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ABSTRACT

The Hamiltonian structure of a set of gauge-free gyrokinetic Vlasov–Maxwell equations is presented in terms of a Hamiltonian functional and a gyrokinetic Vlasov–Maxwell bracket. The bracket is used to show that the gyrokinetic angular momentum conservation law can be expressed in Hamiltonian form. The Jacobi property of the gyrokinetic Vlasov–Maxwell bracket is also demonstrated explicitly.

I. INTRODUCTION

The Hamiltonian structure of several plasma physics models has been a topic of constant interest since the discovery of the Hamiltonian structures for the ideal magnetohydrodynamics[1] and the Vlasov–Maxwell equations.[2–5] The numerical algorithms derived from the Vlasov–Maxwell Hamiltonian structure were explored in several recent papers.[6–12] The generic guiding-center Vlasov–Maxwell bracket was presented by Morrison,[13] while the generic gyrokinetic Vlasov–Maxwell bracket was presented by Burby *et al.*[14–17] The general Hamiltonian formulation for the reduced Vlasov–Maxwell equations was also derived by Lie-transform methods by Brizard *et al.*[18]

The guiding-center Vlasov–Maxwell equations suitable for Hamiltonian formulation were initially presented by Pfirsch and Morrison[19] and recently presented in simplified form (without guiding-center polarization) by Brizard and Tronci,[20] while two gauge-free gyrokinetic Vlasov–Maxwell models suitable for Hamiltonian formulation were presented by Burby and Brizard[21] and Brizard.[22,23]

After having asymptotically eliminated the gyroangle ζ and constructed the gyroaction $J \equiv (mc/q)\,\mu$ as an adiabatic invariant (μ denotes the magnetic moment of a particle of mass m and charge q), a reduced Lagrangian L_g is expressed in general, form as

$$L_{g} = \left(\frac{q}{c}\,\mathbf{A} + \mathbf{\Pi}_{g}\right)\cdot\frac{d\mathbf{X}}{dt} + J\,\frac{d\zeta}{dt} - \left(q\,\Phi + -K_{g}\right), \qquad (1)$$

where the reduced phase-space coordinates $Z^{\alpha} = (\mathbf{X}, p_{||}, J, \zeta)$ include the reduced particle position $\mathbf{X}$ and the parallel kinetic momentum $p_{||}$, the reduced symplectic momentum and kinetic energy are denoted $\mathbf{\Pi}_{g}$ and K_{g}, respectively. We note that, because of the "minimal-coupling" potential terms $(q/c)\mathbf{A}\cdot d\mathbf{X}/dt - q\,\Phi$, the reduced Lagrangian (1) is invariant under an electromagnetic gauge transformation $(\Phi, \mathbf{A}) \rightarrow (\Phi - c^{-1}\partial\chi/\partial t, \mathbf{A} + \nabla\chi)$ since the reduced dynamics is invariant under the Lagrangian gauge transformation[24] $L_{g} \rightarrow L_{g} + (q/c)\,d\chi/dt$, i.e., this transformation does not change the reduced equations of motion obtained from Eq. (1). Hence, a gauge-free reduced Lagrangian formulation is obtained when the reduced symplectic momentum $\mathbf{\Pi}_{g}$ and the reduced kinetic energy K_{g} in Eq. (1) depend on the electric and magnetic fields $(\mathbf{E} \equiv -\nabla\Phi - c^{-1}\partial\mathbf{A}/\partial t, \mathbf{B} \equiv \nabla\times\mathbf{A})$ only. For example, in guiding-center Vlasov–Maxwell theory, Pfirsch and Morrison[19] used $\mathbf{\Pi}_{gc} = p_{||}\,\widehat{\mathsf{b}} + \mathbf{E}\times q\widehat{\mathsf{b}}/\Omega$ and $K_{gc} = \mu\,B + |\mathbf{\Pi}_{gc}|^{2}/2m$, while Brizard and Tronci[20] considered the simpler guiding-center Lagrangian with $\mathbf{\Pi}_{gc} = p_{||}\widehat{\mathsf{b}}$ and $K_{gc} = \mu B + p_{||}^{2}/2m$. The Hamiltonian structure for the Brizard–Tronci version of the guiding-center Vlasov–Maxwell equations was recently presented elsewhere,[25] with an extensive proof of the Jacobi property for the guiding-center Vlasov–Maxwell bracket.[26]

In gyrokinetic Vlasov–Maxwell theory, on the other hand, the electromagnetic potentials in Eq. (1) are decomposed in terms of background and perturbed components: $(\Phi, \mathbf{A}) = (\epsilon\Phi_{1}, \mathbf{A}_{0} + \epsilon\mathbf{A}_{1})$, where $\mathbf{B}_{0} = \nabla\times\mathbf{A}_{0}$ is the time-independent background (unperturbed) magnetic field, and the dimensionless parameter $\epsilon \ll 1$ orders the perturbation amplitudes in a manner consistent with standard gyrokinetic theory.[27] We note that, unless a gyrokinetic Vlasov–Maxwell model is formulated exclusively in terms of the perturbed electric and magnetic fields (with or without the minimal-coupling potential terms), the appearance of potentials in the gyrocenter kinetic energy K_{gy} prevents a Hamiltonian formulation. Hence, standard gyrokinetic Vlasov–Maxwell models[27] are not suitable for a Hamiltonian formulation.

The recent work of Burby and Brizard[21] is suitable for a Hamiltonian formulation, however, since the gyrocenter symplectic momentum $\mathbf{\Pi}_{\rm gy} = p_{\|}\widehat{\sf b}_0$ is expressed only in terms of the unperturbed magnetic field, while the gyrocenter kinetic energy $K_{\rm gy}$ is a function of $(\mathbf{E}_1, \mathbf{B}_1)$ up to second order in ϵ [see Eq. (3) below]. The purpose of the present paper is, therefore, to derive the explicit Hamiltonian structure for the gauge-free gyrokinetic Vlasov–Maxwell equations presented by Burby and Brizard.[21] This direct approach is in contrast to the formal derivation presented by Burby.[15] The case of the gauge-free gyrokinetic equations derived by Brizard,[22,23] in which the gyrocenter symplectic momentum $\mathbf{\Pi}_{\rm gy} = p_{\|}\widehat{\sf b}_0 + \epsilon\,\mathbf{\Pi}_{1\rm gy}(\mathbf{E}_1, \mathbf{B}_1)$ includes first-order electromagnetic corrections, will be considered in future work.

The remainder of this paper is organized as follows. In Sec. II, we present the gauge-free gyrokinetic Vlasov–Maxwell equations derived by Burby and Brizard,[21] which are presented here in the drift-kinetic limit in order to simplify our presentation. In Sec. III, the gyrokinetic Vlasov–Maxwell bracket is explicitly constructed from the Hamiltonian formulation of the gyrokinetic Vlasov–Maxwell equations. This gyrokinetic bracket structure is immediately applied to the proofs that the gyrokinetic entropy functional is a Casimir of the gyrokinetic Vlasov–Maxwell bracket and the gyrokinetic Vlasov–Maxwell toroidal angular momentum conservation law, first derived in variational (Lagrangian) form in Refs. 23 and 28 can be expressed in Hamiltonian form. In Sec. IV, the explicit proof of the Jacobi property of the gyrokinetic Vlasov–Maxwell bracket derived in Sec. III is given, and a summary of our work is presented in Sec. V.

II. GAUGE-FREE GYROKINETIC VLASOV–MAXWELL EQUATIONS

We begin with the gauge-free gyrocenter single-particle Lagrangian,

$$L_{\rm gy} = \left[\frac{q}{c}(\mathbf{A}_0 + \epsilon - \mathbf{A}_1) + p_{\|}\,\widehat{\sf b}_0 - J\,\mathbf{R}_0^*\right]\cdot\frac{d\mathbf{X}}{dt} + J\,\frac{d\zeta}{dt} - (q\,\epsilon\,\Phi_1 + -K_{\rm gy}) \equiv P_\alpha\,\frac{dZ^\alpha}{dt} - H_{\rm gy}, \tag{2}$$

where the higher-order guiding-center corrections $\mathbf{R}_0^* \equiv \mathbf{R}_0 + \frac{1}{2}\nabla\times\widehat{\sf b}_0$ include the background gyrogauge vector field $\mathbf{R}_0$ (which ensures that the guiding-center equations of motion are independent of the gyroangle as well as how the gyroangle is measured[29]) and the guiding-center polarization correction $\frac{1}{2}\nabla\times\widehat{\sf b}_0$.[30] Next, the gyrocenter kinetic energy is expanded up to second order in $\epsilon \ll 1$,[21]

$$\begin{aligned} K_{\rm gy} = {} & \frac{p_{\|}^2}{2m} + \mu\left(B_0 + \epsilon\,B_{1\|} + \frac{\epsilon^2}{2B_0}\,|\mathbf{B}_1|^2\right) \\ & -\epsilon\,\boldsymbol{\pi}_{\rm gc}\cdot\left(\mathbf{E}_1 + \frac{p_{\|}\widehat{\sf b}_0}{mc}\times\mathbf{B}_1\right) \\ & -\epsilon^2\,\frac{mc^2}{2B_0^2}\,|\mathbf{E}_1 + (p_{\|}\widehat{\sf b}_0/mc)\times\mathbf{B}_1|^2, \end{aligned} \tag{3}$$

where $\boldsymbol{\pi}_{\rm gc}$ denotes the guiding-center electric-dipole moment.[30] For the sake of clarity, the gyrokinetic Vlasov–Maxwell model considered here is presented in its drift-kinetic limit, where finite-Larmor-radius (FLR) corrections are retained only through the guiding-center electric-dipole moment $\boldsymbol{\pi}_{\rm gc}$.

