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Continuity of logarithmic capacity

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ABSTRACT

We prove the continuity of logarithmic capacity under Hausdorff convergence of uniformly perfect planar sets. The continuity holds when the Hausdorff distance to the limit set tends to zero at sufficiently rapid rate, compared to the decay of the parameters involved in the uniformly perfect condition. The continuity may fail otherwise.

1. Introduction

The studies of the continuity of set capacity and related quantities have a long history in potential theory. In 1961, Gehring proved that the conformal modulus of planar annuli is continuous under Hausdorff convergence of the boundary components [6]. He then extended this results to the modulus of rings in space [7]. Aseev [2, Theorem 7] proved the continuity of condenser capacity under the Hausdorff convergence of its plates, under the assumption that the plates are uniformly perfect with the same constant α. Aseev and Lazareva [3] proved the analogous continuity result for logarithmic capacity of sets. Ransford, Younsi and Ai [13] recently proved that logarithmic capacity of a set varies continuously under holomorphic motions.

A more general theorem of Aseev [2, Theorem 6] involves the concept of strong convergence. According to [2], a sequence of sets E_n strongly converges to a set E if there exists $\alpha > 0$ such that E_n can be expressed as the union of α-uniformly perfect sets with diameters bounded below by some constant $\delta_\alpha(E_n)$,

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and $d(E_n, E)/\delta_\alpha(E_n) \to 0$. By [2, Theorem 6] the conformal capacities $\mathrm{cap}(E_n^0, E_n^1)$ converge to $\mathrm{cap}(E^0, E^1)$ if $E_n^j \to E^j$ strongly for $j = 0, 1$.

Although Aseev's theorem weakens the assumption of the sets being uniformly perfect, it does not cover the naturally occurring case of uniformly perfect sets with parameters that are not bounded away from 0. This is the setting of the present article. Our main result is Theorem 3.4, which asserts in part that $\mathrm{cap}(E_n) \to \mathrm{cap}(E)$ whenever E_n is α_n-uniformly perfect and the Hausdorff distance $d_n := d_H(E_n, E)$ tends to 0 sufficiently quickly, compared to α_n (these quantities are introduced in Definition 2.2, (2.4), and (2.1)). Specifically, having

$$\log \frac{1}{\alpha_n} \le \frac{1}{24} \frac{\log(1/d_n)}{\log\log(1/d_n)}$$

is sufficient for continuity, by Remark 3.6.

On the other hand, Proposition 4.1 shows that an inequality of the form

$$\log \frac{1}{\alpha_n} \le \frac{C}{d_n}$$

does not ensure that $\mathrm{cap}\, E_n$ converges to $\mathrm{cap}\, E$. In the final section we consider an application of our main result to the NED property [1] of Cantor-type sets.

2. Definitions and preliminary results

Definition 2.1. The *Hausdorff distance* between two nonempty bounded closed sets A and B is defined as

$$d_H(A, B) := \max\left(\sup_{z\in A}\inf_{t\in B}|z-t|, \sup_{z\in B}\inf_{t\in A}|z-t|\right). \tag{2.1}$$

Below we briefly describe some of the key concepts of the potential theory in the complex plane; see [12, 14,17] for more.

Let $E \subset \mathbb{C}$ be a compact set in the complex plane. The collection of all positive unit Borel measures with support in E is denoted $\mathcal{M}(E)$. The *logarithmic energy* of a measure $\nu \in \mathcal{M}(E)$ is defined as

$$I(\nu) := \iint \log \frac{1}{|z-t|} d\nu(z) d\nu(t), \tag{2.2}$$

and the energy V of E by

$$V := \inf\{I(\nu) \colon \nu \in \mathcal{M}(E)\}. \tag{2.3}$$

The energy V takes values in $(-\infty, \infty]$. When it is finite, there is a unique *equilibrium measure* $\nu_E \in \mathcal{M}(E)$ for which the infimum defining V in (2.3) is attained. The quantity

$$\mathrm{cap}(E) := e^{-V} \tag{2.4}$$

is called the *logarithmic capacity* of compact set E. For a general set $E \subset \mathbb{C}$,

$$\mathrm{cap}(E) := \sup\{\mathrm{cap}(K) \colon K \subset E, K \text{ is compact}\} \tag{2.5}$$

which is also known as the *inner capacity* of E.

