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Chow groups and L-derivatives of automorphic motives for unitary groups

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Abstract

In this article, we study the Chow group of the motive associated to a tempered global L-packet π of unitary groups of even rank with respect to a CM extension, whose global root number is -1. We show that, under some restrictions on the ramification of π, if the central derivative $L'(1/2, \pi)$ is nonvanishing, then the π-nearly isotypic localization of the Chow group of a certain unitary Shimura variety over its reflex field does not vanish. This proves part of the Beilinson–Bloch conjecture for Chow groups and L-functions, which generalizes the Birch and Swinnerton-Dyer conjecture. Moreover, assuming the modularity of Kudla's generating functions of special cycles, we explicitly construct elements in a certain π-nearly isotypic subspace of the Chow group by arithmetic theta lifting, and compute their heights in terms of the central derivative $L'(1/2, \pi)$ and local doubling zeta integrals. This confirms the conjectural arithmetic inner product formula proposed by one of us, which generalizes the Gross–Zagier formula to higher dimensional motives.

Contents

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1. Introduction

In 1986, Gross and Zagier [GZ86] proved a remarkable formula that relates the Néron–Tate heights of Heegner points on a rational elliptic curve to the central derivative of the corresponding Rankin–Selberg L-function. A decade later, Kudla [Kud97] revealed another striking relation between Gillet–Soulé heights of special cycles on Shimura curves and derivatives of Siegel Eisenstein series of genus two, suggesting an arithmetic version of theta lifting and the Siegel–Weil formula; see, for example, [Kud02], [Kud03]. This was later further developed in his joint work with Rapoport and Yang [KRY06]. For the higher dimensional case, in a series of papers starting from the late 1990s, Kudla and Rapoport developed the theory of special cycles on integral models of Shimura varieties for GSpin groups in lower rank cases and for unitary groups of arbitrary ranks [KR11], [KR14]. They also studied special cycles on the relevant Rapoport–Zink spaces over non-archimedean local fields. In particular, they formulated a conjecture relating the arithmetic intersection number of special cycles on the unitary Rapoport–Zink space to the first derivative of local Whittaker functions [KR11, Conj. 1.3].

In his thesis work [Liu11a], [Liu11b], one of us studied special cycles as elements in the Chow group of the unitary Shimura variety over its reflex field (rather than in the arithmetic Chow group of a certain integral model) and the Beilinson–Bloch height of the arithmetic theta lifting (rather than the Gillet–Soulé height). In particular, in the setting of unitary groups, he proposed an explicit conjectural formula for the Beilinson–Bloch height in terms of the central L-derivative and local doubling zeta integrals. Such a formula is completely parallel to the Rallis inner product formula [Ral84], which computes the Petersson inner product of the global theta lifting, hence was named *arithmetic inner product formula* in [Liu11a], and can be regarded as a higher dimensional generalization of the Gross–Zagier formula.[1] In the case of $\mathrm{U}(1,1)$ over an arbitrary CM extension, such a conjectural formula was completely confirmed in [Liu11b], while the case for $\mathrm{U}(r,r)$ with $r \geqslant 2$ is significantly harder. Recently,

[1] By "generalization of the Gross–Zagier formula," we simply mean that they are both formulae relating Beilinson–Bloch heights of special cycles and central derivatives of L-functions. However, from a representation-theoretical point of view, the more accurate generalization of the Gross–Zagier formula should be the arithmetic Gan–Gross–Prasad conjecture.

the Kudla–Rapoport conjecture has been proved by W. Zhang and one of us in [LZ]; it has become possible to attack the cases for higher rank groups. In what follows, we will explain our new results on Chow groups of automorphic motives for unitary groups and the arithmetic inner product formula.

Beilinson–Bloch conjecture. Let E be a number field and X a projective smooth scheme over E of odd dimension $2r-1$. We have the L-function $L(s, \mathrm{H}^{2r-1}(X \otimes_E \overline{E}, \mathbb{Q}_\ell(r)))$ for the middle degree ℓ-adic cohomology of X for every rational prime ℓ, which is conjectured to be meromorphic, independent of ℓ, and satisfy a functional equation with center $s=0$. Let $\mathrm{CH}^r(X)^0$ be the group of codimension r Chow cycles on X that are homologically trivial (on $X \otimes_E \overline{E}$). Then the unrefined Beilinson–Bloch conjecture ([Beĭ87, Conj. 5.9] and [Blo84]) predicts that

$$\operatorname{rank} \mathrm{CH}^r(X)^0 = \operatorname{ord}_{s=0} L(s, \mathrm{H}^{2r-1}(X \otimes_E \overline{E}, \mathbb{Q}_\ell(r)))$$

holds for every ℓ, hence in particular, $\mathrm{CH}^r(X)^0$ has finite rank. Note that when X is an elliptic curve, this recovers the (unrefined) Birch and Swinnerton-Dyer conjecture.

In fact, this conjecture can also be formulated in terms of Chow motives. Based on this point of view, we have an equivariant version of the Beilinson–Bloch conjecture as follows. Suppose that X admits an action of an algebra T via étale correspondences. Then T acts on both $\mathrm{CH}^r(X)^0$ and $\mathrm{H}^{2r-1}(X \otimes_E \overline{E}, \mathbb{Q}_\ell(r))$. Let ϱ be a nonzero irreducible finite-dimensional complex representation of T. Then for every ℓ and every embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we have the L-function

$$L(s, \mathrm{Hom}_{\mathrm{T}}(\varrho, \mathrm{H}^{2r-1}(X \otimes_E \overline{E}, \mathbb{Q}_\ell(r))_{\mathbb{C}})).$$

Then it is expected that

(1.1)
$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{T}}(\varrho, \mathrm{CH}^r(X)^0_{\mathbb{C}}) = \operatorname{ord}_{s=0} L(s, \mathrm{Hom}_{\mathrm{T}}(\varrho, \mathrm{H}^{2r-1}(X \otimes_E \overline{E}, \mathbb{Q}_\ell(r))_{\mathbb{C}}))$$

holds, which can be regarded as the Beilinson–Bloch conjecture for the (conjectural Chow) motive $\mathrm{Hom}_{\mathrm{T}}(\varrho, \mathrm{h}^{2r-1}(X)(r)_{\mathbb{C}})$, where $\mathrm{h}^{2r-1}(X)$ is the (conjectural Chow) motive of X of degree $2r-1$.

Now we propose a more specific conjecture for unitary Shimura varieties, guided by the equivariant version of the Beilinson–Bloch conjecture above.

Let E/F be a CM extension of number fields with the complex conjugation c. We fix an embedding $\boldsymbol{\iota}\colon E \hookrightarrow \mathbb{C}$ and regard E as a subfield of $\mathbb{C}$. Take an even positive integer $n = 2r$. We equip $W_r := E^n$ with the skew-hermitian form (with respect to the involution c) given by the matrix $\left(\begin{smallmatrix} & 1_r \\ -1_r & \end{smallmatrix}\right)$. Put $G_r := \mathrm{U}(W_r)$, the unitary group of W_r, which is a quasi-split reductive group

over F. For every non-archimedean place v of F, we denote by $K_{r,v} \subseteq G_r(F_v)$ the stabilizer of the lattice $O_{E_v}^n$, which is a special maximal subgroup.

We first recall the notation of unitary Shimura varieties. Consider a hermitian space V over E of rank n (with respect to the involution c) that has signature $(n-1,1)$ at the real place of F induced by $\boldsymbol{\iota}$ and signature $(n,0)$ at other real places. Put $H \coloneqq \mathrm{U}(V)$ for its unitary group, which is a reductive group over F. Note that for all but finitely many places v of F, $H_v \coloneqq H \otimes_F F_v$ and $G_{r,v} \coloneqq G_r \otimes_F F_v$ are isomorphic as reductive groups over F_v. We have a system $\{X_L\}$ of Shimura varieties[2] of dimension $n-1$ over E indexed by open compact subgroups $L \subseteq H(\mathbb{A}_F^\infty)$. (See Section 4 for more details.)

Let π be a tempered cuspidal automorphic representation of $G_r(\mathbb{A}_F)$. By the endoscopic classification for quasi-split unitary groups, we have the automorphic base change $\mathrm{BC}(\pi)$ of π, which is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ that is an isobaric sum of mutually non-isomorphic (unitary) cuspidal automorphic representations.

CONJECTURE 1.1. *Let π be a tempered cuspidal automorphic representation of $G_r(\mathbb{A}_F)$, and let V be a hermitian space over E of rank n that has signature $(n-1,1)$ at the real place of F induced by $\boldsymbol{\iota}$ and signature $(n,0)$ at other real places. For every irreducible admissible representation $\tilde{\pi}^\infty$ of $H(\mathbb{A}_F^\infty)$ satisfying*

(a) *$\tilde{\pi}_v^\infty \simeq \pi_v$ for all but finitely many non-archimedean places v of F for which $H_v \simeq G_{r,v}$,*

(b) *$\mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde{\pi}^\infty, \varinjlim_L \mathrm{H}_{\mathrm{dR}}^{n-1}(X_L/\mathbb{C})\right) \neq 0$,*

the identity

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde{\pi}^\infty, \varinjlim_L \mathrm{CH}^r(X_L)^0_{\mathbb{C}}\right) = \mathrm{ord}_{s=\frac{1}{2}} L(s, \Pi_{j(\tilde{\pi}^\infty)})$$

holds. Here, $\Pi_{j(\tilde{\pi}^\infty)}$ is the cuspidal factor of $\mathrm{BC}(\pi)$ determined by $\tilde{\pi}^\infty$ (see Lemma 3.15); in particular, $\Pi_{j(\tilde{\pi}^\infty)} = \mathrm{BC}(\pi)$ if $\mathrm{BC}(\pi)$ is already cuspidal.

In relation with (1.1), we take X to be X_L for L such that $(\tilde{\pi}^\infty)^L \neq 0$, T to be the Hecke algebra of $H(\mathbb{A}_F^\infty)$ of level L, and ϱ to be $(\tilde{\pi}^\infty)^L$. Moreover, in this case we know that the L-function on the right-hand side of (1.1) coincides with $L(s, \Pi_{j(\tilde{\pi}^\infty)})$ up to a shift by $\frac{1}{2}$. Thus, Conjecture 1.1 is a special case of (1.1) after taking limit of L.

In the case when $\{X_L\}$ is replaced by classical modular curves, Conjecture 1.1 in fact recovers the (unrefined) Birch and Swinnerton-Dyer conjecture for rational elliptic curves. See [Gro04, §22] for more details from this point of view. Conjecture 1.1 was only known in the case of modular/Shimura curves

[2]When $F = \mathbb{Q}$, we have to replace X_L by its canonical smooth toroidal compactification.

when the analytic rank is at most 1 [GZ86], [Kol90], [Nek07a], [YZZ13], and partially known in the case of Shimura varieties for $\mathrm{U}(2) \times \mathrm{U}(3)$ when the analytic rank is exactly 1 [Xue19].[3]

Remark 1.2. It should be possible to formulate Conjecture 1.1 using totally positive definite incoherent hermitian spaces (that is, totally positive definite hermitian spaces over $\mathbb{A}_E$ that are not base change from E) and incoherent Shimura varieties without fixing an embedding $\boldsymbol{\iota}$. The notion of incoherent spaces was first invented by Kudla (in the quadratic case), which he called an incoherent collection of quadratic spaces over local fields [Kud97, Def. 2.1]. Around the similar time, Gross realized that a Shimura curve can be uniformized at its supersingular points in terms of a collection of quaternion algebras over the base number field. (See [Gro04] and also [GGP12] for generalizations.) In their work [YZZ13], Yuan, S. Zhang, and W. Zhang put this infinite collection of quaternion algebras as a single quaternion algebra over the adèles as a uniform description of the geometry of Shimura curves and the representation theory. This viewpoint was later adapted by W. Zhang [Zha12] and one of us [Liu11a], [Liu11b]. In [Zha19] (which is based on his 2010 talk at Gross birthday conference), S. Zhang summarizes how one can use the notion of incoherent quadratic/hermitian spaces to formulate various conjectures that are arithmetic counterparts of classical period formulae. In particular, there should exists a compatible system of varieties $\{X_L\}$ over (the abstract field) E such that for every embedding $\iota\colon E \hookrightarrow \mathbb{C}$, the system $\{X_L \otimes_E \iota(E)\}$ recovers the usual Shimura varieties defined above Conjecture 1.1; this was explained in more details in [Gro21]. Based on this observation, one should be able to formulate Conjecture 1.1 for the system $\{X_L\}$ associated to totally positive definite incoherent hermitian spaces.

Main results. Our main results in this article prove part of Conjecture 1.1 under certain assumptions on E/F and π. Denote by $\mathtt{V}_F^{(\infty)}$ and $\mathtt{V}_F^{\mathrm{fin}}$ the set of archimedean and non-archimedean places of F, respectively. Denote by $\mathtt{V}_F^{\mathrm{spl}}$, $\mathtt{V}_F^{\mathrm{int}}$, and $\mathtt{V}_F^{\mathrm{ram}}$ the subsets of $\mathtt{V}_F^{\mathrm{fin}}$ of those that are split, inert, and ramified in E, respectively. For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we denote by q_v the residue cardinality of F_v.

Assumption 1.3. Suppose that $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$ and that $\mathtt{V}_F^{\mathrm{spl}}$ contains all 2-adic places. In particular, $[F : \mathbb{Q}]$ is even. We consider a cuspidal automorphic representation π of $G_r(\mathbb{A}_F)$ realized on a space $\mathcal{V}_\pi$ of cusp forms, satisfying the following:

[3] Interestingly, the height formula in [Xue19], which is for the endoscopic case, is obtained by reducing it to the arithmetic inner product formula for $\mathrm{U}(1,1)$.

(1) For every $v \in \mathtt{V}_F^{(\infty)}$, π_v is the holomorphic discrete series representation of Harish-Chandra parameter $\{\frac{1-n}{2}, \frac{3-n}{2}, \dots, \frac{n-3}{2}, \frac{n-1}{2}\}$.
(2) For every $v \in \mathtt{V}_F^{\mathrm{spl}}$, π_v is a principal series.
(3) For every $v \in \mathtt{V}_F^{\mathrm{int}}$, π_v is either unramified or almost unramified (see Remark 1.4 below) with respect to $K_{r,v}$; moreover, if π_v is almost unramified, then v is unramified over $\mathbb{Q}$.
(4) For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, π_v is tempered.

Remark 1.4. We have the following remarks concerning Assumption 1.3:

(1) In (1), by [Sch75, Th. 1.3], the condition for π_v is equivalent to that π_v is a discrete series representation whose restriction to $K_{r,v}$ contains the character $\kappa_{r,v}^r$. (See Notation 2.3(G5), (G6) for the notation.) Moreover, one can also describe π_v as the theta lifting of the trivial representation of the (positive) definite unitary group of rank n; see, for example, [KK07].
(2) Part (2) will only be used in the proof of Lemma 7.3 in order to quote a vanishing result from [CS17]. However, in our second article on this subject [LL21], we have successively removed this assumption by confirming the conjecture in Remark 7.4 by proving a stronger vanishing property.
(3) In (3), the notion of almost unramified representations of $G_r(F_v)$ at $v \in \mathtt{V}_F^{\mathrm{int}}$ is defined in [Liu21b, Def. 5.3]. Roughly speaking, an irreducible admissible representation π_v of $G_r(F_v)$ is almost unramified (with respect to $K_{r,v}$) if $\pi_v^{I_{r,v}}$ contains a particular character as a module over $\mathbb{C}[I_{r,v}\backslash K_{r,v}/I_{r,v}]$, where $I_{r,v}$ is an Iwahori subgroup contained in $K_{r,v}$, and that the Satake parameter of π_v contains the pair $\{q_v, q_v^{-1}\}$; it is *not* unramified. By [Liu21b, Th. 1.2], when q_v is odd, *almost* unramified representations are exactly those representations whose local theta lifting to the non-quasi-split unitary group of the same rank $2r$ has nonzero invariants under the stabilizer of an *almost* self-dual lattice.

Suppose that we are in Assumption 1.3. Denote by

- $L(s, \pi)$ the doubling L-function (see Definition 3.3 for the more precise definition);
- $\mathtt{R}_\pi \subseteq \mathtt{V}_F^{\mathrm{spl}}$ the (finite) subset for which π_v is ramified;
- $\mathtt{S}_\pi \subseteq \mathtt{V}_F^{\mathrm{int}}$ the (finite) subset for which π_v is almost unramified.

Then we have $\varepsilon(\pi) = (-1)^{r[F:\mathbb{Q}]+|\mathtt{S}_\pi|}$ for the global (doubling) root number, so that the vanishing order of $L(s, \pi)$ at the center $s = \frac{1}{2}$ has the same parity as $|\mathtt{S}_\pi|$ since $[F : \mathbb{Q}]$ is even. The cuspidal automorphic representation π determines a hermitian space V_π over $\mathbb{A}_E$ of rank n via local theta dichotomy (such that the local theta lifting of π_v to $\mathrm{U}(V_\pi)(F_v)$ is nontrivial for every place v of F), unique up to isomorphism, which is totally positive definite and

satisfies that for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, the local Hasse invariant $\epsilon(V_\pi \otimes_{\mathbb{A}_F} F_v) = 1$ if and only if $v \notin \mathtt{S}_\pi$; see Proposition 3.6(2).

Now suppose that $|\mathtt{S}_\pi|$ is odd, hence $\varepsilon(\pi) = -1$, which is equivalent to that V_π is not the base change of a hermitian space over E. In this case, we take V to be the hermitian space of E in the context of Conjecture 1.1, unique up to isomorphism, satisfying that $V_v \simeq V_{\pi,v}$ for every $v \in \mathtt{V}_F^{\mathrm{fin}}$. Let $\mathtt{R}$ be a finite subset of $\mathtt{V}_F^{\mathrm{fin}}$. We fix a special maximal subgroup $L^{\mathtt{R}}$ of $H(\mathbb{A}_F^{\infty,\mathtt{R}})$ that is the stabilizer of a lattice $\Lambda^{\mathtt{R}}$ in $V \otimes_F \mathbb{A}_F^{\infty,\mathtt{R}}$. (See Notation 2.2(H6) for more details.) For a field $\mathbb{L}$, we denote by $\mathbb{T}_{\mathbb{L}}^{\mathtt{R}}$ the (abstract) Hecke algebra $\mathbb{L}[L^{\mathtt{R}} \backslash H(\mathbb{A}_F^{\infty,\mathtt{R}}) / L^{\mathtt{R}}]$, which is a commutative $\mathbb{L}$-algebra. When $\mathtt{R}$ contains $\mathtt{R}_\pi$, the cuspidal automorphic representation π gives rise to a character

$$\chi_\pi^{\mathtt{R}} \colon \mathbb{T}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}} \to \mathbb{Q}^{\mathrm{ac}},$$

where $\mathbb{Q}^{\mathrm{ac}}$ denotes the subfield of $\mathbb{C}$ of algebraic numbers; we put $\mathfrak{m}_\pi^{\mathtt{R}} \coloneqq \ker \chi_\pi^{\mathtt{R}}$, which is a maximal ideal of $\mathbb{T}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}}$.

The following is the first main theorem of this article.

THEOREM 1.5. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 1.3 with $|\mathtt{S}_\pi|$ odd, for which we assume Hypothesis 6.6. If $L'(\frac{1}{2}, \pi) \neq 0$, that is, $\mathrm{ord}_{s=\frac{1}{2}} L(s,\pi) = 1$, then as long as $\mathtt{R}$ satisfies $\mathtt{R}_\pi \subseteq \mathtt{R}$ and $|\mathtt{R} \cap \mathtt{V}_F^{\mathrm{spl}}| \geqslant 2$, the nonvanishing*

$$\varinjlim_{L_{\mathtt{R}}} \left(\mathrm{CH}^r(X_{L_{\mathtt{R}} L^{\mathtt{R}}})^0_{\mathbb{Q}^{\mathrm{ac}}}\right)_{\mathfrak{m}_\pi^{\mathtt{R}}} \neq 0$$

holds, where the colimit is taken over all open compact subgroups $L_{\mathtt{R}}$ of $H(F_{\mathtt{R}})$.

Remark 1.6. We have the following remarks concerning Theorem 1.5:

(1) For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, the local doubling L-function $L(s, \pi_v)$ coincides with $L(s, \mathrm{BC}(\pi_v))$, where $\mathrm{BC}(\pi_v)$ denotes the standard base change of π_v to $\mathrm{GL}_n(E_v)$. (See Remark 3.4 for more details.) In particular, combining with the local-global compatibility [KMSW14, Th. 1.7.1], we know that $L(s,\pi)$ coincides with the standard L-function of the automorphic base change of π.

(2) Since $|\mathtt{S}_\pi|$ is odd, by (1) and Remark 3.16, Conjecture 1.1 predicts the nonvanishing

$$\varinjlim_{L_{\mathtt{R}}} \mathrm{CH}^r(X_{L_{\mathtt{R}} L^{\mathtt{R}}})^0_{\mathbb{Q}^{\mathrm{ac}}}[\mathfrak{m}_\pi^{\mathtt{R}}] \neq 0$$

when $\mathrm{ord}_{s=\frac{1}{2}} L(s,\pi) = 1$ (by considering $\tilde{\pi}^\infty$ as the theta lifting of π^∞), which further implies the nonvanishing in our statement. However, it is conjectured that $\mathrm{CH}^r(X_{L_{\mathtt{R}} L^{\mathtt{R}}})^0_{\mathbb{Q}^{\mathrm{ac}}}$ is finite dimensional, which implies that the two types of nonvanishing are equivalent. Thus, our theorem provides evidence toward Conjecture 1.1. See Theorem 1.7(2) below for a stronger result under an extra hypothesis.

(3) Hypothesis 6.6 describes the Galois representation on the π-nearly isotypic subspace of the middle degree ℓ-adic cohomology $\varinjlim_L \mathrm{H}^{2r-1}(X_L \otimes_E \overline{E}, \overline{\mathbb{Q}}_\ell)$. See Remark 6.7 for the status of this hypothesis.

(4) In fact, the nonvanishing property we prove is that

$$\varinjlim_{L_{\mathtt{R}}} \left(\mathrm{SCH}^r(X_{L_{\mathtt{R}}L^{\mathtt{R}}})^0_{\mathbb{Q}^{\mathrm{ac}}}\right)_{\mathfrak{m}^{\mathtt{R}}_\pi} \neq 0,$$

where $\mathrm{SCH}^r(X_{L_{\mathtt{R}}L^{\mathtt{R}}})^0$ denotes the subgroup of $\mathrm{CH}^r(X_{L_{\mathtt{R}}L^{\mathtt{R}}})^0$ generated by special cycles (recalled in Section 4).

(5) It is clear that the field $\mathbb{Q}^{\mathrm{ac}}$ in the statement of the theorem can be replaced by an arbitrary subfield over which π^∞ (hence $\chi^{\mathtt{R}}_\pi$) is defined.

(6) The main reason we assume $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$ is that the local ingredient [LZ] only deals with places that are inert in E; we hope to remove this assumption in the future.

Our remaining results rely on Hypothesis 4.5 on the modularity of Kudla's generating functions of special cycles, hence are conditional at this moment; see Remark 4.6.

THEOREM 1.7. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 1.3 with $|\mathtt{S}_\pi|$ odd, for which we assume Hypothesis 6.6. Assume Hypothesis 4.5 on the modularity of generating functions of codimension r.*

(1) *For every*

- *$\varphi_1 = \otimes_v \varphi_{1v} \in \mathcal{V}_\pi$ and $\varphi_2 = \otimes_v \varphi_{2v} \in \mathcal{V}_\pi$ such that for every $v \in \mathtt{V}_F^{(\infty)}$, φ_{1v} and φ_{2v} have the lowest weight and satisfy $\langle \varphi_{1v}^{\mathtt{c}}, \varphi_{2v}\rangle_{\pi_v} = 1$;*
- *$\phi_1^\infty = \otimes_v \phi_{1v}^\infty \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)$ and $\phi_2^\infty = \otimes_v \phi_{2v}^\infty \in \mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty)$,*

the identity

$$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E} = \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v, V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}})$$

holds. Here,

- *$\Theta_{\phi_i^\infty}(\varphi_i) \in \varinjlim_L \mathrm{CH}^r(X_L)^0_{\mathbb{C}}$ is the arithmetic theta lifting (Definition 4.8), which is only well defined under Hypothesis 4.5;*
- *$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E}$ is the normalized height pairing (Definition 6.11),[4] which is constructed based on Beilinson's notion of height pairing;*

[4] Strictly speaking, $\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E}$ relies on the choice of a rational prime ℓ and is a priori an element in $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. However, the above identity implicitly says that it belongs to $\mathbb{C}$ and is independent of the choice of ℓ.

- $b_{2r}(0)$ *is defined in Notation* 2.1(F4), *which equals* $L(M_r^\vee(1))$ *where* M_r *is the motive associated to* G_r *by Gross* [Gro97] *and is, in particular, a positive real number*;
- $C_r = (-1)^r 2^{-2r}\pi^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)}$, *which is the exact value of a certain archimedean doubling zeta integral*; *and*
- $\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}})$ *is the normalized local doubling zeta integral* (*see Section* 3), *which equals* 1 *for all but finitely many* v.

(2) *In the context of Conjecture* 1.1, *take* $\tilde\pi^\infty$ *to be the theta lifting of* π^∞ *to* $H(\mathbb{A}_F^\infty)$. *If* $L'(\frac12,\pi)\neq 0$, *that is*, $\operatorname{ord}_{s=\frac12}L(s,\pi)=1$, *then*

$$\operatorname{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde\pi^\infty,\varinjlim_L \mathrm{CH}^r(X_L)^0_{\mathbb{C}}\right)\neq 0$$

holds.

Remark 1.8. We have the following remarks concerning Theorem 1.7.

(1) Part (1) verifies the so-called *arithmetic inner product formula*, a conjecture proposed by one of us [Liu11a, Conj. 3.11].

(2) The arithmetic inner product formula in part (1) is perfectly parallel to the classical Rallis inner product formula. In fact, suppose that $|\mathtt{S}_\pi|$ is even, hence $V_\pi\simeq V\otimes_F\mathbb{A}_F$ for a hermitian space V over E. We have the classical theta lifting $\theta_{\phi^\infty}(\varphi)$ where we use standard Gaussian functions at archimedean places. Then the Rallis inner product formula in this case reads as

$$\langle\theta_{\phi_1^\infty}(\varphi_1),\theta_{\phi_2^\infty}(\varphi_2)\rangle_H=\frac{L(\frac12,\pi)}{b_{2r}(0)}\cdot C_r^{[F:\mathbb{Q}]}\cdot\prod_{v\in\mathtt{V}_F^{\mathrm{fin}}}\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}}),$$

in which $\langle\ ,\ \rangle_H$ denotes the Petersson inner product with respect to the *Tamagawa measure* on $H(\mathbb{A}_F)$.

(3) In part (2), the representation $\tilde\pi^\infty$ satisfies (a) of Conjecture 1.1. By Remark 1.6(1) and Remark 3.16, if $\operatorname{ord}_{s=\frac12}L(s,\pi)=1$, then $\tilde\pi^\infty$ satisfies (b) of Conjecture 1.1 as well, and $\Pi_{j(\tilde\pi^\infty)}$ is the unique cuspidal factor of the automorphic base change of π such that $\operatorname{ord}_{s=\frac12}L(s,\Pi_{j(\tilde\pi^\infty)})=1$. In particular, part (2) provides evidence toward Conjecture 1.1, which is more direct than Theorem 1.5 (but is conditional on the modularity of generating functions).

In the case where $\mathtt{R}_\pi=\emptyset$, that is, π_v is either unramified or almost unramified for every $v\in\mathtt{V}_F^{\mathrm{fin}}$, we have a very explicit height formula for test vectors that are new everywhere.

COROLLARY 1.9. *Let* $(\pi,\mathcal{V}_\pi)$ *be as in Assumption* 1.3 *with* $|\mathtt{S}_\pi|$ *odd, for which we assume Hypothesis* 6.6. *Assume Hypothesis* 4.5 *on the modularity*

of generating functions of codimension r. In the situation of Theorem 1.7(1), suppose further that

- $\mathtt{R}_\pi = \emptyset$;
- $\varphi_1 = \varphi_2 = \varphi \in \mathcal{V}_\pi^{[r]\emptyset}$ *(see Notation 2.3(G8) for the precise definition of the one-dimensional space $\mathcal{V}_\pi^{[r]\emptyset}$ of holomorphic new forms) such that for every $v \in \mathtt{V}_F$, $\langle \varphi_v^{\mathtt{c}}, \varphi_v \rangle_{\pi_v} = 1$; and*
- $\phi_1^\infty = \phi_2^\infty = \phi^\infty$ *such that for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\phi_v^\infty = \mathbb{1}_{(\Lambda_v^\emptyset)^r}$.*

Then the identity

$$\langle \Theta_{\phi^\infty}(\varphi), \Theta_{\phi^\infty}(\varphi) \rangle_{X,E}^\natural = (-1)^r \cdot \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{S}_\pi} \frac{q_v^{r-1}(q_v+1)}{(q_v^{2r-1}+1)(q_v^{2r}-1)}$$

holds.

Remark 1.10. Assuming the conjecture on the injectivity of the étale Abel–Jacobi map, one can show that the cycle $\Theta_{\phi^\infty}(\varphi)$ is a primitive cycle of codimension r. By [Beĭ87, Conj. 5.5], we expect $(-1)^r \langle \Theta_{\phi^\infty}(\varphi), \Theta_{\phi^\infty}(\varphi) \rangle_{X,E}^\natural \geqslant 0$ holds, which, in the situation of Corollary 1.9, is equivalent to $L'(\frac{1}{2}, \pi) \geqslant 0$.

Strategy and structure. The main strategy for the proofs of our main results is to adopt Beilinson's notion of height pairing together with various sophisticated uses of Hecke operators. In [Beĭ87], Beilinson constructed, under certain assumptions, a (hermitian) height pairing on $\mathrm{CH}^r(X_L)^0_{\mathbb{C}}$ valued in $\mathbb{C}$. Since those assumptions have not been resolved even today, we are not able to use the full notion of this height pairing. However, after choosing a sufficiently large prime ℓ, Beilinson's construction gives an *unconditional* height pairing on a subspace $\mathrm{CH}^r(X_L)^{\langle \ell \rangle}_{\mathbb{C}}$ (a priori depending on ℓ) of $\mathrm{CH}^r(X_L)^0_{\mathbb{C}}$ valued in $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

The candidates for those nonvanishing elements in Theorem 1.5 are Kudla's special cycles $Z_T(\phi^\infty)$ (which will be recalled in Section 4), which are in general elements in $\mathrm{CH}^r(X_L)_{\mathbb{C}}$. We show that there exists an element $\mathrm{s} \in \mathbb{T}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}} \backslash \mathfrak{m}^{\mathtt{R}}_\pi$ such that s^* annihilates the quotient space $\mathrm{CH}^r(X_L)_{\mathbb{C}} / \mathrm{CH}^r(X_L)^{\langle \ell \rangle}_{\mathbb{C}}$. The existence of such element allows us to consider the modified cycles $\mathrm{s}^* Z_T(\phi^\infty)$ without changing their (non)triviality in the localization of $\mathrm{CH}^r(X_L)_{\mathbb{C}}$ at $\mathfrak{m}^{\mathtt{R}}_\pi$, moreover at the same time to talk about their heights.

More precisely, we consider two modified cycles $\mathrm{s}_1^* Z_{T_1}(\phi_1^\infty)$ and $\mathrm{s}_2^* Z_{T_2}(\phi_2^\infty)$ as above. When $\phi_1^\infty \otimes \phi_2^\infty$ satisfies a certain regularity condition, the two cycles have disjoint support, hence their height pairing (in the sense of Beilinson) has a decomposition into so-called *local indices* according to places u of E. We mention especially that if u is non-archimedean, then the local index at u is defined via a winding number on the ℓ-adic cohomology of $X_L \otimes_E E_u$, which a priori has nothing to do with intersection theory. When $X_L \otimes_E E_u$ has a smooth

integral model, it is well known that such a winding number can be computed as the intersection number of integral extensions of the cycles. However, when $X_L \otimes_E E_u$ does not have smooth reduction, there is no general way to compute the local index. Nevertheless, we show that, under certain assumptions on the ramification and on the representation π, the local index between $\mathrm{s}_1^* Z_{T_1}(\phi_1^\infty)$ and $\mathrm{s}_2^* Z_{T_2}(\phi_2^\infty)$ can be computed in terms of the intersection number of some nice extensions of cycles on some nice regular model, after further suitable translations by elements in $\mathbb{T}^{\mathtt{R}}_{\bar{\mathbb{Q}}^{\mathrm{ac}}} \backslash \mathfrak{m}_\pi^{\mathtt{R}}$. Eventually, all these local indices turn out to be (linear combinations of) Fourier coefficients of derivatives of Eisenstein series (and values of Eisenstein series for finitely many u).

The final ingredient is the Euler expansion of the doubling integral of cusp forms in π against those derivatives of Eisenstein series (and Eisenstein series), which expresses the height pairing in terms of $L'(\frac{1}{2}, \pi)$ and local doubling zeta integrals. (In particular, it belongs to $\mathbb{C}$ and is independent of ℓ.) An apparent technical challenge for this approach is to show that there exist test functions $(\phi_1^\infty, \phi_2^\infty)$ satisfying the regularity condition and yielding nonvanishing local doubling zeta integrals; this is solved in Proposition 3.13. The proofs for Theorem 1.7 and Corollary 1.9 follow from a similar strategy.

In Section 2, we collect setups and notation that are running through the entire article, organized in several groups so that the readers can easily refer to them. In Section 3, we recall the doubling method in the theory of theta lifting and prove all necessary results from the representation-theoretical side. In Section 4, we recall the notation of unitary Shimura varieties, their special cycles and generating functions. We introduce the important hypothesis on the modularity of generating functions, assuming which we define arithmetic theta lifting. In Section 6, we introduce the notion of Beilinson's height, in a restricted but unconditional form, together with the decomposition into local indices. In Section 5, we introduce a variant of unitary Shimura variety that admits moduli interpretation, which will only be used in computing local indices at various places. In Sections 7, 8, 9, and 10, we compute local indices at split, inert with self-dual level, inert with almost self-dual level, and archimedean places, respectively. Finally, in Section 11, we prove our main results. There are two appendices: Appendix A contains two lemmas in Fourier analysis that are only used in the proof of Proposition 3.13, and Appendix B collects some new observations concerning Beilinson's local indices at non-archimedean places.

Notation and conventions.

- When we have a function f on a product set $A_1 \times \cdots \times A_m$, we will write $f(a_1, \dots, a_m)$ instead of $f((a_1, \dots, a_m))$ for its value at an element $(a_1, \dots, a_m) \in A_1 \times \cdots \times A_m$.

- For a set S, we denote by $\mathbb{1}_S$ the characteristic function of S.
- All rings are commutative and unital, and ring homomorphisms preserve units. However, we use the word *algebra* in the general sense, which is not necessarily commutative or unital.
- If a base ring is not specified in the tensor operation $\otimes$, then it is $\mathbb{Z}$.
- For an abelian group A and a ring R, we put $A_R \coloneqq A \otimes R$ as an R-module.
- For an integer $m \geqslant 0$, we denote by 0_m and 1_m the null and identity matrices of rank m, respectively. We also denote by $\mathtt{w}_m$ the matrix $\left(\begin{smallmatrix} & 1_m \\ -1_m & \end{smallmatrix}\right)$.
- We denote by $\mathtt{c}\colon \mathbb{C} \to \mathbb{C}$ the complex conjugation. For an element x in a complex space with a default underlying real structure, we denote by $x^{\mathtt{c}}$ its complex conjugation.
- For a field K, we denote by $\overline{K}$ the abstract algebraic closure of K. However, for aesthetic reason, we will write $\overline{\mathbb{Q}}_p$ instead of $\overline{\mathbb{Q}_p}$ and will denote by $\overline{\mathbb{F}}_p$ its residue field. On the other hand, we denote by $\mathbb{Q}^{\mathrm{ac}}$ the algebraic closure of $\mathbb{Q}$ *inside* $\mathbb{C}$.
- For a number field K, we denote by $\psi_K\colon K\backslash\mathbb{A}_K \to \mathbb{C}^\times$ the standard additive character, namely, $\psi_K \coloneqq \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{K/\mathbb{Q}}$ in which $\psi_{\mathbb{Q}}\colon \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ is the unique character such that $\psi_{\mathbb{Q},\infty}(x) = \mathrm{e}^{2\pi i x}$.
- Throughout the entire article, all parabolic inductions are unitarily normalized.

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2. **Running notation**

In this section, we collect several groups of more specific notation that will be used throughout the remaining sections except appendices.

Notation 2.1. Let E/F be a CM extension of number fields, so that $\mathtt{c}$ is a well-defined element in $\mathrm{Gal}(E/F)$. We continue to fix an embedding $\boldsymbol{\iota}\colon E \hookrightarrow \mathbb{C}$. We denote by $\mathbf{u}$ the (archimedean) place of E induced by $\boldsymbol{\iota}$ and regard E as a subfield of $\mathbb{C}$ via $\boldsymbol{\iota}$.

(F1) We denote by
- $\mathtt{V}_F$ and $\mathtt{V}_F^{\mathrm{fin}}$ the set of all places and non-archimedean places of F, respectively;

- $\mathtt{V}_F^{\mathrm{spl}}$, $\mathtt{V}_F^{\mathrm{int}}$, and $\mathtt{V}_F^{\mathrm{ram}}$ the subsets of $\mathtt{V}_F^{\mathrm{fin}}$ of those that are split, inert, and ramified in E, respectively;
- $\mathtt{V}_F^{(\diamond)}$ the subset of $\mathtt{V}_F$ of places above $\diamond$ for every place $\diamond$ of $\mathbb{Q}$; and
- $\mathtt{V}_E^{?}$ the places of E above $\mathtt{V}_F^{?}$.

Moreover,

- for every place $u \in \mathtt{V}_E$ of E, we denote by $\underline{u} \in \mathtt{V}_F$ the underlying place of F;
- for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we denote by $\mathfrak{p}_v$ the maximal ideal of O_{F_v} and put $q_v \coloneqq |O_{F_v}/\mathfrak{p}_v|$;
- for every $v \in \mathtt{V}_F$, we put $E_v \coloneqq E \otimes_F F_v$ and denote by $|\ |_{E_v} \colon E_v^\times \to \mathbb{C}^\times$ the normalized norm character.

(F2) Let $m \geqslant 0$ be an integer.

- We denote by Herm_m the subscheme of $\mathrm{Res}_{E/F} \mathrm{Mat}_{m,m}$ of m-by-m matrices b satisfying ${}^{\mathrm{t}}b^{\mathtt{c}} = b$. Put $\mathrm{Herm}_m^\circ \coloneqq \mathrm{Herm}_m \cap \mathrm{Res}_{E/F} \mathrm{GL}_m$.
- For every ordered partition $m = m_1 + \cdots + m_s$ with m_i a positive integer, we denote by $\partial_{m_1,\ldots,m_s} \colon \mathrm{Herm}_m \to \mathrm{Herm}_{m_1} \times \cdots \times \mathrm{Herm}_{m_s}$ the morphism that extracts the diagonal blocks with corresponding ranks.
- In addition we denote by $\mathrm{Herm}_m(F)^+$ (resp. $\mathrm{Herm}_m^\circ(F)^+$) the subset of $\mathrm{Herm}_m(F)$ of elements that are totally semi-positive definite (resp. totally positive definite).

(F3) For every $u \in \mathtt{V}_E^{(\infty)}$, we fix an embedding $\iota_u \colon E \hookrightarrow \mathbb{C}$ inducing u (with $\iota_{\mathbf{u}} = \boldsymbol{\iota}$) and identify E_u with $\mathbb{C}$ via ι_u.

(F4) Let $\eta \coloneqq \eta_{E/F} \colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ be the quadratic character associated to E/F. For every $v \in \mathtt{V}_F$ and every positive integer m, put

$$b_{m,v}(s) \coloneqq \prod_{i=1}^{m} L(2s+i, \eta_v^{m-i}).$$

Put $b_m(s) \coloneqq \prod_{v \in \mathtt{V}_F} b_{m,v}(s)$.

(F5) For every element $T \in \mathrm{Herm}_m(\mathbb{A}_F)$, let $\psi_T \colon \mathrm{Herm}_m(\mathbb{A}_F) \to \mathbb{C}^\times$ be given by the formula $\psi_T(b) \coloneqq \psi_F(\mathrm{tr}\, bT)$.

(F6) Let R be a commutative F-algebra. A (skew-)hermitian space over $R \otimes_F E$ is a free $R \otimes_F E$-module V of finite rank, equipped with a (skew-)hermitian form $(\ ,\ )_V$ with respect to the involution $\mathtt{c}$ that is nondegenerate.

Notation 2.2. Throughout the article, we fix an even positive integer $n = 2r$. Let $(V, (\ ,\ )_V)$ be a hermitian space over $\mathbb{A}_E$ of rank n that is totally positive definite.

(H1) For every commutative $\mathbb{A}_F$-algebra R and every integer $m \geqslant 0$, we denote by

$$T(x) \coloneqq ((x_i, x_j)_V)_{i,j} \in \mathrm{Herm}_m(R)$$

the moment matrix of an element $x = (x_1, \dots, x_m) \in V^m \otimes_{\mathbb{A}_F} R$.

(H2) For every $v \in \mathtt{V}_F$, we put $V_v \coloneqq V \otimes_{\mathbb{A}_F} F_v$, which is a hermitian space over E_v, and define the local Hasse invariant of V_v to be $\epsilon(V_v) \coloneqq \eta_v((-1)^r \det V_v) \in \{\pm 1\}$, which equals 1 for all but finitely many v. In what follows, we will abbreviate $\epsilon(V_v)$ as ϵ_v. Recall that V is coherent (resp. incoherent) if $\prod_{v \in \mathtt{V}_F} \epsilon_v = 1$ (resp. $\prod_{v \in \mathtt{V}_F} \epsilon_v = -1$).

(H3) Let v be a place of F and $m \geqslant 0$ an integer.