A. Gyrocenter equations of motion

The gyrocenter equations of motion are first derived from the gyrocenter Lagrangian (2) as Euler–Lagrange equations $\omega_{\alpha\beta}\,dZ^\beta/dt - \partial H_{\rm gy}/\partial Z^\alpha$,

$$0 = \epsilon\,q\mathbf{E}_1 - \nabla K_{\rm gy} + \frac{q}{c}\frac{d\mathbf{X}}{dt}\times\mathbf{B}^* - \frac{dp_{\|}}{dt}\,\widehat{\sf b}_0, \tag{4}$$

$$0 = \widehat{\sf b}_0\cdot\frac{d\mathbf{X}}{dt} - \frac{\partial K_{\rm gy}}{\partial p_{\|}}, \tag{5}$$

$$0 = -\frac{dJ}{dt} - \frac{\partial K_{\rm gy}}{\partial\zeta} \equiv -\frac{dJ}{dt}, \tag{6}$$

$$0 = \frac{d\zeta}{dt} - \frac{\partial K_{\rm gy}}{\partial J} - \mathbf{R}_0^*\cdot\frac{d\mathbf{X}}{dt}, \tag{7}$$

where $\omega_{\alpha\beta}(\mathbf{X}, p_{\|}, \mu) \equiv \partial P_\beta/\partial Z^\alpha - \partial P_\alpha/\partial Z^\beta$. Equation (6) implies that the gyroaction J (and the gyrocenter magnetic moment μ) is a gyrocenter invariant as a result of the gyroangle-independence of the gyrocenter kinetic energy (3), and the gyroangle ζ is an ignorable coordinate since Eq. (7) is decoupled from the reduced gyrocenter equations of motion (4) and (5), which are expressed in Hamiltonian form as

$$\frac{d\mathbf{X}}{dt} = \{\mathbf{X},\, K_{\rm gy}\}_{\rm gy} + q\,\epsilon\,\mathbf{E}_1\cdot\{\mathbf{X}, \mathbf{X}\}_{\rm gy}, \tag{8}$$

$$\frac{dp_{\|}}{dt} = \{p_{\|},\, K_{\rm gy}\}_{\rm gy} + q\,\epsilon\,\mathbf{E}_1\cdot\{\mathbf{X}, p_{\|}\}_{\rm gy}. \tag{9}$$

Here, the gyrocenter Poisson bracket,

$$\{f,\, g\}_{\rm gy} \equiv \frac{\mathbf{B}^*}{B_{\|}^*}\cdot\left(\nabla f\,\frac{\partial g}{\partial p_{\|}} - \frac{\partial f}{\partial p_{\|}}\,\nabla g\right) - \frac{c\widehat{\sf b}_0}{qB_{\|}^*}\cdot\nabla f\times\nabla g, \tag{10}$$

is used without the ignorable gyromotion canonical pair (J, ζ), with

$$\begin{aligned} \mathbf{B}^* &\equiv \mathbf{B}_0^* + \epsilon\,\mathbf{B}_1, \\ B_{\|}^* &\equiv \widehat{\sf b}_0\cdot\mathbf{B}^* = B_{\|0}^* + \epsilon\,B_{1\|}, \end{aligned} \tag{11}$$

where $\mathbf{B}_0^* = \mathbf{B}_0 + (p_{\|}c/q)\,\nabla\times\widehat{\sf b}_0 - (\mu mc^2/q^2)\,\nabla\times\mathbf{R}_0^*$, which is gyrogauge invariant, and $B_{\|0}^* \equiv \widehat{\sf b}_0\cdot\mathbf{B}_0^*$. We note that the gyrocenter equations of motion (8) and (9) are gauge independent since they only involve the perturbed electromagnetic fields $(\mathbf{E}_1, \mathbf{B}_1)$.

Next, we note that the gyrocenter Poisson bracket (10) satisfies the Jacobi property for arbitrary functions (f, g, h),

$$\Big\{\{f, g\}_{\rm gy}, h\Big\}_{\rm gy} + \Big\{\{g, h\}_{\rm gy}, f\Big\}_{\rm gy} + \Big\{\{h, f\}_{\rm gy}, g\Big\}_{\rm gy} = 0, \tag{12}$$

subject to the condition,

$$\nabla\cdot\mathbf{B}^* = 0, \tag{13}$$

which is satisfied by the definition (11). We note that the gyrocenter Poisson bracket can be expressed in divergence form,

$$\{f,\, g\}_{\rm gy} = \frac{1}{B_{\|}^*}\frac{\partial}{\partial Z^\alpha}\Big(B_{\|}^*\,f\,\{Z^\alpha, g\}_{\rm gy}\Big), \tag{14}$$

while the gyrocenter equations of motion (8) and (9) satisfy the gyrocenter Liouville equation,

$$\frac{\partial B_{\|}^*}{\partial t} = \widehat{\mathsf{b}}_0 \cdot \epsilon\,\frac{\partial \mathbf{B}_1}{\partial t} = -c\,\widehat{\mathsf{b}}_0 \cdot \nabla\times\epsilon\,\mathbf{E}_1 = -\nabla\cdot\left(B_{\|}^*\,\frac{d\mathbf{X}}{dt}\right) - \frac{\partial}{\partial p_{\|}}\left(B_{\|}^*\,\frac{dp_{\|}}{dt}\right). \tag{15}$$

B. Gyrokinetic Vlasov–Maxwell equations

With the help of the reduced gyrocenter equations of motion (8) and (9), we now introduce the gyrokinetic Vlasov–Maxwell equations,

$$\frac{\partial F_{\rm gy}}{\partial t} = -\nabla\cdot\left(F_{\rm gy}\,\frac{d\mathbf{X}}{dt}\right) - \frac{\partial}{\partial p_{\|}}\left(F_{\rm gy}\,\frac{dp_{\|}}{dt}\right), \tag{16}$$

$$\frac{\partial \mathbf{D}_{\rm gy}}{\partial t} = c\,\nabla\times\mathbf{H}_{\rm gy} - 4\pi q\int_P F_{\rm gy}\,\frac{d\mathbf{X}}{dt}, \tag{17}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -c\,\nabla\times\mathbf{E}_1, \tag{18}$$

where the gyrocenter phase-space density $F_{\rm gy} \equiv F\,B_{\|}^*$ is defined in terms of the gyrocenter Jacobian $B_{\|}^*$, so that the gyrokinetic Vlasov equation (16) is expressed in divergence form, the symbol $\int_P$ denotes an integration over $(p_{\|},\mu)$ in Eq. (17), and summation over particle species is implied throughout the work.

The macroscopic gyrokinetic fields $(\mathbf{D}_{\rm gy},\mathbf{H}_{\rm gy})$ in Eq. (17) are defined as

$$\begin{pmatrix}\mathbf{D}_{\rm gy}\\ \mathbf{H}_{\rm gy}\end{pmatrix} = \begin{pmatrix}\epsilon\,\mathbf{E}_1 + 4\pi\,\mathbb{P}_{\rm gy}\\ \mathbf{B}_0 + \epsilon\,\mathbf{B}_1 - 4\pi\,\mathbb{M}_{\rm gy}\end{pmatrix}, \tag{19}$$

where the gyrocenter polarization and magnetization are defined in terms of the gyrocenter kinetic energy (3) as

$$\mathbb{P}_{\rm gy} = -\epsilon^{-1}\int_P F_{\rm gy}\,\frac{\partial K_{\rm gy}}{\partial \mathbf{E}_1} \equiv \int_P F_{\rm gy}\,\boldsymbol{\pi}_{\rm gy} = \int_P F_{\rm gy}\left[\boldsymbol{\pi}_{\rm gc} + \epsilon\,\frac{mc^2}{B_0^2}\left(\mathbf{E}_1 + \frac{p_{\|}\widehat{\mathsf{b}}_0}{mc}\times\mathbf{B}_1\right)\right], \tag{20}$$

$$\mathbb{M}_{\rm gy} = -\epsilon^{-1}\int_P F_{\rm gy}\,\frac{\partial K_{\rm gy}}{\partial \mathbf{B}_1} = \int_P F_{\rm gy}\left[-\mu\left(\widehat{\mathsf{b}}_0 + \epsilon\,\frac{\mathbf{B}_1}{B_0}\right) + \boldsymbol{\pi}_{\rm gy}\times\frac{p_{\|}\widehat{\mathsf{b}}_0}{mc}\right], \tag{21}$$

which are expressed in their drift-kinetic dipole-moment form, whereas FLR corrections would involve higher-order multipole moments.

The remaining Maxwell equations,

$$\nabla\cdot\mathbf{D}_{\rm gy} = 4\pi q\int_P F_{\rm gy}, \qquad \nabla\cdot\mathbf{B}_1 = 0, \tag{22}$$

may be viewed as initial conditions for $(\mathbf{D}_{\rm gy},\mathbf{B}_1)$, since $\nabla\cdot(\partial\mathbf{D}_{\rm gy}/\partial t) = 4\pi\,\partial\varrho_{\rm gy}/\partial t = -4\pi\,\nabla\cdot\mathbf{J}_{\rm gy}$ follows from Eq. (17), which is a statement of the gyrokinetic charge conservation law, while $\nabla\cdot(\partial\mathbf{B}_1/\partial t) = 0$ follows from Eq. (18).

