x)$ cannot. The next theorem shows that the finite dimensional distributions of $\{Z_{m,n}(x)\}$ converge to those of $Z(x)$ as $m.n \to \infty$ under some conditions.

Theorem 2.4. *If the correlation function of $Z(x)$ is continuous in $D \times D$, the finite dimensional distributions of $Z_{m,n}(x)$ converge to those of $Z(x)$ as $m, n \to \infty$.*

Proof. First note that the correlation function is square integrable since it is continuous and D is compact. For a fixed arbitrary $x \in \mathbb{R}^2$, the sequence of random variables $Z_{m,n}(x)$ is m.s. Cauchy by Mercer's theorem ([6], Sect. IV.2, and [12], Sect. 6.2) so that $Z_{m,n}(x)$ converges in m.s. to $Z(x)$ as $m,n \to \infty$ and, therefore, in distribution by Chebychev's inequality. Consider now the random vectors $\mathcal{Z}_{m,n} := \big(Z_{m,n}(x_1), \ldots, Z_{m,n}(x_p)\big)$ and $\mathcal{Z} := \big(Z(x_1), \ldots, Z(x_p)\big)$ corresponding to arbitrary p arguments $x_1, \ldots, x_p \in \mathbb{R}^d$. The m.s. convergence of the components of $\mathcal{Z}_{m,n}$ to those of $\mathcal{Z}$ implies $P\big(\|\mathcal{Z}_{m,n} - \mathcal{Z}\| > \varepsilon\big) \leq E\big[\|\mathcal{Z}_{m,n} - \mathcal{Z}\|\big]/\varepsilon \to 0$ for any $\varepsilon > 0$ since $E\big[\|\mathcal{Z}_{m,n} - \mathcal{Z}\|\big] \to 0$ as $m,n \to \infty$, which shows that $\mathcal{Z}_{m,n}$ converges in probability to $\mathcal{Z}$. This convergence implies the convergence of the joint distribution of $\mathcal{Z}_{m,n}$ to that of $\mathcal{Z}$ by the Portmanteau Theorem [2] (Sect. 1.2), so that the finite dimensional distributions of $\mathcal{Z}_{m,n}(x)$ converge to those of $\mathcal{Z}(x)$. The result also follows from [20] (Theorem 18.10) which shows that the convergence in probability of $\mathcal{Z}_{m,n}$ to $\mathcal{Z}$ implies the convergence of the finite dimensional distributions of $\mathcal{Z}_{m,n}$ to those of $\mathcal{Z}$. □

2.1.2. *Weak convergence of FD models.* We develop a practical criterion (Theorem 2.5) for checking whether a family $\{Z_{m,n}(x)\}$ of FD models is tight. If the family of random fields $\{Z_{m,n}(x)\}$ is tight and, in addition, its finite dimensional distributions converge to those of a real-valued m.s.-periodic random field $Z(x)$, then $Z_{m,n}$ converges weakly to Z in the space of continuous functions $C(D)$ [2] (Theorem 8.1). Under these conditions, samples of $Z_{m,n}(x)$ are similar to those of $Z(x)$ on a subset of the sample space Ω of nearly unit measure for sufficiently large (m,n). Theorems 2.6 and 2.7 consider the special case of Gaussian and translation random fields.

Theorem 2.5. *If the series* $\sum_{k,l} s_{kl}^{1/2} < \infty$ *and* $\sum_{k,l} \|\nu_{kl}\|\, s_{kl}^{1/2} < \infty$ *are convergent, the sequence of FD fields* $\{Z_{m.n}(x)\}$ *in Eq. 2.9 is tight.*

Proof. We show that the two conditions of Theorem 8.2 in [2] are satisfied. The first condition requires to show that for $\varepsilon > 0$ there is an $a > 0$ such that $P\big(|Z_{m,n}(0)| > a\big) \leq \varepsilon$, where $Z_{m,n}(0) = \sum_{|k|\leq m, |l|\leq n} U_{kl}$. Then, $P\big(|Z_{m,n}(0)| > a\big) \leq E\big[|\sum_{|k|\leq m,|l|\leq n} U_{kl}|\big]/a \leq \sum_{|k|\leq m,|l|\leq n} E\big[|U_{kl}|\big]/a \leq \sum_{|k|\leq m,|l|\leq n} E\big[U_{kl}^2\big]^{1/2}/a = \sum_{|k|\leq m,|l|\leq n} s_{kl}^{1/2} a \leq \sum_{k,l} s_{kl}^{1/2} a$ so that, if $\sum_{k,l} s_{kl}^{1/2} < \infty$ there exists an $a = \sum_{k,l} s_{kl}^{1/2}/\varepsilon$ with the required property for any given $\varepsilon > 0$.

The second condition requires to show that for any $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that $P\big(W_{m,n}(\delta) > \varepsilon\big) \leq \eta$ for all m,n, where $W_{m,n}(\delta) = \sup_{\|x-y\|\leq\delta} |Z_{m,n}(x) - Z_{m,n}(y)|$ denotes the modulus of continuity of $Z_{m,n}(x)$. Since $|Z_{m,n}(x) - Z_{m,n}(y)|$ is equal to

$$
\begin{aligned}
&\Big| \sum_{|k|\leq m,|l|\leq n} \Big(U_{kl}\, \big(\cos(\nu_{kl}\cdot x) - \cos(\nu_{kl}\cdot y)\big) - V_{kl}\, \big(\sin(\nu_{kl}\cdot x) - \sin(\nu_{kl}\cdot y)\big)\Big)\Big| \\
&= \Big| \sum_{|k|\leq m,|l|\leq n} \Big(-2\, U_{kl}\, \sin\big(\nu_{kl}\cdot(x+y)/2\big)\, \sin\big(\nu_{kl}\cdot(x-y)/2\big) \\
&\quad - 2\, V_{kl}\, \cos\big(\nu_{kl}\cdot(x+y)/2\big)\, \sin\big(\nu_{kl}\cdot(x-y)/2\big)\Big| \\
&= \Big| -2 \sum_{|k|\leq m,|l|\leq n} \sin\big(\nu_{kl}\cdot(x-y)/2\big) \Big(U_{kl}\, \sin\big(\nu_{kl}\cdot(x+y)/2\big) + V_{kl}\, \cos\big(\nu_{kl}\cdot(x+y)/2\big)\Big)\Big| \\
&\leq 2 \sum_{|k|\leq m,|l|\leq n} \big|\sin\big(\nu_{kl}\cdot(x-y)/2\big)\big|\, \big(|U_{kl}| + |V_{kl}|\big),
\end{aligned}
$$

we have

$$W_{m,n}(\delta) \leq \sup_{\|x-y\|\leq\delta} \left(2 \sum_{|k|\leq m, |l|\leq n} \left| \sin\left(\nu_{kl}\cdot(x-y)/2\right)\right| \left(|U_{kl}| + |V_{kl}|\right)\right)$$
$$\leq \delta \sum_{|k|\leq m, |l|\leq n} \|\nu_{kl}\| \left(|U_{kl}| + |V_{kl}|\right) \leq \delta \sum_{k,l} \|\nu_{kl}\| \left(|U_{kl}| + |V_{kl}|\right)$$

by using $\left|\sin\left(\nu_{kl}\cdot(x-y)/2\right)\right| \leq |\nu_{kl}\cdot(x-y)/2| \leq \|\nu_{kl}\|\,\|x-y\| \leq \delta\,\|\nu_{kl}\|$, so that

$$P\big(W_{m,n}(\delta) > \varepsilon\big) \leq \frac{E\big[W_{m,n}(\delta)\big]}{\varepsilon} \leq \frac{\delta\sum_{|k|\leq m,|l|\leq n}\|\nu_{kl}\|\left(E|U_{kl}| + E|V_{kl}|\right)}{\varepsilon}$$
$$\leq \frac{\delta\sum_{k,l}\|\nu_{kl}\|\left(E[U_{kl}^2]^{1/2} + E[V_{kl}^2]^{1/2}\right)}{\varepsilon} = \frac{2\,\delta\sum_{k,l}\|\nu_{kl}\|\,s_{kl}^{1/2}}{\varepsilon}$$

If $\sum_{k,l}\|\nu_{kl}\|\,s_{kl}^{1/2} < \infty$, then, for given $\varepsilon, \eta > 0$, the solution of $2\,\delta\sum_{k,l}\|\nu_{kl}\|\,s_{kl}^{1/2}/\varepsilon = \eta$ gives the required δ. We conclude that the weak convergence $Z_{m,n} \Longrightarrow Z$ holds under the conditions of the theorem. □

Criterion for weak convergence: If the conditions of Theorems 2.4 and 2.5 are satisfied, i.e., (1) the correlation function of $Z(x)$ is continuous and square integrable in $D \times D$ and (2) the series $\sum_{k,l} s_{kl}^{1/2}$ and $\sum_{k,l}\|\nu_{kl}\|\,s_{kl}^{1/2}$ are convergent, then $Z_{m,n} \Longrightarrow Z$ as $m, n \to \infty$ [2] (Theorem 8.1). The conditions of Theorem 2.4 relate to properties of the correlation functions of $Z(x)$. The conditions of Theorem 2.5 relate to the frequencies and the amplitudes of the constitutive random waves of $Z(x)$. Intuitively, they impose restrictions on the amplitudes of the high frequency fluctuations of $Z(x)$.

The criterion applies to both Gaussian and non-Gaussian random fields $Z(x)$. The following two theorems deal with Gaussian fields and continuous mappings of Gaussian fields, referred to as translation fields.

