DIFFERECES BETWEEN TOTIENTS

KEVIN FORD AND SERGEI KONYAGIN

Abstract. We study the set $\mathcal{D}$ of positive integers d for which the equation $\phi(a)-\phi(b)=d$ has infinitely many solution pairs (a,b). We show that $\min \mathcal{D} \leqslant 154$, exhibit a specific A so that every multiple of A is in $\mathcal{D}$, and show that any progression $a \mod d$ with $4|a$ and $4|d$, contains infinitely many elements of $\mathcal{D}$. We also show that the Generalized Elliott-Halberstam Conjecture, as defined in [**6**], implies that $\mathcal{D}$ contains all positive, even integers.

1. Introduction

Let $\mathcal{V} = \{v_1, v_2, \ldots\}$ be the set of totients, that is, $\mathcal{V}$ is the image of Euler's totient function $\phi(n)$. In this paper we study the set $\mathcal{D}$ of positive integers which are infinitely often a difference of two elements of $\mathcal{V}$. A classical conjecture asserts that every even positive integer is infinitely often the difference of two primes, and this implies immediately that $\mathcal{D}$ is the set of all positive, even integers. We are interested in what can be accomplished unconditionally, by leveraging the recent breakthroughs on gaps between consecutive primes by Zhang [**8**], Maynard [**4**], Tao (unpublished) and the PolyMath8b project [**6**]. We let $\mathcal{E}$ be the set of positive even numbers that are infinitely often the difference of two primes. Clearly $\mathcal{E} \subseteq \mathcal{D}$. In this note we prove some results about $\mathcal{D}$ which are not known for $\mathcal{E}$.

The behavior of the smallest elements of $\mathcal{D}$ arose in recent work of Fouvry and Waldschmidt [**2**] concerning representation of integers by cyclotomic forms, and the problem of studying the differences of totients was also posted in a list of open problems by Shparlinski [**7**, Problem 56]. Our paper is a companion of the recent work of the first author [**1**] concerning the equation $\phi(n+k)=\phi(n)$ for fixed k.

It is known [**6**] that $\min \mathcal{E} \leqslant 246$ and thus $\min \mathcal{D} \leqslant 246$. We can do somewhat better.

Theorem 1. *We have* $\min \mathcal{D} \leqslant 154$.

Although there is no specific even integer which is known to be infinitely often the difference of two primes, we give an infinite family of specific numbers that are in $\mathcal{D}$.

Theorem 2. *Let* $a_0 = \prod_{p \leqslant 47} p$ *and* $b_0 = \mathrm{lcm}[1, 2, \ldots, 49]$. *Then every multiple of* $\phi(a_0 b_0) a_0$ *lies in* $\mathcal{D}$.

Granville, Kane, Koukoulopoulos and Lemke-Oliver [**3**] showed that $\mathcal{E}$ has lower asymptotic density at least $\frac{1}{354}$ and thus so does $\mathcal{D}$. We do not know how to prove a better lower bound for the density and leave this as an open problem.

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Central to the works [**4, 5, 6, 8**] is the concept of an admissible set of linear forms. For positive integers a_i and integers b_i, the set of affine-linear forms $a_1x + b_1, \dots, a_kx + b_k$ is admissible if, for every prime p, there is an $x \in \mathbb{Z}$ such that $p \nmid (a_1x + b_1)\cdots(a_kx + b_k)$.

Definition. Hypothesis DHL$[k, m]$ is the statement that for any admissible k-tuple of linear forms $a_in + b_i$, $1 \leqslant i \leqslant k$, for infinitely many n, at least m of them are simultaneously prime.

In this paper we are concerned with the statements DHL$[k, 2]$. The Polymath8b project [**6**], plus subsequent work of Maynard [**5**], established DHL$[50, 2]$ unconditionally.

The Elliott-Halberstam Conjecture implies DHL$[5, 2]$, see [**4**]. The Generalized Elliott-Halberstam Conjecture implies DHL$[3, 2]$ (see [**6**] for details).