The logarithmic capacity of a compact set $E \subset \mathbb{C}$ is equal to its *transfinite diameter* [17, Theorem III.26], which is defined as $d(E) = \lim_{n\to\infty} d_n(E)$ where

$$d_n(E)^{n(n-1)/2} = \sup \left\{ \prod_{1\le k<\ell\le n} |z_k - z_\ell| \colon z_1, \dots, z_n \in E \right\}$$

The quantity $d_n(E)$, $n = 2, 3, \dots$, is called the n-diameter of E. It is clearly continuous with respect to Hausdorff metric. Since $d_{n+1}(E) \le d_n(E)$ by [17, Theorem III.21], it follows that transfinite diameter is upper semicontinuous in the Hausdorff metric d_H. For future reference, we state this in terms of capacity:

$$\operatorname{cap}(E) \ge \limsup_{n\to\infty} \operatorname{cap}(E_n) \quad \text{if } d_H(E_n, E) \to 0. \tag{2.6}$$

Strict inequality may hold in (2.6), for example if $\{E_n\}$ is a sequence of finite sets converging to the closed unit disk.

Let Ω be a domain in the extended complex plane $\overline{\mathbb{C}}$. If the complement of Ω has positive capacity, then for every $w \in \Omega$ there exists Green's function $g(\cdot, w, \Omega)$ with a pole $w \in \Omega$. It is the unique function such that

- $g(\cdot, w, \Omega)$ is harmonic in $\Omega \setminus \{w\}$;
- $g(z, w, \Omega) = -\log|z - w| + O(1)$ as $z \to w$ if w is finite;
- $g(z, w, \Omega) = \log|z| + O(1)$, as $z \to w$ if $w = \infty$;
- $g(z, w, \Omega) \to 0$ as $z \to \zeta$, for all points $\zeta \in \partial\Omega$ except for a set of capacity zero.

The domain Ω is *regular* if $g(z, w, \Omega) \to 0$ holds for all boundary points $\zeta \in \partial\Omega$. It suffices to check this property for one value of w, by [12, Theorem 4.4.9]. When Ω is regular, we let $g(z, w, \Omega) = 0$ for $z \notin \Omega$, to make Green's function continuous on $\overline{\mathbb{C}} \setminus \{w\}$.

Let ν be a finite positive Borel measure of compact support. Its *logarithmic potential* is defined by

$$U^\nu(z) := \int \log \frac{1}{|z-t|} d\nu(t). \tag{2.7}$$

Let Ω be the *outer domain* relative to E, that is the unbounded component of the complement $\overline{\mathbb{C}} \setminus E$. Then we have an identity (e.g., [14, p. 53])

$$g(z, \infty, \Omega) = -U^{\nu_E}(z) + \log \frac{1}{\operatorname{cap}(E)} \tag{2.8}$$

The right hand side of (2.8) has asymptotic behavior $\log|z| + \log \frac{1}{\operatorname{cap}(E)} + o(1)$ as $z \to \infty$. We also use notation

$$g_E(z) = g(z, \infty, \Omega).$$

Our approach requires an explicit Hölder estimate for Green's function of a uniformly perfect set.

Definition 2.2. A closed set $E \subset \mathbb{C}$ is *uniformly perfect* if there exists a constant $\alpha \in (0, 1)$ such that the set $E \cap \{z \colon \alpha r \le |z - a| \le r\}$ is nonempty for every $a \in E$ and every r such that $0 < r \le \operatorname{diam} E$.

To emphasize the value of α, we sometimes call E an α-uniformly perfect set.

By [10, Theorem 1],

$$\operatorname{cap}(E \cap \overline{B}(a,r)) \geq \frac{\alpha^2}{32} r \tag{2.9}$$

for all $0 < r \leq \operatorname{diam} E$. Note that the definition of a uniformly perfect set in [10] requires the set to be unbounded. However, the proof of [10, Theorem 1] applies verbatim to our situation.

