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ABSTRACT

This Letter reports on a metriplectic formulation of a collisional, nonlinear full-*f* electromagnetic gyrokinetic theory compliant with energy conservation and monotonic entropy production. In an axisymmetric background magnetic field, the toroidal angular momentum is also conserved. Notably, a new collisional current, contributing to the gyrokinetic Maxwell–Ampère equation and the gyrokinetic charge conservation law, is discovered.

The theoretical foundations of plasma physics are based on two sets of complementary formulations that are either kinetic or fluid and can represent either collisionless or collisional (dissipative) systems. When the formulations are collisionless, the associated Lagrangian and Hamiltonian structures (see a review by Morrison[1]) play a role in extracting conservation laws and guide, e.g., the development of modern numerical simulation methods.[2–13] When the formulations include collisional effects, the properties of the collision operator ought to guarantee that the irreversible plasma evolution satisfies the laws of thermodynamics.[14]

The modern theory of gyrokinetics,[15] which is used in the challenging task of investigating the turbulent dynamics of a magnetically confined plasma in a realistic geometry, has a solid foundation in the collisionless regime. The Lagrangian (variational) structure of the theory[16,17] enables deriving conservation laws[18–20] that provide useful verification tests for numerical algorithms. The accompanying, rather recently developed Hamiltonian structure,[21,22] on the contrary, establishes a transparent formulation of the full-*f* electromagnetic gyrokinetics in terms of genuine dynamical variables that are the gyrocenter phase-space density distributions F_s of species s, the electromagnetic displacement field $\boldsymbol{D}$, and the perturbation magnetic field $\boldsymbol{B}_1$. The existence of these formalisms owes to the dynamical reduction of the original Vlasov–Maxwell theory with the Lie-transform perturbation method[23–30] at the level of the action integral.

Currently, no similar systematic treatment exists for a gyrokinetic version of the nonlinear Landau collision operator compatible with the gyrokinetic Vlasov–Maxwell system. Despite several attempts at constructing collision operators for gyrokinetic applications,[31–36] no full-*f* gyrokinetic Vlasov–Maxwell–Landau field theory has yet been presented that would conserve energy, produce entropy monotonically, and conserve the toroidal angular momentum functional in an axially symmetric background magnetic field $\boldsymbol{B}_0$. Only the full-*f* electrostatic model has been expressed as a collisional field theory,[37,38] despite there being successful δf-formulations[39–42] that rely on various model linearized collision operators and even a Vlasov–Poisson–Ampère–Landau model[43,44] with explicit collisional conservation laws and transport fluxes. Exploiting the general framework of metriplectic dynamics,[45–52] this Letter proposes a theory for the full-*f* Vlasov–Maxwell–Landau case, securing also the as-of-yet-missing closed form expression for the associated gyrokinetic Landau collision operator. A notable outcome of the new theory is the appearance of a collisional current in the Maxwell–Ampère equation and the gyrokinetic charge conservation law. This new current is shown to be mandatory for the Gauss's law for $\boldsymbol{D}$ and the charge conservation to remain valid. It arises due to the non-zero spatial components of collisional flux in gyrocenter coordinates.[31] Next, the details, leading to these realizations, will be presented.

Following Ref. 21, we assume that (without loss of generality) gyrocenter coordinates have been found such that the single-gyrocenter Hamiltonian is given by $H_s = K_s[\boldsymbol{E}_1, \boldsymbol{B}_1] + q_s\,\Phi$, where $\Phi = \Phi(\boldsymbol{X})$ denotes the electrostatic potential evaluated at the gyrocenter position, and the gyrocenter kinetic energy $K_s[\boldsymbol{E}_1, \boldsymbol{B}_1]$ is a functional of the electromagnetic field. (We refer readers to Ref. 53 for details on how explicit dependence of K_s on the electromagnetic potentials can be avoided.) We then use the gyrocenter kinetic energy $\mathcal{K}[\boldsymbol{E}_1, \boldsymbol{B}_1] = \sum_s \int K_s[\boldsymbol{E}_1, \boldsymbol{B}_1]\, F_s$ to define the gyrokinetic constitutive law,

$$\boldsymbol{D} = \boldsymbol{E}_1 - 4\pi \frac{\delta \mathcal{K}}{\delta \boldsymbol{E}_1}, \tag{1}$$

