MIXING PROPERTIES OF COLORINGS OF THE $\mathbb{Z}^d$ LATTICE

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Abstract. We study and classify proper q-colorings of the $\mathbb{Z}^d$ lattice, identifying three regimes where different combinatorial behavior holds: (1) When $q \leq d+1$, there exist frozen colorings, that is, proper q-colorings of $\mathbb{Z}^d$ which cannot be modified on any finite subset. (2) We prove a strong list-coloring property which implies that, when $q \geq d+2$, any proper q-coloring of the boundary of a box of side length $n \geq d+2$ can be extended to a proper q-coloring of the entire box. (3) When $q \geq 2d+1$, the latter holds for any $n \geq 1$. Consequently, we classify the space of proper q-colorings of the $\mathbb{Z}^d$ lattice by their mixing properties.

1. Introduction

A *proper coloring* of a graph G is an assignment of a color (say a number in $\mathbb{Z}$) to each vertex of G so that adjacent vertices are assigned different colors. For an integer $q \geq 2$, a (proper) *q-coloring* of G is a proper coloring in which all colors belong to a fixed set of size q, e.g., $\{0, 1, \ldots, q-1\}$.

In this work, we mainly consider the d-dimensional integer lattice $\mathbb{Z}^d$ for $d \geq 1$. We view it both as the group and its Cayley graph with respect to the standard generators. Notice that, in this case, the number of neighbors of each vertex is $2d$.

Our first result is about frozen q-colorings. A q-coloring of $\mathbb{Z}^d$ is *frozen* if any q-coloring of $\mathbb{Z}^d$ which differs from it on finitely many sites is identical to it. The existence of a frozen q-coloring precludes the possibility of any reasonable mixing property.

Theorem 1.1. *There exist frozen q-colorings of $\mathbb{Z}^d$ if and only if $2 \leq q \leq d+1$.*

Since there are no frozen q-colorings of $\mathbb{Z}^d$ when $q \geq d+2$, a natural question is how unconstrained are proper colorings in this regime. In order to understand better what happens when $q \geq d+2$, we show that a certain list-coloring property of large boxes in $\mathbb{Z}^d$ holds. A consequence of this property will be that whenever $q \geq d+2$, large boxes in $\mathbb{Z}^d$ have the property that any partial proper q-coloring of their boundary can be extended to a proper q-coloring of (the interior of) the entire box. To state the list-coloring result precisely, we now introduce some definitions.

For a graph G and a function $L : G \to \mathbb{N}$, we say that G is *L-list-colorable* if for any collection of sets – also called *lists* – $\{S_v\}_{v \in G}$ with $|S_v| \geq L(v)$, there exists a proper coloring f of G such that $f(v) \in S_v$ for all $v \in G$.

Denote $[n] := \{1, \ldots, n\}$. Depending on the context, $[n]^d = \{1, \ldots, n\}^d$ may also be interpreted as an induced subgraph of $\mathbb{Z}^d$. Define $L_n^d : [n]^d \to \{2, \ldots, d+2\}$ by

$$L_n^d(\vec{i}) := 2 + |\{1 \leq k \leq d \;:\; 1 < |i_k| < n\}|.$$

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Our second result is the following.

Theorem 1.2. *The graph $[n]^d$ is L^d_n-list-colorable whenever $n \geq d+2$.*

The above results have implications for the mixing properties of the set of all q-colorings. Specifically, we characterise when the set of q-colorings of $\mathbb{Z}^d$ is topologically mixing, strongly irreducible, has the finite extension property and is topologically strong spatial mixing (see Section 4 for definitions and results). Mixing properties have important consequences in statistical physics [10], dynamical systems [3], and the study of constrained satisfaction problems [7]. We discuss some of these aspects in Section 5.

The rest of the paper is organized as follows. Section 2 is dedicated to frozen q-colorings and, in particular, contains the proof of Theorem 1.1. In addition, we also prove a result about non-existence of frozen q-colorings in general graphs satisfying appropriate expansion properties. Section 3 is dedicated to list-colorings and, in particular, contains the proof of Theorem 1.2. In Section 4, we introduce a hierarchy of mixing properties and, using the previous results, show that for a fixed dimension d, there exist two critical numbers of colors, namely $q = d+1$ and $q = 2d$, that determine three different mixing regimes. Finally, in Section 5, we conclude with a discussion and open questions.

We end this section with some notation that will be used throughout the paper. The set of edges of a graph G is denoted by $\mathbb{E}_G$. Given $U, V \subset G$, we denote by $\mathbb{E}(U,V) \subset \mathbb{E}_G$ the set of edges between a vertex of U and a vertex of V. Given a set $U \subset G$, we denote the *external vertex boundary* of U by

$$\partial U := \{v \in G \setminus U \ : \ v \text{ is adjacent to some } u \in U\}.$$

2. Frozen q-colorings

In this section, we prove Theorem 1.1. We split the proof into two parts: existence and non-existence of frozen q-colorings. Theorem 1.1 is a direct consequence of Proposition 2.1 and Corollary 2.3 below. We begin with the existence of frozen q-colorings when $q \leq d+1$. Many constructions have appeared in the past which are similar in principle (see, e.g., [8, Section 8]).

Proposition 2.1. *There exist frozen q-colorings of $\mathbb{Z}^d$ for any $2 \leq q \leq d+1$.*

Proof. First suppose that $q = d+1$ and let $x \in \{0,1,\dots,q-1\}^{\mathbb{Z}^d}$ be given by

$$x_{\vec{i}} := \sum_{k=1}^{d} k i_k \pmod q \qquad \text{for all } \vec{i} \in \mathbb{Z}^d.$$

First, let us verify that x is a q-coloring. Indeed, for $\vec{i} \in \mathbb{Z}^d$ and $\vec{e}_k$ the unit vector in the kth direction ($1 \leq k \leq d$), we have that

$$x_{\vec{i}+\vec{e}_k} = x_{\vec{i}} + k \pmod q.$$

In particular, $x_{\vec{i}+\vec{e}_k} - x_{\vec{i}} = k \neq 0 \pmod q$, and thus adjacent vertices have different colors.