The following theorem of Siciak [15, Theorem 4.1] provides a Hölder estimate for Green's function of a uniformly perfect set. It requires additional notation. Given a compact subset E of a disk $B(a,R)$, let $h(\cdot, E, B(a,R))$ be the Perron solution [12, Def. 4.1.1] of the Dirichlet problem $\Delta u = 0$ in $B(a,R) \setminus E$, $u = 0$ on E, and $u = 1$ on $\partial B(a,R)$. For $1 \leq r < R$ let

$$c(E; B(a,R), \overline{B}(a,r)) = 1 - \sup_{|z-a|=r} h(z, E, B(a,R)).$$

This capacity-like quantity takes values between 0 and 1 and is monotone with respect to E. Finally, let

$$c_E(a,t,r,R) = c(E \cap \overline{B}(a,t); B(a,tR), \overline{B}(a,tr)), \quad 0 \leq t \leq 1$$

which represents the c-capacity of the t-neighborhood of a in E, scaled according to its size.

Theorem 2.3. *[15, Theorem 4.1] Let $1 \leq r < R < \infty$ and let $\{\rho_n\}$ be a sequence of real numbers such that $0 < \rho_n < 1$ and*

$$\frac{R}{r} \leq \frac{\rho_n}{\rho_{n+1}} \leq B < \infty, \quad n \geq 1. \tag{2.10}$$

If a is a point of a compact set E of $\mathbb{C}$ such that $c_E(a, \rho_n, r, R) \geq m > 0$ $(n \geq 1)$, then for every $\rho > 0$ the function $g_{E \cap \overline{B}(a,\rho)}$ is Hölder continuous at a with exponent $\mu = m/\log B$:

$$g_{E \cap \overline{B(a,\rho)}}(z) \leq M \delta^{m/\log B}, \quad |z-a| \leq \delta \leq 1, \tag{2.11}$$

where M depends only on ρ, r, R, m, and B.

Definition 2.4. Let $w \in \mathbb{C}$ be given and let $\Omega_n \subset \overline{\mathbb{C}}$ be domains such that w is an interior point of $\bigcap_{n=1}^{\infty} \Omega_n$. Following [11, p. 13], we say that

$$\Omega_n \to \Omega \text{ as } n \to \infty \text{ with respect to } w$$

in the sense of *kernel convergence* if

- Ω is a domain such that $w \in \Omega$ and some neighborhood of every $z \in \Omega$ lies in Ω_n for large n;
- for $z \in \partial\Omega$ there exist $z_n \in \partial\Omega_n$ such that $z_n \to z$ as $n \to \infty$.

An equivalent definition is found in [5, p. 77] and [8, p. 54]. According to it, Ω is the kernel of $\{\Omega_n\}$ if it is the maximal domain containing w such that every compact subset of Ω belongs to all but finitely many of the domains Ω_n. The convergence to Ω in the sense of kernel requires that every subsequence of $\{\Omega_n\}$ also has the same kernel Ω.

3. Main results

Our first step is to prove a version of the estimate (2.11) for Green's function in which the modulus of continuity has an explicit value of the multiplicative constant M. Such an estimate will be obtained from the formula (2a) in the proof of [15, Theorem 4.1], which states that under the assumptions of Theorem 2.3,

$$h(z, E \cap \overline{B}(a, \rho_n), B(a, R\rho_n)) \le \left(\frac{1}{r\rho_n}\right)^{m/\log B} \delta^{m/\log B} \tag{3.1}$$

for all z with $|z - a| \le \delta \le \min(1, r\rho_{n+1})$.

In order to use (3.1), we need to relate the function h to Green's function g_E.

Lemma 3.1. *Suppose that $E \subset \mathbb{C}$ is a compact set of positive logarithmic capacity. Then for $a \in E$ and for $R > \operatorname{diam} E$ we have*

$$g_E(z) \le \left(\log \frac{R + \operatorname{diam} E}{\operatorname{cap} E}\right) h(z, E, B(a, R)) \tag{3.2}$$

for all z with $|z - a| \le R$.