Theorem 2.6. *If $Z(x)$ is Gaussian and satisfies the conditions of Theorems 2.4 and 2.5, then $Z_{m.n}(x)$ converges almost surely (a.s.) to $Z(x)$ in the metric of $C(D)$ as $m, n \to \infty$.*

Proof. For Gaussian fields $Z(x)$, the FD models $Z_{m,n}(x)$ are sums of independent Gaussian processes, the independent Gaussian processes $U_{kl}\cos(\nu_{kl}\cdot x)$ and $V_{kl}\sin(\nu_{kl}\cdot x)$. Under the conditions of Theorems 3 and 4, we have the weak convergence $Z_{m,n} \Longrightarrow Z$ as $m, n \to \infty$. This implies the almost sure convergence of $Z_{m,n}$ to Z in $C(D)$ by the Itô-Nisio theorem, see [14] (Theorem 2.1.1) and [13]. □

This result implies that the discrepancy between target and FD samples, i.e., samples of $Z(x)$ and $Z_{m,n}(x)$, quantified by the sup-norm metric is small on a subset Ω_0 of the sample space Ω of nearly unit measure for sufficiently large (m, n), a statement similar to that under the weak convergence $Z_{m,n} \Longrightarrow Z$. In addition, the a.s. convergence guarantees that the subset Ω_0 increases with (m, n) so that the accuracy of FD models improves with the truncation level (m, n). This property is illustrated by examples in a subsequent section dealing with extremes of $Z(x)$.

Theorem 2.7. *Let $Z(x) = F^{-1}\circ\Phi\big(G(x)\big)$, where F is a strictly increasing, continuous distribution, Φ denotes the distribution of the standard normal variables and $G(x)$ is a real-valued, D-m.s. periodic homogeneous Gaussian field with zero mean, unit variance and continuous samples. Let $G_{m,n}(x)$ be FD dimensional models of $G(x)$ constructed as in Eq. 2.9. Then, the sequence of FD fields $Z_{m,n}(x) = F^{-1}\circ\Phi\big(G_{m,n}(x)\big)$ converges a.s. to $Z(x)$ in $C(D)$.*

Proof. That $G_{m,n}(x)$ converges a.s. to $G(x)$ in the space of continuous functions $C(D)$ follows from the previous theorem. Since the mapping $h = F^{-1} \circ \Phi$ from the Gaussian to the non-Gaussian fields is continuous, the a.s. convergence of $G_{m,n}(x)$ to $G(x)$ in $C(D)$ implies the a.s. convergence of $Z_{m,n}(x) = h\big(G_{m,n}(x)\big)$ to $Z(x) = h\big(G(x)\big)$ in this space as $m, n \to \infty$ by the continuous mapping theorem [17] (Theorem 1.10). $\square$

Memoryless mappings of Gaussian function constitute a class of non-Gaussian fields and processes, referred to as translation random functions [7]. They are completely defined by the correlation function of their Gaussian image $G(x)$ and the marginal distribution F. The correlation functions and the marginal distributions of translation fields cannot be selected arbitrarily. They must satisfy compatibility conditions [9] which, generally, are rather weak as the correlation functions of $Z(x) = F^{-1} \circ \Phi\big(G(x)\big)$ and $G(x)$ are similar. Also, samples of $Z(x)$ obtained from D-periodic samples $G(x)$ are D-periodic.

2.2. **Bounded frequency range fields.** Consider a material specimen in $D = [-a_1, a_1] \times [-a_2, a_2]$ or $D = [0, T_1] \times [0, T_2]$, where $T_i = 2\,a_i$, $i = 1, 2$, and suppose that a particular material property is described by a real-valued, zero-mean weakly homogeneous random field $\{Z(x),\ x \in \mathbb{R}^2\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with correlation and spectral density functions $c(\xi) = E[Z(x+\xi)\, Z(x)]$ and $s(\nu)$. It is assumed that the random field $Z(x)$ has the properties 1, 2 and 4 of Sect. 2, i.e., it is homogeneous with continuous samples and the support $D_\nu = [-\bar{\nu}_1, \bar{\nu}_1] \times [-\bar{\nu}_2, \bar{\nu}_2]$, $0 < \bar{\nu}_1, \bar{\nu}_2 < \infty$, of its spectral density $s(\nu)$ is a bounded rectangle of $\mathbb{R}^2$. Under these assumptions, the samples of $Z(x)$ are elements of the space of continuous functions $C(D)$ which can be viewed as superposition of random waves with frequencies in D_ν.

The random field $Z(x)$ in this subsection differs essentially from m.s. periodic random fields. In contrast to m.s. periodic fields which have countably infinite sets of frequencies in $\mathbb{R}^d$, they have uncountably infinite sets of frequencies in bounded subsets D_ν of $\mathbb{R}^d$. Truncated Fourier series [19] (Sect. 11.12) of samples of $Z(x)$ or truncated Karhunen-Loève series using eigenfunctions of the correlation function of $Z(x)$ [12] (Sect. 6.2) can be used to construct FD models for the random field $Z(x)$ considered here. These constructions do not use the fact that the support D_ν of the spectral density of $Z(x)$ is bounded. In contrast, the FD models of $Z(x)$ of the subsequent subsection use this feature of the spectral density of $Z(x)$ explicitly.

2.2.1. *Finite dimensional (FD) models.* The presentation of the results of this subsection, which are well-known, follows that in [1] (Sect. 2.4) and [21] (Chap. 3). Let $\{I_{kl}\}$ denote a partition of D_ν in rectangles with centers $\{\nu_{kl} = (\nu_{1,k}, \nu_{2,l})\}$, $|k| \le m$, $|l| \le n$, and sides $\Delta\nu_1 \times \Delta\nu_2$ which decrease with m and n, e.g., $\Delta\nu_1 = \bar{\nu}_1/m$ and $\Delta\nu_2 = \bar{\nu}_2/n$. The FD models $\{Z_{m.n}(x)\}$ of $Z(x)$ are based on partitions $\{I_{kl}\}$ of the support D_ν of the spectral density of $Z(x)$. We denote these FD models and their frequencies as in the previous sections dealing with m.s. periodic random fields, i.e., $Z_{m.n}(x)$ and ν_{kl}, although they have different definitions and meanings.

Consider the family of FD models

$$Z_{m,n}(x) = \sum_{|k| \le m,\ |l| \le n} \big[\Delta U_{kl}\, \cos(\nu_{kl} \cdot x) - \Delta V_{kl}\, \sin(\nu_{kl} \cdot x)\big], \quad x \in \mathbb{R}^2, \tag{2.12}$$

where ΔU_{kl} and ΔV_{kl} are uncorrelated random variables with $E\big[\Delta U_{kl}\big] = E\big[\Delta V_{kl}\big] = 0$ and $s_{kl} = E\big[\Delta U_{kl}^2\big] = E\big[\Delta V_{kl}^2\big] = \int_{I_{kl}} s(\nu)\, d\nu \simeq s(\nu_{kl})\, \Delta\nu_1\, \Delta\nu_2$, where the latter approximation holds for small frequency increments. The FD model in this equation has the same form as that in Eq. 2.9 but, as stated, the random coefficients and frequencies differ. The mean and correlation functions of $Z_{m,n}(x)$ are $E[Z_{m,n}(x)] = 0$ and

$$c_{m,n}(x,y) = E\big[Z_{m,n}(x)\, Z_{m,n}(y)\big] = \sum_{|k|\le m,\ |l|\le n} s_{kl}\, \cos\big(\nu_{kl}\cdot(x-y)\big)$$
$$\simeq \sum_{|k|\le m,\ |l|\le n} s(\nu_{kl})\, \cos\big(\nu_{kl}\cdot(x-y)\big)\, \Delta\nu_1\, \Delta\nu_2$$

which results by considerations as in Eq. 2.10 and shows that $Z_{m,n}(x)$ is a weakly homogeneous random field. The latter expression of $c_{m,n}(x,y)$ shows that the correlation function of $Z_{m,n}(x)$ converges to $c(\xi) = \int_{D_\nu} \cos(\nu\cdot\xi)\, s(\nu)\, d\nu$ as $m,n\to\infty$, $\xi\in\mathbb{R}^2$, so that the spectral density of $Z_{m,n}(x)$ converges to that of $Z(x)$. This convergence implies the convergence of the finite dimensional distributions of $Z_{m,n}(x)$ to those of $Z(x)$ by arguments as in Theorem 2.4.

2.2.2. *Weak convergence of FD models.* It is shown that the FD models $Z_{m,n}(x)$ in Eq. 2.12 converge weakly to $Z(x)$ in $C(D)$ as $m,n\to\infty$. There is no additional requirement for this convergence, in contrast to the FD models for m.s. periodic random fields (see Theorem 2.5). Intuition suggests that the FD models $Z_{m,n}(x)$ have this property since the samples of $Z(x)$ cannot oscillate very fast as their frequencies are confined to D_ν. We prove this statement for random fields $Z(x)$ defined on the real line and outline the steps of the proof for random fields defined on $\mathbb{R}^2$.

Theorem 2.8. *Let $\{Z(x),\ x\in D\}$ be a real-valued weakly homogeneous random field, where D is a bounded rectangle of $\mathbb{R}$, $d=1,2$. If the spectral density $s(\nu)$ has a bounded support D_ν, then the family of FD models in Eq. 2.12 converges weakly to $Z(x)$ in $C(D)$.*

Proof. Case $d=1$: Then, $D=[0,T]\subset\mathbb{R}\}$, $0<T<\infty$, $D_\nu=[-\bar\nu,\bar\nu]$, $0<\bar\nu<\infty$, and the family of FD models of $Z(x)$ has the form

$$Z_n(x) = \sum_{|k|\le n} \big[\Delta U_k\, \cos(\nu_k\, x) - \Delta V_k\, \sin(\nu_k\, x)\big], \tag{2.13}$$

where $\{\Delta U_k\}$ and $\{\Delta V_k\}$ are uncorrelated random variables with means and variances $E[\Delta U_k] = E[\Delta V_k] = 0$ and $s_k = E[\Delta U_k^2] = E[\Delta V_k^2] = \int_{I_k} s(\nu\, d\nu) \simeq s(\nu_k)\,\Delta\nu_k$, where $\{I_k\}$ is a partition of the frequency band $D_\nu = [-\bar\nu,\bar\nu]$, $0<\bar\nu<\infty$, of $Z(x)$ in equal intervals of size $\Delta\nu = \bar\nu/n$ and centers $\{\nu_k\}$. Note that $Z_n(x)$ is a weakly homogeneous random field with mean and correlation functions $E[Z_n(x)] = 0$ and $c_n(x,y) = E[Z_n(x)\, Z_n(y)] = \sum_{|k|\le n} s_k\, \cos\big(\nu_k\,(x-y)\big)$.