Let us now show that x is frozen. Suppose that y is a q-coloring that differs from x on finitely many sites. Among all $\vec{i}$ such that $x_{\vec{i}} \neq y_{\vec{i}}$, we choose one which maximizes $\sum_{k=1}^{d} i_k$. Then, for any $1 \leq k \leq d$,

$$y_{\vec{i}+\vec{e}_k} = x_{\vec{i}+\vec{e}_k} = x_{\vec{i}} + k \pmod q.$$

Therefore,

$$\{y_{\vec{i}+\vec{e}_k} \;:\; 1 \le k \le d\} = \{0,1,\ldots,q-1\} \setminus \{x_{\vec{i}}\}.$$

Since y is a proper q-coloring, it must be that $y_{\vec{i}} = x_{\vec{i}}$, contradicting the choice of $\vec{i}$.

Now, to deal with the case $q < d+1$, notice that by the previous construction, we already have a frozen q-coloring of $\mathbb{Z}^{q-1}$. Thus, it suffices to prove that we can always extend a frozen q-coloring x of $\mathbb{Z}^r$ to $\mathbb{Z}^{r+1}$, as we may then proceed by induction on r. Given a frozen q-coloring x of $\mathbb{Z}^r$, consider the q-coloring y of $\mathbb{Z}^{r+1}$ defined as

$$y_{(i_1,\ldots,i_{r+1})} := x_{(i_1,\ldots,i_{r-1},i_r+i_{r+1})},$$

which is clearly proper, since if $(i_1,\ldots,i_{r+1})$ is adjacent to $(j_1,\ldots,j_{r+1})$ in $\mathbb{Z}^{r+1}$, then $(i_1,\ldots,i_{r-1},i_r+i_{r+1})$ is adjacent to $(j_1,\ldots,j_{r-1},j_r+j_{r+1})$ in $\mathbb{Z}^r$. Notice that y is also frozen, since y restricted to $\mathbb{Z}^r \times \{i_{r+1}\}$ can be seen as a translate of the frozen q-coloring x. □

Next we prove that no frozen q-coloring exists when $q \ge d+2$. While this follows from the list-coloring result given in Theorem 1.2 (see also Theorem 4.4), we give a direct argument here which applies in greater generality. Let us now state the result precisely.

The *edge-isoperimetric constant* of a graph G is

$$h(G) := \inf_F \frac{|\mathbb{E}(F, G\setminus F)|}{|F|},$$

where the infimum is taken over all non-empty finite subsets of vertices F. As for $\mathbb{Z}^d$, a q-coloring of G is *frozen* if any q-coloring of G which differs from it on finitely many sites is identical to it.

Proposition 2.2. *Let G be a graph of maximum degree Δ and $q > \frac{1}{2}\Delta + \frac{1}{2}h(G) + 1$. Then there do not exist frozen q-colorings of G.*

Since, in the $\mathbb{Z}^d$ case, $\Delta = 2d$ and $h(\mathbb{Z}^d) = 0$ (e.g., consider sets of the form $F = [n]^d$ for increasing n), we have the following corollary.

Corollary 2.3. *There do not exist frozen q-colorings of $\mathbb{Z}^d$ for any $q \ge d+2$.*

Proposition 2.2 is sharp in many cases, including $\mathbb{Z}^d$ (Theorem 1.1), the triangular lattice (see Figure 1), the honeycomb lattice (trivially, since a 2-coloring of an infinite connected bipartite graph is always frozen), and regular trees (see [9, 16]).

We remark that finite (non-empty) graphs never admit frozen q-colorings since the colors can always be permuted. Proposition 2.2 is thus trivial for finite graphs. Nevertheless, our argument can provide meaningful information for finite graphs as well. Proposition 2.2 is an immediate consequence of the following.

Given a subset F of vertices of G, we say that a q-coloring of G is *frozen on F* if any q-coloring of G which differs from it on a subset of F is identical to it. Thus, a q-coloring is frozen if and only if it is frozen on every finite set.

Proposition 2.4. *Let G be a graph, $q \ge 2$, and $F \subset G$ finite. If $(q-1)|F| > |\mathbb{E}_F| + |\mathbb{E}(F, G\setminus F)|$, then no q-coloring of G is frozen on F.*

Since $2|\mathbb{E}_F| + |\mathbb{E}(F, G\setminus F)|$ equals the sum of the degrees of vertices in F, the following is an immediate corollary to Proposition 2.4.

Corollary 2.5. *Let G be a graph of maximum degree Δ, $q \ge 2$, and $F \subset G$ finite. If $(q - 1 - \frac{\Delta}{2})|F| > \frac{1}{2}|\mathbb{E}(F, G\setminus F)|$, then no q-coloring of G is frozen on F.*

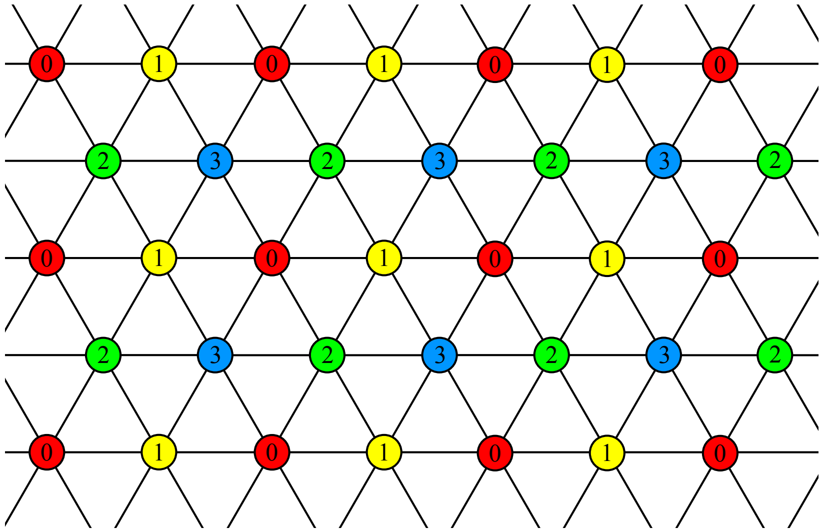

FIGURE 1. A frozen 4-coloring of the triangular lattice.

