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# Finite dimensional (FD) Slepian models for non-Gaussian processes

M. Grigoriu

Cornell University, Ithaca NY 14853-3501, USA



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#### ABSTRACT

Conditions under which samples of continuous stochastic processes X(t) on bounded time intervals  $[0,\tau]$  can be represented by samples of finite dimensional (FD) processes  $X_d(t)$  are augmented such that samples of Slepian models  $S_{d,a}(t)$  of  $X_d(t)$  can be used as surrogates for samples of Slepian models  $S_a(t)$  of X(t). FD processes are deterministic functions of time and  $d < \infty$  random variables. The discrepancy between target and FD samples is quantified by the metric of the space  $C[0,\tau]$  of continuous functions. The numerical illustrations, which include Gaussian/non-Gaussian FD processes and solutions of linear/nonlinear random vibration problems, are consistent with the theoretical findings in the paper.

#### 1. Introduction

Slepian models  $S_a(t)$  describe the evolution of random processes X(t) following crossings of specified levels a with positive slopes, referred to as a-upcrossings. They can be used to (1) characterize extremes of random processes above specified levels, properties of excursions of these processes above these levels and other sample properties of random processes and fields, e.g., extremes of winds, waves and other natural hazards and of responses of dynamical systems to random inputs [1–5], (2) estimate extreme responses in random microstructures [6–10], and (3) assess the performance and/or reliability of dynamical systems from properties of the length, area and extremes of excursions of the states of these systems above safe levels [11–14].

Simple analytical approximations are available for extremes of Gaussian processes. For example,  $P(\sup_{0 \le t \le \tau} \{X(t)\} \le a) \simeq P(X(0) \le a) \exp(-v(a)\tau)$  for a zero-mean, unit-variance, mean square (m.s.) differentiable stationary Gaussian process X(t), where  $v(a) = (\dot{\sigma}/(2\pi)) \exp(-a^2/2)$  denotes the mean a-upcrossing of X(t) and  $\dot{\sigma}$  is the standard deviation of  $\dot{X}(t) = dX(t)/dt$  [15] (Chap. 7). These approximations extend directly to nonstationary Gaussian processes [15] (Sect. 7.3.1) and non-Gaussian translation processes [16] (Chap. 3). They are accurate for relatively large levels a [17] (Chap. 7). There are no similar analytical approximations for the area, duration and other properties of excursions of non-Gaussian processes above levels a. Moments of these random variables are difficult to obtain even for stationary Gaussian processes X(t), see [18](Sects. 10.8 and 13.4).

Numerical algorithms have been developed to estimate properties of excursions of stationary Gaussian processes X(t) above levels a via Slepian models  $S_a(t)$  [17] (Sect. 10.3). These models consists of sums of two terms, a regression which depends on a single random variable, the slope of X(t) at the upcrossing time, and a residual term, which is a zero-mean nonstationary Gaussian process quantifying deviations from the regression term. For example, the lowest order approximation of

the duration of an excursion of X(t) above a results by using only the regression component of  $S_a(t)$  [13].

Slepian models have been originally developed for ergodic Gaussian processes via Palm distributions [17] (Chap. 10). They have been extended to non-ergodic stationary and nonstationary Gaussian processes in [1,19,20] by using horizontal windows to define a-upcrossings of X(t) and letting window widths decrease to zero. The approach is general, i.e., it applies to arbitrary stationary/non-stationary processes whose samples satisfy some smoothness conditions, and deliver a formula for calculating the finite dimensional densities of  $S_a(t)$ . The resulting Slepian models have simple expressions for Gaussian processes but are impractical for non-Gaussian processes.

Our objective is to construct accurate approximations of Slepian models  $S_a(t)$  for non-Gaussian processes X(t) with continuous samples. Since the Slepian models  $S_a(t)$  of non-Gaussian processes do not seem to admit analytical expressions as those for Gaussian processes, we construct numerical Slepian models  $S_{d,a}(t)$  by using finite dimensional (FD) representations  $X_d(t)$  of X(t). These representations are deterministic functions of time t and finite numbers d of random variables, where d is referred to as stochastic dimension. We note that FD random functions, such as  $X_d(t)$ , are essentials for the numerical solution of stochastic problems since, generally, target processes X(t) have infinite stochastic dimensions as uncountable families of random variables indexed by time t [21] (Chaps. 7 and 9).

There are many FD representations which have been designed to capture various features of target random processes. For example, truncated Karhunen–Loève (KL) series are FD processes which can be used to characterize the mean and correlation functions of input random processes and of outputs of linear dynamical systems to these inputs. However, they are inadequate for extremes and other sample properties of non-Gaussian processes. Following are two examples of FD processes  $X_d(t)$  whose samples can be used as surrogates for samples of continuous processes X(t).

The first FD representation of X(t) is defined by linear interpolations  $X_d(t)$  between values  $\{X(t_i)\}$  of this process at the points  $0 = t_0 < t_1 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_6 < t_7 < t_8 < t_8 < t_9 <$ 

E-mail address: mdg12@cornell.edu.

 $\cdots < t_N = \tau$  of partitions of  $[0,\tau]$ . The representation is conceptually simple, depends on d=N+1 random variables and its samples are guaranteed to approach the samples of X(t) as the largest time step of the partition of  $[0,\tau]$  decreases to zero. Two aspects of the representation limit its use in applications. Its stochastic dimension d=N+1 is large for partitions with many time steps and increases with  $\tau$ . In addition, the law of  $X_d(t)$  has to be reconstructed if the partition  $\{t_i\}$  is modified. The second representation views the FD processes  $X_d(t)$  as elements of subsets of linear spaces spanned by basis functions of the space  $L^2[0,\tau]$  of square integrable functions on  $[0,\tau]$ , e.g., the top d eigenfunctions of the correlation function of X(t) which are also used to construct truncated Karhunen–Loève (KL) representations. Generally, this representation is more accurate than the previous one for the same stochastic dimension, i.e. d=N+1. Also, changes in d do not require to reconstruct the law of the process as for the previous representation.

We use the latter FD processes  $X_d(t)$  to represent target non-Gaussian processes X(t) and their Slepian models  $S_a(t)$  on bounded time intervals  $[0,\tau]$ . It is assumed that X(t) is defined on a probability space  $(\Omega,\mathcal{F},P)$  and has continuous samples. It is shown that, under some conditions, the sequence  $\{X_d\}$  of FD processes converges weakly to X as  $d\to\infty$  in the space  $C[0,\tau]$  of real-valued continuous functions, i.e.,  $\lim_{d\to\infty}\sup_{0\le t\le \tau}|X_d(t)-X(t)|=0$  in distribution, a convergence denoted by  $X_d\Longrightarrow X$ . This means that samples of  $X_d(t)$  can be used as substitutes for samples of X(t) on subsets of the sample space  $\Omega$  of nearly unit probability. The convergence  $X_d\Longrightarrow X$  is insufficient to conclude that the Slepian models  $S_{d,a}(t)$  of  $X_d(t)$  converge weakly to  $S_a(t)$  in  $C[0,\tau]$  as  $d\to\infty$ . Additional conditions are established to assure that the convergence  $S_{d,a}\Longrightarrow S_a$  holds. Under these conditions, properties of excursions of X(t) above a can be inferred from samples of  $S_{d,a}(t)$ .

The paper is organized as follows. Finite dimensional representations  $X_d(t)$  of X(t) are discussed in Section 2. The section includes the FD representation  $B_d(t)$  of the Brownian motion process B(t). Slepian models for ergodic and non-ergodic processes are reviewed in Section 3. The main results are in Section 4 which establishes conditions under which samples of  $X_d(t)$  and  $S_{d,a}(t)$  can be used as surrogates for samples of X(t) and  $S_a(t)$ . Section 5 presents two sets of numerical examples. The first deals with Slepian models of Gaussian and non-Gaussian FD processes. The second constructs Slepian models for the stationary solutions of linear and nonlinear random vibration problems. It is shown that samples of  $S_a(t)$  can be approximated by samples of  $S_{d,a}(t)$  under the conditions established in Section 4.

#### 2. Finite dimensional (FD) processes

Let  $\{X(t),\ 0 \le t \le \tau\}$  be a real-valued process with mean E[X(t)] = 0 and correlation function  $c(s,t) = E[X(s)\,X(t)]$  defined on a probability space  $\left(\Omega,\mathcal{F},P\right)$ . We illustrate the construction of finite dimensional (FD) processes  $X_d(t),\ d=1,2,...$ , for X(t) by using the basis functions of the KL series representation of this process. The formulation is applied to develop FD representations for the Brownian motion process.

## 2.1. Processes with continuous correlation function

It is assumed that the correlation function of X(t) is continuous, so that it is square integrable on  $[0,\tau]^2$ , i.e.,  $\int_{[0,\tau]^2} c(s,t)^2 \, ds \, dt < \infty$ . The eigenvalues  $\{\lambda_k\}$ ,  $k=1,2,\ldots$ , of the operator  $\mathcal{A}\varphi(t)=\int_0^\tau c(s,t)\,\varphi(t)\, dt$  are positive reals and the eigenfunctions  $\{\varphi_k(t)\}$ ,  $k=1,2,\ldots$ , of this operator are orthonormal, i.e.,  $\langle\varphi_k,\varphi_l\rangle=\int_0^\tau \varphi_k(t)\,\varphi_l(t)\, dt=\delta_{kl}$ , where  $\delta_{kl}=1$  for k=l and zero otherwise. According to Mercer's theorem [22] (Sect 6.2), the series  $c(s,t)=\sum_{k=1}^\infty \lambda_k\,\varphi_k(s)\,\varphi_k(t)$  converges absolutely and uniformly in  $[0,\tau]^2$  and X(t) admits the Karhunen–Loève (KL) representation  $X(t)=\sum_{k=1}^\infty Z_k\,\varphi_k(t)$ , where the equality holds in the mean square (m.s.) sense and  $\{Z_k\}$  are zero-mean uncorrelated random variables with variances  $\{\lambda_k\}$ .

Consider the family of FD random processes

$$X_d(t) = \sum_{k=1}^d Z_k \, \varphi_k(t), \quad d = 1, 2, \dots,$$
 (1)

where the random coefficients  $\{Z_k\}$  are defined sample-by-sample from samples of X(t) by projection, i.e.,

$$Z_k(\omega) = \int_0^\tau X(t, \omega) \varphi_k(t) dt, \ k \ge 1, \omega \in \Omega.$$
 (2)

Other basis functions can be used to construct FD processes, e.g., other sets of orthogonal functions or polynomials. The eigenfunctions of the correlation functions of X(t) are selected since they minimize the mean square error and are used frequently in applications.

We note that (1) the processes  $\{X_d(t)\}$  are defined completely, (2) the samples  $X_d(t,\omega)$  and  $X(t,\omega)$  of  $X_d(t)$  and X(t) are paired by construction and (3) the processes  $X_d(t)$  have the same mean and correlation functions as the truncated version of the KL series representation of X(t). The latter statement follows from the observations that

$$E[Z_k] = E\left[\int_a^b X(t)\,\varphi_k(t)\,dt\right] = \int_a^b E[X(t)]\,\varphi_k(t)\,dt = 0$$

and

$$\begin{split} E[Z_k | Z_l] &= E\left[\int_{[a,b]^2} X(s) | X(t) | \varphi_k(s) | \varphi_l(t) | ds | dt\right] \\ &= \int_{[a,b]^2} E[X(s) | X(t)] | \varphi_k(s) | \varphi_l(t) | ds | dt \\ &= \int_a^b \left[\int_a^b c(s,t) | \varphi_l(t) | dt\right] | \varphi_k(s) | ds \\ &= \lambda_l \int_a^b \varphi_l(t) | \varphi_k(s) | ds = \lambda_l | \delta_{kl}, \end{split}$$

where the change of order of integration holds by Fubini's theorem. Note also that  $X_d(t)$  converges in m.s. to X(t) as  $d\to\infty$  by Mercer's theorem.

