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On the Asymptotic Behavior of Solutions to the Vlasov–Poisson System

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We prove small data modified scattering for the Vlasov–Poisson system in dimension $d = 3$, using a method inspired from dispersive analysis. In particular, we identify a simple asymptotic dynamics related to the scattering mass.

1 Introduction

1.1 The Vlasov–Poisson system

We consider the Vlasov–Poisson system for a density function $f : \mathbb{R}^3_x \times \mathbb{R}^3_v \times \mathbb{R}_t \to \mathbb{R}_+$:

$$\begin{cases} \left(\partial_t + v \cdot \nabla_x\right) f - q\nabla_x\phi \cdot \nabla_v f = 0, \quad q = \pm 1, \\ -\Delta_x\phi(x,t) = \int_{\mathbb{R}^3} f(x,v,t)dv, \\ f(t=0,x,v) = f_0(x,v). \end{cases} \tag{1.1}$$

This model is relevant in plasma physics (usually for $q = 1$) and in astrophysics (for $q = -1$); we refer to [7, 15] for more background. In dimension $d = 3$, solutions to (1.1) are global in time under rather mild assumptions [14, 19], but a complete understanding of their asymptotic behavior is still elusive. In the case of small data [1] provide decay

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estimates, and modified scattering was established in [4]. Recently, these works have been revisited from different point of views [9, 20, 23, 24] with varying improvements.

The relationship between kinetic and dispersive equations, particularly the Schrödinger equation is classical, see e.g. Section 1.2 of [13] for a compelling presentation. A quantum analog of the Vlasov–Poisson system is the Hartree equation, which can be analyzed effectively using dispersive tools [12]. In this paper, we want to adapt classical methods from dispersive equations to recover a simple proof of small data/modified scattering for (1.1) based on energy estimates and convergence in a weaker norm. In particular, this allows to clarify the role of some of the assumptions and to isolate a particularly simple asymptotic dynamic. We hope that this framework will be useful when considering coupling of kinetic equations and other (particularly dispersive) equations (see [21, 22] for examples of such problems).

1.2 Main result

Assuming that the initial density is a nonnegative function $f \geq 0$ (as opposed to a measure), by the transport nature of (1.1) this condition is propagated along the flow. We may thus introduce $\mu = \sqrt{f}$ and consider the more symmetric equation

$$\left(\partial_t + v \cdot \nabla_x\right) \mu - q\nabla_x\phi \cdot \nabla_v\mu = 0, \qquad -\Delta_x\phi = \int_{\mathbb{R}^3} \mu^2 dv. \tag{1.2}$$

Our main result can thus be stated as follows:

Theorem 1.1. There exists $\varepsilon^* > 0$ such that for any $0 < \varepsilon_0 \leq \varepsilon^*$, the following holds: if μ_0 is a smooth initial data such that

$$\mathcal{E}(\mu_0) := \|x\mu_0\|_{L^2_{x,v}} + \|\mu_0\|_{H^3_{x,v}} \leq \varepsilon_0 \tag{1.3}$$

then there exists a unique solution to (1.2), which is global and scatters. This solution satisfies

$$\|\gamma(t)\|_{L^\infty_v L^2_x} + \|\gamma(t)\|_{L^2_v H^3_x} \lesssim \varepsilon_0, \quad \mathcal{E}(\gamma(t)) \lesssim \varepsilon_0 \ln^3\langle t\rangle(\ln\langle\ln\langle t\rangle\rangle)^6, \quad \langle t\rangle = \sqrt{16+t^2},$$

where $\gamma(x, v, t) = \mu(x + tv, v, t)$. In addition, letting

$$m_\infty(v) := \lim_{t\to\infty} \|\mu(\cdot, -v, t)\|^2_{L^2_x}, \quad \widetilde{E}(v) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\zeta}{|\zeta|^3} m_\infty(\zeta - v) d\zeta,$$

we have modified scattering to a new density function

$$\mu(x + tv + q\ln(t)\widetilde{E}(v), v, t) \to \gamma_\infty(x, v) \quad \text{in } L^\infty_v L^2_x \cap H^1_{x,v}. \tag{1.4}$$

A few remarks are in order:

Remark 1.2.

(1) The main novelty of our theorem lies in the limited assumptions on the initial data and the explicit form of the asymptotic behavior (1.4). We feel however that the importance of this paper lies in the simplicity and versatility in the method developed. In addition it clarifies the relevance of various controls (velocity moments and vector fields seem less relevant, regularity in v seems central). We refer to [5, 18] for subsequent developments based on the present methodology.

(2) Under our assumptions without velocity moments, classical large data global existence [14, 19] do not apply. In particular, our solutions can have infinite physical momentum and energy.

(3) Our assumptions on the initial data are weaker than in most recent works [20, 23]. Although they are distinct from the ones in [1, 4], our assumptions have the scale advantage of requiring three derivatives only in L^2 rather than two in L^∞, and do not require compact support. This is related to the fact that we rely on energy estimates rather than transport bounds that naturally give pointwise bounds without appealing to Sobolev inequality.

(4) If one considers measure initial data (in particular *monokinetic* initial data), the asymptotic behavior can be radically different, thus some amount of regularity is needed.

(5) Moving from the density f to $\mu = \sqrt{f}$ has several advantages: (1) it automatically accounts for the nonnegativity of the density, (2) it allows us to separate further our functions from Dirac masses (the natural space is now $\mu \in L^2_{x,v}$ as opposed to $f \in L^1_{x,v}$), (3) it makes the analogy with cubic dispersive problems (in particular Hartree) more transparent. To take advantage of the Sobolev scale, one might want to work on $\widetilde{\mu} = f^{1/2p}$, $p \geq 2$. However, smoothness of $\widetilde{\mu}$ then carries nontrivial implications for f.

(6) In our setting, we consider solutions that are perturbations of the vacuum since by density we may assume that $\mu_0 \in C^\infty_c(\mathbb{R}^3_x \times \mathbb{R}^3_v)$. There are other natural equilibria with nondecaying density (e.g. BGK solutions [3]). The

analysis of their perturbations is related to the study of the *Landau damping*, see [2, 16, 17] and relies on different ideas.

(7) It is remarkable that one only gets logarithmic growth of the energy $\mathcal{E}(\gamma)$. In addition, for 1 derivative, one obtains optimal growth in the sense that the upper and lower growth rates are equal up to a multiplicative constant (the upper bound in (3.4) is obtained via the sharp decay of $\nabla\phi$, whereas the logarithmic lower bound derives from the logarithmic correction in (1.4) and the convergence in $H^1_{x,v}$—see also Section 4 for more intuition).

(8) The methods presented here have broad application for kinetic equations, however, in general, e.g. for relativistic models, we expect that control on velocity moments would be necessary. Indeed the change of variable associated to dispersion (see (2.7)) would involve losses in v (see [23]).

1.3 Method

Our method extends a series of works on the asymptotic description of small data solutions for dispersive and related equations [6, 8, 10–12], which couple energy estimates with a refined scattering analysis in a weaker norm, here

$$\|\mu\|_Z := \|\mu\|_{L^\infty_v L^2_x}, \tag{1.5}$$

which is associated to conservation laws of the resonant/asymptotic system. In particular, one can observe that this norm is invariant both for the free streaming and for the modified scattering flow. The key technical properties we require of this Z-norm are (1) that it is weak enough so as to remain uniformly bounded throughout the nonlinear evolution and (2) that it is strong enough to provide optimal decay for the main unknown. Our analysis then proceeds over a few steps.

We obtain refined dispersive estimates involving the (minimal) Z-norm. This corresponds to requirement (2) above and is done in Lemma 2.1 below. It can be thought as an analog of similar estimates for the Schrödinger evolution and is a good "proof of concept" for the definition of the Z-norm. As for the Hartree equation, the critical step is however to bound a quartic expression, which appears in Subsection 3.3.

Addressing (1) above, it turns out that for (1.2), it is relatively easy to obtain a uniform bound on the Z-norm and convergence of the *scattering mass*

$$m_t(v) := \int_{\mathbb{R}^3} \mu^2(x,v,t)dx \to m_\infty(v). \tag{1.6}$$

The scattering mass controls how much mass is "seen" by a frame advected by free streaming and controls the asymptotic dynamics. In particular, it allows to define the effective electric field and characteristics

$$\frac{dX}{dt} = V, \quad \frac{dV}{dt} = -\frac{q}{t^2}\widetilde{E}(V), \quad \widetilde{E}(\zeta) := \nabla_\zeta(-\Delta_\zeta^{-1})m_\infty,$$

from which modified scattering follows.

Commuting (1.2) with the corresponding moment or derivative operators, one sees in (3.7) that the energy estimates separate into simple estimates that do not require very sharp bounds and are less related to decay on the one hand (Subsection 3.2) and energy estimates for the velocity regularity, which require sharp control on the unknowns and provide the "fuel" for the decay (which is obtained by trading regularity in v) in Subsection 3.3.

This paper is organized as follows: in Section 2 we introduce our notations and some estimates to control the electric field. The global existence part of Theorem 1.1 is proved through a bootstrap in Section 3. Finally in Section 4, we obtain the modified scattering.