Proof of Proposition 2.4. Suppose towards a contradiction that there exists a q-coloring of G which is frozen on F. Consider the restriction of the coloring to the finite graph $G' := (F \cup \partial F, \mathbb{E}_F \cup \mathbb{E}(F, G \setminus F))$. For distinct colors i and j, let $G'_{i,j}$ be the subgraph of G' consisting of edges between vertices colored i and j. A *bi-color component* is a connected component of any such $G'_{i,j}$. Let $\mathcal{A}$ be the collection of bi-color components. Note that the bi-color components partition the edges of G' so that $\sum_{A \in \mathcal{A}} |\mathbb{E}_A| = |\mathbb{E}_F| + |\mathbb{E}(F, G \setminus F)|$. Note also that, since the q-coloring is frozen on F, each $A \in \mathcal{A}$ contains a vertex in ∂F, as otherwise, the two colors in A could be swapped (contradicting that the coloring is frozen). Hence, each $A \in \mathcal{A}$ contains at most $|\mathbb{E}_A|$ vertices of F. Thus, we have

$$\sum_{A \in \mathcal{A}} |A \cap F| \leq \sum_{A \in \mathcal{A}} |\mathbb{E}_A| = |\mathbb{E}_F| + |\mathbb{E}(F, G \setminus F)|.$$

On the other hand, using again that the q-coloring is frozen on F (and hence also on each individual vertex in F), we see that each vertex in F is contained in exactly $q-1$ bi-color components, and it follows that $\sum_{A \in \mathcal{A}} |A \cap F| = (q-1)|F|$. We thus conclude that $(q-1)|F| \leq |\mathbb{E}_F| + |\mathbb{E}(F, G \setminus F)|$, contradicting the assumption. $\square$

We remark that the proof of Proposition 2.4 shows something stronger, namely, that (under the assumption) any q-coloring of G can be modified by swapping the two colors of a bi-color component (a so-called *Kempe chain move*) contained in F.

We end this section with a short discussion about the existence of *single-site frozen* q-colorings, those which are frozen on every F having $|F| = 1$. Given a graph G of maximum degree Δ, it is clear that there do not exist single-site frozen q-colorings of G whenever $q \geq \Delta + 2$. On the other hand, on $\mathbb{Z}^d$, it is straightforward to check that

$$x_{\vec{i}} := \sum_{k=1}^{d} k i_k \pmod{2d+1}, \qquad \vec{i} \in \mathbb{Z}^d,$$

defines a single-site frozen $(2d+1)$-coloring of $\mathbb{Z}^d$. Similar constructions (like in the proof of Proposition 2.1) yield single-site frozen q-colorings of $\mathbb{Z}^d$ for any $2 \le q \le 2d$. Thus, single-site frozen q-colorings of $\mathbb{Z}^d$ exist if and only if $2 \le q \le 2d+1$.

3. List-colorability

In this section, we prove Theorem 1.2. We will use the main result from [2]. We say that a digraph D is L-list-colorable if the underlying undirected graph is such. Define $L_D(v) := d_D^+(v) + 1$, where $d_D^+(v)$ is the out-degree of v in D. The following is [2, Theorem 1.1] specialized to digraphs having no odd directed cycles (this special case also follows from Richardson's Theorem; see [2, Remark 2.4]).

Theorem 3.1. *A finite digraph D having no odd directed cycles is L_D-list-colorable.*

Thus, Theorem 3.1 allows to prove L-list-colorability of an undirected graph G by exhibiting an orientation of the edges of G so that the out-degree of any vertex v is strictly less than $L(v)$. The following provides a sufficient (and in fact necessary) condition for such an orientation to exist (see the closely related [2, Lemma 3.1]).

Corollary 3.2. *Let G be a finite bipartite graph and let $L : G \to \mathbb{N}$ satisfy*

$$\sum_{v \in H} (L(v) - 1) \ge |\mathbb{E}_H| \qquad \text{for any induced subgraph } H \subset G.$$

Then G is L-list-colorable.

Proof. Denote $G = (V, E)$ and consider the set $\mathcal{V} := \{(v, i) : v \in V, 1 \le i \le L(v) - 1\}$ containing $L(v) - 1$ copies of any vertex v. Let $\mathcal{G}$ be the bipartite graph with bipartition classes E and $\mathcal{V}$ in which $\{e, (v, i)\}$ is an edge in $\mathcal{G}$ if and only if v is incident to e. For any $F \subset E$, letting H be the subgraph of G induced by the endpoints of F, we see that the number of neighbors of F is $\sum_{v \in H} (L(v) - 1) \ge |\mathbb{E}_H| \ge |F|$. Thus, by Hall's theorem, $\mathcal{G}$ contains a matching of size $|E|$. Given such a matching, we obtain a digraph D on V by orienting each edge $e \in E$ from the vertex it is matched to outwards. In this orientation of G, the out-degree of any vertex v is at most $L(v) - 1$ so that $L_D(v) \le L(v)$. Since G is bipartite, D has no odd directed cycles. It follows from Theorem 3.1 that D (and hence also G) is L-list-colorable. □

Proof of Theorem 1.2. Let $\{K_t\}_{t=0}^d$ be the level sets of L_n^d, that is,

$$\begin{aligned} K_t :&= \left\{ \vec{i} \in [n]^d \ : \ L_n^d(\vec{i}) = t + 2 \right\} \\ &= \left\{ \vec{i} \in [n]^d \ : \ |\{1 \le j \le d \ : \ 1 < |i_j| < n\}| = t \right\}. \end{aligned}$$

Let H be an induced subgraph of G. Denote $H_t := H \cap K_t$ for $0 \le t \le d$ and $H_{-1} := \emptyset$. By Corollary 3.2, it suffices to show that

$$\sum_{t=0}^d (t+1)|H_t| \ge \sum_{t=0}^d (|\mathbb{E}_{H_t}| + |\mathbb{E}(H_t, H_{t-1})|).$$

We will in fact prove this inequality term-by-term, namely, that for any $0 \le t \le d$,

$$(t+1)|H_t| \ge |\mathbb{E}_{H_t}| + |\mathbb{E}(H_t, H_{t-1})|. \tag{1}$$

Since this is trivial for $t = 0$ (the right-hand side is zero), we fix $1 \le t \le d$ and aim to show that (1) holds for this t.