C. Hamiltonian gyrokinetic Vlasov–Maxwell equations

In the Hamiltonian formulation of the gyrokinetic Vlasov–Maxwell equations (16)–(18), we begin with the gyrokinetic Hamiltonian functional,[23]

$$\mathcal{H}_{\rm gy} = \int_Z F_{\rm gy}\,K_{\rm gy}(\mathbf{E}_1,\mathbf{B}_1) + \int_X \frac{\epsilon\,\mathbf{E}_1}{4\pi}\cdot\mathbf{D}_{\rm gy} - \frac{1}{8\pi}\int_X\left(\epsilon^2\,|\mathbf{E}_1|^2 - |\mathbf{B}_0 + \epsilon\,\mathbf{B}_1|^2\right), \tag{23}$$

which is derived by Noether method from a Lagrangian formulation of the gyrokinetic Vlasov–Maxwell equations (16)–(18). If we assume that $\mathbf{D}_{\rm gy}$ and $\mathbf{E}_1$ are functionally independent, we then find

$$\frac{\delta\mathcal{H}_{\rm gy}}{\delta\mathbf{D}_{\rm gy}} = \epsilon\,\mathbf{E}_1/4\pi, \qquad \epsilon^{-1}\,\frac{\delta\mathcal{H}_{\rm gy}}{\delta\mathbf{E}_1} = \mathbf{D}_{\rm gy}/4\pi - \epsilon\,\mathbf{E}_1/4\pi - \mathbb{P}_{\rm gy} = 0, \tag{24}$$

where we used the definition (20) for the gyrocenter polarization. As can be seen from the definitions (20)–(21) of the gyrocenter polarization and magnetization, we might conclude that $\mathbf{D}_{\rm gy} = \mathbf{D}_{\rm gy}[F_{\rm gy},\mathbf{E}_1,\mathbf{B}_1]$ and $\mathbf{H}_{\rm gy} = \mathbf{H}_{\rm gy}[F_{\rm gy},\mathbf{E}_1,\mathbf{B}_1]$ might be functionals of $(F_{\rm gy},\mathbf{E}_1,\mathbf{B}_1)$. The gyrokinetic Vlasov–Maxwell equations (16)–(18), however, clearly imply that the correct gyrokinetic fields are $(F_{\rm gy},\mathbf{D}_{\rm gy},\mathbf{B}_1)$, with $\mathbf{E}_1[F_{\rm gy},\mathbf{D}_{\rm gy},\mathbf{B}_1]$ treated as functional in Eq. (23). The reader is invited to consult Morrison's work[13] and its application in gyrokinetic theory[14] to learn how partial functional derivatives can be handled in terms of constitutive relations.

In what follows, we will formulate a Hamiltonian representation of the gyrokinetic Vlasov–Maxwell equations in terms of the gyrokinetic fields $\boldsymbol{\Psi} = (F_{\rm gy},\mathbf{D}_{\rm gy},\mathbf{B}_1)$. Hence, we shall also make use of the functional derivatives (24) and

$$\frac{\delta\mathcal{H}_{\rm gy}}{\delta F_{\rm gy}} = K_{\rm gy}, \tag{25}$$

$$\epsilon^{-1}\,\frac{\delta\mathcal{H}_{\rm gy}}{\delta\mathbf{B}_1} = \int_P F_{\rm gy}\,\epsilon^{-1}\,\frac{\partial K_{\rm gy}}{\partial\mathbf{B}_1} + \frac{1}{4\pi}(\mathbf{B}_0 + \epsilon\,\mathbf{B}_1) = \mathbf{H}_{\rm gy}/4\pi, \tag{26}$$

and we express the gyrokinetic Vlasov–Maxwell equations (16)–(18) in Hamiltonian form,

$$\frac{\partial\Psi^a}{\partial t} \equiv \mathsf{J}_{\rm gy}^{ab}(\boldsymbol{\Psi})\circ\frac{\delta\mathcal{H}_{\rm gy}}{\delta\Psi^b}, \tag{27}$$

where the gyrokinetic Vlasov–Maxwell Poisson operator $\mathsf{J}_{\rm gy}^{ab}(\boldsymbol{\Psi})\circ$ acts on functional derivatives of the gyrokinetic Hamiltonian functional (23),

$$\frac{\partial F_{\rm gy}}{\partial t} = -\frac{\partial}{\partial Z^\alpha}\left(F_{\rm gy}\left\{Z^\alpha,\,\frac{\delta\mathcal{H}_{\rm gy}}{\delta F_{\rm gy}}\right\}_{\rm gy}\right) - \frac{\partial}{\partial Z^\alpha}\left(F_{\rm gy}\,4\pi q\,\frac{\delta\mathcal{H}_{\rm gy}}{\delta\mathbf{D}_{\rm gy}}\cdot\{\mathbf{X},\,Z^\alpha\}_{\rm gy}\right) \equiv \mathsf{J}_{\rm gy}^{Fb}(\boldsymbol{\Psi})\circ\frac{\delta\mathcal{H}_{\rm gy}}{\delta\Psi^b}, \tag{28}$$

$$\frac{\partial \mathbf{D}_{gy}}{\partial t} = 4\pi c\,\nabla\times\left(\epsilon^{-1}\frac{\delta\mathcal{H}_{gy}}{\delta\mathbf{B}_1}\right) - 4\pi q\int_P F_{gy}\,\frac{d\mathbf{X}}{dt} \equiv \mathsf{J}_{gy}^{\mathbf{D}b}(\mathbf{\Psi})\circ\frac{\delta\mathcal{H}_{gy}}{\delta\Psi^b}, \tag{29}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -4\pi c\,\nabla\times\left(\epsilon^{-1}\frac{\delta\mathcal{H}_{gy}}{\delta\mathbf{D}_{gy}}\right) \equiv \mathsf{J}_{gy}^{\mathbf{B}b}(\mathbf{\Psi})\circ\frac{\delta\mathcal{H}_{gy}}{\delta\Psi^b}. \tag{30}$$

The gyrokinetic Vlasov–Maxwell bracket will be constructed in Sec. III from the gyrokinetic Vlasov–Maxwell equations (28)–(30) used in evaluating the time evolution of an arbitrary gyrokinetic functional $\mathcal{F}[F_{gy},\mathbf{D}_{gy},\mathbf{B}_1]$,

$$\frac{\partial\mathcal{F}}{\partial t} = \int_Z \frac{\partial F_{gy}}{\partial t}\frac{\delta\mathcal{F}}{\delta F_{gy}} + \int_X\left(\frac{\partial\mathbf{D}_{gy}}{\partial t}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{D}_{gy}} + \frac{\partial\mathbf{B}_1}{\partial t}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{B}_1}\right) \equiv \left\langle\frac{\delta\mathcal{F}}{\delta\Psi^a}\middle| -\mathsf{J}_{gy}^{ab}(\mathbf{\Psi})\circ\frac{\delta\mathcal{H}_{gy}}{\delta\Psi^b}\right\rangle = \left[\mathcal{F},\mathcal{H}_{gy}\right]_{gy}, \tag{31}$$

where the gyrokinetic Vlasov–Maxwell bracket is applied to functionals of the gyrokinetic fields $\mathbf{\Psi} = (F_{gy},\mathbf{D}_{gy},\mathbf{B}_1)$ and integrations by parts may be performed.

III. GYROKINETIC VLASOV–MAXWELL BRACKET

In Eq. (31), the antisymmetric gyrokinetic Poisson operator $\mathsf{J}_{gy}^{ab}(\mathbf{\Psi})\circ$ guarantees the antisymmetry property: $[\mathcal{F},\mathcal{G}]_{gy} = -[\mathcal{G},\mathcal{F}]_{gy}$; and the bilinearity of Eq. (31) guarantees the Leibniz (product-rule) property: $[\mathcal{F},\mathcal{G}\,\mathcal{K}]_{gy} = [\mathcal{F},\mathcal{G}]_{gy}\,\mathcal{K} + \mathcal{G}\,[\mathcal{F},\mathcal{K}]_{gy}$. The Jacobi property of the gyrokinetic Vlasov–Maxwell bracket is expressed as the requirement that the *Jacobiator*,

$$\mathcal{J}ac[\mathcal{F},\mathcal{G},\mathcal{K}] \equiv \left[[\mathcal{F},\mathcal{G}]_{gy},\mathcal{K}\right]_{gy} + \left[[\mathcal{G},\mathcal{K}]_{gy},\mathcal{F}\right]_{gy} + \left[[\mathcal{K},\mathcal{F}]_{gy},\mathcal{G}\right]_{gy} = 0, \tag{32}$$

must vanish for arbitrary functionals $(\mathcal{F},\mathcal{G},\mathcal{K})$, which imposes constraints on the Poisson operator $\mathsf{J}_{gy}^{ab}(\mathbf{\Psi})$.