2 Notations and Preliminary Estimates

In the following, since all our functions are evaluated at t, we have suppressed the explicit dependence in t of the functions γ, ϕ. To be more thrifty in v derivatives, we will introduce Littlewood–Paley projectors for v-regularity: for φ a typical Littlewood–Paley bump function and $C \in 2^{\mathbb{Z}}$ a dyadic integer, we define

$$P_C^v = \mathcal{F}^{-1}_{\theta\to v}\varphi(C^{-1}\theta)\mathcal{F}_{v\to\theta}. \tag{2.1}$$

Recall Bernstein's estimates and other properties of the Littlewood–Paley projectors:

$$\begin{aligned}\|P_C^v f\|_{L^\infty_v L^2_x} &\lesssim C^{\frac{3}{2}}\|P_C^v f\|_{L^2_{x,v}},\\ \|\nabla_v P_C^v f\|_{L^2_{x,v}} &\simeq C\|P_C^v f\|_{L^2_{x,v}},\\ \|f\|^2_{L^2_{x,v}} &\simeq \sum_C \|P_C^v f\|^2_{L^2_{x,v}}.\end{aligned}$$

As an application, we can observe that the Z-norm defined in (1.5) is bounded by the energy:

$$\|h\|_Z \lesssim \|h\|_{L^2_x L^\infty_v} \lesssim \|h\|_{L^2_x H^2_v},$$
$$\|\nabla_v h_C\|_Z \lesssim \min\{C\|h\|_Z, C^{-\frac{1}{2}}\|h\|_{L^2_x H^3_v}\}, \quad \|\nabla_v h\|_Z \lesssim \|h\|^{\frac{2}{3}}_{L^2_x H^3_v}\|h\|^{\frac{1}{3}}_Z,$$

where we have written $h_C = P^v_C h$ with notations from (2.1). Besides, using that

$$h_M(a, \frac{x-a}{t}) - h_M(a, \frac{x}{t}) = \int_{\theta=0}^{1} \frac{a^j}{t}\partial_{v^j} h_M(a, \frac{x}{t} - \theta\frac{a}{t})d\theta$$

we obtain after summing over M,

$$\begin{aligned}\|h(a, \frac{x-a}{t}) - h(a, \frac{x}{t})\|_{L^2_a L^\infty_x} &\lesssim \sum_M \min\{t^{-1}M^{\frac{5}{2}}\|x\gamma\|_{L^2_{x,v}}, M^{-\frac{1}{2}}\|\gamma\|_{L^2_x H^2_v}\} \\ &\lesssim t^{-\frac{1}{6}}\left\{\|h\|_{L^2_x H^2_v} + \|xh\|_{L^2_{x,v}}\right\}.\end{aligned} \tag{2.2}$$

Because of bounds like (2.2), it turns out that our estimates are more simply stated using a variation of the Z-norm:

$$\|f\|_{Z'} := \|f\|_Z + \langle t\rangle^{-\frac{1}{100}}\left\{\|f\|_{H^2_{x,v}} + \|xf\|_{L^2_{x,v}}\right\}.$$

For the proof of our multilinear estimates we will use the representation

$$\frac{1}{|x|^p} = c_p(\chi)\int_{R=0}^{\infty} R^{-p}\chi(R^{-1}|x|)\frac{dR}{R}, \tag{2.3}$$

valid for some $\chi \in \mathcal{S}$ and $p > 0$.

2.1 Dispersive estimates

We now study the decay properties of a particle density distribution h under the linear flow of (1.2).

Lemma 2.1. For any $x \in \mathbb{R}^3$, there holds that

$$0 \leq \rho_{[h]}(x, t) := \int_{\mathbb{R}^3} h^2(x - tv, v)dv \lesssim \langle t\rangle^{-3}\|h\|^2_{Z'},$$

and

$$\|\nabla_x \phi_{[h]}(x,t)\|_{L^\infty_x} \lesssim \langle t\rangle^{-2}[\|h\|^2_{Z'} + \|h\|^2_{L^2_{x,v}}], \qquad -\Delta\phi_{[h]} = \rho_{[h]}. \tag{2.4}$$

Moreover,

$$\left\|\partial_x^\alpha \nabla_x \phi_{[h]}(x,t)\right\|_{L^\infty_x} \lesssim \langle t\rangle^{-2-|\alpha|} \, \|h\|^2_{L^2_x H_v^{|\alpha|+1}}, \quad |\alpha| \le 2, \tag{2.5}$$

$$\left\|\partial_x^\alpha \nabla_x \phi_{[h]}(x,t)\right\|_{L^2_x} \lesssim \langle t\rangle^{-\frac{1}{2}-|\alpha|} \, \|h\|^2_{L^2_x H_v^{|\alpha|}}, \quad |\alpha| \ge 2. \tag{2.6}$$

Proof of Lemma 2.1. Assume without loss of generality that $t \ge 1$. We change variables and rewrite

$$\begin{aligned}\rho_{[h]}(x,t) &= \int_{\mathbb{R}^3} h(x-tv,v)^2 dv = t^{-3}\int_{\mathbb{R}^3} h^2(a,\frac{x-a}{t})da = t^{-3}\int_{\mathbb{R}^3} h^2(a,\frac{x}{t})da + J(x,t),\\ J(x,t) &:= t^{-3}\int_{\mathbb{R}^3}\left\{h^2(a,\frac{x-a}{t}) - h^2(a,\frac{x}{t})\right\}da.\end{aligned} \tag{2.7}$$

The principal part can be directly bounded in terms of the Z norm,

$$\int_{\mathbb{R}^3} h^2(a,\frac{x}{t})da \le \|h\|^2_Z,$$

so it suffices to show the decay of J. To this end we observe that

$$|J(x,t)| \le t^{-3}\|h\|_{L^2_x L^\infty_v}\|h(a,\frac{x-a}{t}) - h(a,\frac{x}{t})\|_{L^2_a} \lesssim t^{-3-\frac{1}{10}}\,\|h\|^2_{Z'}, \tag{2.8}$$

where in the last inequality we used (2.2).

The second inequality uses similar ideas. Using (2.3), we can decompose

$$\nabla\phi_{[h]}(x,t) = \frac{1}{4\pi}\int_{R=0}^\infty \frac{dR}{R^2}\iint_{\mathbb{R}^3_{y,u}} \nabla_y\left\{\varphi(R^{-1}|x-y|)\right\} h^2(y-tu,u)dydu =: \int_{R=0}^\infty \Phi_R(x,t)\frac{dR}{R^2}. \tag{2.9}$$

On the one hand, we see that

$$|\partial_x^\alpha \Phi_R(x,t)| \lesssim R^{-1-|\alpha|}\iint_{\mathbb{R}^3_{y,u}} h^2(y-tu,u)dydu \lesssim R^{-1-|\alpha|}\|h\|^2_{L^2_{x,v}}, \tag{2.10}$$

which is good enough for large $R \geq t$. On the other hand, for small R we have

$$\begin{aligned}\Phi_R(x,t) &= \iint_{\mathbb{R}^3_{z,u}} \nabla_z \left\{\varphi(R^{-1}|z|)\right\} h^2(x-z-tu,u)dzdu\\ &= t^{-3}\iint_{\mathbb{R}^3_{z,a}} \nabla_z \left\{\varphi(R^{-1}|z|)\right\} h^2(a,\frac{x-z-a}{t})dzda.\end{aligned}$$

Taking derivatives, we see that

$$\begin{aligned}\partial_x^\alpha \Phi_R(x,t) &= t^{-3-|\alpha|}\sum_{|\alpha_1|+|\alpha_2|=|\alpha|,\,|\alpha_1|\leq|\alpha_2|} c_{\alpha_1,\alpha_2}\Phi_R^{\alpha_1,\alpha_2}(x,t),\\ \Phi_R^{\alpha_1,\alpha_2}(x,t) &:= \iint_{\mathbb{R}^3_{z,a}} \nabla_z\left\{\varphi(R^{-1}|z|)\right\}\partial_v^{\alpha_1}h(a,\frac{x-z-a}{t})\partial_v^{\alpha_2}h(a,\frac{x-z-a}{t})dzda. \qquad (2.11)\end{aligned}$$

If $|\alpha|=0$, we estimate, with (2.2),

$$|\Phi_R^{0,0}| \lesssim \|\nabla_z\varphi(R^{-1}|z|)\|_{L^1_z}\|h(a,\frac{x-z-a}{t})\|_{L^\infty_z L^2_a}\|h(a,\frac{x-z-a}{t})\|_{L^\infty_z L^2_a} \lesssim R^2\|h\|^2_{Z'},$$

while if $|\alpha|\geq 1$, using Hölder's inequality, we find that

$$\begin{aligned}|\Phi_R^{\alpha_1,\alpha_2}| &\lesssim \|\nabla_z\varphi(R^{-1}|z|)\|_{L_z^{\frac{6}{5}}}\|\partial_v^{\alpha_1}h(a,\frac{x-z-a}{t})\|_{L^\infty_z L^2_a}\|\partial_v^{\alpha_2}h(a,\frac{x-z-a}{t})\|_{L^6_z L^2_a}\\ &\lesssim R^{\frac{3}{2}}t^{\frac{1}{2}}\|h\|_{L^2_x H_v^{|\alpha_1|+2}}\|h\|_{L^2_x H_v^{|\alpha_2|+1}}.\end{aligned}$$

