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Compact moduli of degree one del Pezzo surfaces

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Abstract
We construct a stable pair compactification of the moduli space of anti-canonically polarized degree one del Pezzo surfaces and relate our constructions to Miranda's GIT quotient and the Hodge theoretic work of Heckman-Looijenga.

1 Introduction

A smooth del Pezzo surface of degree d is a smooth projective surface with $-K_X$ ample and $K_X^2 = d$. These surfaces are at the heart of research in algebraic and arithmetic geometry, and the goal of this paper is to construct a modular geometrically meaningful compactifiction in the degree one case. Celebrated work of Hacking-Keel-Tevelev considered the degree $d \geq 2$ case [12], and so this paper fills in the remaining case.

It is natural to study a del Pezzo surface via its anticanonical linear series. For $d \geq 3$ this is a closed embedding and for $d = 2$ it is a double cover. However, for $d = 1$ the anticanonical pencil is not a morphism; it has a unique basepoint. The blowup of X at this basepoint is a rational elliptic surface $Y \to \mathbb{P}^1$ with section given by the exceptional divisor. Equivalently, X may be obtained as the blowup of $\mathbb{P}^2$ at 8 points in general position, and the anticanonical pencil is the unique pencil of cubics passing through these points. By the Cayley-Bacharach theorem, there is a unique 9^{th} point in the base locus of this pencil which becomes the basepoint of $|-K_X|$. The strategy of this paper is to use the structure of the elliptic fibation $Y \to \mathbb{P}^1$ to construct a geometrically meaningful compactification of the space of degree one del Pezzo surfaces.

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Constructing a compactification of their moduli space is a long standing problem, and has been studied previously using GIT [15], root lattices [26,27], pencils of quadrics [13] (degree 4 case), stable pairs and tropical geometry [12] ($d \geq 2$ case), and Gromov-Hausdorff limits [24]. None of the aforementioned papers obtain a modular compactification when $d = 1$. In [5], Alexeev and Thompson construct a stable pairs compactification for the related, but different, moduli problem of rational elliptic surfaces with a chosen nodal fiber. We prove the following.

Theorem 1.1 *There exists a proper Deligne-Mumford stack $\mathcal{R}$ parametrizing anti-canonically polarized broken del Pezzo surfaces of degree one with the following properties:*

- *The interior $\mathcal{U} \subset \mathcal{R}$ parametrizes degree one del Pezzo surfaces with at worst rational double point singularities.*
- *The complement $\mathcal{R} \setminus \mathcal{U}$ consists of a unique boundary divisor parametrizing 2-Gorenstein slc surfaces with ample anticanonical divisor and exactly two irreducible components.*
- *The locus $\mathcal{R}^\circ \subset \mathcal{R}$ parametrizing surfaces such that every irreducible component is normal is a smooth Deligne-Mumford stack.*

We provide an explicit description of the surfaces parametrized by $\mathcal{R} \setminus \mathcal{U}$ in Theorems 4.5 and 4.8.

While the MMP guarantees the existence of $\mathcal{R}$, the surfaces parametrized by the boundary may a priori have many irreducible components, and the moduli stack might be quite singular with boundary of high codimension. In our case, the moduli stack $\mathcal{R}$ enjoys the perhaps surprising property: the boundary is an irreducible divisor and parametrizes surfaces with at most two irreducible components. It is now natural to ask how $\mathcal{R}$ compares to previously existing compactifications which do not carry a universal family.

1.1 Connnection to GIT

One may associate to any elliptic surface with section a *Weierstrass equation* $y^2 = x^3 + Ax + B$ where A and B are sections of line bundles on the base curve. This is the equation cutting out the *Weierstrass model* obtained by contracting all fibral components that do not meet the section. In the case of rational elliptic surfaces, A and B are degree 4 and 6 homogeneous polynomials on $\mathbb{P}^1$. Using this Weierstrass data, Miranda [22] constructed a GIT compactification W of the moduli space of rational elliptic surfaces (see Sect. 5). We show that our compactification is a certain blowup of W along the strictly semi-stable locus:

Theorem 1.2 *(see Theorem 5.3). Let R be the coarse moduli space of $\mathcal{R}$ and $\Delta \subset R$ the boundary divisor parametrizing non-normal surfaces with $U = R \setminus \Delta$. There is a morphism $R \to W$ to Miranda's GIT compactification such that the following diagram commutes.*

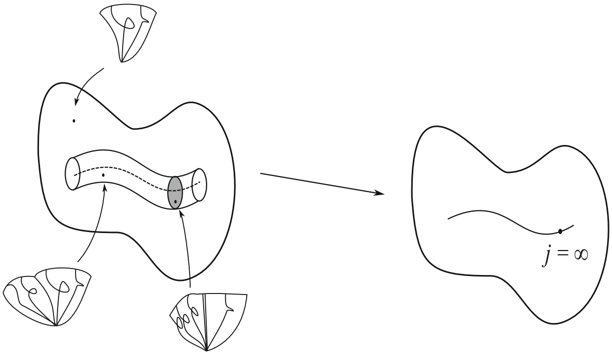

Fig. 1 On the right is Miranda's GIT compactification W, and on the left is $\mathcal{R}$. Note that these spaces are birational. The strictly semistable locus of W is a rational curve with a special point ($j=\infty$). The "tube" depicts the locus of $\mathcal{R}$ parametrizing non-normal surfaces, which is fibered over the strictly semistable locus of W. The shaded in piece depicts the fiber over $j=\infty$. The surfaces parametrized by a generic point of each stratum of $\mathcal{R}$ are depicted

$$\begin{array}{ccccc} \Delta & \hookrightarrow & R & \hookleftarrow & U \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{P}^1 & \longrightarrow & W & \hookleftarrow & W^s \end{array}$$

Here $\Delta \to \mathbb{P}^1$ sends the surface $X \cup Y$ to the j-invariant of the double locus. Then $\mathbb{P}^1 \to W^{sss} \subset W$ maps bijectively onto the strictly semistable locus, and $U \to W^s$ is an isomorphism of the interior of R with the GIT stable locus.

The space $\mathcal{R}$ is the last in a sequence of spaces $\mathcal{R}(a)$ for $\frac{1}{12} < a \leq 1$ arising as compactifications of the space of rational elliptic surfaces. Inspired by Hassett's moduli spaces of weighted stable curves, we construct proper Deligne-Mumford moduli stacks $\mathcal{E}_{\mathcal{A}}$ where $\mathcal{A} = (a_1, \ldots, a_n)$ is a vector of the weights of the marked fibers. As one varies $\mathcal{A}$, the moduli spaces and their universal families are related by divisorial contractions and flips (see [3]). A generic rational elliptic surface has 12 nodal fibers F_i, and so we consider a slice of the moduli space, denoted $\mathcal{E}^s_{\mathcal{A}}$, which compactifies the space of pairs $(f : X \to C, S + F)$, where X is a rational elliptic surface with section S and $F = \sum a_i F_i$. In this paper, we focus on the case where $\mathcal{A} = (a, \ldots, a)$, and so we define the space $\mathcal{R}(a) = \mathcal{E}^s_{\mathcal{A}}/S_{12}$.

When $a \leq \frac{1}{6}$, the section of every surface parametrized by $\mathcal{R}(a)$ must be contracted by the log MMP to form the *pseudoelliptic surfaces* of La Nave [19] (see also Definition 2.14) which in the case of rational elliptic surfaces are exactly the degree one del Pezzo surfaces. In this paper, we explicitly study the moduli spaces $\mathcal{R}(a)$ as well as the wall crossings induced by varying the coefficient a using the theory of twisted stable maps (see [2,6]) combined with the log MMP. In this way, we classify the boundary of the moduli spaces $\mathcal{R}(a)$ for $a \leq 1/6$, which give several geometric compactifications of the moduli space of degree one del Pezzo surfaces. The space denoted by $\mathcal{R}$ above is $\mathcal{R}\left(\frac{1}{12} + \epsilon\right)$ and is, in a sense, the ideal choice for a modular compactification: it is the "smallest compactification" with a universal family, does not depend on any auxiliary choices, and has a particularly nice boundary description.

For various values of $a \leq 1/6$, we are able to relate our compactifications to alternate compactifications which a priori carry less geometric meaning. For example, above we saw how $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$ compared to Miranda's GIT quotient W. It turns out that the space $\mathcal{R}\left(\frac{1}{6}\right)$ also is very much related to both W and two compactifications appearing in the work of Heckman-Looijenga [14]. This reflects the general expectation that wall-crossing morphisms for moduli of stable pairs as in [3] should interpolate between various classical compactifications.

1.2 Connection to Hodge theoretic compactifications

Let $\mathcal{D}^*$ denote the GIT compactification of the space of 12 points in $\mathbb{P}^1$ up to automorphism. To a Weierstrass equation $y^2 = x^3 + Ax + B$, one may associate a discriminant $\mathscr{D} = 4A^3 + 27B^2$. This gives a rational map $W \dashrightarrow \mathcal{D}^*$ and it is natural to ask how close this map is to a morphism. Note that it cannot extend to all of W (for example the Weierstrass equation of a surface with an I_7 fiber is GIT stable but its discriminant is not GIT semistable). We prove that this rational map can be understood via $R(a)$ as follows:

Theorem 1.3 (see Corollary 5.5) *There is a morphism* $R\left(\frac{1}{6}\right) \to \mathcal{D}^*$ *resolving* $W \dashrightarrow \mathcal{D}^*$.

In [14], Heckman-Looijenga study a period map for rational elliptic surfaces. They show that the moduli space of $12I_1$ rational elliptic surfaces is locally a complex hyperbolic variety and identify the rational map $W \dashrightarrow \mathcal{D}^*$ as induced by the period mapping. They describe the normalization of the image in $\mathcal{D}^*$ as the Satake-Baily-Borel (BB) compactification $\mathscr{M}^*$ of a ball quotient, and compare the boundary strata of this compactification with the Miranda's GIT quotient W by introducing a space W^* which dominates both. Note that neither W, $\mathscr{M}^*$, nor W^* carry universal families of surfaces, i.e. they are not coarse moduli spaces of a proper Deligne-Mumford stack with a universal elliptic surface, which is a source of great difficulty in studying their boundary strata.

Theorem 1.4 (Theorem 6.3) *There is a birational morphism* $R\left(\frac{1}{6}\right) \to W^*$ *such that* $\mathcal{R}\left(\frac{1}{6}\right)$ *is the minimal proper Deligne-Mumford stack above both* $\mathscr{M}^*$ *and W extending the universal family on* $\mathscr{M}$.

2 Elliptic surfaces

We begin with a review of the geometry of rational elliptic surfaces (see Miranda's [23]).

Definition 2.1 An irreducible **elliptic surface with section** $(f : X \to C, S)$ is an irreducible surface X together with a surjective proper flat morphism $f : X \to C$ to a smooth curve C and a section S such that:

1. the generic fiber of f is a stable elliptic curve, and

2. the generic point of the section is contained in the smooth locus of f.

We call the pair $(f : X \to C, S)$ **standard** if all of S is contained in the smooth locus of f.

Definition 2.2 A **minimal Weierstrass fibration** is an elliptic surface obtained from a relatively minimal elliptic surface by contracting all fiber components not meeting the section. We call the output of this process a **Weierstrass model**.

Definition 2.3 The **fundamental line bundle** of a standard elliptic surface $(f : X \to C, S)$ is $\mathscr{L} := (f_* \mathscr{N}_{S/X})^{-1}$, where $\mathscr{N}_{S/X}$ denotes the normal bundle of S in X. For an arbitrary elliptic surface we define $\mathscr{L}$ as the line bundle associated to its semi-resolution.

The line bundle $\mathscr{L}$ is invariant under taking a semi-resolution or Weierstrass model of a standard elliptic surface. Furthermore, $\mathscr{L}$ is independent of choice of section S, determines the canonical bundle of X, and $\deg(\mathscr{L})|_C \geq 0$.

If $(f : X \to C, S)$ is a smooth relatively minimal elliptic surface, then f has finitely many singular fibers which are each unions of rational curves with possibly non-reduced components. Recall that the dual graphs are ADE Dynkin diagrams. Furthermore, the possible singular fibers have been classified by Kodaira-Nerón. We refer the reader to [1, Table 1] for the complete classification. However, we point out the definition of the fiber types N_k for $k = 0, 1, 2$, which appear on elliptic surfaces with nodal generic fiber and arise when studying slc surfaces (see [1, Section 5]).

Definition 2.4 N_k are the slc fiber types with Weierstrass equation $y^2 = x^2(x - t^k)$ for $k = 0, 1, 2$.