From the gyrokinetic Vlasov–Maxwell equations (28)–(30), we can now extract the gyrokinetic Vlasov–Maxwell bracket from Eq. (31), which is expressed in terms of two arbitrary gyrocenter functionals $(\mathcal{F},\mathcal{G})$ as

$$\begin{aligned}[\mathcal{F},-\mathcal{G}]_{gy} &= \int_Z F_{gy}\left\{\frac{\delta\mathcal{F}}{\delta F_{gy}},\frac{\delta\mathcal{G}}{\delta F_{gy}}\right\}_{gy} + 4\pi q\int_Z F_{gy}\left(\frac{\delta\mathcal{G}}{\delta\mathbf{D}_{gy}}\cdot\left\{\mathbf{X},\frac{\delta\mathcal{F}}{\delta F_{gy}}\right\}_{gy} - \frac{\delta\mathcal{F}}{\delta\mathbf{D}_{gy}}\cdot\left\{\mathbf{X},\frac{\delta\mathcal{G}}{\delta F_{gy}}\right\}_{gy}\right) + (4\pi q)^2\int_Z F_{gy}\left(\frac{\delta\mathcal{F}}{\delta\mathbf{D}_{gy}}\cdot\{\mathbf{X},\mathbf{X}\}_{gy}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{D}_{gy}}\right)\\ &\quad + 4\pi c\int_X\left[\frac{\delta\mathcal{F}}{\delta\mathbf{D}_{gy}}\cdot\nabla\times\left(\epsilon^{-1}\frac{\delta\mathcal{G}}{\delta\mathbf{B}_1}\right) - \frac{\delta\mathcal{G}}{\delta\mathbf{D}_{gy}}\cdot\nabla\times\left(\epsilon^{-1}\frac{\delta\mathcal{F}}{\delta\mathbf{B}_1}\right)\right].\end{aligned} \tag{33}$$

Here, the first term is the Vlasov sub-bracket, the next three terms (multiplied by first and second powers of $4\pi q$) represent the Interaction sub-bracket, and the last two terms (multiplied by $4\pi c$) represent the Maxwell sub-bracket. We note that the Interaction sub-bracket term proportional to $(4\pi q)^2$ does not appear in the standard Vlasov–Maxwell bracket,[2–5] since the Poisson bracket $\{\mathbf{x},\mathbf{x}\}\equiv 0$ vanishes in particle phase space. The generic form of the gyrokinetic Vlasov–Maxwell bracket was first presented by Burby *et al.*[14] In the electrostatic limit, where $\mathbf{E}_1 = -\nabla\Phi_1$ and $\mathbf{B}_1 = 0$, the gyrokinetic Vlasov–Poisson bracket retains only the contributions from the Vlasov and Interaction sub-brackets.

We postpone the proof of the Jacobi property (32) until Sec. IV and, instead, we now look at two applications of the gyrokinetic Vlasov–Maxwell bracket (33): first, we present the proof that the gyrokinetic entropy functional is a Casimir of the gyrokinetic bracket (33); and, second, we present the proof that the gyrokinetic toroidal angular momentum conservation law can be expressed in Hamiltonian form.

A. Gyrokinetic entropy functional

A Casimir functional $\mathcal{C}$ associated with the gyrokinetic bracket (33) satisfies the equation $[\mathcal{C},\mathcal{K}]_{gy} = 0$, which holds for an arbitrary functional $\mathcal{K}$. It is well known that the gyrokinetic entropy functional,

$$\mathcal{S}_{gy}[F_{gy},\mathbf{B}_1] \equiv -\int_Z F_{gy}\,\ln(F_{gy}/B_{\|}^*), \tag{34}$$

is a Casimir for the gyrokinetic bracket (33), which is one example of the generic form $\mathcal{C}[F_{gy},\mathbf{B}_1] = \int_Z B_{\|}^*\,C(F_{gy}/B_{\|}^*)$[14] for an arbitrary function $C(F)$, where $F = F_{gy}/B_{\|}^*$.

From Eq. (34), using $\delta\mathcal{S}_{gy}/\delta F_{gy} = -1-\ln F$ and $\epsilon^{-1}\delta\mathcal{S}_{gy}/\delta\mathbf{B}_1 = \mathrm{int}_P\,F\,\widehat{\mathsf{b}}_0$, we find

$$\begin{aligned}\left[\mathcal{S}_{gy},-\mathcal{K}\right]_{gy} &= -\int_Z B_{\|}^*\left\{F,\frac{\delta\mathcal{K}}{\delta F_{gy}}\right\}_{gy} - 4\pi q\int_Z\frac{\delta\mathcal{K}}{\delta\mathbf{D}_{gy}}\cdot\left(\mathbf{B}^*\frac{\partial F}{\partial p_{\|}} + \frac{c\widehat{\mathsf{b}}_0}{q}\times\nabla F\right) - 4\pi c\int_Z\frac{\delta\mathcal{K}}{\delta\mathbf{D}_{gy}}\cdot\nabla\times\left(F\,\widehat{\mathsf{b}}_0\right)\\ &= 4\pi c\int_Z F\,\frac{\delta\mathcal{K}}{\delta\mathbf{D}_{gy}}\cdot\left(\frac{q}{c}\frac{\partial\mathbf{B}^*}{\partial p_{\|}} - \nabla\times\widehat{\mathsf{b}}_0\right) = 0,\end{aligned} \tag{35}$$

where the first term vanishes since, according to Eq. (14), it is an exact phase-space divergence, while the remaining terms cancel out.

The concept of gyrokinetic entropy can play an important role in the investigation of magnetized plasma turbulence (see Ref. 31, for example). The gyrokinetic entropy functional (34) can also be used to formulate the gyrokinetic metriplectic evolution of an arbitrary gyrokinetic functional $\mathcal{F}$,[32–34]

$$\frac{\partial\mathcal{F}}{\partial t} = \left[\mathcal{F},\mathcal{H}_{gy}\right]_{gy} + \left(\mathcal{F},\mathcal{S}_{gy}\right)_{gy}, \tag{36}$$

in terms of a self-adjoint collisional bracket $(\,,\,)_{gy}$ that conserves energy and momentum, i.e., $(\mathcal{F},\mathcal{H}_{gy})_{gy} = 0$, and satisfies the second law of thermodynamics: $\partial\mathcal{S}_{gy}/\partial t = (\mathcal{S}_{gy},\mathcal{S}_{gy})_{gy} \geq 0$. This metriplectic formulation[35,36] can assist in the investigation of dissipative turbulent transport in magnetized plasmas based on structure-preserving algorithms.

B. Gyrokinetic Vlasov–Maxwell angular momentum conservation law

The conservation laws of energy-momentum and angular momentum for the gyrokinetic Vlasov–Maxwell (28)–(30) were recently derived by Brizard[23] and Hirvijoki *et al.*[28] As an application of the gyrocenter Vlasov–Maxwell bracket (33), we explore the time evolution of the gyrokinetic Vlasov–Maxwell angular momentum functional,[23]

$$\mathcal{P}_{{\rm gy}\varphi}[F_{\rm gy},\mathbf{D}_{\rm gy},\mathbf{B}_1] \equiv \int_Z F_{\rm gy}\,P_\varphi + \int_X \mathbf{D}_{\rm gy}\times\frac{\epsilon\,\mathbf{B}_1}{4\pi\,c}\cdot\frac{\partial\mathbf{X}}{\partial\varphi}, \tag{37}$$

where the gyrocenter toroidal angular momentum,

$$\begin{aligned} P_\varphi &= \left[\frac{q}{c}\mathbf{A}_0 + p_\parallel\,\widehat{\sf b}_0 - (mc/q)\,\mu\,\mathbf{R}_0^*\right]\cdot\frac{\partial\mathbf{X}}{\partial\varphi} \\ &\equiv \frac{q}{c}\,\mathbf{A}_0^*\cdot\frac{\partial\mathbf{X}}{\partial\varphi}, \end{aligned} \tag{38}$$

includes higher-order guiding-center corrections.[30]

We now evaluate the Hamiltonian evolution of the gyrokinetic functional (37),

$$\begin{aligned} \frac{\partial\mathcal{P}_{{\rm gy}\varphi}}{\partial t} &= \left[\mathcal{P}_{{\rm gy}\varphi},\mathcal{H}_{\rm gy}\right]_{\rm gy} \\ &= \int_Z F_{\rm gy}\left(\frac{dP_\varphi}{dt} - \frac{q}{c}\frac{d\mathbf{X}}{dt}\times\epsilon\mathbf{B}_1\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right) \\ &\quad + \int_X\left(\frac{\epsilon\mathbf{B}_1}{4\pi}\times\frac{\partial\mathbf{X}}{\partial\varphi}\right)\cdot\nabla\times\mathbf{H}_{\rm gy} \\ &\quad - \int_X\frac{\epsilon\mathbf{E}_1}{4\pi}\cdot\nabla\times\left(\frac{\partial\mathbf{X}}{\partial\varphi}\times\mathbf{D}_{\rm gy}\right), \end{aligned} \tag{39}$$

where the functional derivatives of the gyrocenter Hamiltonian functional (23) are given in Eqs. (24)–(26), and the functional derivatives of the gyrocenter Vlasov–Maxwell angular momentum (37) are

$$\begin{pmatrix} \delta\mathcal{P}_{{\rm gy}\varphi}/\delta F_{\rm gy} \\ 4\pi c\,\delta\mathcal{P}_{{\rm gy}\varphi}/\delta\mathbf{D}_{\rm gy} \\ 4\pi c\,\delta\mathcal{P}_{{\rm gy}\varphi}/\delta(\epsilon\,\mathbf{B}_1) \end{pmatrix} = \begin{pmatrix} P_\varphi \\ \epsilon\,\mathbf{B}_1\times\partial\mathbf{X}/\partial\varphi \\ (\partial\mathbf{X}/\partial\varphi)\times\mathbf{D}_{\rm gy} \end{pmatrix}. \tag{40}$$