Write $\deg_v(G)$ for the degree of a vertex v in a graph G. Note that for all $\vec{i} \in K_t$, we have that $\deg_{\vec{i}}(K_t \cup K_{t-1}) = 2t$. Thus,

$$2t|H_t| = \sum_{\vec{i} \in H_t} \deg_{\vec{i}}(K_t \cup K_{t-1}).$$

The right-hand side counts the sum of the number of oriented edges (u, v) such that $u \in H_t$ and $v \in K_t \cup K_{t-1}$. We thus see that

$$2t|H_t| = 2|\mathbb{E}_{H_t}| + |\mathbb{E}(H_t, K_t \setminus H_t)| + |\mathbb{E}(H_t, H_{t-1})| + |\mathbb{E}(H_t, K_{t-1} \setminus H_{t-1})|.$$

Hence, to obtain (1), it suffices to show that

$$2|H_t| + |\mathbb{E}(H_t, K_t \setminus H_t)| + |\mathbb{E}(H_t, K_{t-1} \setminus H_{t-1})| \geq |\mathbb{E}(H_t, H_{t-1})|.$$

Since putting $H_{t-1} = K_{t-1}$ only increases the right-hand side and decreases the left-hand side, it suffices to prove that

$$2|H_t| + |\mathbb{E}(H_t, K_t \setminus H_t)| \geq |\mathbb{E}(H_t, K_{t-1})|.$$

To prove this, we partition $\mathbb{E}(H_t, K_{t-1})$ into two sets, E_1 and E_2, and show that $|E_1| \leq |\mathbb{E}(H_t, K_t \setminus H_t)|$ and $|E_2| \leq 2|H_t|$.

Given $e \in \mathbb{E}(H_t, K_{t-1})$, letting $\vec{i} \in H_t$ and unit vector $\vec{u}$ be such that $e = \{\vec{i}, \vec{i} + \vec{u}\}$, we denote the line in the box $[n]^d$ in the direction e by

$$\begin{aligned} \mathrm{Line}(e) &:= \{\vec{i} - m\vec{u} \in K_t \ : \ m \in \mathbb{Z}\} \\ &= \{\vec{i} - m\vec{u} \ : \ m = 0, \dots, n-3\}. \end{aligned}$$

We now partition $\mathbb{E}(H_t, K_{t-1})$ into the two sets

$$\begin{aligned} E_1 &:= \{e \in \mathbb{E}(H_t, K_{t-1}) \ : \ \mathrm{Line}(e) \not\subset H_t\}, \\ E_2 &:= \{e \in \mathbb{E}(H_t, K_{t-1}) \ : \ \mathrm{Line}(e) \subset H_t\}. \end{aligned}$$

We begin by showing that $|E_1| \leq |\mathbb{E}(H_t, K_t \setminus H_t)|$. To this end, it suffices to construct an injective map $f : E_1 \to \mathbb{E}(H_t, K_t \setminus H_t)$. For $e \in E_1$, define

$$f(e) := \{\vec{i} - (m-1)\vec{u}, \vec{i} - m\vec{u}\},$$

where $\vec{i} \in H_t$ and unit vector $\vec{u}$ are such that $e = \{\vec{i}, \vec{i} + \vec{u}\}$, and where m is the smallest positive integer such that $\vec{i} - m\vec{u} \in K_t \setminus H_t$. It is straightforward to check that f is injective.

We now show that $|E_2| \leq 2|H_t|$. Denote the set of lines contained in H_t by

$$\mathcal{L}_t := \{\mathrm{Line}(e) \ : \ e \in E_2\},$$

and note that $|E_2| = 2|\mathcal{L}_t|$. Observe also that every vertex belongs to at most d lines and that each line in $\mathcal{L}_t$ consists of exactly $n-2$ vertices of H_t. Thus,

$$|\mathcal{L}_t| \leq \frac{d|H_t|}{n-2} \leq |H_t|. \qquad \square$$

Let us make a remark about two aspects of the tightness of Theorem 1.2. The first concerns the assumption that $n \geq d+2$: We show in Section 5 that $[2]^2$ is L_2^2-list-colorable (here $2 = n < d+2 = 4$), while $[2]^3$ is not L_2^3-list-colorable (here $2 = n < d+2 = 5$). Fixing d, the question of the 'optimal' value of n such that $[n]^d$ is L_n^d-list-colorable remains; the use of the word optimal is justified in the next proposition. The second aspect concerns the sizes of the lists: $[n]^d$ may be L-list-colorable for functions L which are pointwise smaller-or-equal than L_n^d. For example, $[n]^2$ is $\min\{L_n^2, 3\}$-list-colorable; see Question 5.7.

Proposition 3.3. *Let $n \geq 2$ and $d \geq 1$. Suppose $[n]^d$ is L_n^d-list-colorable. Then,*

(1) $[n]^{d-1}$ is L_n^{d-1}-list-colorable and
(2) $[n+1]^d$ is L_{n+1}^d-list-colorable.

Proof. Notice that the function L_n^{d-1} is just the restriction $L_n^d|_{[n]^{d-1}\times\{1\}}$ after identifying $[n]^{d-1}$ with $[n]^{d-1} \times \{1\}$. This proves the first part.