These above observations imply that the random vectors  $\mathcal{X}_d = \left(X_d(t_1), \dots, X_d(t_m)\right)$  converge in m.s. to  $\mathcal{X} = \left(X(t_1), \dots, X(t_m)\right)$  as  $d \to \infty$  for any integer  $m \ge 1$  and times  $\{t_i\}$  since, by Mercer's theorem, the components of  $\mathcal{X}_d$  converge to those of  $\mathcal{X}$  in m.s. We conclude that the finite dimensional distributions of  $X_d(t)$  converge to those of X(t) as  $d \to \infty$  since  $\mathcal{X}_d$  converges to  $\mathcal{X}$  in m.s. as  $d \to \infty$ , which implies the convergence in distributions [23] (Theorem 18.10).

#### 2.2. Brownian Motion process

Let  $\{X(t) = B(t), \ 0 \le t \le \tau\}$ ,  $\tau > 0$ , be the standard Brownian motion process defined on a probability space  $(\Omega, \mathcal{F}, P)$ . The eigenvalues and eigenfunctions of its correlation function  $c(s,t) = E[B(s) \ B(t)] = \min(s,t)$ 

$$\lambda_k = \frac{\tau^2}{\pi^2 (k - 1/2)^2} \quad \text{and} \quad \varphi_k(t) = \sqrt{2/\tau} \, \sin((k - 1/2) \, \pi \, t/\tau),$$

$$0 \le t \le \tau, \quad k \ge 1. \tag{3}$$

The eigenfunctions are continuous orthonormal functions which span the space  $L^2[0,\tau]$  of real-valued square integrable functions [24] (Theorem 5.1). The eigenvalues  $\{\lambda_k\}$ , k=1,2,..., are non-negative and converge to zero as  $k\to\infty$  since the correlation function of the Brownian motion is continuous on  $[0,\tau]^2$  [24] (Theorem 5.1).

Consider the family of FD processes (see Eq. (1))

$$B_d(t) = \sum_{k=1}^{d} Z_k \, \varphi_k(t), \quad 0 \le t \le \tau, \quad d = 1, 2, \dots,$$
 (4)

where  $\{\varphi_k\}$  are the top d eigenfunctions, i.e., the eigenfunctions corresponding to the largest d eigenvalues and  $\{Z_k\}$  are random coefficients constructed sample-by-sample from samples  $B(t,\omega)$  of B(t) by projection, i.e.,  $Z_k(\omega) = \langle B(\cdot,\omega), \varphi_k(\cdot) \rangle = \int_0^\tau B(t,\omega) \, \varphi_k(t) \, dt$ . These coefficients are Gaussian variables as integrals of the Brownian motion process B(t)

and have the same first two moments as the random coefficients of the corresponding truncated Karhunen–Loève (KL) representation of B(t) since  $E[Z_k] = 0$  and  $E[Z_k Z_l] = \lambda_k \, \delta_{kl}$ , which result by calculations similar to those of the previous subsection. Accordingly, their correlation functions  $E[B_d(s) B_d(t)] = \sum_{k=1}^d \lambda_k \, \varphi_k(s) \, \varphi_k(t)$  converge absolutely and uniformly in  $[0,\tau]^2$  to the correlation function E[B(s) B(t)] by Mercer's theorem.

A notable difference between  $B_d(t)$  in Eq. (4) and the truncated KL representation of B(t) is that the samples of B(t) are paired with samples of  $B_d(t)$  by construction while those of truncated KL series of B(t) do not have this property. Pairing samples of target and FD processes is essential to establish conditions under which samples of  $B_d(t)$  can be used as surrogates for samples of B(t). The requisite technical condition is that the discrepancy between samples of B(t) and  $B_d(t)$  can be made as small as desired in the metric of the space of continuous functions  $C[0,\tau]$  by increasing the stochastic dimension d.

#### 3. Slepian models for ergodic and non-ergodic processes

Let X(t),  $t \ge 0$ , be a real-valued stochastic process which has an a-upcrossing at time t=0. The evolution of X(t) following an a-upcrossing, i.e., the conditional process  $X(t) \mid \{a - \text{upcrossing at } t = 0\}$  is the Slepian model  $S_a(t)$ ,  $t \ge 0$ , corresponding to the a-upcrossing of X(t) at time t=0.

## 3.1. Ergodic processes

Suppose X(t),  $t \ge 0$ , is an ergodic process and let  $\{s_i\}$  be the times of the a-upcrossings of an infinitely long sample  $X(t,\omega)$  of X(t). The functions  $\{X(s_i+s,\omega)\}$ ,  $s\ge 0$ , are the defining samples of the Slepian model  $S_a(s)$ ,  $s\ge 0$ . They describe the evolution of X(t) following an a-upcrossing. For example, samples  $\{\dot{X}(s_i,\omega)\}$  of the derivatives of  $X(t,\omega)$  at the crossing times can be used to construct the empirical distribution of the slope of X(t) at these times [17] (Sect 10.3).

This formulation is intuitive and can be used to construct Slepian models for any ergodic process. Unfortunately, the ergodicity assumption is rather restrictive in applications. For example, consider the process  $X(t) = Z_1 \cos(\nu t) + Z_2 \sin(\nu t)$ , where  $Z_1, Z_2$  are zero-mean uncorrelated random variables and  $\nu > 0$  denotes a frequency. The process is not ergodic since, e.g., the temporal average of

$$X(t)^{2} = \frac{1 + \cos(2vt)}{2} Z_{1}^{2} + \frac{1 - \cos(2vt)}{2} Z_{2}^{2} + Z_{1} Z_{2} \sin(2vt)$$

is  $\lim_{T\to\infty} (1/T) \int_0^T X(t)^2 dt = (Z_1^2 + Z_2^2)/2$  so that it is sample dependent. Similar arguments hold for the FD representations  $X_d(t)$  of the previous section so that alternative methods have to be employed to construct Slepian models for these processes.

#### 3.2. Non-ergodic processes

Slepian models for non-ergodic processes X(t) are constructed by conditioning on the event of an a-upcrossing. The calculations cannot be performed directly since the probability of the event of an a-upcrossing of X(t) at a given time is zero. To implement this approach, we view crossings as limits of events of non-zero probabilities. For example, an a-upcrossing can be defined by the limit of the sequence of events

$$A_a(\delta) = \{ \text{there exists } t \in [0, \delta] \text{ such that } X(t) = a \text{ and } \dot{X}(t) > 0 \}, \quad \delta > 0,$$

of non-zero probability. The limit of the conditional process  $X(t) \mid A_a(\delta)$  as the width  $\delta$  of the horizontal window  $[0, \delta]$  is shrunk to zero defines the Slepian model  $S_a(t)$  of X(t) [19].

Let X(t) be a real-valued process whose derivative  $\dot{X}(t) = dX(t)/dt$  has finite variance and has continuous samples with probability 1. Suppose that X(t) upcrosses a level a at time t=0 and denote by

 $f(x_1, \ldots, x_n | a$ —upcrossing) the density of  $(X(t_1), \ldots, X(t_n))$ ,  $0 < t_1 < \cdots < t_n$ , conditional on the a-upcrossing event interpreted in the horizontal window sense. This density has the form [1,19,20]

$$f(x_1, \dots, x_n | a - \text{upcrossing}) = \lim_{\delta \to 0} f(x_1, \dots, x_n | A_a(\delta))$$

$$= \frac{\int_0^\infty z f(a, x_1, \dots, x_n, z) dz}{\int_0^\infty z f(a, z) dz},$$
(6)

where  $f(a,x_1,\ldots,x_n,z)$  and f(a,z) denote the densities of  $\big(X(0),X(t_1),\ldots,X(t_n)\,\dot{X}(0)\big)$  and  $\big(X(0),\dot{X}(0)\big)$  and  $A_a(\delta)$  is the event of an a-upcrossing in the horizontal window  $[0,\delta]$ . An alternative form of the above expression is

$$f(x_1, \dots, x_n | a - \text{upcrossing}) = \int_0^\infty g(z|a) f(x_1, \dots, x_n | a, z) dz, \tag{7}$$

where

$$g(z|a) = \frac{z f(a|z)}{\int_0^\infty z f(a|z) dz}$$
(8)

is the density of the slope of X(t) at the time of its a-upcrossing [1].

The conditional approach outlined in this section is general but its implementation is impractical when dealing with non-Gaussian processes since, generally, the functional forms of the joint densities in Eq. (6) to (8) are not known. Practical results can be obtained for stationary and nonstationary Gaussian processes [1,20].

#### 3.2.1. Gaussian processes

If X(t) is a zero-mean, unit-variance stationary Gaussian process, then  $\dot{X}(0)$  and X(0) are independent Gaussian variables so that  $f(z\mid a)=\exp\left(-z^2/(2\,\lambda_2)\right)/\sqrt{2\,\pi\,\lambda_2},\ z\geq 0,\ \int_0^\infty f(\eta\mid a)\,\eta\,d\eta=\sqrt{\lambda_2/(2\,\pi)}$  and  $g(z\mid a)=z/(2\,\lambda_2)\exp\left(-z^2/(2\,\lambda_2)\right),\ z\geq 0$ , where  $\lambda_2$  denotes the variance of  $\dot{X}(0)$ . The finite dimensional distributions of X(t) following an a-upcrossing at time t=0, i.e., the finite dimensional distributions of the Slepian model  $S_a(t)$ , coincides with those of the nonstationary process

$$X_a(t) = a c(t) - \frac{c'(t)}{\lambda_2} Z + \mathcal{K}(t), \quad t \ge 0,$$
 (9)

which is the sum of a regression term  $a c(t) - c'(t) Z/\lambda_2$  and a zero-mean nonstationary Gaussian process  $\mathcal{K}(t)$  whose correlation function depends on the correlation function c(t) = E[X(s)X(s+t)] of X(t). The random variable Z is independent of  $\mathcal{K}(t)$  and denotes the slope of X(t) at the crossing time [17] (Sect 10.3). The Slepian model  $S_a(t)$  and  $X_a(t)$  are versions so that they can be defined on different probability spaces. Similar results are for nonstationary Gaussian processes [1,20].

## 3.2.2. Non-Gaussian processes

The formulation of Eq. (7) to (8) does not deliver simple functional forms for the Slepian models  $S_a(t)$  when dealing with non-Gaussian processes X(t) since, e.g., X(t) and  $\dot{X}(t)$  are dependent non-Gaussian variables. We define  $S_a(t)$  by the subset of samples of X(t) which upcross a at time t=0 in the horizontal window sense, i.e., the samples of X(t) in  $A_a(\delta)$ . This sample-by-sample construction of  $S_a(t)$  is conceptually different from that in [1,20], which uses properties of Gaussian variables to develop versions of  $S_a(t)$ .

We use samples of specialized FD representations  $X_d(t)$  of target processes X(t) to construct FD representations  $S_{d,a}(t)$  of Slepian models  $S_a(t)$  of X(t). There are at least two reasons for using samples of surrogates rather than of target processes. First, it is not possible to generate samples of time-continuous processes X(t) since they are uncountable families of random variables indexed by time  $t \geq 0$ . We can only generate samples of FD processes, see algorithms for generating samples of random processes [25] (Sect 5.3). Second, FD representations of target processes which match only global target statistics, e.g., mean and correlation functions, are insufficient for estimating extremes. We need FD processes  $X_d(t)$  whose samples are similar to those of X(t) for almost all  $\omega \in \Omega$  in a sense to be defined precisely in a subsequent section, so that samples of  $X_d(t)$  and  $S_{d,a}(t)$  can be paired with samples of X(t) and  $S_a(t)$  and constitute accurate surrogates for the samples of target processes and their Slepian models.

#### 4. FD Slepian models

Denote by  $S_{d,a}(t)$  the Slepian model of  $X_d(t)$ . We show that  $S_{d,a}(t)$ can be used as a surrogate/substitute for the Slepian model  $S_a(t)$  of target processes X(t) under some conditions. It is assumed that the target processes X(t) have continuous samples and are defined on bounded time intervals  $[0, \tau]$ . The analysis involves the following two steps. First, conditions are established under which samples of  $X_d(t)$  can be used as substitutes for samples of X(t). This follows from the weak convergence of  $X_d$  to X as  $d \to \infty$  in the space  $C[0, \tau]$  of continuous functions. Second, we establish conditions under which samples of  $S_a(t)$  can be represented by samples of  $S_{d,a}(t)$  so that statistics of, e.g., excursions of target processes above levels a, can be inferred from samples of  $X_d(t)$  and  $S_{d,a}(t)$ . The conditions for the weak convergence of  $X_d$  to X have to be augmented to assure that samples of  $S_{d,a}(t)$  can be used as surrogates for samples of  $S_a(t)$ .