Theorem 3. *We have*

(i) *DHL*$[3, 2]$ *implies that* $\mathcal{D} = \{2, 4, 6, 8, 10, \dots\}$*, the set of all positive even integers.*
(ii) *DHL*$[4, 2]$ *implies that* $\mathcal{D}$ *contains every positive multiple of 4.*
(iii) *DHL*$[5, 2]$ *implies that* $\min \mathcal{D} \leqslant 6$.
(iv) *DHL*$[6, 2]$ *implies* $8 \in \mathcal{D}$.

By contrast, for any $k \geqslant 2$, DHL$[k, 2]$ implies that $\liminf p_{n+1} - p_n \leqslant a_k$, where a_k is the minimum of $h_k - h_1$ over all admissible $k-$tuples $n + h_1, \dots, n + h_k$. We have $a_3 = 6$, $a_4 = 8$, $a_5 = 12$ and $a_6 = 16$.

We show parts of Theorem 3 (ii) and (iii) using a more general result.

Theorem 4. *Assume DHL*$[k, 2]$*, with* $k \geqslant 3$*. Also assume that there are integers* $1 < m_1 < \dots < m_k$ *and* $\ell_{i,j}$ *for* $1 \leqslant i < j \leqslant k$ *and such that*

$$\frac{\ell_{i,j} m_i}{m_j - m_i} \in \mathcal{V}, \quad \frac{\ell_{i,j} m_j}{m_j - m_i} \in \mathcal{V} \qquad (1 \leqslant i < j \leqslant k).$$

Then $L \leqslant 2 \max \ell_{i,j}$*. Moreover, if* $\ell_{i,j} = \ell$ *for all* i, j *then* $2\ell \in \mathcal{D}$.

In Section 3, we give a heuristic argument that there exist numbers $m_1, \dots, m_{50}$ satisfying the hypothesis of Theorem 4 with $\ell_{i,j} = 2$ for all $i < j$. In this case we achieve an unconditional proof that $4 \in \mathcal{D}$. Actually finding such m_i seems computationally difficult, however.

Does every arithmetic progression $a \bmod d$ containing even numbers have infinitely many elements of $\mathcal{D}$? We answer in the affirmative if $4|d$ and $4|a$. The case $a \equiv 2 \pmod 4$ is more difficult; see our Remarks following the proof in Section 4.

Theorem 5. *Let* a, d *be positive integers with* $4|a$*,* $4|d$*. Then the progressions* $a \bmod d$ *contains infinitely many elements of* $\mathcal{D}$.

Observe that, even assuming DHL$[3, 2]$, there is no specific progression $a \bmod d$, not containing 0, which is known to contain a number that is infinitely often the difference of two primes.

When $4|d$ and $a \equiv 2 \pmod 4$ we can sometimes show that $\mathcal{D}$ contains infinitely many elements that are $\equiv a \pmod d$; see the Remarks following the proof of Theorem 5 in Section 4.

2. Proof of Theorems 1–4

Proof of Theorem 1. Define

$$\begin{aligned} S_1 &= \{41, 43, 47, 53, 67, 71\}, \\ S_2 &= \{59, 61, 67, 71, 73, 83, 89, 101, 103, 107, 109, 113, 127, 131, 137, 139\}, \\ S_4 &= \{p \text{ prime} : 127 \leqslant p \leqslant 271\}. \end{aligned}$$

and consider the collection of 50 linear forms

$$n + a \ \ (a \in S_1), \quad 2n + a \ \ (a \in S_2), \quad 4n + a \ \ (n \in S_4).$$

This collection is admissible; indeed if $p < 41$ and $n = 0$ then all of them are coprime to p. For $p > 50$ it is clear that there is an n for which all of them are coprime to p. When $p \in \{41, 43, 47\}$ we take $n = 1, 3, 8$,respectively, and then all of the forms are coprime to p. By DHL$[50, 2]$, there are two of these forms that are simultaneously prime for infinitely many n. If both forms are of the type $n+a$ for $a \in S_1$, then this shows that $\min \mathcal{D} \leqslant 71-41 = 30$. Likewise, if both forms are of the type $2n + a$ for $a \in S_2$ then $\min \mathcal{D} \leqslant 139 - 59 = 80$ and if both forms are of the type $4n + a$ where $a \in S_4$, then $\min \mathcal{D} \leqslant 271 - 127 = 144$. Now suppose for infinitely many n, $n + a$ and $2n + b$ are both prime, where $a \in S_1$, $b \in S_2$. Then