Proof. Both sides of (3.2) are harmonic in the set $\Omega = B(a, R) \setminus E$ and vanish on E up to a polar set. By definition, $h = 1$ on $\partial B(a, R)$. Writing g_E in terms of the potential of the equilibrium measure ν_E, we obtain from (2.8) that for all $z \in \partial B(a, R)$,

$$g_E(z) = \log \frac{1}{\operatorname{cap} E} + \int \log |z - \zeta| \, d\nu_E(\zeta) \le \log \frac{1}{\operatorname{cap} E} + \log(R + \operatorname{diam} E)$$

because $|z-\zeta| \le R + \operatorname{diam} E$ and ν_E is a probability measure. Hence (3.2) holds on $\partial B(a, R)$. The maximum principle completes the proof. $\square$

We also need to translate the geometric property of being α-uniformly perfect into a lower bound on the capacity $c_E(a, \rho)$ required by Theorem 2.3. Lemma 1.7 of [15] states that for $0 < t \le 1$,

$$c_E(a, t, r, R) \ge \left(\log \frac{R - 1}{r + 1}\right) \left(\log \frac{t(R - 1)}{\operatorname{cap}(E \cap \overline{B}(a, t))}\right)^{-1} \tag{3.3}$$

provided that $r < R - 2$. Combining (2.9) and (3.3) we obtain the following: if $E \subset \mathbb{C}$ is an α-uniformly perfect set of diameter 1, then for every $a \in E$ and $0 < t \le 1$,

$$c_E(a, t, r, R) \ge \left(\log \frac{R - 1}{r + 1}\right) \left(\log \frac{32(R - 1)}{\alpha^2}\right)^{-1} \tag{3.4}$$

provided that $r < R - 2$.

Theorem 3.2. *Let E be an α-uniformly perfect bounded set. Let*

$$\begin{aligned} M = M(\alpha) &= \log \frac{384}{\alpha^2}, \\ \beta = \beta(\alpha) &= \frac{\log(9/8)}{2 \log 2} \left(\log \frac{288}{\alpha^2}\right)^{-1}. \end{aligned} \tag{3.5}$$

Then for all points z with $\operatorname{dist}(z, E) \le \operatorname{diam} E$ *we have*

$$g_E(z) \le M \left(\frac{\operatorname{dist}(z, E)}{\operatorname{diam} E} \right)^{\beta}. \tag{3.6}$$

Proof. The problem reduces to the case $\operatorname{diam} E = 1$, $0 \in E$, and $|z| = \operatorname{dist}(z, E)$ by rescaling and translation.

Let $a = 0$, $R = 10$, and $r = 7$ in (3.4):

$$c_E(a, t, 7, 10) \ge m := \left(\log \frac{9}{8} \right) \left(\log \frac{288}{\alpha^2} \right)^{-1}, \quad 0 < t \le 1. \tag{3.7}$$

In Theorem 2.3 choose $B = 2$ and $\rho_n = 2^{-n}$. Then the potential function h associated with the set $E_n = E \cap \overline{B}(0, 2^{-n})$ can be estimated by (3.1) as follows.

$$h(z, E_n, B(0, 10 \cdot 2^{-n})) \le \left(\frac{2^n}{7} \right)^{m/\log 2} \delta^{m/\log 2} \tag{3.8}$$

for all z with $|z| \le \delta \le \min(1, 7 \cdot 2^{-n-1})$. Note that $\operatorname{diam} E_n \le 2^{1-n}$ and $\operatorname{cap} E_n \ge \frac{\alpha^2}{32} 2^{-n}$ by (2.9). Apply Lemma 3.1 to E_n with $a = 0$ and $R = 10 \cdot 2^{-n}$, and then invoke (3.8) to obtain

$$\begin{aligned} g_{E_n}(z) &\le \left(\log \frac{384}{\alpha^2} \right) h(z, E_n, B(0, 10 \cdot 2^{-n})) \\ &\le \left(\log \frac{384}{\alpha^2} \right) \left(\frac{2^n}{7} \right)^{m/\log 2} \delta^{m/\log 2} \end{aligned} \tag{3.9}$$

for all z with $|z| \le \delta \le \min(1, 7 \cdot 2^{-n-1})$.