## 4.1. Sample properties of X(t) and $X_d(t)$

We establish conditions under which samples of FD processes  $X_d(t)$ can be used as approximations of samples of X(t) defined on bounded time intervals  $[0, \tau]$ . It is assumed that X(t) and  $X_d(t)$  have continuous samples. The discrepancy between the samples of these processes is measured by the metric  $\sup_{0 \le t \le \tau} |X(t) - X_d(t)|$  of the space  $C[0, \tau]$  of real-valued continuous functions. We show that  $\sup_{0 \le t \le \tau} |X(t) - X_d(t)|$  $\rightarrow 0$  weakly as  $d \rightarrow \infty$ , a convergence denoted by  $X_d \Longrightarrow X$ .

## 4.1.1. Weak convergence of FD processes For arbitrary $\varepsilon > 0$ , denote by

$$\mathcal{Q}_d(\varepsilon) = \{\omega: \sup_{0 \leq t \leq \tau} |X_d(t,\omega) - X(t,\omega)| > \varepsilon\}, \quad \varepsilon > 0, \tag{10}$$

the subset of  $\Omega$  in which the discrepancy between samples of X(t) and  $X_d(t)$  exceeds  $\varepsilon$  in the metric of  $C[0,\tau]$ . If  $X_d \Longrightarrow X$ , then [26] (Chaps. 8

$$P(\Omega_d(\varepsilon)) \to 0, \quad d \to \infty,$$
 (11)

which means that (1) the samples of  $X_d(t)$  and X(t) in  $\Omega_d(\varepsilon)^c$  differ by less than  $\varepsilon$  in the metric of  $C[0, \tau]$  and (2) the probability measure of the "bad" subset  $\Omega_d(\varepsilon)$  of  $\Omega$  on which these samples differ by at least  $\varepsilon$  can be made as small as desired by increasing d. Note also that, for fixed d, the probability  $P(\Omega_d(\varepsilon))$  increases as  $\varepsilon$  decreases and that  $P(\Omega_d(\varepsilon))$ can be kept constant if a decrease of  $\varepsilon$  is associated with an increase of d.

Let X(t) be a zero-mean, weakly stationary process with one-sided spectral density g(v) of bounded support  $[0, \bar{v}]$ . Consider the family of FD processes

$$X_d(t) = \sum_{k=1}^d \left[ \Delta U_k \cos(v_k t) + \Delta V_k \sin(v_k t) \right], \quad t \in \mathbb{R},$$
 (12)

where  $v_k = (k - 1/2) \Delta v$ ,  $\Delta v = \bar{v}/d$ ,  $\Delta U_k$  and  $\Delta V_k$  are zero-mean uncorrelated random variables,  $E[\Delta U_k] = E[\Delta V_k] = 0$  and  $g_k =$  $E[\Delta U_k^2] = E[\Delta V_k^2] = \int_{I_k} g(\nu) \, d\nu \simeq g(\nu_k) \, \Delta\nu \text{ and } I_k = (\nu_k - \Delta\nu/2, \nu_k + \Delta\nu/2).$ The FD processes  $X_d(\hat{t})$  are frequently used in applications to generate samples of X(t). Note that  $X_d(t)$  in Eq. (12) has continuous samples and is weakly stationary since  $E[X_d(t)] = 0$  and  $c_d(s,t) = E[X_d(s)X_d(t)] =$  $\sum_{k=1}^{d} g_k \cos(\nu_k (s-t)).$ 

**Theorem 1.** If X(t) is a zero-mean, weakly stationary process with continuous samples and one-sided spectral density g(v) of bounded support  $[0, \bar{v}]$ ,  $\bar{\nu} < \infty$ , then the family of FD processes  $\{X_d\}$  in Eq. (12) converges weakly to X in  $C[0, \tau]$  as  $d \to \infty$ .

**Proof.** The complete proof can be found in [27]. We summarize here the main steps of this proof. Since the finite dimensional distributions of  $X_d(t)$  converge to those of X(t) (see Section 2.1), it remains to check the conditions of Theorems 8.1 and 8.2 in [26].

The first condition requires to show that the sequence  $\{X_d(0)\}\$  of random variables is tight, i.e., there exists a > 0 such that  $P(|X_d(0)| >$ a)  $\leq \varepsilon$  for arbitrary  $\varepsilon > 0$ . This follows from

$$\begin{split} E\left[|X_d(0)|\right] &= E\left[\left|\sum_{k=1}^d \Delta U_k\right|\right] \leq E\left[\left(\sum_{k=1}^d \Delta U_k\right)^2\right]^{1/2} \\ &= \left(\int_0^{\bar{v}} g(v)\,dv\right)^{1/2} < \infty \end{split}$$

which holds by the Cauchy–Schwarz inequality and properties of  $\{\Delta U_k\}$ and X(t). This gives  $P(|X_d(0)| > a) \le E[|X_d(0)|]/a \le (\int_0^{\bar{v}} g(v) dv)^{1/2}/a$  $\leq \varepsilon$  by the Chebyshev inequality, so that the condition is satisfied for  $a = \left(\int_0^{\bar{\nu}} g(\nu) \, d\nu\right)^{1/2} / \varepsilon$ .

The second condition requires to show that the sequence  $\{X_d(t)\}$ of processes is tight, i.e., for any  $\varepsilon, \eta > 0$  there exists  $\delta > 0$  such that  $P(W_d(\delta) > \varepsilon) \le \eta$  for all d, where  $W_d(\delta) = \sup_{|s-t| < \delta} |X_d(s) - X_d(t)|$ denotes the modulus of continuity of  $X_d(t)$ ,

$$X_d(s) - X_d(t) = \sum_{k=1}^d \left[ h(\nu_k) \, \Delta U_k + s(\nu_k) \, \Delta V_k \right],$$

 $h(\nu_k) = -2\sin(\alpha \nu_k)\sin(\beta \nu_k), \ s(\nu_k) = -2\cos(\alpha \nu_k)\sin(\beta \nu_k), \ \alpha = (s + 1)\sin(\beta \nu_k)$ t)/2 and  $\beta = (s-t)/2$ . It is shown in [27] via summation by parts that there exists M > 0 such that  $E[W_d(\delta)] \leq \delta M$  for any d and  $P(W_d(\delta) > \varepsilon) \le E[W_d(\delta)]/\varepsilon \le \delta M/\varepsilon$  by the Chebyshev inequality. This means that for given  $\varepsilon, \eta > 0$ , we have  $P(W_d(\delta) > \varepsilon) \leq \eta$  for  $\delta = \varepsilon \eta / M$ . Since the conditions of Theorem 8.2 [26] are satisfied, we conclude that the family of FD processes  $X_d$  is tight. Since the finite dimensional distributions of  $X_d$  converge to those of X, the family  $\{X_d\}$ of FD processes converges weakly to X in  $C[0, \tau]$  as  $d \to \infty$ .

More general results which hold for both stationary and nonstationary processes are available, as illustrated by the following theorem from [28] (Theorem 3.1), which is stated without proof.

**Theorem 2.** If the finite dimensional distributions of  $X_d(t)$  converge to those of X(t), X(t) has continuous samples and continuous correlation function and either (i) or (ii) holds, then

sup  $|X_d(t) - X(t)| \to 0$  in distribution as  $d \to \infty$ .

where 
$$(i) \sum_{k=1}^{\infty} \sqrt{E|Z_k^2|} \, L_k(\tau) < \infty, \text{ where } L_k(\delta) = \sup_{|s-t| \leq \delta} |\varphi_k(s) - \varphi_k(t)|, \\ \delta \in [0,\tau] \text{ and }$$

(ii) There is M>0 such that  $E\left[\sup_{0\leq t\leq \tau}|\dot{X}_d(t)|\right]\leq M$  for all  $d\geq 1$ and  $\{\varphi_k(t)\}\$  are continuously differentiable functions.

#### 4.1.2. Brownian Motion process

The previous theorem can be applied to show that the sequence of FD processes  $\{B_d\}$  in Eq. (3) converges weakly and almost surely (a.s.) to the standard Brownian motion *B* in  $C[0, \tau]$  as  $d \to \infty$ , i.e., the sequence of random variables  $\sup_{0 \le t \le \tau} |B_d(t) - B(t)|$  converges to zero in distribution and almost surely as  $d \to \infty$  [28] (Remark 3.1).

These theoretical observations are supported numerically by the plots of Figs. 1 and 2. The left and right panels of Fig. 1 show with solid and dashed lines five samples of B(t) and the corresponding samples of  $B_d(t)$ ,  $t \in [0,1]$ , for d = 10 and d = 20. The recurrence formula B(t + $\Delta t$ ) =  $B(t) + \sqrt{\Delta t} G$  was used to generated samples of B(t), where  $\Delta t$  = 0.001 and G denotes a standard Gaussian variable independent of B(t). Visual inspection of these plots suggests that the discrepancies between the samples of  $B_d(t)$  and B(t) decreases with d. This observation is consistent with the histograms of the error  $\sup_{0 \leq t \leq 1} |B(t,\omega) - B_d(t,\omega)|$ shown in the left and right panels of Fig. 2 for d = 10 and d = 20. The histograms are based on 5000 independent samples of B(t) and  $B_d(t)$ .

As previously stated, other FD models can be constructed for the target processes. For example, the process  $W_N$  interpolating linearly

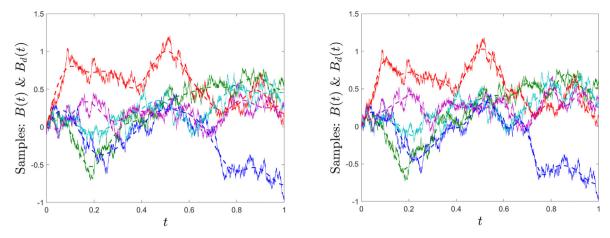


Fig. 1. Five samples of B(t) and corresponding samples of  $B_d(t)$  for d=10 and d=20 (left and right panels).

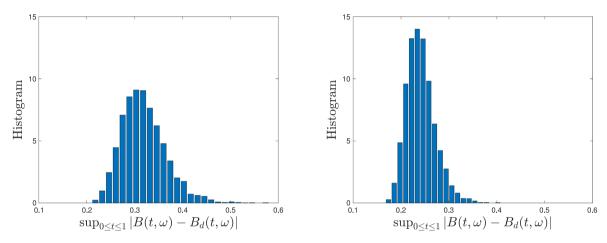


Fig. 2. Histograms of  $\sup_{0 \le t \le 1} |B(t, \omega) - B_n(t, \omega)|$  based on 5000 samples of B(t) and  $B_d(t)$  for d = 10 and d = 20 (left and right panels).

between values of B at equally spaced times  $0 = t_0^N < \cdots < t_i^N <$  $\cdots < t_N^N = 1$  of [0,1] constitutes an alternative FD model for B(t)in [0,1]. We prefer the FD model  $B_d(t)$  over  $W_N(t)$  for two reasons. First, its functional form is unrelated to time discretization, a features which facilitate, e.g., the numerical solution of differential equations with Brownian motion inputs. Second,  $B_d(t)$  is more accurate than  $W_N(t)$  in metrics of interest. For example, the estimates of the mean and standard deviation of the random error  $\sup_{0 \leq t \leq 1} |B(t) - B_d(t)|$  are 0.3191 (0.2414) and 0.0465 (0.0301) for d = N - 1 = 10 (d = N - 1 = 10) 20). The corresponding estimates of the mean and standard deviation of  $\sup_{0 \le t \le 1} |B(t) - W_N(t)|$  are 0.4126 (0.3189) and 0.0697 (0.0465). The estimates of the probabilities  $P(\sup_{0 \le t \le 1} |B(t) - B_d(t)| > a)$  and  $P\left(\sup_{0 \le t \le 1} |B(t) - W_N(t)| > a\right)$  are 0.0364 and 0.4712 for a = 0.41and d = N - 1 = 10 and are 0.0392 and 0.6264 for a = 0.30 and d = N - 1 = 20, which shows that the right tail of the distribution of  $\sup_{0 \le t \le 1} |B(t) - W_N(t)|$  is heavier than that of the distribution of  $\sup_{0 \le t \le 1} |B(t) - B_d(t)|$ . All estimates are based on 5,000 independent samples and 10,000 time steps in [0,1].