$$\phi(4(n + a)) = 2n + 2a - 2, \quad \phi(2n + b) = 2n + b - 1,$$

which shows that $|b - 2a + 1| \in \mathcal{D}$. We have $b - 2a + 1 \neq 0$ for all choices, and the maximum of $|b - 2a + 1|$ is 82, and hence $\min \mathcal{D} \leqslant 82$. Similarly, if for infinitely many n, $2n + a$ and $4n + b$ are both prime, where $a \in S_2$, $b \in S_4$, then $|b - 2a + 1| \in \mathcal{D}$. Hence $\min \mathcal{D} \leqslant 154$. Finally, if for infinitely many n, $n + a$ and $4n + b$ are both prime, where $a \in S_1$, $b \in S_4$, then $|b - 4a + 3| \in \mathcal{D}$ since

$$\phi(8(n + a)) = 4(n + a - 1), \qquad \phi(4n + b) = 4n + b - 1.$$

In all cases $0 < |b - 4a + 3| \leqslant 154$. □

Proof of Theorem 2. Let

$$a_0 = \prod_{p \leqslant 47} p, \quad b_0 = \operatorname{lcm}[1, 2, \ldots, 49].$$

Let $k \in \mathbb{N}$ and consider the admissible set of linear forms $n + ka_0, n + 2ka_0, \ldots, n + 50ka_0$. Since DHL$[50, 2]$ holds, for any $k \in \mathbb{N}$ there exists $j_k \in \{1, \ldots, 49\}$ such that the equation

$$\phi(u) - \phi(v) = u - v = kj_k a_0$$

has infinitely many solutions in primes u, v. Since every prime factor of j_k is a divisor of $a_0 b_0 / j_k$, we have $\phi(a_0 b_0 l / j_k) = \phi(a_0 b_0 l)/j_k$ for any $l \in \mathbb{N}$. Therefore,

$$\phi(a_0 b_0 u / j_k) - \phi(a_0 b_0 v / j_k) = \phi(a_0 b_0)(\phi(u) - \phi(v))/j_k = \phi(a_0 b_0) a_0 k,$$

as required. □

Proof of Theorem 4. The set of forms $m_1n-1,\dots,m_kn-1$ is clearly admissible. By DHL$[k,2]$, for some pair $i<j$ and for infinitely many n , m_in-1 and m_jn-1 are prime. Let $\ell=\ell_{i,j}$ and suppose that x,y satisfy

$$\phi(x)=\frac{\ell m_i}{m_j-m_i},\qquad \phi(y)=\frac{\ell m_j}{m_j-m_i}.$$

Then for sufficiently large n

$$\phi(x(m_jn-1))-\phi(y(m_in-1))=\frac{-2\ell m_i+2\ell m_j}{m_j-m_i}=2\ell. \qquad \square$$

Proof of Theorem 3. (i) Let $h\in\mathbb{N}$ and consider the triple $\{n+1,n+2h+1,2n+2h+1\}$. This is admissible, since when $n=0$, all of the forms are odd, and similarly none are divisible by 3 for some $n\in\{0,1\}$. By DHL$[3,2]$, either (i) $n+1$ and $n+2h+1$ are infinitely often both prime, (ii) $n+1$ and $2n+2h+1$ are infinitely often both prime or (iii) $n+2h+1$ and $2n+2h+1$ are infinitely often both prime. In case (i) we have $\phi(n+2h+1)-\phi(n+1)=2h$, in case (ii) we have $\phi(2n+2h+1)-\phi(4(n+1))=2h$, and in case (iii) we have $\phi(4(n+2h+1))-\phi(2n+2h+1)=2h$.