For the second part, we start by coloring $[n+1]^d \setminus [n]^d$ and then using the given hypothesis to color the portion which is left, that is, $[n]^d$. In order to color $[n+1]^d \setminus [n]^d$, we proceed by decomposing it into copies of $[n]^k$, where $0 \leq k \leq d-1$, and successively coloring them in increasing order of k (using the first part repeatedly). □

4. Mixing properties

In this section, we study various aspects of rigidity and mixing of X_q^d, the set of all proper q-colorings of $\mathbb{Z}^d$ (the color set here is always taken to be $\{1, \ldots, q\}$). Let us discuss some consequences of the above theorems in terms of *mixing properties*. The space X_q^d is an instance of a so-called *shift space* [3], and the following properties, which we define only for X_q^d, are applicable in this more general context (for a general introduction to mixing properties, we would refer the reader to [3] and in the context of graph homomorphisms to [6, 12]).

Given nonempty sets $U, V \subset \mathbb{Z}^d$, we denote

$$\mathrm{dist}(U, V) := \min_{\vec{i}\in U, \vec{j}\in V} |\vec{i} - \vec{j}|_1.$$

Also, we denote by B_n^d the set $\{-n, \ldots, n\}^d$.

Four important mixing properties (in increasing order of strength) are:

(1) X_q^d is *topologically mixing (TM)* if for all $U, V \subset \mathbb{Z}^d$, there exists $n \in \mathbb{N}$ such that for all $\vec{i} \in \mathbb{Z}^d$ for which $d(U+\vec{i}, V) \geq n$ and $x, y \in X_q^d$ there exists $z \in X_q^d$ such that $z|_{U+\vec{i}} = x|_{U+\vec{i}}$ and $z|_V = y|_V$.
(2) X_q^d is *strongly irreducible (SI) with gap n* if for all $x, y \in X_q^d$ and $U, V \subset \mathbb{Z}^d$ for which $\mathrm{dist}(U, V) \geq n$, there exists $z \in X_q^d$ such that $z|_U = x|_V$ and $z|_U = y|_V$.
(3) X_q^d has the *finite extension property (FEP) with distance n* if for any $U \subseteq \mathbb{Z}^d$ and any coloring u of U, if u can be extended to a q-coloring of $U + B_n^d$, then u can be extended to a q-coloring of $\mathbb{Z}^d$.
(4) X_q^d is *topologically strong spatial mixing (TSSM) with gap n* if for all $x, y \in X_q^d$ and $U, V, W \subset \mathbb{Z}^d$ for which $\mathrm{dist}(U, V) \geq n$ and $x|_W = y|_W$, there exists $z \in X_q^d$ such that $z|_{U\cup W} = x|_{U\cup W}$ and $z|_{V\cup W} = y|_{V\cup W}$.

The following implications hold:

$$\text{(TSSM)} \implies \text{(FEP)} \implies \text{(SI)} \implies \text{(TM)}. \tag{2}$$

The first implication follows from [5, Proposition 2.12] and the additional observation that though the FEP property in [5] is seemingly different from ours, it is in fact equivalent (the gap might be different for the two definitions though). The other two implications are straightforward to verify.

A *partial q-coloring of U* is a q-coloring of a subset C of U, i.e., an assignment of colors in $\{1, \ldots, q\}$ to each vertex in C such that any pair of adjacent vertices in C have different colors. We call C the *support* of the partial q-coloring.

We say that X_q^d is *n-fillable* if any partial q-coloring of $\partial[n]^d$ with support C can be extended to a q-coloring of $[n]^d \cup C$. Notice that X_q^d is 1-fillable if and only if given any q-coloring of the neighbors of a vertex, we can always extend it to the vertex itself, and this is true if and only if $q \geq 2d+1$. In [17], this last property was called *single-site fillability (SSF)* and used in the context of shift spaces.

Proposition 4.1. *If $[n]^d$ is L_n^d-list-colorable, then X_q^d is n-fillable for $q \geq d+2$.*

Proof. Fix $q \geq d+2$ and suppose that $[n]^d$ is L_n^d-list-colorable. Let c be a partial q-coloring of $\partial[n]^d$ with support C. Now consider the lists $S : B_n^d \to 2^{\{1,2,\ldots,q\}}$ given by

$$S(\vec{i}) := \{1, 2\ldots, q\} \setminus \{c_{\vec{j}} \ : \ \vec{j} \in C \cap \partial\{\vec{i}\}\}.$$

Then, clearly $|S(\vec{i})| \geq L_n^d(\vec{i})$. Since $[n]^d$ is L_n^d-list-colorable, we have that c extends to a proper coloring of $[n]^d \cup C$. □

For $d = 2$ and $q \geq 4$, n-fillability already followed from [20, Section 4.4] and the analogous property for a 2×2 box is also true (see [4]).

For $q \leq d+1$ and any n, we can also construct an explicit partial q-coloring of $\partial[n]^d$ which cannot be extended to a q-coloring of $[n]^d$. Observe that the frozen q-coloring x defined in the proof of Proposition 2.1 has an additional property: Restricting x to the elements of $\partial[n]^d$ with at least one coordinate zero extends to a q-coloring of $[n]^d$ in a unique manner. Now we let

$$C := \{(i_1, \ldots, i_d) \in \partial[n]^d \ : \ i_k = 0 \text{ for some } k\} \cup \{(n+1, 1, \ldots, 1)\}$$

and define c to be the q-coloring of C which coincides with x everywhere except at $(n+1, 1, \ldots, 1)$, where it is equal to $x_{(n,1,\ldots,1)}$; it is clear that c cannot be extended to a coloring of $[n]^d$.

Now we consider the following generalization to more general shapes.

Proposition 4.2. *If X_q^d is n-fillable, then for any subset $U \subseteq \mathbb{Z}^d$ that can be written as a union of translations of $[n]^d$ and any partial q-coloring c of ∂U, there exists an extension of c to a q-coloring of $U \cup C$, where C is the support of c.*

Proof. Suppose that X_q^d is n-fillable for some $n \in \mathbb{N}$. Consider any set $F \subset \mathbb{Z}^d$ such that $U = \bigcup_{\vec{j} \in F}(\vec{j} + [n]^d)$. Without loss of generality, suppose that F is minimal, i.e., for every proper subset F', $\bigcup_{\vec{j} \in F'}(\vec{j} + [n]^d)$ is a proper subset of U.

