2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)

A Matrix Trickle-Down Theorem on Simplicial Complexes and Applications to Sampling Colorings

Dorna Abdolazimi
Computer Science & Engineering
University of Washington
Seattle, WA, USA
dornaa@cs.washington.edu

Kuikui Liu
Computer Science & Engineering
University of Washington
Seattle, WA, USA
liukui17@cs.washington.edu

Shayan Oveis Gharan
Computer Science & Engineering
University of Washington
Seattle, WA, USA
shayan@cs.washington.edu

***Abstract*—We show that the natural Glauber dynamics mixes rapidly and generates a random proper *edge*-coloring of a graph with maximum degree Δ whenever the number of colors is at least $q \geq \left(\frac{10}{3} + \epsilon\right)\Delta$, where $\epsilon > 0$ is arbitrary and the maximum degree satisfies $\Delta \geq C$ for a constant $C = C(\epsilon)$ depending only on ϵ. For edge-colorings, this improves upon prior work [Vig99; Che+19] which show rapid mixing when $q \geq \left(\frac{11}{3} - \epsilon_0\right)\Delta$, where $\epsilon_0 \approx 10^{-5}$ is a small fixed constant. At the heart of our proof, we establish a matrix trickle-down theorem, generalizing Oppenheim's influential result, as a new technique to prove that a high dimensional simplicial complex is a local spectral expander.**

I. Introduction

Given an (undirected) graph $G = (V, E)$ with $n = |V|$ vertices and with maximum degree $\Delta \geq 1$ can we generate a uniformly random proper coloring of vertices of G using q colors? For $q \leq \Delta$, there is no efficient algorithm (in the sense of an FPRAS) to approximately count proper q-colorings (at least when q is even) unless $\mathsf{NP} = \mathsf{RP}$, even for Δ-regular graphs which are triangle-free [GŠV15]. This fundamental question in the field of counting and sampling has puzzled researchers for decades. One can study a natural Markov Chain (MC), known as the Glauber dynamics, to generate a random proper coloring: Given a proper vertex coloring of G, choose a uniformly random vertex $v \in G$ and re-color v, namely choose a uniformly random color which is not present in any of the neighbors of v.

It is not hard to see that for $q \geq \Delta + 2$ this chain is irreducible and has unique stationary distribution which is uniform over all proper colorings of G. It is conjectured that the Glauber dynamics mixes in time $O(n \log n)$ for q as low as $\Delta + 2$ but despite significant attempts we are still very far from proving this conjecture.

To this date, the best known result for general graphs is due Chen, Delcourt, Moitra, Perarnau and Postle [Che+19] who show that the Glauber dynamics mixes in polynomial time for $q \geq (11/6 - \epsilon)\Delta$ for some universal constant $\epsilon > 0$; this slightly improves on the classical works of Jerrum and Vigoda [Jer95; Vig99] which bounds the mixing time by a polynomial in n for $q \geq (11/6)\Delta$.

Most of the recent analyses of the Glauber dynamics are focused on "locally sparse" graphs [HV03; Mol04; HV05; FV06; FV07; HVV07; Dye+13; Che+21; Fen+21] where it was typically shown how to break the $11/6$ barrier bound when the underlying graph has a large girth. We note that these assumptions are typically very strong as it can be seen that triangle graph free graphs can be colored with as little as $O(\Delta/\log\Delta)$ many colors [Joh96].

The results on locally sparse graphs typically exploit (strong) correlation decay properties: Roughly speaking, they imply that if we color a vertex v with a color c, then the marginal probability of coloring a "far away" vertex u with a color c' does not change, or changes very mildly. Although it is conjectured that vertex coloring exhibits correlation decay, more formally known as "strong spatial mixing" property, for $q \geq \Delta + O(1)$, to this date, we are lacking techniques to establish such a statement (see e.g., [GMP05; Yin14; GKM15; Eft+19]).

In this paper, we study random proper edge coloring of graphs, which equivalently can be seen as a random proper vertex coloring of line graphs. Unlike most recent trends which focus on sparse graphs with large girths, line graphs are very dense locally as they contain induced cliques of size $\Omega(\Delta)$. To the best of our knowledge, the only previous result on edge coloring which goes significantly beyond the $11/6$ barrier is recent work of Delcourt, Heinrich and Perarnau [DHP20] which shows that the Glauber dynamics mixes rapidly when the underlying graph is a tree and $q \geq \Delta + 1$. Note that for a graph with maximum degree Δ, the maximum degree of the line graph could be as large as 2Δ. Therefore, with the $11/6$ barrier, one would need $q \geq 11/3\Delta$ to guarantee polynomial mixing for all edge coloring instances. In our main theorem we prove that this barrier can be broken for edge coloring of any graph with maximum degree Δ.

Theorem I.1. *Let $G = (V, E)$ be a graph of maximum degree Δ. For any $0 < \epsilon \leq \frac{1}{10}$ such that $\frac{\ln^2 \Delta}{\Delta} \leq \frac{\epsilon^3}{15}$, and any collection of color lists $L = \{L(e)\}_{e \in E}$ satisfying $|L(e)| \geq \Delta(e) + (4/3 + 4\epsilon)\Delta$ where $\Delta(e)$ is the number of*

2575-8454/21/$31.00 ©2021 IEEE
DOI 10.1109/FOCS52979.2021.00024

neighbors of e in the line graph of G, the spectral gap of the Glauber dyanmics for sampling proper L-edge-list-colorings on G is $\Omega(n^{-O(1/\epsilon)})$, so the mixing time is $O(n^{O(1/\epsilon)})$. Furthermore, if $\Delta \leq O(1)$, the modified and standard log-Sobolev constants are $\Omega_{\epsilon,\Delta}(1/n)$.

We remark that our general mixing time bound has no dependence on Δ or q. So, the algorithm runs in polynomial time even for graphs of unbounded degree. In a second contribution we show that for any list vertex coloring instance where G is a tree with max degree Δ, and the size of the list of every vertex v is at least $\Delta(v) + \epsilon\Delta$ for $\epsilon = \Omega(\frac{\ln \Delta}{\sqrt{\Delta}})$ the Glauber dynamics mixes rapidly and generates a uniformly random vertex coloring of G. The precise statement of this result is included in the extended version. Although our theorem is not as strong as [MSW04], it shows that Glauber dynamics mixes rapidly even when we have a list coloring problem on a tree and furthermore it gives a possible avenue to exploit our techniques to prove that Glauber dynamics mixes rapidly on any graph when $q \geq (1+\epsilon)\Delta$. We expect that upon further investigation our techniques can be coupled with the extensive literature on random proper colorings of graphs with large girth to break the $11/6 - \epsilon$ barrier.

To establish the above results, we view the Glauber dynamics as a high dimensional walk on a simplicial complex and we prove that this complex is a "local spectral expander" (see Definition I.2 below). By local-to-global theorems, local spectral expansion immediately implies a bound on the spectral gap of the Glauber dynamics and therefore on its mixing time (see Section II-E and Section II-B below). A simplicial complex X is a downward closed collection of sets. Given a graph G and a set of q colors, we build a simplicial complex as follows: We let each maximal set in X be a set of vertex color pairs that denote a proper coloring of vertices of G, then we add all subsets of the maximal sets to X to make it downward closed. By viewing the Glauber dynamics can be viewed as a walk on the maximal sets in the coloring complex X. Unlike recent developments [ALO20; Che+21; Fen+21; CLV20; CLV21; Fri+21] which exploit the correlation decay property to prove local spectral expansion, our proof is a direct method: We use an inductive argument inspired by the Oppehheim's trickle-down theorem to bound local eigenvalues. We expect our technique to find more applications in analysis of Markov chains as well as in other applications of simplicial complexes.

A. Main Technical Contributions

A simplicial complex X on a finite ground set U is a downwards closed collection of subsets of U, i.e. if $\tau \in X$ and $\sigma \subset \tau \subseteq U$, then $\sigma \in X$. The elements of X are called faces, and then maximal faces are called facets. We say a face τ is of dimension k if $|\tau| = k+1$ and write $\dim(\tau) = k$. We write $X(k) = \{\tau \in X : \dim(\tau) = k\}$. We say X is a pure d-dimensional complex if every facet has dimension d. In this paper we only work with pure simplicial complexes. To keep the notation concise, let $X(\leq i) := X(-1) \cup \cdots \cup X(i)$. The co-dimension of a face τ is defined as $\operatorname{codim}(\tau) := d - \dim(\tau)$. For a face τ, define the link of τ as the simplicial complex $X_\tau = \{\sigma \setminus \tau : \sigma \in X, \sigma \supset \tau\}$. Note that X_τ is a $(\operatorname{codim}(\tau) - 1)$- dimensional complex. Let $\pi_{X,d}$ be a distribution on the maximal faces of a pure d-dimensional simplicial complex X; we may drop X from the subscript if it is clear in the context. We will refer to the pair (X, π_d) as a weighted simplicial complex. For each face τ and each integer $-1 \leq i \leq \operatorname{codim}(\tau) - 1$, we write $\pi_{\tau,i}$ for the induced distribution on $X_\tau(i)$ given by

$$\pi_{\tau,i}(\eta) = \frac{1}{\binom{\operatorname{codim}(\tau)}{i+1}} \Pr_{\sigma \sim \pi_d}[\sigma \supset \eta \mid \sigma \supset \tau]. \tag{1}$$

One should view this as a marginal distribution conditioned on τ. We will often view $\pi_{\tau,i}$ as a vector in $\mathbb{R}_{\geq 0}^{X_\tau(i)}$. We remove the subscript τ when $\tau = \emptyset$. We also omit i from the subscript when $i = 0$, i.e $\pi_\tau := \pi_{\tau,0}$.

Given a weighted simplicial complex (X, π_d), there is a natural Markov chain known as the down-up walk (or high-order walk) [KM17; DK17; KO18b] whose stationary distribution is π_d. Starting at a facet $\sigma \in X(d)$, we transition to the next facet $\sigma' \in X(d)$ via the following two-step process:

1) Select a uniformly random element $x \in \sigma$ and remove x from σ.
2) Select a random $\sigma' \in X(d)$ containing $\sigma \setminus \{x\}$ with probability proportional to $\pi_d(\sigma')$.