Integrating the above bounds for $R\leq t$ and (2.10) for $R\geq t$ in (2.5), we obtain (2.4) and (2.9). Finally, for $|\alpha|\geq 2$, we estimate the electric field in L^2. For $R\geq t$ it suffices to note that

$$\left\|\partial_x^\alpha\Phi_R(x,t)\right\|_{L^2_x} \lesssim \left\|\iint_{\mathbb{R}^3_{y,u}} \partial_x^\alpha\nabla_y\left\{\varphi(R^{-1}|x-y|)\right\}h^2(y-tu,u)dydu\right\|_{L^2_x} \lesssim R^{-|\alpha|+\frac{1}{2}}\|h\|^2_{L^2_{x,v}}.$$

For $t\leq R$ we use Hölder and Sobolev inequalities in (2.11) to find (for $2/p_j=|\alpha_j|/|\alpha|$) that

$$\begin{aligned}\left|\Phi_R^{\alpha_1,\alpha_2}(x,t)\right| &\lesssim t^{-|\alpha|-3}\|\nabla_z\left\{\varphi(R^{-1}|z|)\right\}\|_{L^2_z}\|\partial_v^{\alpha_1}h(a,\frac{z}{t})\|_{L^2_a L^{p_1}_z}\|\partial_v^{\alpha_2}h(a,\frac{z}{t})\|_{L^2_a L^{p_2}_z}\\ &\lesssim t^{-|\alpha|-\frac{3}{2}}R^{\frac{1}{2}}\|h\|^2_{L^2_x H_v^{|\alpha|}},\end{aligned}$$

while estimating in L^1_x yields

$$\left\|\partial_x^\alpha \Phi_R(x,t)\right\|_{L^1_x} \lesssim t^{-|\alpha|} \left\|\nabla_z \left\{\varphi(R^{-1}|z|)\right\}\right\|_{L^1_x} \|h\|^2_{L^2_x H_v^{|\alpha|}} \lesssim t^{-|\alpha|} R^2 \|h\|^2_{L^2_x H_v^{|\alpha|}}.$$

By interpolation it thus follows that

$$\left\|\partial_x^\alpha \Phi_R(x,t)\right\|_{L^2_x} \lesssim t^{-|\alpha|-\frac{3}{4}} R^{\frac{5}{4}} \|h\|^2_{L^2_x H_v^{|\alpha|}},$$

and hence

$$\left\|\partial_x^\alpha \nabla_x \phi(t,x)\right\|_{L^2} \lesssim \int_0^\infty \min\{t^{-|\alpha|-\frac{3}{4}} R^{\frac{5}{4}}, R^{-|\alpha|+\frac{1}{2}}\} \frac{dR}{R^2} \cdot \|h\|^2_{L^2_x H_v^{|\alpha|}} \lesssim t^{-|\alpha|-\frac{1}{2}} \|h\|^2_{L^2_x H_v^{|\alpha|}}.$$

■

Remark 2.2. It is important to have a sharp control on the Electric field as in (2.4). The first bound on ρ is here essentially for motivation to help clarify the relationship with the Schrödinger equation. The decomposition (2.7) with (2.8) is one of the main motivations for the definition of the Z-norm. It can be compared to Fraunhoffer's inequality for the Schrödinger flow:

$$\left(e^{-it\Delta} f\right)(x) = \frac{e^{-i\frac{|x|^2}{4t}}}{(2\pi i t)^{\frac{d}{2}}} \widehat{f}(-\frac{x}{2t}) + O_{L^\infty}(t^{-\frac{d}{2}-\frac{1}{4}})$$

valid whenever $f \in \mathcal{S}$, which, when considering uniform estimates, leads to the natural definition for NLS, $\|f\|_Z = \|\widehat{f}\|_{L^\infty}$, see [6, 12].

3 Nonlinear Analysis I: Bootstrap of the Norms

We first integrate the linear flow and define

$$\gamma(x,v,t) := \mu(x+tv,v,t), \quad \rho(x,t) := \int_{\mathbb{R}^3} \mu^2(x,v,t)dv = \int_{\mathbb{R}^3} \gamma^2(x-tv,v,t)dv,$$

and we obtain the new equation

$$\partial_t \gamma(x,v) = q\nabla_x \phi(x+tv) \cdot \left\{\nabla_v - t\nabla_x\right\} \gamma(x,v), \quad -\Delta_x \phi = \rho. \tag{3.1}$$

We can now state our main bootstrap proposition from which global existence and boundedness easily follow.

Proposition 3.1. Let $\delta > 0$ be a fixed small constant. There exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon_0 \le \varepsilon \le \varepsilon^*$, the following holds. Assume that γ solves (3.1) on $0 \le t \le T$, with initial data μ_0 satisfying (1.3) and obeys the bound

$$\begin{aligned} \|\gamma\|_{L^2_{x,v}} &\le \varepsilon, \\ \|\gamma\|_Z + \|\gamma\|_{L^2_v H^3_x} &\le \varepsilon, \\ \|\gamma\|_{L^2_x H^3_v} + \|x\gamma\|_{L^2_{x,v}} &\le \varepsilon \langle t\rangle^{\delta}, \end{aligned} \tag{3.2}$$

then there holds that

$$\begin{aligned} \|\nabla_x \phi\|_{L^\infty_x} &\le C_1 \langle t\rangle^{-2} \varepsilon^2, \\ \|\partial_x^\alpha \nabla_x \phi\|_{L^\infty_x} &\le C_1 \langle t\rangle^{-2-|\alpha|+2\delta} \varepsilon^2, \qquad 1 \le |\alpha| \le 2, \\ \|\partial_x^\alpha \nabla_x \phi\|_{L^2_x} &\le C_1 \langle t\rangle^{-\frac{1}{2}-|\alpha|+2\delta} \varepsilon^2, \qquad 2 \le |\alpha| \le 3, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|\gamma\|_{L^2_{x,v}} &\le \varepsilon_0, \\ \|\gamma\|_Z + \left\|\nabla_x \gamma\right\|_Z + \|\gamma\|_{L^2_v H^3_x} &\le \varepsilon_0 + C_2 \varepsilon^3, \\ \|\gamma\|_{L^2_x H^1_v} + \|x\gamma\|_{L^2_{x,v}} &\le \varepsilon_0 + C_2 \varepsilon^3 \ln\langle t\rangle, \\ \|\gamma\|_{L^2_x H^{|\alpha|}_v} &\le \varepsilon_0 + C_2 \varepsilon^{\frac{5}{2}} (\ln\langle t\rangle)^{|\alpha|} \cdot (\ln\langle \ln\langle t\rangle\rangle)^{2|\alpha|}, \qquad 2 \le |\alpha| \le 3 \end{aligned} \tag{3.4}$$

for some universal constant $C_1, C_2 = C_2(\delta)$. In addition, the scattering mass defined in (1.6) converges uniformly to a limit $m_\infty(v) \in L^1_v \cap L^\infty_v$, and

$$\|m_t - m_\infty\|_{L^\infty_v} \lesssim \varepsilon^3 \langle t\rangle^{-\frac{1}{2}}. \tag{3.5}$$

Lemma 2.1 gives (3.3). The first norm in (3.4) is trivially controlled by conservation laws (see (3.8) below). We give the bounds for the Z norms in Subsection 3.1. Subsections 3.2 and 3.3 then demonstrates the energy estimates, including the most delicate terms: derivatives in velocity.

Remark 3.2 (Regarding the growth rates of derivatives in v). We have made some effort to obtain almost sharp bounds in the v derivative; a slightly simpler analysis would have allowed to propagate slow polynomial growth. For one v derivative, (3.4) gives the sharp growth rate $\ln(t)$, whereas for two or three derivatives there is an additional factor of $(\ln(\ln(t)))^p$. An inspection of the proof shows that if one were to propagate higher regularity in v, this loss could be relegated to higher orders of derivatives in v.

3.1 Propagation of the Z-norm

We can directly compute that, for fixed $v \in \mathbb{R}^3$,

$$\frac{1}{2}\frac{d}{dt}\int \gamma^2 dx = q\int_{\mathbb{R}^3} \nabla_x\phi(x+tv)\gamma(x,v)\nabla_v\gamma(x,v)dx + q\frac{t}{2}\int_{\mathbb{R}^3} \Delta_x\phi(x+tv)\gamma^2(x,v)dx$$

from which we deduce that

$$|\|\gamma(t)\|_Z^2 - \|\gamma_0\|_Z^2| \lesssim \int_{s=0}^t \|\nabla_x\phi(s)\|_{L^\infty_x}\|\gamma(s)\|_Z\|\nabla_v\gamma(s)\|_Z ds + \int_{s=0}^t s\|\Delta\phi(s)\|_{L^\infty}\|\gamma(s)\|_Z^2 ds.$$

Using (3.3), this leads to the bound of the Z norm in (3.4) and to (3.5).