We show that the two condition of Theorem 8.2 in [2] are satisfied. The first conditions requires to find an $a>0$ such that $P\big(|Z_n(0)|>a\big)\le\varepsilon$ for arbitrary $\varepsilon>0$. We have

$$E\big[|Z_n(0)|\big] = E\Big[\Big|\sum_{|k|\le n}\Delta U_k\Big|\Big] \le E\Big[\Big(\sum_{|k|\le n}\Delta U_k\Big)^2\Big]^{1/2} = E\Big[\sum_{|k|,|l|\le n}\Delta U_k\,\Delta U_l\Big]^{1/2}$$
$$= \Big(\sum_{|k|\le n} E\big[\Delta U_k^2\big]\Big)^{1/2} = \Big(\int_{D_\nu} s(\nu)\, d\nu\Big)^{1/2} < \infty$$

by Cauchy-Schwarz inequality and properties of $\{\Delta U_k\}$ and $Z(x)$, so that we have $P\big(|Z_n(0)|>a\big) \le E\big[|Z_n(0)|\big]/a \le \big(\int_{D_\nu} s(\nu)\, d\nu\big)^{1/2}/a \le \varepsilon$, which shows that $a = \big(\int_{D_\nu} s(\nu)\, d\nu\big)^{1/2}/\varepsilon$ satisfies this condition.

The second condition requires to show that for any $\varepsilon,\eta>0$ there exists $\delta>0$ such that $P\big(W_n(\delta)>\varepsilon\big)\le\eta$ for all n, where $W_n(\delta) = \sup_{|x-y|\le\delta}|Z_n(x)-Z_n(y)|$ denotes the modulus of continuity of $Z_n(x)$. For arbitrary but fixed arguments x and y, we have

$$Z_n(x) - Z_n(y) = \sum_{|k|\le n} \big[h(\nu_k)\,\Delta U_k + g(\nu_k)\,\Delta V_k\big],$$

where $h(\nu) = -2\,\sin\big(\alpha\,\nu_k\big)\,\sin\big(\beta\,\nu_k\big),\ g(\nu) = -2\,\cos\big(\alpha\,\nu_k\big)\,\sin\big(\beta\,\nu_k\big),\ \alpha = (x+y)/2$ and $\beta = (x-y)/2$. The first term in the expression of $Z_n(x) - Z_n(y)$ for $k \ge 1$, i.e., $\sum_{k=1}^n h(\nu_k)\,\Delta U_k$, takes the form

$$\sum_{k=1}^n h(\nu_k)\,\Delta U_k = h(\nu_n)\,U_n - h(\nu_1)\,U_0 - \sum_{k=1}^{n-1}\big(h(\nu_{k+1}) - h(\nu_k)\big)\,U_k,$$

by summation by parts, where $U(\nu) = \int_{\eta\le\nu} dU(\eta)$, $U_k = U(\nu_k + \Delta\nu/2)$ and $\Delta U_k = U_k - U_{k-1}$. Similar expressions result for the other terms in the expression of $Z_n(x) - Z_n(y)$. The absolute value of the above summation for $|x-y| \le \delta$, $\delta > 0$, can be bounded by

$$\begin{aligned}\Big|\sum_{k=1}^n h(\nu_n)\,\Delta U_n\Big| &\le |h(\nu_n)|\,|U_n| + |h(\nu_1)|\,|U_0| + \sum_{k=1}^{n-1}\big|h(\nu_{k+1}) - h(\nu_k)\big|\,\big|U_k\big| \\ &\le \delta\,\bar\nu\,\big(|U_n| + |U_0|\big) + \sum_{k=1}^{n-1}\big|h(\nu_{k+1}) - h(\nu_k)\big|\,\big|U_k\big|,\end{aligned}$$

where the bound on the above first two terms holds since $|\sin\big(\nu_k\,(x-y)/2\big)| \le |\nu_k\,(x-y)/2|$, $\nu_k \le \bar\nu$ and $|x-y| \le \delta$ by assumption so that $|h(\nu_k)| = |-2\,\sin\big(\alpha\,\nu_k\big)\,\sin\big(\beta\,\nu_k\big)| \le \bar\nu\,\delta$. Since $h(\nu_{k+1}) - h(\nu_k) = h'(\nu_k^*)\,\Delta\nu$, $\nu_k^* \in [\nu_k, \nu_{k+1}]$, by the mean value theorem and $h'(\nu)/2 = -\alpha\,\cos(\alpha\,\nu)\,\sin(\beta\,\nu) - \beta\,\sin(\alpha\,\nu)\,\cos(\beta\,\nu)$, we have

$$\big|h(\nu_{k+1}) - h(\nu_k)\big| \le 2\,\Delta\nu\,\big(|\alpha|\,|\beta\,\nu_k^*| + |\beta|\big) \le \delta\,\Delta\nu\big(T\,\bar\nu + 1\big)$$

so that

$$\Big|\sum_{k=1}^n h(\nu_k)\,\Delta U_k\Big| \le \delta\,\bar\nu\,\big(|U_n| + |U_0|\big) + \delta\,\Delta\nu\big(T\,\bar\nu + 1\big)\sum_{k=1}^{n-1}\big|U_k\big|.$$

The expectation of $\big|\sum_{k=1}^n h(\nu_k)\,\Delta U_k\big|$ is bounded by δ scaled by a strictly positive finite constant since $E[|U_k|] \le E[U_k^2]^{1/2} \le \big(\int_{-\bar\nu}^{\bar\nu} s(\nu)\,d\nu\big)^{1/2}$ and

$$\sum_{k=1}^{n-1} E[|U_k|]\,\Delta\nu \le \sum_{k=1}^{n-1} E[U_k^2]^{1/2}\,\Delta\nu = \sum_{k=1}^{n-1}\left(\int_{-\bar\nu}^{\bar\nu} s(\nu)\,d\nu\right)^{1/2}\Delta\nu \le \bar\nu\left(\int_{-\bar\nu}^{\bar\nu} s(\nu)\,d\nu\right)^{1/2}$$

is finite. Similar arguments show that the other terms in the expression of $Z_n(x) - Z_n(y)$ admit the same types of bounds so that $E\big[W_n(\delta)\big] \le \delta\,M$ for any n, where $M > 0$ is a finite constant. The Chebyshev inequality gives $P\big(W_n(\delta) > \varepsilon\big) \le E\big[W_n(\delta)\big]/\varepsilon \le \delta\,M/\varepsilon$, so that for given $\varepsilon, \eta > 0$, there exists $\delta = \varepsilon\,\eta/M$ such that $P\big(W_n(\delta) > \varepsilon\big) \le \eta$. Since the conditions of Theorem 8.2 in [2] are satisfied, we conclude that the family of FD models Z_n is tight. Since the finite dimensional distributions of Z_n converge to those of Z, the family $\{Z_n\}$ of FD models converges weakly to Z in $C(D)$ as $n \to \infty$.

Case $d = 2$: The weak convergence of the family $Z_{m,n}(x)$ of FD models in Eq. 2.12 results by similar arguments. For the first condition of Theorem 8.2 in [2], note that

$$\begin{aligned}E\big[|Z_{m,n}(0)|\big] &= E\Big[\Big|\sum_{|k|\le m,\ |l|\le n}\Delta U_{kl}\Big|\Big] \le E\Big[\Big(\sum_{|k|\le m,\ |l|\le n}\Delta U_{kl}\Big)^2\Big]^{1/2} \\ &= \Big(\sum_{|k|\le m,\ |l|\le n} E\big[\Delta U_{kl}^2\big]\Big)^{1/2} = \Big(\int_{D_\nu} s(\nu)\,d\nu\Big)^{1/2} < \infty\end{aligned}$$

by Cauchy-Schwarz inequality and properties of $\{\Delta U_{kl}\}$ and $Z(x)$, so that we have $P\big(|Z_{m,n}(0)| > a\big) \leq E\big[|Z_{m,n}(0)|\big]/a \leq \big(\int_{D_\nu} s(\nu)\, d\nu\big)^{1/2}/a \leq \varepsilon$, which shows that $a = \big(\int_{D_\nu} s(\nu)\, d\nu\big)^{1/2}/\varepsilon$ satisfies this condition.

For the second condition, we need to show that for any $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that $P\big(W_{m,n}(\delta) > \varepsilon\big) \leq \eta$ for all m, n, where $W_{m,n}(\delta) = \sup_{\|x-y\| \leq \delta} |Z_{m,n}(x) - Z_{m,n}(y)|$ denotes the modulus of continuity of $Z_{m,n}(x)$. For arbitrary but fixed $x, y \in \mathbb{R}^2$, we have

$$Z_{m,n}(x) - Z_{m,n}(y) = \sum_{|k| \leq m,\ |l| \leq n} \big[h(\nu_{kl})\, \Delta U_{kl} + g(\nu_{kl})\, \Delta V_{kl}\big],$$

where $h(\nu) = -2 \sin\big(\nu\cdot(x+y)/2\big) \sin\big(\nu\cdot(x-y)/2\big)$ and $g(\nu) = -2 \cos\big(\nu\cdot(x+y)/2\big) \sin\big(\nu\cdot(x-y)/2\big)$. That the expectation $E\big[W_{m,n}(\delta)\big]$ of the modulus of continuity can be bounded by δ scaled with a strictly positive finite constant results by arguments similar to those of Theorem 2.8. The calculations are lengthier since the single summation for the case $d = 1$ is replaced with the double summation $\sum_{k=1,\dots m,\ l=1,\dots,n} h(\nu_{kl})\, \Delta U_{kl}$. $\square$

We conclude with the observations that, if $Z(x)$ is a Gaussian field, the weak convergence $Z_{m,n} \Longrightarrow Z$ implies the a.s. convergence of $Z_{m,n}(x)$ to $Z(x)$ in the space of continuous functions $C(D)$ by Theorem 2.6. If $Z(x)$ is the translation random field in Theorem 2.7 and the spectral density of its Gaussian image $G(x)$ has a bounded support D_ν, then $Z_{m,n}(x)$ converges a.s. to $Z(x)$ in the space of continuous functions $C(D)$.