Now, given an arbitrary $\vec{i} \in F$, proceed by using the n-fillability property to find a coloring a of $\vec{i} + [n]^d$ and considering its boundary to be colored according to c restricted to $C \cap \partial(\vec{i} + [n]^d)$. Next, iterate this process with U', C', and c', where U' is $\bigcup_{\vec{j} \in F \setminus \{\vec{i}\}}(\vec{j} + [n]^d)$, C' is $\partial U'$, and c' is the restriction to C' of the concatenation of the partial colorings a and c. If F is finite, this process finishes in at most $|F|$ steps, and if F is infinite, we can conclude by the compactness of $\{1, \ldots, q\}^U$. □

Proposition 4.3. *If X_q^d is $(2n+1)$-fillable, then it satisfies FEP with distance $2n$.*

Proof. In this proof, we suppress the d in the notation of B_n^d. Consider $U \subseteq \mathbb{Z}^d$ and a coloring u of U that can be extended to a proper q-coloring $\overline{u}$ of $U + B_{2n}$.

Now, consider a family of vectors $\{\vec{k}_\ell\}_{\ell \in \mathbb{N}}$ in $\mathbb{Z}^d$ such that $\{\vec{k}_\ell + B_n\}_{\ell \in \mathbb{N}}$ is a partition of $\mathbb{Z}^d$. For example, $\{n\vec{k} + B_n\}_{\vec{k} \in \mathbb{Z}^d}$ would be enough. Let $S \subseteq \mathbb{N}$ be the set of indices ℓ such that $(\vec{k}_\ell + B_n) \cap U \neq \emptyset$.

Notice that $U \subseteq \bigcup_{\ell \in S}(\vec{k}_\ell + B_n) \subseteq U + B_{2n}$. Indeed, if $\vec{i} \in U$, then, since $\{\vec{k}_\ell + B_n\}_{\ell \in \mathbb{N}}$ is a partition of $\mathbb{Z}^d$, there exists (a unique) $\ell^* \in S$ such that $\vec{i} \in (\vec{k}_{\ell^*} + B_n)$. On the other hand, if $\vec{i} \in (\vec{k}_{\ell^*} + B_n)$ for some $\ell^* \in S$, then there exists $\vec{j} \in (\vec{k}_{\ell^*} + B_n) \cap U$, so $\|\vec{i} - \vec{j}\|_\infty \leq 2n$. Therefore, $\vec{i} \in (\vec{j} + B_{2n}) \subseteq U + B_{2n}$.

Finally, consider u' to be the restriction of $\overline{u}$ to $\bigcup_{\ell \in S}(\vec{k}_\ell + B_n)$. Then, u' is a proper partial coloring and the complement of its support $\{\vec{k}_\ell + B_n\}_{\ell \in \mathbb{N} \setminus S}$ is a (disjoint) union of boxes B_n with perhaps partially defined colorings on its boundary (the restriction of u' to $\partial(\vec{k}_\ell + B_n)$). Then, observing that B_n is just a translation of $[2n+1]^d$, by Proposition 4.2, we conclude. $\square$

From what has been observed in [5, Proposition 2.12], it follows for X_q^d that FEP with distance 0 is equivalent to TSSM. Furthermore, it is easy to see that FEP with distance n implies SI with gap $2n$. Considering all this, we have the following result.

Theorem 4.4. *Let $d \geq 1$. Then,*

(1) For $3 \leq q \leq d+1$, X_q^d is TM, but not SI.
(2) For $d+2 \leq q \leq 2d$, X_q^d is FEP, but not TSSM.
(3) For $2d+1 \leq q$, X_q^d is TSSM.

In the following, $\vec{i} \in \mathbb{Z}^d$ is called *even* or *odd* if the sum of its coordinates is even or odd, respectively. Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{2d}$ be the unit vectors in $\mathbb{Z}^d$, where $\vec{e}_i = -\vec{e}_{2d-i+1}$ for $1 \leq i \leq d$.

Proof. We split the proof into the three cases in the statement.

- **Case 1:** $3 \leq q \leq d+1$**.** In [12], it is shown that, for all d, X_q^d is topologically mixing if and only if $q \geq 3$. However, X_q^d is not SI if $q \leq d+1$. By Theorem 1.1, we have in fact that there exist frozen q-colorings, which is stronger, i.e., if X_q^d has a frozen q-coloring, then it cannot be SI: Let x be a frozen q-coloring and y any q-coloring which differs from x at $\vec{0}$. Let $n \geq 1$, $U = \partial B_n$ and $V = \{\vec{0}\}$. Observe that there does not exists a q-coloring z which agrees with x on U and y on V, while $\mathrm{dist}(U, V) = n$.
- **Case 2:** $d+2 \leq q \leq 2d$. By Proposition 4.1, X_q^d is n-fillable for $q \geq d+2$. Therefore, by Proposition 4.3, X_q^d satisfies FEP with distance n.

  Fix $n \in \mathbb{N}$ and let $S := \partial\{m\vec{e}_1 : m \in \mathbb{Z}\}$, $U := \{(0, \ldots, 0)\}$, and $V := \{n\vec{e}_1\}$. Let $x, y \in X_q^d$ be given by

$$x_{\vec{i}} := \begin{cases} m+t \mod (q-2) & \text{if } \vec{i} = m\vec{e}_1 + \vec{e}_t \text{ for } 2 \leq t \leq 2d-1 \text{ and } m \in \mathbb{Z} \\ q-2 & \text{if } \vec{i} \notin \partial\{m\vec{e}_1 \ : \ m \in \mathbb{Z}\} \text{ is odd} \\ q-1 & \text{if } \vec{i} \notin \partial\{m\vec{e}_1 \ : \ m \in \mathbb{Z}\} \text{ is even} \end{cases}$$

  and

$$y_{\vec{i}} := \begin{cases} m+t \mod (q-2) & \text{if } \vec{i} = m\vec{e}_1 + \vec{e}_t \text{ for } 2 \leq t \leq 2d-1 \text{ and } m \in \mathbb{Z} \\ q-1 & \text{if } \vec{i} \notin \partial\{m\vec{e}_1 \ : \ m \in \mathbb{Z}\} \text{ is odd} \\ q-2 & \text{if } \vec{i} \notin \partial\{m\vec{e}_1 \ : \ m \in \mathbb{Z}\} \text{ is even.} \end{cases}$$