3.1.1 *Higher order Z-norm*

We can even control higher regularity in Z. Using (3.7), we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3_x}(\partial_{x^j}\gamma)^2 dx = q\int_{\mathbb{R}^3_x} \nabla_x\phi(x+tv)\partial_{x^j}\gamma\nabla_v\partial_{x^j}\gamma\, dx + \frac{t}{2}\int_{\mathbb{R}^3_x}\Delta_x\phi(x+tv)(\partial_{x^j}\gamma)^2 dx\\ + q\int_{\mathbb{R}^3_x}\nabla_x\partial_{x^j}\phi(x+tv)\cdot\nabla_v\gamma\cdot\partial_{x^j}\gamma\, dx - qt\int_{\mathbb{R}^3_x}\nabla_x\partial_{x^j}\phi(x+tv)\cdot\nabla_x\gamma\cdot\partial_{x^j}\gamma\, dx.\end{aligned}$$

All but the first terms can be controlled as before. The first term requires a little more work since it contains 2 derivatives; however, we can still integrate the x-derivative by parts to move it to a more favorable position. To decide when to do it, we use the Littlewood–Paley decomposition (2.1) to decompose

$$\int_{\mathbb{R}^3} \nabla_x\phi(x+tv)\partial_{x^j}\gamma\nabla_v\partial_{x^j}\gamma\, dx = \Big(\sum_{C_1>C_2} + \sum_{C_1\le C_2}\Big)\int_{\mathbb{R}^3}\nabla_x\phi(x+tv)\partial_{x^j}\gamma_{C_1}\nabla_v\partial_{x^j}\gamma_{C_2} dx,$$

where on the one hand we estimate

$$\begin{aligned}\left|\sum_{C_1>C_2}\int_{\mathbb{R}^3}\nabla_x\phi(x+tv)\partial_{x^j}\gamma_{C_1}\nabla_v\partial_{x^j}\gamma_{C_2}dx\right| &\lesssim \sum_{C_1>C_2}\left\|\nabla_x\phi\right\|_{L^\infty}\left\|\partial_{x^j}\gamma_{C_1}\right\|_{L^\infty_vL^2_x}\left\|\nabla_v\partial_{x^j}\gamma_{C_2}\right\|_{L^\infty_vL^2_x}\\ &\lesssim \left\|\nabla_x\phi\right\|_{L^\infty}\sum_{C_1>C_2}C_1^{\frac{3}{2}}\left\|\partial_{x^j}\gamma_{C_1}\right\|_{L^2_{x,v}}C_2^{1+\frac{3}{2}}\left\|\partial_{x^j}\gamma_{C_2}\right\|_{L^2_{x,v}}\\ &\lesssim \left\|\nabla_x\phi\right\|_{L^\infty}\sum_{C_1>C_2}\left(\frac{C_2}{C_1}\right)^{\frac{1}{2}}C_1^2\left\|\partial_{x^j}\gamma_{C_1}\right\|_{L^2_{x,v}}C_2^2\left\|\partial_{x^j}\gamma_{C_2}\right\|_{L^2_{x,v}}\lesssim\left\|\nabla_x\phi\right\|_{L^\infty}\left\|\gamma\right\|^2_{H^1_xH^2_v}. \qquad (3.6)\end{aligned}$$

To control the second sum, we integrate by parts in x to get, when $C_1 \le C_2$:

$$\begin{aligned}&\int_{\mathbb{R}^3}\nabla_x\phi(x+tv)\partial_{x^j}\gamma_{C_1}\nabla_v\partial_{x^j}\gamma_{C_2}dx\\ &=-\int_{\mathbb{R}^3}\partial_{x^j}\nabla_x\phi(x+tv)\partial_{x^j}\gamma_{C_1}\nabla_v\gamma_{C_2}dx-\int_{\mathbb{R}^3}\nabla_x\phi(x+tv)\partial^2_{x^j}\gamma_{C_1}\cdot\nabla_v\gamma_{C_2}dx.\end{aligned}$$

The first term leads to a simple sum as before:

$$\left|\sum_{C_1\le C_2}\int_{\mathbb{R}^3}\partial_{x^j}\nabla_x\phi(x+tv)\partial_{x^j}\gamma_{C_1}\nabla_v\gamma_{C_2}dx\right|\lesssim\left\|\partial_{x^j}\nabla_x\phi\right\|_{L^\infty}\left\|\gamma\right\|_{H^1_xH^2_v}\left\|\gamma\right\|_{L^2_xH^3_v},$$

and we can sum the last term as in (3.6) to get

$$\begin{aligned}\left|\sum_{C_1\le C_2}\int_{\mathbb{R}^3}\nabla_x\phi(x+tv)\partial^2_{x^j}\gamma_{C_1}\nabla_v\gamma_{C_2}dx\right| &\lesssim \sum_{C_1\le C_2}\left\|\nabla_x\phi\right\|_{L^\infty}\left\|\partial^2_{x^j}\gamma_{C_1}\right\|_{L^\infty_vL^2_x}\left\|\nabla_v\gamma_{C_2}\right\|_{L^\infty_vL^2_x}\\ \lesssim \left\|\nabla_x\phi\right\|_{L^\infty}\sum_{C_1\le C_2}\left(\frac{C_1}{C_2}\right)^{\frac{1}{2}}C_1\left\|\partial^2_{x^j}\gamma_{C_1}\right\|_{L^2_{x,v}}C_2^3\left\|\gamma_{C_2}\right\|_{L^2_{x,v}} &\lesssim\left\|\nabla_x\phi\right\|_{L^\infty}\left\|\gamma\right\|_{H^2_xH^1_v}\left\|\gamma\right\|_{L^2_xH^3_v}.\end{aligned}$$

Combining the estimates with (3.3), we obtain a control of $\nabla_x\gamma$ in Z as in (3.4).

3.2 Energy Estimates I: simpler energies

We can do energy estimates based on $L^2_{x,v}$ norm. Considering directly (3.1) and deriving with respect to x, we find that (for $\gamma_{x^j} = \partial_{x^j}\gamma$ and $\gamma_{v^j} = \partial_{v^j}\gamma$)

$$\begin{aligned}\partial_t \left\{x_j\gamma\right\} - q\nabla_x\phi(x+tv)\cdot\left\{\nabla_v - t\nabla_x\right\}\left\{x_j\gamma\right\} &= qt\partial_j\phi(x+tv)\cdot\gamma,\\ \partial_t\gamma_{x^j} - q\nabla_x\phi(x+tv)\cdot\left\{\nabla_v - t\nabla_x\right\}\gamma_{x^j} &= q\nabla_x\partial_{x^j}\phi(x+tv)\cdot\left\{\nabla_v - t\nabla_x\right\}\gamma,\\ \partial_t\gamma_{v^j} - q\nabla_x\phi(x+tv)\cdot\left\{\nabla_v - t\nabla_x\right\}\gamma_{v^j} &= qt\nabla_x\partial_{x^j}\phi(x+tv)\cdot\left\{\nabla_v - t\nabla_x\right\}\gamma.\end{aligned} \tag{3.7}$$

We can use this to control the rest of the norms in (3.4). Note that since

$$\mathrm{div}_{x,v}(V) = 0, \quad V(x,v) = (-t\nabla_x\phi(x+tv,t), \nabla_x\phi(x+tv)) \in \mathbb{R}^3_x\times\mathbb{R}^3_v \tag{3.8}$$

the left-hand side is conservative and hence only the right-hand side contributes to changes in $L^2_{x,v}$ norms.

3.2.1 *Position*

We deduce from (3.7) that

$$\frac{d}{dt}\|x\gamma\|^2_{L^2_{x,v}} \lesssim t\|\nabla_x\phi\|_{L^\infty_x}\|\gamma\|_{L^2}\|x\gamma\|_{L^2}.$$

Using (3.2)-(3.3) and Grönwall, we obtain that

$$\|x\gamma(t)\|_{L^2_{x,v}} \le \|x\gamma_0\|_{L^2_{x,v}} + C\varepsilon^2\|\gamma\|_{L^2_{x,v}}\ln t,$$

which leads to the control of $x\gamma$ in (3.4).

3.2.2 *Spatial regularity*

Commuting again (3.7), we see that for $1 \le |\alpha| \le 2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha_x\gamma\|^2_{L^2} &= q\sum_{\beta_1+\beta_2=\alpha,\ |\beta_2|<|\alpha|}\iint \nabla_x\partial_x^{\beta_1}\phi(x+tv)\cdot\left\{\nabla_v - t\nabla_x\right\}\partial_x^{\beta_2}\gamma\cdot\partial^\alpha_x\gamma\\ &\lesssim \sum_{\beta_1+\beta_2=\alpha,\ |\beta_2|<|\alpha|}\|\nabla_x^{|\beta_1|+1}\phi\|_{L^\infty_x}\left\{\|\nabla_v\partial_x^{\beta_2}\gamma\|_{L^2_{x,v}} + t\|\nabla_x\partial_x^{\beta_2}\gamma\|_{L^2_{x,v}}\right\}\|\partial^\alpha_x\gamma\|_{L^2_{x,v}}.\end{aligned}$$

Using (3.2)-(3.3) and Grönwall's Lemma, this remains bounded. The same estimates also work for $|\alpha| = 3$ as long as $|\beta_1| < 3$. If $|\beta_1| = 3$ (and thus $\alpha = \beta_1$, $\beta_2 = 0$) we estimate via

(2.6) to obtain

$$\left|\iint \nabla_x\partial_x^\alpha\phi(x+tv)\cdot\{\nabla_v - t\nabla_x\}\gamma\cdot\partial_x^\alpha\gamma\right| \lesssim \|\nabla_x\partial_x^\alpha\phi\|_{L^2_x}(\|\nabla_v\gamma\|_{L^\infty_x L^2_v}+t\|\nabla_x\gamma\|_{L^\infty_x L^2_v})\|\partial_x^\alpha\gamma\|_{L^2_{x,v}},$$

which is bounded by the bootstrap assumptions (3.2)-(3.3).