2.3. **Numerical illustrations.** The numerical illustration of the statements of the theorems in the previous sections is challenging since the discrepancy $\sup_{x\in D} |Z_{m,n}(x) - Z(x)|$ between target and FD samples cannot be calculated exactly for the following two reasons. First, the samples of the random fields $Z_{m,n}(x)$ and $Z(x)$ can only be calculated at large but finite sets of points $x_i \in D$, $i = 1, \dots, I$. Since $\max_{i=1,\dots,I} |Z_{m,n}(x_i) - Z(x_i)| \leq \sup_{x\in D} |Z_{m,n}(x) - Z(x)|$ a.s., we can only obtained lower bounds on the discrepancy between target and FD samples. These bounds are likely to be tight for sufficiently large $I < \infty$ since $Z_{m,n}(x)$ and $Z(x)$ have continuous samples. Second, the algorithms for generation target and FD samples can handle large but finite sets of frequencies, e.g., the set of frequencies $\{\nu_{kl}\}$, $|k| \leq M$, $|l| \leq N$, with large $M, N < \infty$ corresponding to truncations of the representation of Eq. 2.8 for m.s. periodic fields and to fine discretizations of D_ν as used in Eq. 2.12 for fields with bounded frequency range. The samples for large (M, N) and fine discretizations of D_ν are viewed as actual samples of $Z(x)$.

Example 2.9. Let $\{Z(x),\ x \in D = [0, l]\}$ be a real-valued Gaussian field defined on the real line with spectral density $s(\nu) = 1(-\bar{\nu} \leq \nu \leq \bar{\nu})/(2\,\bar{\nu})$, $0 < \bar{\nu} < \infty$, and correlation function $c(\xi) = E\big[Z(x+\xi)\, Z(x)\big] = \sin\big(\bar{\nu}\, |\xi|\big)/\big(\bar{\nu}\, |\xi|\big)$. According to Theorem 2.8. the family of FD models $Z_n(x)$ in Eq. 2.13 converges weakly to $Z(x)$ in the space of continuous functions $C(D)$ as $n \to \infty$. Since $Z(x)$ is Gaussian, $Z_n(x)$ also converges a.s. to Z in $C(D)$ by Theorem 2.6.

The plots of Figs. 1 and 2 show histograms of the error $\sup_{x\in D} |Z(x) - Z_n(x)|$ and scatter plots of $\big(\sup_{x\in D} |Z(x)|, \sup_{x\in D} |Z_n(x)|\big)$ based on 1000 independent samples of these fields for several values of n. The plots are for $\bar{\nu} = 22$ and the samples of $Z(x)$ are approximated by samples of a discrete spectral representation of this field corresponding to a partition $\{J_r\}$, $r = 1, \dots, N$, of $[0, \bar{\nu}]$ in $N = 5000$ equal intervals. The FD models $Z_n(x)$ are given by Eq. 3.10 and correspond to coarse partitions $\{I_k\}$ of $[0, \bar{\nu}]$ with $n = 5$, $n = 8$ and $n = 10$. The frequencies $\{\nu_k\}$ are the centers of $\{I_k\}$ and the samples of the random coefficients $\{\Delta U_k\}$ and $\{\Delta V_k\}$ are given by the sums of the samples of the corresponding random coefficients of $Z(x)$ in the intervals $\{J_r\}$ which are included in I_k, so that $\{\Delta U_k\}$ and $\{\Delta V_k\}$ have the correct properties and are paired with the

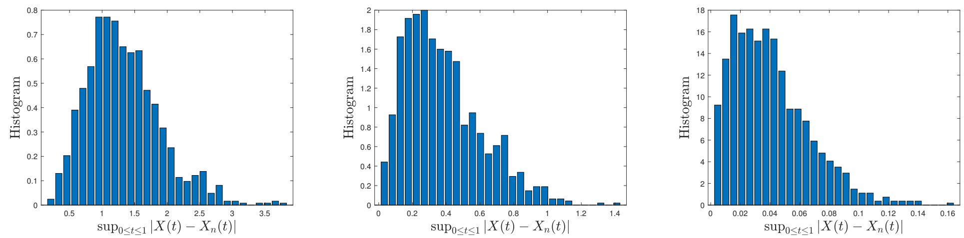

FIGURE 1. Histograms of $\sup_{x\in[0,l]}|Z(x)-Z_n(x)|$ for $\bar{\nu}=22$, and $n=5$, 8 and 10 (left, middle and right panels)

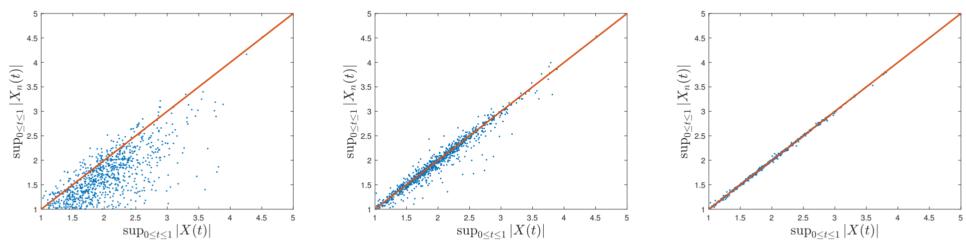

FIGURE 2. Scatter plots of $\big(\sup_{x\in[0,l]}|Z(x)|,\sup_{x\in[0,l]}|Z_n(x)|\big)$ for $\bar{\nu}=22$, and $n=5$, 8 and 10 (left, middle and right panels)

target samples. The plots show that the samples of $Z_n(x)$ approximate accurately the corresponding samples of $Z(x)$ even for small values of n.

3. Vector-valued microstructure models

Let $Z(x)$, $x\in\mathbb{R}^d$, be a zero-mean $\mathbb{R}^q$-valued random field with correlation function $c(\xi)=E[Z(x+\xi)\,Z(x)']$, an (q,q)-matrix for each $\xi\in\mathbb{R}^d$. As previously, it is assumed that $Z(x)$ is weakly homogeneous with continuous samples which is m.s. periodic or consists of superposition of waves with bounded frequencies.

The relationship between the correlation functions $c_{ii}(\xi)$ and the spectral densities $s_{ii}(\nu)$ of the individual components $Z_i(x)$, $i=1,\ldots,q$, of $Z(x)$ is given by Eq. 2.1. As for scalar-valued fields, it is assumed that the spectral distributions of the random fields $Z_i(x)$ are absolutely continuous with respect to the Lebesgue measure so that their spectral densities exist. It turns out that the correlation functions $c_{ij}(\xi)$ and the corresponding spectral densities $s_{ij}(\nu)$, $i\neq j$, satisfy similar relationships, i.e.,

$$c_{ij}(\xi)=\int_{\mathbb{R}^d}e^{\sqrt{-1}\,\nu\cdot\xi}\,s_{ij}(\nu)\,d\nu \quad\text{and}\quad s_{ij}(\nu)=\frac{1}{(2\,\pi)^d}\int_{\mathbb{R}^d}e^{-\sqrt{-1}\,\xi\cdot\nu}\,c_{ij}(\xi)\,d\xi, \tag{3.1}$$

where $s_{ij}(\nu)=\big[s_1(\nu)-\sqrt{-1}\,s_2(\nu)-(1-\sqrt{-1})\,\big(s_{ii}(\nu)+s_{jj}(\nu)\big)\big]$, $s_1(\nu)$ and $s_2(\nu)$ denote the spectral densities of the random fields $Z_i(x)+Z_j(x)$ and $\sqrt{-1}\,Z_i(x)+Z_j(x)$ [4] (Sect. 8.1), and the imaginary unit $\sqrt{-1}$ is written explicitly to avoid confusion with the indices of the components of $Z(x)$. The above relationships coincides with those of Eq. 2.1 for $i=j$. Generally, the spectral densities $s_{ij}(\nu)$ are complex-valued such that $s_{ij}(\nu)=s^*_{ji}(\nu)$, which follows from Eq. 3.1 and the property

$$c_{ij}(\xi)=E\big[Z_i(x+\xi)\,Z_j(x)\big]=E\big[Z_j(x)\,Z_i(x+\xi)\big]=E\big[Z_j(x-\xi)\,Z_i(x)\big]=c_{ji}(-\xi)$$

of the correlation functions of weakly homogeneous random fields. Similar arguments show that $s_{ij}(\nu) = s_{ji}(-\nu)$ and $s_{ij}(\nu)^* = s_{ij}(-\nu)$. The latter relationship implies that the real and the imaginary parts of $s_{ij}(\nu)$ are even and odd functions.