Clearly $x|_S = y|_S$. Suppose that $z \in X_q^d$ is such that $z|_{U\cup S} := x|_{U\cup S}$. We have that

$$\begin{aligned} z_{(0,\ldots,0)} = &\quad x_{(0,\ldots,0)} &= q-1 \\ z_{m\vec{e}_1+\vec{e}_t} = &\quad x_{m\vec{e}_1+\vec{e}_t} &= m+t \mod (q-2) \end{aligned}$$

for $2 \le t \le 2d-1$ and $m \in \mathbb{Z}$. Since $q \le 2d$, it follows that for all $m \in \mathbb{Z}$,

$$\{z_{m\vec{e}_1+\vec{e}_t} \ : \ 2 \le t \le 2d-1\} = \{0, 1, \ldots, q-3\}$$

and hence $z_{m\vec{e}_1} = x_{m\vec{e}_1}$ for all $m \in \mathbb{Z}$ and $z|_V \ne y|_V$. Since n was arbitrary, we have that X_q^d is not TSSM.

- **Case 3:** $2d+1 \le q$**.** In this case, X_q^d is 0-fillable and thus satisfies TSSM. $\square$

5. Discussion

5.1. **Gibbs measures and the influence of boundaries.** One of the key motivations of this paper was to study the influence of a q-coloring of the boundary of a box on the colorings inside. Given $n, d, q \in \mathbb{N}$ and $x \in X_q^d$, let

$$X_{x,n,d,q} := \left\{ y \in X_q^d \ : \ y|_{\mathbb{Z}^d \setminus [n]^d} = x|_{\mathbb{Z}^d \setminus [n]^d} \right\}.$$

If X_q^d is SI, then there exists $n_0 \in \mathbb{N}$ such that for all $x \in X_q^d$ and $n \ge n_0$,

$$\begin{aligned} 1 \quad &\ge \quad \lim_{n\to\infty} \frac{\log |X_{x,n,d,q}|}{\log |\{q\text{-colorings of } [n]^d\}|} \\ &\ge \quad \lim_{n\to\infty} \frac{\log |\{q\text{-colorings of } [1+n_0, n-n_0]^d\}|}{\log |\{q\text{-colorings of } [n]^d\}|} = 1. \end{aligned}$$

It is not difficult to prove that the limit

$$\lim_{n\to\infty} \frac{1}{n^d} \log |\{q\text{-colorings of } [n]^d\}|$$

exists for all d, q and is referred to as the *entropy (of X_q^d)*, denoted by $h_{d,q}$. If the limit

$$\lim_{n\to\infty} \frac{1}{n^d} \log |X_{x,n,d,q}|$$

exists, then it is denoted by $h_{x,d,q}$.

By Theorem 4.4, (2) and the calculation above, we have that for $q \ge d+2$, $h_{x,d,q} = h_{d,q}$ for all $x \in X_q^d$. If $q \le d+1$, then there are frozen q-colorings $x \in X_{d,q}$ by Theorem 1.1. For such x, $h_{x,d,q} = 0$.

Question 5.1. *Given $q \le d+1$, what is the set of possible values $h_{x,d,q}$ for $x \in X_q^d$? Is it the entire interval $[0, h_{d,q}]$?*

This has been established for $q = 3$ in [21] using the "height function" formalism, which is missing for other values of q.

For $q \ge d+2$, one of the main questions that we would like to address is the following. Let $\mu_{x,n,d,q}$ denote the uniform measure on $X_{x,n,d,q}$.

Question 5.2. *For what values of q and d does X_q^d have a unique Gibbs measure? In other words, do the measures $\mu_{x,n,d,q}$ converge weakly as n goes to infinity to the same limit for all $x \in X_q^d$?*

This has been proved in the case when $q \geq 3.6d$ [14] and we suspect that there exists a sequence q_d satisfying $\lim_{d\to\infty} \frac{q_d}{d} = 1$ such that it is true when $q \geq q_d$. Note that the existence of frozen colorings preclude the possibility of a unique Gibbs measure, so that Theorem 1.1 implies that this does not hold when $q \leq d+1$. We also mention that it has recently been shown [19] that there are multiple maximal-entropy Gibbs measures when $d \geq Cq^{10}\log^3 q$ for some absolute constant $C > 0$.

5.2. **Sampling a uniform q-coloring of $[n]^d$.** Suppose that we are to sample a random coloring according to $\mu_{x,n,d,q}$. One way to obtain an approximate such sample is by the Markov Chain Monte Carlo method: construct an ergodic Markov chain on $X_{x,n,d,q}$ whose stationary distribution is $\mu_{x,n,d,q}$, and run it for a long time. A common way to devise such a Markov chain is via the Metropolis–Hastings algorithm for an appropriate set of possible local changes. We mention a couple of such local changes, and address the corresponding ergodicity requirement, namely, whether one can transition between any two elements of $X_{x,n,d,q}$ via the local changes.

Let us fix $d \geq 2$ and $q \geq 3$ for the following discusssion. We refer to [11] for some more details.