- For $T \in \mathrm{Herm}_m(F_v)$, we put $(V_v^m)_T \coloneqq \{x \in V_v^m \mid T(x) = T\}$, and

$$(V_v^m)_{\mathrm{reg}} \coloneqq \bigcup_{T \in \mathrm{Herm}_m^\circ(F_v)} (V_v^m)_T.$$

- We write $\mathscr{S}(V_v^m)$ for the space of (complex valued) Bruhat–Schwartz functions on V_v^m. When $v \in \mathtt{V}_F^{(\infty)}$, we have the Gaussian function $\phi_v^0 \in \mathscr{S}(V_v^m)$ given by the formula $\phi_v^0(x) = \mathrm{e}^{-2\pi \operatorname{tr} T(x)}$.
- We have a Fourier transform map $\widehat{\ }\colon \mathscr{S}(V_v^m) \to \mathscr{S}(V_v^m)$ sending ϕ to $\widehat{\phi}$ defined by the formula

$$\widehat{\phi}(x) \coloneqq \int_{V_v^m} \phi(y) \psi_{E,v}\left(\sum_{i=1}^m (x_i, y_i)_V\right) \mathrm{d}y,$$

  where $\mathrm{d}y$ is the self-dual Haar measure on V_v^m with respect to $\psi_{E,v}$.
- In what follows, we will always use this self-dual Haar measure on V_v^m.

(H4) Let $m \geqslant 0$ be an integer. For $T \in \mathrm{Herm}_m(F)$, we put

$$\mathrm{Diff}(T, V) \coloneqq \{v \in \mathtt{V}_F \mid (V_v^m)_T = \emptyset\},$$

which is a finite subset of $\mathtt{V}_F \setminus \mathtt{V}_F^{\mathrm{spl}}$.

(H5) Take a nonempty finite subset $\mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{fin}}$ that contains $\mathtt{V}_F^{\mathrm{ram}}$. Let $\mathtt{S}$ be the subset of $\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$ consisting of v such that $\epsilon_v = -1$, which is contained in $\mathtt{V}_F^{\mathrm{int}}$.

(H6) We fix a $\prod_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}} O_{E_v}$-lattice $\Lambda^{\mathtt{R}}$ in $V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty,\mathtt{R}}$ such that for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$, $\Lambda_v^{\mathtt{R}}$ is a subgroup of $(\Lambda_v^{\mathtt{R}})^\vee$ of index $q_v^{1-\epsilon_v}$, where

$$(\Lambda_v^{\mathtt{R}})^\vee \coloneqq \{x \in V_v \mid \psi_{E,v}((x, y)_V) = 1 \text{ for every } y \in \Lambda_v^{\mathtt{R}}\}$$

is the $\psi_{E,v}$-dual lattice of $\Lambda_v^{\mathtt{R}}$.

(H7) Put $H \coloneqq \mathrm{U}(V)$, which is a reductive group over $\mathbb{A}_F$.

(H8) Denote by $L^{\mathtt{R}} \subseteq H(\mathbb{A}_F^{\infty,\mathtt{R}})$ the stabilizer of $\Lambda^{\mathtt{R}}$, which is a special maximal subgroup.[5] We have the (abstract) Hecke algebra away from $\mathtt{R}$

$$\mathbb{T}^{\mathtt{R}} \coloneqq \mathbb{Z}[L^{\mathtt{R}} \backslash H(\mathbb{A}_F^{\infty,\mathtt{R}}) / L^{\mathtt{R}}],$$

which is a ring with the unit $\mathbb{1}_{L^{\mathtt{R}}}$, and denote by $\mathbb{S}^{\mathtt{R}}$ the subring

$$\varinjlim_{\substack{\mathtt{T} \subseteq \mathtt{V}_F^{\mathrm{spl}} \backslash \mathtt{R} \\ |\mathtt{T}| < \infty}} \mathbb{Z}[(L^{\mathtt{R}})_{\mathtt{T}} \backslash H(F_{\mathtt{T}}) / (L^{\mathtt{R}})_{\mathtt{T}}] \otimes \mathbb{1}_{(L^{\mathtt{R}})^{\mathtt{T}}}$$

of $\mathbb{T}^{\mathtt{R}}$.

(H9) Suppose that V is *incoherent*, namely, $\prod_{v \in \mathtt{V}_F} \epsilon_v = -1$. For every $u \in \mathtt{V}_E \setminus \mathtt{V}_E^{\mathrm{spl}}$, we fix a *$u$-nearby space* ${}^uV$ of V, which is a hermitian space over E, and an isomorphism ${}^uV \otimes_F \mathbb{A}_F^{\frac{u}{F}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\frac{u}{F}}$. More precisely,

- if $u \in \mathtt{V}_E^{(\infty)}$, then ${}^uV$ is the hermitian space over E, unique up to isomorphism, that has signature $(n-1,1)$ at u and satisfies ${}^uV \otimes_F \mathbb{A}_F^{\frac{u}{F}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\frac{u}{F}}$;
- if $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{\mathrm{spl}}$, then ${}^uV$ is the hermitian space over E, unique up to isomorphism, that satisfies ${}^uV \otimes_F \mathbb{A}_F^{\frac{u}{F}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\frac{u}{F}}$.

Put ${}^uH \coloneqq \mathrm{U}({}^uV)$, which is a reductive group over F. Then ${}^uH(\mathbb{A}_F^{\frac{u}{F}})$ and $H(\mathbb{A}_F^{\frac{u}{F}})$ are identified.

Notation 2.3. Let $m \geqslant 0$ be an integer. We equip $W_m = E^{2m}$ and $\bar{W}_m = E^{2m}$ with the skew-hermitian forms given by the matrices $\mathtt{w}_m$ and $-\mathtt{w}_m$, respectively.

(G1) Let G_m be the unitary group of both W_m and $\bar{W}_m$. We write elements of W_m and $\bar{W}_m$ in the row form, on which G_m acts from the right.

(G2) We denote by $\{e_1, \dots, e_{2m}\}$ and $\{\bar{e}_1, \dots, \bar{e}_{2m}\}$ the natural bases of W_m and $\bar{W}_m$, respectively.

(G3) Let $P_m \subseteq G_m$ be the parabolic subgroup stabilizing the subspace generated by $\{e_{r+1}, \dots, e_{2m}\}$, and let $N_m \subseteq P_m$ be its unipotent radical.

(G4) We have

- a homomorphism $m\colon \mathrm{Res}_{E/F} \mathrm{GL}_m \to P_m$ sending a to

$$m(a) \coloneqq \begin{pmatrix} a & \\ & {}^{\mathrm{t}}a^{\mathtt{c},-1} \end{pmatrix},$$

  which identifies $\mathrm{Res}_{E/F} \mathrm{GL}_m$ as a Levi factor of P_m;
- a homomorphism $n\colon \mathrm{Herm}_m \to N_m$ sending b to

$$n(b) \coloneqq \begin{pmatrix} 1_m & b \\ & 1_m \end{pmatrix},$$

  which is an isomorphism.

[5] When $r \geqslant 2$ (resp. $r = 1$), the set of conjugacy classes of special maximal subgroups of $H(\mathbb{A}_F^{\infty,\mathtt{R}})$ is canonically a torsor over $\mu_2^{\oplus \mathtt{V}_F^{\mathrm{int}} \backslash \mathtt{R}}$ (resp. $\mu_2^{\oplus \mathtt{V}_F^{\mathrm{int}} \backslash (\mathtt{R} \cup \mathtt{S}_\pi)}$).

(G5) We define a maximal compact subgroup $K_m = \prod_{v\in \mathtt{V}_F} K_{m,v}$ of $G_m(\mathbb{A}_F)$ in the following way:
- for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $K_{m,v}$ is the stabilizer of the lattice $O_{E_v}^{2m}$;
- for $v \in \mathtt{V}_F^{(\infty)}$, $K_{m,v}$ is the subgroup of the form

$$[k_1, k_2] \coloneqq \frac{1}{2}\begin{pmatrix} k_1 + k_2 & -ik_1 + ik_2 \\ ik_1 - ik_2 & k_1 + k_2 \end{pmatrix},$$

in which $k_i \in \mathrm{GL}_m(\mathbb{C})$ satisfies $k_i\, {}^{\mathrm{t}}k_i^{\mathtt{c}} = 1_m$ for $i = 1, 2$. Here, we have identified $G_m(F_v)$ as a subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ via the embedding ι_u with $v = \underline{u}$ in Notation 2.1(F3).

(G6) For every $v \in \mathtt{V}_F^{(\infty)}$, we have a character $\kappa_{m,v}\colon K_{m,v} \to \mathbb{C}^\times$ that sends $[k_1, k_2]$ to $\det k_1/\det k_2$.[6]

(G7) For every $v \in \mathtt{V}_F$, we define a Haar measure $\mathrm{d}g_v$ on $G_m(F_v)$ as follows:
- for $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\mathrm{d}g_v$ is the Haar measure under which $K_{m,v}$ has volume 1;
- for $v \in \mathtt{V}_F^{(\infty)}$, $\mathrm{d}g_v$ is the product of the measure on $K_{m,v}$ of total volume 1 and the standard hyperbolic measure on $G_m(F_v)/K_{m,v}$; see, for example, [EL20, §2.1].

Put $\mathrm{d}g = \prod_v \mathrm{d}g_v$, which is a Haar measure on $G_m(\mathbb{A}_F)$.

(G8) We denote by $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the space of both $\mathcal{Z}(\mathfrak{g}_{m,\infty})$-finite and $K_{m,\infty}$-finite automorphic forms on $G_m(\mathbb{A}_F)$, where $\mathcal{Z}(\mathfrak{g}_{m,\infty})$ denotes the center of the complexified universal enveloping algebra of the Lie algebra $\mathfrak{g}_{m,\infty}$ of $G_m \otimes_F F_\infty$. We denote by
- $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F)) \subseteq \mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the maximal subspace on which for every $v \in \mathtt{V}_F^{(\infty)}$, $K_{m,v}$ acts by the character $\kappa_{m,v}^r$;
- $\mathcal{A}^{[r]_{\mathtt{R}}}(G_m(F)\backslash G_m(\mathbb{A}_F)) \subseteq \mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the maximal subspace on which
  - for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus (\mathtt{R} \cup \mathtt{S})$, $K_{m,v}$ acts trivially; and
  - for every $v \in \mathtt{S}$, the standard Iwahori subgroup $I_{m,v}$ acts trivially and $\mathbb{C}[I_{m,v}\backslash K_{m,v}/I_{m,v}]$ acts by the character $\kappa_{m,v}^-$ ([Liu21b, Def. 2.1]);
- $\mathcal{A}_{\mathrm{cusp}}(G_m(F)\backslash G_m(\mathbb{A}_F))$ the subspace of $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$ of cusp forms, and by $\langle\ ,\ \rangle_{G_m}$ the hermitian form on $\mathcal{A}_{\mathrm{cusp}}(G_m(F)\backslash G_m(\mathbb{A}_F))$ given by the Petersson inner product with respect to the Haar measure $\mathrm{d}g$.

For a subspace $\mathcal{V}$ of $\mathcal{A}(G_m(F)\backslash G_m(\mathbb{A}_F))$, we denote by
- $\mathcal{V}^{[r]}$ the intersection of $\mathcal{V}$ and $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F))$;
- $\mathcal{V}^{[r]_{\mathtt{R}}}$ the intersection of $\mathcal{V}$ and $\mathcal{A}^{[r]_{\mathtt{R}}}(G_m(F)\backslash G_m(\mathbb{A}_F))$;
- $\mathcal{V}^{\mathtt{c}}$ the subspace $\{\varphi^{\mathtt{c}} \mid \varphi \in \mathcal{V}\}$.

[6] In fact, neither $K_{m,v}$ nor $\kappa_{m,v}$ depends on the choice of the embedding ι_u for $v = \underline{u} \in \mathtt{V}_F^{(\infty)}$.

Notation 2.4. We review the Weil representation.

(W1) For every $v \in \mathtt{V}_F$, we have the Weil representation $\omega_{m,v}$ of $G_m(F_v) \times H(F_v)$, with respect to the additive character $\psi_{F,v}$ and the trivial splitting character, realized on the Schrödinger model $\mathscr{S}(V_v^m)$. For the readers' convenience, we review the formulae:

- for $a \in \mathrm{GL}_m(E_v)$ and $\phi \in \mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}(m(a))\phi(x) = |\det a|_{E_v}^r \cdot \phi(xa);$$
- for $b \in \mathrm{Herm}_m(F_v)$ and $\phi \in \mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}(n(b))\phi(x) = \psi_{T(x)}(b) \cdot \phi(x)$$
(see Notation 2.1(F5) for $\psi_{T(x)}$);
- for $\phi \in \mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}\left(\mathtt{w}_m\right)\phi(x) = \gamma^m_{V_v,\psi_{F,v}} \cdot \widehat{\phi}(x),$$
where $\gamma_{V_v,\psi_{F,v}}$ is certain Weil constant determined by V_v and $\psi_{F,v}$;
- for $h \in H(F_v)$ and $\phi \in \mathscr{S}(V_v^m)$, we have
$$\omega_{m,v}(h)\phi(x) = \phi(h^{-1}x).$$

We put $\omega_m \coloneqq \otimes_v \omega_{m,v}$ as the adèlic version, realized on $\mathscr{S}(V^m)$.

(W2) For every v of F, we also realize the contragredient representation $\omega_{m,v}^\vee$ on the space $\mathscr{S}(V_v^m)$ as well via the bilinear pairing
$$\langle\ ,\ \rangle_{\omega_{m,v}} \colon \mathscr{S}(V_v^m) \times \mathscr{S}(V_v^m) \to \mathbb{C}$$
defined by the formula
$$\langle \phi^\vee, \phi \rangle_{\omega_{m,v}} \coloneqq \int_{V_v^m} \phi(x)\phi^\vee(x)\,\mathrm{d}x$$
for $\phi, \phi^\vee \in \mathscr{S}(V_v^m)$.

Notation 2.5. For a locally Noetherian scheme X and an integer $m \geqslant 0$, we denote by $\mathrm{Z}^m(X)$ the free abelian group generated by irreducible closed subschemes of codimension m and $\mathrm{CH}^m(X)$ the quotient by rational equivalence. Suppose that X is smooth over a field K of characteristic zero. Let ℓ be a rational prime.

(C1) We denote by $\mathrm{Z}^m(X)^0$ the kernel of the de Rham cycle class map
$$\mathrm{cl}_{X,\mathrm{dR}} \colon \mathrm{Z}^m(X) \to \mathrm{H}^{2m}_{\mathrm{dR}}(X/K)(m),$$
and by $\mathrm{CH}^m(X)^0$ the image of $\mathrm{Z}^m(X)^0$ in $\mathrm{CH}^m(X)$.

(C2) When K is a non-archimedean local field, we denote by $\mathrm{Z}^m(X)^{\langle \ell \rangle}$ the kernels of the ℓ-adic cycle class map
$$\mathrm{cl}_{X,\ell} \colon \mathrm{Z}^m(X) \to \mathrm{H}^{2m}(X, \mathbb{Q}_\ell(m)).$$

(C3) When K is a number field, we define $\mathrm{Z}^m(X)^{\langle \ell \rangle}$ via the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Z}^m(X)^{\langle \ell \rangle} & \longrightarrow & \prod_v \mathrm{Z}^m(X_{K_v})^{\langle \ell \rangle} \\ \downarrow & & \downarrow \\ \mathrm{Z}^m(X) & \longrightarrow & \prod_v \mathrm{Z}^m(X_{K_v}), \end{array}$$

where the product is taken over all non-archimedean places v of K not above ℓ. We denote by $\mathrm{CH}^m(X)^{\langle \ell \rangle}$ the image of $\mathrm{Z}^m(X)^{\langle \ell \rangle}$ in $\mathrm{CH}^m(X)$, which is contained in $\mathrm{CH}^m(X)^0$ by the comparison theorem between de Rham and ℓ-adic cohomology.

3. Doubling method and analytic side

In this section, we review the doubling method and prove several statements on the analytic side of our desired height formula.

We have the doubling skew-hermitian space $W_r^\square := W_r \oplus \bar{W}_r$ (Notation 2.3(G1)). Let $G_r^\square$ be the unitary group of $W_r^\square$, which contains $G_r \times G_r$ canonically. We now take a basis $\{e_1^\square, \dots, e_{4r}^\square\}$ of $W_r^\square$ by the formula

$$e_i^\square = e_i, \quad e_{r+i}^\square = -\bar{e}_i, \quad e_{2r+i}^\square = e_{r+i}, \quad e_{3r+i}^\square = \bar{e}_{r+i}$$

for $1 \leqslant i \leqslant r$, under which we may identify $W_r^\square$ with W_{2r} and $G_r^\square$ with G_{2r}. Put

$$\mathtt{w}_r^\square := \mathtt{w}_{2r}, \quad P_r^\square := P_{2r}, \quad N_r^\square := N_{2r}, \quad K_r^\square := K_{2r}, \quad \omega_r^\square := \omega_{2r}. \tag{3.1}$$

(See Notations 2.3 and 2.4.) We denote by

$$\delta_r^\square \colon P_r^\square \to \mathrm{Res}_{E/F} \mathrm{GL}_1$$

the composition of the Levi quotient map $P_r^\square = P_{2r} \to M_{2r}$, the isomorphism $m^{-1} \colon M_{2r} \to \mathrm{Res}_{E/F} \mathrm{GL}_{2r}$, and the determinant $\mathrm{Res}_{E/F} \mathrm{GL}_{2r} \to \mathrm{Res}_{E/F} \mathrm{GL}_1$. Put

$$\mathbf{w}_r := \begin{pmatrix} & & 1_r & \\ & 1_r & & \\ -1_r & 1_r & & \\ & & 1_r & 1_r \end{pmatrix} \in G_r^\square(F).$$

Then $P_r^\square \mathbf{w}_r (G_r \times G_r)$ is Zariski open in $G_r^\square$.

Let v be a place of F. For $s \in \mathbb{C}$, we have the degenerate principal series of $G_r^\square(F_v)$, which is defined as the normalized induced representation

$$\mathrm{I}_{r,v}^\square(s) := \mathrm{Ind}_{P_r^\square(F_v)}^{G_r^\square(F_v)} (|\ |_{E_v}^s \circ \delta_{r,v}^\square)$$

of $G_r^\square(F_v)$. We denote by $\mathrm{I}_r^\square(s)$ the restricted tensor product of $\mathrm{I}_{r,v}^\square(s)$ for all places v of F with respect to unramified sections.

For every section $f \in \mathrm{I}_r^{\square}(0)$, let $f^{(s)} \in \mathrm{I}_r^{\square}(s)$ be the standard section induced by f. Then we have the Eisenstein series $E(g, f^{(s)})$ for $g \in G_r^{\square}(\mathbb{A}_F)$. We have a $G_r^{\square}(\mathbb{A}_F)$-intertwining map

$$f_\bullet \colon \mathscr{S}(V^{2r}) \to \mathrm{I}_r^{\square}(0)$$

sending Φ to f_Φ defined by the formula $f_\Phi(g) \coloneqq \omega_r^{\square}(g)\Phi(0)$. (See (3.1) for $\omega_r^{\square}$.) In particular, for $\Phi \in \mathscr{S}(V^{2r})$, we have the Eisenstein series

$$E(s, g, \Phi) = E(g, f_\Phi^{(s)}) \coloneqq \sum_{\gamma \in P_r^{\square}(F)\backslash G_r^{\square}(F)} f_\Phi^{(s)}(\gamma g)$$

for $g \in G_r^{\square}(\mathbb{A}_F)$. It is meromorphic in s and holomorphic on the imaginary line.

Assumption 3.1. In what follows, we will consider an irreducible automorphic subrepresentation $(\pi, \mathcal{V}_\pi)$ of $\mathcal{A}_{\mathrm{cusp}}(G_r(F)\backslash G_r(\mathbb{A}_F))$ satisfying that

(1) for every $v \in \mathtt{V}_F^{(\infty)}$, π_v is the (unique up to isomorphism) discrete series representation whose restriction to $K_{r,v}$ contains the character $\kappa_{r,v}^r$;
(2) for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$, π_v is unramified (resp. almost unramified) with respect to $K_{r,v}$ if $\epsilon_v = 1$ (resp. $\epsilon_v = -1$);
(3) for every $v \in \mathtt{V}_F^{\mathrm{fin}}$, π_v is tempered.

We realize the contragredient representation $\pi^\vee$ on $\mathcal{V}_\pi^{\mathtt{c}}$ via the Petersson inner product $\langle\ ,\ \rangle_{G_r}$ (Notation 2.3(G8)). By (1) and (2), we have $\mathcal{V}_\pi^{[r]_{\mathtt{R}}} \neq \{0\}$, where $\mathcal{V}_\pi^{[r]_{\mathtt{R}}}$ is defined in Notation 2.3(G8).

Remark 3.2. By Proposition 3.6(2) below, we know that when $\mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$, V coincides with the hermitian space over $\mathbb{A}_E$ of rank n determined by π via local theta dichotomy.

Definition 3.3. We define the L-function for π as the Euler product $L(s, \pi) \coloneqq \prod_v L(s, \pi_v)$ over all places of F, in which

(1) For $v \in \mathtt{V}_F^{\mathrm{fin}}$, $L(s, \pi_v)$ is the doubling L-function defined in [Yam14, Th. 5.2].
(2) For $v \in \mathtt{V}_F^{(\infty)}$, $L(s, \pi_v)$ is the L-function of the standard base change $\mathrm{BC}(\pi_v)$ of π_v. By Assumption 3.1(1), $\mathrm{BC}(\pi_v)$ is the principal series representation of $\mathrm{GL}_n(\mathbb{C})$ that is the normalized induction of

$$\arg^{n-1} \boxtimes \arg^{n-3} \boxtimes \cdots \boxtimes \arg^{3-n} \boxtimes \arg^{1-n},$$

where $\arg \colon \mathbb{C}^\times \to \mathbb{C}^\times$ is the argument character. In particular, we have

$$L(s + \tfrac{1}{2}, \pi_v) = \left(\prod_{i=1}^{r} 2(2\pi)^{-(s+i)} \Gamma(s+i) \right)^2. \tag{3.2}$$

Remark 3.4. Let v be a place of F.

(1) For $v \in \mathtt{V}_F^{(\infty)}$, the doubling L-function is only well defined up to an entire function without zeros. However, one can show that $L(s, \pi_v)$ satisfies the requirement for the doubling L-function in [Yam14, Th. 5.2].
(2) For $v \in \mathtt{V}_F^{\mathrm{spl}}$, the standard base change $\mathrm{BC}(\pi_v)$ is well defined and we have $L(s, \pi_v) = L(s, \mathrm{BC}(\pi_v))$ by [Yam14, Th. 7.2].
(3) For $v \in \mathtt{V}_F^{\mathrm{int}} \setminus \mathtt{R}$, the standard base change $\mathrm{BC}(\pi_v)$ is well defined and we have $L(s, \pi_v) = L(s, \mathrm{BC}(\pi_v))$ by [Liu21b, Rem. 1.4].

In particular, when $\mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$, we have $L(s, \pi) = \prod_v L(s, \mathrm{BC}(\pi_v))$.

Let v be a place of F. We denote by $\langle\ ,\ \rangle_{\pi_v} \colon \pi_v^\vee \times \pi_v \to \mathbb{C}$ the tautological pairing. For $\varphi_v \in \pi_v$, $\varphi_v^\vee \in \pi_v^\vee$, and a good section $f^{(s)} \in \mathrm{I}_{r,v}^{\square}(s)$ ([Yam14, Def. 3.1]), we have the local doubling zeta integral

$$Z(\varphi_v^\vee \otimes \varphi_v, f^{(s)}) \coloneqq \int_{G_r(F_v)} \langle \pi_v^\vee(g)\varphi_v^\vee, \varphi_v \rangle_{\pi_v} \cdot f^{(s)}(\mathbf{w}_r(g, 1_{2r}))\, \mathrm{d}g,$$

and the normalized version

$$Z^\natural(\varphi_v^\vee \otimes \varphi_v, f^{(s)}) \coloneqq \left(\frac{L(s+\frac{1}{2}, \pi_v)}{b_{2r,v}(s)} \right)^{-1} \cdot Z(\varphi_v^\vee \otimes \varphi_v, f^{(s)}),$$

which is holomorphic in s. In particular, taking $s = 0$, we obtain a functional

$$\mathfrak{Z}_{\pi_v, V_v}^\natural \colon \pi_v^\vee \otimes \pi_v \otimes \mathscr{S}(V_v^{2r}) \to \mathbb{C}$$

such that

$$\mathfrak{Z}_{\pi_v, V_v}^\natural(\varphi_v^\vee, \varphi_v, \Phi_v) = Z^\natural(\varphi_v^\vee \otimes \varphi_v, f_{\Phi_v}^{(0)}) = Z^\natural(\varphi_v^\vee \otimes \varphi_v, f_{\Phi_v}).$$

Remark 3.5. By [Yam14, Lemma 7.2], we know that the integral defining $Z(\varphi_v^\vee \otimes \varphi_v, f^{(0)})$ is absolutely convergent and that

$$\frac{L(s+\frac{1}{2}, \pi_v)}{b_{2r,v}(s)}$$

is finite and invertible at $s = 0$.

PROPOSITION 3.6. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1.*

(1) *For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_r(F_v) \times G_r(F_v)}(\mathrm{I}_{r,v}^{\square}(0), \pi_v \boxtimes \pi_v^\vee) = 1.$$

(2) *For every $v \in (\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}) \cup \mathtt{V}_F^{\mathrm{spl}}$, V_v is the unique hermitian space over E_v of rank $2r$, up to isomorphism, such that $\mathfrak{Z}_{\pi_v, V_v}^\natural \neq 0$.*
(3) *For every $v \in \mathtt{V}_F^{\mathrm{fin}}$, $\mathrm{Hom}_{G_r(F_v)}(\mathscr{S}(V_v^r), \pi_v)$ is irreducible as a representation of $H(F_v)$ and is nonzero if $v \in (\mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}) \cup \mathtt{V}_F^{\mathrm{spl}}$.*

Proof. To ease notation, we will suppress the place v throughout the proof. For a hermitian space $\tilde{V}$ over E of rank $2r$, denote by $\mathrm{R}(0,\tilde{V}) \subseteq \mathrm{I}_r^{\square}(0)$ the subspace spanned by Siegel–Weil sections from $\tilde{V}$ and put $\Theta(\pi,\tilde{V}) \coloneqq \mathrm{Hom}_{G_r(F)}(\mathscr{S}(\tilde{V}^r),\pi)$. By the seesaw identity, we have

$$\mathrm{Hom}_{G_r(F)\times G_r(F)}(\mathrm{R}(0,\tilde{V}),\pi\boxtimes\pi^\vee) \simeq \mathrm{Hom}_{\tilde{H}(F)}(\Theta(\pi,\tilde{V})\otimes\Theta(\pi^\vee,\tilde{V}),\mathbb{1}),$$

where $\tilde{H} \coloneqq \mathrm{U}(\tilde{V})$. Since π is tempered, by (the same argument for) [GI16, Th. 4.1(v)], $\Theta(\pi,\tilde{V})$ is a semisimple representation of $\tilde{H}(F)$. By [GT16, Th. 1.2], we know that $\Theta(\pi,\tilde{V})$ is either zero or irreducible. By the local theta dichotomy [GG11, Th. 1.8] (see also [HKS96, Cor. 4.4] and [Har07, Th. 2.1.7]), there exists exactly one choice $\tilde{V}$, up to isomorphism, such that $\Theta(\pi,\tilde{V}) \neq 0$. Thus, we obtain (1) by [KS97, Ths. 1.2, 1.3].

For (2), there are two cases. If $v \in \mathtt{V}_F^{\mathrm{spl}}$, then it follows from (1) and [KS97, Th. 1.3]. If $v \in \mathtt{V}_F^{\mathrm{int}} \setminus \mathtt{R}$, then the uniqueness follows from (1) and [KS97, Th. 1.2]; the nonvanishing of $\mathfrak{Z}^\natural_{\pi,V}$ follows from [Liu21b, Prop. 5.6, Lemma 6.1].

For (3), the irreducibility of $\Theta(\pi,V) = \mathrm{Hom}_{G_r(F)}(\mathscr{S}(V^r),\pi)$ has already been proved; the nonvanishing follows from (2). □

PROPOSITION 3.7. *Let $(\pi,\mathcal{V}_\pi)$ be as in Assumption 3.1 such that $L(\frac{1}{2},\pi) = 0$. Take*

- *$\varphi_1 = \otimes_v \varphi_{1v} \in \mathcal{V}_\pi^{[r]\mathtt{R}}$ and $\varphi_2 = \otimes_v \varphi_{2v} \in \mathcal{V}_\pi^{[r]\mathtt{R}}$ such that $\langle \varphi_{1v}^{\mathtt{c}}, \varphi_{2v}\rangle_{\pi_v} = 1$ for $v \in \mathtt{V}_F \setminus \mathtt{R}$,[7] and*
- *$\Phi = \otimes_v \Phi_v \in \mathscr{S}(V^{2r})$ such that Φ_v is the Gaussian function (Notation 2.2(H3)) for $v \in \mathtt{V}_F^{(\infty)}$ and $\Phi_v = \mathbb{1}_{(\Lambda_v^{\mathtt{R}})^{2r}}$ for $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$.*

Then we have

$$\begin{aligned}
&\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1) E'(0,(g_1,g_2),\Phi)\,\mathrm{d}g_1\,\mathrm{d}g_2 \\
&= \frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v\in\mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\Phi_v) \\
&= \frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v\in\mathtt{S}} \frac{(-1)^r q_v^{r-1}(q_v+1)}{(q_v^{2r-1}+1)(q_v^{2r}-1)} \cdot \prod_{v\in\mathtt{R}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\Phi_v),
\end{aligned}$$

where

$$C_r \coloneqq (-1)^r 2^{-2r}\pi^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)},$$

and the measure on $G_r(\mathbb{A}_F)$ is the one defined in Notation 2.3(G7).

[7] Strictly speaking, what we fixed is a decomposition $\varphi_1^{\mathtt{c}} = \otimes_v(\varphi_1^{\mathtt{c}})_v$. We have abused notation by writing $\varphi_{1v}^{\mathtt{c}}$ instead of $(\varphi_1^{\mathtt{c}})_v$.

Proof. By the formula derived in [Liu11a, p. 869], we have

$$\iint_{[G_r(F)\backslash G_r(\mathbb{A}_F)]^2} \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1)E'(0,(g_1,g_2),\Phi)\,\mathrm{d}g_1\,\mathrm{d}g_2 = \frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)}\prod_v \mathfrak{Z}^{\natural}_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\Phi_v).$$

For $v\in \mathtt{V}_F^{(\infty)}$, it is clear that $\mathfrak{Z}^{\natural}_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\Phi_v)$ depends only on r, which we denote by C_r. Note that for $g\in G_r(F_v)$,

$$f^{(0)}_{\Phi_v}(\mathbf{w}_r(g,1_{2r})) = (-1)^r\langle \omega_{r,v}(g)\phi_v^0,\phi_v^0\rangle_{\omega_{r,v}},$$

where ϕ_v^0 is the Gaussian function on V_v^r and $\langle\ ,\ \rangle_{\omega_{r,v}}$ is the pairing in Notation 2.4(W2). In particular, $f^{(0)}_{\Phi_v}(\mathbf{w}_r(1_{2r},1_{2r})) = (-1)^r 2^{-2r^2}$. By [EL20, Th. 1.3 , Prop. 3.3.2] (with $n=k=2r$, $a=b=r$, $\tau_1=\cdots=\tau_r=r$, $\nu_1=\cdots=\nu_r=-r$, and $\chi_{\mathrm{ac}}^r=1$), we have

$$Z(0,\varphi_{1v}^{\mathtt{c}}\otimes\varphi_{2v},\Phi_v) = (-1)^r 2^{-2r^2}\cdot 2^{r^2-r}\pi^{r^2}\frac{\Gamma(1)\cdots\Gamma(r)}{\Gamma(r+1)\cdots\Gamma(2r)}.$$

By (3.2) and the formula

$$b_{2r,v}(s) = \left(\prod_{i=1}^{r}\pi^{-(s+i)}\Gamma(s+i)\right)^2,$$

we obtain our formula for C_r.

By [Yam14, Prop. 7.1, (7.2)], we have $\mathfrak{Z}^{\natural}_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\Phi_v)=1$ for $v\in \mathtt{V}_F^{\mathrm{fin}}\setminus(\mathtt{R}\cup\mathtt{S})$. By [Liu21b, Prop. 5.6, Lemma 6.1], we have

$$\mathfrak{Z}^{\natural}_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\Phi_v) = \frac{(-1)^r q_v^{r-1}(q_v+1)}{(q_v^{2r-1}+1)(q_v^{2r}-1)}$$

for $v\in\mathtt{S}$. The proposition is proved. □

Now we study the Eisenstein series $E(s,g,\Phi)$ via Whittaker functions. For every $v\in\mathtt{V}_F$, $T^{\square}\in \mathrm{Herm}_{2r}^{\circ}(F_v)$, and $\Phi_v\in\mathscr{S}(V_v^{2r})$, we define the local Whittaker function on $G_r^{\square}(F_v)$ with parameter $s\in\mathbb{C}$ as

$$W_{T^{\square}}(s,g,\Phi_v) := \int_{\mathrm{Herm}_{2r}(F_v)} f^{(s)}_{\Phi_v}(\mathtt{w}_r^{\square}n(b)g)\psi_{T^{\square}}(b)^{-1}\,\mathrm{d}b \tag{3.3}$$

(see (3.1) for $\mathtt{w}_r^{\square}$) by meromorphic continuation, where $\mathrm{d}b$ is the self-dual measure on $\mathrm{Herm}_{2r}(F_v)$ with respect to $\psi_{F,v}$. By [Liu11a, Lemma 2.8(1)], we know that $W_{T^{\square}}(s,g,\Phi_v)$ is an entire function in the variable s.

Definition 3.8. By the definition of local Whittaker functions (3.3), for every $v \in \mathtt{V}_F$, there exists a unique Haar measure $\mathrm{d}h_v$ on $H(F_v)$ such that for every $T^\square \in \mathrm{Herm}^\circ_{2r}(F_v)$ and every $\Phi_v \in \mathscr{S}(V_v^{2r})$, we have

$$W_{T^\square}(0, 1_{4r}, \Phi_v) = \frac{\gamma_{V_v,\psi_{F,v}}^{2r}}{b_{2r,v}(0)} \int_{H(F_v)} \Phi_v(h_v^{-1}x)\, \mathrm{d}h_v,$$

where x is an arbitrary element in $(V_v^{2r})_{T^\square}$ (Notation 2.2(H3)). For every open compact subgroup L_v of $H(F_v)$, we denote by $\mathrm{vol}(L_v)$ the volume of L_v under the measure $\mathrm{d}h_v$.

By [Tan99, Prop. 3.2], for all but finitely many $v \in \mathtt{V}_F^{\mathrm{fin}}$, a hyperspecial maximal subgroup of $H(F_v)$ has volume 1 under $\mathrm{d}h_v$. In particular, we may define the *normalized measure*

$$\mathrm{d}^\natural h := \frac{1}{b_{2r}(0)} \prod_{v \in \mathtt{V}_F} \mathrm{d}h_v$$

on $H(\mathbb{A}_F)$. In what follows, for an open compact subgroup L of $H(\mathbb{A}_F^\infty)$, we will denote by $\mathrm{vol}^\natural(L)$ the volume of $H(F_\infty)L$ under the measure $\mathrm{d}^\natural h$.

Remark 3.9. Note that when V is coherent, $\mathrm{d}^\natural h$ coincides with the Tamagawa measure on $H(\mathbb{A}_F)$. Later in Definition 6.11, we will use the volume $\mathrm{vol}^\natural(L)$ to scale the normalized height pairing. In view of Remark 1.8(2), this is the most "natural" way.

Proposition 3.10. *Suppose that V is incoherent.*

(1) *Take an element $u \in \mathtt{V}_E \setminus \mathtt{V}_E^{\mathrm{spl}}$, and take ${}^u\Phi = \otimes_v {}^u\Phi_v \in \mathscr{S}({}^uV^{2r} \otimes_F \mathbb{A}_F)$, where we recall from Notation 2.2(H9) that ${}^uV$ is the u-nearby hermitian space, such that $\mathrm{supp}({}^u\Phi_v) \subseteq ({}^uV_v^{2r})_{\mathrm{reg}}$ (Notation 2.2(H3)) for v in a nonempty subset $\mathtt{R}' \subseteq \mathtt{R}$. Then for every $g \in P_r^\square(F_{\mathtt{R}'})G_r^\square(\mathbb{A}_F^{\mathtt{R}'})$, we have*

$$E(0, g, {}^u\Phi) = \sum_{T^\square \in \mathrm{Herm}^\circ_{2r}(F)} \prod_{v \in \mathtt{V}_F} W_{T^\square}(0, g_v, {}^u\Phi_v).$$

(2) *Take $\Phi = \otimes_v \Phi_v \in \mathscr{S}(V^{2r})$ such that $\mathrm{supp}(\Phi_v) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for v in a subset $\mathtt{R}' \subseteq \mathtt{R}$ of cardinality at least 2. Then for every $g \in P_r^\square(F_{\mathtt{R}'})G_r^\square(\mathbb{A}_F^{\mathtt{R}'})$, we have*

$$E'(0, g, \Phi) = \sum_{w \in \mathtt{V}_F \setminus \mathtt{V}_F^{\mathrm{spl}}} \mathfrak{E}(g, \Phi)_w,$$

where

$$\mathfrak{E}(g, \Phi)_w := \sum_{\substack{T^\square \in \mathrm{Herm}^\circ_{2r}(F) \\ \mathrm{Diff}(T^\square, V) = \{w\}}} W'_{T^\square}(0, g_w, \Phi_w) \prod_{v \in \mathtt{V}_F \setminus \{w\}} W_{T^\square}(0, g_v, \Phi_v).$$

Here, $\mathrm{Diff}(T^\square, V)$ is defined in Notation 2.2(H4).

Proof. This is proved in [Liu11b, §2B]. □

Definition 3.11. Suppose that V is incoherent. Take an element $u \in \mathtt{V}_E \setminus \mathtt{V}_E^{\mathrm{spl}}$ and a pair (T_1, T_2) of elements in $\mathrm{Herm}_r(F)$.

(1) For ${}^u\Phi = \otimes_v {}^u\Phi_v \in \mathscr{S}({}^uV^{2r} \otimes_F \mathbb{A}_F)$, we put

$$E_{T_1,T_2}(g, {}^u\Phi) := \sum_{\substack{T^\square \in \mathrm{Herm}^\circ_{2r}(F) \\ \partial_{r,r} T^\square = (T_1, T_2)}} \prod_{v \in \mathtt{V}_F} W_{T^\square}(0, g_v, {}^u\Phi_v).$$

(2) For $\Phi = \otimes_v \Phi_v \in \mathscr{S}(V^{2r})$, we put

$$\mathfrak{E}_{T_1,T_2}(g, \Phi)_u := \sum_{\substack{T^\square \in \mathrm{Herm}^\circ_{2r}(F) \\ \mathrm{Diff}(T^\square, V) = \{\underline{u}\} \\ \partial_{r,r} T^\square = (T_1, T_2)}} W'_{T^\square}(0, g_{\underline{u}}, \Phi_{\underline{u}}) \prod_{v \in \mathtt{V}_F \setminus \{\underline{u}\}} W_{T^\square}(0, g_v, \Phi_v).$$

Here, $\partial_{r,r} \colon \mathrm{Herm}_{2r} \to \mathrm{Herm}_r \times \mathrm{Herm}_r$ is defined in Notation 2.1(F2).

Remark 3.12. The image of $\mathrm{Herm}^\circ_{2r}(F)^+$ under $\partial_{r,r}$ is contained in the subset $\mathrm{Herm}^\circ_r(F)^+ \times \mathrm{Herm}^\circ_r(F)^+$.

The following proposition ensures the sufficient supply of test functions with support in $(V_v^{2r})_{\mathrm{reg}}$. As we have mentioned in Section 1, it solves a key technical challenge for our approach.

PROPOSITION 3.13. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1. Take $v \in \mathtt{V}_F^{\mathrm{fin}}$, and suppose that $\mathfrak{Z}^\natural_{\pi_v, V_v} \neq 0$. Then for every $\varphi_v \in \pi_v$ and $\varphi_v^\vee \in \pi_v^\vee$ that are both nonzero, we can find elements $\phi_v, \phi_v^\vee \in \mathscr{S}(V_v^r)$ such that $\mathrm{supp}(\phi_v \otimes \phi_v^\vee) \in (V_v^{2r})_{\mathrm{reg}}$ and $\mathfrak{Z}^\natural_{\pi_v, V_v}(\varphi_v^\vee, \varphi_v, \phi_v \otimes \phi_v^\vee) \neq 0$.*

Proof. To ease notation, we will suppress the place v throughout the proof. We identify the F-vector space $\mathrm{Herm}_{2r}(F)$ with its dual $\mathrm{Herm}_{2r}(F)^\vee$ via the bilinear form $(x, y) \mapsto \mathrm{tr}\, xy$. Take an element $\Psi \in \mathscr{S}(\mathrm{Herm}_{2r}(F))$. Let $\widehat{\Psi} \in \mathscr{S}(\mathrm{Herm}_{2r}(F))$ be the Fourier transform of Ψ with respect to ψ. Let f_Ψ be the unique section in $\mathrm{I}_r^\square(0)$ such that $f_\Psi(\mathtt{w}_r^\square n(b)) = \widehat{\Psi}(b)$ and $f_\Psi = 0$ outside $P_r^\square(F)\mathtt{w}_r^\square N_r^\square(F)$. Take $\varphi \in \pi$ and $\varphi^\vee \in \pi^\vee$ that are both nonzero. We claim that

$(*)$ There exists an element $\Psi \in \mathscr{S}(\mathrm{Herm}^\circ_{2r}(F))$ such that $Z(\varphi^\vee \otimes \varphi, f_\Psi) \neq 0$.