The components $Z_i(x)$ of $Z(x)$ admit the spectral representations (see Eqs. 2.2-2.3)

$$Z_i(x) = \int_{\mathbb{R}^d} e^{\sqrt{-1}\,\nu\cdot x}\, dW_i(\nu), \quad i = 1,\dots,q, \quad x \in \mathbb{R}^d, \tag{3.2}$$

where $W_i(\nu)$ is a complex-valued field with orthogonal increments whose first two moments are $E[dW_i(\nu)] = 0$ and $E[|dW_i(\nu)|^2] = s_{ii}(\nu)\,d\nu$. The increments of these random fields are related by $E[dW_i(\nu)\,dW_j(\eta)^*] = s_{ij}(\nu)\,\delta(\nu-\eta)\,d\nu$ so that the correlation function of $Z_i(x)$ and $Z_j(x)$, $i \neq j$, has the form

$$c_{ij}(\xi) = E\int_{\mathbb{R}^{2\,d}} \left(e^{\sqrt{-1}\,\nu\cdot(x+\xi)}\, dW_i(\nu)\right)\left(e^{\sqrt{-1}\,\eta\cdot x}\, dW_i(\eta)\right)^* = \int_{\mathbb{R}^d} e^{\sqrt{-1}\,\nu\cdot\xi}\, s_{ij}(\nu)\,d\nu,$$

which coincides with that of Eq. 3.1. Since the random fields $Z_i(x)$ are real-valued, they also admit the representation

$$Z_i(x) = \int_{\mathbb{R}^d} \big(\cos(\nu\cdot x)\,dU_i(\nu) - \sin(\nu\cdot x)\,dV_i(\nu)\big), \quad i = 1,\dots,q, \quad x \in \mathbb{R}^d, \tag{3.3}$$

where $U_i(\nu)$ and $V_i(\nu)$ are zero-mean, real-valued random fields with orthogonal increments of moments $E[dU_i(\nu)\,dU_i(\nu')] = E[dV_i(\nu)\,dV_i(\nu')] = s_{ii}(\nu)\,\delta(\nu-\nu')\,d\nu$ and $E[dU_i(\nu)\,dV_i(\nu')] = 0$, $i = 1,\dots,q$, see Eq. 2.3. The moments of the increments of these processes corresponding to distinct components result by direct calculations under the condition of homogeneity and are $E[dU_i(\nu)\,dU_j(\nu')] = E[dV_i(\nu)\,dV_j(\nu')] = \mathcal{R}[s_{ij}(\nu)]\,\delta(\nu-\nu')\,d\nu$ and $E[dU_i(\nu)\,dV_j(\nu')] = -E[dV_i(\nu)\,dU_j(\nu')] = -\mathcal{I}[s_{ij}(\nu)]\,\delta(\nu-\nu')\,d\nu$, see also [16] (Problem 9, p. 180), where $\mathcal{R}[s_{ij}(\nu)]$ and $\mathcal{I}[s_{ij}(\nu)]$ denote the real and the imaginary parts of $s_{ij}(\nu)$. The correlation function of distinct components of $Z(x)$ results from Eq. 3.3 by using the properties of the random fields $U_i(\nu)$ and $V_i(\nu)$. It has the expression

$$c_{ij}(\xi) = E\big[Z_i(x+\xi)\,Z_j(x)\big] = \int_{\mathbb{R}^d}\left(\cos(\nu\cdot\xi)\,\mathcal{R}[s_{ij}(\nu)] - \sin(\nu\cdot\xi)\,\mathcal{I}[s_{ij}(\nu)]\right)d\nu, \tag{3.4}$$

since $\int_{\mathbb{R}^d}\big(\cos(\nu\cdot\xi)\,\mathcal{I}[s_{ij}(\nu)] + \sin(\nu\cdot\xi)\,\mathcal{R}[s_{ij}(\nu)]\big)\,d\nu = 0$. The result is in agreement with Eq. 3.1 since $\int_{\mathbb{R}^d}\big(\cos(\nu\cdot\xi)\,\mathcal{I}[s_{ij}(\nu)] + \sin(\nu\cdot\xi)\,\mathcal{R}[s_{ij}(\nu)]\big)\,d\nu = 0$ by the properties of $s_{ij}(\nu)$. The formula of Eq. 3.4 simplifies to $c_{ij}(\xi) = \int_{\mathbb{R}^d}\cos(\nu\cdot\xi)\,s_{ij}(\nu)\,d\nu$ if $s_{ij}(\nu)$ is real-valued, e.g., the correlation functions for $i = j$.

3.1. **Mean square periodic fields.** The discussion is limited to two-dimensional specimens, i.e., $d = 2$, so that $Z(x)$ is a q-dimensional random vector at each $x \in \mathbb{R}^2$. We say that the vector-valued random field $Z(x)$ is $D = [-a_1, a_1] \times [-a_2, a_2]$- or (T_1, T_2)-m.s. periodic, $T_r = 2\,a_r$ for $r = 1, 2$, if

$$E\big[\|Z(x_1+T_1, x_2+T_2) - Z(x_1, x_2)\|^2\big] = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{3.5}$$

The definition implies that $Z(x)$ is (T_1, T_2)-m.s. periodic if its components $\{Z_i(x)\}$ are (T_1, T_2)-m.s. periodic random fields, see Eq. 2.4. Accordingly, the samples of $\{Z_i(x)\}$ can be represented by superpositions of waves with frequencies $\nu_{kl} = \big(\nu_{1,k}, \nu_{2,l}\big)$, where $\nu_{r,1} = 2\,\pi/T_r$, $\nu_{r,k} = k\,\nu_{r,1}$, $r = 1, 2$ and k is an integer. This property can be generalized such that different components of $Z(x)$ have different periodicities.

Arguments as in Theorem 2.2 show that the correlations $c_{ij}(\xi) = E\big[Z_i(x+\xi)\,Z_j(x)\big]$ between the components of $Z(x)$ are D-periodic functions since

$$\begin{aligned}&\big|E[Z_i(x_1+\xi_1+T_1, x_2+\xi_2+T_2)\,Z_j(x_1,x_2)] - E[Z_i(x_1+\xi_1, x_2+\xi_2)\,Z_j(x_1,x_2)]\\&= \big|E\big[\big(Z_i(x_1+\xi_1+T_1, x_2+\xi_2+T_2) - Z_i(x_1+\xi_1, x_2+\xi_2)\big)\,Z_j(x_1,x_2)\big]\big|\\&\le E\big[\big(Z_i(x_1+\xi_1+T_1, x_2+\xi_2+T_2) - Z_i(x_1+\xi_1, x_2+\xi_2)\big)^2\big]^{1/2}\,E\big[Z_j(x_1,x_2)^2\big]^{1/2},\end{aligned}$$

and $E\big[\big(Z_i(x_1+\xi_1+T_1, x_2+\xi_2+T_2) - Z_i(x_1+\xi_1, x_2+\xi_2)\big)^2\big] = 0$. If the partial derivatives $\partial c_{ij}(\xi)/\partial\xi_1$, $\partial c_{ij}(\xi)/\partial\xi_2$ and $\partial^2 c_{ij}(\xi)/\partial\xi_1\,\partial\xi_2$ of $c_{ij}(\xi)$ are continuous in D, then $c_{ij}(\xi)$ admits the convergent Fourier series representation

$$c_{ij}(\xi) = \sum_{kl} s_{ij,kl}\, e^{\sqrt{-1}\,\nu_{kl}\cdot\xi} \quad \text{with } s_{ij,kl} = \frac{1}{4\,a_1\,a_2}\int_D c_{ij}(\xi), \quad \xi\in\mathbb{R}^2, \tag{3.6}$$

by Theorem 2.3.

3.1.1. *Finite dimensional (FD) models.* The spectral representation in Eq. 3.3 takes the form

$$Z_i(x) = \sum_{k,l}\big[U_{i,kl}\,\cos(\nu_{kl}\cdot x) - V_{i,kl}\,\sin(\nu_{kl}\cdot x)\big], \quad x\in\mathbb{R}^2, \quad i=1,\dots,q, \tag{3.7}$$

where $\{U_{i,kl}\}$ and $\{V_{i,kl}\}$ are zero-mean random variables with the same properties as the increments of $\{U_i(\nu\}$ and $\{V_i(\nu)\}$ in Eq. 3.3. Consider the family of FD random fields

$$Z_{i,(m,n)}(x) = \sum_{|k|\le m, |l|\le n}\big[U_{i,kl}\,\cos(\nu_{kl}\cdot x) - V_{i,kl}\,\sin(\nu_{kl}\cdot x)\big], \quad x\in\mathbb{R}^2, \tag{3.8}$$

obtained by truncation of the infinite series representation of $Z_i(x)$ in Eq. 3.7, where the summation is over $k = \pm1, \pm2, \dots, \pm m$ and $l = \pm1, \pm2, \dots, \pm n$, see Eqs. 2.8 and 2.9. For simplicity, we use the same truncation levels for the components of $Z(x)$ although different truncation levels may yield more accurate representations [10]. The random fields $Z_{i,(m,n)}(x)$ are weakly homogeneous since their mean and correlation functions are $E[Z_{i,(m,n)}(x)] = 0$ and

$$\begin{aligned}c_{ij,(m,n)}(x,y) &= E\big[Z_{i,(m,n)}(x)\,Z_{j,(m,n)}(y)\big]\\&= \sum_{|k|\le m, |l|\le n}\Big[\mathcal{R}[s_{ij,kl}]\,\cos\big(\nu_{kl}\cdot(x-y)\big) - \mathcal{I}[s_{ij,kl}]\,\sin\big(\nu_{kl}\cdot(x-y)\big)\Big].\end{aligned} \tag{3.9}$$

The correlation functions become $c_{ij,(m,n)}(x,y) = \sum_{|k|\le m, |l|\le n} s_{ij,kl}\,\cos\big(\nu_{kl}\cdot(x-y)\big)$ if the spectral densities $\{s_{ij}(\nu)\}$, $i\ne j$, are real-valued. If the correlation function $c_{ij}(\xi) = E[Z_i(x+\xi)\,Z_j(x)]$ of $Z(x)$ is continuous and square integrable on $D\times D$, then $c_{ij,(m,n)}(x,y)$ converges absolutely and uniformly to $c_{ij}(\xi)$ by Mercer's theorem, see [1], Sect.3.3, [5], Appendix 2 and Sect.6-4, and [12], Sect.6.2.