Given a complex number z with $|z| \le 1$, let n be the smallest integer such that $|z| \le 4^{1-n}$. Since $4^{1-n} < 7 \cdot 2^{-n-1}$, we can choose $\delta = 4^{1-n}$ in (3.9), thus obtaining

$$\begin{aligned} g_{E_n}(z) &\le \left(\log \frac{384}{\alpha^2} \right) \left(\frac{4}{7} 2^{-n} \right)^{m/\log 2} \\ &\le \left(\log \frac{384}{\alpha^2} \right) |z|^{m/(2\log 2)} \end{aligned} \tag{3.10}$$

where the last inequality holds because $4^{-n} < |z|$ by the choice of n. This proves (3.6). $\square$

Remark 3.3. The Hölder exponent in Theorem 3.2 is not optimal. In general it is difficult to determine the precise degree of Hölder regularity of Green's function, but see [4,16] for such results in the context of Cantor-type sets.

We are ready to state and prove our main result.

Theorem 3.4. *Suppose that for each $n \in \mathbb{N}$, E_n is a compact α_n-uniformly perfect subset of $\mathbb{C}$. Furthermore, suppose $E_n \to E \subset \mathbb{C}$ in the Hausdorff metric d_H, where E is a compact set. If the sequence*

$$d_H(E_n, E) \exp \left(24 \log \frac{1}{\alpha_n} \log \log \frac{1}{\alpha_n} \right) \tag{3.11}$$

is bounded, then $\operatorname{cap}(E_n) \to \operatorname{cap}(E)$. *In addition, if* $\operatorname{cap}(E) > 0$, *then Green's functions $g_{E_n}(z)$ converge to $g_E(z)$ uniformly with respect to $z \in \mathbb{C}$ and the unbounded component Ω of $\mathbb{C} \setminus E$ is a regular domain for the Dirichlet problem.*

The proof requires a lemma from [9].

Lemma 3.5. *[9, Lemma 2.1] If $\Omega_n \to \Omega$ in the sense of kernel with respect to $w \in \Omega$, and each domain Ω_n is regular, then*

$$\sup_{z\in\mathbb{C}\setminus\{w\}} (g(z, w, \Omega) - g(z, w, \Omega_n)) \to 0.$$

Although in [9] we assumed all domains to be regular, only the regularity of Ω_n was used in the proof of Lemma 3.5. We also note that when the complements Ω_n^c converge in the sense of Hausdorff metric, the domains Ω converge in the sense of kernel (see e.g. [8, p. 54]) and therefore Lemma 3.5 applies.

Proof of Theorem 3.4. If $\operatorname{cap}(E) = 0$, then the upper semicontinuity of capacity (2.6) implies $\operatorname{cap}(E_n) \to 0$. For the rest of the proof we assume $\operatorname{cap}(E) > 0$. This ensures the existence of Green's function g_E. We write $g = g_E$ and $g_n = g_{E_n}$. Since E_n is a uniformly perfect set, each component of its complement is regular [10, Remark 1], and therefore g_n extends continuously to $\mathbb{C}$.

Since $\operatorname{diam} E > 0$, we may rescale the sets so that $\operatorname{diam} E = 1$. The assumption $d_H(E_n, E) \to 0$ implies $\operatorname{diam} E_n \to 1$. By discarding finitely many terms of the sequence $\{E_n\}$ we can ensure that $d_H(E_n, E) \le 1/3$ and $\operatorname{diam} E_n \ge 2/3$ for all n.

Let $M_n = M(\alpha_n)$ and $\beta_n = \beta(\alpha_n)$ be as in (3.5). Theorem 3.2 yields

$$g_n(z) \le M_n \left(\frac{\operatorname{dist}(z, E_n)}{\operatorname{diam} E_n}\right)^{\beta_n}, \quad \text{if } \operatorname{dist}(z, E_n) \le 2/3. \tag{3.12}$$

Our next step is to show that

$$M_n d_H(E_n, E)^{\beta_n} \to 0. \tag{3.13}$$

Indeed, if α_n is bounded from below by a positive constant α, then $M_n \le M(\alpha)$ and $\beta_n \ge M(\alpha)$, hence (3.13) holds by virtue of $d_H(E_n, E) \to 0$. Consider the case $\alpha_n \to 0$. The logarithm of the left-hand side of (3.13) does not exceed

$$A + \log\log\frac{384}{\alpha_n^2} - C\log\frac{1}{\alpha_n}\log\log\frac{1}{\alpha_n}\left(\log\frac{288}{\alpha_n^2}\right)^{-1} \tag{3.14}$$

where A is some constant and

$$C = 24\frac{\log(9/8)}{2\log 2} > 2.$$

Up to a bounded additive term, the expression (3.14) simplifies to

$$(1 - C/2)\log\log\frac{1}{\alpha_n} \to -\infty$$

which proves (3.13) in this case as well. The case when $\limsup \alpha_n > \liminf \alpha_n = 0$ follows by considering subsequences.