If $P_d^\vee$ denotes the transition probability matrix of this Markov chain, then we may write down its entries as follows.

$$P_d^\vee(\sigma, \sigma') = \begin{cases} \frac{1}{d+1}\pi_{\sigma\cap\sigma',0}(\sigma' \setminus \sigma) & \text{if } |\sigma \cap \sigma'| = d, \\ 0 & \text{otherwise.} \end{cases}$$

One of the beautiful insights of [KM17; DK17; KO18b] is that spectral properties of $P_d^\vee$ may be studied through spectral properties of "local walks", which are defined using only pairwise conditional marginals and are more tractable to analyze. Fix a face τ of co-dimension $k \geq 2$; the local walk for τ is a Markov chain on $X_\tau(0)$ with transition probability matrix $P_{X,\tau} \in \mathbb{R}_{\geq 0}^{X_\tau(0)\times X_\tau(0)}$ described by

$$\begin{aligned} P_{X,\tau}(x,y) &= \frac{\pi_{\tau,1}(x,y)}{2\pi_\tau(x)} = \pi_{\tau\cup\{x\}}(y) \\ &= \frac{1}{k-1} \Pr_{\sigma\sim\pi_d}[y \in \sigma \mid \sigma \supset \tau \cup \{x\}]. \end{aligned} \tag{2}$$

for distinct $x, y \in X_\tau(0)$; we note that $P_{X,\tau}$ has zero diagonal. We drop X in the subscript when it is clear in the context. Note that, by definition, the walk is reversible with stationary distribution π_τ. Furthermore, the row corresponding to element $x \in \operatorname{supp} \pi_\tau$ is precisely $\pi_{\tau\cup\{x\}}$. For our calculations, it is easier to work with matrices $\{P_\tau\}_{X(\leq d-2)}$ if they all live in $\mathbb{R}^{X(0)\times X(0)}$. Therefore, we modify the definition of P_τ as follows: for any $\tau \in X(\leq d-2)$, let $P_\tau \in \mathbb{R}^{X(0)\times X(0)}$ be supported on $X_\tau(0) \times X_\tau(0)$ block and equal to the transition probability matrix of the local walk for τ on this block. Similarly, we often need to see π_τ as a probability distribution over $X(0)$ supported on $X_\tau(0)$ entries. Therefore, π_τ can be seen as a vector in $\mathbb{R}^{X(0)}$. Furthermore, define $\Pi_\tau \in \mathbb{R}^{X(0)\times X(0)}$ as $\Pi_\tau := \operatorname{diag}(\pi_\tau)$. We proceed by introducing some terminology related to the local walks. We say X is totally connected if $\lambda_2(P_\tau) < 1$ for any $\tau \in X(\leq d-2)$, i.e. the local walk for τ is irreducible/connected for any face τ of co-dimension at least 2. Furthermore, we define local spectral expansion as follows.

Definition I.2 (Local Spectral Expansion [DK17; KO18b; Dik+18])**.** *We say a weighted simplicial complex (X, π_d) of dimension-d is a $(\gamma_{-1}, \ldots, \gamma_{d-2})$-local spectral expander if for every k and every $\tau \in X(k)$, $\lambda_2(P_\tau) \leq \gamma_k$.*

Remark I.3. We only work with this "one-sided" version of local spectral expansion. Many other works [KO18b; Opp18; KO18a] consider the stronger two-sided local spectral expansion, which further imposes a lower bound on the smallest eigenvalue of the local walks P_τ.

In almost all applications of high dimensional simplicial complexes, one first needs to to prove that the underlying complex is a local spectral expander and then exploit "local-to-global" theorems to prove global properties of the underlying complex X. Perhaps, the main generic technique to prove that a given complex X is a local spectral expander is the Oppenheim "trickle-down method" [Opp18], where one can show that if $\gamma_{d-2} \ll \frac{1}{d}$ and X is "well connected" then all γ_i's are at most $2\gamma_{d-2}$.

In our main technical contribution, we give recursive framework for bounding the local spectral expansion of a weighted simplicial complex X which significantly generalizes Oppenheim's result.

Theorem I.4 (Inductive Matrix Trickle-Down Method)**.** *Let (X, π_d) be a totally connected weighted simplicial complex. Suppose $\{M_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau \in X(\leq d-2)}$ is a family of symmetric matrices satisfying the following:*

1) ***Base Case:*** *For every τ of co-dimension 2, we have the spectral inequality*
$$\Pi_\tau P_\tau - 2\pi_\tau \pi_\tau^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.$$
2) ***Recursive Condition:*** *For every τ of co-dimension at least $k \geq 3$, M_τ satisfies*
$$M_\tau \preceq \frac{k-1}{2k-1}\Pi_\tau \quad \textit{and}$$
$$\mathbb{E}_{x\sim\pi_\tau} M_{\tau\cup\{x\}} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau.$$

Then $\lambda_2(P_\tau) \leq \rho(\Pi_\tau^{-1} M_\tau)$ for all $\tau \in X(\leq d-2)$, where ρ represents the spectral radius. In particular, (X, π_d) is a $(\mu_{-1}, \ldots, \mu_{d-2})$-local spectral expander with $\mu_k = \max_{\tau\in X(k)} \rho(\Pi_\tau^{-1} M_\tau)$.

Remark I.5. The original "trickle-down method" is a special case of our result by taking the matrix M_τ to be a multiple of Π_τ. We justify this formally through a quick calculation in the extended version of the paper.

Remark I.6. It also turns out for our applications to proper colorings we will need a slight extension of the above theorem. However, one should take the above theorem as the heart of our technical contributions. See Theorem III.5 below and the surrounding discussion for more details on the slight extension.

Typically, it is not difficult to construct some family of matrices $\{M_\tau\}$ satisfying the assumptions of Theorem I.4. The key challenge is choose M_τ's such that one can bound $\rho(\Pi_\tau^{-1} M_\tau) \leq O(1/\operatorname{codim}(\tau))$. One of our second key insights is that the matrices M_τ can be designed to have convenient sparsity patterns depending on (X, π_d) which allow for straightforward bounds on $\rho(\Pi_\tau^{-1} M_\tau)$. For instance, in our application to proper colorings, our matrices M_τ will have rows and columns corresponding to vertex-color pairs vc, and they will be supported on the "proper coloring constraints", namely pairs uc, vc' of vertex-color pairs where the vertices u, v are neighbors and the two colors c, c' are identical. We demonstrate the usefulness of this approach on sampling proper colorings in graphs below.

The rest of the paper is organized as follows. In Section III, we prove Theorem I.4 and its extension. Then we apply these theorems to analyze the Glauber dynamics for sampling a graph coloring. First, to demonstrate our approach better, we focus on vertex-coloring in in Section IV. We start by restricting our attention to when the matrix bounds $\{M_\tau\}_{\tau\in X:\operatorname{codim}(\tau)\geq 2}$ in Theorem I.4 are diagonal matrices. Using diagonal matrix bounds, we show the essence of our technique while retrieving a known result for vertex-coloring. Finally, in Section V, we prove Theorem I.1, our main result for edge-coloring.

II. Preliminaries

First, we fix some notational conventions. When it is clear from context, we write a to denote a singleton $\{a\}$. When we want to add a subscript b to an object denoted by x_a, we write $x_{a,b}$, i.e. $x_{a,b} := (x_a)_b$. We use the same convention for superscripts. Throughout the paper, adding or removing $\emptyset$ as a subscript does not change the object. For an integer q, we denote $\{1, \ldots, q\}$ by $[q]$. For a function $f : D \to R$ and a $D' \subseteq D$, we write $f|_{D'}$ to denote the restriction of f to domain D'.

Matrices and Vectors: Given a set S, we write $v \in \mathbb{R}^S$ and $A \in \mathbb{R}^{S\times S}$ to respectively denote a vector and a matrix indexed by S. We see a probability distribution p over a set S as a vector $p \in \mathbb{R}^S_{\geq 0}$. For a $n \times n$ matrix A, with eigenvalues $\lambda_1, \ldots, \lambda_n$, we write $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$ to denote the spectral radius of A and $\|A\|_\infty = \max_{1\le i\le n}\{\sum_{j=1}^n |A_{ij}|\}$. We write $\preceq$ to denote the Loewner order, i.e. for any symmetric matrices $A, B \in \mathbb{R}^{S\times S}$, we write $A \preceq B$ if $B-A$ is positive semidefinite. For any matrix $A \in \mathbb{R}^{S\times S}$ and any $S' \subseteq S$, $A^{S'} \in \mathbb{R}^{S\times S}$ is defined to be $A^{S'}(x,y) = A(x,y)$ for $x, y \in S'$, and 0 on all other entries. We say a matrix A is diagonal if $A(x,y) = 0$ whenever $x \neq y$. We say matrix $A \in \mathbb{R}^{S\times S}$ is hollow if $A(x,x) = 0$ for all $x \in S$. For any matrix A define A^+ as $A^+(x,y) \coloneqq \max\{0, A(x,y)\}$ and $A^- \coloneqq A - A^+$. Furthermore, $\mathrm{diag}(A)$ is a diagonal matrix whose diagonal elements match those of A, i.e. $A(x,x) = \mathrm{diag}(A)(x,x)$. We define off-$\mathrm{diag}(A) \coloneqq A - \mathrm{diag}(A)$.

Graphs: Fix a graph $G = (V,E)$. We denote the degree of each vertex v by $\Delta_G(v)$ and the maximum degree of the graph by Δ_G. Let $n_G \coloneqq |V|$ and $m_G \coloneqq |E|$. Furthermore, let $G[U]$ be the induced subgraph of G on $U \subseteq V$. We may drop the subscripts when it is clear from context. For any $u, v \in V$, we write $u \sim v$ if $\{u,v\} \in E$. Furthermore, for any $e \in E$, we write $e \sim v$ whenever vertex v is an endpoint of e, and for an edge $f \neq e$ we write $e \sim f$ when e and f share an endpoint. For an edge $e = \{u,v\}$, define $\Delta(e)$ as the number of edges that share an endpoint with e, i.e. $\Delta(e) \coloneqq \Delta(u) + \Delta(v) - 2$. The line graph $L(G)$ of a graph G is defined as follows: every vertex in $L(G)$ corresponds to an edge in G and there is an edge between two vertices in $L(G)$ if their corresponding edges in G share an endpoint.

A. Linear Algebra

Fact II.1. *For any symmetric matrix $A \in \mathbb{R}^{S\times S}$ where $A_{i,j} \neq 0$ only for $i, j \in S' \subseteq S$, we have $A \preceq \|A\|_\infty I^{S'}$.*

Fact II.2. *For matrices $A, B \in \mathbb{R}^{n\times n}$ and positive $\epsilon > 0$, we have the inequalities $AB + BA \preceq \epsilon A^2 + \frac{1}{\epsilon}B^2$ and $(A+B)^2 \preceq (1+\epsilon)A^2 + (1+1/\epsilon)B^2$.*

Lemma II.3. *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices such that $A\cdot(I-\alpha A) \preceq B\cdot(I-\alpha B)$ for a positive real number $\alpha > 0$. If $A, B \preceq \frac{1}{2\alpha}\cdot I$, then $A \preceq B$. Note that we crucially do not require $A, B \succeq 0$.*

The above facts and Lemma II.3 are proved in the extended version of the paper.