(ii) The deduction $4\in\mathcal{D}$ follows from Theorem 4 using $\ell_{i,j}=2$ for all i,j and

$$\{m_1,\dots,m_4\}=\{6,8,9,12\}.$$

Now suppose that d is congruent to 0 or 4 modulo 12, and define a by $d=2(a+1)$. In particular, $(a,6)=1$. Thus, the set of forms m_in-a, $1\leqslant i\leqslant 4$, are admissible. By DHL$[4,2]$, for some pair $i<j$ and for infinitely many n , m_in-a and m_jn-a are prime. Suppose that x,y satisfy

$$\phi(x)=\frac{2m_i}{m_j-m_i},\qquad \phi(y)=\frac{2m_j}{m_j-m_i}.$$

Then for sufficiently large n

$$\phi(x(m_jn-a))-\phi(y(m_in-a))=2(a+1)=d. \qquad \square$$

Hence, $d\in\mathcal{D}$.

Finally, if $d\equiv 8 \pmod{12}$, write $d=2(b-1)$, so that $(b,6)=1$. Similarly, the set of forms m_in+b, $1\leqslant i\leqslant 4$, are admissible and we conclude that $d\in\mathcal{D}$.

(iii) Consider the admissible set of forms $\{f_1(n),\dots,f_5(n)\}=\{n,n+2,2n+1,4n-1,4n+3\}$. Indeed, if $n\equiv 11 \pmod{30}$ then all of the forms are coprime to 30. By DHL$[5,2]$, for some $i<j$ and infinitely many n, $f_i(n)$ and $f_j(n)$ are both prime. Say $f_i(n)=an+b$ and $f_j(n)=cn+d$ with $c/a\in\{1,2,4\}$. Then

$$\phi((2c/a)(an+b))=(c/a)(an+b-1)=cn+(c/a)(b-1)$$

and

$$\phi(cn+d)=cn+d-1.$$

Thus, $|(c/a)(b-1)-(d-1)|\in\mathcal{D}$. In all cases, $|(c/a)(b-1)-(d-1)|\leqslant 6$.

(iv) Use Theorem 4 with the set

$$\{h-72,h-66,h-64,h-63,h-60,h\},\quad h=120193920,$$

and $\ell_{i,j}=4$ for all i,j. We used PARI/GP to verify that $4m_i/(m_j-m_i)\in\mathcal{V}$ and $4m_j/(m_j-m_i)\in\mathcal{V}$ for all $i<j$.

3. A heuristic argument

In this section, we give an argument that there should exist $m_1, \dots, m_{50}$ satisfying the hypothesis of Theorem 4 with $\ell_{i,j} = 2$ for all i, j. We first give a general construction of numbers with $\frac{m_i}{m_j - m_i}$ all integers.

Lemma 3.1. *For any positive integer b and and $k \geqslant 2$ there is a set $\{m_1, \dots, m_k\}$ of positive integers with $m_1 < m_2 < \cdots < m_k$ and with*

$$b \Big| \frac{m_j}{m_j - m_i} \quad (1 \leqslant i < j \leqslant k). \tag{3.1}$$

Proof. Induction on k. When $k = 2$ take $\{m_1, m_2\} = \{2b - 1, 2b\}$. Now assume (3.1) holds for some $k \geqslant 2$. Let M be the least common multiple of the $\binom{k}{2}$ numbers

$$m_j - m_i \quad (1 \leqslant i < j \leqslant k),$$

and let K be the least common multiple of the numbers

$$M, bM - m_1, \dots, bM - m_k.$$

We claim that the set

$$\{m'_1, \dots, m'_{k+1}\} = \{Kb - bM + m_1, \dots, Kb - bM + m_k, Kb\}$$

satisfies (3.1). Indeed, when $1 \leqslant i < j \leqslant k$ we have

$$\frac{m'_j}{m'_j - m'_i} = \frac{Kb - bM + m_j}{m_j - m_i},$$

which by hypothesis is divisible by b. Finally, for any $i \leqslant k$,

$$\frac{m'_{k+1}}{m'_{k+1} - m'_i} = \frac{Kb}{bM - m_i},$$

which is also divisible by b. $\square$

Now let $b = \prod_{p \leqslant 2450} p$, and apply Lemma 3.1 with $k = 50$. There is a set $\{m_1, \dots, m_{50}\}$ such that for all $i < j$,