If we ignore exact spatial derivatives (which vanish when integrated over space), the second term in Eq. (39) yields the non-vanishing terms,

$$\begin{aligned} \left(\frac{\epsilon\mathbf{B}_1}{4\pi}\times\frac{\partial\mathbf{X}}{\partial\varphi}\right)\cdot\nabla\times\mathbf{H}_{\rm gy} &= \mathbf{H}_{\rm gy}\cdot\frac{\epsilon}{4\pi}\frac{\partial\mathbf{B}_1}{\partial\varphi} - \frac{\epsilon\mathbf{B}_1}{4\pi}\cdot\nabla\left(\frac{\partial\mathbf{X}}{\partial\varphi}\right)\cdot\mathbf{H}_{\rm gy} \\ &= -\frac{\epsilon\mathbf{B}_1}{4\pi}\cdot\left(\frac{\partial\mathbf{B}_0}{\partial\varphi} - \widehat{\sf z}\times\mathbf{B}_0\right) \\ &\quad - \mathbb{M}_{\rm gy}\cdot\epsilon\left(\frac{\partial\mathbf{B}_1}{\partial\varphi} - \widehat{\sf z}\times\mathbf{B}_1\right) \\ &= \int_P F_{\rm gy}\,\frac{\partial K_{\rm gy}}{\partial\mathbf{B}_1}\cdot\frac{\partial\mathbf{B}_1}{\partial\varphi} + \widehat{\sf z}\cdot\left(\epsilon\,\mathbf{B}_1\times\mathbb{M}_{\rm gy}\right), \end{aligned} \tag{41}$$

where we used the definitions (19) and (21) for the gyrokinetic H-field and the gyrocenter magnetization, respectively, and we used the axisymmetric vector identity,

$$\partial\mathbf{B}_0/\partial\varphi = \widehat{\sf z}\times\mathbf{B}_0, \tag{42}$$

and the vector identity,

$$\mathbf{VW} : \nabla(\partial\mathbf{X}/\partial\varphi) = \widehat{\sf z}\cdot(\mathbf{W}\times\mathbf{V}), \tag{43}$$

which holds for arbitrary vectors fields $(\mathbf{V},\mathbf{W})$. Next, the third term in Eq. (39) yields the non-vanishing terms,

$$\begin{aligned} -\frac{\epsilon\mathbf{E}_1}{4\pi}\cdot\nabla\times\left(\frac{\partial\mathbf{X}}{\partial\varphi}\times\mathbf{D}_{\rm gy}\right) &= -\frac{\epsilon\mathbf{E}_1}{4\pi}\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\left(\nabla\cdot\mathbf{D}_{\rm gy}\right) - \mathbf{D}_{\rm gy}\cdot\frac{\epsilon}{4\pi}\frac{\partial\mathbf{E}_1}{\partial\varphi} \\ &\quad - \mathbf{D}_{\rm gy}\cdot\nabla\left(\frac{\partial\mathbf{X}}{\partial\varphi}\right)\cdot\frac{\epsilon\mathbf{E}_1}{4\pi} \\ &= \int_P F_{\rm gy}\left(\frac{\partial K_{\rm gy}}{\partial\mathbf{E}_1}\cdot\frac{\partial\mathbf{E}_1}{\partial\varphi} - \epsilon\,q\mathbf{E}_1\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right) \\ &\quad + \widehat{\sf z}\cdot\left(\epsilon\,\mathbf{E}_1\times\mathbb{P}_{\rm gy}\right), \end{aligned} \tag{44}$$

where we used the vector identity (43) and the definition (20) for the gyrocenter polarization, as well as the gyrokinetic Poisson equation in Eq. (22). Hence, by combining Eqs. (41) and (44), Eq. (39) becomes

$$\begin{aligned} \frac{\partial\mathcal{P}_{{\rm gy}\varphi}}{\partial t} &= \int_Z F_{\rm gy}\left[\frac{dP_\varphi}{dt} - \epsilon\,q\left(\mathbf{E}_1 + \frac{1}{c}\frac{d\mathbf{X}}{dt}\times\mathbf{B}_1\right)\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right] \\ &\quad + \int_Z F_{\rm gy}\left(\frac{\partial K_{\rm gy}}{\partial\mathbf{E}_1}\cdot\frac{\partial\mathbf{E}_1}{\partial\varphi} + \frac{\partial K_{\rm gy}}{\partial\mathbf{B}_1}\cdot\frac{\partial\mathbf{B}_1}{\partial\varphi}\right) \\ &\quad + \int_X \widehat{\sf z}\cdot\left(\epsilon\,\mathbf{E}_1\times\mathbb{P}_{\rm gy} + \epsilon\,\mathbf{B}_1\times\mathbb{M}_{\rm gy}\right). \end{aligned} \tag{45}$$

Next, using the explicit expressions for the gyrocenter polarization and magnetization (20) and (21), the last terms in Eq. (45) become the polarization and magnetization torques

$$\begin{aligned} \int_X \widehat{\sf z}\cdot\left(\epsilon\,\mathbf{E}_1\times\mathbb{P}_{\rm gy} + \epsilon\,\mathbf{B}_1\times\mathbb{M}_{\rm gy}\right) &= \int_Z F_{\rm gy}\,\widehat{\sf z}\cdot\left(\mu\,\widehat{\sf b}_0\times\epsilon\mathbf{B}_1 + \epsilon\mathbf{E}_1\times\boldsymbol{\pi}_{\rm gy}\right) \\ &\quad + \int_Z F_{\rm gy}\,\widehat{\sf z}\cdot\left[\epsilon\mathbf{B}_1\times\left(\boldsymbol{\pi}_{\rm gy}\times\frac{p_\parallel\widehat{\sf b}_0}{mc}\right)\right]. \end{aligned} \tag{46}$$

We now write the full expression for $\partial K_{\rm gy}/\partial\varphi$,

$$\frac{\partial K_{\rm gy}}{\partial\varphi} = \frac{\partial' K_{\rm gy}}{\partial\varphi} + \left(\frac{\partial K_{\rm gy}}{\partial\mathbf{E}_1}\cdot\frac{\partial\mathbf{E}_1}{\partial\varphi} + \frac{\partial K_{\rm gy}}{\partial\mathbf{B}_1}\cdot\frac{\partial\mathbf{B}_1}{\partial\varphi}\right), \tag{47}$$

where $\partial' K_{\rm gy}/\partial\varphi$ denotes the derivative of the gyrocenter kinetic energy (3) at constant perturbed fields $(\mathbf{E}_1,\mathbf{B}_1)$,

$$\begin{aligned} \frac{\partial' K_{\rm gy}}{\partial\varphi} &= \widehat{\sf z}\cdot\left(\mu\,\widehat{\sf b}_0\times\epsilon\mathbf{B}_1 + \epsilon\mathbf{E}_1\times\boldsymbol{\pi}_{\rm gy}\right) \\ &\quad - \widehat{\sf z}\cdot\left[\frac{p_\parallel\widehat{\sf b}_0}{mc}\times\left(\epsilon\mathbf{B}_1\times\boldsymbol{\pi}_{\rm gy}\right)\right] \\ &\quad - \widehat{\sf z}\cdot\left[\boldsymbol{\pi}_{\rm gy}\times\left(\frac{p_\parallel\widehat{\sf b}_0}{mc}\times\epsilon\mathbf{B}_1\right)\right], \end{aligned} \tag{48}$$

where we used Eq. (42). If we now combine these expressions in Eq. (45), we find

$$\frac{\partial \mathcal{P}_{\mathrm{gy}\varphi}}{\partial t} = \int_Z F_{\mathrm{gy}} \left[\frac{dP_\varphi}{dt} - \frac{\partial K_{\mathrm{gy}}}{\partial \varphi} - \epsilon\, q \left(\mathbf{E}_1 + \frac{1}{c}\frac{d\mathbf{X}}{dt} \times \mathbf{B}_1 \right) \cdot \frac{\partial \mathbf{X}}{\partial \varphi} \right], \quad (49)$$

after using the vector identity,

$$\mathbf{U} \times (\mathbf{V} \times \mathbf{W}) + \mathbf{V} \times (\mathbf{W} \times \mathbf{U}) + \mathbf{W} \times (\mathbf{U} \times \mathbf{V}) = 0,$$

with $\mathbf{U} = \epsilon \mathbf{B}_1$, $\mathbf{V} = \boldsymbol{\pi}_{\mathrm{gy}}$, and $\mathbf{W} = p_{||}\widehat{\mathsf{b}}_0/mc$. Finally, using the equation of motion (A3) for the gyrocenter azimuthal angular momentum, we arrive at the conservation law,

$$\frac{\partial \mathcal{P}_{\mathrm{gy}\varphi}}{\partial t} = \left[\mathcal{P}_{\mathrm{gy}\varphi}, \mathcal{H}_{\mathrm{gy}} \right]_{\mathrm{gy}} = 0. \quad (50)$$

This conservation law was derived by the Noether method[23] in the variational (Lagrangian) formulation of the gyrokinetic Vlasov–Maxwell equations (16)–(18). Here, we have shown that this conservation law can also be expressed in Hamiltonian form with the help of the gyrokinetic Vlasov–Maxwell bracket (33).

IV. JACOBI PROPERTY OF THE GYROKINETIC VLASOV–MAXWELL BRACKET

We now verify that the gyrokinetic bracket (33) satisfies the Jacobi property (32). The proof will rely on several Poisson-bracket identities derived from the gyrocenter Poisson bracket (10).