B. Markov Chains

Let P be the transition probability matrix of a Markov chain on a finite state space Ω with stationary distribution π. We say P is irreducible if P is connected. We say P is reversible w.r.t. π if for all $x, y \in \Omega$, we have $\pi(x)P(x,y) = \pi(y)P(y,x)$. In this case, the matrix P becomes self-adjoint w.r.t. the natural inner product $\langle\cdot,\cdot\rangle_\pi$ induced by π on $\mathbb{R}^\Omega$ given by $\langle\phi,\psi\rangle_\pi = \mathbb{E}_\pi[\phi\psi]$.

Throughout, we only work with irreducible reversible Markov chains. We will be interested in the mixing of our Markov chains, which quantifies the rate of convergence to stationarity. Specifically, for an initial starting distribution μ on Ω and error parameter $\epsilon > 0$, define

$$t_{\mathrm{mix}}(P,\epsilon,\mu) \stackrel{\mathrm{def}}{=} \min\{t \ge 0 : \left\|\mu P^t - \pi\right\|_{\mathrm{TV}} \le \epsilon\},$$

where $\|\mu-\nu\|_{\mathrm{TV}} = \frac{1}{2}\sum_{x\in\Omega}|\mu(x)-\nu(x)|$ gives the total variation distance between two distributions μ, ν on Ω. We write $t_{\mathrm{mix}}(P,\epsilon) = \sup_\mu t_{\mathrm{mix}}(P,\epsilon,\mu)$, where the supremum is over all possible starting distributions μ. The mixing time of P is defined as $t_{\mathrm{mix}}(P) = t_{\mathrm{mix}}(P, 1/4)$. It is well-known that the mixing time is controlled by various constants arising from classical functional inequalities. To introduce one of these bounds, we first define $\mathcal{E}_P(\phi,\psi) = \langle\phi,(I-P)\psi\rangle_\pi$ for the Dirichlet form of P, and $\mathrm{Var}_\pi(\phi) = \mathbb{E}_\pi[\phi^2] - \mathbb{E}_\pi[\phi]^2$ for the variance of ϕ w.r.t. π. With these in hand, we define the spectral gap of P as $\lambda(P) \stackrel{\mathrm{def}}{=} \inf_{f\neq 0}\frac{\mathcal{E}(f,f)}{\mathrm{Var}_\pi(f)}$.

Proposition II.4. *For an irreducible reversible Markov chain P on a finite state space Ω with stationary distribution π, we have the following inequality:*

$$t_{\mathrm{mix}}(P) \le O\left(\frac{1}{\lambda(P)}\log\frac{1}{\pi_{\min}}\right) \qquad \text{([LPW17])}$$

C. Products of Weighted Simplicial Complexes

Given two pure simplicial complexes X, Y of dimensions d_X, d_Y respectively with disjoint ground sets, we may form another pure simplicial complex Z of dimension-d_Z for $d_Z \coloneqq d_X + d_Y + 1$ called the product $X \times Y$ of X, Y by taking the ground set of $X \times Y$ to be the disjoint union of the ground sets of X, Y, taking the facets of $X \times Y$ to be of the form $\tau \cup \sigma$, where τ, σ are facets of X, Y respectively, and then taking downwards closure. If π_{X,d_X}, π_{Y,d_Y} are distributions on the facets of X, Y respectively, we then form corresponding product distribution $\pi_{Z,d_Z} = \pi_{X,d_X} \times \pi_{Y,d_Y}$ on the facets of $X \times Y$ by taking $\pi_{Z,d_Z}(\tau\cup\sigma) = \pi_{X,d_X}(\tau)\cdot\pi_{Y,d_Y}(\sigma)$ for facets τ, σ of X, Y respectively. A natural example of product of simplicial complexes is the vertex-coloring complex of a disconnected graph. Say $G = (V,E)$ is a graph with n vertex and ℓ connected components $G[U_1], \ldots, G[U_\ell]$ and associated complexes $X_1, \ldots X_\ell$. If (X, π_{n-1}) is the complex associated with G and π_{n-1} is the uniform distribution over its facets, then we can write $(X, \pi_{n-1}) = (X_1, \mu_1) \times \cdots \times (X_l, \mu_\ell)$, where μ_i is the uniform distribution over facets of X_i. Suppose we have associated a matrix $A_\tau \in \mathbb{R}^{X(0)\times X(0)}$ to any non-empty face τ of co-dimension at least 2 and assume that for any $1 \le i \le \ell$, when τ_{-i} and σ_{-i} are two arbitrary colorings of all connected components except i, then $A_{\tau_{-i}} = A_{\sigma_{-i}}$. We associate a block-diagonal matrix

$$f_\times(X, \{A_\tau\}_{\emptyset\subsetneq\tau\in X(\le n-3)}) \coloneqq \sum_{1\le i\le\ell:|U_i|\neq 1} A_{\tau_{-i}}.$$

When X is the edge-coloring complex of a graph G, it is the vertex-coloring complex of the line graph of G, therefore $f_\times(X, \{A_\tau\}_{\emptyset\subsetneq\tau\in X(\le n-3)})$ is given by the above definition.

D. Garland's Method

We will need the following simple facts, which follow simply by applying the Law of Total Probability appropriately and using the definition of the local walks P and local distributions π. Nearly identical equations were first observed and found to be useful by Garland [Gar73] in the context of understanding cohomology of simplicial complexes. They also lie at the heart of understanding expansion phenomena, in particular local spectral expansion, in simplicial complexes [Opp18; KO18b].

Lemma II.5 ([Opp18])**.** *Given a weighted simplicial complex (X, π_d), we may decompose ΠP as*

$$\Pi P = \mathbb{E}_{x\sim\pi}\Pi_x P_x$$

The proof of the above lemma is straightforward and can be found in the extended version of the paper.

Lemma II.6 ([Opp18])**.** *Given a weighted simplicial complex (X, π_d), we may decompose ΠP^2 as*

$$\Pi P^2 = \mathbb{E}_{x\sim\pi}\pi_x\pi_x^\top$$

Proof: The main observation is that the rows of P are precisely the vectors π_x, and that the rows of ΠP are precisely the vectors $\pi(x)\pi_x$. The claim immediately follows. ■

E. Local-to-Global Theorems

As alluded to earlier, one of the beautiful insights of [DK17; KO18b] is that local spectral expansion implies quantitative bounds on the spectral gap of the down-up walk.

Theorem II.7 ([AL20])**.** *Let (X, π_d) be a $(\gamma_{-1}, \ldots, \gamma_{d-2})$-local spectral expander. Then the down-up walk which samples from π_d has spectral gap lower bounded by*

$$\lambda(P_d^\vee) \geq \frac{1}{d}\prod_{j=-1}^{d-2}(1-\gamma_j)$$

Analogous results have also been proved for the decay of entropy [CLV21; GM20; Ali+21]. We state this result here since our main technical result Theorem I.4 is a general method to obtain local spectral expansion for any weighted simplicial complex.

Theorem II.7 has been used successfully in several prior works [Ana+19; ALO20; CLV20; Che+21; Fen+21; Ana+21] to establish polynomial time mixing for various dynamics. However, the exponent of these running times are typically large, depending sensitively on the constants in the γ_j. Recently, it was shown that for weighted simplicial complexes arising from spin systems on bounded-degree graphs, we can do significantly better. We will need the following to establish optimal mixing times in our applications. We state it specifically for colorings, as we do not analyze any other spin systems in this paper.

Theorem II.8 ([CLV21; Bla+21])**.** *Let (X, π_{n-1}) be a weighted simplicial complex arising from the uniform distribution over proper list-colorings of a graph $G = (V, E)$ with $|V| = n$ and maximum degree Δ. If for some constant C independent of n, (X, π_{n-1}) is a $(\gamma_{-1}, \ldots, \gamma_{n-3})$-local spectral expander with $\gamma_k \leq \frac{C}{n-k-1}$ for all k, then the spectral gap, standard and modified log-Sobolev constants are all $\Omega_{C,\Delta}(1/n)$. In particular, the Glauber dynamics mixes in $O_{C,\Delta}(n \log n)$ steps.*

III. A General Matrix Trickle-Down Method

Our goal in this section is to prove Theorem I.4. Towards this, we first elucidate the original "trickle-down method" of Oppenheim [Opp18], which was the beautiful realization that one can bound the second eigenvalue of the local walk P in terms of the second eigenvalues of the local walks $\{P_x\}_{x\in X(0)}$. This "trickle-down" phenomenon naturally yields an inductive method of bounding the second eigenvalues of all local walks. We formalize this as follows.

Theorem III.1 (Trickle Down Theorem [Opp18])**.** *Given a weighted simplicial complex (X, π_d), suppose the following holds:*

1) ***Connectivity:*** *$\lambda_2(P) < 1$, i.e. the local walk P is connected/irreducible.*
2) ***Spectral Bound for Links Above:*** *For some $0 \leq \lambda \leq 1/2$, we have the bound $\lambda_2(P_x) \leq \lambda$ for all $i \in X(0)$.*

Then the local walk P actually satisfies the spectral bound $\lambda_2(P) \leq \frac{\lambda}{1-\lambda}$.

Remark III.2. The original statement in [Opp18] does not have the assumption $\lambda \leq 1/2$, but the two are completely equivalent, since if $\lambda > 1/2$, then $\frac{\lambda}{1-\lambda} > 1$, making the statement vacuously true.

In particular, suppose we know that the top-dimensional links have local walks with second eigenvalue upper bounded by $1/(d+1)$. Then by applying Theorem III.1 in a totally blackbox fashion, it immediately follows that (X, π_d) is a $\left(\frac{1}{2}, \ldots, \frac{1}{d+1}\right)$-local spectral expander and its high-order walk has spectral gap at least $\Omega(1/d^2)$. This result has already had many applications, such as the construction of bounded-degree high-dimensional expanders [KO18a] and sampling algorithms for matroids [Ana+19; CGM19; Ana+21] and other combinatorial structures [AL20].

Unfortunately, when X is the simplicial complex of proper (partial) colorings, and π_{n-1} is the uniform distribution over proper colorings, this standard trickle-down method is not enough. It turns out that in the worst case, the second largest eigenvalue of the local walk of a top-dimensional link is $\Theta\left(\frac{1}{q-\Delta}\right) = \Theta(1)$, which is much too large since Theorem III.1 needs to be applied $n-2$ times.

Our main technical contribution is to provide a framework which significantly generalizes Theorem III.1, and makes the trickle-down theorem applicable to wider classes of weighted simplicial complexes, including those arising from proper colorings. One of our main insights is to replace the hypothesis that $\lambda_2(P_x) \leq \lambda$, which merely provides a uniform bound on all nontrivial eigenvalues of P_x, by a matrix bound "$P_x \preceq M_x$". The hope is that the matrix M_x itself can simultaneously be easily bounded, as well as provide information on where the "bad" eigen-spaces of P_x are. So, roughly speaking, although many of the top dimensional links P_τ's may have a constant second eigenvalue, by carefully choosing M_τ's one can "average out" these bad eigen-spaces to show that the link of a lower dimensional face has small eigenvalues. We formalize this as follows.