$$b \Big| \frac{m_j}{m_j - m_i}. \tag{3.2}$$

Let M be the least common multiple of the $\binom{50}{2}$ numbers

$$m_j - m_i \quad (1 \leqslant i < j \leqslant 50).$$

Then for any $h \in \mathbb{N}$, the set $\{m_1 + hbM, \dots, m_{50} + hbM\}$ has the same property (3.2). The collection of 2450 linear forms (in h)

$$\frac{2(m_i + hbM)}{m_j - m_i} + 1 = \frac{2(m_j + hbM)}{m_j - m_i} - 1 \qquad (1 \leqslant i < j \leqslant 50)$$

and

$$\frac{2(m_j + hbM)}{m_j - m_i} + 1 \qquad (1 \leqslant i < j \leqslant 50)$$

is admissible by (3.2), and the Prime k-tuples conjecture implies that all of these are prime for some h. We need only the existence of one h, and then the hypotheses of Theorem 4 hold with $\ell_{i,j} = 2$ for all i, j, and consequently $4 \in \mathcal{D}$. Discovering such an h, however, appears to be computationally infeasible.

4. Totient gaps in progressions: proof of Theorem 5

Lemma 4.1. *Suppose that $D \in \mathbb{N}$ and $4|a$. Then there exist v_1 and v_2 such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$ or $(v_1 + 1)(v_2 - 1) \equiv a \pmod{D}$.*

Proof. We use the Chinese Remainder Theorem. We will prove that if $D = p^\alpha$ is a prime power, then for $p \neq 3$ we can find a pair v_1 and v_2 such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$ and another pair v_1' and v_2' such that $(D, v_1') = (D, v_2') = 1$ and $(v_1' + 1)(v_2' - 1) \equiv a \pmod{D}$. If $p = 3$ then we will find appropriate v_1 and v_2 such that one of desired congruences hold. This will suffice for the proof of the lemma.

If$p \neq 3$ and for any a, $4|a$, a pair v_1 and v_2 exists such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv a \pmod{D}$, then there also exists a pair v_1' and v_2' as well. Indeed, take any possible D and a. Then, by our supposition, there are v_1 and v_2 such that $(D, v_1) = (D, v_2) = 1$ and $(v_1 - 1)(v_2 - 1) \equiv -a \pmod{D}$. Then for $v_1' = -v_1$ and $v_2' = v_2$ the desired congruence $(v_1' + 1)(v_2' - 1) \equiv a \pmod{D}$ holds.

Consider $p = 2$. Take $v_1 \equiv 3 \pmod{p^\alpha}$ and $v_2 \equiv \frac{a}{2} + 1 \mod p^\alpha$. Then $(v_1 - 1)(v_2 - 1) \equiv a \pmod{p^\alpha}$ and both v_1, v_2 are odd.

Consider $p = 3$. If $a \equiv 0 \pmod{3}$ or $a \equiv 1 \pmod{3}$, let $v_1 \equiv 2 \pmod{3^\alpha}$ and $v_2 \equiv a + 1 \pmod{3^\alpha}$. Then we have $(v_1 - 1)(v_2 - 1) \equiv a \mod 3^\alpha$. If $a \equiv -1 \pmod{3}$ let $v_1 \equiv -2 \pmod{3^\alpha}$ and $v_2 \equiv -a + 1 \pmod{3^\alpha}$. Then we have $(v_1 + 1)(v_2 - 1) \equiv a \pmod{3^\alpha}$.

Consider $p > 3$. Take v_2 so that $v_2 \notin \{0, 1, 1 - a\} \mod p$. Then there is some $v_1 \not\equiv 0 \pmod{p}$ such that $(v_2 - 1)(v_1 - 1) \equiv a \pmod{p^\alpha}$. □

Proof of Theorem 5. Let D be any positive integer satisfying

(a) $d|D$;
(b) $D, D^2, \ldots, D^{49}$ are all in $\mathcal{V}$.