Assuming $(*)$, we continue the proof. Let V' be the other hermitian space over F of rank $2r$ that is not isomorphic to V if E is a field, or the zero space if $E = F \times F$. Let $\mathrm{Herm}^\circ_{2r}(F)_V$ and $\mathrm{Herm}^\circ_{2r}(F)_{V'}$ be the subset of $\mathrm{Herm}^\circ_{2r}(F)$ that is contained in the image of the moment maps from V^{2r} and V'^{2r}, respectively. Then $\mathrm{Herm}^\circ_{2r}(F)_V \cup \mathrm{Herm}^\circ_{2r}(F)_{V'}$ is a disjoint open cover of $\mathrm{Herm}^\circ_{2r}(F)$. Choose Ψ as in the claim, and put $\Psi_V := \Psi \cdot \mathbb{1}_{\mathrm{Herm}^\circ_{2r}(F)_V}$ and $\Psi_{V'} := \Psi \cdot \mathbb{1}_{\mathrm{Herm}^\circ_{2r}(F)_{V'}}$. We may choose elements $\Phi_V \in \mathscr{S}(V_{\mathrm{reg}}^{2r})$ and $\Phi_{V'} \in \mathscr{S}(V'^{2r}_{\mathrm{reg}})$ such that Ψ_V and $\Psi_{V'}$ are the pushforward of Φ_V and $\Phi_{V'}$ along

the moment map $V_{\mathrm{reg}}^{2r} \to \mathrm{Herm}_{2r}^{\circ}(F)_V$ and $V_{\mathrm{reg}}^{\prime 2r} \to \mathrm{Herm}_{2r}^{\circ}(F)_{V'}$, respectively. It is easy to see that $f_{\Phi_V} = f_{\Psi_V}$ and $f_{\Phi_{V'}} = f_{\Psi_{V'}}$. In particular, we have

$$\begin{aligned} Z(\varphi^\vee \otimes \varphi, f_\Psi) &= Z(\varphi^\vee \otimes \varphi, f_{\Psi_V}) + Z(\varphi^\vee \otimes \varphi, f_{\Psi_{V'}}) \\ &= Z(\varphi^\vee \otimes \varphi, f_{\Phi_V}) + Z(\varphi^\vee \otimes \varphi, f_{\Phi_{V'}}). \end{aligned}$$

By Proposition 3.6 and Remark 3.5, we have $Z(\varphi^\vee \otimes \varphi, f_{\Phi_{V'}}) = 0$ if $\mathfrak{Z}_{\pi,V}^\natural \neq 0$. Thus, we have $Z(\varphi^\vee \otimes \varphi, f_{\Phi_V}) \neq 0$, hence $\mathfrak{Z}_{\pi,V}^\natural(\varphi^\vee \otimes \varphi, \Phi_V) \neq 0$. The theorem follows as Φ_V can be written as a finite sum of elements of the form $\phi \otimes \phi^\vee$ satisfying $\mathrm{supp}(\phi \otimes \phi^\vee) \subseteq \mathrm{supp}(\Phi_V) \subseteq V_{\mathrm{reg}}^{2r}$.

It remains to show $(*)$. We identify $\mathrm{Res}_{E/F} \mathrm{Mat}_{r,r} \times \mathrm{Herm}_r \times \mathrm{Herm}_r$ with Herm_{2r} via the assignment

$$(a, u, v) \mapsto \begin{pmatrix} u & {}^{\mathrm{t}}a^{\mathtt{c}} \\ a & v \end{pmatrix}.$$

Define a polynomial function Δ on Herm_{2r} sending (a, u, v) to $\mathrm{Nm}_{E/F} \det a$. Let Ω be the complement of the Zariski closed subset of Herm_{2r} defined by the ideal (Δ). We define a morphism $\iota\colon \Omega \to G_r$ such that

$$\iota(a, u, v) = \begin{pmatrix} 1_r & v \\ 0 & 1_r \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & -{}^{\mathrm{t}}a^{\mathtt{c},-1} \end{pmatrix} \mathtt{w}_r \begin{pmatrix} 1_r & u \\ 0 & 1_r \end{pmatrix},$$

which is an isomorphism onto the Zariski open subset $P_r \mathtt{w}_r N_r$ of G_r. By a direct computation, we have a unique morphism $p\colon \Omega \to P_r^\square$ such that

$$\mathbf{w}_r \iota(a, u, v) = p(a, u, v) \cdot \mathtt{w}_r^\square \cdot n \begin{pmatrix} u & {}^{\mathrm{t}}a^{\mathtt{c}} \\ a & v \end{pmatrix}, \tag{3.4}$$

which satisfies

$$\mathrm{Nm}_{E/F}\, \delta_r^\square(p(a, u, v)) = \mathrm{Nm}_{E/F} \det a = \Delta(a, u, v). \tag{3.5}$$

Define a locally constant function $\xi_{\varphi^\vee,\varphi}$ on $G_r(F)$ by $\xi_{\varphi^\vee,\varphi}(g) \coloneqq \langle \pi^\vee(g)\varphi^\vee, \varphi\rangle_\pi$. Then by (3.4) and (3.5), we have

$$\begin{aligned} &Z(\varphi^\vee \otimes \varphi, f_\Psi) \\ &\quad = \int_{G_r(F)} \xi_{\varphi^\vee,\varphi}(g) f_\Psi(\mathbf{w}_r(g, 1_{2r}))\, \mathrm{d}g \\ &\quad = \int_{P_r(F)\mathtt{w}_r N_r(F)} \xi_{\varphi^\vee,\varphi}(\iota(a, u, v)) |\Delta(a, u, v)|_F^r \widehat{\Psi}(a, u, v) \cdot \mathrm{d}\iota(a, u, v). \end{aligned}$$

We define a locally constant function $\xi_{\varphi^\vee,\varphi}^\flat$ on $\Omega(F)$ by

$$\xi_{\varphi^\vee,\varphi}^\flat(a, u, v) = |\Delta(a, u, v)|_F^{-r} \xi_{\varphi^\vee,\varphi}(\iota(a, u, v)).$$

Note that there exists a unique Haar measure $\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v$ on $\mathrm{Herm}_{2r}(F)$ such that

$$\mathrm{d}\iota(a, u, v) = |\Delta(a, u, v)|_F^{-2r}\, \mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v.$$

Thus, we have

$$Z(\varphi^\vee \otimes \varphi, f_\Psi) = \int_{\Omega(F)} \xi^\flat_{\varphi^\vee,\varphi}(a,u,v)\widehat{\Psi}(a,u,v)\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v. \tag{3.6}$$

As both $\varphi^\vee$ and φ are nonzero, $\xi^\flat_{\varphi^\vee,\varphi}$ is nonzero, which is also locally integrable on $\mathrm{Herm}_{2r}(F)$ by Remark 3.5. The remaining discussion bifurcates.

When E is a field, we have $\xi_{\varphi^\vee,\varphi} \in L^{2+\varepsilon}(G_r(F))$ for every $\varepsilon > 0$, which is equivalent to

$$\int_{\Omega(F)} |\xi^\flat_{\varphi^\vee,\varphi}(a,u,v)|^{2+\varepsilon} |\Delta(a,u,v)|_F^{r\varepsilon}\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v < \infty.$$

Applying Lemma A.1 to $X = \mathrm{Herm}_{2r}(F)$, we obtain an element $\Psi \in \mathscr{S}(V_{\mathrm{reg}}^{2r})$ such that

$$\int_{\Omega(F)} \xi^\flat_{\varphi^\vee,\varphi}(a,u,v)\widehat{\Psi}(a,u,v)\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v \neq 0,$$

which implies $Z(\varphi^\vee \otimes \varphi, f_\Psi) \neq 0$ by (3.6). Thus, $(*)$ is proved.

When $E = F \times F$, we have $\xi_{\varphi^\vee,\varphi} \in L^{2+\varepsilon}(Z_r(F)\backslash G_r(F))$ for every $\varepsilon > 0$, where Z_r denotes the center of G_r, which is equivalent to

$$\int_{F^\times\backslash\Omega(F)} |\xi^\flat_{\varphi^\vee,\varphi}(a,u,v)|^{2+\varepsilon} |\Delta(a,u,v)|_F^{r\varepsilon}\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v < \infty.$$

Here, the action of $F^\times$ on $\mathrm{Herm}_{2r}(F)$ is given as follows: After identifying $\mathrm{Mat}_{r,r}(E)$ with $\mathrm{Mat}_{r,r}(F) \times \mathrm{Mat}_{r,r}(F)$ via the two factors of F under which we write $a = (a_1, a_2)$, $\alpha \in F^\times$ sends $((a_1,a_2),u,v)$ to $((\alpha a_1, \alpha^{-1}a_2),u,v)$. Applying Lemma A.2 to $X = \mathrm{Herm}_{2r}(F)$ with $X_1 = \mathrm{Mat}_{r,r}(F)$, $X_2 = \mathrm{Mat}_{r,r}(F)$, and $X_3 = \mathrm{Herm}_r(F) \oplus \mathrm{Herm}_r(F)$, we obtain an element $\Psi \in \mathscr{S}(V_{\mathrm{reg}}^{2r})$ such that

$$\int_{\Omega(F)} \xi^\flat_{\varphi^\vee,\varphi}(a,u,v)\widehat{\Psi}(a,u,v)\,\mathrm{d}a\,\mathrm{d}u\,\mathrm{d}v \neq 0,$$

which implies $Z(\varphi^\vee \otimes \varphi, f_\Psi) \neq 0$ by (3.6). Thus, $(*)$ is proved. □

To end this section, we recall some constructions concerning the tempered global L-packet given by π, which will be used in Sections 6 and 9.

Notation 3.14. Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1.

(1) Let Π be the automorphic base change of π, that is, the isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that Π_v is the standard base change of π_v for all but finitely many $v \in \mathbb{V}_F^{\mathrm{fin}}$ for which π_v is unramified.[8] By the local-global compatibility [KMSW14, Th. 1.7.1], for every $v \in \mathbb{V}_F^{(\infty)}$, Π_v

[8]The existence of Π follows from [Shi] or more generally [KMSW14], while the uniqueness of Π up to isomorphism is ensured by the strong multiplicity one theorem.

is the normalized induction of $\arg^{n-1} \boxtimes \arg^{n-3} \boxtimes \cdots \boxtimes \arg^{3-n} \boxtimes \arg^{1-n}$ as in Definition 3.3.

(2) Put $\mathfrak{I} := \{n-1, n-3, \ldots, 3-n, 1-n\}$. For each character $\chi\colon \mu_2^{\mathfrak{I}} \to \mathbb{C}^\times$, we define the signature of χ to be the pair (p,q) with $p+q=n$ such that χ takes value 1 on p μ_2-generators and -1 on q μ_2-generators. For such a character χ of signature (p,q), we have a discrete series representation π^χ of $\mathrm{U}(p,q)$. When $(p,q)=(n-1,1)$, we denote by π_∞^χ the representation of ${}^{\mathbf{u}}H(F_\infty)$ that is the inflation of π^χ along the quotient map ${}^{\mathbf{u}}H(F_\infty) \to {}^{\mathbf{u}}H(\mathbb{R}) \simeq \mathrm{U}(n-1,1)$.

(3) We may write $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_s$, in which Π_j is a conjugate-selfdual cuspidal automorphic representation of $\mathrm{GL}_{n_j}(\mathbb{A}_E)$, with $n_1+\cdots+n_s=n$. Then there is a unique partition $\mathfrak{I} = \mathfrak{I}_1 \sqcup \cdots \sqcup \mathfrak{I}_s$ such that $\Pi_{j,w}$ is the normalized induction of $\boxtimes_{i\in\mathfrak{I}_j} \arg^i$ for $1 \leqslant j \leqslant s$.

(4) Let ℓ be a rational prime with an arbitrarily given isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. For every $1 \leqslant j \leqslant s$, we have a (semisimple) Galois representation

$$\rho_{\Pi_j}\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/E) \to \mathrm{GL}_{n_j}(\overline{\mathbb{Q}}_\ell)$$

attached to Π_j as described in [Car12, Th. 1.1].

Here, we recall from Notation 2.1 that we have regarded E as a subfield of $\mathbb{C}$ via $\boldsymbol{\iota}$.

LEMMA 3.15. *For every irreducible admissible representation $\tilde{\pi}^\infty$ of $H(\mathbb{A}_F^\infty)$ such that Π_v is the standard base change of $\tilde{\pi}_v^\infty$ for all but finitely many $v \in \mathbb{V}_F^{\mathrm{fin}}$ for which $\tilde{\pi}_v^\infty$ is unramified, exactly one of the following two cases happens*:

(a) *There does not exist a character $\chi\colon \mu_2^{\mathfrak{I}} \to \mathbb{C}^\times$ of signature $(n-1,1)$ such that $\pi_\infty^\chi \otimes \tilde{\pi}^\infty$ is a cuspidal automorphic representation of ${}^{\mathbf{u}}H(\mathbb{A}_F)$.*

(b) *There is a unique integer $1 \leqslant j(\tilde{\pi}^\infty) \leqslant s$, such that for every $\chi\colon \mu_2^{\mathfrak{I}} \to \mathbb{C}^\times$ of signature $(n-1,1)$, $\pi_\infty^\chi \otimes \tilde{\pi}^\infty$ is a cuspidal automorphic representation of ${}^{\mathbf{u}}H(\mathbb{A}_F)$ if and only if the unique μ_2-generator of $\mu_2^{\mathfrak{I}}$ on which χ takes value -1 is indexed by an element in $\mathfrak{I}_{j(\tilde{\pi}^\infty)}$.*

Moreover, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde{\pi}^\infty, \varinjlim_L \mathrm{H}^{n-1}_{\mathrm{dR}}(X_L/\mathbb{C})\right) = \begin{cases} 0 & \text{if } \tilde{\pi}^\infty \text{ fits in (a)}, \\ n_{j(\tilde{\pi}^\infty)} > 0 & \text{if } \tilde{\pi}^\infty \text{ fits in (b)}, \end{cases}$$

where $\{X_L\}$ is the unitary Shimura variety recalled in Section 4 below.

Proof. The first part of the lemma is a consequence of Arthur's multiplicity formula for tempered global L-packets [KMSW14, Th. 1.7.1].

The second part of the lemma follows from Matsushima's formula and Arthur's multiplicity formula [KMSW14, Th. 1.7.1]. In particular, the number of characters $\chi\colon \mu_2^{\mathfrak{I}} \to \mathbb{C}^\times$ of signature $(n-1,1)$ such that $\pi_\infty^\chi \otimes \tilde{\pi}^\infty$ is a cuspidal automorphic representation of ${}^{\mathbf{u}}H(\mathbb{A}_F)$ equals $n_{j(\tilde{\pi}^\infty)}$. Here, we also use the

well-known fact that the $(\mathfrak{g}, K)$-cohomology of a (cohomological) discrete series representation is one dimensional in the middle degree and vanishes in all other degrees; see, for example, [BW00, II. Th. 5.4]. □

Remark 3.16. Assume that $\tilde{\pi}^\infty := \mathrm{Hom}_{G_r(\mathbb{A}_F^\infty)}(\mathscr{S}(V^r \otimes_F \mathbb{A}_F^\infty), \pi^\infty)$ is nonzero, which is then an irreducible admissible representation of $H(\mathbb{A}_F^\infty)$ by Proposition 3.6(3). By [GI16, Th. 4.1(ii)], the global root number $\varepsilon(\Pi)$ equals -1. Moreover, using Arthur's multiplicity formula [KMSW14, Th. 1.7.1] and Conjecture $(\mathrm{P1})_n$ in [GI16, §4.4] (which is proved in that article), we see that $\tilde{\pi}^\infty$ fits in the situation (b) of Lemma 3.15 if and only if there is exactly one element $j \in \{1, \dots, s\}$ such that $\varepsilon(\Pi_j) = -1$; in this case we must have $j = j(\tilde{\pi}^\infty)$. When $\mathrm{ord}_{s=\frac{1}{2}} L(s, \Pi) = 1$, there is exactly one element $j \in \{1, \dots, s\}$ such that $\varepsilon(\Pi_j) = -1$ as $L(s, \Pi) = \prod_{j=1}^s L(s, \Pi_j)$, so $\tilde{\pi}^\infty$ fits in the situation (b) of Lemma 3.15 automatically.

4. Special cycles and generating functions

In this section, we review the construction of Kudla's special cycles and generating functions. We also introduce the hypothesis on the modularity of generating functions and derive some of its consequences. From now to the end of Section 11, we assume V incoherent.[9]

Recall that we have fixed a $\mathbf{u}$-nearby space ${}^{\mathbf{u}}V$ and an isomorphism ${}^{\mathbf{u}}V \otimes_F \mathbb{A}_F^{\mathbf{u}} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\mathbf{u}}$ from Notation 2.2(H9). For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, we have the Shimura variety X_L associated to $\mathrm{Res}_{F/\mathbb{Q}} {}^{\mathbf{u}}H$ of the level L, which is a smooth quasi-projective scheme over E (which is regarded as a subfield of $\mathbb{C}$ via $\boldsymbol{\iota}$) of dimension $n-1$. We remind the readers of its complex uniformization

$$(X_L \otimes_E \mathbb{C})^{\mathrm{an}} \simeq {}^{\mathbf{u}}H(F) \backslash {}^{\mathbf{u}}\mathfrak{D} \times H(\mathbb{A}_F)/L, \tag{4.1}$$

where ${}^{\mathbf{u}}\mathfrak{D}$ denotes the complex manifold of negative lines in ${}^{\mathbf{u}}V \otimes_E \mathbb{C}$ and the Deligne homomorphism is the one adopted in [LTX+19, §3.2]. In what follows, for a place $u \in \mathfrak{v}_E$, we put $X_{L,u} := X_L \otimes_E E_u$ as a scheme over E_u.

Now we recall the construction of Kudla's special cycles and their generating functions. Take an integer $1 \leqslant m \leqslant n-1$.

Definition 4.1. For every element $x \in V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$, we have the *special cycle* $Z(x)_L \in \mathrm{CH}^m(X_L)_{\mathbb{Q}}$ defined as follows:

- For $T(x) \notin \mathrm{Herm}_m(F)^+$, we set $Z(x)_L = 0$.

[9] At the end, we will take $V = V_\pi$ as in Section 1. We have changed the use of V from Section 1 since in the proofs of the main results, we need to consider all nearby spaces of V_π. In particular, V in Section 1 is now ${}^{\mathbf{u}}V$.

- For $T(x) \in \mathrm{Herm}_m^\circ(F)^+$, we may find elements $x' \in {}^{\mathbf{u}}V^m$ and $h \in H(\mathbb{A}_F^\infty)$ such that $hx = x'$ holds in $V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$. The components of x' span a totally positive definite hermitian subspace $V_{x'}$ of ${}^{\mathbf{u}}V$ of rank m. Put $H^{x'} \coloneqq \mathrm{U}(V_{x'}^\perp)$, which is naturally a subgroup of ${}^{\mathbf{u}}H$, and let $\{X_M^{x'}\}_{M \subseteq H^{x'}(\mathbb{A}_F^\infty)}$ be the associated system of Shimura varieties. Define $Z(x)_L$ to be the image cycle of the composite morphism
$$X^{x'}_{hLh^{-1} \cap H^{x'}(\mathbb{A}_F^\infty)} \to X_{hLh^{-1}} \xrightarrow{\cdot h} X_L.$$
It is straightforward to check that $Z(x)_L$ does not depend on the choice of x' and h. Moreover, $Z(x)_L$ is a well-defined element in $\mathrm{Z}^m(X_L)$.
- For $T(x) \in \mathrm{Herm}_m(F)^+$ in general, we have an element $Z(x)_L \in \mathrm{CH}^m(X_L)_{\mathbb{Q}}$ (not well defined in $\mathrm{Z}^m(X_L)_{\mathbb{Q}}$). We refer the readers to [Liu11a, §3A] for more details as it is not important to us in this article.

For every $\phi^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$ and $T \in \mathrm{Herm}_m(F)$, we put
$$Z_T(\phi^\infty)_L \coloneqq \sum_{\substack{x \in L \backslash V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \\ T(x) = T}} \phi^\infty(x) Z(x)_L.$$
Since the above summation is finite, $Z_T(\phi^\infty)_L$ is a well-defined element in $\mathrm{CH}^m(X_L)_{\mathbb{C}}$.

Remark 4.2. For $T \in \mathrm{Herm}_m^\circ(F)^+$, $Z_T(\phi^\infty)_L$ is even a well-defined element in $\mathrm{Z}^m(X_L)_{\mathbb{C}}$.

Finally, for every $g \in G_m(\mathbb{A}_F)$, Kudla's *generating function* is defined to be
$$Z_{\phi^\infty}(g)_L \coloneqq \sum_{T \in \mathrm{Herm}_m(F)^+} \omega_{m,\infty}(g_\infty)\phi^0_\infty(T) \cdot Z_T(\omega_m^\infty(g^\infty)\phi^\infty)_L$$
as a formal sum valued in $\mathrm{CH}^m(X_L)_{\mathbb{C}}$, where
$$\omega_{m,\infty}(g_\infty)\phi^0_\infty(T) \coloneqq \prod_{v \in \mathtt{V}_F^{(\infty)}} \omega_{m,v}(g_v)\phi^0_v(T).$$
Here, we note that for $v \in \mathtt{V}_F^{(\infty)}$, the function $\omega_{m,v}(g_v)\phi^0_v$ factors through the moment map $V_v^m \to \mathrm{Herm}_m(F_v)$ (see Notation 2.2(H1)), hence $\omega_{m,v}(g_v)\phi^0_v(T)$ makes sense.

LEMMA 4.3. *In Definition 4.1, we have*
$$\omega_{m,\infty}(n(b)m(a))\phi^0_\infty(T) \cdot Z_T(\omega_m(n(b)m(a))\phi^\infty)_L = \phi^0_\infty({}^{\mathrm{t}}a^{\mathtt{c}}Ta) \cdot Z_{{}^{\mathrm{t}}a^{\mathtt{c}}Ta}(\phi^\infty)_L$$
for every $a \in \mathrm{GL}_m(E)$ *and every* $b \in \mathrm{Herm}_m(F)$.

Proof. This is proved in (the proof of) [Liu11a, Th. 3.5]. □

LEMMA 4.4. *In Definition 4.1, we have* ${}^{\mathrm{t}*}Z_T(\phi^\infty)_L = Z_T({}^{\mathrm{t}}\phi^\infty)_L$ *for every* $\mathrm{t} \in \mathbb{T}_{\mathbb{C}}^{\mathtt{R}}$.

Proof. By linearity, we may assume $\mathrm{t} = \mathbb{1}_{LhL}$ for some $h \in H(\mathbb{A}_F^{\infty,\mathtt{R}})$. Then it follows easily from Definition 4.1 that

$$\mathrm{t}^* Z(x)_L = \sum_{h' \in h^{-1}Lh \cap L \backslash L} Z(hh'x)_L$$

for every $x \in V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$. Thus,

$$\begin{aligned}
\mathrm{t}^* Z_T(\phi^\infty)_L &= \sum_{\substack{x \in L \backslash V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \\ T(x) = T}} \phi^\infty(x) \mathrm{t}^* Z(x)_L \\
&= \sum_{\substack{x \in L \backslash V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \\ T(x) = T}} \phi^\infty(x) \sum_{h' \in h^{-1}Lh \cap L \backslash L} Z(hh'x)_L \\
&= \sum_{\substack{x \in L \backslash V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \\ T(x) = T}} \left(\sum_{h' \in h^{-1}Lh \cap L \backslash L} \phi^\infty(h'^{-1}h^{-1}x) \right) Z(x)_L \\
&= \sum_{\substack{x \in L \backslash V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \\ T(x) = T}} (\mathrm{t}\phi^\infty)(x) Z(x)_L = Z_T(\mathrm{t}\phi^\infty)_L.
\end{aligned}$$

The lemma follows. □

Hypothesis 4.5 (Modularity of generating functions of codimension m). *For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, every $\phi^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, and every complex linear map $l\colon \mathrm{CH}^m(X_L)_{\mathbb{C}} \to \mathbb{C}$, the assignment*

$$g \mapsto l(Z_{\phi^\infty}(g)_L)$$

is absolutely convergent and gives an element in $\mathcal{A}^{[r]}(G_m(F) \backslash G_m(\mathbb{A}_F))$. In other words, the function $Z_{\phi^\infty}(-)_L$ defines an element in

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{CH}^m(X_L)_{\mathbb{C}}^\vee, \mathcal{A}^{[r]}(G_m(F) \backslash G_m(\mathbb{A}_F))).$$

Remark 4.6. Hypothesis 4.5 is believed to hold. In fact, in the case of symplectic groups over $\mathbb{Q}$, the analogous statement was first conjectured by Kudla [Kud04] and has been confirmed in [BWR15] based on previous works [Zha09], [YZZ09]. In our situation, Hypothesis 4.5 is proved in [Liu11a, Th. 3.5] for $m = 1$; for $m \geqslant 2$, we know that $l(Z_{\phi^\infty}(g)_L)$ is formally modular by [Liu11a, Th. 3.5].

Note that the natural inclusion

$$\begin{aligned}
&\mathcal{A}^{[r]}(G_m(F) \backslash G_m(\mathbb{A}_F)) \otimes_{\mathbb{C}} \mathrm{CH}^m(X_L)_{\mathbb{C}} \\
&\qquad \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathrm{CH}^m(X_L)_{\mathbb{C}}^\vee, \mathcal{A}^{[r]}(G_m(F) \backslash G_m(\mathbb{A}_F)))
\end{aligned}$$

might be strict *a priori*, since we do not know whether $\mathrm{CH}^m(X_L)_{\mathbb{C}}$ is finite dimensional. However, we have the following result.

PROPOSITION 4.7. *Assume Hypothesis 4.5 on the modularity of generating functions of codimension m.*

(1) *For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$ and every Schwartz function $\phi^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, $Z_{\phi^\infty}(-)_L$ belongs to $\mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F)) \otimes_{\mathbb{C}} \mathrm{CH}^m(X_L)_{\mathbb{C}}$.*

(2) *The map*

$$\mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L \to \mathcal{A}^{[r]}(G_m(F)\backslash G_m(\mathbb{A}_F)) \otimes_{\mathbb{C}} \mathrm{CH}^m(X_L)_{\mathbb{C}}$$

sending ϕ^∞ to $Z_{\phi^\infty}(-)_L$ is $G_m(\mathbb{A}_F^\infty) \times \mathbb{T}_{\mathbb{C}}^{\mathtt{R}}$-equivariant, where $G_m(\mathbb{A}_F^\infty)$ acts on the source via the Weil representation and on the target via the right translation on the first factor, and $\mathbb{T}_{\mathbb{C}}^{\mathtt{R}}$ acts on the source via the Weil representation and on the target via the Hecke correspondences.

Proof. For (1), fix an open compact subgroup $K \subseteq G_m(\mathbb{A}_F^\infty)$ that fixes ϕ^∞, and a set of representatives $\{g^{(1)}, \dots, g^{(s)}\}$ of the finite double coset $G_m(F)\backslash G_m(\mathbb{A}_F^\infty)/K$. For every $1 \leqslant i \leqslant s$, the restriction of $Z_{\phi^\infty}(-)_L$ to $G_m(F_\infty) \times \{g^{(i)}\}$ is given by the hermitian q-expansion

$$f^{(i)}(q) \coloneqq \sum_{T \in \mathrm{Herm}_m(F)^+} Z_T(\phi^{\infty(i)})_L \cdot q^T,$$

where $\phi^{\infty(i)} \coloneqq \omega_m^\infty(g^{(i)})\phi^\infty$. By Hypothesis 4.5, for every $l \in \mathrm{CH}^m(X_L)_{\mathbb{C}}^\vee$, the q-expansion

$$l(f^{(i)})(q) \coloneqq \sum_{T \in \mathrm{Herm}_m(F)^+} l(Z_T(\phi^{\infty(i)})_L) \cdot q^T$$

belongs to $\mathcal{M}_m^{[r]}(\Gamma^{(i)})$, the space of holomorphic hermitian Siegel modular form of G_m of weight $(\kappa_{m,w}^r)_{w \in \mathbf{V}_F^{(\infty)}}$ (Notation 2.3(G6)) and level $\Gamma^{(i)} \coloneqq G_m(F) \cap g^{(i)}K(g^{(i)})^{-1}$. Let $\mathcal{M}^{(i)}$ be the subspace of $\mathrm{CH}^m(X_L)_{\mathbb{C}}$ spanned by $Z_T(\phi^{\infty(i)})_L$ for all $T \in \mathrm{Herm}_m(F)^+$. We claim that

$$\dim_{\mathbb{C}} \mathcal{M}^{(i)} \leqslant \dim_{\mathbb{C}} \mathcal{M}_m^{[r]}(\Gamma^{(i)}) < \infty. \tag{4.2}$$

Take arbitrary elements $l_1, \dots, l_d$ of $\mathrm{CH}^m(X_L)_{\mathbb{C}}^\vee$ with $d > \dim_{\mathbb{C}} \mathcal{M}_m^{[r]}(\Gamma^{(i)})$. Then there exist $c_1, \dots, c_d \in \mathbb{C}$ not all zero, such that $\sum_{j=1}^d c_j l_j(f^{(i)})(q) = 0$; in other words,

$$\sum_{j=1}^d c_j l_j(Z_T(\phi^{\infty(i)})_L) = 0 \qquad \forall T \in \mathrm{Herm}_m(F)^+.$$

Thus, we have $\sum_{j=1}^d c_j l_j|_{\mathcal{M}^{(i)}} = 0$, which implies (4.2). However, (4.2) implies that the subspace of $\mathrm{CH}^m(X_L)_{\mathbb{C}}$ generated by $Z_T(\omega_m^\infty(g^\infty)\phi^\infty)_L$ for all $T \in \mathrm{Herm}_m(F)^+$ and $g^\infty \in G_m(\mathbb{A}_F^\infty)$ is finite dimensional. Thus, (1) follows.

Part (2) is follows from Lemma 4.4 and the construction.

The proposition follows. □

Definition 4.8. Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1. Assume Hypothesis 4.5 on the modularity of generating functions of codimension r. For every $\varphi \in \mathcal{V}_\pi^{[r]}$, every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, and every $\phi^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, we put

$$\Theta_{\phi^\infty}(\varphi)_L \coloneqq \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi^{\mathsf{c}}(g) Z_{\phi^\infty}(g)_L \, \mathrm{d}g,$$

which is an element in $\mathrm{CH}^r(X_L)_{\mathbb{C}}$ by Proposition 4.7. It is clear that the image of $\Theta_{\phi^\infty}(\varphi)_L$ in

$$\mathrm{CH}^r(X)_{\mathbb{C}} \coloneqq \varinjlim_L \mathrm{CH}^r(X_L)_{\mathbb{C}}$$

depends only on φ and ϕ^∞, which we denote by $\Theta_{\phi^\infty}(\varphi)$. Finally, we define the *arithmetic theta lifting* of $(\pi, \mathcal{V}_\pi)$ to V (with respect to $\boldsymbol{\iota}$) to be the complex subspace $\Theta(\pi, V)$ of $\mathrm{CH}^r(X)_{\mathbb{C}}$ spanned by $\Theta_{\phi^\infty}(\varphi)$ for all $\varphi \in \mathcal{V}_\pi^{[r]}$ and $\phi^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$.

5. **Auxiliary Shimura variety**

In this section, we introduce an auxiliary Shimura variety that will *only* be used in the computation of local indices $I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}$ to be introduced in the next section. We continue the discussion from Section 4.

Notation 5.1. We denote by T_0 the torus over $\mathbb{Q}$ such that for every commutative $\mathbb{Q}$-algebra R, we have $\mathrm{T}_0(R) = \{a \in E \otimes_{\mathbb{Q}} R \mid \mathrm{Nm}_{E/F} a \in R^\times\}$.

We choose a CM type Φ of E containing $\boldsymbol{\iota}$ and denote by E' the subfield of $\mathbb{C}$ generated by E and the reflex field of Φ. We also choose a skew hermitian space W over E of rank 1, whose group of rational similitude is canonically T_0. For a (sufficiently small) open compact subgroup L_0 of $\mathrm{T}_0(\mathbb{A}^\infty)$, we have the PEL type moduli scheme Y of CM abelian varieties with CM type Φ and level L_0, which is a smooth projective scheme over E' of dimension 0; see, for example, [Kot92]. In what follows, when we invoke this construction, the data Φ, W, and L_0 will be fixed, hence will not be carried into the notation E' and Y. For every open compact subgroup $L \subseteq H(\mathbb{A}_F^\infty)$, we put

$$X'_L \coloneqq X_L \otimes_E Y$$

as a scheme over E'.

Unlike X_L, the scheme X'_L has a moduli interpretation as first observed in [RSZ20].

Lemma 5.2. *The E'-scheme X'_L represents the functor that assigns to every locally Noetherian scheme S over E' the set of equivalence classes of sextuples $(A_0, \lambda_0, \eta_0; A, \lambda, \eta)$ where*

- (A_0, λ_0, η_0) *is an element in* $Y(S)$;
- (A, λ) *is a unitary* O_E*-abelian scheme of signature type* $n\Phi - \boldsymbol{\iota} + \boldsymbol{\iota}^{\mathsf{c}}$ *over* S (*see* [LTX+19, Defs. 3.4.2, 3.4.3]);
- η *is an* L*-level structure, that is, for a chosen geometric point* s *on every connected component of* S, *a* $\pi_1(S, s)$*-invariant* L*-orbit of isomorphisms*

$$\eta\colon V \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty \to \mathrm{Hom}^{\lambda_0,\lambda}_{E\otimes_{\mathbb{Q}}\mathbb{A}^\infty}(\mathrm{H}_1(A_{0s}, \mathbb{A}^\infty), \mathrm{H}_1(A_s, \mathbb{A}^\infty))$$

of hermitian spaces over $E \otimes_{\mathbb{Q}} \mathbb{A}^\infty = E \otimes_F \mathbb{A}_F^\infty$. (*See* [LTX+19, Construction 3.4.4] *for the hermitian form on the target of* η.)

Two sextuples $(A_0, \lambda_0, \eta_0; A, \lambda, \eta)$ *and* $(A_0', \lambda_0', \eta_0'; A', \lambda', \eta')$ *are equivalent if there are* O_F*-linear quasi-isogenies* $\varphi_0\colon A_0 \to A_0'$ *and* $\varphi\colon A \to A'$ *such that*

- φ_0 *carries* η_0 *to* η_0';
- *there exists* $c \in \mathbb{Q}^\times$ *such that* $\varphi_0^\vee \circ \lambda_0' \circ \varphi_0 = c\lambda_0$ *and* $\varphi^\vee \circ \lambda' \circ \varphi = c\lambda$;
- *the* L*-orbit of maps* $v \mapsto \varphi_* \circ \eta(v) \circ (\varphi_{0*})^{-1}$ *for* $v \in V \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ *coincides with* η'.

Proof. This is shown in [RSZ20, §3.2]. See also [LTX+19, §4.1]. □

Definition 5.3. For every $x \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ with $T(x) \in \mathrm{Herm}_r^\circ(F)^+$, we define a moduli functor $Z'(x)_L$ over E' as follows: for every locally Noetherian scheme S over E', $Z'(x)_L(S)$ is the set of equivalence classes of septuples $(A_0, \lambda_0, \eta_0; A, \lambda, \eta; \tilde{x})$ where

- $(A_0, \lambda_0, \eta_0; A, \lambda, \eta)$ belongs to $X_L'(S)$;
- $\tilde{x}$ is an element in $\mathrm{Hom}_{O_E}(A_0^r, A)_{\mathbb{Q}}$ satisfying $\tilde{x}_* \in \eta(Lx)$.

By Lemma 5.4(1) below, the image of $Z'(x)_L$ defines an element in $\mathrm{Z}^r(X_L')$, which we denote by $Z(x)_L'$.

LEMMA 5.4. *For every* $x \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ *with* $T(x) \in \mathrm{Herm}_r^\circ(F)^+$, *we have*

(1) *the forgetful morphism* $Z'(x)_L \to X_L'$ *is finite and unramified*;
(2) *the restriction of the algebraic cycle* $Z(x)_L$ *to* X_L' *coincides with* $Z(x)_L'$, *as elements in* $\mathrm{Z}^r(X_L')$.

Proof. In the proof below, we will frequently use notation from Definition 4.1. Take $x' \in {}^{\mathbf{u}}V^r$ and $h \in H(\mathbb{A}_F^\infty)$ such that $hx = x'$ holds in $V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$. For both statements, it suffices to show that there is an isomorphism $Z'(x)_L \xrightarrow{\sim} X^{x'}_{hLh^{-1}\cap H^{x'}(\mathbb{A}_F^\infty)} \otimes_E Y$ such that the following diagram commutes:

$$\begin{array}{ccc} Z'(x)_L & \xrightarrow{\ \sim\ } & X^{x'}_{hLh^{-1}\cap H^{x'}(\mathbb{A}_F^\infty)} \otimes_E Y \\ & \searrow \qquad \swarrow_{\cdot h} & \\ & X_L'. & \end{array} \tag{5.1}$$

Let S be a locally Noetherian scheme over E'. We will construct a functorial bijection between $Z'(x)_L(S)$ and $(X^{x'}_{hLh^{-1}\cap H^{x'}(\mathbb{A}_F^\infty)} \otimes_E Y)(S)$.

Take an element $(A_0, \lambda_0, \eta_0; A, \lambda, \eta; \tilde{x}) \in Z'(x)_L(S)$. We may find an O_E-abelian scheme A_1 of signature type $r\Phi - \boldsymbol{\iota} + \boldsymbol{\iota}^{\mathtt{c}}$ over S, and an element $\tilde{x}_1 \in \mathrm{Hom}_{O_E}(A_1, A)_{\mathbb{Q}}$, such that $\tilde{x}_1 \oplus \tilde{x}$ is an isomorphism in $\mathrm{Hom}_{O_E}(A_1 \times A_0^r, A)_{\mathbb{Q}}$, and that the composition $\tilde{x}^\vee \circ \lambda \circ \tilde{x}_1 \in \mathrm{Hom}_{O_E}(A_1, (A_0^r)^\vee)_{\mathbb{Q}}$ equals zero. Put $\lambda_1 \coloneqq \tilde{x}_1^\vee \circ \lambda \circ \tilde{x}_1$. As $\tilde{x}_* \in \eta(Lx)$, we may replace h by an element in hL such that the restriction of $\eta \circ h^{-1}$ to $V_{x'}^\perp \otimes_{\mathbb{Q}} \mathbb{A}^\infty$, which we denote by η_1, is contained in the submodule $\mathrm{Hom}^{\lambda_0,\lambda_1}_{E\otimes_{\mathbb{Q}}\mathbb{A}^\infty}(\mathrm{H}_1(A_{0s}, \mathbb{A}^\infty), \mathrm{H}_1(A_{1s}, \mathbb{A}^\infty))$. Thus, we obtain an element

$$(A_0, \lambda_0, \eta_0; A_1, \lambda_1, \eta_1) \in (X^{x'}_{hLh^{-1}\cap H^{x'}(\mathbb{A}_F^\infty)} \otimes_E Y)(S).$$

By construction, it maps to $(A_0, \lambda_0, \eta_0; A, \lambda, \eta) \in X'_L(S)$ in (5.1).

For the reverse direction, take an element

$$(A_0, \lambda_0, \eta_0; A_1, \lambda_1, \eta_1) \in (X^{x'}_{hLh^{-1}\cap H^{x'}(\mathbb{A}_F^\infty)} \otimes_E Y)(S).$$

Put $A_2 \coloneqq A_0^r$, and let λ_2 be the polarization such that we have an isomorphism

$$\eta_2 \colon V_{x'} \xrightarrow{\sim} \mathrm{Hom}^{\lambda_0,\lambda_2}_{O_E}(A_0, A_2)_{\mathbb{Q}}$$

of hermitian spaces over E. Put $A \coloneqq A_1 \times A_2$, $\lambda \coloneqq \lambda_1 \times \lambda_2$, and $\eta \coloneqq (\eta_1 \oplus \eta_2 \otimes_{\mathbb{Q}} \mathbb{A}^\infty) \circ h$. Then $(A_0, \lambda_0, \eta_0; A, \lambda, \eta)$ is the image of $(A_0, \lambda_0, \eta_0; A_1, \lambda_1, \eta_1)$ in $X'_L(S)$ in (5.1). Let $\tilde{x}$ be the isomorphism in $\mathrm{Hom}_{O_E}(A_0^r, A_2)_{\mathbb{Q}}$ that corresponds to $\eta_2(x')$, which we regard as an element in $\mathrm{Hom}_{O_E}(A_0^r, A)_{\mathbb{Q}}$. Then we obtain an element $(A_0, \lambda_0, \eta_0; A, \lambda, \eta; \tilde{x}) \in Z'(x)_L(S)$ lying above $(A_0, \lambda_0, \eta_0; A, \lambda, \eta)$.

It is straightforward to check that the above two assignments are inverse to each other. The lemma follows. □

The following lemma will only be used in Section 10.

LEMMA 5.5. *For every $u \in \mathtt{V}_E^{(\infty)}$, there exists an isomorphism*

$$\{X_L \otimes_{E,\iota_u} \mathbb{C}\} \simeq \{{}^uX_L \otimes_{\iota_u(E)} \mathbb{C}\}$$

of systems of complex schemes under which $Z(x)_L \otimes_{E,\iota_u} \mathbb{C}$ coincides with ${}^uZ(x)_L \otimes_{\iota_u(E)} \mathbb{C}$ for every $x \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ with $T(x) \in \mathrm{Herm}_r^\circ(F)^+$. Here, ${}^uX_L$ and ${}^uZ(x)_L$ are defined similarly as X_L and $Z(x)_L$ with $\boldsymbol{\iota}$ replaced by ι_u, hence are schemes and cycles over $\iota_u(E)$.

Proof. We choose an isomorphism $\sigma \colon \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ satisfying $\iota_u = \sigma \circ \boldsymbol{\iota}$.

Choose an element $(A_0, \lambda_0, \eta_0) \in Y(\mathbb{C})$. Then by Lemma 5.2, $X_L \otimes_E \mathbb{C}$ has the following moduli interpretation: For every locally Noetherian complex scheme S, $(X_L \otimes_E \mathbb{C})(S)$ is the set of equivalence classes of triples (A, λ, η) as in Lemma 5.2. In particular, (A, λ) is a unitary O_E-abelian scheme of signature type $n\Phi - \boldsymbol{\iota} + \boldsymbol{\iota}^{\mathtt{c}}$ over S. Since $X_L \otimes_E \mathbb{C}$ does not depend on the choice of

Φ, such moduli interpretation holds as long as Φ contains $\boldsymbol{\iota}$. In particular, we may take Φ such that it contains both ι_u and $\boldsymbol{\iota}$. Then we have such moduli interpretation for both Φ and $\sigma^{-1}\Phi$. Using both moduli interpretations, we obtain an isomorphism $\{X_L \otimes_{E,\iota_u} \mathbb{C}\} \simeq \{{}^u X_L \otimes_{\iota_u(E)} \mathbb{C}\}$ of systems of complex schemes. By Lemma 5.4, it follows easily that under such an isomorphism, $Z(x)_L \otimes_{E,\iota_u} \mathbb{C}$ coincides with ${}^u Z(x)_L \otimes_{\iota_u(E)} \mathbb{C}$ for every $x \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ with $T(x) \in \mathrm{Herm}_r^\circ(F)^+$. The lemma is proved. □

Notation 5.6. In Sections 7, 8, and 9, we will consider a place $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{\mathrm{ram}}$. Let p be the underlying rational prime of u. We will fix an isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ under which $\boldsymbol{\iota}$ induces the place u. In particular, we may identify Φ as a subset of $\mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$.