Theorem III.3 (Matrix Trickle-Down Theorem)**.** *Given a weighted simplicial complex (X, π_d), suppose the following holds:*

1) ***Connectivity:*** *$\lambda_2(P) < 1$, i.e. the local walk P is connected/irreducible.*
2) ***Generalized Spectral Bound for Links Above:*** *There is a family of symmetric matrices $\{M_x\}_{x \in X(0)}$ such that*
$$\Pi_x P_x - \alpha \pi_x \pi_x^\top \preceq M_x \preceq \frac{1}{2\alpha + 1} \Pi_x$$
for all $i \in X(0)$.

Then the local walk P actually satisfies the spectral bound $\Pi P - \left(2 - \frac{1}{\alpha}\right) \pi\pi^\top \preceq M$, and in particular $\lambda_2(P) \leq \rho(\Pi^{-1}M)$, where M is any symmetric matrix satisfying $M \preceq \frac{1}{2\alpha}\Pi$ and $\mathbb{E}_{x\sim\pi} M_x \preceq M - \alpha M \Pi^{-1} M$.

Note that by induction, Theorem I.4 follows as an immediate consequence of this generalized trickle-down result. Let us now prove Theorem III.3. To do this, we first need the following lemma.

Lemma III.4. *Let (X, π_d) be a weighted simplicial complex. Suppose for a symmetric matrix M and an $\alpha \geq 1/2$, the matrix inequalities $M, \Pi P - \left(2 - \frac{1}{\alpha}\right)\pi\pi^\top \preceq \frac{1}{2\alpha}\Pi$ hold and*
$$\Pi P - \alpha \Pi P^2 \preceq M - \alpha M \Pi^{-1} M \tag{3}$$
Then we have the bound $\Pi P - \left(2 - \frac{1}{\alpha}\right)\pi\pi^\top \preceq M$.

Proof: Our goal is to apply Lemma II.3 to suitably chosen A, B. Define $Q = P - \left(2 - \frac{1}{\alpha}\right)\mathbf{1}\pi^\top$. A quick calculation shows that $Q - \alpha Q^2 = P - \alpha P^2$, and so by multiplying both sides of Eq. (3) by $\Pi^{-1/2}$, we see that Eq. (3) is equivalent to
$$\begin{aligned}&\Pi^{1/2} Q \Pi^{-1/2} - \alpha \Pi^{1/2} Q^2 \Pi^{-1/2} \\ &\preceq \Pi^{-1/2} M \Pi^{-1/2} - \alpha \Pi^{-1/2} M \Pi^{-1} M \Pi^{-1/2}\end{aligned}$$
Taking $A = \Pi^{1/2} Q \Pi^{-1/2}$ and $B = \Pi^{-1/2} M \Pi^{-1/2}$, we see by assumption that A, B are symmetric matrices satisfying $A, B \preceq \frac{1}{2\alpha} I$ and $A(I - \alpha A) \preceq B(I - \alpha B)$. It follows by Lemma II.3 that $A \preceq B$, which is equivalent to $\Pi P - \left(2 - \frac{1}{\alpha}\right)\pi\pi^\top = \Pi Q \preceq M$ as desired. ■

With this lemma in hand, let us now prove Theorem III.3.

Proof of Theorem III.3: The conclusion follows immediately from Lemma III.4, and so it suffices to verify the conditions of the lemma. By assumption, we already have $M \preceq \frac{1}{2\alpha}\Pi$. Furthermore, $\Pi_x P_x - \alpha \pi_x \pi_x^\top \preceq M_x \preceq \frac{1}{2\alpha+1}\Pi_x$ implies that $\lambda_2(P_x) \leq \frac{1}{2\alpha+1}$. Since $\lambda_2(P) < 1$, by Theorem III.1 (the original Trickle-Down Theorem), $\lambda_2(P) \leq \frac{1}{2\alpha}$. Combined with the inequality $2 - \frac{1}{\alpha} \geq 1 - \frac{1}{2\alpha}$, which holds since $\alpha \geq 1/2$, it follows that $\Pi P - \left(2 - \frac{1}{\alpha}\right)\pi\pi^\top \preceq \frac{1}{2\alpha}\Pi$.

All that remains is to verify Eq. (3). Observe that
$$\begin{aligned}\Pi P &= \mathbb{E}_{x\sim\pi} \Pi_x P_x && \text{(Lemma II.5)}\\ &\preceq \mathbb{E}_{x\sim\pi}\left[\alpha \pi_x \pi_x^\top + M_x\right] && \text{(Assumption)}\\ &= \alpha \Pi P^2 + \mathbb{E}_{x\sim\pi} M_x && \text{(Lemma II.6)}\\ &\preceq \alpha \Pi P^2 + M - \alpha M \Pi^{-1} M && \text{(Assumption)}\end{aligned}$$
Rearranging, we obtain that $\Pi P - \alpha \Pi P^2 \preceq M - \alpha M \Pi^{-1} M$ as desired. ■

A. A Slight Extension of Theorem I.4

Here, we prove an extension of the matrix trickle-down method to take into account when simplicial complexes factor as products of smaller complexes. This will be useful in the context of proper colorings when the input graph is broken into several connect components by coloring some of the vertices.

Theorem III.5. *Let (X, π_d) be a totally connected weighted complex. Suppose $\{M_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau \in X(\leq d-2)}$ is a family of symmetric matrices satisfying the following:*

1) ***Base Case:*** *For every τ of co-dimension 2, we have the spectral inequality*
$$\Pi_\tau P_\tau - 2\pi_\tau \pi_\tau^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau.$$
2) ***Recursive Condition:*** *For every τ of co-dimension at least $k \geq 3$, at least one of the following holds: M_τ satisfies*
$$M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau \quad \text{and}$$
$$\mathbb{E}_{x\sim\pi_\tau} M_{\tau\cup\{x\}} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau. \tag{4}$$
Or, $(X_\tau, \pi_{\tau,k-1})$ is a product of weighted simplicial complexes $(Y_1, \mu_1), \ldots, (Y_t, \mu_t)$ and for every $\eta \in X_\tau(k-1)$,
$$M_\tau = \bigoplus_{1\leq i\leq t : d_{Y_i}\geq 1} \frac{d_{Y_i}(d_{Y_i}+1)}{k(k-1)} M_{\tau\cup\eta_{-i}},$$
where $\eta_{-i} = \eta \setminus Y_i(0)$.

Then for every $\tau \in X(\leq d-2)$, we have the bound $\lambda_2(\Pi_\tau P_\tau) \leq \rho(\Pi_\tau^{-1} M_\tau)$.

A proof of this theorem can be found in the extended version of the paper.

IV. Vertex Coloring

Fix an integer q and graph $G = (V, E)$ and a function L that maps each $v \in V$ to a subset of $[q]$. We call (G, L) a vertex-list-coloring instance. For any $u, v \in V$, we write $u \sim_c v$ when $u \sim v$ and $c \in L(u) \cap L(v)$. Furthermore, we define a β-extra color vertex-list-coloring instance as follows.

Definition IV.1. *We say a vertex-list-coloring instance (G, L) is a β-extra-color instance if for each $v \in V$, $|L(v)| \geq \beta + \Delta_G(v)$.*

We call an assignment $\sigma : V \to [q]$ a L-vertex-list-coloring of G if $\sigma(v) \in L(v)$ for all $v \in V$; we say σ is proper if $\sigma(u) \neq \sigma(v)$ whenever $u \sim v$. When it is clear from context we say σ is a proper coloring to mean it is a proper L-vertex-list-coloring. We say τ is proper partial coloring on $U \subset V$ when it is a proper $L|_U$-vertex-list-coloring for $G[U]$. We may view list-colorings σ as sets of vertex-color pairs (v, c), which we denote by vc for convenience. When the graph G and color lists L are clear from context, we write π_{n-1} for the uniform distribution over proper L-vertex-list-coloring of $G = (V, E)$. For a proper partial coloring on $U \subset V$, $v \in V \setminus U$ and $c \in L(v)$, define

$$p(vc|\tau) := \mathbb{P}_{\sigma \sim \pi_{n-1}}(\sigma(v) = c | \forall u \in U : \sigma(u) = \tau(u)).$$

In order to analyze the Glauber dynamics for an vertex-list-coloring instance (G, L), we can build an $(n-1)$-dimensional simplicial complex such that the down-up walk on its facets is the same as the Glauber dynamics on (G, L). Our aim is to apply Theorem III.5 to bound the second eigenvalue of the transition probability matrix of the local walks and then use Theorem II.7 to bound the second eigenvalue of the transition probability matrix of the global down-up walk on the facets.

Definition IV.2 (Simplicial Complex of a Vertex-List-Coloring Instance)**.** *Given a vertex-list-coloring instance (G, L), let $X(G, L)$ be a pure $(n-1)$-dimensional simplicial complex specified by the following facets: $\{(v, \sigma(v))\}_{v \in V}$ is a facet if and only if σ is a proper L-vertex-list-coloring for G.*

When it is clear from context, we abbreviate $X(G, L)$ to X. Note that for all $0 \leq k \leq n$, any face τ of co-dimension k is a proper partial coloring on a subset of vertices U of size $n-k$, i.e., k vertices remain uncolored. Furthermore, $X_\tau(0)$ can be seen as the set of all vc such that $c \in L(v)$, $v \notin U$ and for any $u \sim v$, $uc \notin \tau$. We define $V_\tau := \{v : \exists c, vc \in X_\tau(0)\}$. Let $G_\tau := G[V_\tau]$ and $\Delta_\tau(.)$ be the degree function of G_τ and Δ_τ be its maximum degree. We define $L_\tau(v) := \{c \in L(v) : vc \in X_\tau(0)\}$ to be the list of remaining colors available to v after coloring vertex u with c for each $uc \in \tau$. Let $l_\tau(v) := |L_\tau(v)|$ and $l_\tau(u, v) := |L_\tau(u) \cap L_\tau(v)|$. We write $v \sim_{\tau,c} u$ for vertices v, u when $v \sim u$ and $c \in L_\tau(v) \cap L_\tau(u)$. Furthermore, for $U \subseteq V \setminus V_\tau$, let $\tau|_U := \{vc \in \tau : v \in U\}$.

Theorem IV.3. *Suppose (G, L) is a $(1+\epsilon)\Delta$-extra-color vertex-list-coloring instance for an $0 < \epsilon \leq 1$ such that $\frac{\ln(\Delta)+2}{\Delta} \leq \frac{\epsilon^2}{40}$, and let (X, π_{n-1}) be its associated weighted simplicial complex. For any $2 \leq k \leq n$ and $\tau \in X$ of co-dimension k we have $\lambda_2(P_\tau) \leq \frac{\frac{5}{2\epsilon}}{k-1}$.*

Combined with local-to-global theorems (Theorems II.7 and II.8), this yields a mixing time of $O(n \log n)$ for bounded-degree graphs, and $n^{O(1/\epsilon)}$ in general, in this setting where we have at least $(1+\epsilon)\Delta$ additional colors available to each vertex. Again, we emphasize that this mixing result in itself is not new; a simple coupling argument can already recover $O(n \log n)$ mixing for $(\Delta+1)$-extra-color vertex-list-coloring instances. However, we will see later on how our proof technique can be used to obtain new mixing results for sampling edge colorings which, to the best of our knowledge, cannot be recovered via simple coupling arguments.