2.1 Rational elliptic surfaces

We are now ready to define when an elliptic surface is *rational*.

Definition 2.5 We say that an irreducible elliptic surface with section $(f : X \to C, S)$ is **rational** if $C \cong \mathbb{P}^1$ and $\deg(\mathscr{L}) = 1$.

The fact that this definition characterizes rational elliptic surfaces is the content of [23, Lemma III.4.6]. All rational elliptic surfaces arise as the blow up of the base locus of a pencil of cubic curves inside $\mathbb{P}^2$ (see [23, Lemma IV.1.2]).

Remark 2.6 Let C_1 and C_2 be two (distinct) smooth cubic curves in $\mathbb{P}^2$. The pencil generated by these curves has 9 base points, and blowing up these 9 points in $\mathbb{P}^2$ gives a morphism $\pi : X \to \mathbb{P}^1$, where X is a relatively minimal (fibered) rational surface with fibers elliptic curves. Moreover, the canonical class of X is $-C_1$ and $K_X^2 = 0$. The section $S \subset X$ is given by the last exceptional divisor. In this case, it is clear that $S^2 = -1$ and so $\deg(\mathscr{L}) = 1$ (so that $\mathcal{O}(1) \cong \mathscr{L}$).

A rational elliptic surface is defined by a Weierstrass form: $y^2 = x^3 + Ax + B$, where A and B are sections of $\mathcal{O}(4)$ and $\mathcal{O}(6)$ respectively, and the *discriminant* $\mathscr{D} = 4A^3 + 27B^2$ is a section of $\mathscr{L}^{\otimes 12} \cong \mathcal{O}(12)$ which is not identically zero (see [23, Section II.5]).

Remark 2.7 In fact, since the number of singular fibers of a Weierstrass fibration over a projective curve C is given by $12\deg(\mathscr{L}) = 12\deg(\mathcal{O}(1)) = 12$ counted properly (see [23, Lemma II.5.7]), a rational elliptic surface has generically 12 (nodal) singular fibers. In this context, *counting properly* means that the singular fiber is weighted by the order of vanishing of the discriminant. Equivalently, the discriminant is a degree 12 divisor of the base rational curve.

2.2 Rational elliptic surfaces and degree one del Pezzo surfaces

Definition 2.8 Recall a **degree** n **del Pezzo surface** is a surface X with at worst canonical singularities such that $-K_X$ is ample and $K_X^2 = n$.

Remark 2.9 It follows by Castelnuovo's Theorem that a del Pezzo surface is necessarily rational.

Given a degree one del Pezzo surface, the anticanonical linear series $|-K_X| : X \dashrightarrow \mathbb{P}^1$ has a unique base point p. Blowing up along p resolves the basepoint producing a morphism $f : Y = Bl_p(X) \to \mathbb{P}^1$ with section S the exceptional divisor. The fibers of f are necessarily K_Y-trivial curves. It follows by the adjunction formula that f is a genus one fibration with section S, i.e. $(f : Y \to \mathbb{P}^1, S)$ is a rational elliptic surface. Conversely, given a rational elliptic surface $(f : Y \to \mathbb{P}^1, S)$ with at worst rational double point singularities and all fibers being irreducible, e.g. having only twisted or Weierstrass fibers (see Definition 2.12), we may blow down the section to obtain a pseudoelliptic surface X. By Kodaira's canonical bundle formula, one can check that the pseudofiber class f is ample and linear equivalent to $-K_X$ so that X is a degree one del Pezzo surface.

This relation between rational elliptic surfaces and degree one del Pezzo surfaces is the main idea behind our construction of our compactification $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$.

Remark 2.10 One can obtain a degree one del Pezzo surface X by blowing up $\mathbb{P}^2$ in 8 (possibly infinitely near) points and then contracting (-2)-curves. By the Cayley-Bacharach theorem, there exists a unique pencil of cubics in $\mathbb{P}^2$ through these 8 points that passes through a unique 9^{th} point p. This becomes the anticanonical pencil of X with basepoint p.

2.3 Preliminaries on twisted stable maps

Twisted stable maps can be used to compute degenerations of fibered surface pairs with all coefficients one (see e.g. [2,6]).

There is a proper moduli stack of twisted stable maps of fixed degree [6] and it can be used to construct stacks of fibered surfaces in the case where the target $\mathcal{M} = \overline{\mathcal{M}}_{g,n}$ [2]. Relevant for us is the space of twisted stable maps to $\overline{\mathcal{M}}_{1,1}$ inducing a degree 12 map on coarse spaces. Indeed given a rational elliptic surface $(f : X \to \mathbb{P}^1, S + F)$ with only I_1 singular fibers all of which are marked with coefficient one, there is a morphism $\mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and we can understand degenerations of the surface by degenerating in the space of twisted stable maps.

2.3.1 Main relevant consequences of TSM

1. Any time two surfaces are attached along fibers, they must either be attached along nodal fibers, or in pairs consisting of $\mathrm{I}_n^*/\mathrm{I}_m^*/\mathrm{N}_1$ fibers, or in pairs $\mathrm{II}/\mathrm{II}^*$, $\mathrm{III}/\mathrm{III}^*$ and $\mathrm{IV}/\mathrm{IV}^*$.
2. The total number of nodal marked fibers in the degeneration of a marked rational elliptic surface must be 12 (counted with multiplicity).

2.4 Moduli spaces of weighted stable elliptic surface pairs

In [3], we define stable pair compactifications (c.f. [18] and [17]) $\mathcal{E}_{\mathcal{A}}$ compactifying the moduli space of lc models $(f : X \to C, S + F_{\mathcal{A}})$ of $\mathcal{A}$-weighted Weierstrass elliptic surface pairs by allowing our surface pairs to degenerate to semi-log canonical (slc) pairs following the log minimal model program. For each admissible weight vector $\mathcal{A}$, we obtain a compactification $\mathcal{E}_{\mathcal{A}}$, which is representable by a proper Deligne-Mumford stack of finite type [3, Theorem 1.1 & 1.2]. These spaces parameterize slc pairs $(f : X \to C, S + F_{\mathcal{A}})$, where $(f : X \to C, S)$ is an slc elliptic surface with section, and $F_{\mathcal{A}} = \sum a_i F_i$ is a weighted sum of marked fibers with $\mathcal{A} = (a_1, \ldots, a_n)$ and $0 < a_i \leq 1$.

Theorem 2.11 [3, Theorem 1.6] *The boundary of the proper moduli space $\mathcal{E}_{v,\mathcal{A}}$ parametrizes $\mathcal{A}$-broken stable elliptic surfaces, which are pairs $(f : X \to C, S+F_{\mathcal{A}})$ consisting of a stable pair $(X, S + F_{\mathcal{A}})$ with a map to a nodal curve C such that:*

- *X is an slc union of elliptic surfaces with section S and marked fibers, as well as*
- *chains of pseudoelliptic surfaces of type I and II (Definition 2.14) contracted by f with marked pseudofibers.*

Definition 2.12 Let $(g : Y \to C, S' + aF')$ be a Weierstrass elliptic surface pair over the spectrum of a DVR and let $(f : X \to C, S + F_a)$ be its relative log canonical model. We say that X has a(n):

1. **twisted fiber** if the special fiber $f^*(s)$ is irreducible and $(X, S+E)$ has (semi-)log canonical singularities where $E = f^*(s)^{red}$;
2. **intermediate fiber** if $f^*(s)$ is a nodal union of an arithmetic genus zero component A, and a possibly non-reduced arithmetic genus one component supported on a curve E such that S meets A along the smooth locus of $f^*(s)$ and the pair $(X, S + A + E)$ is slc.

Given an elliptic surface $f : X \to C$ over a DVR such that X has an *intermediate fiber*, we obtain the *Weierstrass model* of X by contracting E, and we obtain the *twisted model* by contracting A. As such, the intermediate fiber interpolates between the Weierstrass and twisted models (see [1]). This is made precise via the following, where we explicitly run the log MMP on elliptic fibrations.

Theorem 2.13 [3, Theorem 3.19] *Notation as in Definition 2.12. Suppose the special fiber F' of g is either either (a) one of the Kodaira singular fiber types, or (b) g is isotrivial with constant j-invariant ∞ and F' is an N_0 or N_1 fiber (see Definition 2.4).*

1. *If F is a type* I_n *or* N_0 *fiber, then the relative log canonical model is the Weierstrass model for all* $0 \le a \le 1$.
2. *For any other fiber type, there is an* a_0 *such that the relative log canonical model is*

   (i) *the* Weierstrass *model for any* $0 \le a \le a_0$,
   (ii) *a* twisted fiber *consisting of a single non-reduced component supported on a smooth rational curve when* $a = 1$, *and*
   (iii) *an* intermediate fiber *with E a smooth rational curve for any* $a_0 < a < 1$.

   The constant a_0 *is as follows for the other fiber types:*

$$a_0 = \begin{cases} 5/6 & \mathrm{II} \\ 3/4 & \mathrm{III} \\ 2/3 & \mathrm{IV} \\ 1/2 & \mathrm{N}_1 \end{cases}$$

$$a_0 = \begin{cases} 1/6 & \mathrm{II}^* \\ 1/4 & \mathrm{III}^* \\ 1/3 & \mathrm{IV}^* \\ 1/2 & \mathrm{I}_n^* \end{cases}$$

The MMP will contract the section of an elliptic surface if it has non-positive intersection with the lc divisor of the surface to create a pseudoelliptic. There are two types of pseudoelliptics which appear on the boundary. We refer to [3, Definition 4.6, 4.7] for the precise definitions of the two types of pseudoelliptic surfaces. We give abridged versions of the two definitions for brevity.

Definition 2.14 A **pseudoelliptic pair** is a surface pair (Z, F) obtained by contracting the section of an irreducible elliptic surface pair $(f : X \to C, S + F')$. We call F the **marked pseudofibers** of Z. We call $(f : X \to C, S)$ the associated elliptic surface to (Z, F). It is called:

1. **Type II** if it is formed by the log canonical contraction of a section of an elliptic component attached along *twisted* or *stable* fibers.
2. **Type I** appear in *pseudoelliptic trees* attached by gluing an irreducible pseudofiber G_0 on the root component to an arithmetic genus one component E of an intermediate (pseudo)fiber of an elliptic or pseudoelliptic component.

Remark 2.15 We recall the following from [3, Definition 4.6]. Let $(f : X' \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-broken elliptic surface where $X' = X \cup_E Y$ with a marked Type I pseudoelliptic surface glued $(Y, (F_{\mathcal{A}})|_Y)$ glued to the arithmetic genus one component E of an intermediate (pseudo)fiber $E \cup A$ with reduced component A on X. Then if $F_{\mathcal{A}} = \sum a_i F_i$ we have that

$$\mathrm{Coeff}(A, F_{\mathcal{A}}) = \sum_{\mathrm{Supp}(F_{\mathcal{A}}|_Y)} \mathrm{Coeff}(F_i) = \sum_{\mathrm{Supp}(F_{\mathcal{A}}|_Y)} a_i. \tag{1}$$

is a sum of weights of marked fibers on Y.

Remark 2.16 Contracting the section of a component to form a pseudoelliptic corresponds to stabilizing the base curve as an $\mathcal{A}$-stable curve in the sense of Hassett (see [1, Corollaries 6.7 & 6.8]). This gives a forgetful morphism $\mathcal{E}_{v,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ [3, Theorem 1.4].

Remark 2.17 We recall that for an irreducible component with base curve $\mathbb{P}^1$ and $\deg \mathscr{L} = 1$, contracting the section of an elliptic component might *not* be the final step in the minimal model program. In particular, we might need to contract the entire pseudoelliptic component to a curve or a point. This is the content of [1, Proposition 7.4].

2.5 Wall and chamber structure

We now want to understand how the moduli spaces $\mathcal{E}_{\mathcal{A}}$ change as we vary the weight vector $\mathcal{A}$.

Definition 2.18 There are three types of walls:

(I) A wall of **Type** W_{I} is a wall arising from the log canonical transformations seen in Theorem 2.13, i.e. the walls where the fibers of the relative log canonical model transition from twisted, to intermediate, to Weierstrass fibers.
(II) A wall of **Type** W_{II} is a wall at which the morphism induced by the log canonical contracts the section of some components (i.e. the walls appearing in Hassett's space by Remark 2.16).
(III) A wall of **Type** W_{III} is a wall where the morphism induced by the log canonical contracts an entire rational pseudoelliptic component (Remark 2.17).