3.3 Energy Estimates II: velocity regularity

Finally, the most difficult term comes from the v derivative in (3.7) and its higher order versions:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_v^\alpha\gamma\|_{L^2}^2 &= q\sum_{\substack{\beta_1+\beta_2=\alpha\\|\beta_2|<|\alpha|}}\iint_{\mathbb{R}^3_{x,v}}\nabla_x\partial_v^{\beta_1}\phi(x+tv)\cdot\partial_v^{\beta_2}\nabla_v\gamma(x,v)\cdot\partial_v^\alpha\gamma(x,v)dxdv\\ &\quad - q\sum_{\substack{\beta_1+\beta_2=\alpha\\|\beta_2|<|\alpha|}} t\iint_{\mathbb{R}^3_{x,v}}\nabla_x\partial_v^{\beta_1}\phi(x+tv)\cdot\partial_v^{\beta_2}\nabla_x\gamma(x,v)\cdot\partial_v^\alpha\gamma(x,v)dxdv. \end{aligned} \tag{3.9}$$

We treat the first term in (3.9) using (3.3): If $|\alpha|\leq 2$ we can directly estimate each summand

$$\begin{aligned}\left|\iint_{\mathbb{R}^3_{x,v}}\nabla_x\partial_v^{\beta_1}\phi(x+tv)\cdot\partial_v^{\beta_2}\nabla_v\gamma(x,v)\cdot\partial_v^\alpha\gamma(x,v)dxdv\right| &\lesssim t^{|\beta_1|}\left\|\partial_x^{\beta_1}\nabla_x\phi\right\|_{L^\infty_x}\|\gamma\|^2_{L^2_xH_v^{|\alpha|}}\\ &\lesssim \varepsilon^2\langle t\rangle^{2\delta-2}\|\gamma\|^2_{L^2_xH_v^{|\alpha|}},\end{aligned}$$

and this also works when $|\alpha|=3$ and derivatives split: $|\beta_1|,|\beta_2|\leq 2$. Finally, if all derivatives fall on $\nabla_x\phi$ we change variables

$$\begin{aligned} I &= \iint_{\mathbb{R}^3_{x,v}}\nabla_x\partial_v^\alpha\phi(x+tv)\cdot\nabla_v\gamma(x,v)\cdot\partial_v^\alpha\gamma(x,v)dxdv\\ &= t^{|\alpha|-3}\iint_{\mathbb{R}^3_{x,a}}\nabla_x\partial_x^\alpha\phi(x)\cdot\nabla_v\gamma(a,\frac{x-a}{t})\cdot\partial_v^\alpha\gamma(a,\frac{x-a}{t})dxda\end{aligned}$$

and therefore, using (2.2) and (3.3),

$$\begin{aligned}|I| &\lesssim t^{|\alpha|-3}\|\nabla_x\partial_x^\alpha\phi\|_{L^2_x}\|\partial_v^\alpha\gamma(a,\frac{x-a}{t})\|_{L^2_{x,a}}\|\nabla_v\gamma(a,\frac{x-a}{t})\|_{L^\infty_xL^2_a}\\ &\lesssim t^{|\alpha|-\frac{3}{2}}\|\nabla_x\partial_x^\alpha\phi\|_{L^2_x}\|\nabla_v\gamma\|_{Z'}\|\gamma\|_{L^2_xH^3_v}\lesssim\varepsilon^2\langle t\rangle^{2\delta-2}\|\gamma\|^2_{L^2_xH^3_v}.\end{aligned}$$

The same considerations allow to control the second term in (3.9) when $0 \leq t \leq 100$. For $t \geq 100$, due to the extra factor t and the slow growth of $\|\gamma\|_{L^2_x H^3_v}$, more care is needed. We let

$$I_{\beta_1,\beta_2} := \sum_{\substack{\beta_1+\beta_2=\alpha \\ |\beta_2|<|\alpha|}} t \iint_{\mathbb{R}^3_{x,v}} \nabla_x \partial_v^{\beta_1} \phi(x+tv) \cdot \partial_v^{\beta_2} \nabla_x \gamma(x,v) \cdot \partial_v^{\alpha} \gamma(x,v) dxdv.$$

Case $|\alpha| = 1$**.** Here we necessarily have $\beta_1 = \alpha$. Using that $\|\nabla_x \gamma\|_Z$ is uniformly bounded, we can then proceed as follows: We recognize that $I_{\alpha,0}$ can be written as (identifying an operator and its kernel)

$$I_{\alpha,0} = t^2 \iint_{\mathbb{R}^3} \gamma(y-tu,u)\gamma(y-tu,u)\mathcal{M}_{jk}(x-y)\partial_{x^j}\gamma(x-tv,v)\partial_{v^k}\gamma(x-tv,v)dxdydudv,$$

$$\mathcal{M}_{jk} = (-\Delta)^{-1}\partial_j\partial_k = R_j R_k.$$

Now changing variables, we can rewrite this as

$$I_{\alpha,0} = t^{-4} \iint_{\mathbb{R}^3} \left\{ \gamma(a,\frac{y-a}{t})\gamma(a,\frac{y-a}{t}) \right\} \mathcal{M}_{jk}(x-y) \left\{ \partial_{x^j}\gamma(b,\frac{x-b}{t})\partial_{v^k}\gamma(b,\frac{x-b}{t}) \right\} dadbdxdy.$$

Since $\mathcal{M}_{jk}$ is bounded as a map $L^2 \to L^2$, using (2.2), we see that

$$\begin{aligned} |I_{\alpha,0}| &\lesssim t^{-4} \|\gamma(a,\frac{y-a}{t})\gamma(a,\frac{y-a}{t})\|_{L^2_y L^1_a} \|\partial_{x^j}\gamma(b,\frac{x-b}{t})\partial_{v^k}\gamma(b,\frac{x-b}{t})\|_{L^2_x L^1_b} \\ &\lesssim t^{-1}\|\gamma\|_{Z'}\|\nabla_x\gamma\|_{Z'}\|\gamma\|_{L^2_{x,v}}\|\nabla_v\gamma\|_{L^2_{x,v}}, \end{aligned}$$

and using (3.2) and integrating, we find the bound in (3.4).

Case $|\alpha| \geq 2$**.** For higher $|\alpha|$, we use (2.3) to decompose

$$\begin{aligned} I_{\beta_1,\beta_2} &= t^{|\beta_1|+1} \int_{R=0}^{\infty} I_R^{\beta_1,\beta_2} \frac{dR}{R^2} \\ I_R^{\beta_1,\beta_2} &= \iint_{\mathbb{R}^3} \gamma(y-tu,u)\gamma(y-tu,u)\partial_x^{\beta_1}\partial_{x^j}\left\{\chi(R^{-1}|x-y|)\right\} \\ &\qquad \cdot \partial_{x^j}\partial_v^{\beta_2}\gamma(x-tv,v)\partial_v^{\alpha}\gamma(x-tv,v)dxdydudv. \end{aligned}$$

And we claim that, for $t \geq 100$,

$$|I_R^{\beta_1,\beta_2}| \lesssim R^{-1-|\beta_1|}\varepsilon^2\left[(\ln t)^{2|\alpha|-1}\varepsilon^2 + (\ln t)^{-1}\|\gamma\|^2_{L^2_x H^{|\alpha|}_v}\right],$$
$$|I_R^{\beta_1,\beta_2}| \lesssim R^2 t^{-3-|\beta_1|}(t/R)^{\frac{1}{2}}\left[\varepsilon^2\|\gamma\|^2_{L^2_x H^{|\alpha|}_v} + \varepsilon\|\gamma\|^3_{L^2_x H^{|\alpha|}_v}\right], \qquad R \leq t,$$
$$|I_R^{\beta_1,\beta_2}| \lesssim_\delta R t^{-2-|\beta_1|}\varepsilon^2\left[(\ln t)^{2|\alpha|-1}(\ln\ln t)^{4|\alpha|-1}\varepsilon^{\frac{3}{2}} + (\ln t\cdot\ln\ln t)^{-1}\|\gamma\|^2_{L^2_x H^{|\alpha|}_v}\right]. \tag{3.10}$$

We can combine these bounds and Grönwall estimate to obtain the last energy bounds in (3.4). We integrate the first bound for $R \geq t$, the second for $0 \leq R \leq t/(\ln t)^{100}$ and the last for $t/(\ln t)^{100} \leq R \leq t$, to get

$$\frac{d}{dt}\|\gamma\|^2_{L^2_x H^{|\alpha|}_v} \lesssim \varepsilon^4 t^{-1}(\ln t)^{2\alpha} + \frac{\varepsilon^2}{t\ln t}\|\gamma\|^2_{L^2_x H^{|\alpha|}_v} + \frac{\varepsilon}{t(\ln t)^{50}}\|\gamma\|^3_{L^2_x H^{|\alpha|}_v}$$
$$+ \varepsilon^{\frac{7}{2}} t^{-1}(\ln t)^{2|\alpha|-1}(\ln\ln(t))^{4|\alpha|}$$

which lead to (3.4).

To get the first bound in (3.10), we use a crude estimate

$$|I_R^{\beta_1,\beta_2}| \lesssim R^{-1-|\beta_1|}\|\gamma\|^2_{L^2_{x,v}}\|\partial_x\partial_v^{\beta_2}\gamma\|_{L^2_{x,v}}\|\partial_v^\alpha\gamma\|_{L^2_{x,v}} \lesssim R^{-1-|\beta_1|}\|\gamma\|^2_{L^2_{x,v}}\|\gamma\|^{1+\frac{|\beta_2|}{|\alpha|}}_{L^2_x H^{|\alpha|}_v}\|\gamma\|^{\frac{|\beta_1|}{|\alpha|}}_{L^2_v H_x^{\frac{|\alpha|}{|\beta_1|}}}$$
$$\lesssim R^{-1-|\beta_1|}\varepsilon^{2+\frac{|\beta_1|}{|\alpha|}}\|\gamma\|^{1+\frac{|\beta_2|}{|\alpha|}}_{L^2_x H^{|\alpha|}_v}$$

and using convexity, this gives the first estimate in (3.10).