We are now ready to prove that $g_n \to g$ uniformly on Ω. Lemma 3.5 shows that $\limsup_{n\to\infty} \sup_\Omega(g - g_n) \le 0$. Thus, uniform convergence on Ω will be established once we show that

$$\limsup_{n\to\infty} \sup_\Omega (g_n - g) \le 0 \tag{3.15}$$

Because of (3.13), there exists a sequence of positive numbers $\delta_n \in (0, 1/3]$ such that $M_n(d_H(E_n, E) + \delta_n)^{\beta_n} \to 0$. Suppose $z \in \mathbb{C}$ is such that $\operatorname{dist}(z, E) \leq \delta_n$. Since $\operatorname{dist}(z, E_n) \leq d_H(E, E_n) + \delta_n \leq 2/3$, we can apply (3.12) which yields

$$\begin{aligned} g_n(z) &\leq M_n \left(\frac{\operatorname{dist}(z, E_n)}{\operatorname{diam} E_n} \right)^{\beta_n} \\ &\leq M_n \left(\frac{d_H(E, E_n) + \delta_n}{\operatorname{diam} E_n} \right)^{\beta_n} =: \epsilon_n \end{aligned} \tag{3.16}$$

where $\epsilon_n \to 0$.

If $z \in \Omega$ and $\operatorname{dist}(z, E) \leq \delta_n$, then $g_n(z) - g(z) \leq g_n(z) \leq \epsilon_n$ by (3.16). Since the singularity of $g_n - g$ at ∞ is removable, the maximum principle shows that $g_n - g \leq \epsilon_n$ everywhere in Ω. This concludes the proof of (3.15) and of the uniform convergence $g_n \to g$ on Ω.

Using the asymptotic expansion

$$g_E(z) = \log |z| - \log \operatorname{cap}(E) + o(1), \quad z \to \infty,$$

we conclude that $\operatorname{cap}(E_n) \to \operatorname{cap}(E)$.

The uniformity of convergence $g_n \to g$ on Ω allows us to interchange limits with respect to $n \in \mathbb{N}$ and $z \in \Omega$ below: for every $\zeta \in \partial\Omega$

$$\lim_{z\to\zeta} g(z) = \lim_{z\to\zeta} \lim_{n\to\infty} g_n(z) = \lim_{n\to\infty} \lim_{z\to\zeta} g_n(z) = \lim_{n\to\infty} g_n(\zeta) = 0 \tag{3.17}$$

where the last step uses (3.16) and the fact that $\lim_{n\to\infty} \operatorname{dist}(\zeta, E_n) = 0$.

Property (3.17) shows that Ω is a regular domain. Thus, we have $g_E = 0$ on the complement of Ω.

It remains to show that $g_n \to 0$ uniformly on $\mathbb{C} \setminus \Omega$. The continuity of g_n and the estimate (3.16) imply $g_n \leq \epsilon_n$ on $\partial\Omega$. By the maximum principle, $g_n \leq \epsilon_n$ on $\mathbb{C} \setminus \Omega$, which completes the proof. $\square$

Remark 3.6. The boundedness assumption in Theorem 3.4 holds if for all sufficiently large n,

$$\log \frac{1}{\alpha_n} \leq \frac{1}{24} \frac{\log b_n}{\log \log b_n} \tag{3.18}$$

where $b_n = 1/d_H(E, E_n) \to \infty$ as $n \to \infty$.

Indeed, for large n we have $\log \log b_n > 1$, hence (3.18) implies

$$24 \log \frac{1}{\alpha_n} \log \log \frac{1}{\alpha_n} \leq \frac{\log b_n}{\log \log b_n} (\log \log b_n - \log 24) < \log b_n$$

Therefore the sequence (3.11) is bounded by 1.