Note that there are also *boundary* walls given by $a_i = 0$ and $a_i = 1$ at the boundary, and these can be any of the three types above. There are finitely many walls, and location of the Type W_{I} and W_{II} walls have been calculated (see [3, Theorem 6.3]). We summarize the results here:

- Type W_{I} walls at $a_i = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ (see Theorem 2.13;
- Type W_{II} walls at $\sum_{j=1}^{k} a_{i_j} = 1$ for $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ as well as at $\sum_{i=1}^{r} a_i = 2$.

To construct the desired compactification of degree one del Pezzo surfaces, we must classify the Type W_{III} walls, i.e. when the pseudoelliptic components contract. Note that these components can contract to a point or a curve, and it is the latter case that is more difficult. We will see

- W_{III} walls where a pseudoelliptic component contracts to a point at $\sum_{j=1}^{k} a_i = c$ for $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ and $c = a_0$ from Theorem 2.13;
- W_{III} walls when a pseudoelliptic component contracts onto a curve (see Theorem 3.19).

Finally, we recall one of the main result of [3], which states how $\mathcal{E}_{\mathcal{A}}$ changes as we vary $\mathcal{A}$.

Theorem 2.19 [3, Theorem 1.5] *Let* $\mathcal{A}, \mathcal{B} \in \mathbb{Q}^r$ *be weight vectors with* $0 < \mathcal{A} \leq \mathcal{B} \leq 1$. *Then*

1. *If $\mathcal{A}$ and $\mathcal{B}$ are in the same chamber, then the moduli spaces and universal families are isomorphic.*
2. *If $\mathcal{A} \leq \mathcal{B}$ then there are reduction morphisms $\mathcal{E}_{v,\mathcal{B}} \to \mathcal{E}_{v,\mathcal{A}}$ on moduli spaces which are compatible with the reduction morphisms on the Hassett spaces:*
3. *The universal families are related by a sequence of explicit divisorial contractions and flips*

More precisely, across W_{I} *and* $\mathrm{W}_{\mathrm{III}}$ *walls there is a divisorial contraction of the universal family and across a* W_{II} *wall the universal family undergoes a log flip.*

2.6 Moduli of pseudoelliptic surfaces

The relationship between a rational elliptic surface $\left(f : X \to \mathbb{P}^1, S\right)$ and the corresponding degree one del Pezzo surface Z may be rephrased as Z is the pseudoelliptic surface associated to X. For this reason, it will be crucial for us to consider moduli spaces whose generic member parametrizes pseudoelliptic surfaces. In this subsection, we show that the definitions and results of [3] apply to this degenerate case. This is implicit in [3] but not considered directly.

Definition 2.20 An $\mathcal{A}$-**broken pseudoelliptic surface pair** (X, F) is an slc pair obtained by contracting the section of an $\mathcal{A}$-broken elliptic surface pair to a point. In particular, every component of (X, F) is a pseudoelliptic surface pair.

$\mathcal{A}$-broken pseudoelliptic surfaces can be thought of as degenerate versions of $\mathcal{A}$-broken elliptic surfaces when $\mathcal{A}$ leaves the admissible polytope for Hassett space. Note that the polytope of admissible weights for elliptic surfaces is strictly larger than that for Hassett space as $K_X + S + F_{\mathcal{A}}$ can remain big even when $K_C + \sum a_i p_i$ is not. Moreover, we specialize to $g(C) = 0$ as this is the case of interest.

Theorem 2.21 *Fix $d > 0$. Then for any weight vector $\mathcal{A}$ satisfying*

$$\max\{0, 2-d\} < \sum a_i \leq 2 \tag{2}$$

there exists an irreducible, proper, normal algebraic stack $\mathcal{E}_{v,\mathcal{A}}$ and a universal family of $\mathcal{A}$-broken pseudoelliptic surfaces such that

- *for any normal scheme T, $\mathcal{E}_{v,\mathcal{A}}(T)$ is the groupoid of stable families of $\mathcal{A}$-broken pseudoelliptic surfaces, and*
- *there exists a dense open substack of $\mathcal{E}_{v,\mathcal{A}}$ parametrizing irreducible $\mathcal{A}$-weighted pseudoelliptic surface pairs $(Z, F_{\mathcal{A}})$ whose associated elliptic surface $(X \to C, S + F'_{\mathcal{A}})$ is a minimal Weierstrass surface fibered over $C \cong \mathbb{P}^1$ with* $\deg \mathscr{L} = d$.

Moreover, $\mathcal{E}_{v,\mathcal{A}}$ fit into the same wall-crossing structure along with the moduli of broken elliptic surfaces $\mathcal{E}_{v,\mathcal{A}}$ for $\sum a_i > 2$ described in Sect. 2.5.

Remark 2.22 We use the same notation $\mathcal{E}_{v,\mathcal{A}}$ for both the moduli of broken elliptic and broken pseudoelliptic surfaces to emphasize that they all fit within the same framework as compactifications of the moduli space of log canonical models of elliptic surfaces with marked section, $\mathcal{A}$-weighted fibers, and fixed numerical invariants. The only difference is that when $g(C) = 0$, $d > 0$ and $\sum a_i \leq 2$, the generic member is pseudoelliptic rather than elliptic, while when $\sum a_i > 2$ we have the same spaces defined above as considered in [3]. Note moreover that v can be computed explicitly as a function of $\mathcal{A}$ and d (and using $g(C) = 0$) but we continue the practice of keeping v implicit.

Remark 2.23 Note that in the region of weight vectors satisfying (2), there are no W_{II} walls as all components of the base curve have already been contracted to a point by the MMP but there are further W_{III} walls as we determine below.

Proof of Theorem 2.21 Consider the family of minimal Weierstrass elliptic surfaces

$$\left(f : X \to \mathbb{P}^1, S, F_1, \dots, F_n\right)$$

over $\mathbb{P}^1$ with $d = \deg \mathscr{L}$ and n distinct marked fibers. For any $\mathcal{A}$ satisfying (2), the divisor $K_X + S + F_{\mathcal{A}}$ is big but not nef and the log canonical model of $(X, S + F_{\mathcal{A}})$ is a pseudoelliptic surface [1, Corollary 7.2, Propositions 7.3 & 7.4]. Here $F_{\mathcal{A}} = \sum a_i F_i$. There exists a smooth connected quasi-projective B and a locally stable family of pairs $\pi : (\mathcal{X} \to \mathbb{P}^1_B, S, F_i) \to B$ such that every such elliptic surface pair appears as a fiber of this family. Here we may take for example an open subset of the space of pairs of sections of $\mathcal{O}(4d)$ and $\mathcal{O}(6d)$ respectively.

Now for each weight vector, we can run the π-MMP to obtain the relative log canonical model of $(\mathcal{X}, S + F_{\mathcal{A}})/B$ which we denote $\pi' : (\mathcal{Y}_{\mathcal{A}}, F'_{\mathcal{A}}) \to B$. Up to shrinking B, we can assume that π' is the family of fiberwise log canonical models of π. Note that the section S must be contracted by the π-MMP by our condition on $\mathcal{A}$. Then π' is a stable family of $\mathcal{A}$-weighted pseudoelliptic surface pairs of fixed volume and induces a morphism from B to the proper Deligne-Mumford stack parametrizing KSBA stable pairs of fixed volume. Let $\mathcal{E}_{v,\mathcal{A}}$ be the normalization of the scheme theoretic image of this map. Then $\mathcal{E}_{v,\mathcal{A}}$ is an irreducible proper normal Deligne-Mumford stack with a universal family of stable pairs satisfying the second bullet point by assumption. We need to check that it satisfies the first bullet point. In order to do this, we run stable reduction to see that every point of $\mathcal{E}_{v,\mathcal{A}}$ parametrizes an $\mathcal{A}$-broken pseudoelliptic surface.

Toward this end, let $\left(Y^0, {F'_{\mathcal{A}}}^0\right) \to B^0$ be family of $\mathcal{A}$-stable pseudoelliptic surfaces associated to a family of minimal Weierstrass surfaces over $\mathbb{P}^1$ with d as above and suppose $B^0 = B \setminus 0$ and B is a smooth curve. We can blowup Y^0 along the intersection of marked pseudofibers to reintroduce the section and obtain a family of minimal Weierstrass elliptic surfaces $\left(X^0 \to C, S^0 + F^0_{\mathcal{A}}\right)$ over C^0. We can add a generic marked fiber G^0 with coefficient 1 so that $\left(X^0, S^0 + F^0_{\mathcal{A}} + G^0\right)$ is stable. Then by properness of the space $\mathcal{E}_{v,\mathcal{A}'}$ of $\mathcal{A}'$-broken elliptic surfaces where $\mathcal{A}' = (a_1, \dots, a_n, 1)$, we have a stable limit $(X \to C, S + F_{\mathcal{A}} + G) \to B$ with central fiber an $\mathcal{A}'$-broken elliptic surface. Now we run an MMP with scaling as we decrease the coefficients of G from 1 to 0. Every flip and divisorial contraction of this MMP occurs within the central fiber over $0 \in B$ except for the divisorial contraction of the section S. By [3, Theorem A.10], the only steps in the MMP are those of Types W_I, W_{II} and W_{III}. The resulting stable family $\left(Y, F'_{\mathcal{A}}\right)$ is the stable limit of $\left(Y^0, F'_{\mathcal{A}}{}^0\right)$ and the central fiber is the contraction of the section of an $\mathcal{A}$-broken elliptic surface. Thus, every point in the closure of the image of B is an $\mathcal{A}$-broken pseudoelliptic surface.

Now the vanishing of Theorem [3, Theorem 5.1] holds for $\mathcal{A}$-broken pseudoelliptic surfaces. Indeed the proof proceeds by separating the components of a broken surface proving the appropriate vanishing for each component including trees of pseudoelliptic surfaces as a special case. Moreover, since there are no type W_{II} walls in the region (2), then we may apply the proofs of [3, Theorem 7.4 & Proposition 8.7] verbatim to families of pseudoelliptic surfaces to obtain wall-crossing reduction morphisms extending the wall-crossing structure for moduli of broken elliptic surfaces. □

Remark 2.24 The fact that the wall-crossing for moduli of broken elliptic surfaces extends to the case of broken pseudoelliptic surfaces is special cases of a more general wall-crossing formalism of the present authors joint with Inchiostro and Patakfalvi in the forthcoming paper [4] which shows that under mild assumptions, there are wall-crossing morphisms between stable pair compactifications of the space of log canonical models of a given family as one varies the coefficients. In our case above, we have the family of minimal Weierstrass elliptic surfaces over a genus 0 curve and with fixed numerical invariants. As one varies coefficients, the log canonical model may be either elliptic or pseudoelliptic and we obtain the family of compactifications $\mathcal{E}_{v,\mathcal{A}}$ which parametrize either broken elliptic or broken pseudoelliptic surfaces depending on where the weight vector lies within the space of admissible weights.

3 Definition of the moduli space and Birational contractions across walls

From now on we restrict to the case of rational elliptic surfaces. In particular, the base curve is of genus zero and the degree of $\mathscr{L}$ is one so that $C \cong \mathbb{P}^1$ and $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\mathcal{E}_{1,\mathcal{A}}$ denote the stable pairs compactification of the stack of rational elliptic surfaces with 12 marked fibers weighted by $\mathcal{A} = (a_1, \ldots, a_{12})$. Note here that the choice of 12 marked divisors is part of the data. Thus generically $\mathcal{E}_{1,\mathcal{A}}$ is fibered over the stack of rational elliptic surfaces with section and the fibers are open subvarieties of $\mathbb{P}(|-K_X|)^{12}$.

Definition 3.1 Let $\mathcal{E}^s_{1,\mathcal{A}}$ be the closure in $\mathcal{E}_{1,\mathcal{A}}$ of the locus of log canonical models of pairs $(f : X \to C, S + F_{\mathcal{A}})$ where X is a rational elliptic surface and $\mathrm{Supp}(F_{\mathcal{A}})$ consists of 12 I_1 singular fibers.

Equivalently, $\mathcal{E}^s_{1,\mathcal{A}}$ is the closure of the locus of rational elliptic surface pairs with smooth log canonical model and with divisor given by the discriminant of the fibration. Note that when $\mathcal{A} = (a, \ldots, a)$ is a constant weight vector, then S_{12} acts on $\mathcal{E}^s_{1,\mathcal{A}}$ by permuting the marked fibers.

Definition 3.2 For $\mathcal{A} = (a, \ldots, a)$ the constant weight vector, we define $\mathcal{R}(a) := \mathcal{E}^s_{a,\ldots,a}/S_{12}$.