which relates the electric field $\boldsymbol{E}_1$ and the displacement field $\boldsymbol{D}$ in a manner first described in Ref. 54. Going forward, we will always assume $\boldsymbol{E}_1$ is a definite functional of $\boldsymbol{D}$, $\boldsymbol{B}_1$, and F_s, implicitly defined by (1). While no specific expression for the gyrocenter kinetic energy function K is chosen here, it is noted that an explicit expression for $\boldsymbol{E}_1[F, \boldsymbol{D}, \boldsymbol{B}_1]$ is available at the drift-kinetic limit (see, e.g., Ref. 55) The Hamiltonian functional for the gyrokinetic Vlasov–Maxwell system is

$$\mathcal{H}[F, \boldsymbol{D}, \boldsymbol{B}_1] = \sum_s \int_Z F_s K_s[\boldsymbol{E}_1, \boldsymbol{B}_1] + \frac{1}{4\pi}\int_X \boldsymbol{E}_1 \cdot \boldsymbol{D} - \frac{1}{8\pi}\int_X \left(|\boldsymbol{E}_1|^2 - |\boldsymbol{B}_0 + \boldsymbol{B}_1|^2\right), \tag{2}$$

and the associated functional Poisson bracket is[21,22]

$$\begin{aligned}[\mathcal{A}, \mathcal{B}] = {}& \sum_s \int_Z F_s \left\{ \frac{\delta \mathcal{A}}{\delta F_s}, \frac{\delta \mathcal{B}}{\delta F_s} \right\}_s \\ &+ \sum_s 4\pi q_s \int_Z F_s \left(\frac{\delta \mathcal{B}}{\delta \boldsymbol{D}} \cdot \left\{ \boldsymbol{X}, \frac{\delta \mathcal{A}}{\delta F_s} \right\}_s - \frac{\delta \mathcal{A}}{\delta \boldsymbol{D}} \cdot \left\{ \boldsymbol{X}, \frac{\delta \mathcal{B}}{\delta F_s} \right\}_s \right) \\ &+ \sum_s 16\pi^2 q_s^2 \int_Z F_s \frac{\delta \mathcal{A}}{\delta \boldsymbol{D}} \cdot \{\boldsymbol{X}, \boldsymbol{X}\}_s \cdot \frac{\delta \mathcal{B}}{\delta \boldsymbol{D}} \\ &+ 4\pi c \int_X \left(\frac{\delta \mathcal{A}}{\delta \boldsymbol{D}} \cdot \nabla \times \frac{\delta \mathcal{B}}{\delta \boldsymbol{B}_1} - \frac{\delta \mathcal{B}}{\delta \boldsymbol{D}} \cdot \nabla \times \frac{\delta \mathcal{A}}{\delta \boldsymbol{B}_1} \right). \end{aligned} \tag{3}$$

The notation $\int_Z$ refers to integration over the phase-space coordinates $(\boldsymbol{X}, p_\parallel, \mu, \theta)$ while $\int_X$ and $\int_P$ refer to integration only over the spatial and velocity extent, respectively. Notably, the distributional densities F_s contain the phase-space Jacobians: $\int_P F_s(\boldsymbol{Z})$ is the number of gyrocenters within a volume element $d\boldsymbol{X}$. The single-particle Poisson bracket $\{\cdot,\cdot\}$ in (3) can be evaluated with respect to any two phase-space functions according to

$$\{f, g\} = \frac{q}{mc}\left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \mu}\frac{\partial g}{\partial \theta} \right) - \frac{c\hat{\boldsymbol{b}}_0}{qB_\parallel^*} \cdot \nabla^* f \times \nabla^* g + \frac{\boldsymbol{B}^*}{B_\parallel^*} \cdot \left(\nabla^* f \frac{\partial g}{\partial p_\parallel} - \frac{\partial f}{\partial p_\parallel} \nabla^* g \right). \tag{4}$$