Theorems 1 and 2 show that the samples of the FD representations  $X_d(t)$  and  $B_d(t)$  of non-Gaussian processes X(t) and of the Brownian motion process B(t) can be used as substitutes for the samples of these processes provided that d is sufficiently large. This convergence is insufficient to conclude that the samples of the Slepian model  $S_a(t)$  of X(t) can be substitutes by those of the Slepian model  $S_{d,a}(t)$  of  $X_d(t)$  since it provides no information on the rate at which the samples of X(t) and  $X_d(t)$  change in time.

We have at least two options to construct FD Slepian models for non-Gaussian processes X(t). The first is to augment the conditions of Theorems 1 and 2 with requirements assuring that samples of  $S_{d,a}(t)$ 

can be used as substitutes for samples of  $S_a(t)$ . The other option is to require that X(t) and  $X_d(t)$  have differentiable samples and establish conditions under which  $\left(X_d, \dot{X}_d\right)$  converges weakly to  $\left(X, \dot{X}\right)$  as  $d \to \infty$  in the space of continuous functions. We develop the first option.

## 4.2. Sample properties of $S_a(t)$ and $S_{d,a}(t)$

Consider the random variables  $V_{\delta} = \left(X(\delta) - X(0)\right)/\delta$  and  $V_{d,\delta} = \left(X_d(\delta) - X_d(0)\right)/\delta$ ,  $\delta > 0$ , which constitute approximations of the derivatives of X(t) and  $X_d(t)$  at t = 0, The processes X(t) and  $X_d(t)$  have at least an a-upcrossing in  $[0,\delta]$  if  $X(0) < a < X(0) + \delta V_{\delta}$  and  $X_d(0) < a < X_d(0) + \delta V_{d,\delta}$ . The previous subsection gives conditions under which samples of X(t) and  $X_d(t)$  are similar in the metric of  $C[0,\tau]$ . This section establishes conditions under which samples of  $S_d(t)$  and  $S_{d,a}(t)$  are similar in the sense of the same metric.

We first show that, if  $X_d$  converges weakly to X in  $C[0,\tau]$ , the discrepancy between the initial conditions  $\left(X(0),V_\delta\right)$  and  $\left(X_d(0),V_{d,\delta}\right)$  of target and FD processes can be made as small as desired by increasing the stochastic dimension d of  $X_d(t)$ , see Theorem 3. Then, we show that the probability measure of the subset of samples of X(t) and  $X_d(t)$  which upcross a simultaneously in  $[0,\delta]$  can be made as large as desired by increasing d, see Theorem 4.

For given  $\varepsilon > 0$  and  $\delta > 0$ , denote by

$$\mathcal{A}_d(\varepsilon,\delta) = \{ |X_d(0) - X(0)| \le \varepsilon \} \cap \{ |V_{d,\delta} - V_{\delta}| \le \varepsilon \}$$
 (13)

the subset of  $\Omega$  on which the components of the random vectors  $\left(X(0),V_{\delta}\right)$  and  $\left(X_{d}(0),V_{d,\delta}\right)$  differ by less than  $\epsilon$ . On  $A_{d}(\epsilon,\delta)$ , we have  $X(0)-\epsilon\leq X_{d}(0)\leq X(0)+\epsilon$  and  $V_{\delta}-\epsilon\leq V_{d,\delta}\leq V_{\delta}+\epsilon$  so that

 $X_d(\delta)=X_d(0)+\delta\,V_{d,\delta}$  and  $X(\delta)=X(0)+\delta\,V_\delta$  satisfy the inequalities  $X(\delta)-\varepsilon\,(1+\delta)\leq X_d(\delta)\leq X(\delta)+\varepsilon\,(1+\delta).$  For given  $\delta>0$ , the probability measure of  $A_d(\varepsilon,\delta)$  depends on  $\varepsilon$  and d. It can be close to zero for small  $\varepsilon$  and low stochastic dimension d. We also note that the inclusion of the event  $\{|V_{d,\delta}-V_\delta|\leq \varepsilon\}$  in the definition of  $A_d(\varepsilon,\delta)$  is essential for the construction of Slepian models since, otherwise, we may have  $X(0,\omega), X(\delta,\omega) < a$ , so that this sample of X(t) does not upcross a in  $[0,\delta]$ , while  $X_d(0,\omega)=X(0,\omega)-\varepsilon< a$  and  $X_d(\delta,\omega)=X(\delta,\omega)+\varepsilon>a$  so that  $X_d(t,\omega)$  upcrosses a in  $[0,\delta]$ . In this case, the discrepancy  $V_{d,\delta}-V_\delta=2\,\varepsilon/\delta$  between the slopes of the target and FD processes can be very large if  $\delta\ll 1$ .

**Theorem 3.** If  $X_d$  converges weakly to X in  $C[0,\tau]$  as  $d \to \infty$ , then (i)  $P(A_d(\varepsilon,\delta)^c) \to 0$  as  $d \to \infty$  for any  $\varepsilon, \delta > 0$ .

(ii)  $P\left(A_d(\varepsilon,\delta)^c\right)\to 0$  as  $d\to\infty$  and  $\delta\to 0$  for any  $\varepsilon>0$  if, in addition, X(t) and  $X_d(t)$  are m.s. differentiable and  $\dot{X}_d(0)$  converges to  $\dot{X}(0)$  in m.s.

**Proof.** Note first that the probability of  $\mathcal{A}_d(\varepsilon,\delta)^c$  on which at least one of the initial conditions of the target and FD processes differ by more than  $\varepsilon$  can be bounded by

$$P(A_d(\varepsilon,\delta)^c) \le P(|X_d(0) - X(0)| > \varepsilon) + P(|V_{d,\delta} - V_{\delta}| > \varepsilon),$$

and that the first term converges to zero as  $d \to \infty$  by the weak convergence of  $X_d$  to X. It remains to show that  $P\big(|V_{d,\delta}-V_{\delta}|>\varepsilon\big)$  can be made as small as desired.

For (i), we have

$$\begin{split} |V_{d,\delta} - V_{\delta}| &\leq |X_d(\delta) - X(\delta)|/\delta + |X_d(0) - X(0)|/\delta \leq (2/\delta) \\ &\times \sup_{0 \leq t \leq \tau} |X_d(t) - X(t)|, \end{split}$$

so that  $P(|V_{d,\delta} - V_{\delta}| > \epsilon) \le P((2/\delta) \sup_{0 \le t \le \tau} |X_d(t) - X(t)| > \epsilon) \to 0$  as  $d \to \infty$  for any  $\epsilon, \delta > 0$  by the weak convergence of the family of FD processes  $X_d$  to X.

For (ii), note that the m.s. discrepancy between  $V_{d,\delta}$  and  $V_{\delta}$  can be bounded by

$$\begin{split} &E\left[\left(V_{d,\delta}-V_{\delta}\right)^{2}\right] \leq 4 \ \left\{ \ E\left[\left(V_{d,\delta}-\dot{X}_{d}(0)\right)^{2}\right] + E\left[\left(\dot{X}_{d}(0)-\dot{X}(0)\right)^{2}\right] \\ &+ E\left[\left(\dot{X}(0)-V_{\delta}\right)^{2}\right] \ \right\}, \end{split}$$

the first and third terms converge to zero as  $\delta \to 0$  for any  $d \geq 1$  by the m.s. differentiability of  $X_d(t)$  and X(t), and the second term approaches zero as  $d \to \infty$  by the postulated m.s. convergence of the family of random variables  $\dot{X}_d(0)$  to  $\dot{X}(0)$ . This implies the convergence  $P(|V_{d,\delta} - V_{\delta}| > \varepsilon) \to 0$  as  $d \to \infty$  and  $\delta \to 0$  and, therefore,  $P(A_d(\varepsilon,\delta)^c) \to 0$  as  $d \to \infty$  and  $\delta \to 0$  for any  $\varepsilon > 0$ .

The pairs of samples  $X(t,\omega)$  and  $X_d(t,\omega)$  of X(t) and  $X_d(t)$  in  $\mathcal{A}_d(\varepsilon,\delta)$  have similar initial conditions, since the discrepancies between the corresponding samples of  $\left(X(0),V_\delta\right)$  and  $\left(X_d(0),V_{d,\delta}\right)$  are within  $\varepsilon$ . Yet, they may not upcross a in  $[0,\delta]$  simultaneously. We show that the probability measure of the subset of  $\Omega$  containing target and FD samples which do not upcross a in  $[0,\delta]$  simultaneously can be made as small as desired under some conditions. The sets

$$\begin{split} &U(a,\delta) = \{X(0) < a < X(\delta)\} = \{X(0) < a < X(0) + \delta \, V_\delta, V_\delta \geq 0\} \quad \text{and} \\ &U_d(a,\delta) = \{X_d(0) < a < X_d(\delta)\} = \{X_d(0) < a < X_d(0) + \delta \, V_{d,\delta}, V_{d,\delta} \geq 0\} \end{split}$$

define the events of at least an a-upcrossing of X(t) and  $X_d(t)$  in  $[0, \delta]$ . For a sufficiently small  $\delta$ , X(t) and  $X_d(t)$  are likely to have a single a-upcrossing in  $[0, \delta]$  if these processes have, e.g., differentiable samples. On  $U(a, \delta) \cap U_d(a, \delta)$ , a-upcrossings of X(t) in  $[0, \delta]$  are accompanied by a-upcrossings of  $X_d(t)$  in this horizontal window. The pairs of samples of X(t) and  $X_d(t)$  with this property are also samples of the Slepian models  $S_a(t)$  and  $S_{d,a}(t)$ . On  $(U(a, \delta) \cap U_d(a, \delta))^c$ , at least one of the conditions  $X(0) < a < X(\delta)$  and  $X_d(0) < a < X_d(\delta)$  is not satisfied. For example, suppose that a sample  $X(t, \omega)$  of X(t) upcrosses a in  $[0, \delta]$ .

Then  $X(t,\omega)$  is also the sample  $S_a(t,\omega)$  of  $S_a(t)$ . Since  $X_d(t,\omega)$  does not upcross a in  $[0,\delta]$ ,  $S_{d,a}(t,\omega)$  does not exist so that the target Slepian sample  $S_a(t,\omega)$  has no FD pair.

**Theorem 4.** If the random vector  $(X(0), X(\delta))$  has a density  $f(\cdot, \cdot)$  with no atoms, then  $P(U_d(a, \delta)^c \mid U(a, \delta)) \sim O(\epsilon)$  on  $A_d(\epsilon, \delta)$  for any  $\epsilon > 0$ ,  $0 < \delta < 1$  and  $d \ge 1$ .