To prove the above statement, first for any τ of co-dimension 2, we find a diagonal matrix F_τ such that $\Pi_\tau P_\tau \preceq 2\pi_\tau \pi_\tau^\top + \Pi_\tau F_\tau$. Then, for all τ of co-dimension at least 3, we apply Theorem III.5 to $M_\tau = \frac{\Pi_\tau F_\tau}{k-1}$ to find a sufficient condition on the diagonal matrix F_τ based on matrices $F_{\tau'}$ for faces $\tau \subsetneq \tau'$, to get $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau)}{k-1}$. To find F_τ for faces τ of co-dimension 2, we state a more general proposition that is also useful for approaches that use non-diagonal matrix bounds.

Proposition IV.4. *Given a vertex-list-coloring instance (G, L), consider the weighted complex (X, π_{n-1}). For any face τ of co-dimension 2 such that $G_\tau = (\{u, v\}, \{uv\})$ is connected we have,*

$$\Pi_\tau P_\tau - 2\pi_\tau \pi_\tau^\top \preceq \sqrt{\Pi_\tau} \widetilde{M_\tau} \sqrt{\Pi_\tau}, \tag{5}$$

where $\widetilde{M_\tau}$ is a block diagonal matrix with a block $\widetilde{M_\tau^c}$ for every color c such that

$$\widetilde{M_\tau^c} = \begin{pmatrix} \frac{1}{(l_\tau(u)-1)(l_\tau(v)-1)} & \frac{-1}{\sqrt{(l_\tau(u)-1)(l_\tau(v)-1)}} \\ \frac{-1}{\sqrt{(l_\tau(u)-1)(l_\tau(v)-1)}} & \frac{1}{(l_\tau(u)-1)(l_\tau(v)-1)} \end{pmatrix}$$

and all other entries are 0.

Proof: For clarity, we drop τ from all notation in the proof. Write $\widetilde{M} = \widetilde{M_d} + \widetilde{M_o}$, where $\widetilde{M_d} = \text{diag}(\widetilde{M})$ and $\widetilde{M_o} = \text{off-diag}(\widetilde{M})$. First observe that for $c \in L(u)$,

$$\pi(uc) = \begin{cases} \frac{l(v)-1}{2(l(u)l(v)-l(u,v))} & \text{if } c \in L(u) \cap L(v) \\ \frac{l(v)}{2(l(u)l(v)-l(u,v))} & \text{otherwise} \end{cases}$$

and a similar identity holds for any $c \in L(v)$. Also, observe that

$$\Pi P = \frac{J - J^u - J^v}{2(l(u)l(v) - l(u,v))} + \sqrt{\Pi}\widetilde{M}_o\sqrt{\Pi},$$

where J, J^u, J^v are the all-ones matrix, all-ones matrix on uc rows/columns and all-ones matrix on vc rows/columns, respectively. So, subtracting $\widetilde{M}_o$ from both sides of (5) and multiplying by $2(l(u)l(v) - l(u,v))$ it is enough to show

$$J - J^u - J^v - 4(l(u)l(v) - l(u,v))\pi\pi^\top \tag{6}$$

$$\preceq 2(l(u)l(v) - l(u,v))\sqrt{\Pi}\widetilde{M}_d\sqrt{\Pi} =: N_d \tag{7}$$

Write $\ell = l(v)\mathbf{1}^u + l(u)\mathbf{1}^v$. Also, let $s \in \mathbb{R}^{l(u)+l(v)}$ where $s(xc) = 1$ if $c \in L(u) \cap L(v)$ and $s(xc) = 0$ otherwise for $x \in \{u, v\}$. Then, by Fact II.2 we can write,

$$\begin{aligned} 4(l(u)l(v) - l(u,v))\pi\pi^\top &= \frac{(\ell - s)(\ell - s)^\top}{l(u)l(v) - l(u,v)} \\ &\underbrace{\succeq}_{Fact\ II.2} \frac{\ell\ell^\top + ss^\top - \frac{1}{2}\ell\ell^\top - 2ss^\top}{l(u)l(v) - l(u,v)} \\ &\succeq \frac{\ell\ell^\top}{2l(u)l(v)} - \frac{ss^\top}{l(u)l(v) - l(u,v)} \end{aligned}$$

Plugging this into (6) it is enough to show that

$$\begin{aligned} &J - J^u - J^v + \frac{ss^\top}{l(u)l(v) - l(u,v)} \\ &= \mathbf{1}^u\mathbf{1}^{v\top} + \mathbf{1}^v\mathbf{1}^{u\top} + \frac{ss^\top}{l(u)l(v) - l(u,v)} \\ &\preceq \frac{\ell\ell^\top}{2l(u)l(v)} + N_d \end{aligned} \tag{8}$$

First, observe that by another application of Fact II.2, $l^2(v)\mathbf{1}^u\mathbf{1}^{u\top} + l^2(u)\mathbf{1}^v\mathbf{1}^{v\top} \succeq l(u)l(v)(\mathbf{1}^u\mathbf{1}^{v\top} + \mathbf{1}^v\mathbf{1}^{u\top})$. So,

$$\begin{aligned} \frac{\ell\ell^\top}{2l(u)l(v)} &= \frac{(l(v)\mathbf{1}^u + l(u)\mathbf{1}^v)(l(v)\mathbf{1}^u + l(u)\mathbf{1}^v)^\top}{2l(u)l(v)} \\ &\succeq \mathbf{1}^u\mathbf{1}^{v\top} + \mathbf{1}^v\mathbf{1}^{u\top} \end{aligned}$$

Let $I^\cap \in \mathbb{R}^{(l(u)+l(v))\times(l(u)+l(v))}$ be the identity matrix only on entries xc, xc where $x \in \{u, v\}$ and $c \in L(u) \cap L(v)$. Finally, (8) simply follows from the fact that

$$\frac{ss^\top}{l(u)l(v) - l(u,v)} \preceq \frac{l(u,v)}{l(u)l(v) - l(u,v)} I^\cap \preceq N_d$$

where the first inequality uses that the only non-zero rows of $ss^\top$ correspond to a common color and the sum of the entries of any such row is exactly $l(u,v)$ and the last inequality uses that $\frac{l(u,v)}{l(u)l(v) - l(u,v)} \leq \frac{1}{\max\{l(u), l(v)\} - 1}$ and that $N_d(uc, uc) = \frac{1}{l(u)-1}, N_d(vc, vc) = \frac{1}{l(v)-1}$ if $c \in L(u) \cap L(v)$ and it is zero otherwise. ■

Note that in the above proposition, $\widetilde{M}_\tau \preceq (\frac{1}{\beta} + \frac{1}{\beta^2})I^{X_\tau(0)}$, which gives us the diagonal matrix F_τ for any τ of co-dimension 2 such that G_τ is connected. Now, using Theorem III.5, we derive a set of sufficient conditions on the family $\{F_\tau\}_{\tau \in X:\mathrm{codim}(\tau)\geq 2}$ to get $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau)}{k-1}$ for all τ of co-dimension $2 \leq k \leq n$.

Proposition IV.5. *Given a β-extra-color vertex-list-coloring instance (G, L), with corresponding weighted simplicial complex (X, π_{n-1}), suppose $\{F_\tau \in \mathbb{R}^{X(0)\times X(0)}\}_{\tau \in X:\mathrm{codim}(\tau)\geq 2}$ is a family of diagonal matrices supported on $X_\tau(0) \times X_\tau(0)$ such that $F_\tau = f_\times(X_\tau, \{F_{\tau\cup\sigma}\}_{\emptyset\subsetneq\sigma\in X_\tau(\leq \mathrm{codim}(\tau)-3)})$ if G_τ is disconnected and otherwise,*

1) *For all τ of co-dimension 2: $F_\tau(vc, vc) = \frac{1}{\beta} + \frac{1}{\beta^2}$ for all $vc \in X_\tau(0)$.*
2) *For all τ of co-dimension $k \geq 3$: $F_\tau \preceq \frac{(k-1)^2}{3k-1} I^{X_\tau(0)}$ and for all $vc \in X_\tau(0)$*

$$\sum_{uc' \in X_{\tau\cup vc}(0)} p(uc'|\tau \cup vc)F_{\tau\cup uc'}(vc, vc)$$
$$\leq (k-2)F_\tau(vc, vc) - F_\tau^2(vc, vc).$$

Then, for all $k \geq 2$ and τ of co-dimension k, $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau)}{k-1}$.

Proof: We prove that the conditions of Theorem III.5 hold for $M_\tau := \frac{\Pi_\tau F_\tau}{k-1}$ for any face τ of co-dimension at least 2. The desired condition holds for any τ of co-dimension 2 by Proposition IV.4. Now, let $k \geq 3$. First assume that G_τ is disconnected with connected components $G_\tau[U_1], \ldots, G_\tau[U_\ell]$ and associated complexes $Y_1, \ldots Y_\ell$. We can write $(X, \pi_{\tau,k-1}) = (Y_1, \mu_1) \times \cdots \times (Y_\ell, \mu_\ell)$, where μ_i is the uniform distribution over facets of Y_i. For an $\alpha \in X_\tau(k-1)$ let $\alpha_{-i} := \alpha \setminus \alpha|_{U_i}$. Therefore,

$$\begin{aligned} &\sum_{1\leq i\leq\ell: d_{Y_i}\geq 1} \frac{d_{Y_i}(d_{Y_i}+1)}{(k-1)k} M_{\tau\cup\alpha_{-i}} \\ &\underset{\text{def of } M_{\tau\cup\alpha_{-i}}}{=} \sum_{1\leq i\leq\ell: d_{Y_i}\geq 1} \frac{d_{Y_i}(d_{Y_i}+1)}{(k-1)k} \frac{\Pi_{\tau\cup\alpha_{-i}}}{d_{Y_i}} F_{\tau\cup\alpha_{-i}} \\ &= \sum_{1\leq i\leq\ell: d_{Y_i}\geq 1} (\Pi_\tau)^{X_{\tau\cup\alpha_{-i}}(0)} \frac{F_{\tau\cup\alpha_{-i}}}{k-1} \underset{\text{def of } F_\tau}{=} \frac{\Pi_\tau F_\tau}{k-1} = M_\tau \end{aligned}$$

as desired.