The following is clear from the analagous statement for $\mathcal{E}_{1,\mathcal{A}}$:

Proposition 3.3 *$\mathcal{R}(a)$ is a proper Deligne-Mumford stack with coarse moduli space $R(a)$.*

The stack $\mathcal{R}(a)$ and its coarse moduli space are the main subjects of this paper but $\mathcal{E}^s_{1,\mathcal{A}}$ and $\mathcal{E}_{1,\mathcal{A}}$ will be used to understand the wall and chamber structure associated to $\mathcal{R}(a)$.

Remark 3.4 Since $\mathcal{E}^s_{1,\mathcal{A}}$ is defined as the closure of the locus of log canonical models of rational elliptic surface pairs with the 12 I_1 fibers marked, then the marked fibers always appear at the discriminant $\mathscr{D}$ of $f : X \to C$ over the smooth locus C^{sm}. In particular, when $\sum a_i > 2$, the map $\mathcal{E}^s_{1,\mathcal{A}} \to \overline{\mathcal{M}}_{0,\mathcal{A}}$ sends $(f : X \to C, S + F_{\mathcal{A}})$ to C marked by the $\mathcal{A}$-weighted discriminant $\mathscr{D}$ of f.

As mentioned in Sect. 2.5, by [3, Theorem 6.3], we have an explicit description of the location of the walls of Types $\mathrm{W_I}$ and $\mathrm{W_{II}}$. One goal of this section is to determine the location of walls of type $\mathrm{W_{III}}$ for the moduli spaces $\mathcal{E}_{\mathcal{A}}$ and then use this to study the explicit birational contractions $\mathcal{R}(a)$ undergoes as one reduces a.

3.1 Pseudoelliptic contractions of $\mathcal{E}_{\mathcal{A}}$

First, recall that walls of Type $\mathrm{W_{III}}$ (see Definition 2.18) correspond to the contraction of an entire pseudoelliptic component. From Definition 2.14, we noted that pseudoelliptic components of Type I are connected to the arithmetic genus one component E of an intermediate (pseudo)fiber of another component.

Let $(f : X \cup Z \to C, S + F_{\mathcal{A}})$ be an $\mathcal{A}$-broken elliptic surface with pseudoelliptic component Z attached to the arithmetic genus one component E of an intermediate (pseudo)fiber $A \cup E$ on X. Suppose further that Z is rational, otherwise Z never contracts with nonzero coefficients (see [1, Corollary 6.10]). Then the contraction of Z to a point produces a minimal Weierstrass fiber at $A \cup E$. Furthermore, Z contracts if and only if E is contracted in the log canonical model of $(X, (S + F_{\mathcal{A}})|_X)$ (see [1, Proposition 7.4]).

Definition 3.5 Let (X, D) be be a pair with (semi-)log canonical singularities and $A \subset X$ a divisor. The **(semi-)log canonical threshold** $\mathrm{lct}(X, D, A)$ is

$$\mathrm{lct}(X, D, A) := \max\{a \ : \ (X, D + aA) \text{ has (semi-)log canonical singularities }\}.$$

Let $(f : X \cup Z \to C, S + F_{\mathcal{A}})$ be as above and let $p : X \to X'$ be the contraction of the $A \cup E$ intermediate fiber onto its Weierstrass (pseudo)fiber $A' \subset X'$. Let $D' = f_* D \subset X'$ where
$D = (S + F_{\mathcal{A}})|_X - \mathrm{Coeff}(F_{\mathcal{A}}, A)A$ the boundary divisor on X excluding the component A.

Proposition 3.6 *The component Z contracts to a point in the log canonical model of $(f : X \cup Z \to C, S + F_{\mathcal{A}})$ if and only if*

$$\sum_{\mathrm{Supp}(F_{\mathcal{A}}|_Z)} a_i \leq \mathrm{lct}(X', D', A')$$

where the left hand side is a sum over marked pseudofibers on Z.

Corollary 3.7 *There are type* $\mathrm{W_{III}}$ *walls for* $\mathcal{E}_{\mathcal{A}}$ *corresponding to pseudoelliptic components contracting to a point given by* $\sum_{i\in I} a_i = c$ *where* $I \subset \{1, \dots, n\}$ *and c is the log canonical threshold of a minimal Weierstrass cusp.*

Remark 3.8 In the case of type II, III, IV and $\mathrm{N_1}$ Weierstrass cusps, the log canonical threshold c is given by the numbers a_0 in Theorem 2.13.

Remark 3.9 Note there are also type $\mathrm{W_{III}}$ contractions of pseudoelliptics at the boundary walls given by $a_i = 0$.

Proof of Proposition 3.6 The component Z contracts to a point if and only if the curve E it is attached to contracts to a point in the log canonical model of $(X, (S + F_{\mathcal{A}})|_X)$. Note first that $\mathrm{Coeff}(E, (S + F_{\mathcal{A}})|_X) = 1$ since E is in the double locus of $X \cup Z$ and by Equation 1,

$$\sum_{\mathrm{Supp}(F_{\mathcal{A}}|_Z)} a_i = \mathrm{Coeff}(A, (S + F_{\mathcal{A}})|_X).$$

We need to compute at which coefficient of A the component E is contracted in the lc model of $(X, (S + F_{\mathcal{A}})_X)$. Since this is a local question, we may assume X is an elliptic surface with section S, intermediate fiber $A \cup E$ and Weierstrass model $p : X \to X'$ with Weierstrass cusp $p_*(A \cup E) = A'$. Suppose that $a \leq \mathrm{lct}(X', S', A')$ where $S' = p_*S$. Consider the log resolution $p : X \to X'$ of the pair $(X', S' + aA')$. By definition of lc singularities, the log canonical model of $(X, S + aA + E)$ relative to p is X' since $E = \mathrm{Exc}(p)$. Conversely, it is easy to compute from the singularity of X at $A \cap E$ that $\mathrm{lct}(X, S + E, A) = 1$ (see the computations in [1]). If $1 > a > \mathrm{lct}(X', S', A')$, the pair $(X, S + aA + E)$ is log canonical while the contraction of E produces pair that has worse than log canonical singularities and so E cannot be contracted in the lc model. □

3.2 The birational contractions of $\mathcal{R}(a)$

Now we use the above discussion to determine the walls of $\mathcal{R}(a)$ as one decreases a and what birational contractions the moduli space undergoes.

Lemma 3.10 *There are Type* $\mathrm{W_{II}}$ *walls where Type* I *pseudoelliptic surfaces of* $\mathcal{R}(a)$ *form at* $a = \frac{1}{k}$ *for* $k = 1, \dots, 5$.

Proof The flips forming Type I pseudoelliptic curves form when a component of the underlying weighted curve is contracted. Since all weights are the same, this occurs when $ka = 1$ as long as the total weight $12a > 2$ so that the moduli space of weighted stable curves is nontrivial. □

Lemma 3.11 *Let $a > \frac{1}{6}$. Then any pseudoelliptic component on a surface parametrized by $\mathcal{R}(a)$ must be a Type* I *pseudoelliptic glued along a type* II, III, IV, *or* N_1.

Proof Let Z be such a component. Then it is formed by a pseudoelliptic flip corresponding to a type W_{II} wall as in Lemma 3.10. In particular, the number of maked points on the section component that contracted to form Z is at most 5. Since the marked points occur at the discriminant of the elliptic fibration (counted with multiplicity) then Z must be a component with at most 5 singular fibers counted with multiplicity away from the double locus.

Since Z is a rational pseudoelliptic surface the total multiplicity of the discriminant of the corresponding elliptic surface is 12 (see Remark 2.7). Therefore the pseudofiber of Z where it is attached must correspond to a fiber with at discriminant at least 7 so it has to be an I_n^* for $n > 0$, II^*, III^* or IV^*. In the first case, it must be attached to another I_m^* or an N_1 fiber by the balancing condition in 2.3.1. By degree considerations it has to be attached to an N_1 fiber. In the latter case, the balancing condition requires it be attached to a type II, III or IV respectively. ◻

Lemma 3.12 *Let $\mathcal{A} = (a, \ldots, a)$ for $a = \frac{1}{6} + \epsilon$. Then curves C parametrized by $\overline{\mathcal{M}}_{0,\mathcal{A}}$ are either*

1. *a smooth $\mathbb{P}^1$ with 12 marked points, or*
2. *the union of two rational curves, each with 6 marked points.*

Proof It is clear that C can be a smooth $\mathbb{P}^1$. If C is the union of two rational components, then since each point is weighted by $\frac{1}{6}+\epsilon$, and since each curve has to have total weight > 2 including the node, each curve must have six points. Suppose $C = \cup_{i=1}^{3} C_i$, and label the two end components by C_1 and C_3, and the bridge by C_2. Then at least one of C_1 and C_3 will not be stable as $5 \cdot (\frac{1}{6} + \epsilon) < 1$. ◻

Corollary 3.13 *If X is parametrized by $\mathcal{R}\left(\frac{1}{6} + \epsilon\right)$, then X has at most two elliptic components.*

Remark 3.14 X can have many Type I pseudoelliptics mapping onto marked points of C.

Definition 3.15 If X parametrized by $\mathcal{R}(\frac{1}{6} + \epsilon)$ has a single (resp. exactly two) fibered component(s) X_0 (resp. $X_0 \cup X_1$), we call X_0 (resp. $X_0 \cup X_1$) the **main component** of X.

Note in particular that every surface parametrized by $\mathcal{R}(\frac{1}{6} + \epsilon)$ consist of a main component with trees of pseudoelliptics attached along Type II, III, IV and N_1 fibers.

Proposition 3.16 *There is a wall at $a = \frac{1}{6}$ where the entire section contracts and the Hassett moduli space becomes a point. Furthermore:*

1. *If X has an irreducible main component X_0 then X_0 contracts to a degree one del Pezzo surface with trees of pseudoelliptics branching off.*
2. *If X has main component $X_0 \cup X_1$, then it either contracts to the above case or it contracts to a union of Type II pseudoelliptics $Y_0 \cup Y_1$ glued along a twisted pseudofibers with trees of pseudoelliptics branching off.*

In the latter case, $Y_0 \cup Y_1$ are glued along twisted $\mathrm{I}_0^*/\mathrm{I}_0^*$, $\mathrm{I}_0^*/\mathrm{N}_1$ or $\mathrm{N}_1/\mathrm{N}_1$ pseudofibers.

Proof If the main component is irreducible, then every other component lies on a Type I pseudoelliptic tree glued along intermediate II, III, IV or N_1 fibers of X_0 by Lemma 3.10. Otherwise the fibered components are of the form $X_0 \cup_E X_1 \to C$ where $C = C_0 \cup_p C_1$ is the nodal union of two 6-pointed rational curves by Lemma 3.12 and E is a twisted fiber of both X_0 and X_1. If $X_i \to C_i$ is a normal elliptic fibration, then it must have 6 singular fibers counted with multiplicity other than the double locus E. Thus E must contribute 6 to the discriminant and so is an I_0^*. If X_i is a non-normal component then it must be an isotrivial j-invariant ∞ component.

If it is trivial then it contracts onto a nodal fiber of the other component producing a surface with a single main component. If it is nontrivial, then it must be unique $2\mathrm{N}_1$ surface with $\deg(\mathscr{L}) = 1$ with 6 marked fibers counted with multiplicity and glued along a twisted fiber with $\mathbb{Z}/2\mathbb{Z}$ stabilizer. This means the two main components are attached along $\mathrm{N}_1/\mathrm{I}_n^*$ or $\mathrm{N}_1/\mathrm{N}_1$. Again by examining degrees of the discriminant we see in the former $n = 0$. □

Corollary 3.17 *Let $\frac{1}{12} < a \leq \frac{1}{6}$. The surfaces parametrized by $\mathcal{R}(a)$ consist of the following:*

1. *An irreducible pseudoelliptic main component with trees of Type I pseudoelliptics attached to it along* II, III, IV *or* N_1 *pseudofibers,*
2. *A main component consisting of two Type* II *pseudoelliptics glued along twisted* $\mathrm{I}_0^*/\mathrm{I}_0^*$, $\mathrm{N}_1/\mathrm{I}_0^*$ *or* $\mathrm{N}_1/\mathrm{N}_1$ *pseudofibers with Type* I *pseudoelliptic trees attached to it along* II, III, IV *or* N_1 *pseudofibers.*

Theorem 3.18 *The Type* $\mathrm{W}_{\mathrm{III}}$ *walls of* $\mathcal{R}(a)$ *corresponding to the contraction of a Type* I *pseudoelliptic component to a point occur at* $\{a = \frac{a_0}{k}\}$ *for* $2 \leq k \leq 5$ *and* a_0 *is one of the four constants appearing in Theorem 2.13 for fibers of type* II, III, IV, *and* N_1.