3.1.2. *Weak convergence of FD models.* The statements of Theorems 2.4 to 2.6 for real-valued random fields extend directly to vector-valued fields. The mean square convergence $E\big[\|Z_n(x) - Z(x)\|^2\big] \to 0$ in these theorems is equivalent to the m.s. convergence of the components of $Z(x)$, i.e., $E\big[\big(Z_{i,(m,n)}(x) - Z_i(x)\big)^2\big] \to 0$, $i=1,\dots,q$, as $m,n\to\infty$. Similar arguments hold for arbitrary sets of spatial coordinates $x_1,\dots,x_p\in\mathbb{R}^d$ of arbitrary size p since $\{Z_n(x_j),\ j=1,\dots,p\}$ and $\{Z(x_j),\ j=1,\dots,p\}$ can be recast into $p\,q$-dimensional random vectors so that the finite dimensional distributions of Z_n converge to those of $Z(x)$ as $m,n\to\infty$.

The tightness of the sequence of vector-valued random fields $Z_{m,n}(x)$ follows from the tightness of the real-valued random fields $\{Z_{i,(m,n)}(x)\}$, i.e., the components of $Z(x)$. For the first condition of Theorem 8.2 in [2], we have $\|Z_{m,n}(0)\| \leq \sum_{i=1}^{d} |Z_{i,(m,n)}(0)|$ so that $E\big[\|Z_{m,n}(0)\|\big] \leq \sum_{i=1}^{d} E\big[|Z_{i,(m,n)}(0)|\big]$ and the latter summation is finite if the components of $Z_{m,n}(x)$ satisfy the conditions of Theorem 2.5. For the second condition of Theorem 8.2, note that

$$\begin{aligned} W_{m,n}(\delta) &= \sup_{\|x-y\|\leq\delta} \|Z_{m,n}(x) - Z_{m,n}(y)\| \\ &\leq \sum_{i=1}^{d} \sup_{\|x-y\|\leq\delta} |Z_{i,(m,n)}(x) - Z_{i,(m,n)}(y)| = \sum_{i=1}^{d} W_{i,(m,n)}(\delta), \end{aligned}$$

where $\{W_{i,(m,n)}(\delta)\}$ denote the moduli of continuity of the components $\{Z_{i,(m,n)}(x)\}$ of $Z_{m.n}(x)$. Since $E\big[W_{m,n}(\delta)\big] \leq \sum_{i=1}^{d} E\big[W_{i,(m,n)}(\delta)\big]$ and the latter expectations are finite if the components of $Z_{m,n}(x)$ satisfy the conditions of Theorem 2.5, we conclude that the family of vector-valued FD models $\{Z_{m,n}(x)\}$ is tight if their components satisfy the conditions of Theorem 2.5. If in addition the finite dimensional distributions of $Z_{m,n}(x)$ converge to those of $Z(x)$, then $Z_{m,n} \Longrightarrow Z$ as $m,n \to \infty$. If $Z(x)$ is Gaussian, we also have the a.s. convergence of $Z_{m,n}$ to Z in the space of continuous functions $C(D)$.

If $Z(x)$ is not Gaussian, we proceed as in Theorem 2.7 by considering translation random fields $Z(x)$ defined by $Z_i(x) = F_i^{-1} \circ \Phi\big(G_i(x)\big)$, $i = 1, \ldots, q$, where F_i are continuous cumulative distribution functions, Φ is the distribution of the standard normal variable and $G_i(x)$ are zero-mean, unit-variance homogeneous Gaussian fields with correlation functions $\zeta_{ij}(\xi) = E\big[G_i(x+\xi)\, G_j(x)\big]$. It is assumed that the vector-valued random field $G(x) = \big(G_1(x), \ldots, G_q(x)\big)$ is D-m.s. periodic with continuous samples and that the sequence of vector-valued FD Gaussian fields with components $G_{i,(m;n)}(x)$ converges a.s. to $G_i(x)$ in $C(D)$ so that the FD models $Z_{i,(m,n)}(x) = F_i^{-1} \circ \Phi\big(G_{i,(m,n)}(x)\big)$ converge a.s. in $C(D)$ to $Z_i(x)$ as $m,n \to \infty$ by the continuous mapping theorem.

3.2. **Bounded frequency range fields.** It is assumed that the spectral densities of the components $Z_i(x)$ of the $\mathbb{R}^q$-valued random field $Z(x)$ have the same bounded support $D_\nu = [-\bar{\nu}_1, \bar{\nu}_1] \times [-\bar{\nu}_2, \bar{\nu}_2]$, $0 < \bar{\nu}_1, \bar{\nu}_2 < \infty$, see Sect. 2.2.1. Consider a partition of D_ν in small rectangles $\{I_{kl}\}$ with sides $\Delta\nu_1 \times \Delta\nu_2$ and centers $\{\nu_{kl}\}$, $|k| \leq m$, $|l| \leq n$, whose measures decrease with m and n. This partition is used to construct FD models $Z_{m,n}(x)$ of $Z(x)$ and show that they converge weakly to $Z(x)$ as the partition of D_ν is refined.

3.2.1. *Finite dimensional (FD) models.* We construct FD models of $Z(x)$ component-by-component based on the approach in Eq. 2.12. The FD models for the components of $Z(x)$ are defined by

$$Z_{i,(m,n)}(x) = \sum_{|k|\leq m,\ |l|\leq n} \big[\Delta U_{i,kl}\, \cos(\nu_{kl}\cdot x) - \Delta V_{i,kl}\, \sin(\nu_{kl}\cdot x)\big], \quad x \in \mathbb{R}^2, \tag{3.10}$$

where $E[\Delta U_{i,kl}] = E[\Delta V_{i,kl}] = 0$,

$$E[\Delta U_{i,kl}\, \Delta U_{j,pq}] = E[\Delta V_{i,kl}\, \Delta V_{j,pq}] = \int_{I_{kl}} \mathcal{R}[s_{ij}(\nu)]\, d\nu \simeq \mathcal{R}[s_{ij}(\nu_{kl})]\, \Delta\nu_1\, \Delta\nu_2 \quad \text{and}$$

$$E[\Delta U_{i,kl}\, \Delta V_{j,pq}] = -E[\Delta V_{i,kl}\, \Delta U_{j,pq}] = -\int_{I_{kl}} \mathcal{I}[s_{ij}(\nu)]\, d\nu \simeq \mathcal{I}[s_{ij}(\nu_{kl})]\, \Delta\nu_1\, \Delta\nu_2.$$

The latter approximations hold for sufficiently fine partitions of D_ν, i.e., sufficiently large m and n.

3.2.2. *Weak convergence of FD models.* Arguments as in Sect. 3.1.2 show that the sequence of $\mathbb{R}^q$-valued FD fields $Z_{m,n}(x)$ converges weakly to $Z(x)$ in the space of continuous functions defined on D if the components of $Z_{m,n}(x)$ converge weakly to those of $Z(x)$ in $C(D)$ as $m,n \to \infty$. The weak convergence of the real-valued random fields $Z_{i,(m,n)}(x)$ to $Z_i(x)$ holds by Theorem 2.8.

In summary, the theorems of the previous sections show that the family of FD models constructed for real- and vector-valued m.s.-periodic random fields $Z(x)$ may or may not converge weakly in $C(D)$ depending on the amplitudes of the constitutive harmonics of $Z(x)$. In contrast, the FD models of real- and vector-valued random fields $Z(x)$ with spectral densities of bounded support converge weakly to $Z(x)$. These theorems are consistent with our intuition. For example, the FD models may be incapable to characterize accurately the samples of m.s.-periodic random fields if their high frequency components have sizable amplitudes. On the other hand, the samples of random fields with bounded frequencies are much smoother than those of m.s.-periodic fields since their constitutive waves have bounded frequencies. The FD models of these fields are expected to be accurate even for relatively low truncation levels (m,n).

3.3. **Compliance/stiffness tensor.** Let $A(x)$, $x \in D$, be the compliance tensor of a linear elastic random material in a bounded subset D of $\mathbb{R}^2$ which is defined on a probability space $(\Omega,\mathcal{F},P)$. The matrix-valued random field $A(x)$ can be viewed as an infinite family of (p,p)-symmetric, positive definite random matrices indexed by $x \in D$. Since $A(x)$ can be recast into a vector-valued random field, previous developments on the construction of FD models for vector-valued random fields and the weak/a.s. convergence of these models apply to matrix-valued random fields, such as $A(x)$.

To satisfy physical constraints, the matrix-valued random field $A(x)$ must be positive definite in the sense that $\inf_{x\in D}\{\zeta'\, A(x)\,\zeta\} > 0$ almost surely (a.s.) for all $\zeta \in \mathbb{R}^p$. The requirement $\zeta'\, A(x)\,\zeta > 0$, $\forall \zeta \in \mathbb{R}^p$, a.s. at each $x \in D$ is insufficient since, although $\Omega_x = \{\omega \in \Omega : \zeta'\, A(x,\omega)\,\zeta \le 0\} \in \mathcal{F}$ is an event with $P\big(\Omega_x\big) = 0$ for $\forall \zeta \in \mathbb{R}^p$, $\zeta \ne 0$, and each $x \in D$, the uncountable union $\cup_{x\in D}\Omega_x$ of events $\{\Omega_x\}$ may not be in $\mathcal{F}$ and, if it is, its probability may not be zero, see Example 3.12 in [8].

3.3.1. *Probabilistic models.* Two physically consistent models of $A(x)$ are briefly discussed. The first, referred to as the eigenvalue/rotation model, ensembles $A(x)$ from translation fields developed for the eigenvalues and the eigenvectors of the correlation function of $A(x)$. The second, referred to as the triangular matrix model, represents $A(x)$ by the product of a triangular matrix-valued random field with its transposition.