Now, assume that G_τ is connected. Note that since each entry of F_τ is at most $\frac{(k-1)^2}{3k-1}$, we have $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$. Therefore, it only remains to show that $\mathbb{E}_{vc\sim\pi_\tau} M_{\tau\cup vc} \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_\tau^{-1} M_\tau$. This is equivalent to showing that

$$\Pi_\tau^{-1}\mathbb{E}_{uc'\sim\pi_\tau}\left[\Pi_{\tau\cup uc'}\frac{F_{\tau\cup uc'}}{k-2}\right] \preceq \frac{F_\tau}{k-1} - \frac{F_\tau^2}{(k-2)(k-1)}.$$

One can check that

$$\mathbb{E}_{uc'\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup uc'}\frac{F_{\tau\cup vc}}{k-2}\right](vc, vc)$$

$$= \frac{\sum_{uc' \in X_{\tau \cup vc}(0)} p(uc'|\tau \cup vc) F_{\tau \cup uc'}(vc, vc)}{(k-1)(k-2)}.$$

Therefore, it is enough that

$$\frac{\sum_{uc' \in X_{\tau \cup vc}(0)} p(uc'|\tau \cup vc) F_{\tau \cup uc'}(vc, vc)}{(k-1)(k-2)} \leq \frac{F_\tau(vc, vc)}{k-1} - \frac{F_\tau^2(vc, vc)}{(k-1)(k-2)},$$

which holds by assumption. ■

Now, to complete the proof of Theorem IV.3 it only remains to find $\{F_\tau\}_{\tau \in X(\leq n-2)}$ that satisfies the above conditions. We find this family of diagonal matrices in the extended version of the paper. Let us remark why we need the assumption $\beta > \Delta$ in this proof. Consider the worst case example, where G is a complete graph with $\Delta + 1$ vertices. In that case, by symmetry, $F_\tau(vc, vc) = \frac{1}{\beta} + \frac{1}{\beta^2}$ for all faces of co-dimension 2, and every matrix F_τ is a multiple of identity on $X_\tau(0) \times X_\tau(0)$. So, the conditions on F_τ reduces to the following systems of inequalities:

$$f(1) = \frac{1}{\beta} + \frac{1}{\beta^2} \quad \text{and}$$
$$\forall 3 \leq k \leq \Delta,$$
$$(k-1)f(k-2) \leq (k-2)f(k-1) - f(k-1)^2$$

It is not hard to see that such a system does not have a solution up to $k = \Delta + 1$ when $\beta \leq \Delta$.

V. Edge Coloring

Consider a graph $G = (V, E)$ and a function $L : E \to 2^{[q]}$. The pair (G, L) is called an edge-list-coloring instance. For a vertex v and an edge e, we write $e \sim_c v$ when $e \sim v$ and $c \in L(e)$. Furthermore, for any $e, f \in E$, we write $e \sim_c f$ when $e \sim f$ and $c \in L(e) \cap L(f)$. Furthermore, we define a β-extra-color edge-list-coloring instance as follows.

Definition V.1. *We say an edge-list-coloring instance (G, L) is a β-extra-color instance if for each $e \in E$, $|L(e)| \geq \beta + \Delta_G(e)$.*

An assignment $\sigma : E \to [q]$ is a L-edge-list-coloring of G if $\sigma(e) \in L(e)$ for all $e \in E$. We say σ is proper if $\sigma(e) \neq \sigma(f)$ whenever $e \sim f$. When it is clear from context we say σ is a proper coloring to mean it is a proper L-edge-list-coloring. We say τ is proper partial coloring on $H \subset E$ when it is a proper $L|_H$-edge-list-coloring for (V, H). We may view a proper coloring as a set of edge-color pairs (e, c) which we denote by ec for simplicity of notation. We denote the uniform distribution over proper L-edge-list-colorings of G by π_{m-1} when (G, L) are clear from context. For a proper partial coloring on $H \subset E$ and $e \in E \setminus H$, define

$$p(ec|\tau) := \mathbb{P}_{\sigma \sim \pi_{m-1}}(\sigma(e) = c | \forall f \in H : \sigma(f) = \tau(f)).$$

To analyze the Glauber dynamics on an edge-list-coloring instance we associate a simplicial complex to it.

Definition V.2 (Simplicial Complex of an Edge-List-Coloring Instance)**.** *Given an edge-list-coloring instance (G, L), let $X(G, L)$ be a pure $(m-1)$-dimensional simplicial complex specified by the following facets: $\{(e, \sigma(e))\}_{e \in E}$ is a facet if and only if σ is a proper L-edge-list-coloring for G.*

When it is clear from context, we abbreviate $X(G, L)$ to X. Note that for all $0 \leq k \leq m$, any face τ of co-dimension k is a partial coloring on a subset of edges H of size $m - k$ (k edges remain uncolored). Furthermore, $X_\tau(0)$ can be seen as the set of all ec such that $c \in L(e)$, $e \notin H$ and for any $f \sim e$, $fc \notin \tau$. Analogous to vertex-list-colorings, the Glauber dynamics on (G, L) is the down-up walk on the facets of (X, π_{m-1}). So, as we did before for vertex-list-colorings, our aim is to apply Theorem III.5 to the simplicial complex to bound the second eigenvalue of the transition probability matrix of the local walks and then apply Theorem II.7 to get a bound for the transition probability matrix of the down-up walk on the facets. The following is the main theorem of this section.

Theorem V.3. *Let (G, L) be a $(\frac{4}{3} + 4\epsilon)\Delta$-extra-color edge-list-coloring instance for some $0 < \epsilon \leq \frac{1}{10}$ such that $\frac{\ln^2(\Delta)}{\Delta} \leq \frac{\epsilon^3}{15}$, and let (X, π_{m-1}) be its associated weighted simplicial complex. For any $2 \leq k \leq m$ and $\tau \in X$ of co-dimension k we have $\lambda_2(P_\tau) \leq \frac{\epsilon + \frac{1}{\epsilon}}{k-1}$.*

We remark that our analysis here is not tight and we expect that the factor $4/3$ can be improved with a more careful analysis.

We proceed by introducing some notation and definitions. Given a face $\tau \in X$, let E_τ be the set of uncolored edges, i.e. $E_\tau := \{e : \exists c, ec \in X_\tau(0)\}$. Let $G_\tau = (V, E_\tau)$ and $\Delta_\tau(.)$ be the degree function of G_τ. Similarly, if $e = \{u, v\}$, define $\Delta_\tau(e)$ to be number of edges in G_τ that share an endpoint with e, i.e. $\Delta_\tau(e) = \Delta_\tau(u) + \Delta_\tau(v) - 2$. We define $L_\tau(e) := \{c \in L(e) : ec \in X_\tau(0)\}$. Let $l_\tau(e) := |L_\tau(e)|$ and $l_\tau(e, f) := |L_\tau(e) \cap L_\tau(f)|$. Furthermore, we write $e \sim_{\tau,c} v$ when $e \sim v$ and $c \in L_\tau(e)$. Similarly, define $e \sim_{\tau,c} f$ for edges e and f. Finally, for any matrix $B \in \mathbb{R}^{X(0) \times X(0)}$, define the restriction of B to $v \in V$ as $B^v(ec, fc) := B(ec, fc)$ for any $e, f \sim v$, and 0 on all other entries. Let $B^c \in \mathbb{R}^{X(0) \times X(0)}$ be defined as $B^c(ec, fc) := B(ec, fc)$ for all $e \sim_c f$, and 0 on all other entries.

To be able to improve upon Theorem IV.3 and prove Theorem V.3, we allow the matrix bounds $\{M_\tau\}_{\tau \in X : \text{codim}(\tau) \geq 2}$ in Theorem III.5 to be non-diagonal matrices. For any $k \geq 2$ and face τ of co-dimension k, we assume that M_τ is of the form

$$M_\tau = \Pi_\tau \frac{F_\tau}{k-1} + \sqrt{\Pi_\tau} \frac{A_\tau}{k-1} \sqrt{\Pi_\tau}, \tag{9}$$

for a diagonal matrix F_τ and a hollow matrix A_τ. The goal is again to find F_τ and A_τ such that M_τ satisfies the conditions of Theorem III.5. For $k=2$, Proposition IV.4 gives us such matrices. For $k \geq 3$, as opposed to what we did for vertex-coloring of trees, we let $\sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau}$ deviate from $\mathbb{E}_{vc\sim\pi_\tau}\sqrt{\Pi_{\tau\cup vc}}\frac{A_{\tau\cup vc}}{k-2}\sqrt{\Pi_{\tau\cup vc}}$ in order to control the growth of F_τ.

Definition V.4 (Family of Matrices $\{A_{\tau,\epsilon}\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$). *Let (G,L) be a β-extra-color edge-list-coloring instance, and let(X,π_{m-1}) be its associated weighted complex. For $\epsilon>0$, define $\{A_{\tau,\epsilon}\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$ as follows: let $A_{\tau,\epsilon} := f_\times(X,\{A_{\tau\cup\sigma,\epsilon}\}_{\emptyset\subsetneq\sigma\in X_\tau(\mathrm{codim}(\tau)-3)})$ if the line graph of G_τ is disconnected and otherwise,*

1) *For any face τ of co-dimension 2, let $A_{\tau,\epsilon}\in\mathbb{R}^{X(0)\times X(0)}$ be a hollow block diagonal matrix with a block for every color such that*

$$A_{\tau,\epsilon}(ec,fc)=A_{\tau,\epsilon}(fc,ec):= -\frac{1}{\sqrt{(l_\tau(e)-1)(l_\tau(f)-1)}},$$

for $e,f\in E_\tau$ and any $c\in L_\tau(e)\cap L_\tau(f)$, and all other entries are 0.

2) *For any $k\geq 3$ and a face τ of co-dimension k, define $A_{\tau,\epsilon}:=\bar{A}_{\tau,\epsilon}+\frac{\text{off-diag}(S_{\tau,\epsilon})}{k-2}$, where $\bar{A}_{\tau,\epsilon}$ and $S_{\tau,\epsilon}$ are defined as follows:*

$$\bar{A}_{\tau,\epsilon}:= \frac{k-1}{k-2}\Pi_\tau^{-1/2}\left(\mathbb{E}_{gc\sim\pi_\tau}\Pi_{\tau\cup gc}^{1/2}A_{\tau\cup gc,\epsilon}\Pi_{\tau\cup gc}^{1/2}\right)\Pi_\tau^{-1/2} \tag{10}$$

$$S^v_{\tau,\epsilon}:= \begin{cases}4(1+\epsilon)\left((\bar{A}^{+,v}_{\tau,\epsilon})^2+(\bar{A}^{-,v}_{\tau,\epsilon})^2\right) & \text{if } \Delta_\tau(v)\leq\frac{\beta}{4(1+\epsilon)},\\ 2(1+\epsilon)(\bar{A}^v_{\tau,\epsilon})^2 & \text{otherwise.}\end{cases} \tag{11}$$

and $S_{\tau,\epsilon}=\sum_v S^v_{\tau,\epsilon}$.