The terms $\boldsymbol{B}^* = \nabla \times \boldsymbol{A}^*$ and $B_\parallel^* = \boldsymbol{B}^* \cdot \hat{\boldsymbol{b}}_0$ are constructed from the so-called modified vector potential

$$\boldsymbol{A}^* = \boldsymbol{A}_0 + (p_\parallel c/q)\hat{\boldsymbol{b}}_0 - (mc^2/q^2)\mu \boldsymbol{R}_0^* + \boldsymbol{A}_1 \equiv \boldsymbol{A}_0^* + \boldsymbol{A}_1 \tag{5}$$

in the usual way with $\hat{\boldsymbol{b}}_0 = \boldsymbol{B}_0/|\boldsymbol{B}_0|$ the background magnetic field unit vector. The modified gradient operator is $\nabla^* = \nabla + \boldsymbol{R}_0^* \partial/\partial\theta$, where $\boldsymbol{R}_0^* = \boldsymbol{R}_0 + \frac{1}{2}\nabla \times \hat{\boldsymbol{b}}_0$, and Littlejohn's gyrogauge vector $\boldsymbol{R}_0$ is constructed from the background magnetic field. The label s is needed to remain mindful of the species-dependent particle mass and charge.

In the absence of collisions, the Hamiltonian functional (2) and the functional Poisson bracket (3) determine the temporal evolution of any functional $\Psi[F, \boldsymbol{D}, \boldsymbol{B}_1]$ via the differential equation

$$\frac{d\Psi}{dt} = [\Psi, \mathcal{H}]. \tag{6}$$

Because the bracket (3) is antisymmetric, the Hamiltonian $\mathcal{H}$ is trivially conserved $d\mathcal{H}/dt = [\mathcal{H}, \mathcal{H}] = 0$. As demonstrated explicitly in Ref. 22, the toroidal angular momentum functional

$$\mathcal{P}_\varphi = \sum_s \int_Z F_s p_{\varphi 0,s} + \frac{1}{4\pi c}\int_X \boldsymbol{D} \times \boldsymbol{B}_1 \cdot (\hat{\boldsymbol{z}} \times \boldsymbol{X}), \tag{7}$$

where $p_{\varphi 0} = (q/c)\boldsymbol{A}_0^* \cdot (\hat{\boldsymbol{z}} \times \boldsymbol{X})$ is the guiding-center single-particle angular momentum, is also conserved, i.e., $d\mathcal{P}_\varphi/dt = [\mathcal{P}_\varphi, \mathcal{H}] = 0$, on the condition that the background magnetic field $\boldsymbol{B}_0$ is axially symmetric. Finally, the entropy functional

$$\mathcal{S}[F, \boldsymbol{B}_1] = -\sum_s \int_Z F_s \ln\left(F_s / B_{\parallel s}^* \right) \tag{8}$$

is a Casimir of the functional Poisson bracket, i.e., $[\mathcal{S}, \mathcal{A}] = 0$ with respect to any arbitrary functional $\mathcal{A}$. For details regarding the derivation of momentum and entropy conservation, see Refs. 21 and 22.

Previously in Ref. 38, a symmetric, so-called metric bracket representative of collisions was found for electrostatic gyrokinetic theory, completing the dissipationless Hamiltonian formulation into a metriplectic formulation consistent with thermodynamics. Just like Poisson brackets, metric brackets are bilinear maps from the space of functionals to real numbers but, unlike Poisson brackets, describe dissipation. Examples are found in dissipative extended magnetohydrodynamics,[56] in simple ideal gas thermodynamics,[57] in port-Hamiltonian formalism describing district-heating networks,[58] and in the numerous applications of open nonequilibrium thermodynamic systems.[59] No further introduction to the concept is given here, and the reader is referred specifically to Ref. 52 for an excellent introduction and examples. To construct a metric bracket representative of Coulomb collisions for electromagnetic gyrokinetics, the previous works in electrostatic theory[37,38] are closely followed: we seek a symmetric positive semidefinite functional bracket with the structure

$$(\mathcal{A}, \mathcal{B}) = \frac{1}{2}\sum_{s\bar{s}} \int_Z \int_{\bar{Z}} F_s(\boldsymbol{Z}) F_{\bar{s}}(\bar{\boldsymbol{Z}})\, \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A}; \boldsymbol{Z}, \bar{\boldsymbol{Z}}) \cdot \mathsf{Q}_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{B}; \boldsymbol{Z}, \bar{\boldsymbol{Z}}), \tag{9}$$

where the 3-by-3 matrix $\mathsf{Q}_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}})$ is defined as

$$\mathsf{Q}_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}}) = \nu_{s\bar{s}}\, \delta_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}})\, \mathbb{Q}(\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{H}; \boldsymbol{Z}, \bar{\boldsymbol{Z}})). \tag{10}$$