We further require that Φ in Notation 5.1 is *admissible* in the following sense: if $\Phi_v \subseteq \Phi$ denotes the subset inducing the place v for every $v \in \mathtt{V}_F^{(p)}$, then it satisfies

(1) when $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}}$, Φ_v induces the same place of E above v, which we denote by v_c and by v_e its conjugate;
(2) when $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{int}}$, Φ_v is the pullback of a CM type of the maximal subfield of E_v unramified over $\mathbb{Q}_p$.

To release the burden of notation, we denote by K the subfield of $\overline{\mathbb{Q}}_p$ generated by E_u and the reflex field of Φ, by k its residue field, and by $\breve{K}$ the completion of the maximal unramified extension of K in $\overline{\mathbb{Q}}_p$ with the residue field $\overline{\mathbb{F}}_p$. It is clear that admissible CM type always exists, and that when $\mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}} = \emptyset$, the field K is unramified over E_u.

We also choose a (sufficiently small) open compact subgroup L_0 of $\mathrm{T}_0(\mathbb{A}^\infty)$ such that $L_{0,p}$ is maximal compact. We denote by $\mathcal{Y}$ the integral model of Y over O_K such that for every $S \in \mathrm{Sch}'_{/O_K}$, $\mathcal{Y}(S)$ is the set of equivalence classes of triples $(A_0, \lambda_0, \eta_0^p)$ where

- (A_0, λ_0) is a unitary O_E-abelian scheme over S of signature type Φ such that λ_0 is a p-principal polarization;
- η_0^p is an L_0^p-level structure. (See [LTX+19, Def. 4.1.2] for more details.)

By [How12, Prop. 3.1.2], $\mathcal{Y}$ is finite and étale over O_K.

6. **Height pairing and geometric side**

In this section, we introduce the notion of a height pairing after Beilinson and initiate the study of the geometric side of our desired height formula. We continue the discussion from Section 4. From this moment, we will further assume $F \neq \mathbb{Q}$, which implies that X_L is projective.

We apply Beilinson's construction of the height pairing in [Beĭ87, §4] to obtain a map

$$\langle\ ,\ \rangle^{\ell}_{X_L,E}\colon \mathrm{CH}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}} \times \mathrm{CH}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}} \to \mathbb{C}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$$

(see Notation 2.5(C3) for the notation) that is complex linear in the first variable, and conjugate symmetric. Here, ℓ is a rational prime such that $X_{L,u}$ has smooth projective reduction for every $u\in \mathtt{V}_E^{(\ell)}$. For a pair (c_1,c_2) of elements in $\mathrm{Z}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}} \times \mathrm{Z}^r(X_L)^{\langle\ell\rangle}_{\mathbb{C}}$ with disjoint supports, we have

$$\langle c_1,c_2\rangle^{\ell}_{X_L,E} = \sum_{u\in \mathtt{V}_E^{(\infty)}} 2\langle c_1,c_2\rangle_{X_{L,u},E_u} + \sum_{u\in \mathtt{V}_E^{\mathrm{fin}}} \log q_u\cdot \langle c_1,c_2\rangle^{\ell}_{X_{L,u},E_u}, \tag{6.1}$$

in which

- q_u is the residue cardinality of E_u for $u\in \mathtt{V}_E^{\mathrm{fin}}$;
- $\langle c_1,c_2\rangle^{\ell}_{X_{L,u},E_u}\in \mathbb{C}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ is the non-archimedean local index (B.1) recalled in Appendix B for $u\in \mathtt{V}_E^{\mathrm{fin}}$ (see Remark B.11 when u is above ℓ), which equals zero for all but finitely many u;
- $\langle c_1,c_2\rangle_{X_{L,u},E_u}\in\mathbb{C}$ is the archimedean local index for $u\in \mathtt{V}_E^{(\infty)}$, which will be recalled when we compute it in Section 10.

Definition 6.1. We say that a rational prime ℓ is $\mathtt{R}$-*good* if ℓ is unramified in E and satisfies $\mathtt{V}_F^{(\ell)}\subseteq \mathtt{V}_F^{\mathrm{fin}}\setminus(\mathtt{R}\cup\mathtt{S})$.

Definition 6.2. For every open compact subgroup $L_{\mathtt{R}}$ of $H(F_{\mathtt{R}})$ and every subfield $\mathbb{L}$ of $\mathbb{C}$, we define

(1) $(\mathbb{S}^{\mathtt{R}}_{\mathbb{L}})^0_{L_{\mathtt{R}}}$ to be the ideal of $\mathbb{S}^{\mathtt{R}}_{\mathbb{L}}$ (Notation 2.2(H8)) of elements that annihilate
$$\bigoplus_{i\neq 2r-1} \mathrm{H}^i_{\mathrm{dR}}(X_{L_{\mathtt{R}}L^{\mathtt{R}}}/E)\otimes_{\mathbb{Q}}\mathbb{L};$$

(2) for every rational prime ℓ, $(\mathbb{S}^{\mathtt{R}}_{\mathbb{L}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$ to be the ideal of $\mathbb{S}^{\mathtt{R}}_{\mathbb{L}}$ of elements that annihilate
$$\bigoplus_{u\in \mathtt{V}_E^{\mathrm{fin}}\setminus \mathtt{V}_E^{(\ell)}} \mathrm{H}^{2r}(X_{L_{\mathtt{R}}L^{\mathtt{R}},u},\mathbb{Q}_\ell(r))\otimes_{\mathbb{Q}}\mathbb{L}.$$

Here, $L^{\mathtt{R}}$ is defined in Notation 2.2(H8).

Definition 6.3. Consider a nonempty subset $\mathtt{R}'\subseteq\mathtt{R}$, an $\mathtt{R}$-good rational prime ℓ, and an open compact subgroup L of $H(\mathbb{A}_F^\infty)$ of the form $L_{\mathtt{R}}L^{\mathtt{R}}$, where $L^{\mathtt{R}}$ is defined in Notation 2.2(H8). An $(\mathtt{R},\mathtt{R}',\ell,L)$-*admissible sextuple* is a sextuple $(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)$ in which

- for $i=1,2$, $\phi_i^\infty=\otimes_v\phi_{iv}^\infty\in\mathscr{S}(V^r\otimes_{\mathbb{A}_F}\mathbb{A}_F^\infty)^L$ in which $\phi_{iv}^\infty=\mathbb{1}_{(\Lambda_v^{\mathtt{R}})^r}$ for $v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\mathtt{R}$, satisfying that $\mathrm{supp}(\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}})\subseteq(V_v^{2r})_{\mathrm{reg}}$ for $v\in\mathtt{R}'$;
- for $i=1,2$, s_i is a product of two elements in $(\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$;

- for $i = 1, 2$, g_i is an element in $G_r(\mathbb{A}_F^{\mathtt{R}'})$.

For an $(\mathtt{R}, \mathtt{R}', \ell, L)$-admissible sextuple $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ and every pair (T_1, T_2) of elements in $\mathrm{Herm}_r^\circ(F)^+$, we define

(1) The global index $I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_L^\ell$ to be

$$\big\langle \omega_{r,\infty}(g_{1\infty})\phi_\infty^0(T_1) \cdot \mathrm{s}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)\phi_1^\infty)_L, \\ \omega_{r,\infty}(g_{2\infty})\phi_\infty^0(T_2) \cdot \mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L \big\rangle_{X_L,E}^\ell$$

as an element in $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. We note that for $i = 1, 2$, $\mathrm{s}_i^* Z_{T_i}(\omega_r^\infty(g_i^\infty)\phi_i^\infty)_L$ belongs to $\mathrm{CH}^r(X_L)_{\mathbb{C}}^{\langle \ell \rangle}$ by Definition 6.2(2);

(2) For every $u \in \mathtt{V}_E^{\mathrm{fin}}$, the local index $I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}^\ell$ to be

$$\big\langle \omega_{r,\infty}(g_{1\infty})\phi_\infty^0(T_1) \cdot \mathrm{s}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)\phi_1^\infty)_L, \\ \omega_{r,\infty}(g_{2\infty})\phi_\infty^0(T_2) \cdot \mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L \big\rangle_{X_{L,u},E_u}^\ell$$

as an element in $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, in view of Remark 4.2 and Lemma 6.4(2) below.

(3) For every $u \in \mathtt{V}_E^{(\infty)}$, the local index $I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}$ to be

$$\big\langle \omega_{r,\infty}(g_{1\infty})\phi_\infty^0(T_1) \cdot \mathrm{s}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)\phi_1^\infty)_L, \\ \omega_{r,\infty}(g_{2\infty})\phi_\infty^0(T_2) \cdot \mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L \big\rangle_{X_{L,u},E_u}$$

as an element in $\mathbb{C}$, in view of Remark 4.2 and Lemma 6.4(2) below.

LEMMA 6.4. *Let* $\mathtt{R}$, $\mathtt{R}'$, ℓ, *and* L *be as in Definition* 6.3. *Let* (T_1, T_2) *be a pair of elements in* $\mathrm{Herm}_r^\circ(F)^+$.

(1) *For* $x_1, x_2 \in V \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ *satisfying* $T(x_1) = T_1$, $T(x_2) = T_2$, *and such that* $(L_v x_{1v}, L_v x_{2v}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ *for some* $v \in \mathtt{R}'$, *the algebraic cycles* $Z(x_1)_L$ *and* $Z(x_2)_L$ *in* $\mathrm{Z}^r(X_L)_{\mathbb{C}}$ *have disjoint supports.*

(2) *For every* $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuple* $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$, *the algebraic cycles* $\mathrm{s}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)\phi_1^\infty)_L$ *and* $\mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L$ *in* $\mathrm{Z}^r(X_L)_{\mathbb{C}}$ *have disjoint supports.*

Proof. It is clear that (2) follows from (1).

For (1), it suffices to check that they are disjoint under complex uniformization (4.1). By definition, for $i = 1, 2$, the support of $Z(x_i)_L$ consists of points $(z_i, h_i' h_i)$ in the double coset (4.1), where $h_i x_i = x_i'$ with $x_i' \in {}^{\mathbf{u}}V^r$, z_i is perpendicular to $V_{x_i'}$, and h_i' acts trivially on $V_{x_i'}$. Suppose that the supports of $Z(x_1)_L$ and $Z(x_2)_L$ are not disjoint. Then we may find $\gamma \in {}^{\mathbf{u}}H(F)$ such that $z_1 = \gamma z_2$ and $h_1' h_1 L = \gamma h_2' h_2 L$. In particular, $V_{x_1'} \cap \gamma V_{x_2'} \neq \{0\}$, which implies that the subspace of ${}^{\mathbf{u}}V^{2r}$ generated by $(x_1', \gamma x_2')$ is a proper subspace. Thus, $(h_1 x_1, \gamma h_2 x_2) \notin (V_v^{2r})_{\mathrm{reg}}$ for every $v \in \mathtt{R}'$. On the other hand, we have

$(h_1x_1, \gamma h_2 x_2) = (h_1' h_1 x_1, \gamma h_2' h_2 x_2)$, which implies that $(L_v x_{1v}, L_v x_{2v})$ is not contained in $(V_v^{2r})_{\mathrm{reg}}$, which is a contradiction. Thus, (1) follows. □

The following definition will be used in the future.

Definition 6.5. Let p be a rational prime. We say that an element $\phi^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$ for some integer $m \geqslant 1$ is *p-basic* if it is of the form $\phi^\infty = \otimes_v \phi_v^\infty$ in which $\phi_v^\infty = \mathbb{1}_{(\Lambda_v^{\mathtt{R}})^m}$ for every $v \in \mathtt{V}_F^{(p)} \setminus (\mathtt{R} \cup \mathtt{V}_F^{\mathrm{spl}})$.

Recall from Notation 3.14(1) that Π is the automorphic base change of π.

Hypothesis 6.6. *Let ℓ be a rational prime with an arbitrarily given isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. For every irreducible admissible representation $\tilde{\pi}^\infty$ of $H(\mathbb{A}_F^\infty)$ such that Π_v is the standard base change of $\tilde{\pi}_v^\infty$ for all but finitely many $v \in \mathtt{V}_F^{\mathrm{fin}}$ for which $\tilde{\pi}_v^\infty$ is unramified, if we are in the situation* (b) *of Lemma* 3.15*, then the semisimplification of the representation*

$$\rho[\pi^\infty] := \mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\tilde{\pi}^\infty, \varinjlim_L \mathrm{H}^{2r-1}(X_L \otimes_E \mathbb{Q}^{\mathrm{ac}}, \overline{\mathbb{Q}}_\ell)\right)$$

of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/E)$ *is isomorphic to* $\rho^{\mathtt{c}}_{\Pi_{j(\tilde{\pi}^\infty)}}$*, where* ρ_{Π_j} *is introduced in Notation* 3.14(4).

Remark 6.7. Concerning Hypothesis 6.6, we have

(1) when $n = 2$, it has been confirmed in [Liu21a, Th. D.6];
(2) when Π is cuspidal (that is, $s = 1$ in Notation 3.14(3)), it will be confirmed in [KSZ] (under the help of [Mok15, KMSW14]);
(3) in general, it will follow from [KSZ] as long as the full endoscopic classification for unitary groups is obtained.

Definition 6.8. Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1. We define a character

$$\chi_\pi^{\mathtt{R}} \colon \mathbb{T}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}} \to \mathbb{Q}^{\mathrm{ac}}$$

as follow. Let $\mathcal{H}^{\mathtt{R}}_{W_r}$ be the restricted tensor product of commutative complex algebras $\mathcal{H}^{\pm}_{W_{r,v}}$ if $\epsilon_v = \pm 1$ over $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$, where $\mathcal{H}^{\pm}_{W_{r,v}}$ is defined in [Liu21b, Def. 2.5] for $v \in \mathtt{V}_F^{\mathrm{int}}$ and is simply the spherical Hecke algebra for $v \in \mathtt{V}_F^{\mathrm{spl}}$. Using the construction in [Liu21b, Def. 2.8], we have a canonical surjective homomorphism $\theta^{\mathtt{R}} \colon \mathcal{H}^{\mathtt{R}}_{W_r} \to \mathbb{T}^{\mathtt{R}}_{\mathbb{C}}$ of complex commutative algebras. Since π_v is unramified (resp. almost unramified) when $\epsilon_v = 1$ (resp. $\epsilon_v = -1$) by Assumption 3.1(2), the algebra $\mathcal{H}^{\mathtt{R}}_{W_r}$ acts on $\mathcal{V}_\pi^{[r]_{\mathtt{R}}}$ by a character $\chi^{\mathtt{R}}_{\pi, W_r} \colon \mathcal{H}^{\mathtt{R}}_{W_r} \to \mathbb{C}$, which factors through $\theta^{\mathtt{R}}$ by [Liu21b, Def. 5.3, Th. 1.1(1)]. Since π is cohomological by Assumption 3.1(1), hence has algebraic Satake parameters, there exists a unique character $\chi_\pi^{\mathtt{R}} \colon \mathbb{T}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}} \to \mathbb{Q}^{\mathrm{ac}}$ such that $\chi^{\mathtt{R}}_{\pi, W_r} = (\chi_\pi^{\mathtt{R}} \otimes_{\mathbb{Q}^{\mathrm{ac}}} \mathbb{C}) \circ \theta^{\mathtt{R}}$.

We put $\mathfrak{m}_\pi^{\mathtt{R}} := \ker \chi_\pi^{\mathtt{R}}$, which is a maximal ideal of $\mathbb{T}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}}$.

PROPOSITION 6.9. *Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1. For every open compact subgroup $L_{\mathbb{R}}$ of $H(F_{\mathbb{R}})$, we have*

(1) $(\mathbb{S}^{\mathbb{R}}_{\mathbb{Q}^{\mathrm{ac}}})^0_{L_{\mathbb{R}}} \setminus \mathfrak{m}^{\mathbb{R}}_\pi$ *is nonempty;*

(2) *under Hypothesis 6.6,* $(\mathbb{S}^{\mathbb{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle \ell \rangle}_{L_{\mathbb{R}}} \setminus \mathfrak{m}^{\mathbb{R}}_\pi$ *is nonempty.*

Proof. For (1), by Matsushima's formula, we know that the localization of the $\mathbb{S}^{\mathbb{R}}_{\mathbb{C}}$-module $\mathrm{H}^i_{\mathrm{dR}}(X_{L_{\mathbb{R}}L^{\mathbb{R}}}/E) \otimes_{\mathbb{Q}} \mathbb{C}$ at $\mathfrak{m}^{\mathbb{R}}_\pi$ is isomorphic to the direct sum of $\mathrm{H}^i(\tilde{\pi}_\infty) \otimes \tilde{\pi}^\infty$ for all cuspidal automorphic representations $\tilde{\pi}$ of ${}^{\mathrm{u}}H(\mathbb{A}_F)$ such that the standard base change of $\tilde{\pi}_v$ is isomorphic to Π_v for all but finitely many $v \in \mathtt{V}_F^{\mathrm{spl}}$, where $\mathrm{H}^i(\tilde{\pi}_\infty)$ denotes the $(\mathfrak{g}, K)$-cohomology of $\tilde{\pi}_\infty$. By [Ram18, Th. A], we know that Π must be the automorphic base change of $\tilde{\pi}$. By [KMSW14, Th. 1.7.1], we know that $\tilde{\pi}_\infty$ is tempered, hence $\mathrm{H}^i(\tilde{\pi}_\infty)$ vanishes for $i \neq 2r-1$. Therefore, (1) follows as $\mathrm{H}^i_{\mathrm{dR}}(X_{L_{\mathbb{R}}L^{\mathbb{R}}}/E)$ is of finite dimension.

For (2), note that for all but finitely many $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{(\ell)}$, the natural map

$$\mathrm{H}^{2r}(X_{L_{\mathbb{R}}L^{\mathbb{R}},u}, \mathbb{Q}_\ell(r)) \to \mathrm{H}^{2r}(X_{L_{\mathbb{R}}L^{\mathbb{R}},u} \otimes_{E_u} \overline{E_u}, \mathbb{Q}_\ell(r))$$

is injective by the Hochschild–Serre spectral sequence and the Weil conjecture. As an $\mathbb{S}^{\mathbb{R}}_{\mathbb{Q}^{\mathrm{ac}}}$-module, we have

$$\mathrm{H}^{2r}(X_{L_{\mathbb{R}}L^{\mathbb{R}},u} \otimes_{E_u} \overline{E_u}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} (E \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}}) \simeq \mathrm{H}^{2r}_{\mathrm{dR}}(X_{L_{\mathbb{R}}L^{\mathbb{R}}}/E) \otimes_{\mathbb{Q}} (\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}}).$$

By (1), we know that there exist elements in $\mathbb{S}^{\mathbb{R}}_{\mathbb{Q}^{\mathrm{ac}}} \setminus \mathfrak{m}^{\mathbb{R}}_\pi$ that annihilate

$$\mathrm{H}^{2r}(X_{L_{\mathbb{R}}L^{\mathbb{R}},u}, \mathbb{Q}_\ell(r))$$

for all but finitely many $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{(\ell)}$. Thus, it remains to show that for every given $u \in \mathtt{V}_E^{\mathrm{fin}} \setminus \mathtt{V}_E^{(\ell)}$ and every embedding $\mathbb{Q}^{\mathrm{ac}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, the localization of $\mathrm{H}^{2r}(X_{L_{\mathbb{R}}L^{\mathbb{R}},u}, \overline{\mathbb{Q}}_\ell(r))$ at $\mathfrak{m}^{\mathbb{R}}_\pi$ vanishes. By (1) and the Hochschild–Serre spectral sequence, it suffices to show that

$$\mathrm{H}^1(E_u, \mathrm{H}^{2r-1}(X_{L_{\mathbb{R}}L^{\mathbb{R}}} \otimes_E \mathbb{Q}^{\mathrm{ac}}, \overline{\mathbb{Q}}_\ell(r))_{\mathfrak{m}^{\mathbb{R}}_\pi}) = 0.$$

By Hypothesis 6.6, it suffices to show that $\mathrm{H}^1(E_u, \rho^{\mathtt{c}}_{\Pi_j}(r)) = 0$ for every j. As shown in the proof of [Car12, Th. 7.4], the associated Weil–Deligne representation of $\rho_{\Pi_j}(r)$ at u is pure (of weight not zero), which implies $\mathrm{H}^1(E_u, \rho^{\mathtt{c}}_{\Pi_j}(r)) = 0$ by [Nek07b, Prop. 4.2.2(1)].

The proposition is proved. □

Until the end of this section, let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 3.1, and assume Hypothesis 4.5 on the modularity of generating functions of codimension r.

PROPOSITION 6.10. *In the situation of Definition 4.8 (and suppose that $F \neq \mathbb{Q}$), suppose that L has the form $L_{\mathbb{R}}L^{\mathbb{R}}$, where $L^{\mathbb{R}}$ is defined in Notation 2.2(H8). For arbitrary elements $\varphi \in \mathcal{V}_\pi^{[r]_{\mathbb{R}}}$ and $\phi^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, we have*

(1) $\mathrm{s}^*\Theta_{\phi^\infty}(\varphi)_L = \chi_\pi^{\mathtt{R}}(\mathrm{s})^{\mathtt{c}} \cdot \Theta_{\phi^\infty}(\varphi)_L$ *for every* $\mathrm{s} \in \mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}}$;
(2) $\Theta_{\phi^\infty}(\varphi)_L \in \mathrm{CH}^r(X_L)^0_{\mathbb{C}}$;
(3) *under Hypothesis 6.6,* $\Theta_{\phi^\infty}(\varphi)_L \in \mathrm{CH}^r(X_L)^{\langle \ell \rangle}_{\mathbb{C}}$ *for every* $\mathtt{R}$*-good rational prime* ℓ.

Proof. For (1), by Lemma 4.4, we have

$$\mathrm{s}^*\Theta_{\phi^\infty}(\varphi)_L = \Theta_{\mathrm{s}\phi^\infty}(\varphi)_L = \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi^{\mathtt{c}}(g) Z_{\mathrm{s}\phi^\infty}(g)_L \, \mathrm{d}g,$$

which, by [Liu11a, Prop. A.5] for split places (see also [Ral82, p. 511]), equals

$$\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} (\pi^\vee(\mathrm{s})\varphi^{\mathtt{c}})(g) Z_{\phi^\infty}(g)_L \, \mathrm{d}g,$$

which equals

$$\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} (\chi_\pi^{\mathtt{R}}(\mathrm{s})^{\mathtt{c}} \cdot \varphi^{\mathtt{c}})(g) Z_{\phi^\infty}(g)_L \, \mathrm{d}g = \chi_\pi^{\mathtt{R}}(\mathrm{s})^{\mathtt{c}} \cdot \Theta_{\phi^\infty}(\varphi)_L.$$

Part (2) is a consequence of (1) and Proposition 6.9(1).
Part (3) is a consequence of (1) and Proposition 6.9(2). □

We now define the *normalized height pairing* between the cycles $\Theta_{\phi^\infty}(\varphi)$ in Definition 4.8, under Hypothesis 6.6.

Definition 6.11. Under Hypothesis 6.6, for arbitrary elements $\varphi_1, \varphi_2 \in \mathcal{V}_\pi^{[r]}$ and $\phi_1^\infty, \phi_2^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$, we define the *normalized height pairing*

$$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2) \rangle^\natural_{X,E} \in \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

to be the unique element[10] such that for every $L = L_{\mathtt{R}} L^{\mathtt{R}}$ as in Proposition 6.10 (with $\mathtt{R}$ possibly enlarged) satisfying $\varphi_1, \varphi_2 \in \mathcal{V}_\pi^{[r]\mathtt{R}}$, $\phi_1^\infty, \phi_2^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$, and that ℓ is $\mathtt{R}$-good, we have

$$\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2) \rangle^\natural_{X,E} = \mathrm{vol}^\natural(L) \cdot \langle \Theta_{\phi_1^\infty}(\varphi_1)_L, \Theta_{\phi_2^\infty}(\varphi_2)_L \rangle^\ell_{X_L,E},$$

where $\mathrm{vol}^\natural(L)$ is introduced in Definition 3.8,[11] and $\langle \Theta_{\phi_1^\infty}(\varphi_1)_L, \Theta_{\phi_2^\infty}(\varphi_2)_L \rangle^\ell_{X_L,E}$ is well defined by Proposition 6.10(3). Note that by the projection formula, the right-hand side of the above formula is independent of L.

[10] The readers may notice that we have dropped ℓ in the notation $\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2) \rangle^\natural_{X,E}$. This is because for those normalized height pairings we are able to compute in this article, the value will turn out to be in $\mathbb{C}$ and is independent of the choice of ℓ.

[11] In fact, it is a good exercise to show that the total degree of the Hodge line bundle on X_L is equal to $2\,\mathrm{vol}^\natural(L)^{-1}$.

7. Local indices at split places

In this section, we compute local indices at all but finitely many places in $\mathtt{V}_E^{\mathrm{spl}}$. Our goal is to prove the following proposition.

PROPOSITION 7.1. *Let* $\mathtt{R}$, $\mathtt{R}'$, ℓ, *and* L *be as in Definition 6.3 such that the cardinality of* $\mathtt{R}'$ *is at least* 2. *Let* $(\pi, \mathcal{V}_\pi)$ *be as in Assumption 3.1, for which we assume Hypothesis 6.6. For every* $u \in \mathtt{V}_E^{\mathrm{spl}}$ *such that*

(a) *the representation* $\pi_{\underline{u}}$ *is a (tempered) principal series,*
(b) $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$ *where* p *is the underlying rational prime of* u,

there exist elements $\mathrm{s}_1^u, \mathrm{s}_2^u \in \mathbb{S}_{\overline{\mathbb{Q}}^{\mathrm{ac}}}^{\mathtt{R}} \setminus \mathfrak{m}_\pi^{\mathtt{R}}$ *such that*

$$I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1^u \mathrm{s}_1, \mathrm{s}_2^u \mathrm{s}_2, g_1, g_2)_{L,u}^{\ell} = 0$$

for every $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuple* $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ *and every pair* (T_1, T_2) *in* $\mathrm{Herm}_r^\circ(F)^+$. *Moreover, we may take* $\mathrm{s}_1^u = \mathrm{s}_2^u = 1$ *if* $\underline{u} \notin \mathtt{R}$.

Since $\underline{u}$ splits in E, we may fix an isomorphism $H \otimes_{\mathbb{A}_F} F_{\underline{u}} \simeq \mathrm{GL}_{n,F_{\underline{u}}}$ such that $L_{\underline{u}}$ is contained in $\mathrm{GL}_n(O_{F_{\underline{u}}})$ and, moreover, equal if $\underline{u} \notin \mathtt{R}$. For every integer $m \geqslant 0$, denote by $L_{\underline{u},m} \subseteq \mathrm{GL}_n(O_{F_{\underline{u}}})$ the principal congruence subgroup of level m.

From now to the end of this section, we assume $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$. We invoke Notation 5.1 together with Notation 5.6, which is possible since $\mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}} = \emptyset$. To ease notation, we put $X_m \coloneqq X'_{L_{\underline{u},m} L^{\underline{u}}} \otimes_{E'} K$ for $m \geqslant 0$. The isomorphism $\mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ in Notation 5.6 identifies $\mathrm{Hom}(E, \mathbb{C})$ with $\mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$. For every $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}} \setminus \{\underline{u}\}$, let $\{v_c, v_e\}$ be the two places of E above v from Notation 5.6. Identify $H \otimes_{\mathbb{A}_F} F_v$ with $\mathrm{GL}(V \otimes_{\mathbb{A}_F} E_{v_e})$.

Let S be a locally Noetherian scheme over O_K and (A, λ) a unitary O_E-abelian scheme of signature type $n\Phi - \iota_w + \iota_w^{\mathtt{c}}$ over S. Then the p-divisible group $A[p^\infty]$ admits a decomposition $A[p^\infty] = \prod_{v \in \mathtt{V}_F^{(p)}} A[v^\infty]$.

For every integer $m \geqslant 0$, we define a moduli functor $\mathcal{X}_m$ over O_K as follows: For every locally Noetherian scheme S over O_K, $\mathcal{X}_m(S)$ is the set of equivalence classes of tuples $(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p, \{\eta_v\}_{v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}} \setminus \{\underline{u}\}}, \eta_{u,m})$, where

- $(A_0, \lambda_0, \eta_0^p)$ is an element in $\mathcal{Y}(S)$;
- (A, λ) is a unitary O_E-abelian scheme of signature type $n\Phi - \iota_w + \iota_w^{\mathtt{c}}$ over S, such that
  - for every $v \in \mathtt{V}_F^{(p)}$, $\lambda[v^\infty]$ is an isogeny (rather than a quasi-isogeny) whose kernel has order $q_v^{1-\epsilon_v}$;

  - $\mathrm{Lie}(A[u^{\mathtt{c},\infty}])$ is of rank 1 on which the action of O_E is given by the embedding $\iota_w^{\mathtt{c}}$;[12]
- η^p is an L^p-level structure, analogous to the one in Lemma 5.2;
- for every $v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}} \setminus \{\underline{u}\}$, η_v is an L_v-level structure, that is, an L_v-orbit of E_{v_e}-linear isomorphisms

$$\eta_v \colon V \otimes_{\mathbb{A}_F} E_{v_e} \xrightarrow{\sim} \mathrm{Hom}_{O_{E_{v_e}}}(A_0[v_e^\infty], A[v_e^\infty]) \otimes_{O_{E_{v_e}}} E_{v_e}$$

of E_{v_e}-sheaves over S;
- $\eta_{u,m} \colon (\mathfrak{p}_{\underline{u}}^{-m}/O_{F_{\underline{u}}})^n \to \underline{\mathrm{Hom}}_{O_{F_{\underline{u}}}}(A_0[u^{\mathtt{c},\infty}][\mathfrak{p}_{\underline{u}}^m], A[u^{\mathtt{c},\infty}][\mathfrak{p}_{\underline{u}}^m])$ is a Drinfeld level-m structure; see [RSZ20, §4.3] for more details.

By [RSZ20, Th. 4.5], for every $m \geqslant 0$, $\mathcal{X}_m$ is a regular scheme, flat (smooth, if $m = 0$) and projective over O_K, and it admits a canonical isomorphism $\mathcal{X}_m \otimes_{O_K} K \simeq X_m$ of schemes over K.[13] Note that for every integer $m \geqslant 0$, $\mathbb{S}^{\mathtt{R} \cup \mathtt{V}_F^{(p)}}$ naturally gives a ring of étale correspondences of $\mathcal{X}_m$.

We first prove the follow lemma, which addresses the easy part of Proposition 7.1 as a warm-up.

LEMMA 7.2. *Let the situation be as in Proposition 7.1. Suppose that* $\underline{u} \notin \mathtt{R}$. *Then we have*

$$I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,u}^{\ell} = 0$$

for every $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuple* $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ *and every pair* (T_1, T_2) *in* $\mathrm{Herm}_r^\circ(F)^+$.

Proof. It suffices to show that for every $x_1, x_2 \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ satisfying $T(x_1), T(x_2) \in \mathrm{Herm}_r^\circ(F)^+$ and $(L_v x_{1v}, L_v x_{2v}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for some $v \in \mathtt{R}'$, we have

$$\langle Z(x_1)_L, Z(x_2)_L \rangle_{X_{L,u}, E_u}^{\ell} = 0.$$

Since $\underline{u} \notin \mathtt{R}$, by Lemmas B.3 and 5.4, it suffices to show that

$$\langle Z(x_1)'_L, Z(x_2)'_L \rangle_{X_0,K}^{\ell} = 0. \tag{7.1}$$

We use the integral model $\mathcal{X}_0$ just constructed above, which is smooth and projective over O_K of relative dimension $n-1$. For $i = 1, 2$, let $\mathcal{Z}(x_i)'_L$ be the Zariski closure of $Z(x_i)'_L$ in $\mathcal{X}_0$. We claim that $\mathcal{Z}(x_1)'_L$ and $\mathcal{Z}(x_2)'_L$ have empty intersection. By Proposition B.10, we obtain (7.1).

[12] Since Φ is admissible (Notation 5.6), the Eisenstein condition at $v \neq \underline{u}$ is implied by the Kottwitz condition, and at $\underline{u}$ it is implied by the Kottwitz condition and that $\mathrm{Lie}(A[u^{\mathtt{c},\infty}])$ is of rank 1 on which the action of O_E is given by the embedding $\iota_w^{\mathtt{c}}$.

[13] Here, we have to use the fact that K is unramified over E_u to conclude that $\mathcal{X}_m$ is regular when $m > 0$.

For the claim, we assume the converse. Then we can find a point

$$(A_0, \lambda_0, \eta_0^p; A, \lambda, \eta^p, \{\eta_v\}_{v \in \mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{spl}} \setminus \{\underline{u}\}}) \in \mathcal{X}_0(\overline{\mathbb{F}}_p)$$

that is in the supports of both $\mathcal{Z}(x_1)'_L$ and $\mathcal{Z}(x_2)'_L$. In particular, for $i = 1, 2$, we can find an element $\tilde{x}_i \in \mathrm{Hom}_{O_E}(A_0^r, A)_{\mathbb{Q}}$ satisfying $\tilde{x}_{i,*} \in \eta^p(L^p x_i^p)$. As $(L_v x_{1v}, L_v x_{2v}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for some $v \in \mathtt{R}'$, we know that A is quasi-isogenous to A_0^{2r}, which is impossible by [RSZ20, Lemma 8.7]. It follows that the supports of $Z(x_1)'_L$ and $Z(x_2)'_L$ have nonempty intersection, which is a contradiction to Lemma 6.4(1). Thus, the claim and hence the lemma are proved. □

To study the general case, we need the following vanishing result.

LEMMA 7.3. *Let the situation be as in Proposition 7.1 with $p \neq \ell$. Then for every integer $m \geqslant 0$, we have*

$$\left(\mathrm{H}^{2r}(\mathcal{X}_m, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}}\right)_{\mathfrak{m}} = 0,$$

where $\mathfrak{m} := \mathfrak{m}_\pi^{\mathtt{R}} \cap \mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R} \cup \mathtt{V}_F^{(p)}}$.

Proof. For every integer $m \geqslant 0$, put $Y_m := \mathcal{X}_m \otimes_{O_K} k$, $Y_{m,0} := Y_m^{\mathrm{red}}$, and for $0 \leqslant j \leqslant n-1$, denote by $Y_{m,j}$ the Zariski closed subset of Y_m on which the formal part of $A[u^{\mathtt{c},\infty}]$ has height at least $j+1$. By the similar argument of [HT01, Cor. III.4.4], we know that $Y_{m,j}^\circ := Y_{m,j} \setminus Y_{m,j+1}$ is smooth over k of pure dimension $n-1-j$.[14] Applying Corollary B.15(2) to $\mathbb{S} = \mathbb{S}^{\mathtt{R} \cup \mathtt{V}_F^{(p)}}$, $\mathbb{L} = \mathbb{Q}^{\mathrm{ac}}$, and $\mathfrak{m} = \mathfrak{m}_\pi^{\mathtt{R}} \cap \mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R} \cup \mathtt{V}_F^{(p)}}$, it suffices to show that for every $m \geqslant 0$,

(1) $(\mathrm{H}^{2r}(X_m, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}})_{\mathfrak{m}} = 0$; and
(2) $(\mathrm{H}^i(Y_{m,j}^\circ \otimes_k \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} \mathbb{Q}^{\mathrm{ac}})_{\mathfrak{m}} = 0$ for every $i \leqslant 2r - 2(j+1)$ and every $0 \leqslant j \leqslant n-1$.

Part (1) has already been proved in Proposition 6.9(2) (as we have assumed Hypothesis 6.6).

Part (2) follows from the following stronger statement:

(3) For an arbitrary embedding $\mathbb{Q}^{\mathrm{ac}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, $\mathrm{H}^i(Y_{m,j}^\circ \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} = 0$ for every $m \geqslant 0$ unless $j = 0$ and $i = 2r-1$.

The argument for (3) is similar to the proof of [CS17, Th. 6.3.1]. For $m \geqslant 0$ and $0 \leqslant j \leqslant n-1$, let $I_{m,j}$ be the Igusa variety (of the first kind) so that $Y_{m,j}^\circ$ is the disjoint union of finitely many $I_{m,j}$; see [HT01, §IV.1]. For each j, we obtain a projective system $\{I_{m,j} \mid m \geqslant 0\}$ with finite étale transition morphisms. If $\mathrm{H}^i(Y_{m,j}^\circ \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} = 0$ for all $m \geqslant 0$, i, and j, then we are done. Otherwise, let j be the maximal integer such that $\mathrm{H}^i(Y_{m,j}^\circ \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} \neq 0$

[14] In the notation of [HT01, §III.4], our $Y_{m,j}^\circ$ is parallel to $\overline{X}_{U^p,m}^{(n-1-j)}$.

for some m and i. Then $\varinjlim_m \mathrm{H}^i(I_{m,j} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} \neq 0$. Now we would like to apply [CS17, Cor. 6.1.4], where in our case, the set $B(G, \mu^{-1})$ is identified with $\{0, \dots, n-1\}$ under which $d = n - j = 2r - 1 - j$, and Ig^j is the perfection of $\mathscr{I}^j_{\mathrm{Mant}} \coloneqq \varprojlim_m \mathscr{I}^j_{\mathrm{Mant},m}$ [CS17, Prop. 4.3.8] in which $\mathscr{I}^j_{\mathrm{Mant},m}$ is a finite Galois cover of $I_{m,j}$.[15] Then we have $\mathrm{H}^i(\mathrm{Ig}^j \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} \neq 0$, which, by [CS17, Cor. 6.1.4][16] (for the coefficients $\overline{\mathbb{Q}}_\ell$), implies that i can only be $2r - 1 - j$. In particular, combining with the Poincaré duality, we have $[\mathrm{H}_c(\mathscr{I}^j_{\mathrm{Mant}} \otimes_k \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_\ell)]_{\mathfrak{m}} \neq 0$ (where we adopted the notation from [CS17, Th. 5.5.7]). By the local-global compatibility at split places ([Shi, Th. 1.1] or more generally [KMSW14, Th. 1.7.1]), we have $\Pi_u \simeq \pi_{\underline{u}}$, where we recall that Π is the automorphic base change of π in Notation 3.14(1). In particular, Π_u is a tempered principal series by (a). Then by [CS17, Th. 5.5.7] (together with the modification in the proof of [LTX+19, Th. D.1.3]) and the very strong multiplicity one property [Ram18, Th. A], we must have $j = 0$ and hence $i = 2r - 1$. Thus, (3) follows.

The lemma is proved. □

Remark 7.4. In fact, we conjecture that Lemma 7.3 remains true without condition (a) in Proposition 7.1. If this is confirmed, then we may remove condition (2) in Assumption 1.3.

Proof of Proposition 7.1. The last part of the proposition has been confirmed in Lemma 7.2. We prove the first part. We may assume $p \neq \ell$ since otherwise it has been covered in Lemma 7.2. Fix an integer $m \geqslant 0$ such that $L_{\underline{u}}$ contains $L_{\underline{u},m}$.

It suffices to show that there exists $\mathrm{s} \in \mathbb{S}^{\mathtt{R} \cup \mathtt{V}_F^{(p)}} \setminus \mathfrak{m}^{\mathtt{R}}_\pi$ such that for every $x_1, x_2 \in V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty$ satisfying $T(x_1), T(x_2) \in \mathrm{Herm}_r^\circ(F)^+$ and $(L_v x_{1v}, L_v x_{2v}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for some $v \in \mathtt{R}' \setminus \{\underline{u}\}$ (which is nonempty as we assume $|\mathtt{R}'| \geqslant 2$), we have

$$\langle \mathrm{s}^* Z(x_1)_L, \mathrm{s}^* Z(x_2)_L \rangle^\ell_{X_{L,u}, E_u} = 0.$$

By Lemmas B.3 and 5.4, it suffices to have

$$\langle \mathrm{s}^* Z(x_1)'_L, \mathrm{s}^* Z(x_2)'_L \rangle^\ell_{X_m, K} = 0. \tag{7.2}$$

To compute the local index on X_m, we use the model $\mathcal{X}_m$ constructed above. Take $\mathrm{s} \in \mathbb{S}^{\mathtt{R} \cup \mathtt{V}_F^{(p)}}_{\mathbb{Q}^{\mathrm{ac}}}$ that is an ℓ-tempered $\mathbb{Q}^{\mathrm{ac}}$-étale correspondence of $\mathcal{X}_m$, which exists by Lemma 7.3 and Corollary B.15(1). Then by Proposition B.13, we

[15] The Galois cover comes from the fact that in the definition of $\mathscr{I}^j_{\mathrm{Mant},m}$, there is also a level structure on the formal part of $A[u^{\mathrm{c},\infty}]$.

[16] Strictly speaking, the authors assumed that the level at p is hyperspecial maximal. In our case, we only require that $L_{\underline{u}}$ is hyperspecial. However, by our special signature condition, the argument of [CS17] works in our case verbatim.

have

$$\langle \mathrm{s}^*Z(x_1)'_L, \mathrm{s}^*Z(x_2)'_L\rangle^{\ell}_{X_m,K} = [\mathrm{s}^*\mathcal{Z}(x_1)'_L].[\mathrm{s}^*\mathcal{Z}(x_2)'_L],$$

where $\mathcal{Z}(x_i)'_L$ is the Zariski closure of $Z(x_i)'_L$ in $\mathcal{X}_m$ for $i = 1, 2$. By the similar argument used in the proof of Lemma 7.2, $\mathrm{s}^*\mathcal{Z}(x_1)'_L$ and $\mathrm{s}^*\mathcal{Z}(x_2)'_L$ have disjoint supports, which implies $\mathrm{s}^*\mathcal{Z}(x_1)'_L.\mathrm{s}^*\mathcal{Z}(x_2)'_L = 0$. Thus (7.2) and hence the proposition hold with $\mathrm{s}^u_1 = \mathrm{s}^u_2 = \mathrm{s}$. □

8. **Local indices at inert places: unramified case**

In this section, we compute local indices at places in $\mathtt{V}_E^{\mathrm{int}}$ that are not above $\mathtt{R} \cup \mathtt{S}$. Our goal is to prove the following proposition.