According to the Bracket theorem,[3,13] the proof of the Jacobi property involves only the explicit dependence of the gyrocenter Vlasov–Maxwell bracket (33), where the gyrokinetic Vlasov–Maxwell Poisson operator $\mathsf{J}^{ab}_{\mathrm{gy}}(F_{\mathrm{gy}}, \mathbf{B}_1) \circ$ is independent of the gyrokinetic displacement field $\mathbf{D}_{\mathrm{gy}}$, where we note that the dependence on the magnetic field $\mathbf{B}_1$ enters through Eq. (11) appearing in the gyrocenter Poisson bracket (10). Hence, we can write the double-bracket involving three arbitrary gyrocenter functionals $(\mathcal{F}, \mathcal{G}, \mathcal{K})$,

$$\begin{aligned} \left[[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}, \mathcal{K} \right]^P_{\mathrm{gy}} &= \int_Z F_{\mathrm{gy}} \left\{ \frac{\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}}{\delta F_{\mathrm{gy}}}, \frac{\delta \mathcal{K}}{\delta F_{\mathrm{gy}}} \right\}_{\mathrm{gy}} \\ &\quad + 4\pi q \int_Z F_{\mathrm{gy}} \frac{\delta \mathcal{K}}{\delta \mathbf{D}_{\mathrm{gy}}} \cdot \left\{ \mathbf{X}, \frac{\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}}{\delta F_{\mathrm{gy}}} \right\}_{\mathrm{gy}} \\ &\quad - 4\pi c \int_X \frac{\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}}{\epsilon \delta \mathbf{B}_1} \cdot \nabla \times \frac{\delta \mathcal{K}}{\delta \mathbf{D}_{\mathrm{gy}}}, \end{aligned} \quad (51)$$

where the terms involving $\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}/\delta \mathbf{D}_{\mathrm{gy}}$ vanish on the basis of the Bracket theorem.

Here, the Poisson variation δ^P of the bracket (33) only involves variations with respect to $(F_{\mathrm{gy}}, \mathbf{B}_1)$,

$$\begin{aligned} \delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}} &= \int_Z \left(\delta F_{\mathrm{gy}} - \frac{F_{\mathrm{gy}}}{B^*_{||}} \widehat{\mathsf{b}}_0 \cdot \epsilon\, \delta \mathbf{B}_1 \right) \\ &\quad \times \left[\{f, g\}_{\mathrm{gy}} + 4\pi q \left(\mathbf{G} \cdot \{\mathbf{X}, f\}_{\mathrm{gy}} - \mathbf{F} \cdot \{\mathbf{X}, g\}_{\mathrm{gy}} \right) \right. \\ &\quad \left. + (4\pi q)^2 \mathbf{F} \cdot \{\mathbf{X}, \mathbf{X}\}_{\mathrm{gy}} \cdot \mathbf{G} \right] + \int_Z F_{\mathrm{gy}} \frac{\epsilon\, \delta \mathbf{B}_1}{B^*_{||}} \\ &\quad \cdot \left[(\nabla f - 4\pi q\, \mathbf{F}) \frac{\partial g}{\partial p_{||}} - (\nabla g - 4\pi q\, \mathbf{G}) \frac{\partial f}{\partial p_{||}} \right], \end{aligned} \quad (52)$$

where we use the notation $(f, g, k) \equiv (\delta\mathcal{F}/\delta F_{\mathrm{gy}}, \delta\mathcal{G}/\delta F_{\mathrm{gy}}, \delta\mathcal{K}/\delta F_{\mathrm{gy}})$ and $(\mathbf{F}, \mathbf{G}, \mathbf{K}) \equiv (\delta\mathcal{F}/\delta \mathbf{D}_{\mathrm{gy}}, \delta\mathcal{G}/\delta \mathbf{D}_{\mathrm{gy}}, \delta\mathcal{K}/\delta \mathbf{D}_{\mathrm{gy}})$. In addition, $\delta B^*_{||} = \widehat{\mathsf{b}}_0 \cdot \epsilon \delta \mathbf{B}_1$ represents the variation of the Jacobian, which appears in the denominator of the gyrocenter Poisson bracket (10), while $\delta \mathbf{B}^* = \epsilon\, \delta \mathbf{B}_1$ appears only in the first term of Eq. (10). In the first two terms of Eq. (51), we therefore have the Vlasov sub-bracket,

$$\begin{aligned} &\left\{ \frac{\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}}{\delta F_{\mathrm{gy}}}, \frac{\delta \mathcal{K}}{\delta F_{\mathrm{gy}}} \right\}_{\mathrm{gy}} \\ &= \left\{ \{f, g\}_{\mathrm{gy}}, k \right\}_{\mathrm{gy}} + 4\pi q \left\{ \left(\mathbf{G} \cdot \{\mathbf{X}, f\}_{\mathrm{gy}} - \mathbf{F} \cdot \{\mathbf{X}, g\}_{\mathrm{gy}} \right), k \right\}_{\mathrm{gy}} \\ &\quad + (4\pi q)^2 \left\{ \mathbf{F} \cdot \{\mathbf{X}, \mathbf{X}\}_{\mathrm{gy}} \cdot \mathbf{G}, k \right\}_{\mathrm{gy}}, \end{aligned} \quad (53)$$

and the Interaction sub-bracket,

$$\begin{aligned} &4\pi q \frac{\delta \mathcal{K}}{\delta \mathbf{D}_{\mathrm{gy}}} \cdot \left\{ \mathbf{X}, \frac{\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}}{\delta F_{\mathrm{gy}}} \right\}_{\mathrm{gy}} \\ &= 4\pi q\, \mathbf{K} \cdot \left\{ \mathbf{X}, \{f, g\}_{\mathrm{gy}} \right\}_{\mathrm{gy}} + (4\pi q)^2\, \mathbf{K} \\ &\quad \cdot \left\{ \mathbf{X}, \left(\mathbf{G} \cdot \{\mathbf{X}, f\}_{\mathrm{gy}} - \mathbf{F} \cdot \{\mathbf{X}, g\}_{\mathrm{gy}} \right) \right\}_{\mathrm{gy}} \\ &\quad + (4\pi q)^3\, \mathbf{K} \cdot \left\{ \mathbf{X}, \mathbf{F} \cdot \{\mathbf{X}, \mathbf{X}\}_{\mathrm{gy}} \cdot \mathbf{G} \right\}_{\mathrm{gy}}, \end{aligned} \quad (54)$$

while the third term in Eq. (51) is the Maxwell sub-bracket,

$$\begin{aligned} &-4\pi c \int_X \frac{\delta^P[\mathcal{F},\mathcal{G}]_{\mathrm{gy}}}{\epsilon\, \delta \mathbf{B}_1} \cdot \nabla \times \frac{\delta \mathcal{K}}{\delta \mathbf{D}_{\mathrm{gy}}} \\ &= -4\pi q \int_Z \frac{c F_{\mathrm{gy}}}{q B^*_{||}} \nabla \times \mathbf{K} \\ &\quad \cdot \left[(\nabla f - 4\pi q\, \mathbf{F}) \frac{\partial g}{\partial p_{||}} - (\nabla g - 4\pi q\, \mathbf{G}) \frac{\partial f}{\partial p_{||}} \right] \\ &\quad + 4\pi q \int_Z F_{\mathrm{gy}} \frac{c \widehat{\mathsf{b}}_0}{q B^*_{||}} \cdot \nabla \times \mathbf{K} \left[\{f, g\}_{\mathrm{gy}} + 4\pi q \right. \\ &\quad \left. \times \left(\mathbf{G} \cdot \{\mathbf{X}, f\}_{\mathrm{gy}} - \mathbf{F} \cdot \{\mathbf{X}, g\}_{\mathrm{gy}} \right) \right] \\ &\quad + (4\pi q)^3 \int_Z F_{\mathrm{gy}} \frac{c \widehat{\mathsf{b}}_0}{q B^*_{||}} \cdot \nabla \times \mathbf{K} \left(\mathbf{F} \cdot \{\mathbf{X}, \mathbf{X}\}_{\mathrm{gy}} \cdot \mathbf{G} \right). \end{aligned} \quad (55)$$

From these terms, it is clear that the proof of the Jacobi property (32) must hold separately for each power of $4\pi q$,

$$\mathcal{J}ac[\mathcal{F},\mathcal{G},\mathcal{K}] \equiv \sum_{n=0}^{3} (4\pi q)^n \int_Z F_{\mathrm{gy}}\, \mathsf{Jac}_n[\mathcal{F},\mathcal{G},\mathcal{K}], \quad (56)$$

where each Jacobiator term $\mathsf{Jac}_n[\mathcal{F},\mathcal{G},\mathcal{K}]$ involves the gyrocenter Poisson bracket (10). At zeroth order, for example, the Vlasov sub-bracket (53) yields

$$\begin{aligned} \mathsf{Jac}_0[\mathcal{F},\mathcal{G},\mathcal{K}] &= \left\{ \{f, g\}_{\mathrm{gy}}, k \right\}_{\mathrm{gy}} + \left\{ \{g, k\}_{\mathrm{gy}}, f \right\}_{\mathrm{gy}} \\ &\quad + \left\{ \{k, f\}_{\mathrm{gy}}, g \right\}_{\mathrm{gy}} = 0, \end{aligned} \quad (57)$$

which vanishes because of the Jacobi property (12) of the gyrocenter Poisson bracket (10). In what follows, we will use cyclic permutations (denoted by $\circlearrowleft$) of the functionals $(\mathcal{F}, \mathcal{G}, \mathcal{K})$ in order to combine similar terms from the sub-brackets (53)–(55) at the next three orders in $4\pi q$.