The delta-function $\delta_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}}) = \delta^3(\boldsymbol{y}_s(\boldsymbol{Z}) - \boldsymbol{y}_{\bar{s}}(\bar{\boldsymbol{Z}}))$ in the matrix (10) enforces the collisions to be local in spatial coordinates via the position $\boldsymbol{y}_s(\boldsymbol{Z}) = \boldsymbol{X} + \boldsymbol{\rho}_{0s}$ of a particle of species s, where $\boldsymbol{\rho}_0$ is the lowest-order Larmor radius evaluated in terms of the background magnetic field. The Landau matrix $\mathbb{Q}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-1}(\mathbb{I} - \boldsymbol{\xi}\boldsymbol{\xi}/|\boldsymbol{\xi}|^2)$ is a scaled projection matrix and the coefficient $\nu_{s\bar{s}} = 2\pi q_s^2 q_{\bar{s}}^2 \ln \Lambda_{s\bar{s}}$ contains the Coulomb logarithm $\Lambda_{s\bar{s}}$, both familiar from the Landau collision operator. The bracket structure (9) supplemented with (10), regardless of the expression for $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A}; \boldsymbol{Z}, \bar{\boldsymbol{Z}})$, guarantees that the Hamiltonian functional $\mathcal{H}$ is an annihilator element of the metric bracket: $(\mathcal{H}, \mathcal{A}) = 0$, with respect to any functional $\mathcal{A}$, credited to the projection property $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{H};\boldsymbol{Z},\bar{\boldsymbol{Z}})\cdot\mathbb{Q}(\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{H};\boldsymbol{Z},\bar{\boldsymbol{Z}}))=\boldsymbol{0}$. While the bracket (9) and the expression (10) appear the same as in the electrostatic theory,[38] the detailed expression for the vector-valued operator $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A};\boldsymbol{Z},\bar{\boldsymbol{Z}})$ differs. In the following paragraphs, an effort is made to justify our choice.

In the electrostatic case,[38] one starts from the particle phase-space collision operator, exploits the single-particle Poisson brackets to transform the velocity derivatives in the collision operator to gyrocenter phase-space[31] obtaining a collision operator for electrostatic gyrokinetics,[37] and, finally, symmetrizes the result utilizing partial integration and the general properties of the single-particle Poisson bracket in a similar manners as one does in deriving the particle phase-space metric bracket from the particle phase-space collision operator.[60] In the end, this process results in an expression $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A};\boldsymbol{Z},\bar{\boldsymbol{Z}})=\{\boldsymbol{y}_s,\delta\mathcal{A}/\delta F_s\}_s(\boldsymbol{Z})-\{\boldsymbol{y}_{\bar{s}},\delta\mathcal{A}/\delta F_{\bar{s}}\}_{\bar{s}}(\bar{\boldsymbol{Z}})$, where the bracket is the guiding-center Poisson bracket and does not contain the magnetic field perturbation $\boldsymbol{B}_1$. As the particle velocity in terms of the gyrocenter coordinates in electrostatic theory can be expressed as $\dot{\boldsymbol{y}}_s=\{\boldsymbol{y}_s,\delta\mathcal{H}/\delta F_s\}_s$, evaluating the vector-valued operator with respect to the Hamiltonian of the electrostatic system results in $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{H};\boldsymbol{Z},\bar{\boldsymbol{Z}})=\dot{\boldsymbol{y}}_s-\dot{\boldsymbol{y}}\bar{s}$ and describes the difference of velocities of two colliding particles, an expression that is needed in the matrix $\mathbb{Q}$ in the Landau operator. There is no $\boldsymbol{B}_1$ in the electrostatic case, and, therefore, in the case of an axially symmetric $\boldsymbol{B}_0$, one has the identity $\{\boldsymbol{y}_s,\delta\mathcal{P}_\varphi/\delta F_s\}_s=\hat{\boldsymbol{z}}\times\boldsymbol{y}_s$, with $\hat{\boldsymbol{z}}$ the unit vector for the axis of rotational symmetry in $\boldsymbol{B}_0$, and consequently $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{P}_\varphi;\boldsymbol{Z},\bar{\boldsymbol{Z}})=\hat{\boldsymbol{z}}\times(\boldsymbol{y}_s(\boldsymbol{Z})-\boldsymbol{y}_{\bar{s}}(\bar{\boldsymbol{Z}}))$. Together with the localizing $\delta_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$, this then guarantees that the toroidal momentum functional is conserved in axially symmetric $\boldsymbol{B}_0$ in collisional electrostatic gyrokinetics and that it is an annihilator element of the associated metric bracket in the sense of $(\mathcal{P}_\varphi,\mathcal{A})=0$ with respect to an arbitrary $\mathcal{A}$.[37,38]