4. Examples and applications

If a sequence of α_n-uniformly perfect sets E_n has $\alpha_n \to 0$ much faster than $d_H(E_n, E) \to 0$, the logarithmic capacity of E_n may fail to converge to the logarithmic capacity of E. The following proposition presents a concrete form of this observation.

Proposition 4.1. *In Theorem 3.4, the sequence (3.11) cannot be replaced by $d_H(E_n, E) \log \alpha_n$. More precisely, there exists a sequence of compact sets E_n which are α_n-uniformly perfect and converge to $\mathbb{T}$ in the uniform* *metric in such a way that $d_H(E_n, E)\log\alpha_n$ is bounded, yet the uniform convergence of Green's functions fails.*

Proof. The idea of this example goes back to Ahlfors and Beurling [1, Theorem 17]. Given a sequence of numbers $L_n \in (0,\pi)$, we construct a sequence of compact subsets of the unit circle $\mathbb{T}$ as follows:

$$E_n = \{z \in \mathbb{T} \colon |\arg(z^n)| \le L_n\}$$

where arg is the principal branch of the argument, taking values between $-\pi$ and π. The set E_n consists of n uniformly distributed arcs of length $2L_n/n$. The gaps between these arcs have length $2(\pi - L_n)/n$, which implies that

$$d_H(E_n, \mathbb{T}) = 2\sin\frac{\pi - L_n}{2n} \tag{4.1}$$

Hence $E_n \to \mathbb{T}$ in the Hausdorff metric.

The diameter of each connected component of E_n is $2\sin(L_n/n)$ and the distance from a component to the rest of E_n is $2\sin((\pi - L_n)/n)$. Suppose that an annulus $\{z \colon r < |z-a| < R\}$ separates E_n. Since the disk $\{z \colon |z-a| \le r\}$ contains a connected component of E_n, we have $r \ge \sin(L_n/n)$. Since also $R - r \le 2\sin((\pi - L_n)/n)$, it follows that

$$\frac{R}{r} \le 1 + \frac{2\sin((\pi - L_n)/n)}{\sin(L_n/n)}$$

Hence, E_n is α_n-uniformly perfect with

$$\alpha_n \ge \left(1 + \frac{2\sin((\pi - L_n)/n)}{\sin(L_n/n)}\right)^{-1}$$

If $L_n \to 0$, this bound on α_n is asymptotic to $L_n/(2\pi)$.

The logarithmic capacity of the circular arc $\Gamma_n = \{e^{it} \colon |t| \le L_n\}$ is equal to $\sin(L_n/2)$ (see e.g. [12, Ch. 5, Table 5.1, p. 135]). Since the set E_n is the preimage of Γ_n under the polynomial $z \mapsto z^n$, it follows that ([12, Theorem 5.2.5, p. 134])

$$\operatorname{cap} E_n = (\operatorname{cap}\Gamma_n)^{1/n} = (\sin(L_n/2))^{1/n}$$

Thus, $\operatorname{cap} E_n \to \operatorname{cap}\mathbb{T} = 1$ if and only if $\log(1/L_n) = o(n)$ as $n \to \infty$.

For example, the choice $L_n = \exp(-n)$ results in $\operatorname{cap} E_n \not\to \operatorname{cap}\mathbb{T}$, which also indicates the failure of uniform convergence of Green's functions. With this choice we have $\log 1/\alpha_n$ asymptotic to n and $d_H(E_n, \mathbb{T}) \le \pi/n$ by virtue of (4.1). Thus the product $d_H(E_n, \mathbb{T})\log\alpha$ is bounded. $\square$

There remains a substantial gap between the assumptions of Theorem 3.4 and Proposition 4.1. As an application of Theorem 3.4 we consider the NED property of Cantor-type sets. The notion of an *NED set* is an important function-theoretic concept of a removability, introduced by Ahlfors and Beurling in [1]. For example, NED sets are removable for holomorphic functions f with finite Dirichlet integral $\int |f'|^2$ and for extremal distances. We do not state the general definition of NED sets here, because the following theorem of Ahlfors and Beurling [1, Theorem 14] suffices for other purposes: a compact subset K of an interval I is NED if and only if

$$\operatorname{cap}(I \setminus K) = \operatorname{cap}(I). \tag{4.2}$$

The left hand side of (4.2) is the inner capacity (2.5) of the non-compact set $I \setminus K$.