On the other hand, we can change variables and integrate by parts to get

$$I_R^{\beta_1,\beta_2} = t^{-6}\iint_{\mathbb{R}^3}\gamma(a,\frac{y-a}{t})\gamma(a,\frac{y-a}{t})\partial_x^{\beta_1}\partial_{x^j}\left\{\chi(R^{-1}|x-y|)\right\}$$
$$\cdot\,\partial_{x^j}\partial_v^{\beta_2}\gamma(b,\frac{x-b}{t})\partial_v^\alpha\gamma(b,\frac{x-b}{t})dxdydadb$$
$$= t^{-6-|\beta_1|}\sum_{\theta_1+\theta_2=\beta_1,\ \theta_1\leq\theta_2} c_{\theta_1,\theta_2} I_R^{\theta_1,\theta_2,\beta_2},$$
$$I_R^{\theta_1,\theta_2,\beta_2} := \iint_{\mathbb{R}^3}\partial_v^{\theta_1}\gamma(a,\frac{y-a}{t})\partial_v^{\theta_2}\gamma(a,\frac{y-a}{t})\partial_{x^j}\left\{\chi(R^{-1}|z|)\right\}$$
$$\cdot\,\partial_{x^j}\partial_v^{\beta_2}\gamma(b,\frac{z+y-b}{t})\partial_v^\alpha\gamma(b,\frac{z+y-b}{t})dzdydadb.$$

In case $\theta_1 = \beta_2 = 0$, $\theta_2 = \beta_1 = \alpha$, we see that

$$|I_R^{0,\alpha,0}| \lesssim R^2 t^3 \|\gamma\|_{Z'} \|\partial_x \gamma\|_{Z'} \|\partial_v^\alpha \gamma\|_{L^2_{x,v}}^2 \lesssim R^2 t^3 \varepsilon^2 \|\partial_v^\alpha \gamma\|_{L^2_{x,v}}^2.$$

In case $\theta_1 = 0$, $\theta_2 = \beta_1$, $\beta_2 \neq 0$, Hölder's inequality gives

$$\begin{aligned}
|I_R^{0,\theta_2,\beta_2}| &\lesssim R^{-1} \|\gamma(a, \frac{y-a}{t})\|_{L^\infty_y L^2_a} \|\partial_v^{\theta_2} \gamma(a, \frac{y-a}{t})\|_{L^6_y L^2_a} \|(\nabla\chi)(R^{-1}|x-y|)\|_{L_y^{\frac{6}{5}}} \\
&\quad \times \|\partial_{x^j} \partial_v^{\beta_2} \gamma(b, \frac{x-b}{t})\|_{L^2_{x,b}} \|\partial_v^\alpha \gamma(b, \frac{x-b}{t})\|_{L^2_{x,b}} \\
&\lesssim R^{-1} \|\gamma\|_{Z'} \cdot t^{\frac{1}{2}} \|\gamma\|_{L^2_x H_v^{|\beta_1|+1}} \cdot R^{\frac{5}{2}} \cdot t^3 \cdot \|\nabla_x \partial_v^{\beta_2} \gamma\|_{L^2_{x,v}} \|\partial_v^\alpha \gamma\|_{L^2_{x,v}} \\
&\lesssim R^2 t^3 (t/R)^{\frac{1}{2}} \cdot \|\gamma\|_{Z'} \cdot \|\gamma\|_{L^2_x H_v^{|\alpha|}}^{1+\frac{|\beta_1|+1}{|\alpha|}+\frac{|\beta_2|}{|\alpha|}} \|\gamma\|_{L^2_x H_x^{|\alpha|}}^{1-\frac{1}{|\alpha|}} \\
&\lesssim R^2 t^3 (t/R)^{\frac{1}{2}} \cdot \varepsilon^{2-\frac{1}{|\alpha|}} \|\gamma\|_{L^2_x H_v^{|\alpha|}}^{2+\frac{1}{|\alpha|}}
\end{aligned}$$

while if $\beta_2 = 0$, $\theta_1 \neq 0$, we proceed similarly

$$\begin{aligned}
|I_R^{\theta_1,\theta_2,0}| &\lesssim R^{-1} \|\partial_v^{\theta_1} \gamma(a, \frac{y-z-a}{t})\|_{L^2_{y,a}} \|\partial_v^{\theta_2} \gamma(a, \frac{y-z-a}{t})\|_{L^6_z L^2_a} \|(\nabla\chi)(R^{-1}|z|)\|_{L_z^{\frac{6}{5}}} \\
&\quad \times \|\nabla_x \gamma(b, \frac{y-b}{t})\|_{L^\infty_y L^2_b} \|\partial_v^\alpha \gamma(b, \frac{y-b}{t})\|_{L^2_{y,b}} \\
&\lesssim R^2 t^3 (t/R)^{\frac{1}{2}} \cdot \varepsilon^{2-\frac{1}{|\alpha|}} \|\gamma\|_{L^2_x H_v^{|\alpha|}}^{2+\frac{1}{|\alpha|}}.
\end{aligned}$$

Finally, if $\theta_1 \neq 0$, $\beta_2 \neq 0$, then $|\alpha| = 3$ and we obtain

$$\begin{aligned}
|I_R^{\theta_1,\theta_2,\beta_2}| &\lesssim R^2 \|\partial_v \gamma(a, \frac{y-a}{t})\|_{L^6_y L^2_a}^2 \|\partial_{x^j} \partial_v \gamma(b, \frac{z+y-b}{t})\|_{L^6_y L^2_b} \|\partial_v^\alpha \gamma(b, \frac{z+y-b}{t})\|_{L^2_{y,b}} \\
&\lesssim R^2 t^3 \|\gamma\|_{L^2_x H^2_v}^2 \|\gamma\|_{H^1_x H^2_v} \|\gamma\|_{L^2_x H^3_v} \lesssim R^2 t^3 \varepsilon \|\gamma\|_{L^2_x H^3_v}^3.
\end{aligned}$$

We now obtain improved bounds in the regime $R \sim t$. In this case, we leave one additional derivative on the kernel to get

$$\begin{aligned}
I_R^{\beta_1,\beta_2} &= t^{-5-|\beta_1|} \sum_{\theta_1+\theta_2+\theta_3=\beta_1,\ \theta_1 \leq \theta_2} c_{\theta_1,\theta_2,\theta_3} I_R^{\theta_1,\theta_2,\theta_3,\beta_2}, \\
I_R^{\theta_1,\theta_2,\theta_3,\beta_2} &= \iint_{\mathbb{R}^3} \partial_v^{\theta_1} \gamma(a, \frac{y-a}{t}) \partial_v^{\theta_2} \gamma(a, \frac{y-a}{t}) \partial_x^{\theta_3} \partial_{x_i} \left\{ \chi(R^{-1}|x-y|) \right\} \\
&\qquad \cdot \partial_{x^j} \partial_v^{\beta_2} \gamma(b, \frac{x-b}{t}) \partial_v^\alpha \gamma(b, \frac{x-b}{t}) dxdydadb
\end{aligned}$$

with $|\theta_3| = 1$, $|\theta_1| + |\theta_2| + |\theta_3| + |\beta_2| = |\alpha|$. If $|\theta_1| = |\theta_2| = 0$, we can directly compute

$$|I_R^{0,0,\theta_3,\beta_2}| \lesssim R\|\gamma\|_{Z'}^2 \|\gamma\|_{H_x^1 H_v^{|\alpha|-1}} \|\gamma\|_{L_x^2 H_v^{|\alpha|}} \lesssim R\varepsilon^2 \|\gamma\|_{H_x^1 H_v^{|\alpha|-1}} \|\gamma\|_{L_x^2 H_v^{|\alpha|}}.$$

If $|\theta_1| = |\beta_2| = 0$, then we can proceed similarly

$$|I_R^{0,\theta_2,\theta_3,0}| \lesssim R\|\gamma\|_{Z'} \|\nabla_x \gamma\|_{Z'} \|\gamma\|_{L_x^2 H_v^{|\alpha|-1}} \|\gamma\|_{L_x^2 H_v^{|\alpha|}} \lesssim R\varepsilon^2 \|\gamma\|_{H_x^1 H_v^{|\alpha|-1}} \|\gamma\|_{L_x^2 H_v^{|\alpha|}}.$$

Finally, if $\theta_2 \neq 0$, $\beta_2 \neq 0$, $|\alpha| = 3$, $|\theta_1| = 0$ and we decompose in Littlewood–Paley pieces:

$$\begin{aligned} I_R^{0,\theta_2,\theta_3,\beta_2} &= \sum_{C_1,C_2} I_{R,C_1,C_2}^{0,\theta_2,\theta_3,\beta_2}, \\ I_{R,C_1,C_2}^{0,\theta_2,\theta_3,\beta_2} &:= \iint_{\mathbb{R}^3} \gamma(a, \frac{y-a}{t}) \partial_v^{\theta_2} \gamma_{C_1}(a, \frac{y-a}{t}) \partial_x^{\theta_3} \partial_{x_i} \left\{ \chi(R^{-1}|x-y|) \right\} \\ &\qquad \cdot \partial_{x^j} \partial_v^{\beta_2} \gamma_{C_2}(b, \frac{x-b}{t}) \partial_v^{\alpha} \gamma(b, \frac{x-b}{t}) dxdydadb. \end{aligned}$$