PROPOSITION 8.1. *Let* $\mathtt{R}$, $\mathtt{R}'$, ℓ, *and* L *be as in Definition 6.3. Take an element* $u \in \mathtt{V}_E^{\mathrm{int}}$ *such that* $\underline{u} \notin \mathtt{S}$ *and whose underlying rational prime* p *is odd and satisfies* $\mathtt{V}_F^{(p)} \cap \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{spl}}$. *Then we have*

$$\log q_u \cdot \mathrm{vol}^{\natural}(L) \cdot I_{T_1,T_2}(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)^{\ell}_{L,u} = \mathfrak{E}_{T_1,T_2}((g_1, g_2), \Phi^0_{\infty} \otimes (\mathrm{s}_1\phi_1^{\infty} \otimes (\mathrm{s}_2\phi_2^{\infty})^{\mathtt{c}}))_u$$

for every $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuple* $(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ *and every pair* (T_1, T_2) *in* $\mathrm{Herm}_r^{\circ}(F)^+$, *where the right-hand side is defined in Definition 3.11 with the Gaussian function* $\Phi^0_{\infty} \in \mathscr{S}(V^{2r} \otimes_{\mathbb{A}_F} F_{\infty})$ (*Notation 2.2*(H3)), *and* $\mathrm{vol}^{\natural}(L)$ *is defined in Definition 3.8.*

To prove Proposition 8.1, we may rescale the hermitian form on V and hence assume that $\psi_{F,v}$ is unramified and that $\Lambda_v^{\mathtt{R}}$ is either a self-dual or an almost self-dual lattice of V_v for every $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}}$.

LEMMA 8.2. *Let the situation be as in Proposition 8.1. If the weaker version of Proposition 8.1 where we only consider* $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuples* $(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ *in which* $g_{1v} = g_{2v} = 1_{2r}$ *for every* $v \in \mathtt{V}_F^{(\infty)} \cup \mathtt{V}_F^{(p)}$ *holds, then the original Proposition 8.1 holds.*

Proof. Take an arbitrary $(\mathtt{R}, \mathtt{R}', \ell, L)$-admissible sextuple $(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$. For $i = 1, 2$, we may find elements $a_i \in \mathrm{GL}_r(E)$ and $b_i \in \mathrm{Herm}_r(F)$ such that $m(a_i)^{-1}n(b_i)^{-1}g_{iv} \in K_{r,v}$ for every $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{R}'$. For $i = 1, 2$, put

$$\tilde{T}_i := {}^{\mathrm{t}}a_i^{\mathtt{c}} T_i a_i, \quad \tilde{\phi}_i^{\infty} := \prod_{v \in \mathtt{R}'} \omega_{r,v}(m(a_i)^{-1}n(b_i)^{-1})\phi_i^{\infty},$$

and let $\tilde{g}_i$ be the away-from-$(\mathtt{R}' \cup \mathtt{V}_F^{(\infty)} \cup \mathtt{V}_F^{(p)})$-component of $m(a_i)^{-1}n(b_i)^{-1}g_i$. Then $(\tilde{\phi}_1^{\infty}, \tilde{\phi}_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, \tilde{g}_1, \tilde{g}_2)$ is an $(\mathtt{R}, \mathtt{R}', \ell, L)$-admissible sextuple.

By Lemma 4.3, we have

$$I_{T_1,T_2}(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)^{\ell}_{L,u} = C \cdot I_{\tilde{T}_1,\tilde{T}_2}(\tilde{\phi}_1^{\infty}, \tilde{\phi}_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, \tilde{g}_1, \tilde{g}_2)^{\ell}_{L,u}$$

in which

$$C = \left(\frac{\omega_{r,\infty}(m(a_1)^{-1}n(b_1)^{-1}g_{1\infty})\phi^0_\infty(\tilde{T}_1)}{\phi^0_\infty(\tilde{T}_1)}\right) \cdot \left(\frac{\omega_{r,\infty}(m(a_2)^{-1}n(b_2)^{-1}g_{2\infty})\phi^0_\infty(\tilde{T}_2)}{\phi^0_\infty(\tilde{T}_2)}\right)^{\mathtt{c}}.$$

On the other hand, from Definition 3.11, we have

$$\mathfrak{E}_{T_1,T_2}((g_1,g_2),\Phi^0_\infty\otimes(\mathrm{s}_1\phi_1^\infty\otimes(\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_u = C\cdot\mathfrak{E}_{\tilde{T}_1,\tilde{T}_2}((\tilde{g}_1,\tilde{g}_2),\Phi^0_\infty\otimes(\mathrm{s}_1\tilde{\phi}_1^\infty\otimes(\mathrm{s}_2\tilde{\phi}_2^\infty)^{\mathtt{c}}))_u$$

with the same C. The lemma follows. □

In order to deal with spherical Hecke operators, we consider the projective system of Shimura varieties $\{X_{\tilde{L}}\}$ indexed by open compact subgroups $\tilde{L}\subseteq L$ satisfying $\tilde{L}_v = L_v$ for $v\in \mathtt{V}_F^{(p)}\setminus\mathtt{V}_F^{\mathrm{spl}}$.

We invoke Notation 5.1 together with Notation 5.6, which is possible since $\mathtt{V}_F^{(p)}\cap\mathtt{V}_F^{\mathrm{ram}}=\emptyset$. There is a projective system $\{\mathcal{X}_{\tilde{L}}\}$ of smooth projective schemes over O_K (see [LZ, §11.2]) with

$$\mathcal{X}_{\tilde{L}}\otimes_{O_K}K = X'_{\tilde{L}}\otimes_{E'}K = \left(X_{\tilde{L}}\otimes_E Y\right)\otimes_{E'}K,$$

and finite étale transition morphisms. In particular, $\mathbb{S}^{\mathtt{R}}$ is naturally a ring of étale correspondences of $\mathcal{X}_L$.

Lemma 8.3. *If* s *is a product of two elements in* $(\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$, *then it gives an* ℓ*-tempered* $\mathbb{Q}^{\mathrm{ac}}$*-étale correspondence of* $\mathcal{X}_L$ (*Definition* B.12).

Proof. We have a short exact sequence

$$\mathrm{H}^{2r}_{\mathcal{X}_L\otimes_{O_K}k}(\mathcal{X}_L,\mathbb{Q}_\ell(r))\to\mathrm{H}^{2r}(\mathcal{X}_L,\mathbb{Q}_\ell(r))\to\mathrm{H}^{2r}(X'_L,\mathbb{Q}_\ell(r)), \tag{8.1}$$

in which we have

$$\mathrm{H}^{2r}_{\mathcal{X}_L\otimes_{O_K}k}(\mathcal{X}_L,\mathbb{Q}_\ell(r))\simeq\mathrm{H}^{2r-2}(\mathcal{X}_L\otimes_{O_K}k,\mathbb{Q}_\ell(r-1))$$

by the absolute purity theorem [Fuj02], since $\mathcal{X}_L$ is smooth over O_K. By the Hochschild–Serre spectral sequence and the Weil conjecture, the natural maps

$$\mathrm{H}^{2r}(X'_L,\mathbb{Q}_\ell(r))\to\mathrm{H}^{2r}(X'_L\otimes_K\overline{\mathbb{Q}}_p,\mathbb{Q}_\ell(r))$$
$$\mathrm{H}^{2r-2}(\mathcal{X}_L\otimes_{O_K}k,\mathbb{Q}_\ell(r-1))\to\mathrm{H}^{2r-2}(\mathcal{X}_L\otimes_{O_K}\overline{\mathbb{F}}_p,\mathbb{Q}_\ell(r-1))$$

are both injective.

By definition, every element in $(\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$ annihilates $\mathrm{H}^{2r}(X'_L\otimes_K\overline{\mathbb{Q}}_p,\mathbb{Q}_\ell(r))\otimes_{\mathbb{Q}}\mathbb{Q}^{\mathrm{ac}}$. By the Poincaré duality and the smooth proper base change theorem, every element in $(\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$ also annihilates $\mathrm{H}^{2r-2}(\mathcal{X}_L\otimes_{O_K}\overline{\mathbb{F}}_p,\mathbb{Q}_\ell(r-1))\otimes_{\mathbb{Q}}\mathbb{Q}^{\mathrm{ac}}$. In particular, s annihilates $\mathrm{H}^{2r}(\mathcal{X}_L,\mathbb{Q}_\ell(r))\otimes_{\mathbb{Q}}\mathbb{Q}^{\mathrm{ac}}$ as each factor annihilates a graded piece in the two-step filtration of $\mathrm{H}^{2r}(\mathcal{X}_L,\mathbb{Q}_\ell(r))$ given by (8.1). The lemma is proved. □

We first recall the uniformization of $\{\mathcal{X}_{\tilde{L}}\}$ along the supersingular locus from [LZ, §13.1]. Fix a complete maximal unramified extension $\breve{K}$ of K. Recall that we have fixed a u-nearby space ${}^uV$ and an isomorphism ${}^uV \otimes_F \mathbb{A}_F^{u} \simeq V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{u}$ from Notation 2.2(H9). We have a compatible system of isomorphisms

$$\mathcal{X}_{\tilde{L}}^{\wedge} \simeq \left({}^uH(F)\backslash \mathcal{N} \times H(\mathbb{A}_F^{\infty,u})/\tilde{L}^{u}\right) \times_{\operatorname{Spf} O_K} \mathcal{Y}^{\wedge} \tag{8.2}$$

of formal schemes over $O_{\breve{K}}$ for every $\tilde{L} \subseteq L$ considered as before. Here, $\mathcal{X}_{\tilde{L}}^{\wedge}$ denotes the completion of $\mathcal{X}_{\tilde{L}} \otimes_{O_K} O_{\breve{K}}$ along its supersingular locus, $\mathcal{N}$ is the relative unitary Rapoport–Zink space over $\operatorname{Spf} O_{\breve{K}}$ as considered in [LZ, §2.1], and $\mathcal{Y}^{\wedge}$ denotes the completion of $\mathcal{Y}$ along its special fiber.

We then recall the notion of integral special cycles. Take an integer $m \geqslant 1$ and an element $\phi^{\infty} \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty})^L$ that is p-basic (Definition 6.5).

For every element $T \in \operatorname{Herm}_m^{\circ}(F)^+$, there are following constructions.

- We have a cycle $Z_T(\phi^{\infty})_L \in \mathrm{Z}^m(X_L)_{\mathbb{C}}$ (Definition 4.1). When ϕ^{∞} is the characteristic function of some open compact subset of $V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty}$, we have a morphism
$$Z'_T(\phi^{\infty})_L \to X'_L$$
defined as the disjoint union of finite and unramified morphisms $Z'(x)_L \to X'_L$ (Definition 5.3) for $x \in L \backslash \operatorname{supp}(\phi^{\infty})$, whose induced cycle coincides with the restriction of $Z_T(\phi^{\infty})_L$ to X'_L (Lemma 5.4). By moduli interpretation, the morphism $Z'_T(\phi^{\infty})_L \to X'_L$ extends naturally to a finite and unramified morphism
$$\mathcal{Z}_T(\phi^{\infty})_L \to \mathcal{X}_L$$
([LZ, §13.3]).
- For $\phi^{\infty} = \phi_1^{\infty} \otimes \cdots \otimes \phi_r^{\infty}$, where each $\phi_j^{\infty} \in \mathscr{S}(V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty})^L$ is p-basic and is the characteristic function of some open compact subset of $V \otimes_{\mathbb{A}_F} \mathbb{A}_F^{\infty}$, we denote by ${}^{\mathrm{K}}\mathcal{Z}_T(\phi^{\infty})_L$ the component of
$$\mathscr{O}_{\mathcal{Z}_{t_1}(\phi_1^{\infty})_L} \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{X}_L}} \cdots \overset{\mathbb{L}}{\otimes}_{\mathscr{O}_{\mathcal{X}_L}} \mathscr{O}_{\mathcal{Z}_{t_r}(\phi_r^{\infty})_L}$$
supported on $\mathcal{Z}_T(\phi^{\infty})_L$,[17] regarded as an element in $\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}_L)_{\mathbb{C}}$ (see Appendix B for the notion of the K-group), where $(t_1, \ldots, t_r)$ is the diagonal of T and $\mathcal{Z}$ denotes the image of $\mathcal{Z}_T(\phi^{\infty})_L$ in $\mathcal{X}_L$. In general, we may always write ϕ^{∞} as a finite complex linear combination (possibly after shrinking L away from $\mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}}$) of those as above, and we define ${}^{\mathrm{K}}\mathcal{Z}_T(\phi^{\infty})_L$ by linearity. By [GS87, Prop. 5.5], ${}^{\mathrm{K}}\mathcal{Z}_T(\phi^{\infty})_L$ belongs to $\mathrm{F}^r\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}_L)_{\mathbb{C}}$, and hence is an extension of $Z_T(\phi^{\infty})'_L$ (Definition B.9).

[17] Here, we note that $\mathcal{Z}_T(\phi^{\infty})_L$ is an open and closed subscheme of $\mathcal{Z}_{t_1}(\phi^{\infty})_L \times_{\mathcal{X}_L} \cdots \times_{\mathcal{X}_L} \mathcal{Z}_{t_r}(\phi^{\infty})_L$.

- We denote by $\mathcal{Z}_T(\phi^\infty)^\wedge_L$ the restriction of $\mathcal{Z}_T(\phi^\infty)_L$ to $\mathcal{X}^\wedge_L$. Then we have the following description:

$$\begin{aligned}\mathcal{Z}_T(\phi^\infty)^\wedge_L = &\sum_{\substack{x\in {}^uH(F)\backslash {}^uV^m\\ T(x)=T}} \sum_{h\in H^x(F)\backslash H(\mathbb{A}_F^{\infty,\underline{u}})/L^{\underline{u}}} \phi^{\infty,\underline{u}}(h^{-1}x)\\ &\cdot (\mathcal{N}(x),h)_L \times_{\operatorname{Spf} O_K} \mathcal{Y}^\wedge,\end{aligned} \tag{8.3}$$

where $\mathcal{N}(x)$ is the special cycle of $\mathcal{N}$ indexed by x ([KR11, Def. 3.2] or [LZ, §2.3]), $(\mathcal{N}(x),h)_L$ denotes the corresponding double coset in the expression (8.2), and H^x is the subgroup of ${}^uH$ of elements that fix every component of x.

In what follows, for $x=(x_1,\dots,x_m)\in {}^uV^m$ with $T(x)\in \mathrm{Herm}^\circ_m(F_{\underline{u}})$, we put

$${}^{\mathrm{K}}\mathcal{N}(x) \coloneqq [\mathcal{N}(x_1)]\cup\cdots\cup[\mathcal{N}(x_m)]$$

as an element in $\mathrm{K}_0^{\mathcal{N}(x)}(\mathcal{N})$. See [Zha21, App. B] for the analogue of Gillet–Soulé K-groups for formal schemes; we denote similarly by [ ] the associated element in the K-group.

Proof of Proposition 8.1. By Lemma B.3 and Definition 3.11, it suffices to show that for every pair of p-basic elements $\phi_1^\infty,\phi_2^\infty\in\mathscr{S}(V^m\otimes_{\mathbb{A}_F}\mathbb{A}_F^\infty)^L$ satisfying that $\mathrm{supp}(\phi_{1v}^\infty\otimes(\phi_{2v}^\infty)^{\mathtt{c}})\subseteq (V_v^{2r})_{\mathrm{reg}}$ for $v\in\mathtt{R}'$, and every pair of elements $\mathrm{s}_1,\mathrm{s}_2$ each of which is a product of two elements in $(\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}})^{\langle\ell\rangle}_{L_{\mathtt{R}}}$, we have

$$\begin{aligned}&\frac{\mathrm{vol}^\natural(L)}{\deg(Y/K)}\Phi^0_\infty(T_1,T_2)\langle \mathrm{s}_1^*Z_{T_1}(\phi_1^\infty)'_L,\mathrm{s}_2^*Z_{T_2}(\phi_2^\infty)'_L\rangle_{X'_L,K}\\ &\quad=\sum_{\substack{T^\square\in\mathrm{Herm}^\circ_{2r}(F)^+\\ \mathrm{Diff}(T^\square,V)=\{\underline{u}\}\\ \partial_{r,r}T^\square=(T_1,T_2)}}\frac{1}{\log q_u}W'_{T^\square}(0,1_{4r},\mathbb{1}_{(\Lambda^{\mathtt{R}}_{\underline{u}})^{2r}})\\ &\qquad\cdot\prod_{v\neq\underline{u}}W_{T^\square}(0,1_{4r},(\Phi^0_\infty\otimes(\mathrm{s}_1\phi_1^\infty\otimes(\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_v).\end{aligned} \tag{8.4}$$

Now using $\mathrm{vol}^\natural(L_{\underline{u}})=1$ and $\prod_{v\in\mathtt{V}_F\setminus\{\underline{u}\}}\gamma^{2r}_{V_v,\psi_{F,v}}=1$, (8.4) is equivalent to

$$\begin{aligned}&\frac{\mathrm{vol}(H(F_\infty)L^{\underline{u}})}{\deg(Y/K)}\Phi^0_\infty(T_1,T_2)\langle \mathrm{s}_1^*Z_{T_1}(\phi_1^\infty)'_L,\mathrm{s}_2^*Z_{T_2}(\phi_2^\infty)'_L\rangle_{X'_L,K}\\ &\quad=\sum_{\substack{T^\square\in\mathrm{Herm}^\circ_{2r}(F)^+\\ \mathrm{Diff}(T^\square,V)=\{\underline{u}\}\\ \partial_{r,r}T^\square=(T_1,T_2)}}\frac{b_{2r,\underline{u}}(0)}{\log q_u}W'_{T^\square}(0,1_{4r},\mathbb{1}_{(\Lambda^{\mathtt{R}}_{\underline{u}})^{2r}})\\ &\qquad\cdot\prod_{v\neq\underline{u}}\frac{b_{2r,v}(0)}{\gamma^{2r}_{V_v,\psi_{F,v}}}W_{T^\square}(0,1_{4r},(\Phi^0_\infty\otimes(\mathrm{s}_1\phi_1^\infty\otimes(\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_v).\end{aligned} \tag{8.5}$$

By Proposition B.13 and Lemma 8.3, we have

$$\begin{aligned}\langle \mathrm{s}_1^* Z_{T_1}(\phi_1^\infty)'_L, \mathrm{s}_2^* Z_{T_2}(\phi_2^\infty)'_L\rangle_{X'_L,K} &= \left(\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L\right).\left(\mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}(\phi_2^\infty)_L\right)\\ &= \chi\left(\pi_*\left(\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L \cup \mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}((\phi_2^\infty)^{\mathtt{c}})_L\right)\right),\end{aligned}$$

where $\pi\colon \mathcal{X}_L \to \operatorname{Spec} O_K$ denotes the structure morphism. As $\operatorname{supp}(\phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for $v \in \mathtt{R}'$, the support of $\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L \cup \mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}((\phi_2^\infty)^{\mathtt{c}})_L$ is contained in the supersingular locus of $\mathcal{X}_L$. Moreover, since s_1^* and s_2^* preserve the supersingular locus, we have

$$\begin{aligned}&\chi\left(\pi_*\left(\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L \cup \mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}((\phi_2^\infty)^{\mathtt{c}})_L\right)\right)\\ &\qquad = \chi\left(\pi_*^\wedge\left(\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L^\wedge \cup \mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}((\phi_2^\infty)^{\mathtt{c}})_L^\wedge\right)\right),\end{aligned}$$

where $\pi^\wedge\colon \mathcal{X}_L^\wedge \to \operatorname{Spf} O_{\breve{K}}$ denotes the structure morphism. To summarize, the left-hand side of (8.5) equals

(8.6)
$$\frac{\mathrm{vol}(H(F_\infty)L^{\underline{u}})}{\deg(Y/K)}\Phi_\infty^0(T_1,T_2)\cdot\chi\left(\pi_*^\wedge\left(\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L^\wedge \cup \mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}((\phi_2^\infty)^{\mathtt{c}})_L^\wedge\right)\right).$$

From (8.3), it is straightforward to see that

$$\begin{aligned}&\mathrm{s}_i^{*\,\mathrm{K}}\mathcal{Z}_{T_i}(\phi_i^\infty)_L^\wedge\\ &= \sum_{\substack{x_i\in {}^uH(F)\backslash {}^uV^r\\ T(x_i)=T_i}}\ \sum_{h_i\in H^{x_i}(F)\backslash H(\mathbb{A}_F^{\infty,\underline{u}})/L^{\underline{u}}} (\mathrm{s}_i\phi_i^{\infty,\underline{u}})(h_i^{-1}x_i)\cdot({}^{\mathrm{K}}\mathcal{N}(x_i),h_i)_L\times_{\operatorname{Spf} O_K}\mathcal{Y}^\wedge\end{aligned}$$

for $i=1,2$. It follows that

$$\begin{aligned}\mathrm{s}_1^{*\,\mathrm{K}}\mathcal{Z}_{T_1}(\phi_1^\infty)_L^\wedge \cup \mathrm{s}_2^{*\,\mathrm{K}}\mathcal{Z}_{T_2}((\phi_2^\infty)^{\mathtt{c}})_L^\wedge &= \sum_{\substack{T^\square\in \mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\ \sum_{\substack{x\in {}^uH(F)\backslash {}^uV^{2r}\\ T(x)=T^\square}}\\ \cdot \sum_{h\in H(\mathbb{A}_F^{\infty,\underline{u}})/L^{\underline{u}}} &(\mathrm{s}_1\phi_1^{\infty,\underline{u}}\otimes(\mathrm{s}_2\phi_2^{\infty,\underline{u}})^{\mathtt{c}})(h^{-1}x)\cdot({}^{\mathrm{K}}\mathcal{N}(x),h)_L\times_{\operatorname{Spf} O_K}\mathcal{Y}^\wedge.\end{aligned}$$

Now by [LZ, Th. 3.4.1, Rem. 3.4.2], we have

$$\chi\left(\pi_*^{\wedge\,\mathrm{K}}\mathcal{N}(x)\right) = \frac{b_{2r,\underline{u}}(0)}{\log q_u}W'_{T^\square}(0,1_{4r},\mathbb{1}_{(\Lambda_{\underline{u}}^{\mathtt{R}})^{2r}})$$

if $T(x)=T^\square$. Thus, we have

$$\begin{aligned}(8.6) = \mathrm{vol}(H(F_\infty)L^{\underline{u}})\cdot\Phi_\infty^0(T_1,T_2)\cdot &\sum_{\substack{T^\square\in \mathrm{Herm}_{2r}^\circ(F)^+\\ \partial_{r,r}T^\square=(T_1,T_2)}}\ \sum_{\substack{x\in {}^uH(F)\backslash {}^uV^{2r}\\ T(x)=T^\square}}\ \sum_{h\in H(\mathbb{A}_F^{\infty,\underline{u}})/L^{\underline{u}}}\\ &(\mathrm{s}_1\phi_1^{\infty,\underline{u}}\otimes(\mathrm{s}_2\phi_2^{\infty,\underline{u}})^{\mathtt{c}})(h^{-1}x)\cdot\left(\frac{b_{2r,\underline{u}}(0)}{\log q_u}W'_{T^\square}(0,1_{4r},\mathbb{1}_{(\Lambda_{\underline{u}}^{\mathtt{R}})^{2r}})\right).\end{aligned}$$

By Definition 3.8, we have

$$\mathrm{vol}(H(F_v))\cdot\Phi_v^0(T_1,T_2)=\frac{b_{2r,v}(0)}{\gamma_{V_v,\psi_{F,v}}^{2r}}W_{T^\square}(0,1_{4r},(\Phi_\infty^0\otimes(\mathrm{s}_1\phi_1^\infty\otimes(\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_v)$$

for $v\in\mathtt{V}_F^{(\infty)}$. By Definition 3.8, for (unique) $x\in {}^uH(F)\backslash {}^uV^{2r}$ with $T(x)=T^\square$, we have

$$\begin{aligned}&\mathrm{vol}(L_v)\sum_{h_v\in H(F_v)/L_v}(\mathrm{s}_1\phi_1^{\infty,\underline{u}}\otimes(\mathrm{s}_2\phi_2^{\infty,\underline{u}})^{\mathtt{c}})_v(h_v^{-1}x)\\&=\frac{b_{2r,v}(0)}{\gamma_{V_v,\psi_{F,v}}^{2r}}W_{T^\square}(0,1_{4r},(\Phi_\infty^0\otimes(\mathrm{s}_1\phi_1^\infty\otimes(\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_v)\end{aligned}$$

for $v\in\mathtt{V}_F^{\mathrm{fin}}\setminus\{\underline{u}\}$.

Therefore, we obtain (8.5) and hence (8.4). The proposition is proved. □

9. Local indices at inert places: almost unramified case

In this section, we compute local indices at places in $\mathtt{V}_E^{\mathrm{int}}$ above $\mathtt{S}$. Our goal is to prove the following proposition.

PROPOSITION 9.1. *Let* $\mathtt{R}$, $\mathtt{R}'$, ℓ, *and* L *be as in Definition 6.3. Let* $(\pi,\mathcal{V}_\pi)$ *be as in Assumption 3.1, for which we assume Hypothesis 6.6. Take an element* $u\in\mathtt{V}_E^{\mathrm{int}}$ *such that* $\underline{u}\in\mathtt{S}$ *and whose underlying rational prime* p *is odd, unramified in* E, *and satisfies* $\mathtt{V}_F^{(p)}\cap\mathtt{R}\subseteq\mathtt{V}_F^{\mathrm{spl}}$. *Recall that we have fixed a* u*-nearby space* ${}^uV$ *and an isomorphism* ${}^uV\otimes_F\mathbb{A}_F^{\underline{u}}\simeq V\otimes_{\mathbb{A}_F}\mathbb{A}_F^{\underline{u}}$ *from Notation 2.2*$(H9)$. *We also fix a* $\psi_{E,\underline{u}}$*-self-dual lattice* $\Lambda_{\underline{u}}^\star$ *of* ${}^uV_{\underline{u}}$. *Then there exist elements* $\mathrm{s}_1^u,\mathrm{s}_2^u\in\mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}}\setminus\mathfrak{m}_\pi^{\mathtt{R}}$ *such that*

$$\begin{aligned}&\log q_u\cdot\mathrm{vol}^\natural(L)\cdot I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1^u\mathrm{s}_1,\mathrm{s}_2^u\mathrm{s}_2,g_1,g_2)_{L,u}^\ell\\&\quad=\mathfrak{E}_{T_1,T_2}((g_1,g_2),\Phi_\infty^0\otimes(\mathrm{s}_1^u\mathrm{s}_1\phi_1^\infty\otimes(\mathrm{s}_2^u\mathrm{s}_2\phi_2^\infty)^{\mathtt{c}}))_u\\&\qquad-\frac{\log q_u}{q_u^r-1}E_{T_1,T_2}((g_1,g_2),\Phi_\infty^0\otimes(\mathrm{s}_1^u\mathrm{s}_1\phi_1^{\infty,\underline{u}}\otimes(\mathrm{s}_2^u\mathrm{s}_2\phi_2^{\infty,\underline{u}})^{\mathtt{c}})\otimes\mathbb{1}_{(\Lambda_{\underline{u}}^\star)^{2r}})\end{aligned}$$

for every $(\mathtt{R},\mathtt{R}',\ell,L)$*-admissible sextuple* $(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1,\mathrm{s}_2,g_1,g_2)$ *and every pair* (T_1,T_2) *in* $\mathrm{Herm}_r^\circ(F)^+$, *where the right-hand side is defined in Definition 3.11 with the Gaussian function* $\Phi_\infty^0\in\mathscr{S}(V^{2r}\otimes_{\mathbb{A}_F}F_\infty)$ *(Notation 2.2*(H3)*), and* $\mathrm{vol}^\natural(L)$ *is defined in Definition 3.8.*

To prove Proposition 9.1, we may rescale the hermitian form on V, hence assume that $\psi_{F,v}$ is unramified and that $\Lambda_v^{\mathtt{R}}$ is either a self-dual or an almost self-dual lattice of V_v for every $v\in\mathtt{V}_F^{(p)}\backslash\mathtt{V}_F^{\mathrm{spl}}$, and moreover that $\Lambda_{\underline{u}}^\star$ is a self-dual lattice of ${}^uV_{\underline{u}}$.

In order to deal with spherical Hecke operators, we consider the projective system of Shimura varieties $\{X_{\tilde{L}}\}$ indexed by open compact subgroups $\tilde{L} \subseteq L$ satisfying $\tilde{L}_v = L_v$ for $v \in \mathtt{V}_F^{(p)} \setminus \mathtt{V}_F^{\mathrm{spl}}$.

We invoke Notation 5.1 together with Notation 5.6, which is possible since $\mathtt{V}_F^{(p)} \cap \mathtt{V}_F^{\mathrm{ram}} = \emptyset$. There is a projective system $\{\mathcal{X}_{\tilde{L}}\}$ of strictly semistable projective schemes over O_K (see [LZ, §11.3])[18] with

$$\mathcal{X}_{\tilde{L}} \otimes_{O_K} K = X'_{\tilde{L}} \otimes_{E'} K = \left(X_{\tilde{L}} \otimes_E Y\right) \otimes_{E'} K$$

and finite étale transition morphisms. In particular, $\mathbb{S}^{\mathtt{R}}$ is naturally a ring of étale correspondences of $\mathcal{X}_L$.

LEMMA 9.2. *Let the situation be as in Proposition 9.1. Then there exists an element in $\mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}} \setminus \mathfrak{m}^{\mathtt{R}}_{\pi}$ that gives an ℓ-tempered $\mathbb{Q}^{\mathrm{ac}}$-étale correspondence of $\mathcal{X}_L$ (Definition B.12).*

Proof. The proof relies on Arthur's multiplicity formula for tempered global L-packets [KMSW14, Th. 1.7.1], which we first recall, using the language for unitary groups adopted in [GGP12, §25]. Recall that $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_s$ from Notation 3.14(3), which is the automorphic base change of π as in Assumption 3.1. Put $A_\Pi := \mu_2^{\{1,\dots,s\}}$. For every place $v \in \mathtt{V}_F$,

- Π_v determines a conjugate-symplectic representation M_v of $\mathrm{WD}(E_v)$ of dimension n;
- there is a finite abelian 2-group A_{M_v} attached to M_v;
- every character $\chi_v \colon A_{M_v} \to \mathbb{C}^\times$ gives a pair $(V^{\chi_v}, \pi^{\chi_v})$ in the Langlands–Vogan packet of M_v, unique up to isomorphism, in which V^{χ_v} is a hermitian space over F_v of rank n and π^{χ_v} is an irreducible admissible representation of $\mathrm{U}(V^{\chi_v})(F_v)$;
- we have a homomorphism $\alpha_v \colon A_\Pi \to A_{M_v}$.

Denote by $\alpha \colon A_\Pi \to \prod_{v \in \mathtt{V}_F} A_{M_v}$ the product of α_v for $v \in \mathtt{V}_F$. We say that a collection $\chi = \{\chi_v \mid v \in \mathtt{V}_F\}$ of characters in which all but finitely many are trivial is coherent (resp. incoherent) if the character $\prod_{v \in \mathtt{V}_F} \chi_v \circ \alpha \colon A_\Pi \to \mathbb{C}^\times$ is trivial (resp. nontrivial). Then Arthur's multiplicity formula states that

(a) If χ is incoherent, then either $\otimes_v V^{\chi_v}$ is incoherent or it is coherent but $\otimes_v \pi^{\chi_v}$ does not appear in the discrete spectrum. If χ is coherent, then there exists a hermitian space V^χ over E, unique up to isomorphism, such that $V_v^\chi \simeq V^{\chi_v}$ for every $v \in \mathtt{V}_F$; the representation $\otimes_v \pi^{\chi_v}$ appears in the discrete spectrum of $\mathrm{U}(V^\chi)$ with multiplicity one. Moreover, every discrete

[18] This is the place where we need the assumption that u is unramified over $\mathbb{Q}$ (and that K is unramified over E_u).

automorphic representation of $\mathrm{U}(\tilde{V})(\mathbb{A}_F)$ for some hermitian space $\tilde{V}$ over E of rank n with Π its automorphic base change is obtained from this way.

Now we take a special look at the places w and $\underline{u}$.

(b) We may canonically identify A_{M_w} with $\mu_2^{\mathfrak{I}}$ from Notation 3.14(2). Then the homomorphism $\alpha_w\colon \mu_2^{\{1,\dots,s\}} \to \mu_2^{\mathfrak{I}}$ is the one induced by the map $\mathfrak{I} \to \{1,\dots,s\}$ given by the partition $\mathfrak{I} = \mathfrak{I}_1 \sqcup \cdots \sqcup \mathfrak{I}_s$.

(c) By (a), Π_u is the standard base change of $\pi_{\underline{u}}$. By [LTX+19, Lemma C.2.3], we have $M_{\underline{u}} = M_{\underline{u}}^2 + M_{\underline{u}}^{n-2}$, where $M_{\underline{u}}^2$ corresponds to the Steinberg representation of $\mathrm{GL}_2(E_u)$ and $M_{\underline{u}}^{n-2}$ corresponds to a tempered unramified principal series of $\mathrm{GL}_{n-2}(E_u)$, which implies $A_{M_{\underline{u}}} = A_{M_{\underline{u}}^2} \times A_{M_{\underline{u}}^{n-2}}$ in which $A_{M_{\underline{u}}^2} = \mu_2$.

(d) Without lost of generality, we may assume that Π_{1u} is ramified. Then the composition of $\alpha_{\underline{u}}$ with the projection $A_{M_{\underline{u}}} \to A_{M_{\underline{u}}^2} = \mu_2$ coincides with the projection $\mu_2^{\{1,\dots,s\}} \to \mu_2$ to the first factor.

Next, we recall some facts from [LTX+19, §§5.1, 5.2, 5.4, 5.5][19] about the reduction of the scheme $\mathcal{X}_{\tilde{L}}$. Denote by $L_{\underline{u}}^\star \subseteq {}^uH(F_{\underline{u}})$ the stabilizer of $\Lambda_{\underline{u}}^\star$, which is a hyperspecial maximal subgroup. We have a decomposition

$$\mathcal{X}_{\tilde{L}} \otimes_{O_K} \overline{\mathbb{F}}_p = Y_{\tilde{L}}^\circ \bigcup Y_{\tilde{L}}^\bullet$$

that is compatible with changing $\tilde{L}$, in which

- $Y_{\tilde{L}}^\circ$ is a $\mathbb{P}^{2r-1}$-fibration over

$$ {}^uH(F)\backslash {}^uH(\mathbb{A}_F^\infty)/\tilde{L}^{\underline{u}} L_{\underline{u}}^\star \times (\mathcal{Y} \otimes_{O_K} \overline{\mathbb{F}}_p);$$

- $Y_{\tilde{L}}^\bullet$ is proper and smooth over $\overline{\mathbb{F}}_p$ of dimension $2r-1$;
- the intersection $Y_{\tilde{L}}^\dagger := Y_{\tilde{L}}^\circ \cap Y_{\tilde{L}}^\bullet$ is a relative Fermat hypersurface in $Y_{\tilde{L}}^\circ$.

By Corollary B.15 and Proposition 6.9(2), it suffices to show that for an arbitrary embedding $\mathbb{Q}^{\mathrm{ac}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have

(1) $\mathrm{H}^i(Y_L^\circ, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} = 0$ for $i \leqslant 2r-2$,
(2) $\mathrm{H}^i(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} = 0$ for $i \leqslant 2r-2$,
(3) $\mathrm{H}^i(Y_L^\dagger, \overline{\mathbb{Q}}_\ell)_{\mathfrak{m}} = 0$ for $i \leqslant 2r-3$,

where $\mathfrak{m} := \mathfrak{m}_\pi^{\mathtt{R}} \cap \mathbb{S}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}}$. Note that we have used the Gysin exact sequence and the absolute purity theorem [Fuj02] to switch the cohomology from open strata to closed strata.

[19] Strictly speaking, [LTX+19] has more conditions on the place $\underline{u}$ and the level at p. However, for those facts we will use in this proof, it is straightforward to remove those extra conditions.

$q \geqslant 2r+1$	$0 \longrightarrow \mathrm{H}^q(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \longrightarrow 0$		
$q = 2r$	$\mathrm{H}^{2r-2}(Y_L^\dagger, \overline{\mathbb{Q}}_\ell(-1))_\mathfrak{m} \xrightarrow{\mathrm{d}_{1,\mathfrak{m}}^{-1,2r}} \mathrm{H}^{2r}(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \longrightarrow 0$		
$q = 2r-1$	$0 \longrightarrow \mathrm{H}^{2r-1}(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \longrightarrow 0$		
$q = 2r-2$	$0 \longrightarrow \mathrm{H}^{2r-2}(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \xrightarrow{\mathrm{d}_{1,\mathfrak{m}}^{0,2r-2}} \mathrm{H}^{2r-2}(Y_L^\dagger, \overline{\mathbb{Q}}_\ell)_\mathfrak{m}$		
$q \leqslant 2r-3$	$0 \longrightarrow \mathrm{H}^q(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \longrightarrow 0$		
$\mathrm{E}_1^{p,q}$	$p=-1$	$p=0$	$p=1$

Table 1.

For (1), we have $\mathrm{H}^i(Y_L^\circ, \overline{\mathbb{Q}}_\ell) = 0$ when i is odd. When i is even, $\mathrm{H}^i(Y_L^\circ, \overline{\mathbb{Q}}_\ell)_\mathfrak{m}$ is a direct sum of $L^{\underline{u}} \times L_{\underline{u}}^\star$-invariants of π' for finitely many cuspidal automorphic representation π' of ${}^{\underline{u}}H(\mathbb{A}_F)$ satisfying that π'_∞ is trivial and that $\pi'_v \simeq \pi_v$ for all but finitely many $v \in \mathtt{V}_F^{\mathrm{spl}}$. For every such π', let Π' be its automorphic base change, which is isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ [KMSW14, Th. 1.7.1]. Since $\Pi'_u \simeq \Pi_u$ for all but finitely many $u \in \mathtt{V}_E^{\mathrm{spl}}$, we must have $\Pi' \simeq \Pi$ by [Ram18, Th. A]. Therefore, we have

$$\mathrm{H}^i(Y_L^\circ, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} \simeq \bigoplus_{\substack{\chi=\{\chi^v\} \\ \chi\circ\alpha=1 \\ V^\chi \simeq {}^{\underline{u}}V}} \left(\left(\otimes_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}} \pi^{\chi_v} \right)^{L^{\underline{u}}} \otimes (\pi^{\chi_{\underline{u}}})^{L_{\underline{u}}^\star} \right)^{\oplus \deg(Y/K)}.$$

However, since $\Pi_{\underline{u}}$ is ramified, we have $(\pi^{\chi_{\underline{u}}})^{L_{\underline{u}}^\star} = 0$ for every $\chi_{\underline{u}}$. Thus, (1) follows.

For (3), by the Lefschetz hyperplane theorem and the Poincaré duality, we have $\mathrm{H}^i(Y_L^\dagger, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$ if $i \neq 2r-2$ by (1). Thus, (3) follows.

For (2), we consider the weight spectral sequence $\mathrm{E}^{p,q}$ abutting to the cohomology $\mathrm{H}^{p+q}(\mathcal{X}_{\tilde{L}} \otimes_{O_K} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell)$, after localization at $\mathfrak{m}$. We write down the first page $\mathrm{E}_{1,\mathfrak{m}}^{p,q}$ as in Table 1. By Proposition 6.9(1), we have $\mathrm{H}^i(\mathcal{X}_{\tilde{L}} \otimes_{O_K} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$ for $i \neq 2r-1$, which implies that $\mathrm{H}^q(Y_L^\bullet, \overline{\mathbb{Q}}_\ell)_\mathfrak{m} = 0$ for $q \leqslant 2r-3$ and that $\mathrm{d}_{1,\mathfrak{m}}^{0,2r-2}$ is injective. The spectral sequence then degenerates at the second page and we have $\mathrm{im}(\mathrm{d}_{1,\mathfrak{m}}^{0,2r-2}) = \ker(\mathrm{E}_{1,\mathfrak{m}}^{1,2r-2} \to \mathrm{E}_{\infty,\mathfrak{m}}^{1,2r-2})$. Thus, it

remains to show that the canonical quotient map $\mathrm{E}_{1,\mathfrak{m}}^{1,2r-2} \to \mathrm{E}_{\infty,\mathfrak{m}}^{1,2r-2}$ is an isomorphism. Consider an arbitrary collection $\chi^{\infty,\underline{u}} = \{\chi^v \mid v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}\}$ in which all but finitely many are trivial and such that $V^{\chi_v} \simeq V_v$ for every $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}$. Put $\pi^{\chi^{\infty,\underline{u}}} := \otimes_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}} \pi^{\chi_v}$. By a similar argument for (1), it suffices to show the following statement:

(4) The canonical quotient map $\mathrm{E}_1^{1,2r-2}[\pi^{\chi^{\infty,\underline{u}}}] \to \mathrm{E}_\infty^{1,2r-2}[\pi^{\chi^{\infty,\underline{u}}}]$ is an isomorphism.