Eigenvalue/rotation random fields: Let $\{\Lambda_k(x,\omega)\}$ and $\{V_k(x,\omega)\}$ denote the eigenvalues and eigenvectors of a sample $A(x,\omega)$ of $A(x)$ at a fixed but arbitrary $x \in D$, i.e., the solutions of $\det\big(A(x,\omega) - \Lambda_k(x,\omega)\,I\big) = 0$ and $A(x,\omega)\,V_k(x,\omega) = \Lambda_k(x,\omega)\,V_k(x,\omega)$, $k = 1,\ldots,p$, where I denotes the identity matrix. If $\inf_{x\in D}\{\Lambda_k(x,\omega)\} > 0$ for almost all samples of $A(x)$, then $\inf_{x\in D}\{\zeta'\, A(x,\omega)\,\zeta\} > 0$ a.s. for $\zeta \in \mathbb{R}^p$ arbitrary.

The random field $A(x)$ admits the representation $A(x) = V(x)\,\Lambda(x)\,V(x)'$, where $V(x) = [V_1(x),\ldots,V_p(x)]$ is an (p,p) matrix whose columns are the eigenvectors $\{V_k(x)\}$ and $\Lambda(x)$ is an (p,p) diagonal matrix whose non-zero entries are the eigenvalues $\{\Lambda_k(x)\}$. The eigenvectors $\{V_k(x)\}$ of $A(x)$ define the rotation fields $\{\Theta_k(x)\}$ for the principal directions of $A(x)$ at each $x \in D$.

The random fields $\{\Lambda_k(x)\}$ and $\{\Theta_k(x)\}$ are continuous functions of the entries of $A(x)$ so that they are dependent random elements defined on the probability space $(\Omega,\mathcal{F},P)$ of $A(x)$. If $A(x)$ has continuous samples, then the real-valued random fields $\{\Lambda_k(x)\}$ and $\{\Theta_k(x)\}$ have continuous samples. Since the samples of the random fields $\{\Lambda_k(x)\}$ and $\{\Theta_k(x)\}$ must take values in bounded intervals, these fields are non-Gaussian.

Triangular matrix random fields: The random field $A(x)$ admits the decomposition $A(x) = L(x)\,L(x)'$ at each $x \in D$ since it is symmetric and positive definite, where $L(x)$ is a lower triangular matrix. If almost all samples of the real-valued random fields $L_{ii}(x)$, $i = 1, \ldots, p$, are positive, then $A(x)$ is a.s. positive definite at each $x \in D$. The constraints and the construction on these fields is similar to that of the random fields of the previous model. Various algorithms can be used to construct $L(x)$. For example,

$$L_{ij}(x) = \frac{A_{ij}(x) - \sum_{s=1}^{j-1} A_{is}(x)\,A_{js}(x)}{A_{jj}(x) - \sum_{s=1}^{j-1} A_{js}(x)^2} =: \frac{h_{ij}\big(A(x)\big)}{g_{ij}\big(A(x)\big)}, \quad 1 \le j \le i \le m, \tag{3.11}$$

by the Cholesky decomposition, where the convention $\sum_{r=1}^{0} A_{ir}(x)\,A_{jr}(x) = 0$ is used [8] (Sect. 5.2). This shows that the real-valued random fields $\{L_{ij}\}$ are dependent with continuous samples.

3.3.2. *Random compliance tensor for 2D specimens.* Let $A(x)$, $x \in D \subset \mathbb{R}^2$, be the compliance tensor of a two-dimensional linear elasticity problem and denote by $\Sigma(x) = \big[\Sigma_{11}(x), \Sigma_{22}(x), \Sigma_{12}(x)\big]'$ and $S(x) = \big([S_{11}(x), S_{22}(x), S_{12}(x)\big]'$ the vector-valued stress and strain random fields. The stress-strain relationship has the form

$$S(x) = \begin{bmatrix} S_{11}(x) \\ S_{22}(x) \\ S_{12}(x) \end{bmatrix} = \begin{bmatrix} A_{11}(x) & A_{12}(x) & A_{13}(x) \\ A_{12}(x) & A_{22}(x) & A_{23}(x) \\ A_{13}(x) & A_{23}(x) & A_{33}(x) \end{bmatrix} \begin{bmatrix} \Sigma_{11}(x) \\ \Sigma_{22}(x) \\ \Sigma_{12}(x) \end{bmatrix} = A(x)\,\Sigma(x), \quad x \in D, \tag{3.12}$$

where $\{A(x),\ x \in D\}$ is an a.s. symmetric, positive definite matrix-valued random field, which means that almost all samples of $A(x)$ are symmetric, positive definite matrices in D.

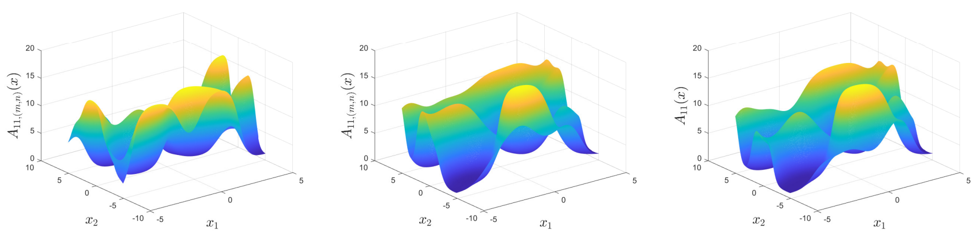

FIGURE 3. A sample of $A_{11}(x)$ (right panel) and corresponding samples of $A_{11,(m,n)}(x)$ for $(m,n) = (5,5)$ and $(20,20)$ (left and middle panels)

For a numerical illustration, we construct the random field $A(x)$ from a lower triangular matrix-valued random field $L(x)$ whose non-zero entries are real-valued random fields defined on $D = [-a_1, a_1] \times [-a_2, a_2]$. In addition, the random fields $L_{ii}(x)$ on the diagonal of $L(x)$ have positive samples. The following algorithm has been used to construct $L(x)$. First, we construct the vector-valued Gaussian field $G(x) = [G_1(x), \ldots, G_6(x)]'$ via the linear transformation $G(x) = a\,N(x)$, where $N(x) = [N_1(x), \ldots, N_6(x)]'$ is homogeneous Gaussian field with independent zero-mean, unit-variance components with spectral densities $s_r(\nu) \propto 1(\nu \in D_\nu)\,\exp\big[-(\nu_1^2 - 2\,\rho_r\,\nu_1\,\nu_2 + \nu_2^2)/\big(2\,(1-\rho_r^2)\big)\big]$, $|\rho_r| < 1$, for $r = 1, \ldots, 6$, with the bounded support $D_\nu = [-\bar{\nu}_1, \bar{\nu}_1] \times [-\bar{\nu}_2, \bar{\nu}_2]$, $0 < \bar{\nu}_1, \bar{\nu}_2 < \infty$. Second, the first three components, or any other three components, of the Gaussian field $G(x)$ are mapped into, e.g., Beta translation fields $\{L_{11}(x), L_{22}(x), L_{33}(x)\}$ with samples in bounded intervals of $(0, \infty)$. The latter three components of $G(x)$ can be left unchanged or modified arbitrarily. The compliance tensor is defined by $A(x) = L(x)\,L(x)'$, where the non-zero entries of $L(x)$ are $L_{rr}(x)$, $r = 1, 2, 3$, $L_{21}(x) = G_4(x)$, $L_{31}(x) = G_5(x)$ and

$L_{32}(x) = G_6(x)$. It satisfies physical constraints in the sense that the random variable $\inf_{x\in D} \zeta' A(x) \zeta > 0$ a.s. for all $\zeta \in \mathbb{R}^3 \setminus \{0\}$.

The FD models $\{N_{r,(m,n)}(x)\}$ and $\{G_{r,(m,n)}(x)\}$ of $\{N_r(x)\}$ and $\{G_r(x)\}$ result from Eqs. 2.12 and 3.10. They converge weakly to $\{N_r(x)\}$ and $\{G_r(x)\}$ in $C(D)$ as $m, n \to \infty$ by Theorem 2.8 and considerations in Sect. 3.2.2. Since the fields are Gaussian, the above weak convergence implies the a.s. convergence of $\{N_{r,(m,n)}(x)\}$ and $\{G_{r,(m,n)}(x)\}$ to $\{N_r(x)\}$ and $\{G_r(x)\}$ in the space of continuous functions $C(D)$ by the Itô-Nisio theorem [13], see also Theorem 2.6. These properties persists for the matrix-valued random fields $L(x)$ and $A(x)$ by the continuous mapping theorem, see also Theorem 2.7. Accordingly, it is expected that the samples of the FD models of the compliance field $A(x)$ are similar to those of this field for sufficiently large m and n.

The following numerical values are for $\bar{\nu}_1 = \bar{\nu}_2 = 5$, $\rho_r = 0.1 : 0.1 : 0.6$, $a_1 = 5$, $a_2 = 10$ and $\gamma = \{\gamma_{ij}\}$ with entries $\gamma_{ii} = 1$ and $\gamma_{ij} = 0.7$, $i \neq j$. The supports of the Beta distributions for $\{L_{11}(x), L_{22}(x), L_{33}(x)\}$ are $(1, 4)$, $(2, 6)$ and $(1, 3)$ and their shape parameters are equal to one. The right panel of Fig. 3 shows a sample of $A_{11}(x)$, i.e., a sample of a discrete version of the spectral representation of the random fields $\{N_r(x)\}$, $r = 1, \ldots, 6$, constructed from a partition of D_ν in equal rectangles $\{I_{kl}\}$ with sides $\bar{\nu}_i/100$, $i = 1, 2$, see Eq. 3.10. Further refinements have not been used since they result in insignificant changes in the samples of these fields. The left and middle panels of this figure show the corresponding samples of the FD models $A_{11,(m,n)}(x)$ which are based on partitions of D_ν in rectangles of sizes $\bar{\nu}_i/5$, $i = 1, 2$ and $\bar{\nu}_i/20$, $i = 1, 2$. Visual inspection of the plots suggests that the accuracy of the FD models improves with (m, n) as they can capture additional high frequency details of the target samples. This observation is consistent with the weak and a.s convergence of the FD models of $A(x)$. The errors $\sup_{x\in D} |A_{ij}(x) - A_{ij,(m,n)}(x)|$ for the samples $A_{11,m.n}(x)$ in the figure decrease from 14.36 for $(m, n) = (5, 5)$ to 7.91 for $(m, n) = (20, 20)$.