In case $\min\{C_1, C_2\} \leq 1$, the derivative is favorable and one can proceed as above. From now on, we may assume that the sums are over dyadic $C_1, C_2 \geq 1$. Proceeding as above, we can bound

$$\begin{aligned} |I_{R,C_1,C_2}^{0,\theta_2,\theta_3,\beta_2}| \lesssim Rt^3 \|\gamma\|_{Z'} \|\gamma\|_{L_x^2 H_v^{|\alpha|}} \cdot \min\{ & C_2^{-1} \|\partial_v \gamma_{C_1}(a, \frac{y-a}{t})\|_{L_y^\infty L_a^2} \|\gamma\|_{H_x^1 H_v^2}, \\ & C_1 C_2 \|\gamma_{C_1}\|_{L_{x,v}^2} \|\nabla_x \gamma\|_{Z'}\} \end{aligned}$$

using that

$$\begin{aligned} \|\partial_v \gamma_C(a, \frac{y-a}{t})\|_{L_y^\infty L_a^2} &\lesssim \min\{C\|\gamma\|_{Z'}, C^{-\frac{3}{2}} \|\gamma\|_{L_x^2 H_v^3}\} \lesssim \min\{C\varepsilon, C^{-\frac{3}{2}} \|\gamma\|_{L_x^2 H_v^3}\} \\ \|\gamma_C\|_{L_{x,v}^2} &\lesssim \min\{\|\gamma\|_{L_{x,v}^2}, C^{-3} \|\gamma\|_{L_x^2 H_v^3}\} \lesssim \min\{\varepsilon, C^{-3} \|\gamma\|_{L_x^2 H_v^3}\} \end{aligned}$$

and summing the bounds above and using interpolation, one finds that

$$\sum_{C_1\le C_2}|I^{0,\theta_2,\theta_3,\beta_2}_{R,C_1,C_2}|\lesssim Rt^3\varepsilon\|\gamma\|_{L^2_xH^{|\alpha|}_v}\|\gamma\|_{H^1_xH^2_v}\sum_{C_1}\min\{\varepsilon,C_1^{-\frac{1}{2}}\|\gamma\|_{L^2_xH^3_v}\}$$

$$\lesssim Rt^3\varepsilon^{\frac{7}{3}}\|\gamma\|^{\frac{5}{3}}_{L^2_xH^{|\alpha|}_v}\ln\langle\|\gamma\|_{L^2_xH^{|\alpha|}_v}\rangle,$$

$$\sum_{C_2<C_1}|I^{0,\theta_2,\theta_3,\beta_2}_{R,C_1,C_2}|\lesssim Rt^3\varepsilon^2\|\gamma\|_{L^2_xH^{|\alpha|}_v}\sum_{C_1}\min\{\varepsilon C_1^2,C_1^{-1}\|\gamma\|_{L^2_xH^3_v}\}$$

In total, this gives, for any $\delta>0$,

$$|I^{0,\theta_2,\theta_3,\beta_2}_{R}|\lesssim Rt^3\varepsilon^2\left\{\frac{1}{\ln\langle t\rangle\cdot\ln\langle\ln\langle t\rangle\rangle}\|\gamma\|^2_{L^2_xH^{|\alpha|}_v}+\varepsilon^{\frac{3}{2}}(\ln\langle t\rangle)^5(\ln\langle\ln\langle t\rangle\rangle)^{11}\right\},$$

which leads to an acceptable contribution in (3.4)

4 Nonlinear Analysis II: Asymptotic Flow and Strong Convergence

Once we have isolated the scattering mass in (1.6), we can simplify the dynamics along rays by studying the electric field $\nabla_x\phi$. We compute

$$\nabla_x\phi(x+tv)=-\frac{1}{4\pi}\iint_{\mathbb{R}^3}\frac{x+tv-y}{|x+tv-y|^3}\gamma^2(y-tu,u)dudy$$
$$=\frac{1}{4\pi}\frac{1}{t^3}\iint_{\mathbb{R}^3}\frac{z}{|z|^3}\gamma^2(a,\frac{x-a+z}{t}+v)dadz.$$

When x remains in a bounded set, the main contribution to the electric field will come from

$$E_{main}(v,t):=\frac{1}{4\pi}\frac{1}{t^3}\iint_{\mathbb{R}^3}\frac{z}{|z|^3}\gamma^2(a,\frac{z}{t}+v)dadz=\frac{1}{4\pi}\frac{1}{t^2}\int_{\mathbb{R}^3}\frac{\zeta}{|\zeta|^3}m_t(\zeta-v)d\zeta.$$

This expression only involves the scattering mass that converges. We thus define

$$\widetilde{E}(v):=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\zeta}{|\zeta|^3}m_\infty(\zeta-v)d\zeta.$$

Note that $m_\infty\in L^1\cap L^\infty$, so that $\widetilde{E}(v)$ is well defined and $E_{main}=t^{-2}\widetilde{E}+o(t^{-2})$. Inspired by the model characteristics of

$$\partial_tf(x,v,t)=qE(v,t)\cdot\left\{\nabla_v-t\nabla_x\right\}f(x,v,t)$$

we define for $t \geq 1$,

$$\sigma(x, v, t) = \gamma(X, v, t), \qquad X := x + q\ln(t)\widetilde{E}(v). \tag{4.1}$$

Proposition 4.1. Let $\gamma(x, v, t)$ be a global solution of (3.1) as in Proposition 3.1. Then with X, σ as in (4.1) there exists $\sigma_\infty(x, v) \in Z \cap H^1_{x,v}$ such that

$$\left\|\sigma(x, v, t) - \sigma_\infty(x, v)\right\|_{Z\cap H^1_{x,v}} \lesssim \varepsilon^3 t^{-\frac{1}{200}}.$$

Proof. In the following, we may assume $t \geq 100$. From (4.1) we compute that

$$\partial_{x^j}\sigma = \partial_{x^j}\gamma, \qquad \partial_{v^j}\sigma = \partial_{v^j}\gamma + q\ln(t)\partial_{v^j}\widetilde{E}^k\partial_{x^k}\gamma, \qquad \partial_t\sigma = \partial_t\gamma + \frac{q}{t}\widetilde{E}^p\partial_{x^p}\gamma,$$

from which we obtain the equation

$$\partial_t\sigma(x, v) = q\partial_{x^k}\phi(X + tv)(\partial_{v^k}\gamma)(X, v) + q\left\{\frac{1}{t}\widetilde{E}^p - t\partial_{x^p}\phi(X + vt)\right\}\cdot\partial_{x^p}\gamma(X, v). \tag{4.2}$$

We claim that this is integrable in time in both Z and $H^1_{x,v}$.

We start with the first term in (4.2). For $1 \leq j, k \leq 3$, we compute

$$\begin{aligned}
\left\|\partial_{x^k}\phi(X + tv)(\partial_{v^k}\gamma)(X, v)\right\|_Z &\lesssim \|\nabla\phi\|_{L^\infty}\left\|\nabla_v\gamma\right\|_Z,\\
\left\|\partial_{x^j}[\partial_{x^k}\phi(X + tv)(\partial_{v^k}\gamma)(X, v)]\right\|_{L^2_{x,v}} &\lesssim \left\|\partial_{x^j}\nabla\phi\right\|_{L^\infty}\|\gamma\|_{L^2_xH^1_v} + \|\nabla\phi\|_{L^\infty}\|\gamma\|_{H^1_xH^1_v},\\
\left\|\partial_{v^j}[\partial_{x^k}\phi(X + tv)(\partial_{v^k}\gamma)(X, v)]\right\|_{L^2_{x,v}} &\lesssim t\left\|\partial_{x^j}\nabla\phi\right\|_{L^\infty}\|\gamma\|_{L^2_xH^1_v} + \|\nabla\phi\|_{L^\infty}\|\gamma\|_{L^2_xH^2_v}\\
&\quad + \ln(t)\|\nabla\phi\|_{L^\infty}\|\nabla_v\widetilde{E}(v)\|_{L^4}\left\|\nabla_v\gamma\right\|_{H^1_xL^4_v}.
\end{aligned}$$

By boundedness of the Riesz transform, we see that

$$\partial_{v^j}\widetilde{E}^k(v) = R_jR_k\tilde{m}_\infty \in L^p, \quad 1 < p < \infty, \qquad \tilde{m}_\infty(x) = m_\infty(-x)$$

so together with the bounds (3.4) on γ and the decay (3.3) of $\nabla\phi$, time integrability of the first term follows. For the second term in (4.2) we compute that

$$\begin{aligned}
&\mathcal{DE} := \frac{1}{t}\widetilde{E}^p - t\partial_{x^p}\phi(X+vt) = \frac{1}{t}\nabla(-\Delta)^{-1}\left\{M_1+M_2+M_3\right\},\\
&M_1(\zeta) := m_\infty(\zeta) - m_t(\zeta), \qquad M_2(\zeta) := \int_{\mathbb{R}^3}\left\{\gamma^2(a,\zeta)-\gamma^2(a,\zeta-\frac{a}{t})\right\}da\\
&M_3(\zeta) := \int_{\mathbb{R}^3}\left\{\gamma^2(a,\zeta-\frac{a}{t})-\gamma^2(a,\zeta+\frac{X-a}{t})\right\}da.
\end{aligned}$$