Now we show (4). Without lost of generality, we may replace K by a finite unramified extension in $\overline{\mathbb{Q}}_p$ such that Y is a finite disjoint union of $\operatorname{Spec} K$. Define the character $\chi_{\underline{u}}^+$ (resp. $\chi_{\underline{u}}^-$) to be the inflation of the trivial (resp. nontrivial) character of $A_{M_{\underline{u}}^2} = \mu_2$ along the quotient homomorphism $A_{M_{\underline{u}}} \to A_{M_{\underline{u}}^2}$. Then we have $V^{\chi_{\underline{u}}^+} \simeq {}^uV_{\underline{u}}$ and $V^{\chi_{\underline{u}}^-} \simeq {}^{\mathbf{u}}V_{\underline{u}}$. If $\mathrm{E}_1^{1,2r-2}[\pi^{\chi^{\infty,\underline{u}}}] = 0$, then we are done. Otherwise, we have $\mathrm{H}^{2r-2}(Y_L^\dagger, \overline{\mathbb{Q}}_\ell)[\pi^{\chi^{\infty,\underline{u}}}] \neq 0$. By (the proof of) [LTX+19, Prop. 5.5.4], $\pi^{\chi^{\infty,\underline{u}}}$ can be complemented to a cuspidal automorphic representation π' of ${}^uH(\mathbb{A}_F)$ such that π'_∞ is trivial and that π'_u is almost unramified with respect to the hyperspecial subgroup $L_{\underline{u}}^\star$. In other words, the collection $\{\chi_v = 1 \mid v \in \mathtt{V}_F^{(\infty)}\} \cup \chi^{\infty,\underline{u}} \cup \{\chi_{\underline{u}}^+\}$ is coherent, and we have

(9.1)
$$\mathrm{E}_1^{1,2r-2}[\pi^{\chi^{\infty,\underline{u}}}] = \mathrm{H}^{2r-2}(Y_L^\dagger, \overline{\mathbb{Q}}_\ell)[\pi^{\chi^{\infty,\underline{u}}}] \simeq \left(\left(\otimes_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}} \pi^{\chi_v}\right)^{L^{\underline{u}}}\right)^{\oplus \deg(Y/K)}.$$

On the other hand, since the representation $\pi^{\chi_{\underline{u}}^-}$ is the only member in the Langlands–Vogan packet of $M_{\underline{u}}$ realized on ${}^{\mathbf{u}}V_{\underline{u}}$ that has nonzero invariants under $L_{\underline{u}}^{\mathtt{R}}$, we have an isomorphism

$$\mathrm{H}^{2r-1}(\mathcal{X}_{\tilde{L}} \otimes_{O_K} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell)[\pi^{\chi^{\infty,\underline{u}}}] \simeq \left(\rho[\tilde{\pi}^\infty]|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)}\right) \otimes \left(\left(\otimes_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}} \pi^{\chi_v}\right)^{L^{\underline{u}}}\right)^{\oplus \deg(Y/K)}$$

of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$, where $\tilde{\pi}^\infty = \pi^{\chi^{\infty,\underline{u}}} \otimes \pi^{\chi_{\underline{u}}^-}$. By (a)–(d) above, it is easy to see that, in the notation of Lemma 3.15(b), we must have $j(\tilde{\pi}^\infty) = 1$, hence that the semisimplification of $\rho[\tilde{\pi}^\infty]$ is isomorphic to $\rho_{\Pi_1}^{\mathtt{c}}$ by Hypothesis 6.6. Thus, $\rho[\tilde{\pi}^\infty]|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)}$ has nontrivial monodromy, which implies that the dimension of $\mathrm{E}_\infty^{1,2r-2}[\pi^{\chi^{\infty,\underline{u}}}]$ is at least the dimension of

$$\left(\left(\otimes_{v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \{\underline{u}\}} \pi^{\chi_v}\right)^{L^{\underline{u}}}\right)^{\oplus \deg(Y/K)}.$$

Therefore, (4) follows from (9.1). The lemma is proved. □

Proof of Proposition 9.1. The proof of Proposition 9.1 is parallel to that of Proposition 8.1. Take elements $\mathrm{s}_1^u, \mathrm{s}_2^u \in \mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}} \setminus \mathfrak{m}_\pi^{\mathtt{R}}$ that give ℓ-tempered $\mathbb{Q}^{\mathrm{ac}}$-étale correspondences of $\mathcal{X}_L$, which is possible by Lemma 9.2.

By Lemma B.3 and Definition 3.11, it suffices to show that for every pair of p-basic elements $\phi_1^\infty, \phi_2^\infty \in \mathscr{S}(V^m \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)^L$ satisfying that $\mathrm{supp}(\phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for $v \in \mathtt{R}'$, and every pair of elements $\mathrm{s}_1, \mathrm{s}_2 \in (\mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}})_{L_{\mathtt{R}}}^{\langle \ell \rangle}$ that give ℓ-tempered $\mathbb{Q}^{\mathrm{ac}}$-étale correspondences of $\mathcal{X}_L$, we have

$$\begin{aligned}(9.2)\quad &\frac{\mathrm{vol}^\natural(L)}{\deg(Y/K)} \Phi_\infty^0(T_1,T_2) \langle \mathrm{s}_1^* Z_{T_1}(\phi_1^\infty)'_L, \mathrm{s}_2^* Z_{T_2}(\phi_2^\infty)'_L \rangle_{X'_L,K} \\ &= \sum_{\substack{T^\square \in \mathrm{Herm}_{2r}^\circ(F)^+ \\ \mathrm{Diff}(T^\square,V)=\{\underline{u}\} \\ \partial_{r,r} T^\square = (T_1,T_2)}} \frac{1}{\log q_u} \left(W'_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda_{\underline{u}}^{\mathtt{R}})^{2r}}) - \frac{\log q_u}{q_u^r - 1} W_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda_{\underline{u}}^\star)^{2r}}) \right) \\ &\qquad \times \prod_{v \neq \underline{u}} W_{T^\square}(0, 1_{4r}, (\Phi_\infty^0 \otimes (\mathrm{s}_1 \phi_1^\infty \otimes (\mathrm{s}_2 \phi_2^\infty)^{\mathtt{c}}))_v).\end{aligned}$$

As $\mathrm{vol}(L_{\underline{u}}) = (q_{\underline{u}}+1)(q_{\underline{u}}^{2r}-1)^{-1}$, (9.2) is equivalent to

$$\begin{aligned}(9.3)\quad &\frac{\mathrm{vol}(H(F_\infty)L^{\underline{u}})}{\deg(Y/K)} \Phi_\infty^0(T_1,T_2) \langle \mathrm{s}_1^* Z_{T_1}(\phi_1^\infty)'_L, \mathrm{s}_2^* Z_{T_2}(\phi_2^\infty)'_L \rangle_{X'_L,K} \\ &= \sum_{\substack{T^\square \in \mathrm{Herm}_{2r}^\circ(F)^+ \\ \mathrm{Diff}(T^\square,V)=\{\underline{u}\} \\ \partial_{r,r} T^\square = (T_1,T_2)}} \\ &\qquad \times \frac{b_{2r,\underline{u}}(0)}{\log q_u} \left(\frac{q_{\underline{u}}^{2r}-1}{q_{\underline{u}}+1} W'_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda_{\underline{u}}^{\mathtt{R}})^{2r}}) - \frac{\log q_u}{q_{\underline{u}}+1} W_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda_{\underline{u}}^\star)^{2r}}) \right) \\ &\qquad \times \prod_{v \neq \underline{u}} \frac{b_{2r,v}(0)}{\gamma_{V_v,\psi_{F,v}}^{2r}} W_{T^\square}(0, 1_{4r}, (\Phi_\infty^0 \otimes (\mathrm{s}_1 \phi_1^\infty \otimes (\mathrm{s}_2 \phi_2^\infty)^{\mathtt{c}}))_v),\end{aligned}$$

parallel to (8.5).

The proof of (9.3) is same to that of (8.5) except that now we have

$$\begin{aligned}&\chi\left(\pi_*^{\wedge_{\mathrm{K}}} \mathcal{N}(x)\right) \\ &= \frac{b_{2r,\underline{u}}(0)}{\log q_u} \left(\frac{q_{\underline{u}}^{2r}-1}{q_{\underline{u}}+1} W'_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda_{\underline{u}}^{\mathtt{R}})^{2r}}) - \frac{\log q_u}{q_{\underline{u}}+1} W_{T^\square}(0, 1_{4r}, \mathbb{1}_{(\Lambda_{\underline{u}}^\star)^{2r}}) \right)\end{aligned}$$

if $T(x) = T^\square$, by [LZ, Th. 10.5.1, Rem. 10.5.4]. The proposition is proved. □

10. **Local indices at archimedean places**

In this section, we compute local indices at places in $\mathtt{V}_E^{(\infty)}$.

Proposition 10.1. *Let* $\mathtt{R}$, $\mathtt{R}'$, ℓ, *and* L *be as in Definition 6.3. Let* $(\pi, \mathcal{V}_\pi)$ *be as in Assumption 3.1. Take an element* $u \in \mathtt{V}_E^{(\infty)}$. *Consider an* $(\mathtt{R}, \mathtt{R}', \ell, L)$*-admissible sextuple* $(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)$ *and an element* $\varphi_1 \in \mathcal{V}_\pi^{[r]\mathtt{R}}$. *Let* $K_1 \subseteq G_r(\mathbb{A}_F^\infty)$ *be an open compact subgroup that fixes both* ϕ_1^∞ *and* φ_1, *and let* $\mathfrak{F}_1 \subseteq$

$G_r(F_\infty)$ be a Siegel fundamental domain for the congruence subgroup $G_r(F) \cap g_1^\infty K_1 (g_1^\infty)^{-1}$. Then for every $T_2 \in \mathrm{Herm}_r^\circ(F)^+$, we have

(10.1)

$$\begin{aligned}&\mathrm{vol}^\natural(L) \cdot \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1) \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, \tau_1 g_1, g_2)_{L,u}\, \mathrm{d}\tau_1 \\ &= \frac{1}{2} \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1) \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} \mathfrak{E}_{T_1,T_2}((\tau_1 g_1, g_2), \Phi_\infty^0 \otimes (\mathrm{s}_1 \phi_1^\infty \otimes (\mathrm{s}_2 \phi_2^\infty)^{\mathtt{c}}))_u\, \mathrm{d}\tau_1,\end{aligned}$$

in which both sides are absolutely convergent. Here, the term $\mathfrak{E}_{T_1,T_2}$ is defined in Definition 3.11 with the Gaussian function $\Phi_\infty^0 \in \mathscr{S}(V^{2r} \otimes_{\mathbb{A}_F} F_\infty)$ (Notation 2.2(H3)), and $\mathrm{vol}^\natural(L)$ is defined in Definition 3.8.

Remark 10.2. The relation between I_{T_1,T_2} and $\mathfrak{E}_{T_1,T_2}$ for each individual pair (T_1, T_2) in the style of Proposition 8.1 is much more complicated, which involves the so-called holomorphic projection. (See [Liu11b, §6A] for the case where $r = 1$.) The main technical innovation in the archimedean computation in this article is that we do not need to compare I_{T_1,T_2} and $\mathfrak{E}_{T_1,T_2}$ in order to obtain the main theorems; it suffices for us to compare both sides after taking summation and convolution for any of the two variables as in Proposition 10.1, which does not require holomorphic projection.

As we have promised in Section 6, we start by recalling the definition of the archimedean local index in the decomposition (6.1).

Let X be a smooth projective complex scheme of pure dimension $n-1$. For an element $Z \in \mathrm{Z}^r(X)_{\mathbb{C}}$, recall that a *Green current* for Z is an $(r-1, r-1)$-current g_Z on $X(\mathbb{C})$ that is smooth away from the support of Z and satisfies

$$\mathrm{dd}^{\mathtt{c}} \mathrm{g}_Z + \delta_Z = [\omega_Z]$$

for a unique smooth (r,r)-form ω_Z on $X(\mathbb{C})$, which we call the *tail form* of g_Z. If $Z \in \mathrm{Z}^r(X)^0_{\mathbb{C}}$, then we say that a Green current g_Z for Z is *harmonic* if $\omega_Z = 0$, and we use $\mathrm{g}_Z^\heartsuit$ to indicate a harmonic Green current. For two elements $Z_1, Z_2 \in \mathrm{Z}^r(X)^0_{\mathbb{C}}$ with disjoint supports, we define

$$\langle Z_1, Z_2 \rangle_{X,\mathbb{C}} \coloneqq \frac{1}{2} \int_{X(\mathbb{C})} \mathrm{g}_{Z_1}^\heartsuit \wedge \delta_{Z_2^{\mathrm{c}}}, \tag{10.2}$$

which is independent of the choice of harmonic Green current $\mathrm{g}_{Z_1}^\heartsuit$.

By Lemma 5.5, we may assume $u = \mathbf{u}$ without lost of generality in the proof of Proposition 10.1. Now we apply the above discussion to the complex scheme $X_L \otimes_E \mathbb{C}$. For $i = 1, 2$,

- we denote by $\mathrm{g}_{T_i}^\heartsuit(\phi_i^\infty, \mathrm{s}_i, g_i^\infty)_L$ a harmonic Green current for the algebraic cycle $\mathrm{s}_i^* Z_{T_i}(\omega_r^\infty(g_i^\infty)\phi_i^\infty)_L$ on $X_L \otimes_E \mathbb{C}$;

- for every element $g_i \in G_r(\mathbb{A}_F)$ with finite part g_i^∞, there is a particular Green current for $\mathrm{s}_i^* Z_{T_i}(\omega_r^\infty(g_i^\infty)\phi_i^\infty)_L$ on $X_L \otimes_E \mathbb{C}$, known as the *Kudla–Milson Green current* (see the proof of [Liu11a, Th. 4.20]), denoted by $\mathrm{g}^{\mathrm{KM}}_{T_i}(\phi_i^\infty, \mathrm{s}_i, g_i)_L$, with the tail form $\omega^{\mathrm{KM}}_{T_i}(\phi_i^\infty, \mathrm{s}_i, g_i)_L$.

Proof of Proposition 10.1. As we have pointed out, it suffices to prove the proposition for $u = \mathbf{u}$. By (10.2), we have

$$\begin{aligned} &I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)_{L,\mathbf{u}} \\ &= \frac{1}{2} C_{T_1,T_2}(g_{1\infty}, g_{2\infty}) \int_{X_L(\mathbb{C})} \mathrm{g}^{\heartsuit}_{T_1}(\phi_1^\infty, \mathrm{s}_1, g_1^\infty)_L \wedge \delta_{(\mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L)^{\mathsf{c}}}, \end{aligned} \tag{10.3}$$

where

$$C_{T_1,T_2}(g_{1\infty}, g_{2\infty}) := \omega_{r,\infty}(g_{1\infty})\phi^0_\infty(T_1) \cdot (\omega_{r,\infty}(g_{2\infty})\phi^0_\infty(T_2))^{\mathsf{c}}.$$

We need a variant of (10.3). Put

$$\begin{aligned} &I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)^{\mathrm{KM}}_{L,\mathbf{u}} \\ &:= \frac{1}{2} C_{T_1,T_2}(g_{1\infty}, g_{2\infty}) \left(\int_{X_L(\mathbb{C})} \mathrm{g}^{\mathrm{KM}}_{T_1}(\phi_1^\infty, \mathrm{s}_1, g_1)_L \wedge \delta_{(\mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L)^{\mathsf{c}}} \right. \\ &\qquad \left. + \int_{X_L(\mathbb{C})} \omega^{\mathrm{KM}}_{T_1}(\phi_1^\infty, \mathrm{s}_1, g_1)_L \wedge \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^\infty, \mathrm{s}_2, g_2)^{\mathsf{c}}_L \right). \end{aligned} \tag{10.4}$$

By [Liu11a, Th. 4.20][20], we have

$$\begin{aligned} &\mathrm{vol}^\natural(L) \cdot I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \mathrm{s}_1, \mathrm{s}_2, g_1, g_2)^{\mathrm{KM}}_{L,\mathbf{u}} \\ &\qquad = \frac{1}{2} \mathfrak{E}_{T_1,T_2}((g_1, g_2), \Phi^0_\infty \otimes (\mathrm{s}_1\phi_1^\infty \otimes (\mathrm{s}_2\phi_2^\infty)^{\mathsf{c}}))_{\mathbf{u}}. \end{aligned} \tag{10.5}$$

We first check the absolute convergence of the two sides of (10.1). It is clear that the assignment

$$\tau_1 \mapsto \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} \left| \mathfrak{E}_{T_1,T_2}((\tau_1 g_1, g_2), \Phi^0_\infty \otimes (\mathrm{s}_1\phi_1^\infty \otimes \mathrm{s}_2\phi_2^\infty))_{\mathbf{u}} \right|$$

is slowly increasing on $\mathfrak{F}_1$, which implies that the right-hand side of (10.1) is absolutely convergent since φ is a cusp form.

[20] There is a sign error in [Liu11a, Th. 4.20]: the correct sign should be $\prod_v \gamma_{\mathbb{V}_v}^{2n}$, which is 1, rather than $\prod_v \gamma_{\mathbb{V}_v}$, which is -1. (The root of this sign error is that in the formula for $\omega_\chi(w_r)$ on [Liu11a, p. 858], the constant γ_V should really be γ_V^r.) This result was later reproved in [GS19, Cor. 5.12] by a different method.

For the left-hand side, by (10.5), it suffices to show that the expression

$$\sum_{T_1\in \mathrm{Herm}_r^\circ(F)^+} \Big| I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1,\mathrm{s}_2,\tau_1 g_1,g_2)^{\mathrm{KM}}_{L,\mathbf{u}} - I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1,\mathrm{s}_2,\tau_1 g_1,g_2)_{L,\mathbf{u}}\Big|$$

is absolutely convergent and is slowly increasing on τ_1. For short, put

$$\eta := -\mathrm{g}^\heartsuit_{T_2}(\phi_2^\infty,\mathrm{s}_2,g_2^\infty)^{\mathfrak{c}}_L,$$

which is an $(r-1,r-1)$-current on $X_L(\mathbb{C})$ satisfying

$$\mathrm{dd}^{\mathfrak{c}}\eta = \delta_{(\mathrm{s}_2^* Z_{T_2}(\omega_r^\infty(g_2^\infty)\phi_2^\infty)_L)^{\mathfrak{c}}}. \tag{10.6}$$

Then for every $T_1\in \mathrm{Herm}_r^\circ(F)^+$,

$$\begin{aligned}
&J_{T_1}(\tau_1)\\
&\quad := I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1,\mathrm{s}_2,\tau_1 g_1,g_2)^{\mathrm{KM}}_{L,\mathbf{u}} - I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\mathrm{s}_1,\mathrm{s}_2,\tau_1 g_1,g_2)_{L,\mathbf{u}}\\
&\quad = C_{T_1,T_2}(\tau_1 g_{1\infty},g_{2\infty})\int_{X_L(\mathbb{C})} \left(\mathrm{g}^{\mathrm{KM}}_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L - \mathrm{g}^\heartsuit_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L\right)\wedge \mathrm{dd}^{\mathfrak{c}}\eta\\
&\qquad + C_{T_1,T_2}(\tau_1 g_{1\infty},g_{2\infty})\int_{X_L(\mathbb{C})} \omega^{\mathrm{KM}}_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L\wedge \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^\infty,\mathrm{s}_2,g_2)^{\mathfrak{c}}_L\\
&\quad = C_{T_1,T_2}(\tau_1 g_{1\infty},g_{2\infty})\int_{X_L(\mathbb{C})} \mathrm{dd}^{\mathfrak{c}}\left(\mathrm{g}^{\mathrm{KM}}_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L - \mathrm{g}^\heartsuit_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L\right)\wedge \eta\\
&\qquad + C_{T_1,T_2}(\tau_1 g_{1\infty},g_{2\infty})\int_{X_L(\mathbb{C})} \omega^{\mathrm{KM}}_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L\wedge \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^\infty,\mathrm{s}_2,g_2)^{\mathfrak{c}}_L\\
&\quad = C_{T_1,T_2}(\tau_1 g_{1\infty},g_{2\infty})\int_{X_L(\mathbb{C})} \omega^{\mathrm{KM}}_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L\wedge \left(\eta + \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^\infty,\mathrm{s}_2,g_2)^{\mathfrak{c}}_L\right).
\end{aligned}$$

By the claim $(*)$ below and the fact that $\eta+\mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^\infty,\mathrm{s}_2,g_2)^{\mathfrak{c}}_L$ is a smooth form on $X_L(\mathbb{C})$, we know that $\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+}|J_{T_1}(\tau_1)|$ is convergent and is slowly increasing in $\tau_1\in\mathfrak{F}_1$, which implies that the left-hand side of (10.1) is absolutely convergent as we have pointed out.

We claim that

$(*)$ The summation

$$\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+} C_{T_1,T_2}(\tau_1 g_{1\infty},g_{2\infty})\cdot \omega^{\mathrm{KM}}_{T_1}(\phi_1^\infty,\mathrm{s}_1,\tau_1 g_1)_L$$

is convergent in the space of smooth (r,r)-form on $X_L(\mathbb{C})$ with respect to the C^∞-topology, and is locally uniformly slowly increasing in $\tau_1\in\mathfrak{F}_1$.

Note that since $X_L(\mathbb{C})$ is a finite disjoint union compact quotients of the real Lie group ${}^{\mathbf{u}}H(\mathbb{R})$, the C^∞-topology on $X_L(\mathbb{C})$, which is a Fréchet topology, can defined by a natural family of semi-norms given by the upper bound of $|Df|$ on a compact neighbourhood of the identity in ${}^{\mathbf{u}}H(\mathbb{R})$, where

D runs through invariant differential operators on ${}^{\mathbf{u}}H(\mathbb{R})$. Take such a semi-norm $\|\ \|$. By the construction of the Kudla–Millson form ([Mil85, §III.1] or [KM86, §3]), it suffices to consider semi-norms $\|\ \|$ satisfying that there exists $\phi_\infty \in \mathscr{S}({}^{\mathbf{u}}V^r \otimes_F F_\infty)$ such that

$$\Big\|\omega_{r,\infty}(\tau_1 g_{1\infty})\phi^0_\infty(T_1)\cdot \omega^{\mathrm{KM}}_{T_1}(\phi^\infty_1, \mathrm{s}_1, \tau_1 g_1)_L\Big\| = \sup_{h\in {}^{\mathbf{u}}H(F)\backslash {}^{\mathbf{u}}H(\mathbb{A}_F)} \big\{\big|\theta_{\phi_\infty\otimes \mathrm{s}_1\phi^\infty_1, T_1}(\tau_1 g_1, h)\big|\big\}$$

for every $\tau_1 \in \mathfrak{F}_1$ and every $T_1 \in \mathrm{Herm}^\circ_r(F)^+$, where $\theta_{\phi_\infty\otimes \mathrm{s}_1\phi^\infty_1, T_1}$ denotes the T_1-component of the classical theta function of $\phi_\infty \otimes \mathrm{s}_1\phi^\infty_1$. Now since ${}^{\mathbf{u}}H(F)\backslash {}^{\mathbf{u}}H(\mathbb{A}_F)$ is compact, and the assignment

$$\tau_1 \mapsto \sum_{T_1\in \mathrm{Herm}^\circ_r(F)^+} \big|\theta_{\phi_\infty\otimes \mathrm{s}_1\phi^\infty_1, T_1}(\tau_1 g_1, h)\big|$$

is slowly increasing on $\mathfrak{F}_1$, locally uniformly in h, the claim follows.

Now we continue to prove (10.1). By (10.5), it suffices to show that

$$\begin{aligned}(10.7)\quad &\int_{\mathfrak{F}_1} \varphi^{\mathsf{c}}(\tau_1 g_1) \sum_{T_1\in \mathrm{Herm}^\circ_r(F)^+} I_{T_1,T_2}(\phi^\infty_1, \phi^\infty_2, \mathrm{s}_1, \mathrm{s}_2, \tau_1 g_1, g_2)^{\mathrm{KM}}_{L,\mathbf{u}}\,\mathrm{d}\tau_1 \\ &\quad = \int_{\mathfrak{F}_1} \varphi^{\mathsf{c}}(\tau_1 g_1) \sum_{T_1\in \mathrm{Herm}^\circ_r(F)^+} I_{T_1,T_2}(\phi^\infty_1, \phi^\infty_2, \mathrm{s}_1, \mathrm{s}_2, \tau_1 g_1, g_2)_{L,\mathbf{u}}\,\mathrm{d}\tau_1,\end{aligned}$$

in which we have already known that both sides are absolutely convergent.

Take an element $T_1 \in \mathrm{Herm}^\circ_r(F)^+$. We put

$${}^{\varphi}\mathbf{g}^{\heartsuit}_{T_1} := \int_{\mathfrak{F}_1} \varphi^{\mathsf{c}}(\tau_1 g_1) C_{T_1,T_2}(\tau_1 g_{1\infty}, g_{2\infty})\mathbf{g}^{\heartsuit}_{T_1}(\phi^\infty_1, \mathrm{s}_1, g^\infty_1)_L\,\mathrm{d}\tau_1,$$

which is a harmonic Green current for

$${}^{\varphi}Z_{T_1} := \left(\int_{\mathfrak{F}_1} \varphi^{\mathsf{c}}(\tau_1 g_1) C_{T_1,T_2}(\tau_1 g_{1\infty}, g_{2\infty})\,\mathrm{d}\tau_1\right)\cdot \mathrm{s}_1^* Z_{T_1}(\omega^\infty_r(g^\infty_1)\phi^\infty_1)_L.$$

We also put

$${}^{\varphi}\mathbf{g}^{\mathrm{KM}}_{T_1} := \int_{\mathfrak{F}_1} \varphi^{\mathsf{c}}(\tau_1 g_1) C_{T_1,T_2}(\tau_1 g_{1\infty}, g_{2\infty})\mathbf{g}^{\mathrm{KM}}_{T_1}(\phi^\infty_1, \mathrm{s}_1, \tau_1 g_1)_L\,\mathrm{d}\tau_1,$$

which is a Green current for ${}^{\varphi}Z_{T_1}$, whose tail form is

$${}^{\varphi}\omega^{\mathrm{KM}}_{T_1} := \int_{\mathfrak{F}_1} \varphi^{\mathsf{c}}(\tau_1 g_1) C_{T_1,T_2}(\tau_1 g_{1\infty}, g_{2\infty})\omega^{\mathrm{KM}}_{T_1}(\phi^\infty_1, \mathrm{s}_1, \tau_1 g_1)_L\,\mathrm{d}\tau_1.$$

Then by (10.3), (10.4), and (10.6), we have

$$\begin{aligned}
{}^{\varphi}J_{T_1} &\coloneqq \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1) I_{T_1,T_2}(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, \tau_1 g_1, g_2)^{\mathrm{KM}}_{L,\mathbf{u}} \,\mathrm{d}\tau_1 \\
&\quad - \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1) I_{T_1,T_2}(\phi_1^{\infty}, \phi_2^{\infty}, \mathrm{s}_1, \mathrm{s}_2, \tau_1 g_1, g_2)_{L,\mathbf{u}} \,\mathrm{d}\tau_1 \\
&= \int_{X_L(\mathbb{C})} ({}^{\varphi}\mathrm{g}^{\mathrm{KM}}_{T_1} - {}^{\varphi}\mathrm{g}^{\heartsuit}_{T_1}) \wedge \mathrm{dd}^{\mathtt{c}}\eta + \int_{X_L(\mathbb{C})} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1} \wedge \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^{\infty}, \mathrm{s}_2, g_2)^{\mathtt{c}}_L \\
&= \int_{X_L(\mathbb{C})} \mathrm{dd}^{\mathtt{c}}({}^{\varphi}\mathrm{g}^{\mathrm{KM}}_{T_1} - {}^{\varphi}\mathrm{g}^{\heartsuit}_{T_1}) \wedge \eta + \int_{X_L(\mathbb{C})} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1} \wedge \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^{\infty}, \mathrm{s}_2, g_2)^{\mathtt{c}}_L \\
&= \int_{X_L(\mathbb{C})} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1} \wedge \left(\eta + \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^{\infty}, \mathrm{s}_2, g_2)^{\mathtt{c}}_L\right).
\end{aligned}$$

Therefore, the difference between the two sides of (10.7) equals

(10.8)

$$\sum_{T_1 \in \mathrm{Herm}_r^{\circ}(F)^+} {}^{\varphi}J_{T_1} = \int_{X_L(\mathbb{C})} \left(\sum_{T_1 \in \mathrm{Herm}_r^{\circ}(F)^+} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1} \right) \wedge \left(\eta + \mathrm{g}^{\mathrm{KM}}_{T_2}(\phi_2^{\infty}, \mathrm{s}_2, g_2)^{\mathtt{c}}_L\right).$$

Here, to validate the exchange of summation and integration, it suffices to show that the summation $\sum_{T_1 \in \mathrm{Herm}_r^{\circ}(F)^+} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1}$ is convergent in the space of smooth (r,r)-form on $X_L(\mathbb{C})$ with respect to the C^{∞}-topology, since $X_L(\mathbb{C})$ is compact. However, this follows from the claim $(*)$.

We then continue by computing the right-hand side of (10.8). Since $\mathrm{supp}(\phi_{1v}^{\infty}) \subseteq (V_v^r)_{\mathrm{reg}}$ for some $v \in \mathtt{R}'$, we have

$$\sum_{T_1 \in \mathrm{Herm}_r^{\circ}(F)^+} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1} = \sum_{T_1 \in \mathrm{Herm}_r(F)^+} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1},$$

where ${}^{\varphi}\omega^{\mathrm{KM}}_{T_1}$ for $T_1 \in \mathrm{Herm}_r(F)^+ \setminus \mathrm{Herm}_r^{\circ}(F)^+$ is defined similarly. However,

$$\sum_{T_1 \in \mathrm{Herm}_r(F)^+} {}^{\varphi}\omega^{\mathrm{KM}}_{T_1} = (\omega_{r,\infty}(g_{2\infty})\phi^0_{\infty}(T_2))^{\mathtt{c}} \cdot \int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1)\omega^{\mathrm{KM}}(\tau_1 g_1)\,\mathrm{d}\tau_1,$$

where $\omega^{\mathrm{KM}}(g_1)$ is the Kudla–Milson form for the generating function $Z_{\mathrm{s}_1\phi_1^{\infty}}(g_1)_L$. By [Mil85, Th. III.2.1], we know that

$$\int_{\mathfrak{F}_1} \varphi^{\mathtt{c}}(\tau_1 g_1)\omega^{\mathrm{KM}}(\tau_1 g_1)\,\mathrm{d}\tau_1 = \int_{\Gamma_1 \backslash G_r(F_{\infty})} \varphi^{\mathtt{c}}(g_1' g_1)\omega^{\mathrm{KM}}(g_1' g_1)\,\mathrm{d}g_1' \tag{10.9}$$

is a harmonic (r,r)-form on $X_L(\mathbb{C})$. Since $Z_{\mathrm{s}_1\phi_1^{\infty}}(g_1)_L$ is cohomologically trivial, the cohomology class of (10.9) is also trivial, which implies that (10.9) vanishes. Therefore, we obtain (10.7). The proposition is proved. □

11. **Proof of main results**

In this section, we prove our main results in Section 1. Thus, we put ourselves in Assumption 1.3. In particular, we have $\mathtt{V}_F^{\mathrm{ram}} = \emptyset$, $\mathtt{V}_F^{(2)} \subseteq \mathtt{V}_F^{\mathrm{spl}}$.

Let $(\pi, \mathcal{V}_\pi)$ be as in Assumption 1.3 with $|\mathtt{S}_\pi|$ odd, for which we assume Hypothesis 6.6. Take

- a totally positive definite hermitian space V over $\mathbb{A}_E$ of rank $2r$ as in Notation 2.2 satisfying that $\epsilon(V_v) = -1$ if and only if $v \in \mathtt{S}_\pi$ (so that V is incoherent and $V = V_\pi$ as in Section 1);[21]
- $\mathtt{S} = \mathtt{S}_\pi$ (so that every underlying rational prime of $\mathtt{S}$ is unramified in E);
- $\mathtt{R}$ a finite subset of $\mathtt{V}_F^{\mathrm{spl}}$ containing $\mathtt{R}_\pi$ and of cardinality at least 2, and $\mathtt{R}' = \mathtt{R}$;
- an $\mathtt{R}$-good rational prime ℓ (Definition 6.1);
- for $i = 1, 2$, a nonzero element $\varphi_i = \otimes_v \varphi_{iv} \in \mathcal{V}_\pi^{[r]\mathtt{R}}$ satisfying that $\langle \varphi_{1v}^{\mathtt{c}}, \varphi_{2v} \rangle_{\pi_v} = 1$ for $v \in \mathtt{V}_F \setminus \mathtt{R}$;
- for $i = 1, 2$, an element $\phi_i^\infty = \otimes_v \phi_{iv}^\infty \in \mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty)$ satisfying
  - $\phi_{iv}^\infty = \mathbb{1}_{(\Lambda_v^{\mathtt{R}})^r}$ for $v \in \mathtt{V}_F^{\mathrm{fin}} \setminus \mathtt{R}$;
  - $\mathrm{supp}(\phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}) \subseteq (V_v^{2r})_{\mathrm{reg}}$ for $v \in \mathtt{R}$;
- an open compact subgroup L of $H(\mathbb{A}_F^\infty)$ of the form $L_{\mathtt{R}} L^{\mathtt{R}}$, where $L^{\mathtt{R}}$ is defined in Notation 2.2(H8), that fixes both ϕ_1^∞ and ϕ_2^∞;
- an open compact subgroup $K \subseteq G_r(\mathbb{A}_F^\infty)$ that fixes $\varphi_1, \varphi_2, \phi_1^\infty, \phi_2^\infty$;
- a set of representatives $\{g^{(1)}, \dots, g^{(s)}\}$ of the double coset

$$G_r(F) \backslash G_r(\mathbb{A}_F^\infty) / K$$

  satisfying $g^{(j)} \in G_r(\mathbb{A}_F^{\infty,\mathtt{R}})$ for $1 \leqslant j \leqslant s$, together with a Siegel fundamental domain $\mathfrak{F}^{(j)} \subseteq G_r(F_\infty)$ for the congruence subgroup $G_r(F) \cap g^{(j)} K (g^{(j)})^{-1}$ for each $1 \leqslant j \leqslant s$;
- for $i = 1, 2$, s_i a product of two elements in $(\mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}})_{L_{\mathtt{R}}}^{\langle \ell \rangle}$ satisfying $\chi_\pi^{\mathtt{R}}(\mathrm{s}_i) = 1$ (which is possible by Proposition 6.9(2));
- for $i = 1, 2$, an element $\mathrm{s}_i^u \in (\mathbb{S}_{\mathbb{Q}^{\mathrm{ac}}}^{\mathtt{R}})_{L_{\mathtt{R}}}^{\langle \ell \rangle}$ for every $u \in \mathtt{V}_E^{\mathrm{spl}} \cup \mathtt{S}_E$, where $\mathtt{S}_E$ denotes the subset of $\mathtt{V}_E^{\mathrm{int}}$ above $\mathtt{S}$, as in Propositions 7.1 and 9.1, satisfying $\chi_\pi^{\mathtt{R}}(\mathrm{s}_i^u) = 1$ and that $\mathrm{s}_i^u = 1$ for all but finitely many u.

In what follows, we put

$$\tilde{\mathrm{s}}_i := \mathrm{s}_i \cdot \prod_{u \in \mathtt{V}_E^{\mathrm{spl}} \cup \mathtt{S}_E} \mathrm{s}_i^u$$

for $i = 1, 2$.

LEMMA 11.1. *Let the situation be as above.*

[21] We have changed the use of V from Section 1 since in the proofs below, we need to consider all nearby spaces of V_π. In particular, V in Section 1 is now ${}^{\mathtt{u}}V$.

(1) *For every* $T_2 \in \mathrm{Herm}_r^\circ(F)^+$ *and every* $\mathrm{t} \in \mathbb{T}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}}$, *the identity*

$$\begin{aligned}
&\mathrm{vol}^\natural(L) \sum_{j=1}^{s} \int_{\mathfrak{F}^{(j)}} \varphi_1^{\mathtt{c}}(\tau^{(j)} g^{(j)}) \\
&\quad \cdot \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\mathrm{t}\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j)} g^{(j)}, g_2)^\ell_L \, \mathrm{d}\tau^{(j)} \\
&= \chi_\pi^{\mathtt{R}}(\mathrm{t})^{\mathtt{c}} \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_1^{\mathtt{c}}(g_1) E'(0, (g_1, g_2), \Phi_\infty^0 \otimes \Phi^\infty)_{-,T_2} \, \mathrm{d}g_1 \\
&\quad - \chi_\pi^{\mathtt{R}}(\mathrm{t})^{\mathtt{c}} \sum_{u \in \mathtt{S}_E} \frac{\log q_u}{q_u^r - 1} \\
&\quad \cdot \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_1^{\mathtt{c}}(g_1) E(0, (g_1, g_2), \Phi_\infty^0 \otimes \Phi^{\infty,\underline{u}} \otimes \mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}})_{-,T_2} \, \mathrm{d}g_1
\end{aligned}$$

holds for every $g_2 \in G_r(\mathbb{A}_F^{\mathtt{R}})$, *where* $\Phi^\infty := \tilde{\mathrm{s}}_1 \phi_1^\infty \otimes (\tilde{\mathrm{s}}_2 \phi_2^\infty)^{\mathtt{c}}$, $\Lambda^\star_{\underline{u}}$ *is the lattice in Proposition 9.1, and* $E(s, (g_1, g_2), \Phi)_{-,T_2}$ *denotes the* T_2*-Siegel Fourier coefficient of the Eisenstein series* $E(s, (g_1, g_2), \Phi)$ *with respect to the second variable* g_2.

(2) *The identity*

$$\begin{aligned}
&\mathrm{vol}^\natural(L) \sum_{j_2=1}^{s} \sum_{j_1=1}^{s} \int_{\mathfrak{F}^{(j_2)}} \int_{\mathfrak{F}^{(j_1)}} \varphi_2(\tau^{(j_2)} g^{(j_2)}) \varphi_1^{\mathtt{c}}(\tau^{(j_1)} g^{(j_1)}) \sum_{T_2 \in \mathrm{Herm}_r^\circ(F)^+} \\
&\quad \cdot \sum_{T_1 \in \mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j_1)} g^{(j_1)}, \tau^{(j_2)} g^{(j_2)})^\ell_L \, \mathrm{d}\tau^{(j_1)} \, \mathrm{d}\tau^{(j_2)} \\
&= \frac{L'(\frac{1}{2}, \pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v \in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v, V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}})
\end{aligned}$$

holds.

Proof. For (1), pick an element $\mathrm{h} \in \mathcal{H}_{W_r}^{\mathtt{R}}$ such that $\theta^{\mathtt{R}}(\mathrm{h}) = \mathrm{t}$ as in Definition 6.8. Then there exist finitely many pairs $(c_k, h_k) \in \mathbb{C} \times G_r(\mathbb{A}_F^{\infty,\mathtt{R}})$ such that $\mathrm{h}\phi_1^\infty = \sum_k c_k \omega_r^\infty(h_k)\phi_1^\infty$ and $\mathrm{h}\varphi_1 = \sum_k c_k \pi(h_k)\varphi_1$. By [Liu21b, Th. 1.1] for inert places and [Liu11a, Prop. A.5] for split places (see also [Ral82, p. 511]), we have

$$\mathrm{t}\phi_1^\infty = \mathrm{h}\phi_1^\infty = \sum_k c_k \omega_r^\infty(h_k)\phi_1^\infty.$$

Thus, we have

$$\begin{aligned}
&I_{T_1,T_2}(\mathrm{t}\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j)} g^{(j)}, g_2)^\ell_L \\
&\qquad = \sum_k c_k I_{T_1,T_2}(\mathrm{t}\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j)} g^{(j)} h_k, g_2)^\ell_L.
\end{aligned} \tag{11.1}$$

By Lemma 6.4, we have

$$\begin{aligned} &I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\tilde{\mathrm{s}}_1,\tilde{\mathrm{s}}_2,\tau^{(j)}g^{(j)}h_k,g_2)_L^\ell \\ &\quad=\sum_{u\in \mathtt{V}_E^{(\infty)}} 2I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\tilde{\mathrm{s}}_1,\tilde{\mathrm{s}}_2,\tau^{(j)}g^{(j)}h_k,g_2)_{L,u} \\ &\quad+\sum_{u\in \mathtt{V}_E^{\mathrm{fin}}} \log q_u\cdot I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\tilde{\mathrm{s}}_1,\tilde{\mathrm{s}}_2,\tau^{(j)}g^{(j)}h_k,g_2)_{L,u}^\ell. \end{aligned} \tag{11.2}$$

Combining (11.1), (11.2), and Propositions 7.1, 8.1, 9.1, and 10.1, we have

$$\begin{aligned} &\mathrm{vol}^\natural(L)\sum_{j=1}^{s}\int_{\mathfrak{F}^{(j)}}\varphi_1^{\mathtt{c}}(\tau^{(j)}g^{(j)}) \\ &\qquad\cdot\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\mathrm{t}\phi_1^\infty,\phi_2^\infty,\tilde{\mathrm{s}}_1,\tilde{\mathrm{s}}_2,\tau^{(j)}g^{(j)},g_2)_L^\ell\,\mathrm{d}\tau^{(j)} \\ &=\sum_k\sum_{j=1}^{s}c_k\int_{\mathfrak{F}^{(j)}}\varphi_1^{\mathtt{c}}(\tau^{(j)}g^{(j)}) \\ &\qquad\cdot\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+}\mathfrak{E}^{\mathtt{S}}_{T_1,T_2}((\tau^{(j)}g^{(j)}h_k,g_2),\Phi^0_\infty\otimes\Phi^\infty), \end{aligned} \tag{11.3}$$

where we put

$$\begin{aligned} &\mathfrak{E}^{\mathtt{S}}_{T_1,T_2}((g_1,g_2),\Phi^0_\infty\otimes\Phi^\infty) \\ &\qquad:=\sum_{u\in\mathtt{V}_E\setminus\mathtt{V}_E^{\mathrm{spl}}}\mathfrak{E}_{T_1,T_2}((g_1,g_2),\Phi^0_\infty\otimes\Phi^\infty)_u \\ &\qquad\quad-\sum_{u\in\mathtt{S}_E}\frac{\log q_u}{q_u^r-1}E_{T_1,T_2}((g_1,g_2),\Phi^0_\infty\otimes\Phi^{\infty,\underline{u}}\otimes\mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}}). \end{aligned}$$

By Proposition 3.10 and Remark 3.12, we have

$$\begin{aligned} &\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+}\mathfrak{E}^{\mathtt{S}}_{T_1,T_2}((\tau^{(j)}g^{(j)}h_k,g_2),\Phi^0_\infty\otimes\Phi^\infty) \\ &\qquad=E'(0,(\tau^{(j)}g^{(j)}h_k,g_2),\Phi^0_\infty\otimes\Phi^\infty)_{-,T_2} \\ &\qquad\quad-\sum_{u\in\mathtt{S}_E}\frac{\log q_u}{q_u^r-1}E(0,(\tau^{(j)}g^{(j)}h_k,g_2),\Phi^0_\infty\otimes\Phi^{\infty,\underline{u}}\otimes\mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}})_{-,T_2} \end{aligned}$$

for every $1 \leqslant j \leqslant s$ and every k. It follows that

$$\begin{aligned}(11.3) &= \sum_k c_k \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_1^{\mathtt{c}}(g_1) E'(0,(g_1h_k,g_2),\Phi^0_\infty\otimes\Phi^\infty)_{-,T_2}\,\mathrm{d}g_1 \\ &\quad - \sum_{u\in\mathtt{S}_E}\frac{\log q_u}{q_u^r-1}\sum_k c_k \\ &\quad \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \varphi_1^{\mathtt{c}}(g_1) E(0,(g_1h_k,g_2),\Phi^0_\infty\otimes\Phi^{\infty,\underline{u}}\otimes\mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}})_{-,T_2}\,\mathrm{d}g_1 \\ &= \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} (\mathrm{h}\varphi_1^{\mathtt{c}})(g_1) E'(0,(g_1,g_2),\Phi^0_\infty\otimes\Phi^\infty)_{-,T_2}\,\mathrm{d}g_1 - \sum_{u\in\mathtt{S}_E}\frac{\log q_u}{q_u^r-1} \\ &\quad \cdot \int_{G_r(F)\backslash G_r(\mathbb{A}_F)} (\mathrm{h}\varphi_1^{\mathtt{c}})(g_1) E(0,(g_1,g_2),\Phi^0_\infty\otimes\Phi^{\infty,\underline{u}}\otimes\mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}})_{-,T_2}\,\mathrm{d}g_1.\end{aligned}$$

Part (1) follows as $\mathrm{h}\varphi_1^{\mathtt{c}} = \chi_\pi^{\mathtt{R}}(\mathrm{t})^{\mathtt{c}}\cdot\varphi_1^{\mathtt{c}}$.