The random field $\{A(x),\ x \in D\}$ in the above illustration is completely defined by the marginal distributions of the real-valued random fields $\{L_{ij}(x)\}$, the mapping $G(x) = a\, N(x)$ and the correlation functions of the components of $N(x)$. The model can match exactly specified marginal distributions but not joint distributions. The results of Sect. 3 can be used to construct FD models for general positive definite, matrix-valued non-Gaussian fields of the type in [18] and prove that they converge weakly/a.s. to these target non-Gaussian fields.

4. Response of random microstructure

Consider a material specimen in a bounded subset D of $\mathbb{R}^d$ whose properties are described by a scalar-/vector-/matrix-valued random field $\{Z(x),\ x \in D\}$. The specimen is subjected to some actions and boundary conditions which, for simplicity, are assumed to be deterministic. The material response is a random field $U(x)$ which satisfies a stochastic differential equation whose coefficients depend on $Z(x)$, so that it is a functional of $Z(x)$. Analytical solutions of these stochastic equations are possible in few special cases [3]. Generally, only numerical approximations of $U(x)$ can be obtained based on FD models of $Z(x)$. The numerical solutions are useful if they converge in some sense to the target solutions $U(x)$, e.g., in the sense of the theorems of the previous sections.

We consider a simple stochastic problem, the one-dimensional transport equation for materials whose properties vary continuously in space according to a real-valued random field $Z(x)$, $x \in \mathbb{R}$. Let $\{Z_n(x)\}$ be a family of FD models of $Z(x)$ and denote by $U(x)$ and $U_n(x)$ the solutions of the transport equation with material properties described by $Z(x)$ and $Z_n(x)$. We show that, if Z_n converges a.s. to Z in the space of continuous functions $C(D)$, then U_n also converges a.s. to U in $C(D)$ under some conditions. This means that samples of $U_n(x)$ can be used as substitutes for samples of $U(x)$ for sufficiently large n and that extremes of $U(x)$ can be inferred from samples of $U_n(x)$.

The random fields $U(x)$ and $U_n(x)$ satisfy the one-dimensional stochastic transport differential equations

$$\frac{d}{dx}\left(Z(x)\,\frac{dU(x)}{dx}\right)=0,\quad x\in D=(0,l),\quad U(0)=\alpha,\ U(l)=\beta \quad \text{and} \tag{4.1}$$

$$\frac{d}{dx}\left(Z_n(x)\,\frac{dU_n(x)}{dx}\right)=0,\quad x\in D=(0,l),\quad U_n(0)=\alpha,\ U_n(l)=\beta, \tag{4.2}$$

where $Z(x)$ denotes the random conductivity field and $Z_n(x)$ is an FD model of this field. It is assumed that almost all samples of the random fields $Z(x)$ and $Z_n(x)$ take values in bounded intervals of $(0,\infty)$ and are differentiable, so that the strong solutions of the above equations exist and are

$$U(x)=\alpha+(\beta-\alpha)\,I(x)/I(l),\quad \text{where } I(x)=\int_0^x dy/Z(y) \quad \text{and} \tag{4.3}$$

$$U_n(x)=\alpha+(\beta-\alpha)\,I_n(x)/I_n(l),\quad \text{where } I_n(x)=\int_0^x dy/Z_n(y). \tag{4.4}$$

Since $U(x)$ and $U_n(x)$ take values in bounded intervals, they have finite moments.

Theorem 4.1. *If almost all samples of the random fields $Z(x)$ and $Z_n(x)$ are differentiable with values in bounded intervals of $(0,\infty)$ and Z_n converges a.s. to Z in $C(D)$ as $n\to\infty$, then U_n converges a.s. to U in $C(D)$ as $n\to\infty$.*

Proof. The discrepancy between the integrals $I(x)$ and $I_n(x)$ can be bounded by

$$\begin{aligned}|I(x)-I_n(x)| &= \left|\int_0^x \frac{dy}{Z(y)}-\int_0^x \frac{dy}{Z_n(y)}\right| = \left|\int_0^x \frac{Z_n(y)-Z(y)}{Z(y)\,Z_n(y)}\,dy\right| \\ &\le \int_0^x \left|\frac{Z_n(y)-Z(y)}{Z(y)\,Z_n(y)}\right| dy \le \sup_{y\in D}|Z_n(y)-Z(y)| \int_0^l \frac{dy}{Z(y)\,Z_n(y)},\quad x\in\mathbb{R}.\end{aligned}$$

Since $Z_n(x)$ converges a.s. to $Z(x)$ in the space of continuous functions as $n\to\infty$ and $\int_0^l dy/(Z(y)\,Z_n(y))$ is bounded a.s. by assumption, then $\sup_{x\in D}|I(x)-I_n(x)|\to 0$ almost surely in $C(D)$ as $n\to\infty$.

The discrepancy between the exact and the FD solutions can be bounded by

$$\begin{aligned}|U(x)-U_n(x)|/(\beta-\alpha) &\le \frac{I_n(l)\,|I(x)-I_n(x)|+I_n(x)\,|I_n(l)-I(l)|}{I(l)\,I_n(l)} \\ &\le \left[\frac{2}{I(l)}\int_0^l \frac{dy}{Z(y)\,Z_n(y)}\right]\sup_{y\in D}|Z_n(y)-Z(y)|.\end{aligned}$$

Since the term in the square brackets is bounded a.s. and Z_n converges a.s. to Z in $C(D)$, then $U_n(x)$ converges a.s. to $U(x)$ in the space of continuous functions. □

Example 4.2. Let $Z(x)=a+(b-a)\,F^{-1}\circ\Phi\big(G(x)\big)=h\big(G(x)\big)$, $x\in D=[0,l]$, be a translation random field, where F denotes a standard Beta distribution with support $(0,1)$ and shape parameters (p,q) and $G(x)$ is homogeneous, zero-mean, unit-variance, and one-sided spectral density $g(\nu)\propto 1(0\le\nu\le\bar{\nu})\,(\lambda^2+\nu)^{-2}$, $0\le\nu\le\bar{\nu}<\infty$. Since the spectral density of $G(x)$ has bounded support, the FD models $G_n(x)$ converge a.s. to $G(x)$ in $C(D)$ as $n\to\infty$ so do the corresponding FD models $Z_n(x)=h\big(G_n(x)\big)$ of $Z(x)$, see Theorems 2.5-2.7.

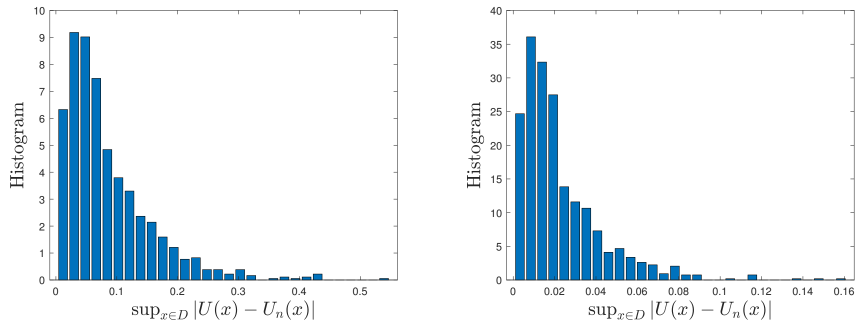

FIGURE 4. Histogram of $\sup_{x\in R}|U(x)-U_n(x)|$ for $\lambda=8$, $\bar{\nu}=20$ and $n=10$ and $n=40$ (left and right panels)

As previously discussed, samples of $Z(x)$ cannot be obtained exactly. They are generated from a representation of the type in Eq. 2.12 for a very fine partition of the frequency band $[0,\bar{\nu}]$ in N equal intervals. The FD models $Z_n(x)$ are also based on this equation but correspond to coarse partitions of $[0,\bar{\nu}]$ in $n \ll N$ equal intervals. The plots of Fig. 4 are for $N=5000$ and $(\lambda,\bar{\nu})=(8,20)$. The left and right panels of this figure show histograms of the discrepancy $\sup_{x\in D}|U(x)-U_n(x)|$ between target and FD solution samples for $n=10$ and $n=40$. Note that the histograms are drawn at different scales. The samples and the histograms show that the FD models are satisfactory even for small values of n, and that their accuracy improves with n.

5. Comments

Analytical solutions of stochastic problems are available in few cases which, generally, are of limited practical interest. Numerical methods have to be employed to solve most stochastic problems encountered in applications. These methods require to discretize the physical space, e.g., by using the finite difference/element methods, and the probability space, e.g., by using finite dimensional (FD) models. It was assumed that the material properties vary continuously in space so that the random fields $Z(x)$ describing these properties and their FD models have continuous samples. We have considered two types of random fields for material properties, mean square periodic fields and fields whose spectral densities have bounded supports.

The focus was on the accuracy of FD models of random material properties $Z(x)$. We have also examined the accuracy of FD-based approximations of material responses $U(x)$. Conditions were established under which samples of FD material models and corresponding material responses can be used as substitutes for samples of $Z(x)$ and $U(x)$. These theoretical results have been illustrated by numerical examples quantifying the discrepancy between target and FD samples.

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