**Proof.** On  $A_d(\varepsilon, \delta)$ , the complement of  $U_d(a, \delta)$  satisfies the inequality

$$\begin{split} &U_d(a,\delta)^c = \{X_d(0) > a\} \cup \{X_d(\delta) < a\} \subseteq \{X(0) > a - \varepsilon\} \\ &\cup \{X(\delta) < a + \varepsilon \, (1 + \delta)\} \end{split}$$

so that

$$U_d(a,\delta)^c \cap U(a,\delta) \subseteq \{a - \varepsilon < X(0) < a, X(\delta) > a\}$$
  
 
$$\cup \{X(0) < a, a < X(\delta) < a + \varepsilon (1 + \delta)\}$$

since  $U_d(a,\delta)^c$  is bounded by a union of the events  $\{X(0)>a-\epsilon\}$  and  $\{X(\delta)< a+\epsilon\,(1+\delta)\}$ . The subset  $U_d(a,\delta)^c\cap U(a,\delta)$  of  $\mathcal{A}_d(\epsilon,\delta)$  consists of samples  $X(t,\omega)$  of X(t) which upcross a in  $[0,\delta]$  while their FD pairs  $X_d(t,\omega)$  do not upcross this level in  $[0,\delta]$ . We have

$$\begin{split} P\big(U_d(a,\delta)^c \cap U(a,\delta)\big) &\leq P\big(a-\varepsilon < X(0) < a, X(\delta) > a\big) \\ &\quad + P\big(X(0) < a-\varepsilon, a < X(\delta) < a+\varepsilon \, (1+\delta)\big) \end{split}$$

and

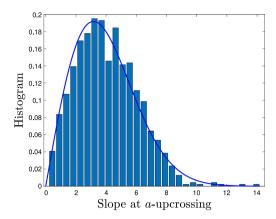
$$\begin{split} &P\big(a-\varepsilon < X(0) < a, X(\delta) > a\big) = \int_{a-\varepsilon}^a dx \, \int_a^\infty dy \, f(x,y) \\ &\leq \int_{a-\varepsilon}^a dx \, \int_{-\infty}^\infty dy \, f(x,y) \\ &= \int_{a-\varepsilon}^a f_{X(0)}(x) \, dx \sim O(\varepsilon) \\ &P\big(X(0) < a-\varepsilon, a < X(\delta) < a+\varepsilon \, (1+\delta)\big) = \int_a^{a+\varepsilon \, (1+\delta)} dx \, \int_{-\infty}^a dy \, f(x,y) \\ &\leq \int_a^{a+\varepsilon \, (1+\delta)} dy \, \int_{-\infty}^\infty dx \, f(x,y) = \int_a^{a+\varepsilon \, (1+\delta)} f_{X(\delta)}(y) \, dy \sim O(\varepsilon), \end{split}$$

where  $f_{X(0)}$  and  $f_{X(\delta)}$  denote the densities of X(0) and  $X(\delta)$ . The above bounds show that, on  $\mathcal{A}_d(\varepsilon,\delta)$  with arbitrary  $\varepsilon>0$ ,  $0<\delta<1$  and  $d\geq 1$ , we have  $P\left(U_d(a,\delta)^c\cap U(a,\delta)\right)\sim O(\varepsilon)$  and, therefore,  $P\left(U_d(a,\delta)^c\mid U(a,\delta)\right)\sim O(\varepsilon)$ .

In summary, Theorem 4 show that (1) the subset  $A_d(\varepsilon, \delta)$  on which the initial conditions for the target and FD Slepian models differs by less than a given  $\varepsilon > 0$  nearly fills the sample space  $\Omega$  for any  $\delta > 0$ provided that the stochastic dimension d is sufficiently large and (2) the fraction of samples of  $X_d(t)$  and X(t) in  $A_d(\varepsilon, \delta)$  which do not upcross a simultaneously in  $[0,\delta]$  is of order  $\varepsilon>0$  for any  $0<\delta<1$  and  $d \ge 1$ . This means that for given  $\varepsilon > 0$ , which controls the discrepancy between the initial conditions for the Slepian models  $S_{d,q}(t)$  and  $S_{q}(t)$ , and given  $\delta > 0$ , which defines the width of the horizontal window used to calculate a-upcrossings of  $X_d(t)$  and X(t), samples of  $S_{d,a}(t)$  can be used as surrogates for samples of  $S_a(t)$  provided that the stochastic dimension d is sufficiently large for the following two reasons. First, the subset  $A_d(\varepsilon, \delta)$  on which the initial conditions for the target and FD Slepian models differs by less that a given  $\varepsilon > 0$  nearly fills the sample space  $\Omega$  for any  $\delta>0$ . Second, subsets  $\mathcal{A}_d(\varepsilon,\delta)$  of sufficiently large stochastic dimensions, the fraction of samples of  $X_d(t)$  and X(t) which do not upcross a simultaneously in  $[0, \delta]$  is of order  $\varepsilon > 0$ . Since most of the samples of X(t) and  $X_d(t)$  are similar in the metric of  $C[0, \tau]$  by the weak convergence  $X_d \Longrightarrow X$  and upcross a simultaneously in  $[0, \delta]$ , most of samples of  $S_a(t)$  and  $S_{d,a}(t)$  can be paired and are similar.

## 5. Numerical illustrations

Two sets of examples are presented. The first constructs Slepian models for FD processes with stochastic dimension d = 4 which have



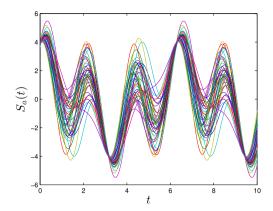


Fig. 3. Gaussian process: Histogram of  $\dot{X}(t)$ , t = 0, conditional on a-upcrossing in  $[0, \delta]$  and samples of  $S_a(t)$  for a = 4 (left and right panels).

independent Gaussian and dependent non-Gaussian coefficients  $\{Z_k\}$ . The second set examines the stationary solution of a linear oscillator subjected to a non-Gaussian process and of a nonlinear oscillator subjected to Gaussian white noise. It is shown that these processes satisfy the conditions of the above theorems so that it is expected that samples of  $X_d(t)$  and  $S_{d,a}(t)$  can be used as substitutes for samples of X(t) and  $S_a(t)$ . This observation is consistent with the numerical illustrations presented in this section.

#### 5.1. Slepian models for FD Gaussian and non-Gaussian processes

Let

$$X(t) = Z_1 \cos(vt) + Z_2 \sin(vt) + Z_3 \cos(3vt) + Z_4 \sin(3vt)$$
 (14)

be a real-valued FD process with stochastic dimension d=4. We consider two versions of this process corresponding to coefficients  $\{Z_k\}$  which are independent N(0,1) and coefficients  $\{Z_k\}$  which are zero-mean, unit-variance dependent non-Gaussian variables. The two target processes X(t) are FD processes with d=4 and random coefficients  $(Z_1,Z_2,Z_3,Z_4)$  of known distributions.

The following algorithm is used to generate samples of the Slepian models  $S_a(t)$  and of their slopes at the initial time. The index d is dropped for simplicity since  $X(t) = X_d(t)$  and  $S_a(t) = S_{d,a}(t)$ . For small  $\delta > 0$ , we generate samples of  $(X(0), X(\delta))$ , which result from samples of  $(Z_1, Z_2, Z_3, Z_4)$ , and retain the first n samples satisfying the condition  $X(0) < a < X(\delta)$ , where n is a specified sample size. The selected samples of  $(Z_1, Z_2, Z_3, Z_4)$  are subsequently used to produce samples of X(t) which exhibit a-upcrossings in  $[0, \delta]$ . They are the defining samples of  $S_a(t)$ . The corresponding samples of  $X(t) = v(Z_2 + 3Z_4)$  are used to construct histograms of the slope of X(t), t = 0, at a-upcrossings.

## 5.1.1. Independent Gaussian coefficients

The numerical results in Fig. 3 are for a time interval  $[0,\tau]$ ,  $\tau=10$ ,  $n_t=200$  time steps, v=1, a=4 and  $\delta=\Delta t=\tau/n_t$ . Smaller values of  $\delta$  have not change results. The left panel of the figure shows a histogram of  $\dot{X}(t)$  at the times of a-upcrossings based on n=1000 samples. The solid line is the Rayleigh density of this conditional random variable, which is known for stationary Gaussian processes, see [17] (Sect 10.3). The right panel shows samples of  $S_a(t)$  corresponding to a-upcrossings of X(t) at t=0.

# 5.1.2. Dependent non-Gaussian coefficients

Let  $Y(t) = (A\cos(vt) + B\sin(vt))^3$ , where A and B are independent N(0,1) variables. This process has the functional form of X(t) in Eq. (14) with coefficients  $\tilde{Z}_1 = (3/4) \left(A^3 + A B^2\right)$ ,  $\tilde{Z}_2 = (3/4) \left(A^2 B + B^3\right)$ ,  $\tilde{Z}_3 = (1/4) \left(A^3 - 3 A B^2\right)$  and  $\tilde{Z}_4 = (1/4) \left(3 A^2 B - B^3\right)$ . These coefficients are uncorrelated but dependent random variables, as functions of the Gaussian variables A and B, and non-Gaussian, as nonlinear functions

of A and B. Consider the process X(t) in Eq. (14) whose random coefficients  $\{Z_k\}$  are obtained from  $\{\tilde{Z}_k\}$  by scaling, i.e.,  $Z_k = \tilde{Z}_k/\mathrm{Std}[Z_k]$ ,  $k = 1, \ldots, 4$ , so that they have unit variances.

The plots in Fig. 4 are similar to those in Fig. 3. They are for a time interval  $[0,\tau]$ ,  $\tau=10$ ,  $n_t=200$  time steps,  $\nu=1$ , a=4 and  $\delta=\Delta t=\tau/n_t$ . The left panel of Fig. 4 shows a histogram of  $\dot{X}(t)$  conditional on a-upcrossings in  $[0,\delta]$  which is constructed from n=1000 samples. The solid line is the Rayleigh density of this conditional random variable corresponding to stationary Gaussian processes, see [17] (Sect 10.3). As expected, this density differs significantly from the histogram of the slope of this non-Gaussian process at the a-upcrossing time. The right panel shows samples of  $S_a(t)$  corresponding to a-upcrossings of X(t) in  $[0,\delta]$ . The ranges of the Slepian processes in Figs. 3 and 4 differ since, although the coefficients  $\{Z_k\}$  are uncorrelated and have the same means and variances, their joint distributions differ significantly. The coefficients  $\{Z_k\}$  are independent Gaussian and dependent non-Gaussian variables in Figs. 3 and 4.

**Construction of FD Slepian models:** As noted, the target and FD processes coincide so that  $X(t) = X_d(t)$  and  $S_a(t) = S_{d,a}(t)$ , the functional form of X(t) is given by Eq. (14) and the distribution of the random coefficients  $\{Z_k\}$  is known. Large sets of samples of these coefficients can be generated efficiently since they are independent Gaussian variables or nonlinear functions of independent Gaussian variables.

Samples  $\{Z_k(\omega)\}$  of  $\{Z_k\}$  can be mapped into samples  $X(t,\omega)$  of X(t) by elementary calculations via Eq. (14). The subset of these samples with the property  $X(0,\omega) < a < X(\delta,\omega)$  are the corresponding samples of  $S_a(t)$ , where  $\delta > 0$  denotes the width of the horizontal window used to define a-upcrossings of X(t).

## 5.2. Target and FD Slepian models for dynamical system states

The first target process X(t) is the solution of a linear dynamical system subjected to a polynomial of a Gaussian process. The second process X(t) is the solution of an Itô stochastic differential equation driven by Brownian motion. We construct FD processes  $X_d(t)$  for X(t), Slepian models  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$  for X(t) and  $X_d(t)$  and assess the accuracy of the FD Slepian models  $S_{d,a}(\alpha)$ , where  $\alpha = t - t_0 \ge 0$  and  $t_0$  denotes the time from which X(t) can be assumed to be stationary.

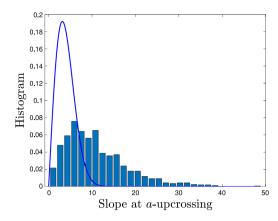
## 5.2.1. Linear dynamical systems

Let X(t),  $t \ge 0$ , be the solution of

$$\ddot{X}(t) + 2\zeta v_0 \dot{X}(t) + v_0^2 X(t) = Y(t)^2, \quad t \ge 0,$$
(15)

with zero initial conditions X(0) = 0 and  $\dot{X}(0) = 0$ , where  $\zeta \in (0, 1)$ ,  $v_0 > 0$ , Y(t) is the stationary solution of

$$dY(t) = -\rho Y(t) dt + \sqrt{2\rho} dB(t), \quad \rho > 0, \quad t \ge 0,$$
 (16)



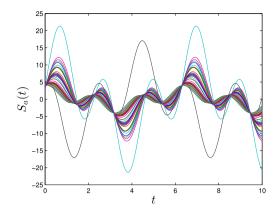


Fig. 4. Non-Gaussian process: Histogram of  $\dot{X}(t)$ , t = 0, conditional on a-upcrossing in  $[0, \delta]$  and samples of  $S_a(t)$  for a = 4 (left and right panels).

and B(t),  $t \ge 0$ , denotes the standard Brownian motion. The solution of the latter equation for the initial condition  $Y(0) \sim N(0,1)$  assumed to be independent of B(t) is a stationary Gaussian process with mean E[Y(t)] = 0 and correlation function  $E[Y(s)Y(t)] = \exp(-\rho |s-t|)$ . The stationary solution X(t) of Eq. (15) is a non-Gaussian process which can have large skewness and kurtosis coefficients [25] (Example 7.23).