In the electromagnetic case, trying to proceed as in the electrostatic case leads to a dead end. First of all, the single-particle velocity, expressed in the gyrocenter coordinates, no longer has the same expression as in the electrostatic case. Second, because the single-gyrocenter Poisson bracket now contains also the time-dependent magnetic perturbation $\boldsymbol{B}_1$, the identity relied upon for toroidal momentum conservation in the electrostatic case with an axially symmetric $\boldsymbol{B}_0$ no longer holds up. These issues were further discussed also in Ref. 61, unfortunately to no avail. The puzzle begins to unravel upon using a definition for the particle velocity that is compatible with the Hamiltonian formulation of the gyrokinetic Vlasov–Maxwell system, namely

$$\frac{d\boldsymbol{y}_s}{dt}=\left\{\boldsymbol{y}_s,\frac{\delta\mathcal{H}}{\delta F_s}\right\}_s+4\pi q_s\frac{\delta\mathcal{H}}{\delta\boldsymbol{D}}\cdot\{\boldsymbol{X},\boldsymbol{y}_s\}_s, \tag{11}$$

and simultaneously discovering an identity that holds in the case of an axially symmetric $\boldsymbol{B}_0$, namely

$$\frac{d\boldsymbol{y}_s}{d\varphi}=\left\{\boldsymbol{y}_s,\frac{\delta\mathcal{P}_\varphi}{\delta F_s}\right\}_s+4\pi q_s\frac{\delta\mathcal{P}_\varphi}{\delta\boldsymbol{D}}\cdot\{\boldsymbol{X},\boldsymbol{y}_s\}_s=\hat{\boldsymbol{z}}\times\boldsymbol{y}_s. \tag{12}$$

We note that both identities hold even if the modified gyrogauge vector $\boldsymbol{R}_0^*$ is dropped from the modified vector potential (5), as is often customary. Considering how the operator $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A};\boldsymbol{Z},\bar{\boldsymbol{Z}})$ is defined in the electrostatic case, and that evaluating it with respect to the system's Hamiltonian should result in an expression that represents the difference of the colliding particle velocities also in the electromagnetic case, we, therefore, propose the following modified expression:

$$\begin{aligned}\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{A};\boldsymbol{Z},\bar{\boldsymbol{Z}})=&\left\{\boldsymbol{y}_s,\frac{\delta\mathcal{A}}{\delta F_s}\right\}_s(\boldsymbol{Z})+4\pi q_s\frac{\delta\mathcal{A}}{\delta\boldsymbol{D}(\boldsymbol{X})}\cdot\{\boldsymbol{X},\boldsymbol{y}_s\}_s(\boldsymbol{Z})\\&-\left\{\boldsymbol{y}_{\bar{s}},\frac{\delta\mathcal{A}}{\delta F_{\bar{s}}}\right\}_{\bar{s}}(\bar{\boldsymbol{Z}})-4\pi q_{\bar{s}}\frac{\delta\mathcal{A}}{\delta\boldsymbol{D}(\bar{\boldsymbol{X}})}\cdot\{\boldsymbol{X},\boldsymbol{y}_{\bar{s}}\}_{\bar{s}}(\bar{\boldsymbol{Z}}).\end{aligned} \tag{13}$$

One can confirm from Eq. (11) that $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{H};\boldsymbol{Z},\bar{\boldsymbol{Z}})=\dot{\boldsymbol{y}}_s(\boldsymbol{Z})-\dot{\boldsymbol{y}}\bar{s}(\bar{\boldsymbol{Z}})$ becomes the desired difference in the particle velocities of species s and $\bar{s}$ required in the matrix $\mathbb{Q}$ of the Landau collision operator. Furthermore, in an axially symmetric background field $\boldsymbol{B}_0$, one confirms $\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{P}_\varphi;\boldsymbol{Z},\bar{\boldsymbol{Z}})=\hat{\boldsymbol{z}}\times(\boldsymbol{y}_s(\boldsymbol{Z})-\boldsymbol{y}_{\bar{s}}(\bar{\boldsymbol{Z}}))$ which, together with $\delta_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$ in the matrix (10), guarantees that the toroidal momentum functional (7) is an annihilator element of the metric bracket (9) in the sense of $(\mathcal{P}_\varphi,\mathcal{A})=0$, with respect to an arbitrary functional $\mathcal{A}$, just like in the electrostatic case.