Observe that all three matrices $\bar{A}_{\tau,\epsilon},S_{\tau,\epsilon},A_{\tau,\epsilon}$ are symmetric and hollow. When it is clear from context, we drop ϵ from the subscripts of matrices defined above .

In order to find diagonal matrices $\{F_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$ such that $\{M_\tau\}_{\tau\in X:\mathrm{codim}(\tau)\geq 2}$ as defined by Eq. (9) satisfies the conditions of Theorem III.5, we would need to have a bound on the entries of $\{A_{\tau,\epsilon}\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$ and $\{S_{\tau,\epsilon}\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$.

Proposition V.5. *Given a β-extra-color edge-list-coloring instance (G,L) where $\beta=(\frac{4}{3}+4\epsilon)\Delta$ for an $0<\epsilon\leq\frac{1}{10}$ such that $2\epsilon^{-2}\leq\Delta$,*

1) *For any $\tau\in X$ with $\mathrm{codim}(\tau)\geq 2$, $v\in G$, and $e,f\sim_{\tau,c}v$, $|A_{\tau,\epsilon}(ec,fc)|\leq\frac{1}{2\epsilon\Delta}\leq\frac{\epsilon}{4}$.*
2) *For any $\tau\in X$ with $\mathrm{codim}(\tau)\geq 3$, $v\in G$, and $e,f\sim_{\tau,c}v$,*

$$S_{\tau,\epsilon}(ec,fc)\leq\frac{(1+\epsilon)(\Delta_\tau(v)-2)}{2\epsilon^2\Delta^2}$$

.

3) *For any $\tau\in X$ with $\mathrm{codim}(\tau)\geq 3$, color c, and $(e=\{u,v\},c)\in X_\tau(0)$,*

$$S_{\tau,\epsilon}(ec,ec)\leq\frac{(1+\epsilon)(\Delta_\tau(v)+\Delta_\tau(u)-2)}{2\epsilon^2\Delta^2}.$$

Proposition V.5 is proved in the extended version of the paper.

The following lemma is a crucial part of our proof as it will help us bound the term $M_\tau\Pi_\tau^{-1}M_\tau$ in Eq. (4) effectively.

Lemma V.6. *Consider a graph $G=(V,E)$, and some weight function $w:E\to\mathbb{R}_{\geq 0}$. Let A be the weighted adjacency matrix of its line graph. Then*

$$A^2\preceq 2\sum_{v\in V}(A^v)^2,$$

where $A^v(e,f)=A(e,f)$ if $e,f\sim v$ and 0 otherwise.

Proof: It is enough to show that for all $x\in\mathbb{R}^E$, $x^\top A^2x\leq 2\sum_{v\in V}x^\top(A^v)^2x$. We have

$$x^\top A^2x=\|Ax\|_2^2=\sum_{e\in E}(Ax(e))^2=\sum_{e\in E}\langle A_e,x\rangle^2$$

where A_e is the row indexed by e. Now, let $e=\{u,v\}\in E$. We can write $\langle A_e,x\rangle=\langle(A^u)_e,x\rangle+\langle(A^v)_e,x\rangle$. Therefore, by an application of Fact II.2

$$\sum_{e\in E}\langle A_e,x\rangle^2\preceq 2\sum_{e=\{u,v\}\in E}\langle(A^u)_e,x\rangle^2+\langle(A^v)_e,x\rangle^2 = 2\sum_v\sum_{e\sim v}(A^vx(e))^2=2\sum_{v\in V}x^\top(A^v)^2x.$$

■

Now, we apply Theorem III.5 to derive sufficient conditions on the family $\{F_\tau\}_{\tau\in X,\mathrm{codim}(\tau)\geq 2}$ to get $\lambda_2(P_\tau)\leq\frac{\rho(F_\tau+A_\tau)}{k-1}$ for all τ of co-dimension $2\leq k\leq m$.

Proposition V.7. *Let (G,L) be a $(\frac{4}{3}+4\epsilon)\Delta$-extra-color edge-list-coloring instance such that $0\leq\epsilon\leq\frac{1}{10}$ and $\Delta\geq 2\epsilon^{-2}$, and let (X,π_{m-1}) be its associated weighted simplicial complex. Suppose that $\{F_\tau\in\mathbb{R}^{X(0)\times X(0)}\}_{\tau\in X:\mathrm{codim}(X)\geq 2}$ is a family of diagonal matrices supported on $X_\tau(0)\times X_\tau(0)$ such that $F_\tau=f_\times(X_\tau,\{F_{\tau\cup\sigma}\}_{\emptyset\subsetneq\sigma\in X_\tau(\leq\mathrm{codim}(\tau)-3)})$ if the line graph of G_τ is connected and otherwise,*

1) *For all τ of co-dimension 2: F_τ is defined as $F_\tau(ec,ec)=\frac{1}{(\frac{4}{3}+4\epsilon)^2\Delta^2}=\frac{1}{\beta^2}$ for $ec\in X_\tau(0)$ and 0 on all other entries.*
2) *For all τ of co-dimension $k\geq 3$: $F_\tau\preceq(\frac{(k-1)^2}{3k-1}-\frac{1}{2\epsilon\Delta})I^{X_\tau(0)}$, and for any $ec\in X_\tau(0)$*

$$\sum_{gc'\in X_{\tau\cup ec}(0)}p(gc'|\tau\cup ec)F_{\tau\cup gc'}(ec,ec)$$

$$\leq (k-2)F_\tau(ec,ec) - \left(\frac{2+\epsilon}{\epsilon}\right)F_\tau^2(ec,ec) - \gamma_\tau(ec), \tag{12}$$

where $\gamma_\tau(ec) = \frac{(1+\epsilon)\Delta_\tau(e)}{2\epsilon^2\Delta^2} + \frac{(1+\epsilon)^2(2+3\epsilon+\epsilon^2)}{\epsilon^5\Delta^2}$.

Then for all $k \geq 2$ *and* τ *of co-dimension* k, $\lambda_2(P_\tau) \leq \frac{\rho(F_\tau+A_\tau)}{k-1}$, *where* A_τ *is defined in Definition V.4.*

Proof: We prove that the conditions of Theorem III.5 hold for $\{M_\tau\}_{\tau\in X(\leq m-3)}$ defined as follows:

$$M_\tau := \Pi_\tau \frac{F_\tau}{k-1} + \sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau},$$

for all $\tau \in X, k = \operatorname{codim}(\tau) \geq 2$. Note that the condition of the theorem holds for any τ of co-dimension 2 by definition. So, we prove the statement for τ of co-dimension at least 3. Assume the line graph of G_τ is disconnected. Using the definition of A_τ and our assumption about F_τ, the proof of this case is similar to what we argued in Proposition IV.5. Now, assume that the line graph of G_τ is connected. Note that by Proposition V.5, the absolute value of every off-diagonal entry of A_τ is at most $\frac{1}{2\epsilon\Delta}$ and that there are at most $(k-1)$ non-zero entries per row. Therefore, $\sqrt{\Pi_\tau}\frac{A_\tau}{k-1}\sqrt{\Pi_\tau} \preceq \frac{1}{2\epsilon\Delta}\Pi_\tau$. Combined with the bound on entries of diagonal matrix F_τ, this implies that $M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau$. Therefore, it only remains to show that $\mathbb{E}_{gc\sim\pi_\tau} M_{\tau\cup gc} \preceq M_\tau - \frac{k-1}{k-2}M_\tau\Pi_\tau^{-1}M_\tau$. This is equivalent to showing that

$$\Pi_\tau^{-1/2}\mathbb{E}_{gc\sim\pi_\tau}\left[\Pi_{\tau\cup gc}\frac{F_{\tau\cup gc}}{k-2} + \Pi_{\tau\cup gc}^{1/2}\frac{A_{\tau\cup gc}}{k-2}\Pi_{\tau\cup gc}^{1/2}\right]\Pi_\tau^{-1/2} \preceq \frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{(F_\tau+A_\tau)^2}{(k-2)(k-1)}. \tag{13}$$

We proceed by first proving a lowerbound on the RHS. By two applications of Fact II.2, we can write

$$\begin{aligned}(F_\tau+A_\tau)^2 &\preceq \left(1+\frac{2}{\epsilon}\right)F_\tau^2 + \left(1+\frac{\epsilon}{2}\right)A_\tau^2 \\ &\preceq \left(1+\frac{2}{\epsilon}\right)F_\tau^2 + (1+\epsilon)\bar{A}_\tau^2 \\ &\quad + \frac{(3+\epsilon+2/\epsilon)\operatorname{off-diag}(S_\tau)^2}{(k-2)^2}.\end{aligned} \tag{14}$$

We proceed by finding a diagonal matrix to upperbound $\bar{A}_\tau^2$. For any $c \in [q]$, $\bar{A}_\tau^c$ is the weighted adjacency matrix of a line graph. Therefore, by Lemma V.6, $(\bar{A}_\tau^c)^2 \preceq 2\sum_{v\in V}(\bar{A}_\tau^{c,v})^2$. Since $\bar{A}_\tau^2 = \sum_{c\in[q]}(\bar{A}_\tau^c)^2$, we get that

$$\bar{A}_\tau^2 \preceq 2\sum_{v\in V}(\bar{A}_\tau^v)^2 \preceq 4\sum_{v\in V}((\bar{A}_\tau^{+,v})^2 + (\bar{A}_\tau^{-,v})^2).$$

where in the second inequality we used Fact II.2. Therefore, by definition of S_τ (see Eq. (11)),

$$(1+\epsilon)\bar{A}_\tau^2 \preceq S_\tau = (k-2)(A_\tau - \bar{A}_\tau) + \operatorname{diag}(S_\tau).$$

So, by (14), we can lowerbound the RHS of (13) as follows

$$\begin{aligned}&\frac{F_\tau}{k-1} + \frac{A_\tau}{k-1} - \frac{(F_\tau+A_\tau)^2}{(k-2)(k-1)} \\ &\succeq \frac{F_\tau}{k-1} + \frac{\bar{A}_\tau}{k-1} - \frac{(1+\frac{2}{\epsilon})F_\tau^2}{(k-1)(k-2)} \\ &\quad - \frac{\operatorname{diag}(S_\tau)}{(k-1)(k-2)} - \frac{(3+\epsilon+\frac{2}{\epsilon})\operatorname{off-diag}(S_\tau)^2}{(k-1)(k-2)^3}.\end{aligned}$$

On the other hand, by definition of $\bar{A}_\tau$ (see (10)), the LHS of (13) is equal to

$$\mathbb{E}_{gc\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup gc}\frac{F_{\tau\cup gc}}{k-2}\right] + \frac{\bar{A}_\tau}{k-1},$$

and

$$\begin{aligned}&\mathbb{E}_{gc\sim\pi_\tau}\left[\Pi_\tau^{-1}\Pi_{\tau\cup gc}\frac{F_{\tau\cup gc}}{k-2}\right](ec,ec) \\ &= \sum_{gc'\in X_{\tau\cup ec}(0)} p(gc'|\tau\cup ec)F_{\tau\cup gc'}(ec,ec).\end{aligned}$$