For (2), we apply (1) to $\mathrm{t}=1$ and $g_2 = \tau^{(j_2)}g^{(j_2)}$ for $1\leqslant j_2\leqslant s$, and hence obtain

(11.4)

$$\begin{aligned}&\mathrm{vol}^\natural(L)\sum_{j_2=1}^s\sum_{j_1=1}^s \int_{\mathfrak{F}^{(j_2)}}\int_{\mathfrak{F}^{(j_1)}} \varphi_2(\tau^{(j_2)}g^{(j_2)})\varphi_1^{\mathtt{c}}(\tau^{(j_1)}g^{(j_1)}) \sum_{T_2\in\mathrm{Herm}_r^\circ(F)^+}\sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+} \\ &\quad \cdot I_{T_1,T_2}(\phi_1^\infty,\phi_2^\infty,\tilde{\mathrm{s}}_1,\tilde{\mathrm{s}}_2,\tau^{(j_1)}g^{(j_1)},\tau^{(j_2)}g^{(j_2)})^\ell_L r d\tau^{(j_1)}\,\mathrm{d}\tau^{(j_2)} \\ &= \iint_{[G_r(F)\backslash G_r(\mathbb{A}_F)]^2} \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1)E'(0,(g_1,g_2),\Phi^0_\infty\otimes\Phi^\infty)\,\mathrm{d}g_1\,\mathrm{d}g_2 \\ &\quad - \sum_{u\in\mathtt{S}_E}\frac{\log q_u}{q_u^r-1}\iint_{[G_r(F)\backslash G_r(\mathbb{A}_F)]^2} \\ &\quad \cdot \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1)E(0,(g_1,g_2),\Phi^0_\infty\otimes\Phi^{\infty,\underline{u}}\otimes\mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}})\,\mathrm{d}g_1\,\mathrm{d}g_2.\end{aligned}$$

By the classical Rallis inner product formula (see, e.g., [Liu11a, (2)–(6)]) and Proposition 3.6(2), we have

$$\begin{aligned}&\int_{G_r(F)\backslash G_r(\mathbb{A}_F)}\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \\ &\qquad \cdot \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1)E(0,(g_1,g_2),\Phi^0_\infty\otimes\Phi^{\infty,\underline{u}}\otimes\mathbb{1}_{(\Lambda^\star_{\underline{u}})^{2r}})\,\mathrm{d}g_1\,\mathrm{d}g_2 = 0\end{aligned}$$

for every $u\in\mathtt{S}_E$. Together with $\chi_\pi^{\mathtt{R}}(\tilde{\mathrm{s}}_1)=\chi_\pi^{\mathtt{R}}(\tilde{\mathrm{s}}_2)=1$, we have

$$\begin{aligned}(11.4) &= \int_{G_r(F)\backslash G_r(\mathbb{A}_F)}\int_{G_r(F)\backslash G_r(\mathbb{A}_F)} \\ &\quad \cdot \varphi_2(g_2)\varphi_1^{\mathtt{c}}(g_1)E'(0,(g_1,g_2),\Phi^0_\infty\otimes(\phi_1^\infty\otimes(\phi_2^\infty)^{\mathtt{c}}))\,\mathrm{d}g_1\,\mathrm{d}g_2.\end{aligned} \tag{11.5}$$

By Proposition 3.7, we have

$$(11.5) = \frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v\in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^{\natural}_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^{\infty}\otimes(\phi_{2v}^{\infty})^{\mathtt{c}}).$$

Part (2) is proved. □

Proof of Theorem 1.5. First, it suffices to prove the theorem for $\mathtt{R}$ satisfying $\mathtt{R}_\pi \subseteq \mathtt{R} \subseteq \mathtt{V}_F^{\mathrm{fin}}$ and $|\mathtt{R}| \geqslant 2$. Take an element $w \in \mathtt{V}_F^{(\infty)}$, and put ourselves in the setup of Section 4. We prove that the localization of the $\mathbb{T}^{\mathtt{R}}_{\mathbb{C}}$-module $\mathrm{CH}^r(X_L)^0_{\mathbb{C}}$ at $\mathfrak{m}_\pi^{\mathtt{R}}$, is nonvanishing.

Assume the converse. Then for every element $T_2 \in \mathrm{Herm}_r^\circ(F)^+$, we can find $\mathrm{t}_{T_2} \in \mathbb{T}^{\mathtt{R}}_{\mathbb{Q}^{\mathrm{ac}}}$ satisfying $\chi_\pi^{\mathtt{R}}(\mathrm{t}_{T_2}) = 1$ and $\mathrm{t}_{T_2}^* Z_{T_2}(\omega_r^\infty(g^{(j)})\phi_2^\infty)_L = 0$ for every $1 \leqslant j \leqslant s$. Let $\hat{\mathrm{t}}_{T_2}$ be the adjoint of t_{T_2}. Then we have

$$\begin{aligned}
&\langle \tilde{\mathrm{s}}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)(\hat{\mathrm{t}}_{T_2}\phi_1^\infty))_L, \tilde{\mathrm{s}}_2^* Z_{T_2}(\omega_r^\infty(g^{(j)})\phi_2^\infty)_L\rangle^\ell_{X_L,E}\\
&\quad = \langle \hat{\mathrm{t}}_{T_2}^*\tilde{\mathrm{s}}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)\phi_1^\infty)_L, \tilde{\mathrm{s}}_2^* Z_{T_2}(\omega_r^\infty(g^{(j)})\phi_2^\infty)_L\rangle^\ell_{X_L,E}\\
&\quad = \langle \tilde{\mathrm{s}}_1^* Z_{T_1}(\omega_r^\infty(g_1^\infty)\phi_1^\infty)_L, \mathrm{t}_{T_2}^*\tilde{\mathrm{s}}_2^* Z_{T_2}(\omega_r^\infty(g^{(j)})\phi_2^\infty)_L\rangle^\ell_{X_L,E} = 0
\end{aligned}$$

for every $T_1 \in \mathrm{Herm}_r^\circ(F)^+$, $g_1^\infty \in G_r(\mathbb{A}_F^{\infty,\mathtt{R}})$, and $1 \leqslant j \leqslant s$. In particular, we have

$$I_{T_1,T_2}(\hat{\mathrm{t}}_{T_2}\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j)}g^{(j)}, g_2)^\ell_L = 0$$

for every $g_2 \in G_r(\mathbb{A}_F^{\mathtt{R}})$ with $g_2^\infty \in \{g^{(1)},\dots,g^{(s)}\}$. It follows that

$$\begin{aligned}
&\mathrm{vol}^{\natural}(L)\sum_{j=1}^{s}\int_{\mathfrak{F}^{(j)}} \varphi_1^{\mathtt{c}}(\tau^{(j)}g^{(j)})\\
&\qquad\cdot \sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\hat{\mathrm{t}}_{T_2}\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j)}g^{(j)}, g_2)^\ell_L \,\mathrm{d}\tau^{(j)} = 0
\end{aligned}$$

for every $g_2 \in G_r(\mathbb{A}_F^{\mathtt{R}})$ with $g_2^\infty \in \{g^{(1)},\dots,g^{(s)}\}$ and every $T_2 \in \mathrm{Herm}_r^\circ(F)^+$. Now applying Lemma 11.1(1) twice with $\mathrm{t} = \hat{\mathrm{t}}_{T_2}$ and $\mathrm{t} = 1$, respectively, we obtain

$$\begin{aligned}
&\mathrm{vol}^{\natural}(L)\sum_{j=1}^{s}\int_{\mathfrak{F}^{(j)}} \varphi_1^{\mathtt{c}}(\tau^{(j)}g^{(j)})\\
&\qquad\cdot \sum_{T_1\in\mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j)}g^{(j)}, g_2)^\ell_L \,\mathrm{d}\tau^{(j)} = 0
\end{aligned}$$

for every $g_2 \in G_r(\mathbb{A}_F^{\mathtt{R}})$ with $g_2^\infty \in \{g^{(1)},\dots,g^{(s)}\}$ and every $T_2 \in \mathrm{Herm}_r^\circ(F)^+$. By Lemma 11.1(2), we obtain

$$\frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \cdot \prod_{v\in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^{\natural}_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}},\varphi_{2v},\phi_{1v}^{\infty}\otimes(\phi_{2v}^{\infty})^{\mathtt{c}}) = 0;$$

that is,

$$\prod_{v\in \mathtt{S}} \frac{(-1)^r q_v^{r-1}(q_v+1)}{(q_v^{2r-1}+1)(q_v^{2r}-1)} \cdot \prod_{v\in \mathtt{R}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}) = 0.$$

Now by Proposition 3.13, we may choose $\varphi_1, \varphi_2, \phi_1^\infty, \phi_2^\infty$ such that for every $v \in \mathtt{R}$,

$$\mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}) \neq 0.$$

As $L'(\frac{1}{2},\pi) \neq 0$, we obtain a contradiction. The theorem is proved. □

Proof of Theorem 1.7 and Corollary 1.9. By Definition 6.11 and Proposition 6.10(1), we have

$$\begin{aligned}
&\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E} \\
&\quad = \mathrm{vol}^\natural(L) \sum_{j_2=1}^{s}\sum_{j_1=1}^{s} \int_{\mathfrak{F}^{(j_2)}}\int_{\mathfrak{F}^{(j_1)}} \varphi_2(\tau^{(j_2)} g^{(j_2)})\varphi_1^{\mathtt{c}}(\tau^{(j_1)} g^{(j_1)}) \sum_{T_2\in \mathrm{Herm}_r^\circ(F)^+} \\
&\qquad \cdot \sum_{T_1\in \mathrm{Herm}_r^\circ(F)^+} I_{T_1,T_2}(\phi_1^\infty, \phi_2^\infty, \tilde{\mathrm{s}}_1, \tilde{\mathrm{s}}_2, \tau^{(j_1)} g^{(j_1)}, \tau^{(j_2)} g^{(j_2)})^\ell_L \, \mathrm{d}\tau^{(j_1)} \, \mathrm{d}\tau^{(j_2)}.
\end{aligned}$$

By Lemma 11.1(2) and Proposition 3.7, we obtain

$$\begin{aligned}
\langle \Theta_{\phi_1^\infty}(\varphi_1), \Theta_{\phi_2^\infty}(\varphi_2)\rangle^\natural_{X,E} &= \frac{L'(\frac{1}{2},\pi)}{b_{2r}(0)} \cdot C_r^{[F:\mathbb{Q}]} \\
&\quad \cdot \prod_{v\in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}).
\end{aligned} \tag{11.6}$$

By Proposition 3.13, we may choose $\varphi_1, \varphi_2, \phi_1^\infty, \phi_2^\infty$ such that

$$\prod_{v\in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v,V_v}(\varphi_{1v}^{\mathtt{c}}, \varphi_{2v}, \phi_{1v}^\infty \otimes (\phi_{2v}^\infty)^{\mathtt{c}}) \neq 0.$$

Now we claim that (11.6) holds for arbitrary vectors $\varphi_1, \varphi_2, \phi_1^\infty, \phi_2^\infty$ (not necessarily those in the beginning of this section) as in the statement of Theorem 1.7(1). This is a consequence of Proposition 3.6(1) as both sides of (11.6) give elements in the space

$$\bigotimes_{v\in \mathtt{V}_F^{\mathrm{fin}}} \mathrm{Hom}_{G_r(F_v)\times G_r(F_v)}(\mathrm{I}^\square_{r,v}(0), \pi_v \boxtimes \pi_v^\vee),$$

which is of dimension one. Thus, Theorem 1.7(1) follows.

By Proposition 6.10(2), the assignment $(\varphi, \phi^\infty) \mapsto \Theta_{\phi^\infty}(\varphi)$ gives an element in

$$\mathrm{Hom}_{H(\mathbb{A}_F^\infty)}\left(\mathrm{Hom}_{G_r(\mathbb{A}_F^\infty)}(\mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty), \pi^\infty), \varinjlim_L \mathrm{CH}^r(X_L)^0_{\mathbb{C}}\right),$$

in which $\mathrm{Hom}_{G_r(\mathbb{A}_F^\infty)}(\mathscr{S}(V^r \otimes_{\mathbb{A}_F} \mathbb{A}_F^\infty), \pi^\infty)$ is simply the theta lifting of π^∞ to $H(\mathbb{A}_F^\infty)$ by Proposition 3.6(3). Thus, Theorem 1.7(2) is a consequence of (11.6) and the fact that $\prod_{v\in \mathtt{V}_F^{\mathrm{fin}}} \mathfrak{Z}^\natural_{\pi_v,V_v}$ is nontrivial.

Finally, Corollary 1.9 is a consequence of (11.6) and Proposition 3.7 (where one may take $\mathtt{R} = \emptyset$). □

Appendix A. Two lemmas in Fourier analysis

In this appendix, we prove two lemmas in Fourier analysis that are only used in the proof of Proposition 3.13. Both the lemmas and their proofs are variants of [AN04, Th. 1] (in the non-archimedean setting).

Let F be a non-archimedean local field (of arbitrary characteristic). Denote the maximal ideal of O_F by $\mathfrak{p}_F$, and put $q \coloneqq |O_F/\mathfrak{p}_F|$. We fix a nontrivial additive character $\psi\colon F \to \mathbb{C}^\times$ that is used to define the Fourier transform.

Lemma A.1. *Consider a finite-dimensional F-vector space X, a nonzero homogeneous polynomial Δ on X, and a real number $r \geqslant 0$. Let f be a nonzero locally constant function on an open subset $\Omega \subseteq X$ on which Δ is nonvanishing. Suppose that f is locally integrable on X and satisfies that for every $\varepsilon > 0$, we have*

$$\int_\Omega |f(x)|^{2+\varepsilon} |\Delta(x)|_F^{r\varepsilon}\, \mathrm{d}x < \infty.$$

Then the support of the Fourier transform of f, as a distribution on $X^\vee$, cannot be contained in an analytic hypersurface.

Proof. Let $n \geqslant 1$ be the dimension of X. Without lost of generality, we may identify both X and $X^\vee$ with F^n, take $\mathrm{d}x$ to be the measure that gives O_F volume 1, and assume that ψ_F has conductor O_F. For every integer N, we put $B_N^n \coloneqq (\mathfrak{p}_F^N)^n$, which is an open compact subset of F^n.

Let u be the Fourier transform of f. For every integer $N \geqslant 0$, put $\chi_N \coloneqq q^{Nn} \mathbb{1}_{B_N^n} \in \mathscr{S}(F^n)$, and put $u_N \coloneqq u * \chi_N$, which is a locally constant function on F^n. Take two real numbers $0 \leqslant \delta < 1$ and $\varepsilon > 0$ to be determined later. Let $p \geqslant 2$ satisfy $\frac{1}{2-\delta} + \frac{1}{p} = 1$. Since the Fourier transform is a bounded operator from $L^{2-\delta}(F^n)$ to $L^p(F^n)$, we have

$$\begin{aligned} \|u_N\|_p^{2-\delta} &\leqslant C_\delta \int_{F^n} |f(x)|^{2-\delta} |\widehat{\chi_N}(x)|^{2-\delta}\, \mathrm{d}x \\ &= C_\delta \int_{F^n} |f(x)|^{2-\delta} |\mathbb{1}_{B_{-N}^n}(x)|^{2-\delta}\, \mathrm{d}x \\ &= C_\delta \int_{B_{-N}^n} |f(x)|^{2-\delta}\, \mathrm{d}x \end{aligned}$$

for some constant $C_\delta > 0$ depending only on δ. By Hölder's inequality, we have

$$\|u_N\|_p^{2-\delta} \leqslant C_\delta \left(\int_{B_{-N}^n} |\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}\, \mathrm{d}x \right)^{\frac{\delta+\varepsilon}{2+\varepsilon}} \left(\int_{B_{-N}^n} |f(x)|^{2+\varepsilon} |\Delta(x)|_F^{r\varepsilon}\, \mathrm{d}x \right)^{\frac{2-\delta}{2+\varepsilon}}.$$

Let d be the degree of Δ. There exists a real number $0 < \rho_\Delta < n/d$ depending only on Δ such that as long as $r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon} < \rho_\Delta$, the function $|\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}$ is locally integrable. In this case, there exists a constant $C_{\delta,\varepsilon} > 0$ such that

$$\int_{B^n_{-N}} |\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}\,\mathrm{d}x = C_{\delta,\varepsilon}\cdot q^{N\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)}.$$

By the integrability condition on f, there is a new constant $C'_{\delta,\varepsilon} > 0$ depending only on δ and ε such that

$$\|u_N\|_p^{2-\delta} \leqslant C'_{\delta,\varepsilon}\cdot q^{N\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon}} \tag{A.1}$$

holds for every $N \geqslant 0$.

Now suppose that the support of u is contained in an analytic hypersurface U. For $N \geqslant 0$, put $U_N \coloneqq U + B^n_N \subseteq F^n$ as a tubular neighbourhood of U, which contains the support of u_N. Then for every $g \in \mathscr{S}(F^n)$, we have

$$\lim_{N\to\infty} q^N \int_{U_N} g(x)\,\mathrm{d}x = \int_U g(y)\,\mathrm{d}y. \tag{A.2}$$

Then by Hölder's inequality, (A.1), and (A.2), we have

$$\begin{aligned}
|\langle u, g\rangle|^{2-\delta} &= \lim_{N\to\infty} |\langle u_N, g\rangle|^{2-\delta} \\
&\leqslant \lim_{N\to\infty} \|u_N\|_p^{2-\delta}\cdot \int_{U_N} |g(x)|^{2-\delta}\,\mathrm{d}x \\
&\leqslant C'_{\delta,\varepsilon}\cdot \lim_{N\to\infty} q^{N\left(\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon}-1\right)}\cdot q^N \int_{U_N} |g(x)|^{2-\delta}\,\mathrm{d}x \\
&= C'_{\delta,\varepsilon}\cdot \int_U |g(y)|^{2-\delta}\,\mathrm{d}y \cdot \lim_{N\to\infty} q^{N\left(\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon}-1\right)}.
\end{aligned}$$

Choose suitable δ, ε such that

$$r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon} < \rho_\Delta, \qquad n - dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon} < \frac{2+\varepsilon}{\delta+\varepsilon}.$$

Then the above limit is zero; that is, $\langle u, g\rangle = 0$ for every $g \in \mathscr{S}(F^n)$. Thus, we have $u = 0$. The lemma is proved. □

LEMMA A.2. *Consider a finite-dimensional F-vector space X, a nonzero homogeneous polynomial Δ on X, and a real number $r \geqslant 0$. Denote by $\Omega \subseteq X$ the nonvanishing locus of Δ. Let f be a nonzero locally constant function on Ω that is locally integrable on X, satisfying the following condition: there exists a decomposition $X = X_1 \oplus X_2 \oplus X_3$ with $\dim_F X_1 = \dim_F X_2 > 0$ such that*

(1) *Ω is disjoint from $X_1 \oplus X_3 \cup X_2 \oplus X_3$;*
(2) *$|\Delta(\alpha x_1, \alpha^{-1}x_2, x_3)|_F = |\Delta(x_1, x_2, x_3)|_F$ for every $\alpha \in F^\times$ and $x_i \in X_i$;*
(3) *$|f(\alpha x_1, \alpha^{-1}x_2, x_3)| = |f(x_1, x_2, x_3)|$ for every $\alpha \in F^\times$ and $x_i \in X_i$;*

(4) *for every* $\varepsilon > 0$, *we have* $\int_{F^\times\backslash\Omega} |f(x)|^{2+\varepsilon}|\Delta(x)|_F^{r\varepsilon}\,\mathrm{d}x < \infty$, *where the action of* $\alpha \in F^\times$ *on* Ω *is given by* $\alpha\cdot(x_1,x_2,x_3) = (\alpha x_1, \alpha^{-1}x_2, x_3)$.

Then the support of the Fourier transform of f, *as a distribution on* $X^\vee$, *cannot be contained in an analytic hypersurface.*

Proof. Let $n \geqslant 1$ be the dimension of X. Without lost of generality, we may identify the decomposition $X = X_1 \oplus X_2 \oplus X_3$ with $F^n = F^m \oplus F^m \oplus F^{n-2m}$, identify $X^\vee$ with F^n, take $\mathrm{d}x$ to be the measure that gives O_F volume 1, and assume that ψ_F has conductor O_F. For integers N and $l \geqslant 0$, we put $B_N^l := (\mathfrak{p}_F^N)^l$ and $A_N^l := B_N^l \setminus B_{N+1}^l$, which are open compact subsets of F^l. It is clear that the natural map $\varpi^{\mathbb{Z}} \times (A_0^m \times F^m \times F^{n-2m}) \to F^n$ given by the action in (4) is injective, and by (1) that Ω is contained in $\varpi^{\mathbb{Z}}\cdot(A_0^m \times F^m \times F^{n-2m})$.

Let u be the Fourier transform of f. For integers $N \geqslant 0$, put $\chi_N := q^{Nn}\mathbb{1}_{B_N^n} \in \mathscr{S}(F^n)$, and put $u_N := u * \chi_N$, which is a locally constant function on F^n. Take three real numbers $0 \leqslant \delta < \gamma < 1$ and $\varepsilon > 0$ to be determined later. Let $p > 2$ satisfy $\frac{1}{2-\gamma} + \frac{1}{p} = 1$. Since the Fourier transform is a bounded operator from $L^{2-\gamma}(F^n)$ to $L^p(F^n)$, we have

$$\begin{aligned}
\|u_N\|_p^{2-\gamma} &\leqslant C_\gamma \int_{F^n} |f(x)|^{2-\gamma} |\widehat{\chi_N}(x)|^{2-\gamma}\,\mathrm{d}x \\
&= C_\gamma \int_{F^n} |f(x)|^{2-\gamma} |\mathbb{1}_{B_{-N}^n}(x)|^{2-\gamma}\,\mathrm{d}x \\
&= C_\gamma \int_{B_{-N}^n} |f(x)|^{2-\gamma}\,\mathrm{d}x
\end{aligned}$$

for some constant $C_\gamma > 0$ depending only on γ. By (3) and Hölder's inequality, we have

$$\begin{aligned}
&\int_{B_{-N}^n} |f(x)|^{2-\gamma}\,\mathrm{d}x \\
&= \sum_{i=-2N}^{\infty} (i+2N+1) \int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2-\gamma}\,\mathrm{d}x \\
&\leqslant \sum_{i=-2N}^{\infty} (i+2N+1) \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} \mathrm{d}x\right)^{\frac{\gamma-\delta}{2-\delta}} \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2-\delta}\,\mathrm{d}x\right)^{\frac{2-\gamma}{2-\delta}} \\
&\leqslant \sum_{i=-2N}^{\infty} (i+2N+1)(q^{-im}q^{N(n-2m)})^{\frac{\gamma-\delta}{2-\delta}} \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2-\delta}\,\mathrm{d}x\right)^{\frac{2-\gamma}{2-\delta}} \\
&= C_{\gamma,\delta}\cdot q^{Nn\frac{\gamma-\delta}{2-\delta}} \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2-\delta}\,\mathrm{d}x\right)^{\frac{2-\gamma}{2-\delta}}
\end{aligned}$$

for some constant $C_{\gamma,\delta} > 0$. Together, we obtain

$$\|u_N\|_p^{2-\delta} \leqslant C'_{\gamma,\delta} \cdot q^{Nn\frac{\gamma-\delta}{2-\gamma}} \cdot \int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2-\delta}\,\mathrm{d}x \tag{A.3}$$

for a new constant $C'_{\gamma,\delta} > 0$ depending only on γ and δ. By Hölder's inequality, we have

$$\begin{aligned}
&\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2-\delta}\,\mathrm{d}x \\
&\leqslant \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}\,\mathrm{d}x\right)^{\frac{\delta+\varepsilon}{2+\varepsilon}} \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2+\varepsilon}|\Delta(x)|_F^{r\varepsilon}\,\mathrm{d}x\right)^{\frac{2-\delta}{2+\varepsilon}} \\
&\leqslant \left(\int_{B_{-2N}^n} |\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}\,\mathrm{d}x\right)^{\frac{\delta+\varepsilon}{2+\varepsilon}} \left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2+\varepsilon}|\Delta(x)|_F^{r\varepsilon}\,\mathrm{d}x\right)^{\frac{2-\delta}{2+\varepsilon}}.
\end{aligned}$$

Let d be the degree of Δ. There exists a real number $0 < \rho_\Delta < n/d$ depending only on Δ such that as long as $r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon} < \rho_\Delta$, the function $|\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}$ is locally integrable. In this case, there exists a constant $C_{\delta,\varepsilon} > 0$ such that

$$\int_{B_{-2N}^n} |\Delta(x)|_F^{-r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}}\,\mathrm{d}x = C_{\delta,\varepsilon} \cdot q^{2N\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)}.$$

On the other hand, by (4), we have

$$\left(\int_{A_0^m \times A_i^m \times B_{-N}^{n-2m}} |f(x)|^{2+\varepsilon}|\Delta(x)|_F^{r\varepsilon}\,\mathrm{d}x\right)^{\frac{2-\delta}{2+\varepsilon}} \leqslant C'_{\delta,\varepsilon}$$

for a constant $C'_{\delta,\varepsilon} > 0$. Thus, continuing (A.3), we have a constant $C_{\gamma,\delta,\varepsilon} > 0$ depending only on $\gamma, \delta, \varepsilon$ such that

$$\|u_N\|_p^{2-\delta} \leqslant C_{\gamma,\delta,\varepsilon} \cdot q^{N\left(n\frac{\gamma-\delta}{2-\gamma}+2\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon}\right)} \tag{A.4}$$

holds for all $N \geqslant 0$.

Now suppose that the support of u is contained in an analytic hypersurface U. For $N \geqslant 0$, put $U_N \coloneqq U + B_N^n \subseteq F^n$ as a tubular neighbourhood of U, which contains the support of u_N. Then for every $g \in \mathscr{S}(F^n)$, we have

$$\lim_{N\to\infty} q^N \int_{U_N} g(x)\,\mathrm{d}x = \int_U g(y)\,\mathrm{d}y. \tag{A.5}$$

Then by Hölder's inequality, (A.4), and (A.5), we have

$$\begin{aligned}
|\langle u, g\rangle|^{2-\delta} &= \lim_{N\to\infty} |\langle u_N, g\rangle|^{2-\delta} \\
&\leqslant \lim_{N\to\infty} \|u_N\|_p^{2-\delta} \cdot \left(\int_{U_N} |g(x)|^{2-\gamma}\,\mathrm{d}x\right)^{\frac{2-\delta}{2-\gamma}} \\
&\leqslant C_{\gamma,\delta,\varepsilon} \cdot \lim_{N\to\infty} q^{N\left(n\frac{\gamma-\delta}{2-\gamma}+2\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon}-\frac{2-\delta}{2-\gamma}\right)} \\
&\quad \cdot \left(q^N \int_{U_N} |g(x)|^{2-\gamma}\,\mathrm{d}x\right)^{\frac{2-\delta}{2-\gamma}} \\
&= C_{\gamma,\delta,\varepsilon} \cdot \left(\int_{U} |g(y)|^{2-\gamma}\,\mathrm{d}y\right)^{\frac{2-\delta}{2-\gamma}} \\
&\quad \cdot \lim_{N\to\infty} q^{N\left(n\frac{\gamma-\delta}{2-\gamma}+2\left(n-dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon}-\frac{2-\delta}{2-\gamma}\right)}.
\end{aligned}$$

Choose suitable $\gamma, \delta, \varepsilon$ such that

$$r\frac{(2-\delta)\varepsilon}{\delta+\varepsilon} < \rho_\Delta, \qquad n\frac{\gamma-\delta}{2-\gamma} + 2\left(n - dr\frac{(2-\delta)\varepsilon}{\delta+\varepsilon}\right)\frac{\delta+\varepsilon}{2+\varepsilon} < \frac{2-\delta}{2-\gamma}.$$

Then the above limit is zero; that is, $\langle u, g\rangle = 0$ for every $g \in \mathscr{S}(F^n)$. Thus, we have $u = 0$. The lemma is proved. □

Appendix B. Remarks on Beilinson's non-archimedean local indices

In this appendix, we review Beilinson's notion of non-archimedean local indices between algebraic cycles [Beĭ87] and make some complementary remarks.

Let K be a non-archimedean local field, with the ring of integers O_K and the residue field k. Take a rational prime ℓ that is invertible on k. Let X be a smooth projective scheme over K of pure dimension $n-1$. For every integer $d \geqslant 0$, we have the cycle class map

$$\mathrm{cl}_{X,\ell}\colon \mathrm{Z}^d(X) \to \mathrm{H}^{2d}(X, \mathbb{Q}_\ell(d)),$$

whose kernel we denote by $\mathrm{Z}^d(X)^{\langle \ell\rangle}$.

Remark B.1. A priori, $\mathrm{Z}^d(X)^{\langle \ell\rangle}$ depends on the rational prime ℓ. However, if K is of characteristic zero and we assume the monodromy-weight conjecture for X, then one can replace $\mathrm{cl}_{X,\ell}$ by the geometric cycle class map, hence $\mathrm{Z}^d(X)^{\langle \ell\rangle}$ does not depend on ℓ.

For a Zariski closed subset Z of X, we denote by $\mathrm{Z}^d_Z(X)$ the subgroup of $\mathrm{Z}^d(X)$ consisting of cycles whose support is contained in Z.

Definition B.2. For every pair of integers $d_1, d_2 \geqslant 0$ satisfying $d_1 + d_2 = n$, we define the subgroups

$$\mathrm{Z}^{d_1,d_2}(X) \coloneqq \sum_{Z_1,Z_2} \mathrm{Z}^{d_1}_{Z_1}(X)_{\mathbb{C}} \times \mathrm{Z}^{d_2}_{Z_2}(X)_{\mathbb{C}} \subseteq \mathrm{Z}^{d_1}(X)_{\mathbb{C}} \times \mathrm{Z}^{d_2}(X)_{\mathbb{C}},$$

$$\begin{aligned}\mathrm{Z}^{d_1,d_2}(X)^{\langle \ell \rangle} &\coloneqq \sum_{Z_1,Z_2} (\mathrm{Z}^{d_1}_{Z_1}(X)_{\mathbb{C}} \cap \mathrm{Z}^{d_1}(X)^{\langle \ell \rangle}_{\mathbb{C}}) \times (\mathrm{Z}^{d_2}_{Z_2}(X)_{\mathbb{C}} \cap \mathrm{Z}^{d_2}(X)^{\langle \ell \rangle}_{\mathbb{C}}) \\ &\subseteq \mathrm{Z}^{d_1}(X)_{\mathbb{C}} \times \mathrm{Z}^{d_2}(X)_{\mathbb{C}},\end{aligned}$$

where the sum is taken over all pairs (Z_1, Z_2) of disjoint Zariski closed subsets of X. It is clear that $\mathrm{Z}^{d_1,d_2}(X)^{\langle \ell \rangle}$ is stable under switching the two factors.

Take a pair of integers $d_1, d_2 \geqslant 0$ satisfying $d_1 + d_2 = n$. In [Beĭ87, §2], Beilinson defined a map

$$\langle\ ,\ \rangle^{\ell}_{X,K} \colon \mathrm{Z}^{d_1,d_2}(X)^{\langle \ell \rangle} \to \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \tag{B.1}$$

called *local index*, satisfying the following properties:

- its restriction to every subspace
$$(\mathrm{Z}^{d_1}_{Z_1}(X)_{\mathbb{C}} \cap \mathrm{Z}^{d_1}(X)^{\langle \ell \rangle}_{\mathbb{C}}) \times (\mathrm{Z}^{d_2}_{Z_2}(X)_{\mathbb{C}} \cap \mathrm{Z}^{d_2}(X)^{\langle \ell \rangle}_{\mathbb{C}})$$
is complex linear in the first variable;
- $\langle\ ,\ \rangle^{\ell}_{X,K}$ is conjugate symmetric.

We briefly recall the definition. Take a pair $(c_1, c_2) \in \mathrm{Z}^{d_1,d_2}(X)^{\langle \ell \rangle}$. By linearity, we may assume $c_1 \in \mathrm{Z}^{d_1}_{Z_1}(X)$ and $c_2 \in \mathrm{Z}^{d_2}_{Z_2}(X)$ with $Z_1 \cap Z_2 = \emptyset$. For $i = 1, 2$, put $U_i \coloneqq X \setminus Z_i$. Then we have the refined cycle class $\mathrm{cl}^{Z_i}_{X,\ell}(c_i) \in \mathrm{H}^{2d_i}_{Z_i}(X, \mathbb{Q}_{\ell}(d_i))$, which goes to 0 under the natural map $\mathrm{H}^{2d_i}_{Z_i}(X, \mathbb{Q}_{\ell}(d_i)) \to \mathrm{H}^{2d_i}(X, \mathbb{Q}_{\ell}(d_i))$. Thus, we can choose a class $\gamma_i \in \mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_{\ell}(d_i))$ that goes to $\mathrm{cl}^{Z_i}_{X,\ell}(c_i)$ under the coboundary map $\mathrm{H}^{2d_i-1}(U_i, \mathbb{Q}_{\ell}(d_i)) \to \mathrm{H}^{2d_i}_{Z_i}(X, \mathbb{Q}_{\ell}(d_i))$. Then we define $\langle c_1, c_2 \rangle^{\ell}_{X,K}$ to be the image of $\gamma_1 \cup \gamma_2$ under the composite map

$$\mathrm{H}^{2n-2}(U_1 \cap U_2, \mathbb{Q}_{\ell}(n)) \to \mathrm{H}^{2n-1}(X, \mathbb{Q}_{\ell}(n)) \xrightarrow{\mathrm{Tr}_X} \mathrm{H}^1(\operatorname{Spec} K, \mathbb{Q}_{\ell}(1)) = \mathbb{Q}_{\ell},$$

in which the first map is the coboundary in the Mayer–Vietoris exact sequence for the covering $X = U_1 \cup U_2$. It is easy to check that $\langle c_1, c_2 \rangle^{\ell}_{X,K}$ does not depend on the choices of γ_1, γ_2, and that $\langle c_1, c_2 \rangle^{\ell}_{X,K} = \langle c_2, c_1 \rangle^{\ell}_{X,K}$.

Lemma B.3. *Take a pair* $(c_1, c_2) \in \mathrm{Z}^{d_1,d_2}(X)^{\langle \ell \rangle}$.

(1) *Let* K' *be a finite extension of* K. *Put* $X' \coloneqq X \otimes_K K'$ *regarded as a scheme over* K'. *Then we have* $(c_1', c_2') \in \mathrm{Z}^{d_1,d_2}(X')^{\langle \ell \rangle}$ *and* $\langle c_1', c_2' \rangle^{\ell}_{X',K'} = \langle c_1, c_2 \rangle^{\ell}_{X,K}$, *where* c_i' *is the restriction of* c_i *on* X' *for* $i = 1, 2$.

(2) *Let* $u \colon X' \to X$ *be a finite étale morphism. Then we have* $(c_1', c_2') \in \mathrm{Z}^{d_1,d_2}(X')^{\langle \ell \rangle}$ *and* $\langle c_1', c_2' \rangle^{\ell}_{X',K} = \deg u \cdot \langle c_1, c_2 \rangle^{\ell}_{X,K}$, *where* c_i' *is the restriction of* c_i *on* X' *for* $i = 1, 2$.

Proof. In both statements, it is clear that $(c_1', c_2') \in \mathrm{Z}^{d_1,d_2}(X')^{\langle \ell \rangle}$.

Part (1) follows from the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^{2n-2}(U_1 \cap U_2, \mathbb{Q}_\ell(n)) & \longrightarrow & \mathrm{H}^{2n-1}(X, \mathbb{Q}_\ell(n)) & \xrightarrow{\mathrm{Tr}_X} & \mathrm{H}^1(\operatorname{Spec} K, \mathbb{Q}_\ell(1)) & = & \mathbb{Q}_\ell \\
\downarrow & & \downarrow & & \downarrow & & \| \\
\mathrm{H}^{2n-2}(U_1' \cap U_2', \mathbb{Q}_\ell(n)) & \longrightarrow & \mathrm{H}^{2n-1}(X', \mathbb{Q}_\ell(n)) & \xrightarrow{\mathrm{Tr}_{X'}} & \mathrm{H}^1(\operatorname{Spec} K', \mathbb{Q}_\ell(1)) & = & \mathbb{Q}_\ell,
\end{array}$$

in which U_i' is the restriction of U_i on X' and the construction of the local index.

Part (2) follows from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^{2n-2}(U_1 \cap U_2, \mathbb{Q}_\ell(n)) & \longrightarrow & \mathrm{H}^{2n-1}(X, \mathbb{Q}_\ell(n)) & \xrightarrow{\mathrm{Tr}_X} & \mathrm{H}^1(\operatorname{Spec} K, \mathbb{Q}_\ell(1)) \\
\downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle \deg u \cdot \mathrm{id}} \\
\mathrm{H}^{2n-2}(U_1' \cap U_2', \mathbb{Q}_\ell(n)) & \longrightarrow & \mathrm{H}^{2n-1}(X', \mathbb{Q}_\ell(n)) & \xrightarrow{\mathrm{Tr}_{X'}} & \mathrm{H}^1(\operatorname{Spec} K, \mathbb{Q}_\ell(1)),
\end{array}$$

in which U_i' is the restriction of U_i on X' and the construction of the local index. □

In what follows, $\pi\colon \mathcal{X} \to \operatorname{Spec} O_K$ is a projective morphism such that $\mathcal{X} \otimes_{O_K} K = X$. We put $Y \coloneqq \mathcal{X} \otimes_{O_K} k$.

Lemma B.4. *Consider elements $c_1 \in \mathrm{Z}^{d_1}_{Z_1}(X)$ and $c_2 \in \mathrm{Z}^{d_2}_{Z_2}(X)$ with $Z_1 \cap Z_2 = \emptyset$. For every $\beta_1 \in \mathrm{H}^{2d_1}_{Y \cup Z_1}(\mathcal{X}, \mathbb{Q}_\ell(d_1))$ whose image in $\mathrm{H}^{2d_1}_{Z_1}(X, \mathbb{Q}_\ell(d_1))$ coincides with $\mathrm{cl}^{Z_1}_{X,\ell}(c_1)$ and whose image in $\mathrm{H}^{2d_1}(\mathcal{X}, \mathbb{Q}_\ell(d_1))$ vanishes, and for every $\beta_2 \in \mathrm{H}^{2d_2-2(n-1)}_{Y \cup Z_2}(\mathcal{X}, \pi^! \mathbb{Q}_\ell(d_2 - n + 1))$ whose image in*

$$\mathrm{H}^{2d_2-2(n-1)}_{Z_2}(X, \pi^! \mathbb{Q}_\ell(d_2 - n + 1)) = \mathrm{H}^{2d_2}_{Z_2}(X, \mathbb{Q}_\ell(d_2))$$

coincides with $\mathrm{cl}^{Z_2}_{X,\ell}(c_2)$, the image of $\beta_1 \cup \beta_2 \in \mathrm{H}^2_Y(\mathcal{X}, \pi^! \mathbb{Q}_\ell(1))$ under the trace map $\mathrm{H}^2_Y(\mathcal{X}, \pi^! \mathbb{Q}_\ell(1)) \to \mathrm{H}^2_{\operatorname{Spec} k}(\operatorname{Spec} O_K, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$ coincides with $\langle c_1, c_2 \rangle^\ell_{X,K}$.

This is claimed in [Beĭ87, Lemma-definition 2.1.1] without proof. For completeness, we include a proof here (though it is straightforward).

Proof. Before the proof, let us make a remark on cup products. Let S be a Noetherian scheme on which ℓ is invertible. Given $F, G, H \in \mathrm{D}^b(S, \mathbb{Q}_\ell)$, the bounded derived category of ℓ-adic sheaves on S, together with a map

$$\kappa\colon F \overset{\mathbb{L}}{\otimes} G \to H,$$

we have a cup product map

$$\cup_\kappa\colon \mathrm{H}^i(S, F) \times \mathrm{H}^j(S, G) \to \mathrm{H}^{i+j}(S, H)$$

for integers i and j, which is the composition of the cup product for (hyper)cohomology

$$\mathrm{H}^i(S,F)\times\mathrm{H}^j(S,G)\to\mathrm{H}^{i+j}(S,F\overset{\mathbb{L}}{\otimes}G)$$

and the induced map $\mathrm{H}^{i+j}(S,\kappa)\colon \mathrm{H}^{i+j}(S,F\overset{\mathbb{L}}{\otimes}G)\to\mathrm{H}^{i+j}(S,H)$. In particular, if we have maps $f\colon F\to F'$ and $h\colon H\to H'$ rendering the diagram

$$\begin{array}{ccc} F\overset{\mathbb{L}}{\otimes}G & \xrightarrow{\kappa} & H \\ {\scriptstyle f\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle h} \\ F'\overset{\mathbb{L}}{\otimes}G & \xrightarrow{\kappa'} & H' \end{array}$$

commutative (in $\mathrm{D}^b(S,\mathbb{Q}_\ell)$), then the induced diagram

$$\begin{array}{ccc} \mathrm{H}^i(S,F)\times\mathrm{H}^j(S,G) & \xrightarrow{\cup_\kappa} & \mathrm{H}^{i+j}(S,H) \\ {\scriptstyle \mathrm{H}^i(S,f)\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{H}^{i+j}(S,h)} \\ \mathrm{H}^i(S,F')\times\mathrm{H}^j(S,G) & \xrightarrow{\cup_{\kappa'}} & \mathrm{H}^{i+j}(S,H') \end{array}$$

commutes.