Proof By Lemma 3.10 and Corollary 3.17, any Type I pseudoelliptic consists of a surface with $2 \leq k \leq 5$ marked fibers (counted with multiplicity) and a II^*, III^*, IV^*, I_n^* $(n > 0)$, or N_1 fiber attached to a type II, III, IV, N_1 or N_1 respectively. By Corollary 3.7 these surfaces contract when $ka = c$ for c the log canonical threshold of type II, III, IV or N_1 minimal Weierstrass cusp respectively. The log canonical thresholds are the a_0 in Theorem 2.13. □

Theorem 3.19 (see Example 3.20) *There are walls of type* $\mathrm{W}_{\mathrm{III}}$ *at* $a = \frac{1}{k}$ *for* $2 \leq k \leq 9$. *Where a trivial component of j-invariant infinity contracts onto its attaching (pseudo)fiber.*

Proof Trivial j-invariant infinity components appear when marked fibers collide and carry the number of markings that collide to form the component. If such a component $Z \subset X$ has $k \leq 6$ marked fibers, then it must contract onto the fiber direction at the Type W_{II} walls $a = \frac{1}{k}$ where the corresponding section contracts to a point.

Suppose $X = Z \cup_E Y$ with Z carrying $k \geq 7$ marked fibers. Then at coefficients $a = \frac{1}{6} + \epsilon$, the surface Z is the main component and Y is a Type I pseudoelliptic tree. In particular the trivial component Z is blown up at the point where the fiber E meets the section. Then at $a = \frac{1}{6}$ the section contracts and so the main component Z becomes a nontrivial $\mathbb{P}^1$ bundle over the nodal curve E and the marked pseudofibers become sections of the projection $Z \to E$ and the flipped curve A in the intermediate pseudofiber $A \cup E$ becomes a fiber of this projection. Then one may compute that when $a = \frac{1}{k}$ the restriction of the log canonical divisor to Z is linearly equivalent to A and so the component Z contracts along the projection $Z \to E$. Finally $k \leq 9$ because $k \geq 10$ fibers on a rational elliptic surface cannot collide (see e.g. Persson's classification of singular fibers [25]). □

Example 3.20 (see Fig. 2) Suppose X_η is a smooth rational elliptic surface with 12 (I_1) fibers appearing as the general fiber of a family $\mathscr{X} \to B$. We will compute the stable limit when 7 of the nodal fibers collide for all weights a. Let X_a^0 denote the special fiber of $\mathscr{X} \to B$. We begin with the twisted stable maps limit at $a = 1$. The surface X_1^0 is the union of two surfaces, $X_1^0 = Z \cup_{\mathrm{I}_7} Y$, where Z is a trivial nodal elliptic surface $\mathbb{P}^1 \times E$ and seven marked fibers, glued to Y along an I_7 fiber.

At $a = \frac{1}{5}$ the section of Y contracts to obtain $X_{\frac{1}{5}}^0 = Z \cup_{\mathrm{I}_7} \tilde{Y}$, where $\tilde{Y}$ is the pseudoelliptic surface corresponding to Y. Decreasing weights so that $\mathcal{A} = 1/5 - \epsilon$ we cross a wall of type W_{II} and a flip occurs in the family to obtain $X_{\frac{1}{5}-\epsilon}^0 = \hat{Z} \cup_{\mathrm{I}_7'} \tilde{Y}$, where we blow up a point on Z corresponding to the contraction of the section of Y. We note that I_7' is an intermediate fiber on $\hat{Z}$ and is the union of a genus one component E with a genus zero component A.

At $a = \frac{1}{6}$ the section of $\hat{Z}$ contracts to form $\tilde{Z}$. Since $\tilde{Z}$ is the blowdown of the strict transform of the section of the blowup of a trivial surface $\mathbb{P}^1 \times E$, then $\tilde{Z}$ is a $\mathbb{P}^1$ nontrivial $\mathbb{P}^1$-bundle over E. The component A becomes a fiber of the projection $\tilde{Z} \to E$ and the marked pseudofibers become sections. At $a = 1/7$, the surface $\tilde{Z}$ contracts onto the E component (i.e. I_7 pseudofiber of Y) and we are left with $X_{\frac{1}{7}}^0$, a single pseudoelliptic component with an I_7 pseudofiber and five I_1 pseudofibers.

Remark 3.21 The above example occurs when $6 \leq k \leq 9$ nodal fibers collide, and the final wall is at $\frac{1}{k}$. Note that for numerical reasons, we cannot have $10 \leq n \leq 12$ nodal fibers collide (see [25]).

Example 3.22 (see Fig. 3) Suppose X_1^0 is the twisted stable maps limit of a family $\mathscr{X} \to B$ and $X_1^0 = Y \cup_{\mathrm{I}_0^*} Z$, where there are six marked fibers on both irreducible components, and there is a $\mathbb{Z}/2\mathbb{Z}$ stabilizer at node of the stable base curve corresponding to the double locus of X_1^0. Suppose further that $j(\mathrm{I}_0^*) = \infty$. Then the marked j-invariant infinity fibers on Y or Z can collide into the double locus and we obtain new isotrivial components of j-invariant infinity.

If $1 \leq n \leq 5$ marked fibers on Y collide onto the double locus, the stable limit will be a new surface $Y' \cup W \cup Z$, where each component has $(6 - n)$, n and 6 marked fibers respectively, Y' now has an I_n^* fiber where the markings collided, and the component W is isotrivial of j-invariant infinity. By examining the stabilizer of the twisted stable map at the nodes, we see that W has an N_1 fiber glued to the I_0^* fiber of Z and another N_1 fiber glued to the I_n^* fiber of Y'. When the coefficients decrease past

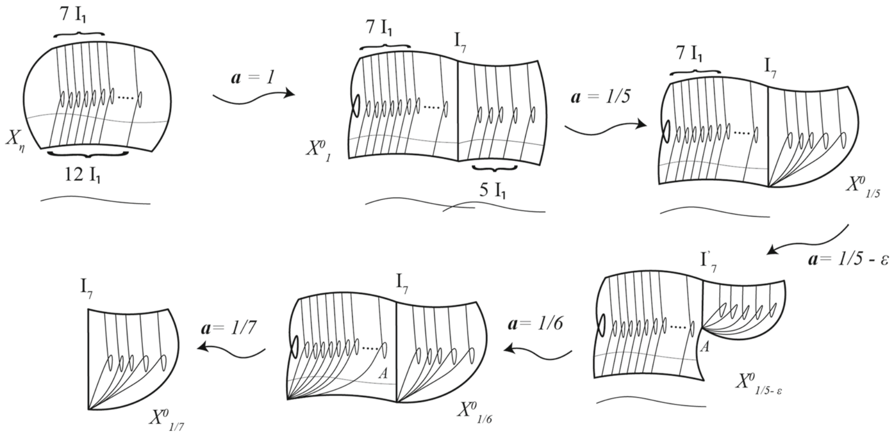

Fig. 2 Example 3.20 illustrating Theorem 3.19

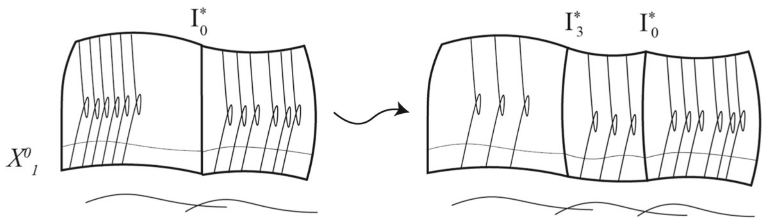

Fig. 3 Example 3.22 with 3 I_1 fibers colliding into the $\mathrm{I}_0^*/\mathrm{I}_0^*$ double locus

a Type W_{II} wall $a = \frac{1}{6-n}$, Y' undergoes a pseudoelliptic flip explaining how isotrivial main components appear in Proposition 3.16.

Remark 3.23 In Example 3.22 there can be a chain of isotrivial j-invariant infinity surfaces sandwiched between the two non-isotrivial surfaces if I_n type fibers collide into the double locus from multiple sides. However, each end surface must have ≥ 2 singular marked fibers (counted with multiplicity) and each isotrivial surface must have ≥ 1 and so the maximum length of a chain is 8.

4 Explicit moduli of del Pezzo surfaces of degree one

We begin by defining a compactification $\mathcal{DP}^1$ following [9]. Later on, we will see that the space $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$ is a slice inside $\mathcal{DP}^1$, allowing us to apply the methods of [9] to $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$.

Definition 4.1 *(c.f. [9, Definition 2.8]).* Let X be a surface and D a $\mathbb{Q}$-Cartier divisor on X. Then (X, D) is a **Hacking stable**, or H-stable for short, degree one del Pezzo pair if:

1. $\left(X, \left(\frac{1}{12}+\epsilon\right) D\right)$ is slc and $K_X + \left(\frac{1}{12}+\epsilon\right) D$ is ample,

2. the divisor $12K_X + D$ is linearly equivalent to 0, and
3. (X, D) admits a $\mathbb{Q}$-Gorenstein deformation to a smooth del Pezzo surface of degree one

The stack of such objects we will denote by $\mathcal{DP}^1$ (see [9, Section 4]).

Remark 4.2 It follows that for an H-stable pair, $-K_X$ is ample (see [9, Proposition 2.13]). In particular, Definition 4.1 is an slc generalization of a degree one del Pezzo surface X marked by D the discriminant (weighted with multiplicity) of the anticanonical linear series $|-K_X|$.

We now turn to $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$, and show that pairs parametrized by $\mathcal{R}(\frac{1}{12}+\epsilon)$ are H-stable, so that $\mathcal{R}(\frac{1}{12}+\epsilon)$ embeds into $\mathcal{DP}^1$. First, we define a special locus inside $\mathcal{R}(a)$.

Definition 4.3 Let $\mathcal{R}^{\circ}(a)$ denote the locus inside $\mathcal{R}(a)$ parametrizing surfaces without isotrivial j-invariant infinity components.

4.1 Stable degree one del Pezzo surfaces and $\mathcal{R}(\frac{1}{12}+\epsilon)$

In Section 3 we computed the type W_{III} walls for $\mathcal{R}(a)$ – these occur at $\frac{1}{k}$ for $2 \leq k \leq 9$ (see Theorem 3.19) and $\frac{a_0}{k}$ for $2 \leq k \leq 5$ where a_0 is a constant appearing in Theorem 2.13 depending on the Kodaira type of the fiber a Type I pseudoelliptic is attached to (see Theorem 3.18). Furthermore, recall the type W_{I} walls where fibers become Weierstrass are at $\frac{5}{6}$, $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{1}{2}$ and type W_{II} agree with those of Hassett space.

In particular, when $a \leq \frac{1}{6}$, all sections are contracted so that $\mathcal{R}(a)$ is a moduli space of pseudoelliptic surfaces. Since the contraction of the section of a rational elliptic surface yields a degree one del Pezzo surface whose pseudofibers are anticanonical curves, we see the following:

Lemma 4.4 *Let $\frac{1}{12} < a \leq \frac{1}{6}$. Then $\mathcal{R}(a)$ is a compactification of a moduli space of degree one del Pezzo surfaces with canonical singularities and marked anticanonical curves.*

We can be more precise about the marking on a del Pezzo surface on the interior of $\mathcal{R}(a)$. If (X, F_a) is a normal surface parametrized by $\mathcal{R}(a)$, then it is the blowdown of the section of a rational Weierstrass fibration. The boundary divisor consists of the singular fibers weigthed by a counted with multiplicity. As each fiber of the Weierstrass fibration becomes an anticanonical curve upon blowing down the section, $F_a \sim 12af$ where $f \in |-K_X|$ is a pseudofiber class. We may conclude that $F_a \in |-\alpha K_X|$ with $1 < \alpha \leq 2$. In particular, the necessarily ample log canonical divisor satisfies $K_X + F_a \sim_{\mathbb{Q}} -\delta K_X$ for $0 < \delta \leq 1$. In particular (X, F_a) can be thought of as an anticanonically polarized degree one del Pezzo surface with at worst rational double point singularities.

We now characterize the two types of surfaces parametrized by the boundary of $\mathcal{R}^0(a)$.