Various methods can be used to construct FD processes  $X_d(t)$  of X(t). For example, these processes can be obtained by projecting samples of X(t) on the eigenfunctions of the correlation function of the stationary solution of Eq. (15). The correlation function of X(t) can be obtained numerically or as the Fourier transform of the spectral density of this process, which is given by the product of the spectral density of  $Y(t)^2$  and the frequency response function of the defining equation of X(t) [15] (Section 5.2.2). FD processes  $X_d(t)$  can also be obtained from FD representations  $B_d(t)$  of the Brownian motion process B(t) by solving Eq. (16) with  $B_d(t)$  in place of B(t) to obtain an FD model  $Y_d(t)$  of Y(t) and, then, solving Eq. (15) with  $Y_d(t)$  in place of Y(t).

An alternative method is used. We construct FD processes  $Y_d(t)$  of Y(t) and develop corresponding FD processes of the input  $Y(t)^2$ , FD processes for the solution  $X_d(t)$  of Eq. (15) to  $Y_d(t)^2$  and Slepian models  $S_{d,a}(t)$ . Then, we show that the families of FD processes  $X_d(t)$  and Slepian models  $S_{d,a}(t)$  converge weakly to X(t) and  $S_d(t)$  in  $C[0,\tau]$  as  $d\to\infty$ . The FD processes  $Y_d(t)$  have the form

$$Y_d(t) = \sum_{k=1}^d Z_k \, \varphi_k(t), \quad 0 \le t \le \tau, \tag{17}$$

where  $\varphi_k(t)$  are the eigenfunctions of the stationary correlation function of Y(t) in  $[0,\tau]$  and the samples of the random coefficients  $\{Z_k\}$  are obtained by projecting samples of Y(t) on the basis functions  $\{\varphi_1,\ldots\varphi_d\}$ , see Eq. (2). Note that the large sets of samples of the random coefficients  $\{Z_k\}$  can be obtained efficiently since they result from input samples, i.e., samples of Y(t). The remainder of this subsection provides technical details and numerical results.

**Property 1.** The processes Y(t),  $Y(t)^2$  and X(t) have continuous samples almost surely

**Proof.** The increment Y(t+h) - Y(t) is a zero-mean Gaussian variable with variance  $2\left(1-\exp(-\rho|h|)\right)$  so that  $E\left[\left(Y(t+h)-Y(t)\right)^{\alpha}\right] \le c \ h^{1+\beta}$  for  $\alpha=4,\ \beta=2$  and  $c=12\ \rho^2$ . Since Y(t) is separable, almost all samples of Y(t) are continuous in any finite time interval [29] (Proposition 4.2). Since  $Y(t)^2$  is a continuous mapping of Y(t) and X(t) is obtained from  $Y(t)^2$  by integration, these processes also have continuous samples.  $\blacktriangle$ 

**Property 2.** The FD processes  $Y_d(t)$  and  $Y_d(t)^2$  converge weakly and a.s. to Y(t) and  $Y(t)^2$  in  $C[0, \tau]$  as  $d \to \infty$ 

**Proof.** We show as in the proof of Theorem 3 that  $\{Y_d(t)\}$  satisfies the conditions of Theorem 12.3 in [26]. Note first that the sequence of

random variables  $\{Y_d(0) = \sum_{k=1}^d Z_k \varphi_k(0)\}$  is tight since

$$P\big(|Y_d(0)|>a\big)\leq \frac{E\big[Y_d(0)^2\big]}{a^2}\leq \frac{E\big[Y(0)^2\big]}{a^2}=\frac{1}{a^2}$$

so that for any  $\varepsilon > 0$  there is an  $a = 1/\sqrt{\varepsilon}$  for which  $P(|Y_d(0)| > a) \le \varepsilon$  for all d > 1.

Since  $Y_d(s) - Y_d(t) = \sum_{k=1}^d Z_k \psi_k(s,t) \sim N(0, \sum_{k=1}^d \lambda_k \psi_k(s,t)^2)$  with  $\psi_k(s,t) = \varphi_k(s) - \varphi_k(t)$  and  $\lambda_k = E[Z_k^2]$ , we have

$$E[(Y_d(s) - Y_d(t))^4] = 3\left(\sum_{k=1}^d \lambda_k \, \psi_k(s, t)^2\right)^2$$

$$\leq 3\left(\sum_{k=1}^\infty \lambda_k \, \psi_k(s, t)^2\right)^2 = 12\left(1 - e^{\rho \, |s - t|}\right)^2$$

$$\leq 12\left(\rho \, |s - t|\right)^2 = 12\,\rho^2 \, (s - t)^2 = \left(h(s) - h(t)\right)^2,$$

where  $h(s) = \sqrt{12 \rho^2} \, s$  is a continuous increasing function, i.e.,  $E\left[\left(Y_d(s) - Y_d(t)\right)^{\gamma}\right] \le \left(h(s) - h(t)\right)^{\alpha}$  for  $\gamma = 4$  and  $\alpha = 2$ . We conclude that  $Y_d$  converges weakly to Y in  $C[0,\tau]$  as  $d \to \infty$ . Since these processes are Gaussian, we also have a.s. convergence by the Itô-Nisio lemma [30]. This implies the weak and almost sure convergence of  $Y_d(t)^2$  to  $Y(t)^2$  by the continuous mapping theorem [26].

**Property 3.** The FD processes  $X_d(t)$  converge weakly and a.s. to X(t) in  $C[0,\tau]$  as  $d\to\infty$ 

**Proof.** The solutions of the oscillator in Eq. (15) to  $Y(t)^2$  and  $Y_d(t)^2$  are

$$X(t) = \int_{0}^{t} \phi(t-s) Y(s)^{2} ds$$
 and  $X_{d}(t) = \int_{0}^{t} \phi(t-s) Y_{d}(s)^{2} ds$  (18)

where  $\phi(u) = \exp(-\zeta v_0 u) \sin(v_d u)/v_d$  and  $v_d = v_0 \sqrt{1 - \zeta^2}$ . The discrepancy between X(t) and  $X_d(t)$  can be bounded by

$$\begin{split} |X(t) - X_d(t)| &\leq \int_0^t |\phi(t - s)| \, |Y(s)^2 - Y_d(s)^2| \, ds \\ &\leq \frac{1}{v_d} \, \int_0^t |Y(s)^2 - Y_d(s)^2| \, ds \end{split}$$

so that

$$\begin{split} \sup_{0 \le t \le \tau} |X(t) - X_d(t)| &\leq \frac{1}{\nu_d} \int_0^\tau |Y(s)^2 - Y_d(s)^2| \, ds \\ &\leq \frac{\tau}{\nu_d} \sup_{0 \le t \le \tau} |Y(t)^2 - Y_d(t)^2|. \end{split}$$

This implies that  $X_d$  converges weakly and a.s. to X in  $C[0,\tau]$  as  $d\to\infty$  by Property 2. Similar arguments hold for the stationary solutions which result from Eq. (18) in which the integration interval is extended from [0,t] to  $(-\infty,t]$ .

The above properties show that the conditions of Theorems 4 and 5 are satisfied so that  $P(\mathcal{A}_d(\epsilon, \delta)^c)$  can be made as small as desired by

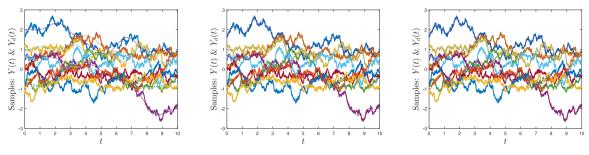


Fig. 5. Ten samples of Y(t) (solid lines) and  $Y_d(t)$  (dashed lines) for d = 10, 40 and 80 (left, middle and right panels).

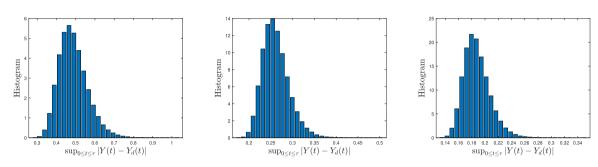


Fig. 6. Histograms of the error  $\sup_{0 \le t \le r} |Y(t) - Y_d(t)|$  for d = 10, 40 and 80 (left, middle and right panels).

increasing d and, on  $A_d(\varepsilon,\delta)$  most of the samples of X(t) and  $X_d(t)$  upcross a simultaneously in  $[0,\delta]$ . Accordingly, we expect that the samples of  $S_{d,a}(t)$  can be used as surrogates for samples of  $S_a(t)$  for sufficiently large d since the fraction of samples X(t) and  $X_d(t)$  which do not upcross a simultaneously in  $[0,\delta]$  is of order  $\varepsilon>0$ , which can be selected arbitrarily small.

The following numerical results are for  $\rho=0.12,\ \tau=10,\ v_0=3,\ \zeta=0.28$  and a=1.3. The statistics are based on 100,000 independent samples of  $Y(t),\ Y_d(t),\ X(t)$  and  $X_d(t)$  generated at a time step  $\Delta t=0.01$ . The a-upcrossings of X(t) and  $X_d(t)$  are recorded in the horizontal window  $[t_0,t_0+\delta]$  with  $t_0=5$ . These values  $t_0$  and  $\delta$  assure that  $X(t),\ t\geq t_0$ , can be viewed as stationary and that the width of the horizontal window is adequate for the frequency content of X(t). Since  $X_d$  converges a.s. to X in  $C[0,\tau]$  as  $d\to\infty$  by the a.s. convergence of  $Y_d$  and  $Y_d^2$  to Y and  $Y_s^2$ , it is expected that the FD representations  $X_d(t)$  of X(t) and  $S_{d,\delta}(t)$  of  $S_a(t)$  will improve with the stochastic dimension d. This theoretical statement is in agreement with the numerical results of Figs. 5 to 9.

Fig. 5 shows with solid and dashed lines ten samples of Y(t) and the corresponding samples of  $Y_d(t)$ . Histograms of the error  $\sup_{0 \le t \le r} |Y(t) - Y_d(t)|$  are in Fig. 6 for d=10, 40 and 80 (left, middle and right panels). The plots show that the accuracy of  $Y_d(t)$  improves with its stochastic dimension d in agreement with the above theoretical arguments. Note that the histograms are at different scales.

The plots of Figs. 7 and 8 are similar to those of Figs. 5 and 6 but are for the solutions X(t) and  $X_d(t)$  of Eq. (15) to Y(t) and  $Y_d(t)$ . They show with solid and dashed lines ten samples of X(t) and the corresponding samples of  $X_d(t)$  defined by Eq. (18) and histograms of the error  $\sup_{0 \le t \le \tau} |X(t) - X_d(t)|$  for d = 10, 40 and 80 (left, middle and right panels). The accuracy of  $X_d(t)$  improves with its stochastic dimension d in agreement with the above theoretical arguments. For example, the ranges of the histograms in Fig. 8 are [0,0.45],  $[1,12] \times 10^{-3}$  and  $[0.5,2.5] \times 10^{-3}$  for d = 10, 40 and 80.