We will often use the convolution structure. Sobolev inequality directly gives that

$$\frac{1}{t}\|\nabla(-\Delta)^{-1}M_j\|_{L^\infty_x} \lesssim t^{-1}\|M_j\|_{L^2\cap L^4}.$$

This allows to bound the contribution of M_1 using (3.5). We can treat M_2 similarly since

$$\begin{aligned}
\|\mathfrak{1}_{\{|a|\ge t^{\frac{1}{2}}\}}\left\{\gamma^2(a,\zeta)-\gamma^2(a,\zeta-\frac{a}{t})\right\}\|_{L^1_{\zeta,a}} &\lesssim t^{-\frac{1}{2}}\|\gamma\|_{L^2_{x,v}}\|x\gamma\|_{L^2_{x,v}} \lesssim t^{-\frac{1}{3}}\varepsilon^2,\\
\|\mathfrak{1}_{\{|a|\le t^{\frac{1}{2}}\}}\left\{\gamma^2(a,\zeta)-\gamma^2(a,\zeta-\frac{a}{t})\right\}\|_{L^1_{\zeta,a}} &\lesssim t^{-\frac{1}{2}}\|\gamma\|_{L^2_{x,v}}\|\nabla_v\gamma\|_{L^2_{x,v}} \lesssim t^{-\frac{1}{3}}\varepsilon^2,\\
\|\gamma^2(a,\zeta)-\gamma^2(a,\zeta-\frac{a}{t})\|_{L^\infty_\zeta L^1_a} &\lesssim \|\gamma\|^2_{H^2_{x,v}} \lesssim \varepsilon^2 t^{2\delta}
\end{aligned}$$

For M_3, we observe the bounds

$$\begin{aligned}
\|\gamma^2(a,\zeta-\frac{a}{t})-\gamma^2(a,\zeta+\frac{X-a}{t})\|_{L^1_{\zeta,a}} &\lesssim t^{-1}|X|\|\gamma\|_{L^2_{x,v}}\|\nabla_v\gamma\|_{L^2_{x,v}} \lesssim t^{\delta-1}|X|\varepsilon^2,\\
\|\gamma^2(a,\zeta-\frac{a}{t})-\gamma^2(a,\zeta+\frac{X-a}{t})\|_{L^1_{\zeta,a}} &\lesssim \|\gamma\|^2_{L^2_{x,v}} \lesssim \varepsilon^2,\\
\|\gamma^2(a,\zeta-\frac{a}{t})-\gamma^2(a,\zeta+\frac{X-a}{t})\|_{L^\infty_\zeta L^1_a} &\lesssim \|\gamma\|^2_{Z'} \lesssim \varepsilon^2
\end{aligned}$$

These bounds are enough to control the Z-norm. Indeed, we see that

$$\|\frac{1}{t}\nabla(-\Delta)^{-1}M_j\cdot\partial_{x^p}\gamma(X,v)\|_Z \lesssim \|\frac{1}{t}\nabla(-\Delta)^{-1}M_j\|_{L^\infty}\|\partial_{x^p}\gamma(X,v)\|_Z$$

and we can use this when $j = 1, 2$, while for M_3, we use that

$$
\begin{aligned}
\|\frac{1}{t}\nabla(-\Delta)^{-1}M_3 \cdot \partial_{x^p}\gamma(X,v)\|_Z \lesssim \|\frac{1}{t}\nabla(-\Delta)^{-1}M_3 \cdot \mathfrak{1}_{\{|X|\leq t^{\frac{1}{6}}\}}\partial_{x^p}\gamma(X,v)\|_Z \\
+ \|\frac{1}{t}\nabla(-\Delta)^{-1}M_3 \cdot \mathfrak{1}_{\{|X|\geq t^{\frac{1}{6}}\}}\partial_{x^p}\gamma(X,v)\|_Z \\
\lesssim \varepsilon^2 t^{-1-\frac{1}{20}}\|\partial_{x^p}\gamma(X,v)\|_Z + \varepsilon^2 t^{-1-\frac{1}{100}}\||x|^{\frac{1}{8}}\nabla_x\gamma\|_Z
\end{aligned}
$$

and again, this gives an acceptable contribution using (4.3) below. The control of $L^2_v H^1_x$ is similar since

$$
\begin{aligned}
&\partial_{x^k}\left\{\left\{\frac{1}{t}\widetilde{E}^p - t\partial_{x^p}\phi(X+vt)\right\}\cdot\partial_{x^p}\gamma(X,v)\right\} \\
&\quad= \frac{1}{t}\sum_j \partial_{x^p}(-\Delta)^{-1}M_j \cdot \partial_{x^p}\partial_{x^k}\gamma(X,v) - t\partial^2_{x^p x^k}\phi(X+tv)\partial_{x^p}\gamma(X,v)
\end{aligned}
$$

with a new term that can be treated as follows:

$$
t\|\nabla^2\phi(X+tv)\nabla_x\gamma(X,v)\|_{L^2_{x,v}} \lesssim t\|\nabla^2\phi\|_{L^\infty}\|\nabla_x\gamma\|_{L^2_{x,v}} \lesssim t^{\delta-2}\varepsilon^3.
$$

Finally, the control of $L^2_x H^1_v$ follows along similar lines, but requires a little more care. Indeed

$$
\begin{aligned}
&\partial_{v^k}\left\{\left\{\frac{1}{t}\widetilde{E}^p - t\partial_{x^p}\phi(X+vt)\right\}\cdot\partial_{x^p}\gamma(X,v)\right\} \\
&\quad= \frac{1}{t}\sum_j \partial_{x^p}(-\Delta)^{-1}M_j \cdot \partial_{x^p}\partial_{v^k}\gamma(X,v) + \frac{\ln t}{t}\nabla(-\Delta)^{-1}M_j \cdot \partial_{x^p}\partial_{x^\ell}\gamma(X,v)\cdot\partial_{v^k}\widetilde{E}^\ell \\
&\quad+ \frac{1}{t}\sum_j \partial_{x^p}\partial_{x^k}(-\Delta)^{-1}M_j \cdot \partial_{x^p}\partial_{v^k}\gamma(X,v)
\end{aligned}
$$

The last term is slightly singular. We can use the boundedness of the Riesz transform to control

$$
\|\partial_{x^p}\partial_{x^k}(-\Delta)^{-1}M_j \cdot \partial_{x^p}\partial_{v^k}\gamma(X,v)\|_{L^2_{x,v}} \lesssim \|M_j\|_{L^4_v}\|\partial_{x^p}\partial_{v^k}\gamma(X,v)\|_{L^4_v L^2_x}
$$

and this is enough for M_1, M_2, and for M_3, we use the same decomposition to get

$$\|\partial_{x^p}\partial_{x^k}(-\Delta)^{-1}M_3\cdot\partial_{x^p}\partial_{v^k}\gamma(X,v)\|_{L^2_{x,v}\{|X|\le t^{\frac{1}{10}}\}}\lesssim\varepsilon^2 t^{-\frac{1}{100}}\|\partial_{x^p}\partial_{v^k}\gamma(X,v)\|_{L^4_v L^2_x}$$

$$\|\partial_{x^p}\partial_{x^k}(-\Delta)^{-1}M_3\cdot\partial_{x^p}\partial_{v^k}\gamma(X,v)\|_{L^2_{x,v}\{|X|\ge t^{\frac{1}{10}}\}}\lesssim\|M_3\|_{L^6_v}\|\partial_{x^p}\partial_{v^k}\gamma(X,v)\|_{L^3_v L^2_x\{|X|\ge t^{\frac{1}{10}}\}}$$

and we can bound the last term with (4.3).

To finish the proof, it suffices to show that

$$\||x|^{\frac{1}{8}}\nabla_x\gamma\|_Z+\||x|^{\frac{1}{8}}\nabla_{x,v}\gamma\|_{L^2_{x,v}}\lesssim\||x|^{\frac{1}{8}}\gamma\|_{H^1_x H_v^{\frac{13}{8}}}\lesssim\|x\gamma\|_{L^2}+\|\gamma\|_{H^3_{x,v}}\lesssim\varepsilon t^{\delta}. \tag{4.3}$$

The first inequality follows from Sobolev embedding; the second inequality follows directly if γ is supported on $\{|x|\le 1\}$ or is localized at small frequencies in x or in v; in the other cases, we introduce a Littlewood–Paley decomposition as in (2.1) in x (P^x_A) and in v (P^v_B) to get

$$\begin{aligned}\||x|^{\frac{1}{8}}\mathfrak{1}_{\{|x|\sim R\}}P^x_A P^v_B\gamma\|_{H^1_x H_v^{\frac{13}{8}}}&\lesssim R^{\frac{1}{8}}AB^{\frac{13}{8}}\|\mathfrak{1}_{\{|x|\sim R\}}P^x_A P^v_B\gamma\|_{L^2_{x,v}}\\&\lesssim R^{\frac{1}{8}}AB^{\frac{13}{8}}\min\{R^{-1},A^{-3},B^{-3}\}\cdot\left[\|x\gamma\|_{L^2_{x,v}}+\|\gamma\|_{H^3_{x,v}}\right]\end{aligned}$$

and we can sum this over dyadic $A,B,R\gtrsim 1$. ■

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