Theorem 4.5 *The surfaces parametrized by $\mathcal{R}^{\circ}(a)$ for $\frac{1}{12} < a \leq \frac{1}{6}$ are either:*

1. *normal degree one del Pezzo surfaces with canonical Gorenstein singularities and all singular pseudofibers being Weierstrass of type* I_n, II, III *or* IV, *or*
2. *the slc union of two degree one del Pezzo surfaces with canonical Gorenstein singularities glued along twisted* I_0^* *pseudofibers such that* $2K_X$ *is Cartier and all other singular pseudofibers as above.*

In both cases, $K_X + D \sim_{\mathbb{Q}} -\delta K_X$ *for* $0 < \delta \leq 1$ *so that* $-K_X$ *is ample and* (X, D) *is anticanonically polarized. We call case (1) surfaces Type A and case (2) surfaces Type B.*

Proof It follows from Corollary 3.13 that the surfaces parametrized by $\mathcal{R}\left(\frac{1}{6}+e\right)$ have at most two elliptic components. Since $a \leq \frac{1}{6}$ the section of every component contracts. Suppose an elliptic fibration $(X \to C, F_{\mathcal{A}})$ in $\mathcal{R}^0\left(\frac{1}{6}+e\right)$ has only one elliptic component, possibly with pseudoelliptic components of Type I attached to it. The base rational curve marked by the $\left(\frac{1}{6}+\epsilon\right)$-weighted discriminant $\left(\frac{1}{6}+\epsilon\right)\mathscr{D}$ is an irreducible Hassett stable curve. In particular, the order of vanishing $v_q(\mathscr{D}) \leq 5$ for every $q \in C$. Any unstable fiber on the elliptic component is type II, III or IV. In particular, any type I pseudoelliptic tree is attached along an intermediate II, III or IV fiber.

By Theorem 3.18, any such pseudoelliptic surface is contracted to a point in the log canonical model for $a \leq \frac{1}{6}$ so every surface in $\mathcal{R}^0(a)$ arising from a surface in $\mathcal{R}^0\left(\frac{1}{6}+\epsilon\right)$ with a unique elliptic component is irreducible. The contraction of the pseudoelliptic components yields ADE singularities by [1, Page 230] since such contractions produce minimal Weierstrass models. The fact that they are del Pezzo surfaces, in the sense that $-K_X$ is ample, follows from calculation preceding the theorem as we saw that $K_X + F_a \sim_{\mathbb{Q}} -\delta K_X$, which is ample. This gives case (1).

Now we discuss Case (2). By Proposition 3.16, the only way to obtain multiple elliptic components in $\mathcal{R}^0(\frac{1}{6}+\epsilon)$ is if there are two components each with six marked fibers glued along $\mathrm{I}_0{}^*$ fibers. By considering stability of the base Hassett curve, we see that $v_q(\mathscr{D}) \leq 5$ so any type I pseudoelliptic trees attached to these components contract to a point by $a = \frac{1}{6}$. Furthermore the section of each component contracts so we obtain two pseudoelliptic surfaces of Type II glued along twisted I_0^* fibers but with all other fibers Weierstrass. In particular each component again has only ADE singularities, a single twisted I_0^* pseudofiber, and else all Weierstrass pseudofibers of types I_n, II, III and IV. Let (X, F_a, E) be such a component with markings F_a and double locus E marked by 1. Then F_a consists of 6 (counted with multiplicity) pseudofibers weighted by a and E is a reduced pseudofiber underlying a twisted I_0^* pseudofiber. Thus as before we may compute $F_a + E \sim_{\mathbb{Q}} -(6a+\frac{1}{2})K_X < -K_X$ with $\frac{1}{12} < a \leq \frac{1}{6}$. Thus $K_X + F_a + E$, the log canonical restricted to each component, satisfies $K_X + F_a + E \sim_{\mathbb{Q}} -\delta K_X$ for $\delta > 0$. In particular $-K_X$ is ample and $K_X^2 = 1$ since $-K_X$ is the class of a pseudofiber so each component is a degree one del Pezzo surface. □

Lemma 4.6 *In the setting above, surfaces of Type A are Gorenstein and surfaces of Type B are* $\mathbb{Q}$*-Gorenstein of index 2.*

Proof Surfaces of Type A are Gorenstein since the singularities are of ADE type ([1, Pg 230]). Surfaces of Type B are Gorenstein away from the double locus as well where the double locus is double normal crossings. Thus we need only check around the points of the double locus where the normalization is singular. There are four such points where each component has an A_1 singularity. Locally around each point the surface is a quotient of a nodal surface by $\mathbb{Z}/2\mathbb{Z}$ since the double locus is a twisted I_0^*. Thus each of these points is 2-Gorenstein so the surface has index 2. □

We have seen already that the surfaces of Type A are anticanonically polarized so it remains to see the same is true for 2-Gorenstein surfaces of Type B.

Lemma 4.7 *Surfaces of Type B are anti-canonically polarized.*

Proof Denote the surface by $X = X_1 \cup X_2$. Let $\nu : \tilde{X} \to X$ denote the normalization, and let ν_i denote the normalization restricted to the preimage of X_i. Then $\nu_i^*(K_X + F_a) = K_{\tilde{X}_i} + \tilde{F}_a|_{X_i} + E$, where E is the preimage of the double locus. We calculate:

$$\nu_i^*(K_X + F_a) \sim_{\mathbb{Q}} -f + 6af + 1/2f = -f + 1/2f + \delta f + 1/2f = \delta f \sim_{\mathbb{Q}} -\delta K_{X_i}$$

for some $\delta > 0$. Here f is a pseudofiber class. On the other hand,

$$\nu_i^*(-K_X) = -K_{X_i} - E \sim_{\mathbb{Q}} -3/2K_{X_i}.$$

So $K_X + F_a \sim_{\mathbb{Q}} -\alpha K_X$ for some $\alpha > 0$, since ν^* is injective on $\mathrm{Pic} \otimes \mathbb{Q}$ as $\cap X_i$ is a reduced $\mathbb{P}^1$ (see [16]) so the pair (X, F_a) is anticanonically polarized. □

For the final chamber $a = \frac{1}{12} + \epsilon$ such a description actually extends to all of $\mathcal{R}(\frac{1}{12} + \epsilon)$:

Theorem 4.8 *The surfaces parametrized by* $\mathcal{R}\left(\frac{1}{12} + \epsilon\right) \setminus \mathcal{R}^0\left(\frac{1}{12} + \epsilon\right)$ *are either the union of*

1. *an isotrivial j-invariant infinity surface and a surface of Type A, glued along twisted* $\mathrm{N}_1/\mathrm{I}_n^*$ *pseudofibers,*
2. *or of two isotrivial j-invariant infinity surfaces glued along twisted* N_1 *pseudofibers,*

In both cases the surfaces are anticanonically polarized with index two. We call the surfaces in (1) Type C and in (2) Type D.

Proof By examining the twisted stable maps degenerations one sees that the only way to obtain isotrivial components of j-invariant ∞ is by marked fibers colliding, or a marked fiber colliding with the double locus as in Examples 3.20 & 3.22 respectively. Any isotrivial components appearing as in Example 3.20 undergo a pseudoelliptic contraction at $\frac{1}{k}$ for $k = 3, \ldots, 9$ so such components do not appear in the surfaces parametrized by $\mathcal{R}\left(\frac{1}{12} + \epsilon\right)$.

Suppose we are in the case of Example 3.22. Then at $a = \frac{1}{6} + \epsilon$ as there are only two fibered components $X \cup Y$ along with some trees of type I pseudoelliptics attached.

The pseudoelliptics contract at walls $\frac{a_0}{k}$ for $k = 2, \ldots, 5$ and $\frac{1}{k}$ for $k = 3, \ldots, 9$. In particular, all of these components have contracted at $a = \frac{1}{12} + \epsilon$. Furthermore, the section of main components $X \cup Y$ contract to type II pseudoelliptics. At least one or both of X and Y are isotrivial $j = \infty$.

If only one is, suppose X, then X has a twisted N_1 fiber attached to a twisted I_n^* fiber of Y for some $n > 0$. If both are isotrivial $j = \infty$, then they are attached along twisted $\mathrm{N}_1/\mathrm{N}_1$ fibers. Then the corresponding pseudoelliptics are attached along $\mathrm{N}_1/\mathrm{I}_n^*$ respectively $\mathrm{N}_1/\mathrm{N}_1$ pseudofibers. Furthermore, locally around a point of the attaching fiber, by definition of $\mathrm{N}_1/\mathrm{I}_n^*$ fibers, the surface looks like the quotient of a family of nodal curves over a nodal curve modulo a $\mathbb{Z}/2\mathbb{Z}$ action. As a family of nodal curves over a nodal curve is Gorenstein, our surface must be 2-Gorenstein. □

Theorem 4.9 *There is an embedding of* $\mathcal{R}\left(\frac{1}{12} + \epsilon\right)$ *into* $\mathcal{DP}^1$. *Furthermore, the locus* $\mathcal{R}^0(\frac{1}{12} + \epsilon)$ *is a section of its image in the stack of unmarked degree one del Pezzo surfaces under the forgetful morphism* $(X, D) \to X$, *where* (X, D) *is an H-stable pair.*

Proof Given $(X, F_a) \in \mathcal{R}\left(\frac{1}{12} + \epsilon\right)$, let $D = \frac{1}{a} F_a$ where $a = \frac{1}{12} + \epsilon$. Then D is a sum of 12 pseudofibers counted with multiplicty. Let f be a pseudofiber class, then $f \sim_{\mathbb{Q}} -K_X$ since X is a pseudoelliptic corresponding to a rational elliptic surface. Thus $12K_X + D \sim_{\mathbb{Q}} 0$ verifying Definition 4.1 (2). Condition (1) is true since (X, F_a) is a stable pair and (3) follows from the definition of $\mathcal{R}(a)$ as the closure of the component parmetrizing smooth rational (pseudo)elliptic surfaces with only I_1 fibers. Over the locus $\mathcal{R}^0(\frac{1}{12} + \epsilon)$, the divisor D is the discriminant of the elliptic fibration pushed forward along the pseudoelliptic contraction (excluding the fiber along which two components are glued in the case that X is on the boundary). Thus sending X to the discriminant of its anticanonical pencil gives a section of the projection map with image $\mathcal{R}^0\left(\frac{1}{12} + \epsilon\right)$ over the locus where X is normal. □

4.2 Smoothness properties of the moduli space $\mathcal{DP}^1$

Our proof follows Hacking [9]. In particular, since $-K_X$ is ample, to show that $\mathcal{R}^0(\frac{1}{12} + \epsilon)$ is smooth, it suffices to show that the $\mathbb{Q}$-Gorenstein deformations of the surfaces of Type A and B are unobstructed, and that $H^1(X, \mathcal{O}_X(D)) = 0$ (see [9, Theorem 3.12 & Lemma 3.14]).

Proposition 4.10 (See [9, Theorem 8.2]) *Let X be a surface of Type A. Then X has unobstructed* $\mathbb{Q}$*-Gorenstein deformations.*

Proof Following Hacking, the obstructions are contained in $T^2_{QG,X}$. Since there is a spectral sequence $E_2^{pq} = H^p(\mathscr{T}^q_{QG,X}) \implies T^{p+q}_{QG,X}$, it is sufficient to show that $H^p(\mathscr{T}^q_{QG,X}) = 0$ for $p + q = 2$.

The sheaf $\mathscr{T}^1_{QG,X}$ is supported on a finite set (the singular locus of X), so $H^1(\mathscr{T}^1_{QG,X}) = 0$. The surface X is lci since the singularities are ADE so $\mathscr{T}^2_{QG,X} = 0$ and $H^0(\mathscr{T}^2_{QG,X}) = 0$. Therefore, it suffices to show that $H^2(\mathscr{T}^0_{QG,X}) = H^2(\mathscr{T}_X) = 0$. This follows by combining the proof of [21, Theorem 21] and [28, Lemma 1.11].

Namely, let $\sigma : S \to X$ be the minimal resolution of X. Then since the singularities of X are quotient singularities, $\sigma_*\Omega^1_S = (\Omega^1_X)^{\vee\vee}$ by [28, Lemma 1.11]. Therefore, $H^0((\Omega^1_X)^{\vee\vee}) = H^0(\Omega^1_S) = 0$, as S is a rational surface. Let $s \neq 0$ be a section of $\mathcal{O}_X(-K_X)$. Then s yields a dual injective morphism $s^v : \mathcal{O}_X(K_X) \to \mathcal{O}_X$. Composing with s^v shows that $\mathrm{Hom}(\mathcal{T}_X, \mathcal{O}_X(K_X)) = 0$ and so by Serre Duality (X is Gorenstein!) $H^2(\mathcal{T}_X) = 0$. □

Proposition 4.11 (see [9, Theorem 9.1]) *Let X be the a surface of Type B. Then X has unobstructed* $\mathbb{Q}$*-Gorenstein deformations.*

Proof Again, it suffices to show that $H^p(\mathcal{T}^q_{QG,X}) = 0$ for $p + q = 2$. The surface X has local canonical covering by a local complete intersection $\pi : Z \to X$ so that $\mathcal{T}^2_Z = 0$. If μ_n is the covering group of π, we have $\mathcal{T}^2_{QG,Z} = \pi_*(\mathcal{T}^2_Z)^{\mu_n} = 0$ so $H^0\left(\mathcal{T}^2_{QG,X}\right) = 0$. The sheaf $\mathcal{T}^1_{QG,X}$ is supported on the singular locus of X which consists of the pseudofiber along which the surfaces are glued as well isolated ADE singularities. We note that the (induced reduced structure of the) gluing fiber is $\mathbb{P}^1$, and we let $i : \mathbb{P}^1 \hookrightarrow X$ denote the inclusion of this fiber in X. By [10, Lemma 3.6], $\mathcal{T}^1_{QG,X} = i_*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{Q}$ where $\mathcal{Q}$ is supported at isolated points, and so $H^1(\mathcal{T}^1_{QG,X}) = 0$.