The solid and dashed lines in Fig. 9 are the subset of samples of X(t) and  $X_d(t)$  which upcross a=1.3 through the horizontal window  $[t_0,t_0+\delta]$  with  $\delta=10$   $\Delta t$  for d=10, 40 and 80 (left, middle and right panels). They constitute the samples of the Slepian models  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$ . These samples are shown in Fig. 9 for d=10, 40 and 80. The defining samples of  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$  nearly coincide for  $d\geq 40$ . However, the Slepian models  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$  differ for d=10 since,

if a sample  $X(t, \omega)$  of X(t) upcrosses a in  $[t_0, t_0 + \delta]$ , the corresponding sample of  $X_d(t, \omega)$  of  $X_d(t)$  may not upcross a in this time interval. The fractions of samples of  $X_d(t)$  which upcross and do not upcross a in  $[t_0, t_0 + \delta]$  when X(t) upcrosses a in this time interval are 0.4444 and 0.5556. In contrast, these fractions are 1 and 0 for  $d \ge 40$ . This shows that the relatively small discrepancy between the samples of X(t) and  $X_d(t)$  for d=10 illustrated and quantified in the left panels of Figs. 5 and 6, which may not be relevant for estimating moments and other global properties of X(t), is insufficient to guarantee that samples of  $S_{d,q}(t)$  can be used as substitutes for samples of  $S_q(t)$ . The plots of Fig. 10 show plots as in Fig. 9 but for a horizontal window of size  $\delta = 5 \Delta t$  rather than  $\delta = 10 \Delta t$ . As expected, the number of Slepian samples is smaller for this window since fewer samples of X(t) and  $X_d(t)$  upcross a in  $[0, \delta = 5 \Delta t]$ . Also, the samples of  $S_a(t)$  and  $S_{d,a}(t)$ nearly originate at a for all stochastic dimensions and the samples of  $S_{d,a}(t)$  for d = 40,80 coincide with those of  $S_{a}(t)$ .

**Construction of FD Slepian models:** Consider a samples  $Y(t,\omega)$  of the input process Y(t) defined by Eq. (16) and denote by  $X(t,\omega)$  the solution of Eq. (15) to this samples. The samples of X(t) with the property  $X(t_0,\omega) < a < X(t_0+\delta,\omega)$  are the samples  $S_a(t,\omega)$  of the Slepian model  $S_a(t)$ ,  $t \ge t_0$ , i.e., the subset of samples of X(t) which upcross a in the time interval  $[t_0,t_0+\delta]$ , where  $t_0>0$  is an arbitrary time and  $\delta>0$  denotes the width of the horizontal window used to define a-upcrossings of X(t) and  $X_a(t)$ 

The FD pairs  $X_d(t,\omega)$  and  $S_{d,a}(t,\omega)$  of the samples  $X(t,\omega)$  and  $S_a(t,\omega)$  are constructed in three steps. First, samples  $\{Z_k(\omega)\}$  of the random coefficients  $\{Z_k\}$  of the FD representation  $Y_d(t)$  of Y(t) in Eq. (17) are obtained by projecting the samples  $Y(t,\omega)$  on the basis functions  $\{\varphi_k(t)\}$ . Second, the resulting samples of  $\{Z_k\}$  are mapped into samples of  $X_d(t,\omega) = \sum_{k,l=1}^d Z_k(\omega) Z_l(\omega) \theta_{kl}(t)$ , where the deterministic functions  $\theta_{kl}(t) = \int_0^t \phi(t-s) \varphi_k(s) \varphi_l(s) \, ds$  can be precalculated and stored. The construction of these samples involves elementary calculations. Third, the subset of samples of  $X_d(t)$  with the property  $X_d(t_0,\omega) < a < X_d(t_0 + \delta, \omega)$  defines the samples  $S_{d,a}(t,\omega)$  of the FD Slepian model of  $X_d(t)$ , which are the pairs of the samples  $S_a(t,\omega)$  of the Slepian model of X(t).

## 5.2.2. Nonlinear dynamical systems

Suppose that the target process X(t) is a component of the vector-valued stationary solution of a stochastic differential equation subjected

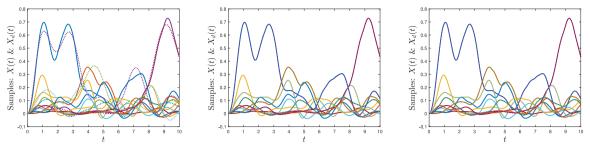


Fig. 7. Ten samples of X(t) (solid lines) and  $X_d(t)$  (dashed lines) for d = 10, 40 and 80 (left, middle and right panels).

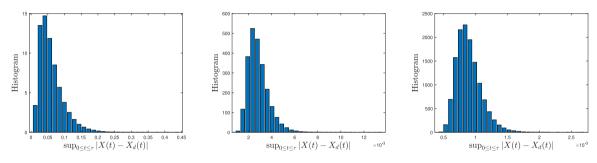


Fig. 8. Histograms of the error  $\sup_{0 \le t \le r} |X(t) - X_d(t)|$  for d = 10, 40 and 80 (left, middle and right panels).

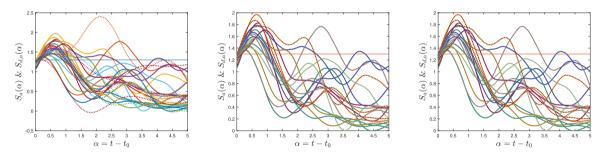


Fig. 9. Slepian models  $S_a(\alpha)$  (solid lines) and  $S_{d,a}(\alpha)$  (dashed lines),  $\alpha=t-t_0$ , of X(t) and  $X_d(t)$  for  $\alpha=1.3$  for d=10, 40 and 80 (left, middle and right panels) and  $\delta=10$   $\Delta t$ .

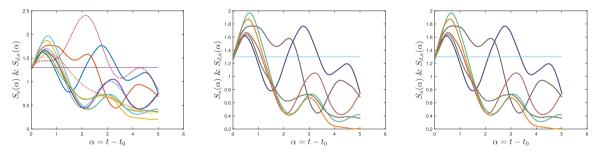


Fig. 10. Slepian models  $S_a(\alpha)$  (solid lines) and  $S_{d,a}(\alpha)$  (dashed lines),  $\alpha=t-t_0$ , of X(t) and  $X_d(t)$  for a=1.3 for d=10, 40 and 80 (left, middle and right panels) and  $\delta=5 \Delta t$ .

to Gaussian white noise, which is interpreted as the formal derivative of the Brownian motion B(t). We construct FD representations  $X_d(t)$  of X(t) and Slepian models  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$  for X(t) and  $X_d(t)$  by following the approach in the previous example and show that samples of X(t) and  $S_a(t)$  can be substituted by samples of  $X_d(t)$  and  $S_{d,a}(t)$  under some conditions. FD representations  $X_d(t)$  of X(t) are constructed from solutions of Eq. (19) with  $B_d(t)$  in place of B(t), where  $B_d(t)$  are FD processes of the type in Eq. (4), which converge weakly and a.s. to B(t) in the space of continuous functions, see Theorem 3. We first show that the solutions  $X_d(t)$  converges weakly and a.s. to X(t) in the space of continuous functions. Then, we show that the conditions of Theorem 4 are satisfied for the numerical versions of Slepian models so that FD Slepian processes can be used as surrogates for target Slepian processes.

For simplicity, the arguments are restricted to real-valued diffusion processes. The extension to vector-valued diffusion processes is direct [25] (Chap. 7). Let X(t) be a real-valued diffusion process defined by the Itô stochastic differential equation

$$dX(t) = a(X(t), t) dt + b(X(t), t) dB(t), \quad X(0) = X_0, \quad t \in [0, \tau],$$
 (19)

where  $X_0$  is a random variable independent of the Brownian motion process B(t). We assume that the drift and diffusion coefficients, a and b, are such that Eq. (19) has a unique strong solution [25] (Sect 4.7.1.1).

The solution of Eq. (19) can be obtained by driving this equation with colored rather than white Gaussian noise provided that it is interpreted in the Stratonovich sense [31]. The standard approximation of B(t) is the FD process  $W_N(t)$  which interpolates linearly between values of B(t) at the points of a partition  $0 = t_0^N < \cdots < t_i^N < \cdots < t_$ 

 $t_N^N=\tau$  of  $[0,\tau]$ . The process  $W_N(t)$  has continuous samples of finite variation on compacts whose derivatives are piecewise continuous. The sequences of processes  $\{W_N\}$  converges to B a.s. and uniformly in  $[0,\tau]$  as the partition is refined, i.e.,  $\max_{1\leq i\leq N}(t_i^N-t_{i-1}^N)\to 0$  as  $N\to\infty$  [25] (Sect 4.7.1.2).

The Wong–Zakai theorem [31] states that the solution  $Y_N(t)$  of the differential equation

$$dY_N(t) = a^* (Y_N(t), t) dt + b(Y_N(t), t) dW_N(t), \quad Y_N(0) = X_0, \quad t \in [0, \tau],$$
(20)

with

$$a^*(y,t) = a(y,t) - \frac{1}{2}b(y,t)\frac{\partial b(y,t)}{\partial y}$$
 (21)

converges to X(t) a.s. and uniformly in [0,1] as  $N\to\infty$ . The defining equation of  $Y_N(t)$  results from Eq. (19) by correcting its drift via Eq. (21) and by replacing the Brownian motion in Eq. (20) with  $W_N(t)$ . Since Eq. (20) is of the Stratonovich type, ordinary calculus can be used to solve for  $Y_N(t)$ .

We first show that the solution  $X_d(t)$  of Eq. (20) with  $B_d(t)$  in place of  $W_N(t)$ , i.e., the equation

$$\begin{split} dX_d(t) &= a^* \left( X_d(t), t \right) dt + b \left( X_d(t), t \right) dB_d(t), \quad X_d(0) = X_0, \\ & t \in [0, \tau], \quad \text{or, equivalently,} \end{split}$$

$$\dot{X}_d(t) = a^* \left( X_d(t), t \right) + b \left( X_d(t), t \right) \dot{B}_d(t), \quad X_d(0) = X_0, \quad t \in [0, \tau], \quad (22)$$

with  $a^*$  given by Eq. (21) and  $\dot{B}_d(t)=dB_d(t)/dt=\sum_{k=1}^n Z_k\,\dot{\varphi}_k(t)$ , also converges to X(t) a.s. and uniformly in [0,1] as  $d\to\infty$ . We prefer the FD process  $B_d(t)$  since it is more efficient than  $W_N$ , as discussed in Section 4.1.2. That the solution  $X_d(t)$  has the stated properties results from the fact that the FD process  $B_d(t)$  satisfies the four conditions of the Wong–Zakai theorem in [31]. We now state these conditions and show that they are satisfied for  $\tau=1$  without loss of generality.

**Condition 1.** For almost all  $\omega$ ,  $B_d(t,\omega) \to B(t,\omega)$  for all  $t \in [0,1]$  and its samples are continuous of bounded variation.

Since the basis functions  $\{\varphi_k(t)\}$  are continuous and differentiable, almost all samples of  $B_d(t)$  are continuous and of bounded variation. For a fixed time t,  $B_d(t)$  is the sum  $\sum_{k=1}^d Z_k \, \varphi_k(t)$  of the independent Gaussian variables  $\{Z_k \, \varphi_k(t)\}$ ,  $k=1,\ldots,d$ , which converges in m.s. to the random variable  $B(t) \sim N(0,t)$ . Since this convergence implies the convergence in probability and the convergence in probability is equivalent to the almost sure convergence for sums of independent random variables by the Lévy theorem [32] (Theorem 7.3.2), we have  $B_d(t) \stackrel{\mathrm{a.s.}}{\longrightarrow} B(t)$  for all  $t \in [0,1]$ .

**Condition 2.** Condition 1 and there exists  $k(\omega) > 0$  and  $n_0(\omega)$  both finite such that  $|B_d(t,\omega)| \le k(\omega)$  a.s. for all  $n > n_0(\omega)$  and all  $t \in [0,1]$ .

Since  $B_d(t)$  and B(t) have continuous samples, we can find  $l_d(\omega) > 0$  and  $l(\omega)$  finite such that  $\sup_{0 \le t \le 1} |B_d(t,\omega)| \le l_d(\omega)$  and  $\sup_{0 \le t \le 1} |B(t,\omega)| \le l(\omega)$  for almost all samples. The convergence of the samples of  $B_d(t)$  to those of B(t) in the metric of C[0,1] (see Condition 4 below) means that for almost all  $\omega \in \Omega$  and given  $\varepsilon > 0$ , there exists  $n_0(\omega)$  such that  $\sup_{0 \le t \le 1} |B_d(t,\omega) - B(t,\omega)| < \varepsilon$  for  $n \ge n_0(\omega)$  so that  $\sup_{0 \le t \le 1} |B_d(t,\omega)| \le \sup_{0 \le t \le 1} |B_d(t,\omega) - B(t,\omega)| + \sup_{0 \le t \le 1} |B(t,\omega)| < \varepsilon + l(\omega) := k(\omega)$  for  $n \ge n_0(\omega)$ .