Comparing this with the assumption (see Eq. (12)), and letting $\gamma_\tau(ec) = 0$ for all $ec \notin X_\tau(0)$, it is enough to show that

$$\operatorname{diag}(\gamma_\tau) \succeq \operatorname{diag}(S_\tau) + \frac{(3+\epsilon+\frac{2}{\epsilon})\operatorname{off-diag}(S_\tau)^2}{(k-2)^2}.$$

First, notice,

$$\begin{aligned}\frac{\operatorname{off-diag}(S_\tau)^2}{(k-2)^2} &\preceq \frac{\|\operatorname{off-diag}(S_\tau)\|_\infty^2 \, I^{X_\tau(0)}}{(k-2)^2} \\ &\preceq \frac{(1+\epsilon)^2(\Delta-2)^2 4(\Delta-1)^2}{4\epsilon^4\Delta^4(k-2)^2} I^{X_\tau(0)} \\ &\preceq \frac{(1+\epsilon)^2}{\epsilon^4\Delta^2} I^{X_\tau(0)},\end{aligned}$$

where the second inequality is by Fact II.1, noting that by part (ii) of Proposition V.5, every off-diagonal entry of S_τ is at most $\frac{(1+\epsilon)(\Delta-2)}{2\epsilon^2\Delta^2}$ and that there are at most $2(\Delta-1)$ non-zero entries per row. Finally, the statement follows from part (iii) of Proposition V.5 which shows $S_\tau(ec,ec) \leq \frac{(1+\epsilon)\Delta_\tau(e)}{2\epsilon^2\Delta^2}$ for any $ec \in X_\tau(0)$. ■

With this in hand, to prove Theorem V.3, it is enough to find a family of matrices $\{F_\tau\}_{\tau\in X:\operatorname{codim}(X)\geq 2}$ that satisfy Proposition V.7. We leave this to the extended version of the paper.

References

[AL20] Vedat Levi Alev and Lap Chi Lau. "Improved Analysis of Higher Order Random Walks and Applications". In: *Proceedings of the 52nd Annual ACM Symposium on Theory of Computing (STOC)*. 2020.

[Ali+21] Yeganeh Alimohammadi, Nima Anari, Kirankumar Shiragur, and Thuy-Duong Vuong. "Fractionally Log-Concave and Sector-Stable Polynomials: Counting Planar Matchings and More". In: *Proceedings of the 53rd Annual ACM Symposium on Theory of Computing (STOC)*. 2021.

[ALO20] Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. "Spectral Independence in High-Dimensional Expanders and Applications to the Hardcore Model". In: *arXiv preprint arXiv:2001.00303* (2020).

[Ana+19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-Concave Polynomials II: High-Dimensional Walks and an FPRAS for Counting Bases of a Matroid". In: *Proceedings of the 51st Annual ACM SIGACT Symposium on the Theory Computing*. STOC 2019. Phoenix, AZ, USA: Association for Computing Machinery, 2019, pp. 1–12. ISBN: 9781450367059. DOI: 10.1145/3313276.3316385.

[Ana+21] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, Cynthia Vinzant, and Thuy-Duong Vuong. "Log-Concave Polynomials IV: Approximate Exchange, Tight Mixing Times, and Near-Optimal Sampling of Forests". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on the Theory Computing*. STOC 2021. New York, NY, USA: Association for Computing Machinery, 2021.

[Bla+21] Antonio Blanca, Pietro Caputo, Zongchen Chen, Daniel Parisi, Daniel Štefankovič, and Eric Vigoda. "On Mixing of Markov Chains: Coupling, Spectral Independence, and Entropy Factorization". In: *arXiv preprint arXiv:2103.07459* (2021).

[CGM19] M. Cryan, H. Guo, and G. Mousa. "Modified log-Sobolev Inequalities for Strongly Log-Concave Distributions". In: *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. 2019, pp. 1358–1370.

[Che+19] Sitan Chen, Michelle Delcourt, Ankur Moitra, Guillem Perarnau, and Luke Postle. "Improved Bounds for Randomly Sampling Colorings via Linear Programming". In: *SODA*. Ed. by Timothy M. Chan. SIAM, 2019, pp. 2216–2234.

[Che+21] Zongchen Chen, Andreas Galanis, Daniel Stefankovic, and Eric Vigoda. "Rapid Mixing for Colorings via Spectral Independence". In: *SODA*. Ed. by Dániel Marx. SIAM, 2021, pp. 1548–1557.

[CLV20] Z. Chen, K. Liu, and E. Vigoda. "Rapid Mixing of Glauber Dynamics up to Uniqueness via Contraction". In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. 2020, pp. 1307–1318.

[CLV21] Z. Chen, K. Liu, and E. Vigoda. "Optimal Mixing of Glauber Dynamics: Entropy Factorization via High-Dimensional Expansion". In: *Proceedings of the 53rd Annual ACM Symposium on Theory of Computing (STOC)*. 2021.

[DHP20] Michelle Delcourt, Marc Heinrich, and Guillem Perarnau. "The Glauber dynamics for edge-colorings of trees". In: 57 (4 Dec. 2020), pp. 1050–1076.

[Dik+18] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. "Boolean Function Analysis on High-Dimensional Expanders". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2018)*. Vol. 116. Leibniz International Proceedings in Informatics (LIPIcs). 2018, 38:1–38:20. ISBN: 978-3-95977-085-9.

[DK17] I. Dinur and T. Kaufman. "High Dimensional Expanders Imply Agreement Expanders". In: *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*. 2017, pp. 974–985.

[Dye+13] Martin Dyer, Alan Frieze, Thomas P. Hayes, and Eric Vigoda. "Randomly coloring constant degree graphs". In: *Random Structures & Algorithms* 43.2 (2013), pp. 181–200.

[Eft+19] Charilaos Efthymiou, Andreas Galanis, Thomas P. Hayes, Daniel Stefankovic, and Eric Vigoda. "Improved Strong Spatial Mixing for Colorings on Trees". In: *APPROX/RANDOM*. Ed. by Dimitris Achlioptas and László A. Végh. Vol. 145. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, 48:1–48:16.

[Fen+21] Weiming Feng, Heng Guo, Yitong Yin, and Chihao Zhang. "Rapid Mixing from Spectral Independence beyond the Boolean Domain". In: *SODA*. Ed. by Dániel Marx. SIAM, 2021, pp. 1558–1577.

[Fri+21] Tobias Friedrich, Andreas Göbel, Martin S. Krejca, and Marcus Pappik. "A spectral independence view on hard spheres via block dynamics". In: *ICALP*. 2021.

[FV06] Alan Frieze and Juan Vera. "On randomly colouring locally sparse graphs". In: *Discrete Mathematics and Theoretical Computer Science* 8 (2006).

[FV07] Alan Frieze and Eric Vigoda. "A survey on the use of Markov chains to randomly sample colourings". In: *Oxford Lecture Series in Mathematics and its Applications* (2007). 34:53.

[Gar73] Howard Garland. "p-Adic Curvature and the Cohomology of Discrete Subgroups of p-Adic Groups". In: 97.3 (1973), pp. 375–423.

[GKM15] D. Gamarnik, D. Katz, and S. Misra. "Strong spatial mixing for list coloring of graphs". In: *Random Structures & Algorithms* 46.4 (2015), pp. 599–613.

[GM20] Heng Guo and Giorgos Mousa. "Local-to-Global Contraction in Simplicial Complexes". In: *arXiv preprint arXiv:2012.14317* (2020).

[GMP05] L.A. Goldberg, R. Martin, and M. Paterson. *Strong spatial mixing with fewer colors for lattice graphs*. 2005.

[GŠV15] Andreas Galanis, Daniel Štefankovič, and Eric Vigoda. "Inapproximability for Antiferromagnetic Spin Systems in the Tree Nonuniqueness Region". In: *J. ACM* 62.6 (Dec. 2015), 50:1–50:60. ISSN: 0004-5411.

[HV03] Thomas P Hayes and Eric Vigoda. "A non-Markovian coupling for randomly sampling colorings". In: *FOCS*. 2003, pp. 618–627.

[HV05] Tom Hayes and Eric Vigoda. "Coupling with the Stationary Distribution and Improved Sampling for Colorings and Independent Sets". In: *SODA*. Vancouver, British Columbia, 2005, pp. 971–979. ISBN: 0-89871-585-7.

[HVV07] Thomas P. Hayes, Juan C. Vera, and Eric Vigoda. "Randomly coloring planar graphs with fewer colors than the maximum degree". In: *STOC*. 2007, pp. 450–458.

[Jer95] Mark Jerrum. "A very simple algorithm for estimating the number of k-colorings of a low-degree graph". In: *Random Structures & Algorithms* 7.2 (1995), pp. 157–165.

[Joh96] A. Johansson. "Asymptotic choice number for triangle free graphs". Unpublished manuscript. 1996.

[KM17] Tali Kaufman and David Mass. "High Dimensional Random Walks and Colorful Expansion". In: *ITCS*. 2017, 4:1–4:27.

[KO18a] Tali Kaufman and Izhar Oppenheim. "Construction of new local spectral high dimensional expanders". In: *STOC*. Ed. by Ilias Diakonikolas, David Kempe, and Monika Henzinger. ACM, 2018, pp. 773–786.

[KO18b] Tali Kaufman and Izhar Oppenheim. "High Order Random Walks: Beyond Spectral Gap". In: *APPROX/RANDOM*. 2018, 47:1–47:17.

[LPW17] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times*. 2nd ed. American Mathematical Society, 2017.

[Mol04] Michael Molloy. "The Glauber dynamics on colorings of a graph with high girth and maximum degree". In: *SIAM Journal on Computing* 33.3 (2004), pp. 721–737.

[MSW04] Fabio Martinelli, Alistair Sinclair, and Dror Weitz. "Fast Mixing for Independent Sets, Colorings and Other Models on Trees". In: *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '04. New Orleans, Louisiana, 2004, pp. 456–465. ISBN: 0-89871-558-X.

[Opp18] Izhar Oppenheim. "Local spectral expansion approach to high dimensional expanders part I: Descent of spectral gaps". In: *Discrete and Computational Geometry* 59.2 (2018), pp. 293–330.

[Vig99] Eric Vigoda. "Improved bounds for sampling colorings". In: *FOCS*. IEEE, 1999, pp. 51–59.

[Yin14] Yitong Yin. "Spatial Mixing of Coloring Random Graphs". In: *Automata, Languages, and Programming*. Ed. by Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias. Springer Berlin Heidelberg, 2014, pp. 1075–1086.