Let $I = [0, 1]$. Given a sequence of numbers $\epsilon_n \in (0, 1)$, let $K_0 = I$ and inductively construct the sets $K_1 \supset K_2 \supset \dots$ so that K_n is obtained by removing the middle ϵ_n-part of each connected component of K_{n-1}. The intersection $K = \bigcap_{n=0}^{\infty} K_n$ is a Cantor-type set which becomes the standard middle-third Cantor set if $\epsilon_n = 1/3$ for all n. Let $E_n = \overline{I \setminus K_n}$ for $n = 1, 2, \cdots$. It is easy to show that $E_n \to [0, 1]$ in the Hausdorff distance; see the proof of Theorem 4.2 below. By the definition of inner capacity, property (4.2) holds if and only if $\operatorname{cap}(E_n) \to \operatorname{cap}(I)$ as $n \to \infty$. This leads us to the following result.

Theorem 4.2. *Suppose K is a Cantor-type set determined by a sequence of numbers $\epsilon_n \in (0, 1)$ such that*

$$\log \frac{1}{\epsilon_n} \le \frac{Cn}{\log n}, \quad n \ge 2, \tag{4.3}$$

for some constant $C < 1/(24 \log 2)$. Then (4.2) holds, and consequently K is an NED set.

Proof. Since K_n consists of 2^n disjoint segments of equal length, each of them has length at most 2^{-n}. Therefore, the 2^{-n-1} neighborhood of E_n covers I. It follows that $d_H(E_n, I) \le 2^{-n-1}$.

We claim that the set E_n is α_n-uniformly perfect where $\alpha_n = \frac{1}{2} \min_{k \le n} \epsilon_k$. Since E_1 is an interval, it suffices to consider $n \ge 2$. Note that the set E_k is constructed by inserting an interval in the middle of each component of $[0, 1] \setminus E_{k-1}$; the length of this interval is $\epsilon_k \ell$ where ℓ is the length of the component. Therefore, the distance from the inserted interval to E_{k-1} is $(1 - \epsilon_k)\ell/2$. It follows that every connected component J of the set E_n satisfies

$$\operatorname{dist}(J, E_n \setminus J) \le \frac{1 - \alpha_n}{2\alpha_n} \operatorname{diam} J. \tag{4.4}$$

Suppose that $a \in E_n$, $0 < r \le \operatorname{diam} E$, and the annulus $\{z : \alpha r \le |z - a| \le r\}$ is disjoint from E. Let k be the smallest index such that $E_k \cap B(a, \alpha r)$ is nonempty. If $k = 1$, then $B(a, \alpha r)$ contains $[(1 - \epsilon_1)/2, (1 + \epsilon_1)/2]$, hence $\alpha r \ge \epsilon_1/2$. And since $r \le \operatorname{diam} E_n \le 1$, it follows that $\alpha \ge \epsilon_1/2 \ge \alpha_n$ as claimed.

Suppose $k \ge 2$. If $B(a, \alpha r)$ contained more than one component of E_k, then it would also contain a component of E_{k-1} situated between those, contrary to the choice of k. Thus, the set $J = E_k \cap B(a, \alpha r)$ is connected. Since $\operatorname{diam} J \le 2\alpha_n r$, the estimate (4.4) implies

$$(1 - \alpha)r \le \operatorname{dist}(J, E_k \setminus B(a, \alpha r)) = \operatorname{dist}(J, E_k \setminus J) \le (1 - \alpha_n)r,$$

hence $\alpha \ge \alpha_n$. This completes the proof that E_n is α_n-perfect.

To justify the application of Theorem 3.4, we use Remark 3.6. Indeed, in the inequality (3.18) we have $\log b_n = \log(1/d_H(E_n, I)) \ge (n + 1) \log 2$, which in view of (4.3) implies that (3.18) holds. Thus, $\operatorname{cap}(E_n) \to \operatorname{cap}(I)$. $\square$

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