A *boundary pivot move* is a pair $(x, y) \in X_q^d \times X_q^d$ such that they differ at most on a single site. We say that X_q^d has the *boundary pivot property* if for all $x \in X_q^d$, $n \in \mathbb{N}$, and $y \in X_{x,n,d,q}$ there exists a sequence of boundary pivot moves from x to y contained in $X_{x,n,d,q}$. It is well-known that X_q^d has the boundary pivot property when $q = 3$ and it is quite easy to prove it for $q \geq 2d+2$ [13, Proposition 3.4]. For $d+2 \leq q \leq 2d+1$, a weaker property holds: A *boundary N-pivot move* is a pair $(x, y) \in X_q^d \times X_q^d$ such that they differ at most on a translate of $[N]^d$. We say that X_q^d has the *generalized boundary pivot property* if there exists $N \in \mathbb{N}$ such that for all $x \in X_q^d$, $n \in \mathbb{N}$, and $y \in X_{x,n,d,q}$, there exists a sequence of boundary N-pivot moves from x to y contained in $X_{x,n,d,q}$. The space X_d^q has the generalized boundary pivot property – this is a consequence of the n-fillability property (which holds by Theorem 1.2 and Proposition 4.1). The proof of this implication follows from the ideas in [11, Proposition 0.1] (look also at the proof of [4, Lemma 4.6] for similar proof).

Question 5.3. *For which q and d does X_q^d satisfy the generalized boundary pivot property?*

We remark that we do not know of any value of (q, d) for which X_q^d does not satisfy the generalized boundary pivot property. To apply the generalized boundary pivot property, we will still need to be able to sample a uniform coloring on a smaller, but still ostensibly large, box. It will then help to know if another property holds in this case in which such a sampling is not necessary. A *Kempe move* is a pair $(x, y) \in X_q^d \times X_q^d$ such that y is obtained from x by swapping the colors on a bicolor component. We say that X_q^d is *Kempe move connected* if for all $x \in X_q^d$, $n \in \mathbb{N}$, and $y \in X_{x,n,d,q}$, there exists a sequence of Kempe moves from x to y contained in $X_{x,n,d,q}$.

Question 5.4. *For which q and d is X_q^d Kempe move connected?*

Again, we are not aware of any (q, d) for which X_q^d is not Kempe move connected.

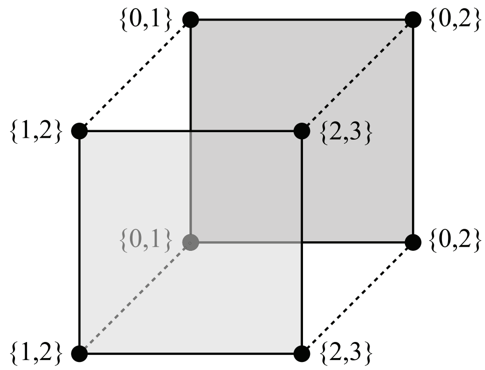

FIGURE 2. The lists $S|_{[2]^2\times\{2\}}$ (front) and $S|_{[2]^2\times\{1\}}$ (back).

5.3. **Extension of a q-coloring of $\partial[n]^d$ to $[n]^d$ for $q \le d+1$.** It was indicated after the end of the proof of Proposition 4.1 that when $3 \le q \le d+1$, there exist q-colorings of $\partial[n]^d$ which do not have an extension into $[n]^d$.

Question 5.5. *Characterise q-colorings of $\partial[n]^d$ which have an extension to $[n]^d$. What is the complexity of determining (in terms of n, q and d) whether an extension is possible?*

For $q = 3$, using the "height functions" formalism, the complexity is known to be of the order $n \log n$ for $d = 2$ (it follows from arguments very similar to those in [18]) and $n^{2(d-1)}$ for higher dimensions.

5.4. **Optimizing the parameters of list-colorability.** We would be interested to determine for which n and L it holds that $[n]^d$ is L-list-colorable. We have shown that $[n]^d$ is L-list-colorable when $n \ge d+2$ and $L = L_n^d$, but we have not tried to optimize n or L. We raise two questions, one concerning the optimal n, and the other related to improving L.

It can be shown (e.g., by Theorem 3.1) that $[2]^2$ is L_2^2-list-colorable. On the other hand, $[2]^3$ is not L_3^2-list-colorable. To see this, consider the lists $S : [2]^3 \to 2^{\{0,1,2,3\}}$ given by Figure 2. The two possible list-colorings of $[2]^2 \times \{1\}$ are $\begin{smallmatrix}1&0\\0&2\end{smallmatrix}$ and $\begin{smallmatrix}0&2\\1&0\end{smallmatrix}$, which are incompatible with those of $[2]^2 \times \{2\}$, that are $\begin{smallmatrix}1&2\\2&3\end{smallmatrix}$ and $\begin{smallmatrix}2&3\\1&2\end{smallmatrix}$.

Recall that, due to Proposition 3.3, the property of being L_n^d-list-colorable is monotone in n, so it is natural to ask the following question.

Question 5.6. *What is the smallest n for which $[n]^d$ is L_n^d-list-colorable?*

Regarding improvements to the function L, we note that it might be possible to decrease L_n^d pointwise and still find that $[n]^d$ is list-colorable. For example, in dimension $d = 2$, it is not hard to show using Theorem 3.1, that $[n]^2$ is L-list-colorable for the function $L := \min\{L_n^2, 3\}$ (simply orient the external face in a cycle and all other edges in the positive direction; note that $L = L_n^2 \equiv 2$ if $n = 2$, but otherwise $L \ne L_n^2$).

Question 5.7. *What is the smallest k for which $[n]^d$ is $\min\{L_n^d, k\}$-list-colorable for large enough n?*

Other functions L which one may wonder about are the constant functions. We say that G is k-list-colorable if it is L-list-colorable for the constant function $L \equiv k$.

Though k-list-colorability is slightly less related to mixing properties, it is still an interesting combinatorial problem.

Question 5.8. *What is the smallest k for which $[n]^d$ is k-list-colorable for all n?*

Using the Lovász local lemma, one may show that $k \geq Cd/\log d$ suffices for some constant $C > 0$ (this holds for any triangle-free graph with maximum degree d [15]; see also [1]). We remark that this easily leads to a proof that X_q^d is 2-fillable when $q \geq d + Cd/\log d$, and that this approach to showing fillability can also be useful for proper colorings of other graphs besides $\mathbb{Z}^d$.

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