**Condition 3.** Condition 2 and  $B_d(t,\omega)$  has piecewise continuous derivatives.

The basis functions  $\{\varphi_k(t)\}$  are continuous and differentiable and so is  $B_d(t)$ .

**Condition 4.** Condition 3 and  $B_d(t,\omega) \to B(t,\omega)$  uniformly in [0,1]. This follows from the almost sure convergence of  $B_d$  to B in C[0,1] as  $d \to \infty$ .

We conclude that the solution  $X_d(t)$  of Eq. (22) converges a.s. and uniformly in [0,1] to the solution X(t) of Eq. (19) as  $d \to \infty$  so that samples of  $X_d(t)$  can be used as substitutes for samples of X(t) for a sufficiently large stochastic dimension.

Let X(t),  $t \ge 0$ , be a vector-valued process defined by the Itô differential equation

$$dX(t) = a(X(t)) dt + b dB(t), \quad t \ge 0,$$
(23)

where  $X(t) = \left[X_1(t) = Y(t), X_2(t) = \dot{Y}(t)\right]', \ a\left(X(t)\right) = \left[X_2(t), -u'\left(X_1(t)\right) - \rho X_2(t)\right]', \ b = \left[0, \sqrt{\pi \, g_0}\right]'$  are two-dimensional column vectors and Y(t) is the solution of

$$\ddot{Y}(t) + \rho \dot{Y}(t) + u'(Y(t)) = W(t), \quad t \ge 0, \tag{24}$$

driven by a zero mean Gaussian white noise W(t) with one-sided spectral density  $g(v)=g_0,\ v\geq 0$ . The derivative u'(y) of u(y) defines the elastic restoring force of the oscillator. The density of the stationary solution X(t) of Eq. (23) has the expression

$$f(x) = f(x_1, x_2) = c \exp\left(-\frac{2\rho}{\pi g_0} u(x_1)\right) \exp\left(-\frac{2\rho}{\pi g_0} \frac{x_2^2}{2}\right),$$

$$x = (x_1, x_2) \in \mathbb{R}^2,$$
(25)

where c > 0 is a normalization constant [25] (Example 7.42). If the initial condition X(0) has the density f(x) in Eq. (25), the solution X(t) of Eq. (23) is a stationary process with the marginal density f(x). Note also that  $X_2(t) = \dot{X}(t)$  is a zero-mean Gaussian variable with finite variance  $\pi g_0/(2\rho)$  which is independent of  $X_1(t) = X(t)$ .

Let  $X_d(t)$  be the solution of Eq. (23) with  $B_d(t)$  in place of B(t). The differential equations of X(t) and  $X_d(t)$  have the same form since the input B(t) is additive. The above considerations show that the solution  $X_d(t)$  of Eq. (23) with  $B_d(t)$  in place of B(t) converges a.s. and uniformly in any bounded interval to X(t) as  $d \to \infty$  so that samples of  $X_d(t)$  can be used as surrogates for samples of X(t) provided that the stochastic dimension is sufficiently large.

Since  $X_d$  converges weakly to X, we have  $P\left(A_d(\varepsilon,\delta)^c\right) \to 0$  as  $d \to \infty$  for any  $\varepsilon, \delta > 0$  by Theorem 3 so that the initial conditions of  $S_a(t)$  and  $S_{d,a}(t)$  differ by less than  $\varepsilon$  for almost all samples as  $d \to \infty$ . The distribution of the stationary solution X(t) of Eq. (23) is continuous by Eq. (25) so that the probability that a-upcrossings of X(t) in  $[0,\delta]$ ,  $0 < \delta < 1$ , are accompanied by a-upcrossings of  $X_d(t)$  with nearly unit probability for sufficiently large d, see Theorem 4. These observations suggest that samples of  $X_d(t)$  and  $S_{d,a}(t)$  can be used as surrogates for samples of X(t) and  $S_a(t)$  for sufficiently large d and that the accuracy of these FD representations improves with their stochastic dimensions.

The following numerical results are for  $\rho=0.5$ ,  $u'(x)=\alpha x+\beta x^3$ ,  $\alpha=\beta=1$ ,  $g_0=1$ , zero initial conditions, stochastic dimension d=20, 40 and 60, a time interval [0, 10] and 100,000 independent samples of X(t). Fig. 11 shows with solid and dashed lines ten samples of X(t) and the corresponding samples of Y(t) in the time interval [0, 10]. The left, middle and right panels of Fig. 12 show histograms of the discrepancy  $\sup_{0 \le t \le \tau} |X(t) - X_d(t)|$  for t=20, 40 and 60. Note that the histograms have different scales. The plots of these figures show visually and quantitatively that the accuracy of the FD processes Y(t) improves with t=100 in the metric of the space of continuous functions T(t=1)0, T(t=1)1 in agreement with theoretical considerations.

The left, middle and right panels of Fig. 13 show with solid and dashed lines the samples of the Slepian models  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$ ,  $\alpha=t-t_0$ , for d=20, 40 and 60 for a=3.5 and  $t_0=5$ . Of the samples of  $S_a(\alpha)$  corresponding to those in the previous two figures, i.e., the samples of X(t) which upcross a in the time interval  $[t_0,t_0+\delta]$ ,  $\delta=3\Delta t$ , only a fraction of 0.1579 samples of  $X_a(t)$  have this property for d=20. This fraction increases to 0.5789 and 0.7895 for d=40 and 60 and is nearly unity for d=100. For  $\delta=\Delta t$  these fractions are 0, 0.2857, 0.7143 and nearly 1 for d=20, 40, 60 and 100. The plots of Fig. 14 are those of Fig. 13 for a horizontal window of size  $\delta=\Delta t$ , rather than  $\delta=3\Delta t$ . The numbers of samples of X(t) and  $X_a(t)$  which upcross a in this shorter

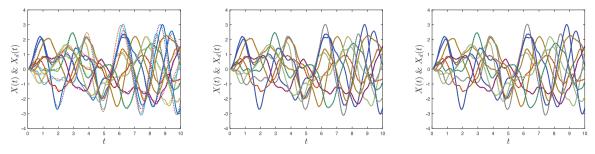


Fig. 11. Ten samples of X(t) (solid lines) and  $X_d(t)$  (dashed lines) for d = 20, 40 and 60 (left, middle and right panels).

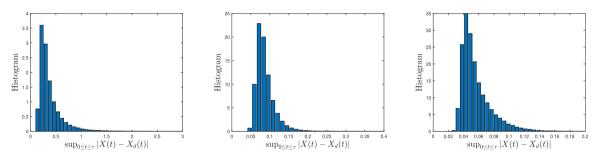


Fig. 12. Histograms of the error  $\sup_{0 \le t \le \tau} |X(t) - X_d(t)|$  for d = 20, 40 and 60 (left, middle and right panels).

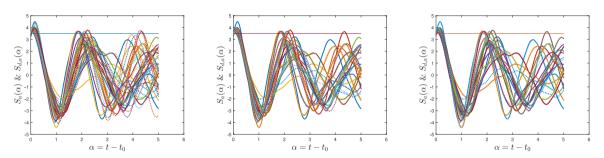


Fig. 13. Samples of  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$  (solid and dashed lines),  $\alpha = t - t_0$ , for d = 20, 40 and 60 (left, middle and right panels) for  $\delta = 3 \Delta t$ .

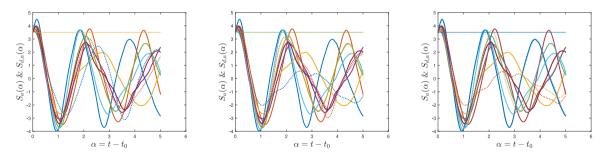


Fig. 14. Samples of  $S_a(\alpha)$  and  $S_{d,a}(\alpha)$  (solid and dashed lines),  $\alpha = t - t_0$ , for d = 20, 40 and 60 (left, middle and right panels) or  $\delta = \Delta t$ .

window decreases and the Slepian samples originate at approximately a=3.5.

**Construction of FD Slepian models:** Consider a samples  $B(t,\omega)$  of the input B(t) to Eq. (23) and denote by  $X(t,\omega)$  the solution of this equation to this input. The samples of X(t) with the property  $X(t_0,\omega) < a < X(t_0 + \delta, \omega)$  for times  $t \ge t_0$  are the samples  $S_a(\alpha, \omega)$  of the Slepian model  $S_a(\alpha)$ , where  $\alpha = t - t_0$ .

The FD pairs  $X_d(t,\omega)$  and  $S_{d,a}(\alpha,\omega)$  of the samples  $X(t,\omega)$  and  $S_a(\alpha,\omega)$  are constructed in three steps. First, samples  $\{Z_k(\omega)\}$  of the random coefficients  $\{Z_k\}$  of the FD representation  $B_d(t)$  of B(t) in Eq. (4) are obtained by projecting the input samples  $B(t,\omega)$  on the basis functions  $\{\varphi_k(t)\}$  in this equations. Note that large sets of samples of  $\{Z_k\}$  can be obtained with a negligible computational effort. Second, the resulting input samples are mapped into samples of  $X_d(t)$  by solving

Eq. (22). The calculation of samples of  $X_d(t)$  is computationally more demanding than in the previous example since their functional form is unknown. They have to be obtained by numerical integration. Third, the subset of samples of  $X_d(t)$  with the property  $X_d(t_0,\omega) < a < X_d(t_0+\delta,\omega)$  defines the samples  $S_{d,a}(\alpha,\omega)$  of the FD Slepian model of  $X_d(t)$  which correspond to the samples  $S_a(\alpha,\omega)$  of the Slepian model of X(t).

## 6. Comments

Analytical formulations are available for the Slepian models of stationary and non-stationary Gaussian processes X(t). The resulting Slepian models are versions of the processes defined by the trajectories

of X(t) following a-upcrossings, i.e., crossings of a of X(t) with positive slopes. The extension of these formulations to non-Gaussian processes X(t) seems to be impractical. This paper has developed Slepian models for continuous processes X(t) by using finite dimensional (FD) representations  $X_d(t)$  of X(t), i.e., deterministic functions of time and sets of  $d < \infty$  random variables. The Slepian models  $S_a(t)$  and  $S_{d,a}(t)$  characterize the evolution of X(t) and  $X_d(t)$  following their a-upcrossings.

It was noted that most stochastic problems do not admit analytical solutions and that numerical solutions of these problems are possible only if the random processes in their definitions are FD processes. The types of FD representations depends on the objective of the analysis, e.g., FD representations given by truncated KL series characterize accurately the mean and correlation functions of target processes but not their sample property. The class of FD processes  $X_d(t)$  and their Slepian models  $S_{d,a}(t)$  in the paper are such that their samples, which can be generated by standard algorithms, can be used as surrogates for the samples of X(t) and  $S_a(t)$ .

Conditions have been established under which FD and target processes have similar samples on bounded time intervals  $[0,\tau]$  in the sense of the metric of the space  $C[0,\tau]$  of continuous functions. Under these conditions, samples of  $X_d(t)$  can be used as substitutes for samples X(t). These conditions have been augmented to assure that samples of  $S_{d,a}(t)$  can be used as surrogates for samples of  $S_a(t)$  for sufficiently large stochastic dimensions d.

Two sets of examples have been presented to illustrate the construction of Slepian models and examine consistency with the theoretical considerations in the paper. The first set includes Gaussian and non-Gaussian FD processes with the same functional form and first two moments which depend on d=4 Gaussian and d=4 non-Gaussian random variables. The target processes X(t) in the second set of examples are solutions of linear and nonlinear random vibration problems with non-Gaussian and Gaussian inputs so that they are non-Gaussian. It was shown that the processes X(t) satisfy the conditions under which target and FD processes and their Slepian models have similar samples. The numerical illustrations are consistent with theoretical predictions.

# Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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