Finally, we must show that $H^2\left(\mathcal{T}^0_{QG,X}\right) = H^2(\mathcal{T}_X) = 0$. Let (X_i, E_i) for $i = 1, 2$ denote the two components with $E_i = E|_{X_i}$ denoting the restriction of the double locus. Following [9, Lemma 9.4], to show that $H^2(\mathcal{T}_X) = 0$, it suffices to show that $H^2(\mathcal{T}_{X_i}(-E_i)) = 0$, which is equivalent to showing that $\mathcal{O}_{X_i}(-K_{X_i} - E_i)$ has a non-zero global section. Note that $-K_{X_i} \sim 2E_i$ since E_i is the support of a multiplicity 2 nonreduced pseudofiber, and so $-K_{X_i} - E_i \sim E_i$. Thus the reflexive sheaf $\mathcal{O}_{X_i}(-K_{X_i} - E_i) = \mathcal{O}(E_i)$ has a section, namely the one cutting out E_i. □

Lemma 4.12 *If* (X, D) *is an* H*-stable pair parametrized by* $\mathcal{R}(\frac{1}{12} + \epsilon)$*, then* $H^1(\mathcal{O}_X(D)) = 0$.

Proof Either X or X^v has canonical singularities. Therefore, it suffices to show that $-(K_X - D)$ is ample, as then the result follows from [9, Lemma 3.14]. Note we have that $12K_X + D \sim_{\mathbb{Q}} 0$ since (X, D) is H-stable and so $D \sim_{\mathbb{Q}} -12K_X$. Thus $-(K_X - D) \sim_{\mathbb{Q}} -13K_X$ is ample. □

Theorem 4.13 *Let* $(X, D) \in \mathcal{R}^0\left(\frac{1}{12} + \epsilon\right) \subset \mathcal{DP}^1$*. Then the stack of* $\mathbb{Q}$*-Gorenstein del Pezzo surfaces of degree one is smooth if a neighborhood of X and the projection from* $\mathcal{DP}^1$ *given by* $(X, D) \to X$ *is smooth. In particular,* $\mathcal{DP}^1$ *is smooth in a neighborhood of* $\mathcal{R}^0\left(\frac{1}{12} + \epsilon\right)$*, and the locus* $\mathcal{R}^0\left(\frac{1}{12} + \epsilon\right)$ *is smooth.*

Proof By Prop 4.10 and 4.11, the $\mathbb{Q}$-Gorenstein deformations of X are unobstructed. By Lemma 4.12 and [9, Theorem 3.12 & Lemma 3.14], the projection from $\mathcal{DP}^1$ given by $(X, D) \to X$ is a smooth morphism since given a $\mathbb{Q}$-Gorenstein deformation of X, deformations of D are unobstructed. This proves that $\mathcal{DP}^1$ is smooth in a neighborhood of $(X, D) \in \mathcal{R}^0\left(\frac{1}{12} + \epsilon\right) \subset \mathcal{DP}^1$. Finally, $\mathcal{R}^0\left(\frac{1}{12} + \epsilon\right)$ is a section of the projection $(X, D) \to X$ over its image by Theorem 4.9 so $\mathcal{R}^0\left(\frac{1}{12} + \epsilon\right)$ is smooth. □

Remark 4.14 We note that the above proof method will not work to show the entire $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$ is smooth, as in the j-invariant ∞ locus, the divisor can move freely, and we need to cut out only those deformations which do not smooth the divisor.

5 Miranda's GIT construction of the moduli space of Weierstrass fibrations

5.1 Overview of Miranda's construction

In [22], Miranda uses GIT to construct a coarse moduli space of *Weierstrass fibrations*. These fibrations arise naturally as follows: let $\widetilde{p}:\widetilde{X}\to Y$ be a minimal elliptic surface with section S. One obtains a normal surface called a Weierstrass fibration $X\to Y$ by contracting each component of the fibers of $\widetilde{p}$ which do not meet the section S. This fibration has only rational double point singularities, and is uniquely determined by $\widetilde{X}$. Let W denote the GIT quotient, and let W^{sss} denote the strictly semistable locus.

Notation 5.1 *Let $\Gamma_n=\Gamma(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(n))$. For the Weierstrass fibration of a rational elliptic surface, we think of X as being the closed subscheme of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(3)\oplus\mathcal{O}_{\mathbb{P}^1})$ defined by the equation $y^2z=x^3+Axz^2+Bz^3$, where $A\in\Gamma_4$, $B\in\Gamma_6$, and*

1. $4A(q)^3+27B(q)^2=0$ *precisely at the (finitely many) singular fibers X_q,*
2. *and for each $q\in\mathbb{P}^1$ we have $v_q(A)\le 3$ or $v_q(B)\le 5$.*

Theorem 5.2 [22, Theorem 6.2, Proposition 8.2, Theorem 8.3]. *Let X be a rational Weierstrass fibration represented by W. Then X is stable if and only if X has smooth generic fiber and the associated elliptic surface $\widetilde{X}$ has only reduced fibers. Furthermore, X is strictly semistable if and only if the associated elliptic surface $\widetilde{X}$ has a fiber of type I_N^* for some $N\ge 0$. Moreover, two strictly semistable elliptic surfaces correspond to the same point in W^{sss} if and only if the j-invariant of the I_N^* fibers are the same.*

In particular, there is a stratification $W=W^s\sqcup\mathbb{A}^1\sqcup\infty$ where the strictly semistable locus is the j-line $\mathbb{A}^1\sqcup\infty$, with $\mathbb{A}^1$ the j-invariant of the I_0^* fibers and ∞ corresponding to I_N^* for $N\ge 1$.

5.2 Relation between W and $\mathcal{R}(\frac{1}{12}+\epsilon)$

We now compare W to $\mathcal{R}(\frac{1}{12}+\epsilon)$.

Theorem 5.3 *Let $R=R\left(\frac{1}{12}+\epsilon\right)$ be the coarse moduli space of $\mathcal{R}\left(\frac{1}{12}+\epsilon\right)$ and $\Delta\subset R$ the boundary divisor parametrizing non-normal surfaces with $U=R\setminus\Delta$. There is a morphism $R\to W$ to Miranda's GIT compactification such that the following diagram commutes.*

$$\begin{array}{ccccc} \Delta & \hookrightarrow & R & \hookleftarrow & U \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{P}^1 & \longrightarrow & W & \hookleftarrow & W^s \end{array}$$

Here $\Delta \to \mathbb{P}^1$ sends the surface $X \cup Y$ to the j-invariant of the double locus, $\mathbb{P}^1 \to W^{sss} \subset W$ maps bijectively onto the strictly semistable locus, and $U \to W^s$ is an isomorphism.

Proof Let $\mathcal{U} \subset \mathcal{R}(\frac{1}{12}+\epsilon)$ be the open locus of normal surfaces, i.e. smooth surfaces in $\mathcal{R}(\frac{1}{12}+\epsilon)$ and surfaces of Type A. Consider the PGL_2-torsor

$$\mathcal{F} = \{(X, s, t) \mid X \in \mathcal{U}, (s,t) \in H^0(-K_X) \text{ where } s, t \text{ span } H^0(-K_X)\}/\sim,$$

where we quotient by scaling. The image of $|-K_X|$ is a $\mathbb{P}^1$ with coordinates (s,t), and the linear series $|-K_X|$ induces the elliptic fibration: the blowup of its base point gives an elliptic fibration (with section), and thus a Weierstrass equation in coordinates s and t. In particular, this Weierstrass coefficients (A, B) are unique up to the scaling of the $\mathbb{G}_m$ action $(A, B) \mapsto (\lambda^4 A, \lambda^6 B)$.

Furthermore by Theorem 5.2 and the characterization of surfaces of Type A (Theorem 4.5), the forms (A, B) are contained in the stable locus $V^s \subset V$. Therefore, we obtain a PGL_2-equivariant morphism $\mathcal{F} \to V^s$ which induces a morphism $\phi : \mathcal{U} \to W$. By comparing the characterization of the type A surfaces parametrized $\mathcal{U}$ and Miranda's stable surfaces, we see that $\mathcal{U} \to W$ must be an isomorphism onto the stable locus. Suppose X is a surface parametrized by the boundary Δ and $\mathscr{X} \to B$ is a 1-parameter family so that $\mathscr{X} \in \mathcal{U}$ for $b \neq 0$ and $\mathscr{X}_0 = X$. Then by [8, Theorem 7.3], to exhibit the existence of a morphism $R \to W$, it suffices to show that $\lim_{b\to 0}(\mathscr{X}_b)$ depends only on X and not on the choice of family. However this follows by Theorem 5.2: the surface X contains a fiber of type I_0^*, namely the gluing fiber, and so the family of Weierstrass data (A_b, B_b) corresponding to $\mathscr{X} \to B$ limits to the unique point $j(I_0^*) \in W^{sss}$ which is well defined since the j-invariant of the attaching fiber is the same on each component of X. Therefore, the morphism ϕ extends to a morphism on all of R and we obtain the desired morphism $R \to W$. Commutativity of the diagram above follows by construction. □

Remark 5.4 Given Weierstrass data (A, B), we can consider the discriminant $\mathscr{D} \in \Gamma_{12}$. If $\mathcal{D}^*$ is the GIT quotient of the space of degree 12 forms on $\mathbb{P}^1$ by automorphisms of $\mathbb{P}^1$, then it is natural to ask if $(A, B) \mapsto \mathscr{D}$ induces a morphism $W \to \mathcal{D}^*$. There is clearly a rational map $W \dashrightarrow \mathcal{D}^*$ but this map *cannot* extend. Indeed W parametrizes surfaces with I_n fibers for $n > 6$ which have discriminant vanishing to order $n > 6$. Such a discriminant is GIT unstable. However, we will see below that the space $R(\frac{1}{6})$ resolves this rational map.

Corollary 5.5 *There are morphisms $R\left(\frac{1}{6}\right) \to W$ and $R\left(\frac{1}{6}\right) \to \mathcal{D}^*$ where $\mathcal{D}^*$ is the GIT moduli space for 12 points in $\mathbb{P}^1$. Furthermore, the diagram*

$$\begin{array}{ccc} & R(\frac{1}{6}) & \\ \swarrow & & \searrow \\ W & \dashrightarrow & \mathcal{D}^* \end{array}$$

commutes where $W \dashrightarrow \mathcal{D}^*$ is the rational map induced by $(A, B) \mapsto \mathscr{D} = 4A^3 + 27B^2$

Proof There is a morphism $R\left(\frac{1}{6}\right) \to W$ induced by composing the morphism in Theorem 5.3 with the reduction morphism $R\left(\frac{1}{6}\right) \to R\left(\frac{1}{12}+\epsilon\right)$ (Theorem 2.19). By [3, Theorem 1.4], there is a morphism $\mathcal{R}\left(\frac{1}{6}+\epsilon\right) \to \mathcal{M}_{\frac{1}{6}+\epsilon}/S_{12}$ which sends a $\left(\frac{1}{6}+\epsilon\right)$-weighted stable rational elliptic surface marked with its singular fibers to the $(\frac{1}{6}+\epsilon)$-weighted 12-pointed stable curve marked by the discriminant of the elliptic fibration. By [11, Section 8], there is a morphism $\mathcal{M}_{\frac{1}{6}+\epsilon}/S_{12} \to \mathcal{D}^*$ which induces $R(\frac{1}{6}+\epsilon) \to \mathcal{D}^*$ by composing and taking coarse moduli space. To obtain the factorization $R\left(\frac{1}{6}+\epsilon\right) \to R\left(\frac{1}{6}\right) \to \mathcal{D}^*$, it suffices by [8, Theorem 7.3] to show that the image of a point under $R\left(\frac{1}{6}+\epsilon\right) \to \mathcal{D}^*$ depends only on the image of that point in $R\left(\frac{1}{6}\right)$. I.e, we must show that given a $\left(\frac{1}{6}+\epsilon\right)$-weighted broken rational elliptic surface, the equivalence class in the GIT moduli space of its discriminant only depends on the $\frac{1}{6}$-weighted stable replacement of the surface.