The new metriplectic formulation for the gyrokinetic Vlasov–Maxwell–Landau theory therefore evolves arbitrary functionals $\Psi[F,\boldsymbol{D},\boldsymbol{B}_1]$ according to the differential equation

$$\frac{d\Psi}{dt}=[\Psi,\mathcal{H}]+(\Psi,\mathcal{S}). \tag{14}$$

This guarantees energy conservation $d\mathcal{H}/dt=[\mathcal{H},\mathcal{H}]+(\mathcal{H},\mathcal{S})=0$ and, in an axially symmetric magnetic background field, also toroidal angular momentum conservation $d\mathcal{P}_\varphi/dt=[\mathcal{P}_\varphi,\mathcal{H}]+(\mathcal{P}_\varphi,\mathcal{S})=0$ on the basis of both $\mathcal{H}$ and $\mathcal{P}_\varphi$ being annihilator elements of the metric bracket. The formalism also guarantees monotonic entropy production $d\mathcal{S}/dt=[\mathcal{S},\mathcal{H}]+(\mathcal{S},\mathcal{S})=(\mathcal{S},\mathcal{S})\geq 0$ on the basis of $\mathcal{S}$ being a Casimir of the Poisson bracket and the metric-bracket being positive semi-definite.

The kinetic equation for the test-particle phase-space density F_s is found by choosing a functional $\Psi(\boldsymbol{Z},t)=\int_{Z'}\delta^6(\boldsymbol{Z}'-\boldsymbol{Z})F_s(\boldsymbol{Z}',t)$ and evaluating the equation $\partial_t\Psi=[\Psi,\mathcal{H}]+(\Psi,\mathcal{S})$. This results in

$$\partial_t F_s+\partial_\alpha(F_s V_s^\alpha)=\sum_{\bar{s}} C_{s\bar{s}}[F_s,F_{\bar{s}}], \tag{15}$$

where $V^\alpha=\{Z^\alpha,K\}+q\boldsymbol{E}_1\cdot\{\boldsymbol{X},Z^\alpha\}$ is the Hamiltonian phase-space vector field and the nonlinear collision operator is given by

$$\begin{aligned}C_{s\bar{s}}[F_s,F_{\bar{s}}]&=-\partial_\alpha(\boldsymbol{\gamma}_{s\bar{s}}\cdot\{\boldsymbol{y}_s,Z^\alpha\}_s)\\&=-\partial_\alpha\left(K^\alpha_{s\bar{s}}F_s-B^*_{\|s}D^{\alpha\beta}_{s\bar{s}}\partial_\beta\left(F_s/B^*_{\|s}\right)\right).\end{aligned} \tag{16}$$

The collisional-flux-related term $\boldsymbol{\gamma}_{s\bar{s}}$, a three-component vector

$$\begin{aligned}\boldsymbol{\gamma}_{s\bar{s}}(\boldsymbol{Z})&=\int_{\bar{Z}}\mathsf{Q}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})\,F_s(\boldsymbol{Z})F_{\bar{s}}(\bar{\boldsymbol{Z}})\cdot\boldsymbol{\Gamma}_{s\bar{s}}(\mathcal{S};\boldsymbol{Z},\bar{\boldsymbol{Z}})\\&=F_s\mathbf{K}_{s\bar{s}}-B^*_{\|s}\{\boldsymbol{y}_s,F_s/B^*_{\|s}\}_s\cdot\mathsf{D}_{s\bar{s}},\end{aligned} \tag{17}$$

and the phase-space diffusion and friction coefficients

$$D^{\alpha\beta}_{s\bar{s}}(\boldsymbol{Z})=\{\boldsymbol{y}_s,Z^\alpha\}_s\cdot\mathsf{D}_{s\bar{s}}(\boldsymbol{Z})\cdot\{\boldsymbol{y}_s,Z^\beta\}_s, \tag{18}$$