A. First-order Jacobi property

By using the Leibniz property of the gyrocenter Poisson bracket (10), the cyclic permutations of the Vlasov sub-bracket (53) include the first-order terms,

$$\begin{aligned} & \{\mathbf{K}, f\}_{\rm gy} \cdot \{\mathbf{X}, g\}_{\rm gy} - \{\mathbf{K}, g\}_{\rm gy} \cdot \{\mathbf{X}, f\}_{\rm gy} \\ & \quad + \mathbf{K} \cdot \left(\left\{ \{\mathbf{X}, g\}_{\rm gy}, f \right\}_{\rm gy} + \left\{ \{f, \mathbf{X}\}_{\rm gy}, g \right\}_{\rm gy} \right), \end{aligned} \tag{58}$$

while the cyclic permutations of the Interaction sub-bracket (54) include

$$\mathbf{K} \cdot \left\{ \mathbf{X}, \{f, g\}_{\rm gy} \right\}_{\rm gy} = \mathbf{K} \cdot \left\{ \{g, f\}_{\rm gy}, \mathbf{X} \right\}_{\rm gy}, \tag{59}$$

where we used the antisymmetry of the gyrocenter Poisson bracket. Finally, the first-order terms in cyclic permutations of the Maxwell sub-bracket (55) are

$$\frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \cdot \nabla \times \mathbf{K} \{f, g\}_{\rm gy} - \frac{c}{qB_\parallel^*} \nabla \times \mathbf{K} \cdot \left(\nabla f \frac{\partial g}{\partial p_\parallel} - \nabla g \frac{\partial f}{\partial p_\parallel} \right). \tag{60}$$

Using the gyrocenter Poisson bracket identity

$$\frac{c}{qB_\parallel^*} \left(\nabla f \frac{\partial g}{\partial p_\parallel} - \nabla g \frac{\partial f}{\partial p_\parallel} \right) = \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \{f, g\}_{\rm gy} + \{\mathbf{X}, f\}_{\rm gy} \times \{\mathbf{X}, g\}_{\rm gy}.$$

Equation (60) becomes

$$\begin{aligned} & \nabla \times \mathbf{K} \cdot \{\mathbf{X}, g\}_{\rm gy} \times \{\mathbf{X}, f\}_{\rm gy} \\ & \quad = \{\mathbf{K}, g\}_{\rm gy} \cdot \{\mathbf{X}, f\}_{\rm gy} - \{\mathbf{K}, f\}_{\rm gy} \cdot \{\mathbf{X}, g\}_{\rm gy}, \end{aligned} \tag{61}$$

where we used the identity

$$\{\mathbf{U}, v\}_{\rm gy} \equiv \{\mathbf{X}, v\}_{\rm gy} \cdot \nabla \mathbf{U}, \tag{62}$$

for a vector field $\mathbf{U}$ assumed to be independent of $p_\parallel$, while $v(\mathbf{X}, p_\parallel)$ is an arbitrary gyrocenter function.

By combining Eqs. (58), (59), and (61), we obtain the first-order term in the Jacobiator (56),

$$\begin{aligned} \mathsf{Jac}_1[\mathcal{F}, \mathcal{G}, \mathcal{K}] = K_i \Big[& \left\{ \{X^i, g\}_{\rm gy}, f \right\}_{\rm gy} + \left\{ \{f, X^i\}_{\rm gy}, g \right\}_{\rm gy} \\ & + \left\{ \{g, f\}_{\rm gy}, X^i \right\}_{\rm gy} \Big] + \circlearrowleft = 0, \end{aligned} \tag{63}$$

which vanishes because of the Jacobi property (12) of the gyrocenter Poisson bracket (10).

B. Second-order Jacobi property

Cyclic permutations of the Vlasov sub-bracket (53) include the second-order terms,

$$\begin{aligned} \left\{ \mathbf{K} \cdot \{\mathbf{X}, \mathbf{X}\}_{\rm gy} \cdot \mathbf{F}, g \right\}_{\rm gy} & = F_i K_j \left\{ \{X^j, X^i\}_{\rm gy}, g \right\}_{\rm gy} \\ & \quad + \{\mathbf{K}, g\}_{\rm gy} \cdot \{\mathbf{X}, \mathbf{X}\}_{\rm gy} \cdot \mathbf{F} \\ & \quad + \mathbf{K} \cdot \{\mathbf{X}, \mathbf{X}\}_{\rm gy} \cdot \{\mathbf{F}, g\}_{\rm gy}, \end{aligned} \tag{64}$$

where summation over repeated indices is implied in the first term. Next, using the identity (62), the second and third terms become

$$\{\mathbf{X}, g\}_{\rm gy} \cdot \left(\nabla \mathbf{K} \cdot \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \times \mathbf{F} - \nabla \mathbf{F} \cdot \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \times \mathbf{K} \right), \tag{65}$$

where we used the Poisson-bracket identity

$$\mathbf{U} \cdot \{\mathbf{X}, \mathbf{X}\}_{\rm gy} \cdot \mathbf{V} = -\frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \cdot \mathbf{U} \times \mathbf{V}, \tag{66}$$

for any two vector fields $(\mathbf{U}, \mathbf{V})$.

Cyclic permutations of the Interaction sub-bracket (54) include the second-order terms,

$$\begin{aligned} & \mathbf{F} \cdot \left\{ \mathbf{X}, \mathbf{K} \cdot \{\mathbf{X}, g\}_{\rm gy} \right\}_{\rm gy} - \mathbf{K} \cdot \left\{ \mathbf{X}, \mathbf{F} \cdot \{\mathbf{X}, g\}_{\rm gy} \right\}_{\rm gy} \\ & \quad = F_i K_j \left(\left\{ \{g, X^j\}_{\rm gy}, X^i \right\}_{\rm gy} + \left\{ \{X^i, g\}_{\rm gy}, X^j \right\}_{\rm gy} \right) \\ & \quad + \left(\mathbf{F} \cdot \{\mathbf{X}, \mathbf{K}\}_{\rm gy} - \mathbf{K} \cdot \{\mathbf{X}, \mathbf{F}\}_{\rm gy} \right) \cdot \{\mathbf{X}, g\}_{\rm gy}, \end{aligned} \tag{67}$$

where the last two terms can be expressed as

$$\left(\frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \times \mathbf{K} \cdot \nabla \mathbf{F} - \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \times \mathbf{F} \cdot \nabla \mathbf{K} \right) \cdot \{\mathbf{X}, g\}_{\rm gy}. \tag{68}$$

Hence, by combining Eqs. (65) and (68), we obtain

$$\begin{aligned} & \{\mathbf{X}, g\}_{\rm gy} \cdot \left[\left(\frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \times \mathbf{F} \right) \times \nabla \times \mathbf{K} - \left(\frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \times \mathbf{K} \right) \times \nabla \times \mathbf{F} \right] \\ & \quad = \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \cdot \left[\nabla \times \mathbf{K} \left(\mathbf{F} \cdot \{\mathbf{X}, g\}_{\rm gy} \right) - \left(\mathbf{F} \cdot \nabla \times \mathbf{K} \right) \frac{\mathbf{B}^*}{B_\parallel^*} \frac{\partial g}{\partial p_\parallel} \right] \\ & \quad - \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \cdot \left[\nabla \times \mathbf{F} \left(\mathbf{K} \cdot \{\mathbf{X}, g\}_{\rm gy} \right) - \left(\mathbf{K} \cdot \nabla \times \mathbf{F} \right) \frac{\mathbf{B}^*}{B_\parallel^*} \frac{\partial g}{\partial p_\parallel} \right], \end{aligned} \tag{69}$$

where we used the Poisson-bracket identity $(c\widehat{\mathsf{b}}_0/qB_\parallel^*) \cdot \{\mathbf{X}, g\}_{\rm gy} = (c/qB_\parallel^*)\, \partial g/\partial p_\parallel$.

Cyclic permutations of the Maxwell sub-bracket (55) include the second-order terms,

$$\begin{aligned} & -\frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \cdot \left[\nabla \times \mathbf{K} \left(\mathbf{F} \cdot \{\mathbf{X}, g\}_{\rm gy} \right) - \left(\mathbf{F} \cdot \nabla \times \mathbf{K} \right) \frac{\mathbf{B}^*}{B_\parallel^*} \frac{\partial g}{\partial p_\parallel} \right] + \frac{c\widehat{\mathsf{b}}_0}{qB_\parallel^*} \\ & \quad \cdot \left[\nabla \times \mathbf{F} \left(\mathbf{K} \cdot \{\mathbf{X}, g\}_{\rm gy} \right) - \left(\mathbf{K} \cdot \nabla \times \mathbf{F} \right) \frac{\mathbf{B}^*}{B_\parallel^*} \frac{\partial g}{\partial p_\parallel} \right]. \end{aligned} \tag{70}$$

which exactly cancels the terms in Eq. (69), so that the second-order Jacobi term in Eq. (56) is obtained by adding Eqs. (64), (67), and (70), which yields the second-order term in the Jacobiator (56),

$$\mathrm{Jac}_2[\mathcal{F},\mathcal{G},\mathcal{K}] = F_i K_j \left[\left\{ \{X^i, g\}_{\rm gy}, X^j \right\}_{\rm gy} + \left\{ \{X^j, X^i\}_{\rm gy}, g \right\}_{\rm gy} + \left\{ \{g,\, X^j\}_{\rm gy}, X^i \right\}_{\rm gy} \right] + \circlearrowleft = 0, \tag{71}$$

which vanishes because of the Jacobi property (12) of the gyrocenter Poisson bracket (10).