This is clear on the locus where $R\left(\frac{1}{6}+\epsilon\right) \to R\left(\frac{1}{6}\right)$ is an isomorphism. The morphism $R\left(\frac{1}{6}+\epsilon\right) \to R\left(\frac{1}{6}\right)$ causes the base curve to contract, and this is an isomorphism on moduli spaces away from the contraction of a trivial j-invariant ∞ component. By Theorem 3.19, the only $\mathrm{W_{III}}$ wall occurring at $a = \frac{1}{6}$ comes from contracting an isotrivial component glued along an I_6 fiber. In this case the base curve of the corresponding surface parametrized by $R\left(\frac{1}{6}+\epsilon\right)$ had to have two components, each with six marked points. Therefore any surface of this form gets mapped to the unique minimal strictly semistable orbit of $\mathcal{D}^*$, which arises precisely from two points each of multiplicity six, and so does not depend on the choice of surface. There also may be the contraction of pseudoelliptic trees of type I to points. However, the discriminant depends only on the main component(s), and not on the pseudoelliptic trees. Indeed since the main components survive under the reduction morphism $R\left(\frac{1}{6}+\epsilon\right) \to R\left(\frac{1}{6}\right)$, we see that the stable replacement inside $\mathcal{D}^*$ only depends on the image of the corresponding point in $R(\frac{1}{6})$. Lastly, commutativity is immediate by construction. □

6 Heckman-Looijenga's compactification

Recall that to a rational elliptic Weierstrass fibration we can associate its *discriminant divisor* $\mathscr{D}$ which is described in the previous section in terms of Weierstrass equation.

Equivalently, for a smooth minimal elliptic surface, $\mathscr{D}$ is given by assigning to any point on the base curve the Euler characteristic of its fiber, and yields an effective divisor of degree 12. When the discriminant is reduced, there are 12 singular fibers of type I_1 – in this case the projective equivalence class of the discriminant divisor determines the surface up to isomorphism.

We can describe a compactification using the fact that the discriminant can be used to classify generic elliptic surfaces. Using this approach, Heckman and Looijenga showed that the moduli space of rational elliptic surfaces can be interpreted as a locally complex hyperbolic variety, and studied its Satake-Baily-Borel compactification (see [14] and [20, Section 7]).

Recall that for the GIT compactification $\mathcal{D}^*$ of the space of 12 points in $\mathbb{P}^1$ up to automorphism, a collection of points is stable (resp. semistable) if there are no points of multiplicity ≥ 6 (resp. ≥ 7). Let $\mathscr{M}$ denote the moduli space of rational elliptic surfaces with *reduced discriminant*, and let $\mathcal{D} \subset \mathcal{D}^*$ denote the SL_2 orbit space of 12 element subsets of $\mathbb{P}^1$. Taking the discriminant of a generic elliptic surface yields a closed embedding $\mathscr{M} \hookrightarrow \mathcal{D}$ (see [14, Proposition 2.1]). While rational elliptic surfaces have 8 dimensional moduli, the dimension of $\mathcal{D}$ is 9, and so the space of rational elliptic surfaces defines an SL_2-invariant hypersurface. This hypersurface corresponds to the 12 element subsets of $\mathbb{P}^1$ that admit an equation which is the sum of a cube and a square.

They obtain a compactification $\mathscr{M}^*$ of $\mathscr{M}$ by taking the normalization of the closure of $\mathscr{M}$ inside $\mathcal{D}^*$. Since they cannot compare $\mathscr{M}^*$ and W directly (see Remark 6.1), they also define two auxilliary compactifications: W^* – the normalization of the closure of the diagonal embedding of $\mathcal{M} \hookrightarrow W \times \mathcal{D}^*$, and $\mathcal{M}^K$ which is a compactification via Kontsevich stable maps . The space $\mathcal{M}^K$ is essentially the image in the Kontsevich space of maps to $\mathbb{P}^1$ of the space of twisted stable maps to $\overline{\mathcal{M}}_{1,1}$ given by composing with the coarse space map $\overline{\mathcal{M}}_{1,1} \to \mathbb{P}^1$, see [2,6].

In [14], the authors compare these compactifications and show, using work of Deligne-Mostow [7], that $\mathscr{M}^*$ can be interpreted as the BB compactification of a complex hyperball quotient.

Remark 6.1 (1) The birational map between W and $\mathscr{M}^*$ *does not* extend to a morphism

(2) $\mathscr{M}^*$ is not the coarse moduli space of some proper Deligne-Mumford stack of elliptic surfaces.

(3) The various compactifications fit together into a diagram as follows:

$$\begin{array}{ccc} & \mathcal{M}^K & \\ & \downarrow & \\ & W^* & \\ \swarrow & & \searrow \\ W & \dashrightarrow & \mathscr{M}^* \end{array} \qquad (3)$$

Table 1 Dimension of boundary components

F	$\dim \mathscr{M}^*(F)$	$\dim W(F)$	$\dim W^*(F)$
I_2	7	7	7
I_k	$9-k$	$9-k$	$9-k$
I_6	0	3	3
I_7	5	2	7
I_8	6	1	7
I_9	7	0	7
II	7	7	7
III	6	6	6
IV	5	5	5
I_0^*	0	1	1
$\mathrm{I}_{l,l'}^*$	$l+l'-1$	0	$l+l'-1$

The following theorem of [14] describes the boundary of $\mathscr{M}$ inside W, W^*, and $\mathscr{M}^*$.

Theorem 6.2 [14, Section 3.3] *The boundary of $\mathscr{M}$ inside W, W^*, and $\mathscr{M}^*$ is the union of irreducible components denoted by $W(F)$ (resp. $W^*(F)$ and $\mathscr{M}^*(F)$), where F runs over the various Kodaira symbols as in Table 1.*

We have $k \leq 5$, and $l, l' \in \{1, 2, 3, 4\}$. The dimension of these components inside each space are in Table 1. We now briefly describe the type of surfaces corresponding to the generic point of the boundary loci $W^*(F)$ labeled by Kodaira symbols F in the above theorem (see [14, Section 3.3]).

6.0.1 Boundary loci

$\mathrm{I}_{k\geq 2}$ The surface has two components, one isotrivial j-invariant ∞ component with k marked fibers, glued to a non-isotrivial component along an I_k fiber. See Example 3.20.

II The surface has two irreducible components, a $10\mathrm{I}_1$ II component glued to a $2\mathrm{I}_1$ II^* component along a $\mathrm{II}/\mathrm{II}^*$ twisted fibers.

III Similar to above but with $\mathrm{III}/\mathrm{III}^*$ twisted fibers.

IV Similar to above but with $\mathrm{IV}/\mathrm{IV}^*$ twisted fibers.

I_0^* The surface has two irreducible components of type $6\mathrm{I}_1$ I_0^* and the surfaces are glued along $\mathrm{I}_0^*/\mathrm{I}_0^*$ twisted fibers. Compare with surfaces of Type A in Theorem 4.5.

$\mathrm{I}_{l,l'}^*$ The surface has three components $X \cup Y \cup Z$. Y is isotrivial j-invariant ∞ with $l + l'$ marked nodal fibers as well as two twisted N_1 fibers. X has $6 - l$ type I_1 fibers and a twisted I_l^* glued along one of the N_1 fiber and Z is similar with l' instead of l. See Example 3.22.

Roughly speaking, the map $W^* \to W$ takes one of the above surfaces to the equivalence class of semistable orbits in Miranda's space associated to the Weierstrass equation of the "main component" of the surface. Similarly, the map $W^* \to \mathcal{M}^*$ takes such a surface to the GIT semistable replacement of the base curve marked by the discriminant divisor.

Theorem 6.3 *There is a projective birational morphism $R\left(\frac{1}{6}\right) \to W^*$ from the coarse moduli space of $\mathcal{R}\left(\frac{1}{6}\right)$ which is an isomorphism away from the $W^*(\mathrm{I}_0^*)$ and $W^*\left(\mathrm{I}_{l,l'}^*\right)$ loci. The universal family of $\mathcal{R}\left(\frac{1}{6}\right)$ over these loci parametrizes surfaces of the type described in Section 6.0.1. Furthermore, $\mathcal{R}\left(\frac{1}{6}\right)$ is the minimal space above both $\mathscr{M}^*$ and W extending the universal family on $\mathscr{M}$.*

Proof W^* is universal for dominant morphisms $X \to W$ and $X \to \mathscr{M}^*$ from a normal variety X that agree over $\mathscr{M}$. By construction $R\left(\frac{1}{6}\right)$ is normal and so the existence of $\varphi : R\left(\frac{1}{6}\right) \to W^*$ follows.

Next one can check by the explicit description of limits in $\mathcal{R}\left(\frac{1}{6}\right)$ given in Sects. 3 and 4.1 that φ is a bijection over strata $W^*(F)$ for $F \neq \mathrm{I}_0^*, \mathrm{I}_{l,l'}$. Indeed for $F = \mathrm{I}_k$, $2 \leq k \leq 6$, II, III and IV the stratum $W^*(F)$ is the coarse moduli space of Weierstrass surfaces containing an F singular fiber since these correspond to irreducible surfaces whose Weierstrass equation is GIT stable. Thus φ is a bijection over these strata as $R(\frac{1}{6})$ is a coarse moduli space of surfaces parametrized by $\mathcal{R}(\frac{1}{6})$.

The strata $W^*(\mathrm{I}_k)$ for $k = 7, 8, 9$ parametrize surfaces $X \cup Y$ where X is trivial with k marked fibers and Y is an I_k Weierstrass surface. The configuration of marked fibers on X is GIT stable in $\mathcal{D}^*$ and Y is GIT stable in W. Therefore $W^*(\mathrm{I}_k)$ is a coarse moduli space for such surfaces. Over this locus $\mathcal{R}\left(\frac{1}{6}\right)$ parametrizes pseudoelliptic models of these same surfaces as in Example 3.20 and so φ is bijective on this locus on the level of coarse moduli spaces. By ZMT, φ is an isomorphism on this locus where it is bijective. Over the $W^*(\mathrm{I}_0^*)$, the $\mathcal{R}\left(\frac{1}{6}\right)$ parametrizes pseudoelliptic surfaces $X \cup Y$ glued along a twisted I_0^* pseudofiber and the map φ takes such a surface to the j-invariant of the I_0^* fiber. In particular, this stratum in $R\left(\frac{1}{6}\right)$ is the 7-dimensional coarse moduli space for rational elliptic surfaces glued along an I_0^* fiber while $W^*(\mathrm{I}_0^*)$ is a 1-dimensional stratum parametrizing only the j-invariant. Thus the universal family of $\mathscr{M}$ does not extend over this locus. Over the $W^*(\mathrm{I}_{l,l'}^*)$ locus, $R\left(\frac{1}{6}\right)$ is the coarse moduli space for surfaces $X \cup Y \cup Z$ as in Example 3.22 and Remark 3.23 where X and Z are I_l^* and $\mathrm{I}_{l'}^*$ pseudoelliptic surfaces and Z is a chain of isotrivial j-∞ pseudoelliptic surfaces glued along twisted N_1 fibers. The map φ takes such a surface to the GIT semistable replacement of the configuration of marked fibers on the components Z. In particular, it forgets the information of X and Y so again the locus $W^*(\mathrm{I}_{l,l'}^*)$ is not a coarse moduli space for the type of surfaces it corresponds to and the universal family over $\mathscr{M}$ does not extend.

This exhausts the list of strata and shows that $R\left(\frac{1}{6}\right) \to W^*$ is an isomorphism away from the locus where W^* is not a coarse moduli space of surfaces. Furthermore, over this locus $R\left(\frac{1}{6}\right)$ is a coarse moduli space for precisely the surfaces the strata in W^* correspond to and so $\mathcal{R}\left(\frac{1}{6}\right)$ is the minimal stack over which the universal family of surfaces extends. □

Remark 6.4 $R\left(\frac{1}{6}\right)$ and W^* are isomorphic along the boundary component corresponding to $\mathrm{I}_{4,4}^*$, but the universal families are different. In the universal family of $\mathcal{R}\left(\frac{1}{6} + \epsilon\right)$ such surfaces have contracted to a point, but there is a *unique* rational elliptic surface with an I_4^* fiber (see [25]).

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