$$K^\alpha_{s\bar{s}}(\boldsymbol{Z})=\{\boldsymbol{y}_s,Z^\alpha\}_s\cdot\mathbf{K}_{s\bar{s}}(\boldsymbol{Z}), \tag{19}$$

are expressed in terms of the guiding-center and gyrocenter phase-space transformations of the Fokker–Planck diffusion and friction coefficients that are functionals of the field-particle density $F_{\bar{s}}$ (and the electromagnetic fields)

$$\mathsf{D}_{s\bar{s}}(\boldsymbol{Z}) = \int_{\bar{Z}} \mathsf{Q}_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}}) F_{\bar{s}}(\bar{\boldsymbol{Z}}), \tag{20}$$

$$\mathbf{K}_{s\bar{s}}(\boldsymbol{Z}) = \int_{\bar{Z}} \mathsf{Q}_{s\bar{s}}(\boldsymbol{Z}, \bar{\boldsymbol{Z}}) \cdot B^*_{\parallel,\bar{s}}(\bar{\boldsymbol{Z}}) \left\{ \boldsymbol{y}_{\bar{s}}, F_{\bar{s}}/B^*_{\parallel,\bar{s}} \right\}_{\bar{s}} (\bar{\boldsymbol{Z}}). \tag{21}$$

In the electrostatic limit, the result agrees with the gyrokinetic collision operator summarized in Eqs. (22)–(25) in Ref. 37, evident from the expressions (17), (20), and (21). In the absence of both electromagnetic and electrostatic fluctuations ($\boldsymbol{E}_1 = 0 = \boldsymbol{B}_1$), i.e., at the limit of only guiding-center motion in a given magnetic background, expressions for the phase-space diffusion and friction coefficients were given in Ref. 31 for Maxwellian field-particle distributions in a nonuniform background magnetic field. The spatial diffusion coefficient $D^{\boldsymbol{XX}} = (D_\mu B/m\Omega^2 + D_\perp/m^2\Omega^2)(\mathbf{1} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})$, where $D_\mu = \mu(D_\parallel - D_\perp)/(2mE)$ and $D_E = D_\parallel/m$ represent classical transport in a magnetized plasma,[14] while the spatial components $(K^{\boldsymbol{X}}, D^{\boldsymbol{X}\mu}, D^{\boldsymbol{X}E}) = (\nu, D_\mu, D_E)\,\hat{\boldsymbol{b}} \times \boldsymbol{v}_{gc}/\Omega$ are orientated in the direction of the guiding-center polarization shift $\hat{\boldsymbol{b}} \times \boldsymbol{v}_{gc}/\Omega$ involving magnetic gradient and curvature in nonuniform magnetic field.[31]

As the metric bracket (9) operates on functionals depending on $\boldsymbol{D}$, it also contributes to the gyrokinetic Maxwell–Ampère equation. Choosing a test functional $\Psi(\boldsymbol{X}, t) = \int_{X'} \delta^3(\boldsymbol{X} - \boldsymbol{X}')\boldsymbol{D}(\boldsymbol{X}', t)$, and evaluating $\partial_t \Psi = [\Psi, \mathcal{H}] + (\Psi, \mathcal{S})$, we find the gyrokinetic Maxwell–Ampère equation

$$\frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\sum_s \int_P q_s F_s V_s^{\boldsymbol{X}} + \frac{4\pi}{c}\boldsymbol{j}_C = \nabla \times \boldsymbol{H}, \tag{22}$$

where $\boldsymbol{H}$ is defined in the standard manner

$$\boldsymbol{H} = \boldsymbol{B}_0 + \boldsymbol{B}_1 + 4\pi \frac{\delta \mathcal{K}}{\delta \boldsymbol{B}_1}, \tag{23}$$

and the collisional contribution to the current density, $\boldsymbol{j}_C$, is given by

$$\boldsymbol{j}_C = \sum_{s\bar{s}} \int_P q_s \gamma_{s\bar{s}} \cdot \{\boldsymbol{y}_s, \boldsymbol{X}\}_s = \sum_{s\bar{s}} \int_P q_s \left(K^{\boldsymbol{X}}_{s\bar{s}} F_s - B^*_{\parallel s} D^{\boldsymbol{X}\beta}_{s\bar{s}} \partial_\beta \left(F_s / B^*_{\parallel s} \right) \right). \tag{24}$$