C. Third-order Jacobi property

At the third order in $4\pi q$, we note that only the Interaction and Maxwell sub-brackets (54) and (55) contribute terms in $\mathrm{Jac}_3[\mathcal{F},\mathcal{G},\mathcal{K}]$. First, the third-order terms in cyclic permutations of Eqs. (54) and (55) are

$$\mathbf{K}\cdot\left\{\mathbf{X}, \mathbf{F}\cdot\{\mathbf{X},\mathbf{X}\}_{\rm gy}\cdot\mathbf{G}\right\}_{\rm gy} + \circlearrowleft = \mathrm{Jac}_3[\mathcal{F},\mathcal{G},\mathcal{K}] + \left[\mathbf{K}\cdot\{\mathbf{X},\mathbf{X}\}_{\rm gy}\cdot\mathbf{G}\left(\frac{c\widehat{\mathsf{b}}_0}{qB_\|^*}\cdot\nabla\times\mathbf{F}\right) + \circlearrowleft\right] \tag{72}$$

and

$$\mathbf{G}\cdot\{\mathbf{X},\mathbf{X}\}_{\rm gy}\cdot\mathbf{K}\left(\frac{c\widehat{\mathsf{b}}_0}{qB_\|^*}\cdot\nabla\times\mathbf{F}\right) + \mathbf{K}\cdot\{\mathbf{X},\mathbf{X}\}_{\rm gy}\cdot\mathbf{F}\left(\frac{c\widehat{\mathsf{b}}_0}{qB_\|^*}\cdot\nabla\times\mathbf{G}\right) + \mathbf{F}\cdot\{\mathbf{X},\mathbf{X}\}_{\rm gy}\cdot\mathbf{G}\left(\frac{c\widehat{\mathsf{b}}_0}{qB_\|^*}\cdot\nabla\times\mathbf{K}\right), \tag{73}$$

which when combined, using the antisymmetry property of Eq. (66), introduces simple cancelations and yields the result

$$\mathbf{K}\cdot\left\{\mathbf{X}, \mathbf{F}\cdot\{\mathbf{X},\mathbf{X}\}_{\rm gy}\cdot\mathbf{G}\right\}_{\rm gy} + \circlearrowleft = \mathrm{Jac}_3[\mathcal{F},\mathcal{G},\mathcal{K}],$$

where the third-order term in the Jacobiator (56),

$$\mathrm{Jac}_3[\mathcal{F},\mathcal{G},\mathcal{K}] = F_i G_j K_\ell \left[\left\{\{X^i,X^j\}_{\rm gy},X^\ell\right\}_{\rm gy} + \left\{\{X^j,X^\ell\}_{\rm gy},X^i\right\}_{\rm gy} + \left\{\{X^\ell,X^i\}_{\rm gy},X^j\right\}_{\rm gy}\right] = 0, \tag{74}$$

vanishes because of the Jacobi property (12) of the gyrocenter Poisson bracket (10).

V. SUMMARY

The explicit Hamiltonian structure of the gauge-free gyrokinetic Vlasov–Maxwell equations was constructed directly from Eqs. (28)–(30) in terms of a gyrokinetic Hamiltonian functional (23) and the gyrocenter Poisson bracket (10), which resulted in the gyrokinetic Vlasov–Maxwell bracket (33). The gauge-free gyrokinetic equations were presented here in their drift-kinetic limit, which simplified the expressions for the gyrocenter polarization and magnetization (20) and (21). Future work will consider extensions of our gyrokinetic Hamiltonian formulation to include higher-order FLR effects.

As simple applications of our gauge-free gyrokinetic Hamiltonian formulation, we demonstrated in Sec. III that the gyrokinetic entropy function (34) is a Casimir functional for the gyrokinetic Vlasov–Maxwell bracket (33), and that the gyrokinetic toroidal angular momentum conservation law can be expressed in Hamiltonian form (50).

In Sec. IV, we presented an explicit proof that the gyrokinetic Vlasov–Maxwell bracket (33) satisfies the Jacobi property (32). While this may seem to be an academic exercise, our proof follows similar proofs for several Vlasov–Maxwell models presented in the Appendix of Ref. 13, for example. In analogy to the recent more extensive proof[26] of the Jacobi property of the guiding-center Vlasov–Maxwell bracket, the proof of the Jacobi property (32) relies on identities derived from the gyrocenter Poisson bracket (10) and the vanishing gyrokinetic Jacobiator (32) is inherited from the Jacobi property (12) of the gyrocenter Poisson bracket.

Future work will consider an alternate gauge-free gyrokinetic Vlasov–Maxwell model[23] in which the gyrocenter polarization drift $\epsilon\,\mathbf{\Pi}_1\cdot d\mathbf{X}/dt$ is added to the symplectic part of the gyrocenter Lagrangian (2), where $\mathbf{\Pi}_1 \equiv [\mathbf{E}_1 + (p_\|\widehat{\mathsf{b}}_0/mc)\times\mathbf{B}_1]\times q\widehat{\mathsf{b}}_0/\Omega_0$. This extension will explicitly introduce electric-field terms in the gyrocenter Poisson bracket (10) and introduce new terms in the Poisson variation (52).

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts of interest to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX: GYROCENTER ANGULAR MOMENTUM EQUATION OF MOTION

In this Appendix, we derive the equation of motion for the gyrocenter angular momentum (38) in an axisymmetric background magnetic field.

We begin with the covariant azimuthal (toroidal) component of the Euler–Lagrange equation (4),

$$\epsilon q\left(\mathbf{E}_1 + \frac{1}{c}\frac{d\mathbf{X}}{dt}\times\mathbf{B}_1\right)\cdot\frac{\partial\mathbf{X}}{\partial\varphi} - \frac{\partial K_{\rm gy}}{\partial\varphi} = \frac{dp_\|}{dt}\left(\widehat{\mathsf{b}}_0\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right) + \frac{q}{c}\mathbf{B}_0^*\times\frac{d\mathbf{X}}{dt}\cdot\frac{\partial\mathbf{X}}{\partial\varphi}, \tag{A1}$$

where the toroidal derivative of the gyrocenter kinetic energy $\partial K_{\rm gy}/\partial\varphi = (\partial\mathbf{X}/\partial\varphi)\cdot\nabla K_{\rm gy}$ is given in Eq. (47).

Next, we explicitly evaluate the gyrocenter time derivative of P_φ,

$$\begin{aligned}\frac{dP_\varphi}{dt} &= \frac{dp_{\|}}{dt}\left(\widehat{\mathsf{b}}_0 \cdot \frac{\partial \mathbf{X}}{\partial \varphi}\right) + \frac{q}{c}\frac{d\mathbf{X}}{dt} \cdot \left[\nabla \mathbf{A}_0^* \cdot \frac{\partial \mathbf{X}}{\partial \varphi} + \nabla\left(\frac{\partial \mathbf{X}}{\partial \varphi}\right) \cdot \mathbf{A}_0^*\right] \\ &= \frac{dp_{\|}}{dt}\left(\widehat{\mathsf{b}}_0 \cdot \frac{\partial \mathbf{X}}{\partial \varphi}\right) + \frac{q}{c}\mathbf{B}_0^* \cdot \left(\frac{d\mathbf{X}}{dt} \times \frac{\partial \mathbf{X}}{\partial \varphi}\right) \\ &\quad + \widehat{\mathsf{z}} \times \mathbf{A}_0^* \cdot \frac{q}{c}\frac{d\mathbf{X}}{dt} + \widehat{\mathsf{z}} \cdot \left(\frac{d\mathbf{X}}{dt} \times \frac{q}{c}\,\mathbf{A}_0^*\right) \\ &= \frac{dp_{\|}}{dt}\left(\widehat{\mathsf{b}}_0 \cdot \frac{\partial \mathbf{X}}{\partial \varphi}\right) + \frac{q}{c}\mathbf{B}_0^* \times \frac{d\mathbf{X}}{dt} \cdot \frac{\partial \mathbf{X}}{\partial \varphi},\end{aligned} \tag{A2}$$

where we used the gyrocenter invariance of the magnetic moment μ and we used $\partial \mathbf{A}_0^*/\partial\varphi \equiv \widehat{\mathsf{z}} \times \mathbf{A}_0^*$ under the assumption of axisymmetry of the background magnetic field, while we used the identity $\mathbf{W} \cdot \nabla(\partial \mathbf{X}/\partial\varphi) \cdot \mathbf{V} \equiv \widehat{\mathsf{z}} \cdot (\mathbf{W} \times \mathbf{V})$, which holds for arbitrary vectors $(\mathbf{V}, \mathbf{W})$. Hence, Eq. (A1) yields the gyrocenter angular momentum equation of motion,

$$\frac{dP_\varphi}{dt} = \epsilon q\left(\mathbf{E}_1 + \frac{1}{c}\frac{d\mathbf{X}}{dt} \times \mathbf{B}_1\right) \cdot \frac{\partial \mathbf{X}}{\partial \varphi} - \frac{\partial K_{\mathrm{gy}}}{\partial \varphi}. \tag{A3}$$

We note that in the electrostatic limit, where $\mathbf{E}_1 = -\nabla\Phi_1$ and $\mathbf{B}_1 = 0$, Eq. (A3) yields the standard equation $dP_\varphi/dt = -\partial H_{\mathrm{gy}}/\partial\varphi$, where $H_{\mathrm{gy}} = \epsilon q\,\Phi_1 + K_{\mathrm{gy}}$.

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