For example, let P be the largest prime factor of d, γ sufficiently large and

$$D = d \prod_{p \leqslant P} p^\gamma.$$

Indeed, if

$$D = \prod_{p \leqslant P} p^{\alpha(p)}, \qquad \prod_{p \leqslant P} (p - 1) = \prod_{p \leqslant P} p^{\beta(p)},$$

then, assuming $\gamma \geqslant \max \beta(p)$, for all $j \geqslant 1$ we have

$$\phi\Bigg(\prod_{p \leqslant P} p^{j\alpha(p) - \beta(p) + 1} \Bigg) = D^j.$$

Now take any D satisfying (a) and (b) above, and let v be coprime to D. Then the set

$$f_j(x) = D^j x - v, \quad j = 1, \ldots, 50,$$

of linear forms is admissible. Indeed, if $p \nmid v$, then $f_1(0)\cdots f_{50}(0) = v^{50} \not\equiv 0 \pmod p$, and if $p|v$ then $p \nmid D$ and $f_1(1)\cdots f_k(1) \equiv D^{1225} \not\equiv 0 \pmod p$. Since DHL$[50,2]$ holds, there are $j_1 < j_2$ such that for infinitely many positive integers x both numbers $p_1 = D^{j_1}x - v$, $p_2 = D^{j_2}x - v$ are primes. Denote $j = j_2 - j_1$. There exists l such that $\phi(l) = D^j$. If x is large enough, then $(p_1, l) = (p_2, l) = 1$. We have

$$\phi(p_2) - \phi(p_1 l) = (p_2 - 1) - (p_1 - 1)D^j = (v+1)(D^j - 1). \tag{4.1}$$

Let v_1, v_2 be as in Lemma 4.1, and let v satisfy

$$\begin{cases} v \equiv -v_1 \pmod D, v > 0 & \text{if } (v_1 - 1)(v_2 - 1) \equiv a \pmod D, \\ v \equiv v_1 \pmod D, v < -1 & \text{otherwise.} \end{cases}$$

Fix a prime $q \equiv v_2 \bmod D$ with $(q, l) = 1$, and assume that $p_1, p_2 > q$. Then

$$\phi(p_2 q) - \phi(p_1 l q) = (q-1)(v+1)(D^j - 1). \tag{4.2}$$

Thus, $|(q-1)(v+1)(D^j-1)| \in \mathcal{D}$. The right side of (4.2) is $\equiv (q-1)(-v-1) \equiv (v_2-1)(-v-1) \pmod D$ and has sign equal to the sign of v. If $(v_1 - 1)(v_2 - 1) \equiv a \pmod D$, then $v > 0$ and thus the right side of (4.2) is positive and congruent to $(v_2 - 1)(v_1 - 1) \equiv a \pmod D$. Otherwise, $v < 0$ and the right side of (4.2) is negative and congruent to $(v_2 - 1)(-v_1 - 1) \equiv -a \pmod D$. By varying q, we find that there are infinitely many elements of $\mathcal{D}$ in the residue class $a \bmod d$. □

Remarks. Equation (4.1) holds for any v coprime to D. Thus, if $a \equiv 2 \pmod 4$ and either $(a+1, D) = 1$ or $(a-1, D) = 1$ then the residue class $a \mod d$ contains infinitely many elements of $\mathcal{D}$; take $v \equiv -a - 1 \pmod d, v > 0$ if $(a+1, D) = 1$ and $v \equiv a - 1 \pmod d, v < -1$ if $(a-1, D) = 1$. Thus, if d has at most two distinct prime factors, and (b) holds for some D composed only of the primes dividing d, then every residue class $a \mod d$, with $2|a$ contains infinitely many elements of $\mathcal{D}$. Note that in this case, for all a with $2|a$, either $(a+1, d) = 1$ or $(a-1, d) = 1$. In particular this holds with d of the form 2^k, $2^k 3^\ell$, or $2^k 5^\ell$ with $k \geqslant 2$, since in each case (a) and (b) hold with $D = d$. Item (b) also holds with $d = D = 28$ (verified with PARI/GP). We do not know how to derive the same conclusion if d has 3 or more prime factors, e.g. $d = 60$.

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Department of Mathematics, 1409 West Green Street, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
Email address: ford@math.uiuc.edu

Steklov Institute of Mathematics, 8 Gubkin Street, Moscow, 119991, Russia
Email address: konyagin@mi-ras.ru