Notably, the new collisional term is mandatory to guarantee that the kinetic equation and the Maxwell–Ampère equation remain consistent with the Gauss's law for the displacement field. The consistency can be verified by taking spatial divergence of the new Maxwell–Ampère equation (22), and using the kinetic equation (15) and the collision operator (16), providing that the time derivative of the familiar Gauss's law for the displacement field is exactly zero. The Gauss's law itself,

$$\nabla \cdot \boldsymbol{D} = 4\pi \sum_s \int_P q_s F_s, \tag{25}$$

therefore serves as an initial condition for the displacement field, just like in the particle phase-space Vlasov–Maxwell–Landau system and, if it holds initially, it holds at later times as well. Consequently, the gyrokinetic charge conservation law also has the same new collisional contribution. Indeed, by considering the gyrocenter charge-density functional $\varrho(\boldsymbol{X}, t) = \sum_s \int_{Z'} \delta^3(\boldsymbol{X} - \boldsymbol{X}')\, q_s\, F_s(\boldsymbol{Z}')$, evaluating $\partial_t \varrho = [\varrho, \mathcal{H}] + (\varrho, \mathcal{S})$ results in

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot \left(\sum_s \int_P q_s F_s V_s^{\boldsymbol{X}} + \boldsymbol{j}_C \right) = 0. \tag{26}$$

Finally, the Maxwell–Faraday equation is derived by choosing a functional $\Psi(\boldsymbol{X}, t) = \int_{X'} \delta^3(\boldsymbol{X} - \boldsymbol{X}')\boldsymbol{B}_1(\boldsymbol{X}', t)$ which provides the standard expression

$$\frac{1}{c}\frac{\partial \boldsymbol{B}_1}{\partial t} + \nabla \times \boldsymbol{E}_1 = 0. \tag{27}$$

Treating inconsistently either the collisional current $\boldsymbol{j}_C$ in the new Maxwell–Ampère equation or the collision operator in the kinetic equation cannot be done in the present theory without breaking the metriplectic structure. Although $\boldsymbol{j}_C$ is formally small, scaling effectively as the collision frequency over the cyclotron frequency, it does not vanish exactly and neglecting it would lead to trouble. Trying to remove the spatial contribution of the collision operator, the source of the current density $\boldsymbol{j}_C$, would, e.g., immediately break the identity needed in guaranteeing the toroidal momentum conservation in an axially symmetric background field. To get rid of such limiting details and enable, e.g., straightforward transport analyses, it would be ideal to find a systematic way for deriving reduced collision operators, order by order, with the metriplectic structure remaining intact. Unfortunately, no such method exists yet.

The new formulation presented in this Letter nevertheless establishes a theoretical foundation for gyrokinetic Vlasov–Maxwell–Landau theory. Exploiting the metriplectic formalism enables not only derivation of the gyrokinetic Landau operator but also retaining the energy and toroidal canonical angular momentum conservation accompanied by monotonic entropy production. While it has long been understood that collisions in the gyrocenter coordinates affect also the spatial $\boldsymbol{X}$ coordinates,[31,62] the new metriplectic formalism uncovers the resulting implications also for the Maxwell-Ampère equation and the charge conservation law in a straightforward manner. It is expected that the new theory could be useful also in finding structure-preserving discretizations, similarly as in Ref. 63. Finally, the reader ought to keep in mind that the theory presented here does contain the displacement current and, therefore, also the accompanying fast transverse electromagnetic waves. In contrast to the standard gyrokinetic approach, where the vacuum permittivity is formally set to zero, such assumption is not taken here. Doing so in the action integral would leave the parallel electric field undetermined from the action principle. For gyrokinetics to become a genuine field theory based on an action principle or a metricplectic formulation, with no fast waves, something akin to the Hamiltonian reduction of the Vlasov–Maxwell system to a dark slow manifold described in Ref. 64 is needed on top of the standard Lie-transform theory used to remove the fast gyromotion.

The supplementary material contains further details regarding derivations of the equations of motion (15) and (22) as well as the identity (12) that guarantees toroidal momentum conservation in axially symmetric $\boldsymbol{B}_0$.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Eero Hirvijoki: Conceptualization (equal); Methodology (equal); Writing – review and editing (equal). **Joshua W. Burby:** Conceptualization (equal); Methodology (equal); Writing – review and editing (equal). **Alain J. Brizard:** Conceptualization (equal); Methodology (equal); Writing – review and editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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