Put $U_i\coloneqq X\setminus Z_i$ for $i=1,2$ as before. For $i=1,2$, choose a class $\gamma_i\in\mathrm{H}^{2d_i-1}(U_i,\mathbb{Q}_\ell(d_i))$ that goes to $\mathrm{cl}^{Z_i}_{X,\ell}(c_i)$ under the coboundary map

$$\mathrm{H}^{2d_i-1}(U_i,\mathbb{Q}_\ell(d_i))\to\mathrm{H}^{2d_i}_{Z_i}(X,\mathbb{Q}_\ell(d_i)).$$

Denote by $\langle\gamma_1,c_2\rangle$, the image of

$$\gamma_1\cup\mathrm{cl}^{Z_2}_{X,\ell}(c_2)\in\mathrm{H}^{2n-1}_{Z_2}(U_1,\mathbb{Q}_\ell(n))=\mathrm{H}^{2n-1}_{Z_2}(X,\mathbb{Q}_\ell(n))$$

under the composite map

$$\mathrm{H}^{2n-1}_{Z_2}(X,\mathbb{Q}_\ell(n))\to\mathrm{H}^{2n-1}(X,\mathbb{Q}_\ell(n))\xrightarrow{\mathrm{Tr}_X}\mathrm{H}^1(\mathrm{Spec}\,K,\mathbb{Q}_\ell(1))=\mathbb{Q}_\ell.$$

We break the proof into two steps:

(1) $\langle\gamma_1,c_2\rangle=\langle c_1,c_2\rangle^\ell_{X,K}$;

(2) the image of $\beta_1\cup\beta_2\in\mathrm{H}^2_Y(\mathcal{X},\pi^!\mathbb{Q}_\ell(1))$ under the trace map

$$\mathrm{H}^2_Y(\mathcal{X},\pi^!\mathbb{Q}_\ell(1))\to\mathbb{Q}_\ell$$

coincides with $\langle\gamma_1,c_2\rangle$.

For (1), it is easy to see that the coboundary map $\mathrm{H}^{2n-2}(U_1\cap U_2,\mathbb{Q}_\ell(n))\to\mathrm{H}^{2n-1}(X,\mathbb{Q}_\ell(n))$ in the Mayer–Vietoris exact sequence is the composition of the coboundary map $\delta\colon\mathrm{H}^{2n-2}(U_1\cap U_2,\mathbb{Q}_\ell(n))\to\mathrm{H}^{2n-1}_{Z_2}(U_1,\mathbb{Q}_\ell(n))$ in the Gysin sequence and the natural map $\mathrm{H}^{2n-1}_{Z_2}(U_1,\mathbb{Q}_\ell(n))=\mathrm{H}^{2n-1}_{Z_2}(X,\mathbb{Q}_\ell(n))\to$

$\mathrm{H}^{2n-1}(X, \mathbb{Q}_\ell(n))$. Thus, it suffices to show that the following diagram commutes:

(B.2)

$$\begin{array}{ccc} \mathrm{H}^{2d_1-1}(U_1, \mathbb{Q}_\ell(d_1)) \times \mathrm{H}^{2d_2-1}(U_2, \mathbb{Q}_\ell(d_2)) & \longrightarrow & \mathrm{H}^{2n-2}(U_1 \cap U_2, \mathbb{Q}_\ell(n)) \\ \downarrow{\scriptstyle \mathrm{id}\times\delta} & & \downarrow{\scriptstyle \delta} \\ \mathrm{H}^{2d_1-1}(U_1, \mathbb{Q}_\ell(d_1)) \times \mathrm{H}^{2d_2}_{Z_2}(X, \mathbb{Q}_\ell(d_2)) & \longrightarrow & \mathrm{H}^{2n-1}_{Z_2}(U_1, \mathbb{Q}_\ell(n)). \end{array}$$

Denote morphisms $\iota_i \colon U_i \to X$ and $\jmath_i \colon Z_i \to X$ for $i = 1, 2$, and $\iota \colon U_1 \cap U_2 \to X$. Then in view of the remark on the cup products, the first row of (B.2) is induced by the natural map

$$\iota_{1*}\iota_1^*\mathbb{Q}_\ell(d_1) \overset{\mathbb{L}}{\otimes} \iota_{2*}\iota_2^*\mathbb{Q}_\ell(d_2) \to \iota_*\iota^*\mathbb{Q}_\ell(n),$$

and the second row of (B.2) is induced by the map

$$\iota_{1*}\iota_1^*\mathbb{Q}_\ell(d_1) \overset{\mathbb{L}}{\otimes} \jmath_{2!}\jmath_2^!\mathbb{Q}_\ell(d_2) \to \jmath_{2!}\jmath_2^!\iota_1^*\mathbb{Q}_\ell(n),$$

which is the cone (of columns) of the natural commutative diagram

$$\begin{array}{ccc} \iota_{1*}\iota_1^*\mathbb{Q}_\ell(d_1) \overset{\mathbb{L}}{\otimes} \mathbb{Q}_\ell(d_2) & \longrightarrow & \iota_{1*}\iota_1^*\mathbb{Q}_\ell(n) \\ \downarrow & & \downarrow \\ \iota_{1*}\iota_1^*\mathbb{Q}_\ell(d_1) \overset{\mathbb{L}}{\otimes} \iota_{2*}\iota_2^*\mathbb{Q}_\ell(d_2) & \longrightarrow & \iota_*\iota^*\mathbb{Q}_\ell(n) \end{array}$$

in $\mathrm{D}^b(X, \mathbb{Q}_\ell)$. It follows that (B.2) commutes. In particular, $\langle c_1, c_2\rangle^\ell_{X,K}$ does not depend on the choices of γ_1 and γ_2, which justifies the notation.

For (2), we may assume that β_1 is the coboundary of γ_1. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(X, \pi^!\mathbb{Q}_\ell(1)) & \xrightarrow{\mathrm{Tr}_X} & \mathrm{H}^1(\operatorname{Spec} K, \mathbb{Q}_\ell(1)) \\ \downarrow & & \downarrow \\ \mathrm{H}^2_Y(\mathcal{X}, \pi^!\mathbb{Q}_\ell(1)) & \xrightarrow{\mathrm{Tr}_{\mathcal{X}}} & \mathrm{H}^2_{\operatorname{Spec} k}(\operatorname{Spec} O_K, \mathbb{Q}_\ell(1)) \end{array}$$

for the trace maps. Thus, as $\mathrm{H}^1(X, \pi^!\mathbb{Q}_\ell(1)) \simeq \mathrm{H}^{2n-1}(X, \mathbb{Q}_\ell(n))$, it remains to show that the diagram

$$\begin{array}{ccc} \mathrm{H}^{2d_1-1}(U_1, \mathbb{Q}_\ell(d_1)) \times \mathrm{H}^{2d_2-2(n-1)}_{Y\cup Z_2}(\mathcal{X}, \pi^!\mathbb{Q}_\ell(d_2-n+1)) & \longrightarrow & \mathrm{H}^1_{Z_2}(U_1, \pi^!\mathbb{Q}_\ell(1)) \\ \downarrow{\scriptstyle \delta\times\mathrm{id}} & & \downarrow{\scriptstyle \delta} \\ \mathrm{H}^{2d_1}_{Y\cup Z_1}(\mathcal{X}, \mathbb{Q}_\ell(d_1)) \times \mathrm{H}^{2d_2-2(n-1)}_{Y\cup Z_2}(\mathcal{X}, \pi^!\mathbb{Q}_\ell(d_2-n+1)) & \longrightarrow & \mathrm{H}^2_Y(\mathcal{X}, \pi^!\mathbb{Q}_\ell(1)) \end{array}$$

commutes. The argument is similar to (1), which we leave to the readers.

The lemma is proved. □

Remark B.5. In Lemma B.4, when $\mathcal{X}$ is regular, the natural map

$$\pi^!\mathbb{Q}_\ell[2-2n](1-n) \to \mathbb{Q}_\ell$$

is an isomorphism, which is a consequence of the absolute purity theorem [Fuj02].

Now we provide a refined method to compute (B.1) in the presence of a regular model of X. Until the end of this section, $\mathcal{X}$ will be regular.

We first review some constructions from [GS87]. For every Zariski closed subset $\mathcal{Z}$ of $\mathcal{X}$, we have the K-group $\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X})$ of complexes with support in $\mathcal{Z}$ defined in [GS87, §1.1], equipped with the codimension filtration

$$\cdots \supset \mathrm{F}^{d-1}\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \supset \mathrm{F}^{d}\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \supset \mathrm{F}^{d+1}\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \supset \cdots .$$

We have

- the pushforward map $\pi_*\colon \mathrm{K}_0^{Y}(\mathcal{X}) \to \mathrm{K}_0^{\operatorname{Spec} k}(\operatorname{Spec} O_K) = \mathrm{K}_0(\operatorname{Spec} k)$;
- for $\mathcal{Z}' \subseteq \mathcal{Z}$, a natural linear map $\mathrm{K}_0^{\mathcal{Z}'}(\mathcal{X}) \to \mathrm{K}_0^{\mathcal{Z}}(\mathcal{X})$, which preserves the codimension filtration;
- a cup-product map $\cup\colon \mathrm{K}_0^{\mathcal{Z}_1}(\mathcal{X}) \times \mathrm{K}_0^{\mathcal{Z}_2}(\mathcal{X}) \to \mathrm{K}_0^{\mathcal{Z}_1\cap\mathcal{Z}_2}(\mathcal{X})$;
- a natural linear map

$$[\ \ ]\colon \bigoplus_{d'\geqslant d} \mathrm{Z}_{\mathcal{Z}}^{d'}(\mathcal{X}) \to \mathrm{F}^{d}\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \tag{B.3}$$

  sending a closed subscheme $\mathcal{Z}'$ of $\mathcal{X}$ contained in $\mathcal{Z}$ to the class of the structure sheaf $\mathscr{O}_{\mathcal{Z}'}$.

See [GS87, §§1, 5] for more details.

Note that since $\mathcal{X}$ is regular, $\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X})$ coincides with Quillen's K-theory with support. (See the proof of [GS87, Th. 8.2].) Then by [Gil81, Def. 2.34(ii)] in which we take the base scheme S to be $\operatorname{Spec} O_K$ and Γ to be the ℓ-adic cohomology theory, we obtain the d-th Chern class map

$$\mathrm{cl}^{\mathcal{Z}}_{\mathcal{X},\ell}\colon \mathrm{F}^{d}\mathrm{K}_0^{\mathcal{Z}}(\mathcal{X}) \to \mathrm{H}^{2d}_{\mathcal{Z}}(\mathcal{X}, \mathbb{Q}_\ell(d))$$

for every integer $d \geqslant 0$.

For the generic fiber X, we have K-groups $\mathrm{K}_0^{Z}(X)$ for Zariski closed subsets Z of X with similar properties as well. The following lemma is probably known, but we cannot find an exact reference.

Lemma B.6. *For every $c \in \mathrm{Z}^d_Z(X)$, the element $\mathrm{cl}^{Z}_{X,\ell}([c]) \in \mathrm{H}^{2d}_Z(X, \mathbb{Q}_\ell(d))$ coincides with the refined cycle class $\mathrm{cl}^{Z}_{X,\ell}(c)$ of c.*

Proof. We may assume Z irreducible and $c = Z$. Let Z' be the smooth locus of Z over K and put $X' \coloneqq X\setminus(Z\setminus Z')$. As a consequence of the semi-purity theorem [Fuj02, §8], the restriction map $\mathrm{H}^{2d}_Z(X, \mathbb{Q}_\ell(d)) \to \mathrm{H}^{2d}_{Z'}(X', \mathbb{Q}_\ell(d))$ is an

isomorphism. Thus, we may assume Z smooth over K as well. Then the lemma follows from [Gil81, Th. 3.1] (with $S = \operatorname{Spec} K$ and $k = q = 0$). □

Definition B.7. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two Zariski closed subsets $\mathcal{Z}$ of $\mathcal{X}$ satisfying $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subseteq Y$. We define a pairing

$$\begin{aligned} \mathrm{K}_0^{\mathcal{Z}_1}(\mathcal{X})_{\mathbb{C}} \times \mathrm{K}_0^{\mathcal{Z}_2}(\mathcal{X})_{\mathbb{C}} &\to \mathbb{C} \\ (\mathcal{C}_1, \mathcal{C}_2) &\mapsto \mathcal{C}_1.\mathcal{C}_2 \end{aligned}$$

that is complex linear in the first variable, conjugate complex linear in the second variable, and such that for $\mathcal{C}_i \in \mathrm{K}_0^{\mathcal{Z}_i}(\mathcal{X})$ with $i = 1, 2$, we have

$$\mathcal{C}_1.\mathcal{C}_2 \coloneqq \chi\left(\pi_*(\mathcal{C}_1 \cup \mathcal{C}_2)\right),$$

where χ denotes the Euler characteristic function on $\mathrm{K}_0(\operatorname{Spec} k)$. Note that as $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subseteq Y$, $\mathcal{C}_1 \cup \mathcal{C}_2$ can be regarded as element in $\mathrm{K}_0^Y(\mathcal{X})_{\mathbb{C}}$.

Lemma B.8. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two Zariski closed subsets of $\mathcal{X}$ satisfying $\mathcal{Z}_1 \cap \mathcal{Z}_2 \subseteq Y$. For $\mathcal{C}_i \in \mathrm{F}^{d_i}\mathrm{K}_0^{\mathcal{Z}_i}(\mathcal{X})$ with $i = 1, 2$, $\mathcal{C}_1.\mathcal{C}_2$ coincides with the image of $\mathrm{cl}_{\mathcal{X},\ell}^{\mathcal{Z}_1}(\mathcal{C}_1) \cup \mathrm{cl}_{\mathcal{X},\ell}^{\mathcal{Z}_2}(\mathcal{C}_2) \in \mathrm{H}^{2n}_{\mathcal{Z}_1\cap\mathcal{Z}_2}(\mathcal{X}, \mathbb{Q}_\ell(n))$ under the natural composite map*

$$\begin{aligned} \mathrm{H}^{2n}_{\mathcal{Z}_1\cap\mathcal{Z}_2}(\mathcal{X}, \mathbb{Q}_\ell(n)) &\to \mathrm{H}^{2n}_Y(\mathcal{X}, \mathbb{Q}_\ell(n)) \\ &\xrightarrow{\pi_*} \mathrm{H}^2_{\operatorname{Spec} k}(\operatorname{Spec} O_K, \mathbb{Q}_\ell(1)) = \mathrm{H}^0(\operatorname{Spec} k, \mathbb{Q}_\ell) = \mathbb{Q}_\ell. \end{aligned}$$

Proof. By [GS87, Prop. 5.5], we have $\mathcal{C}_1 \cup \mathcal{C}_2 \in \mathrm{F}^n\mathrm{K}_0^{\mathcal{Z}_1\cap\mathcal{Z}_2}(\mathcal{X})_{\mathbb{Q}}$. By (the same proof of) [Gil81, Prop. 2.35], $\mathrm{cl}_{\mathcal{X},\ell}^{\mathcal{Z}_1}(\mathcal{C}_1) \cup \mathrm{cl}_{\mathcal{X},\ell}^{\mathcal{Z}_2}(\mathcal{C}_2)$ and $\mathrm{cl}_{\mathcal{X},\ell}^{\mathcal{Z}_1\cap\mathcal{Z}_2}(\mathcal{C}_1 \cup \mathcal{C}_2)$ have the same image in $\mathrm{H}^{2n}_{\mathcal{Z}_1\cap\mathcal{Z}_2}(\mathcal{X}, \mathbb{Q}_\ell(n))$.[22] Since the map $\mathrm{Z}^n(\mathcal{X})_{\mathbb{Q}} \to \mathrm{F}^n\mathrm{K}_0^Y(\mathcal{X})_{\mathbb{Q}}$ is surjective, the diagram

$$\begin{array}{ccc} \mathrm{F}^n\mathrm{K}_0^Y(\mathcal{X})_{\mathbb{Q}} & \xrightarrow{\mathrm{cl}_{\mathcal{X},\ell}^Y} & \mathrm{H}^{2n}_Y(\mathcal{X}, \mathbb{Q}_\ell(n)) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{F}^1\mathrm{K}_0^{\operatorname{Spec} k}(\operatorname{Spec} O_K)_{\mathbb{Q}} & \xrightarrow{\mathrm{cl}^{\operatorname{Spec} k}_{\operatorname{Spec} O_K,\ell}} & \mathrm{H}^2_{\operatorname{Spec} k}(\operatorname{Spec} O_K, \mathbb{Q}_\ell(1)) \end{array}$$

[22] Although [Gil81, Prop. 2.35] only implies the statement when $\mathcal{Z}_1 = \mathcal{Z}_2$, its proof works more generally. In fact, in the proof of [Gil81, Prop. 2.35], if $\mathcal{C}_i$ is represented by a map $[\alpha_i] \colon \mathscr{S}^0_{\mathcal{Z}_i} \to \Omega\mathrm{B}\mathscr{Q}\mathscr{P}_X$, then the product $\mathcal{C}_1 \cup \mathcal{C}_2$ is represented by the composite map

$$\mathscr{S}^0_{\mathcal{Z}_1\cap\mathcal{Z}_2} = \mathscr{S}^0_{\mathcal{Z}_1} \wedge \mathscr{S}^0_{\mathcal{Z}_2} \to \Omega\mathrm{B}\mathscr{Q}\mathscr{P}_X \wedge \Omega\mathrm{B}\mathscr{Q}\mathscr{P}_X \xrightarrow{\mu} \Omega\mathrm{B}\mathscr{Q}\mathscr{P}_X;$$

the remaining argument is same.

commutes. Thus, the proposition follows since the diagram

$$\begin{array}{ccc}
\mathrm{F}^1\mathrm{K}_0^{\mathrm{Spec}\,k}(\mathrm{Spec}\,O_K)_{\mathbb{Q}} & \xrightarrow{\mathrm{cl}^{\mathrm{Spec}\,k}_{\mathrm{Spec}\,O_K,\ell}} & \mathrm{H}^2_{\mathrm{Spec}\,k}(\mathrm{Spec}\,O_K,\mathbb{Q}_\ell(1)) \\
= \downarrow & & \downarrow = \\
\mathrm{K}_0(\mathrm{Spec}\,k)_{\mathbb{Q}} & \xrightarrow{\mathrm{cl}_{\mathrm{Spec}\,k,\ell}} & \mathrm{H}^0(\mathrm{Spec}\,k,\mathbb{Q}_\ell) \\
\chi \downarrow & & \downarrow = \\
\mathbb{Q} & \longrightarrow & \mathbb{Q}_\ell
\end{array}$$

commutes. □

Definition B.9. For an element $c \in \mathrm{Z}^d(X)_{\mathbb{C}}$, we say that an element $\mathcal{C} \in \mathrm{F}^d\mathrm{K}_0^{Y\cup\mathrm{supp}(c)}(\mathcal{X})_{\mathbb{C}}$ is an *extension* of c if $\mathcal{C}|_X \in \mathrm{F}^d\mathrm{K}_0(X)_{\mathbb{C}}$ coincides with $[c]$ under the map (B.3), and that $\mathcal{C}$ is an *ℓ-flat extension* if the image of $\mathrm{cl}^{Y\cup\mathrm{supp}(c)}_{\mathcal{X},\ell}(\mathcal{C})$ in $\mathrm{H}^{2d}(\mathcal{X},\mathbb{Q}_\ell(d))\otimes_{\mathbb{Q}}\mathbb{C}$ vanishes.

Proposition B.10. *Consider a pair $(c_1,c_2) \in \mathrm{Z}^{d_1,d_2}(X)^{\langle\ell\rangle}$ satisfying $\mathrm{supp}(c_i)\cap\mathrm{supp}(c_2)=\emptyset$. If $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are two Zariski closed subsets of $\mathcal{X}$ satisfying $\mathcal{Z}_1\cap\mathcal{Z}_2\subseteq Y$, and $\mathcal{C}_i \in \mathrm{F}^{d_i}\mathrm{K}_0^{\mathcal{Z}_i}(\mathcal{X})_{\mathbb{C}}$ is an extension of c_i for $i=1,2$ in which at least one is ℓ-flat, then we have*

$$\langle c_1,c_2\rangle^{\ell}_{X,K} = \mathcal{C}_1.\mathcal{C}_2.$$

In particular, when π is smooth, we can take $\mathcal{C}_i$ to be the one given by the Zariski closure of c_i in $\mathcal{X}$ via (B.3), hence $\langle c_1,c_2\rangle^{\ell}_{X,K}$ belongs to $\mathbb{C}$ and is independent of ℓ.

Proof. By Lemma B.6, for $i=1,2$, the image in the cycle class of $\mathrm{cl}^{\mathcal{Z}_i}_{\mathcal{X},\ell}(\mathcal{C}_i)$ in $\mathrm{H}^{2d_i}_{\mathcal{Z}_i\cap X}(X,\mathbb{Q}_\ell(d_i))\otimes_{\mathbb{Q}}\mathbb{C}$ coincides with $\mathrm{cl}^{\mathcal{Z}_i\cap X}_{X,\ell}(c_i)$. Without lost of generality, we assume that $\mathcal{C}_1$ is ℓ-flat. Then the lemma follows from Lemma B.8, and Lemma B.4 with $\beta_i := \mathrm{cl}^{\mathcal{Z}_i}_{\mathcal{X},\ell}(\mathcal{C}_i)$ for $i=1,2$ (together with Remark B.5). □

Remark B.11. Suppose X admits smooth projective reduction over O_K. Then the source of (B.1) is independent of ℓ. Moreover, by Proposition B.10, the map (B.1) takes value in $\mathbb{C}$ and is independent of ℓ. Thus, in this case, (B.1) makes sense for an arbitrary rational prime ℓ of which it is independent.

In the remaining discussion, we only consider the case where $n=2r$ for some integer $r\geqslant 1$, and $d_1=d_2=r$. We say that a correspondence

$$t\colon \mathcal{X} \xleftarrow{p} \mathcal{X}' \xrightarrow{q} \mathcal{X}$$

of $\mathcal{X}$ is *étale* if both p and q are finite étale. In what follows, we take a subfield $\mathbb{L}$ of $\mathbb{C}$.

Definition B.12. We say that an $\mathbb{L}$-étale correspondence t, that is, an $\mathbb{L}$-linear combination of étale correspondences, of $\mathcal{X}$ is *ℓ-tempered* if t^* annihilates $\mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$.

Proposition B.13. *Let t be an ℓ-tempered $\mathbb{L}$-étale correspondence of $\mathcal{X}$. Then for every pair $(c_1, c_2) \in \mathrm{Z}^r(X)_{\mathbb{C}} \times \mathrm{Z}^r(X)_{\mathbb{C}}$ satisfying $\mathrm{supp}(t^*c_1) \cap \mathrm{supp}(t^*c_2) = \emptyset$, we have $(t^*c_1, t^*c_2) \in \mathrm{Z}^{r,r}(X)^{\langle \ell \rangle}$ and*

$$\langle t^*c_1, t^*c_2 \rangle^{\ell}_{X,K} = t^*\mathcal{C}_1 . t^*\mathcal{C}_2,$$

where $\mathcal{C}_i \in \mathrm{F}^r\mathrm{K}_0^{Y \cup \mathrm{supp}(c_i)}(\mathcal{X})_{\mathbb{C}}$ is an arbitrary extension of c_i in $\mathcal{X}$ for $i = 1, 2$. In particular, we have $\langle t^*c_1, t^*c_2 \rangle^{\ell}_{X,K} \in \mathbb{C}$.

Proof. For $i = 1, 2$, put $Z_i \coloneqq \mathrm{supp}(c_i)$, $Z_i^t \coloneqq \mathrm{supp}(t^*c_i)$, and

$$\beta_i \coloneqq \mathrm{cl}^{Y \cup Z_i}_{\mathcal{X},\ell}(\mathcal{C}_i) \in \mathrm{H}^{2r}_{Y \cup Z_i}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Note that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{2r}_{Y \cup Z_i}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L} & \longrightarrow & \mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L} \\ {\scriptstyle t^*}\downarrow & & \downarrow{\scriptstyle t^*} \\ \mathrm{H}^{2r}_{Y \cup Z_i^t}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L} & \longrightarrow & \mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L} \end{array}$$

induced by t. Since t is ℓ-tempered, the image of $t^*\beta_i$ in $\mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{C}$ vanishes. Now for $i = 1, 2$, since $t^*\beta_i = \mathrm{cl}^{Y \cup Z_i^t}_{\mathcal{X},\ell}(t^*\mathcal{C}_i)$, we know that $t^*\mathcal{C}_i$ is an ℓ-flat extension of t^*c_i. In particular, we have $(t^*c_1, t^*c_2) \in \mathrm{Z}^{r,r}(X)^{\langle \ell \rangle}$. Finally, the formula for $\langle t^*c_1, t^*c_2 \rangle^{\ell}_{X,K}$ follows from Proposition B.10. □

Now we provide a criterion for an $\mathbb{L}$-étale correspondence to be ℓ-tempered.

Proposition B.14. *Put $Y_0 \coloneqq Y^{\mathrm{red}}$, the induced reduced subscheme of Y. Suppose that we have a finite stratification $Y_0 \supset Y_1 \supset \cdots$ of Zariski closed subsets such that $Y_j^\circ \coloneqq Y_j \setminus Y_{j+1}$ is regular and has pure codimension $n_j \geqslant 1$ in $\mathcal{X}$ for $j \geqslant 0$. If t is an $\mathbb{L}$-étale correspondence of $\mathcal{X}$ stabilizing the stratification $Y_0 \supset Y_1 \supset \cdots$ and such that*

(1) *t^* annihilates $\mathrm{H}^{2r}(X, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$;*
(2) *t^* annihilates $\mathrm{H}^i(Y_j^\circ \otimes_k \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} \mathbb{L}$ for every integer $i \leqslant 2r - 2n_j$ and every j,*

then some positive power of t annihilates $\mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$.

Proof. It suffices to prove that $(t^m)^*$ annihilates $\mathrm{H}^{2r}_Y(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$ for some integer $m \geqslant 1$, since then t^{m+1} is ℓ-tempered.

We prove by decreasing induction on j that $(t^{m_j})^*$ annihilates the space $\mathrm{H}^{2r}_{Y_j}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$ for some integer $m_j \geqslant 1$. We have

$$\mathrm{H}^{2r}_{Y_{j+1}}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L} \to \mathrm{H}^{2r}_{Y_j}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L} \to \mathrm{H}^{2r}_{Y_j^\circ}(\mathcal{X} \setminus Y_{j+1}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}.$$

As Y_j° is a regular closed subscheme of the regular scheme $\mathcal{X} \setminus Y_{j+1}$, by the absolute purity theorem [Fuj02], we have

$$\mathrm{H}^{2r}_{Y_j^\circ}(\mathcal{X} \setminus Y_{j+1}, \mathbb{Q}_\ell(r)) \simeq \mathrm{H}^{2r-2n_j}(Y_j^\circ, \mathbb{Q}_\ell(r-n_j)).$$

By condition (2) and the Hochschild–Serre spectral sequence, we know that $(t^2)^*$ annihilates $\mathrm{H}^{2r-2n_j}(Y_j^\circ, \mathbb{Q}_\ell(r-n_j)) \otimes_{\mathbb{Q}} \mathbb{L}$. Thus, we may take $m_j = m_{j+1} + 2$. In particular, $(t^{m_0})^*$ annihilates $\mathrm{H}^{2r}_{Y_0}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$, which is same as $\mathrm{H}^{2r}_{Y}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$. □

Corollary B.15. *Let $\mathcal{X}$ and $\mathbb{L}$ be as above. Let $\mathbb{S}$ be a ring of étale correspondences of $\mathcal{X}$, and $\mathfrak{m}$ a maximal ideal of $\mathbb{S}_{\mathbb{L}}$.*

(1) *If $(\mathrm{H}^{2d}(\mathcal{X}, \mathbb{Q}_\ell(d)) \otimes_{\mathbb{Q}} \mathbb{L})_{\mathfrak{m}} = 0$, then there exists an ℓ-tempered element in $\mathbb{S}_{\mathbb{L}} \setminus \mathfrak{m}$.*

(2) *Suppose that we have a finite stratification $Y_0 \supset Y_1 \supset \cdots$ of Zariski closed subsets that is stabilized by the action of $\mathbb{S}$, such that $Y_j^\circ \coloneqq Y_j \setminus Y_{j+1}$ is regular and has pure codimension $n_j \geqslant 1$ in $\mathcal{X}$ for $j \geqslant 0$. If*
   - *$(\mathrm{H}^{2r}(X, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L})_{\mathfrak{m}} = 0$ and*
   - *$(\mathrm{H}^{i}(Y_j^\circ \otimes_k \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} \mathbb{L})_{\mathfrak{m}} = 0$ for every integer $i \leqslant 2r - 2n_j$ and every j,*

   then $(\mathrm{H}^{2d}(\mathcal{X}, \mathbb{Q}_\ell(d)) \otimes_{\mathbb{Q}} \mathbb{L})_{\mathfrak{m}} = 0$.

Proof. For (1), since $\mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r))$ is of finite dimension over $\mathbb{Q}_\ell$, it follows that $\mathrm{H}^{2r}(\mathcal{X}, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$ is a finitely generated $\mathbb{S}_{\mathbb{L}}$-module. Then (1) follows from Definition B.12.

For (2), since both $\mathrm{H}^{2r}(X, \mathbb{Q}_\ell(r))$ and $\bigoplus_{i<2r-1} \bigoplus_j \mathrm{H}^i(Y_j^\circ \otimes_k \overline{\mathbb{F}}_p, \mathbb{Q}_\ell)$ are of finite dimension over $\mathbb{Q}_\ell$, it follows that both $\mathrm{H}^{2r}(X, \mathbb{Q}_\ell(r)) \otimes_{\mathbb{Q}} \mathbb{L}$ and $\bigoplus_{i<2r-1} \bigoplus_j \mathrm{H}^i(Y_j^\circ \otimes_k \overline{\mathbb{F}}_p, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} \mathbb{L}$ are finitely generated $\mathbb{S}_{\mathbb{L}}$-modules. Then there exists $t \in \mathbb{S}_{\mathbb{L}} \setminus \mathfrak{m}$ satisfying the two conditions in Proposition B.14. By the same proposition, some power of t annihilates $\mathrm{H}^{2d}(\mathcal{X}, \mathbb{Q}_\ell(d)) \otimes_{\mathbb{Q}} \mathbb{L}$, which implies $(\mathrm{H}^{2d}(\mathcal{X}, \mathbb{Q}_\ell(d)) \otimes_{\mathbb{Q}} \mathbb{L})_{\mathfrak{m}} = 0$. Thus, (2) follows. □

References

[AN04] M. L. Agranovsky and E. K. Narayanan, L^p-integrability, supports of Fourier transforms and uniqueness for convolution equations, *J. Fourier Anal. Appl.* **10** no. 3 (2004), 315–324. MR 2066426. Zbl 1083.42005. https://doi.org/10.1007/s00041-004-0986-4.

[Beĭ87] A. Beĭlinson, Height pairing between algebraic cycles, in *Current Trends in Arithmetical Algebraic Geometry* (Arcata, Calif., 1985), *Contemp. Math.* **67**, Amer. Math. Soc., Providence, RI, 1987, pp. 1–24. MR 0902590. Zbl 0624.14005. https://doi.org/10.1090/conm/067/902590.

[Blo84] S. Bloch, Algebraic cycles and values of L-functions, *J. Reine Angew. Math.* **350** (1984), 94–108. MR 0743535. Zbl 0527.14008. https://doi.org/10.1515/crll.1984.350.94.

[BW00] A. Borel and N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, second ed., *Math. Surv. Monogr.* **67**, Amer. Math. Soc., Providence, RI, 2000. MR 1721403. Zbl 0980.22015. https://doi.org/10.1090/surv/067.

[BWR15] J. H. Bruinier and M. Westerholt-Raum, Kudla's modularity conjecture and formal Fourier-Jacobi series, *Forum Math. Pi* **3** (2015), e7, 30. MR 3406827. Zbl 1376.11032. https://doi.org/10.1017/fmp.2015.6.

[Car12] A. Caraiani, Local-global compatibility and the action of monodromy on nearby cycles, *Duke Math. J.* **161** no. 12 (2012), 2311–2413. MR 2972460. Zbl 1405.22028. https://doi.org/10.1215/00127094-1723706.

[CS17] A. Caraiani and P. Scholze, On the generic part of the cohomology of compact unitary Shimura varieties, *Ann. of Math.* (2) **186** no. 3 (2017), 649–766. MR 3702677. Zbl 1401.11108. https://doi.org/10.4007/annals.2017.186.3.1.

[EL20] E. Eischen and Z. Liu, Archimedean zeta integrals for unitary groups, 2020. arXiv 2006.04302.

[Fuj02] K. Fujiwara, A proof of the absolute purity conjecture (after Gabber), in *Algebraic Geometry* 2000, *Azumino* (Hotaka), *Adv. Stud. Pure Math.* **36**, Math. Soc. Japan, Tokyo, 2002, pp. 153–183. MR 1971516. Zbl 1059.14026. https://doi.org/10.2969/aspm/03610153.

[GGP12] W. T. Gan, B. H. Gross, and D. Prasad, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups, in *Sur les Conjectures de Gross et Prasad. I*, *Astérisque* **346**, Math. Soc. France, Paris, 2012, pp. 1–109. MR 3202556. Zbl 1280.22019.

[GI16] W. T. Gan and A. Ichino, The Gross-Prasad conjecture and local theta correspondence, *Invent. Math.* **206** no. 3 (2016), 705–799. MR 3573972. Zbl 1358.11061. https://doi.org/10.1007/s00222-016-0662-8.

[GT16] W. T. Gan and S. Takeda, A proof of the Howe duality conjecture, *J. Amer. Math. Soc.* **29** no. 2 (2016), 473–493. MR 3454380. Zbl 1342.11051. https://doi.org/10.1090/jams/839.

[GS19] L. E. Garcia and S. Sankaran, Green forms and the arithmetic Siegel-Weil formula, *Invent. Math.* **215** no. 3 (2019), 863–975. MR 3935034. Zbl 1423.14181. https://doi.org/10.1007/s00222-018-0839-4.

[Gil81] H. Gillet, Riemann-Roch theorems for higher algebraic K-theory, *Adv. in Math.* **40** no. 3 (1981), 203–289. MR 0624666. Zbl 0478.14010. https://doi.org/10.1016/S0001-8708(81)80006-0.

[GS87] H. Gillet and C. Soulé, Intersection theory using Adams operations, *Invent. Math.* **90** no. 2 (1987), 243–277. MR 0910201. Zbl 0632.14009. https://doi.org/10.1007/BF01388705.

[GG11] Z. Gong and L. Grenié, An inequality for local unitary theta correspondence, *Ann. Fac. Sci. Toulouse Math.* (6) **20** no. 1 (2011), 167–202. MR 2830396. Zbl 1254.11049. https://doi.org/10.5802/afst.1289.

[Gro97] B. H. Gross, On the motive of a reductive group, *Invent. Math.* **130** no. 2 (1997), 287–313. MR 1474159. Zbl 0904.11014. https://doi.org/10.1007/s002220050186.

[Gro04] B. H. Gross, Heegner points and representation theory, in *Heegner Points and Rankin L-Series*, *Math. Sci. Res. Inst. Publ.* **49**, Cambridge Univ. Press, Cambridge, 2004, pp. 37–65. MR 2083210. Zbl 1126.11032. https://doi.org/10.1017/CBO9780511756375.005.

[Gro21] B. H. Gross, Incoherent definite spaces and Shimura varieties, in *Relative Trace Formulas*, Springer, Cham, 2021, Simons Symp., pp. 187–215. Zbl 7390070. https://doi.org/10.1007/978-3-030-68506-5_5.

[GZ86] B. H. Gross and D. B. Zagier, Heegner points and derivatives of L-series, *Invent. Math.* **84** no. 2 (1986), 225–320. MR 0833192. Zbl 0608.14019. https://doi.org/10.1007/BF01388809.

[Har07] M. Harris, Cohomological automorphic forms on unitary groups. II. Period relations and values of L-functions, in *Harmonic Analysis*, *Group Representations*, *Automorphic Forms and Invariant Theory*, *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **12**, World Sci. Publ., Hackensack, NJ, 2007, pp. 89–149. MR 2401812. Zbl 1390.11088. https://doi.org/10.1142/9789812770790_0005.

[HKS96] M. Harris, S. S. Kudla, and W. J. Sweet, Theta dichotomy for unitary groups, *J. Amer. Math. Soc.* **9** no. 4 (1996), 941–1004. MR 1327161. Zbl 0870.11026. https://doi.org/10.1090/S0894-0347-96-00198-1.

[HT01] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, *Ann. of Math. Stud.* **151**, Princeton Univ. Press, Princeton, NJ, 2001, with an appendix by Vladimir G. Berkovich. MR 1876802. Zbl 1036.11027. https://doi.org/10.1515/9781400837205.

[How12] B. Howard, Complex multiplication cycles and Kudla-Rapoport divisors, *Ann. of Math.* (2) **176** no. 2 (2012), 1097–1171. MR 2950771. Zbl 1327.14126. https://doi.org/10.4007/annals.2012.176.2.9.

[KMSW14] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White, Endoscopic classification of representations: Inner forms of unitary groups, 2014. arXiv 1409.3731.

[KSZ] M. Kisin, S. W. Shin, and Y. Zhu, Cohomology of certain Shimura varieties of Abelian type (temporary title), in preparation.

[Kol90] V. A. Kolyvagin, Euler systems, in *The Grothendieck Festschrift*, *Vol. II*, *Progr. Math.* **87**, Birkhäuser Boston, Boston, MA, 1990, pp. 435–483. MR 1106906. Zbl 0742.14017.

[KK07] T. Konno and K. Konno, On doubling construction for real unitary dual pairs, *Kyushu J. Math.* **61** no. 1 (2007), 35–82. MR 2317282. Zbl 1218.22012. https://doi.org/10.2206/kyushujm.61.35.

[Kot92] R. E. Kottwitz, Points on some Shimura varieties over finite fields, *J. Amer. Math. Soc.* **5** no. 2 (1992), 373–444. MR 1124982. Zbl 0796.14014. https://doi.org/10.2307/2152772.

[KR11] S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties I. Unramified local theory, *Invent. Math.* **184** no. 3 (2011), 629–682. MR 2800697. Zbl 1229.14020. https://doi.org/10.1007/s00222-010-0298-z.

[KR14] S. Kudla and M. Rapoport, Special cycles on unitary Shimura varieties II: Global theory, *J. Reine Angew. Math.* **697** (2014), 91–157. MR 3281653. Zbl 1316.11050. https://doi.org/10.1515/crelle-2012-0121.

[Kud97] S. S. Kudla, Central derivatives of Eisenstein series and height pairings, *Ann. of Math.* (2) **146** no. 3 (1997), 545–646. MR 1491448. Zbl 0990.11032. https://doi.org/10.2307/2952456.

[Kud02] S. S. Kudla, Derivatives of Eisenstein series and generating functions for arithmetic cycles, in *Séminaire Bourbaki*, *Vol.* 1999/2000, *Astérisque* **276**, Math. Soc. France, Paris, 2002, pp. 341–368. MR 1886765. Zbl 1066.11026. Available at http://www.numdam.org/item/SB_1999-2000__42__341_0/.

[Kud03] S. S. Kudla, Modular forms and arithmetic geometry, in *Current Developments in Mathematics*, 2002, Int. Press, Somerville, MA, 2003, pp. 135–179. MR 2062318. Zbl 1061.11020. https://doi.org/10.4310/CDM.2002.v2002.n1.a4.

[Kud04] S. S. Kudla, Special cycles and derivatives of Eisenstein series, in *Heegner Points and Rankin L-series*, *Math. Sci. Res. Inst. Publ.* **49**, Cambridge Univ. Press, Cambridge, 2004, pp. 243–270. MR 2083214. Zbl 1073.11042. https://doi.org/10.1017/CBO9780511756375.009.

[KM86] S. S. Kudla and J. J. Millson, The theta correspondence and harmonic forms. I, *Math. Ann.* **274** no. 3 (1986), 353–378. MR 0842618. Zbl 0594.10020. https://doi.org/10.1007/BF01457221.

[KRY06] S. S. Kudla, M. Rapoport, and T. Yang, *Modular Forms and Special Cycles on Shimura Curves*, *Ann. of Math. Stud.* **161**, Princeton Univ. Press, Princeton, NJ, 2006. MR 2220359. Zbl 1157.11027. https://doi.org/10.1515/9781400837168.

[KS97] S. S. Kudla and W. J. Sweet, Jr., Degenerate principal series representations for $\mathrm{U}(n,n)$, *Israel J. Math.* **98** (1997), 253–306. MR 1459856. Zbl 0896.22007. https://doi.org/10.1007/BF02937337.

[LL21] C. Li and Y. Liu, Chow groups and L-derivatives of automorphic motives for unitary groups, II, 2021. arXiv 2101.09485.

[LZ] C. Li and S. Zhang, Kudla–Rapoport cycles and derivatives of local densities, to appear in *J. Amer. Math. Soc.*; published electronically September 10, 2021. https://doi.org/10.1090/jams/988.

[Liu11a] Y. Liu, Arithmetic theta lifting and L-derivatives for unitary groups, I, *Algebra Number Theory* **5** no. 7 (2011), 849–921. MR 2928563. Zbl 1258.11060. https://doi.org/10.2140/ant.2011.5.849.

[Liu11b] Y. Liu, Arithmetic theta lifting and L-derivatives for unitary groups, II, *Algebra Number Theory* **5** no. 7 (2011), 923–1000. MR 2928564. Zbl 1258.11061. https://doi.org/10.2140/ant.2011.5.923.

[Liu21a] Y. Liu, Fourier–Jacobi cycles and arithmetic relative trace formula (with an appendix by Chao Li and Yihang Zhu), *Camb. J. Math.* **9** no. 1 (2021), 1–147. Zbl 7374198. https://doi.org/10.4310/CJM.2021.v9.n1.a1.

[Liu21b] Y. Liu, Theta correspondence for almost unramified representations of unitary groups, 2021, to appear in *J. Number Theory*. arXiv 2105.14223.

[LTX+19] Y. Liu, Y. Tian, L. Xiao, W. Zhang, and X. Zhu, On the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives, 2019. arXiv 1912.11942v2.

[Mil85] J. J. Millson, Cycles and harmonic forms on locally symmetric spaces, *Canad. Math. Bull.* **28** no. 1 (1985), 3–38. MR 0778258. Zbl 0575.10022. https://doi.org/10.4153/CMB-1985-001-x.

[Mok15] C. P. Mok, Endoscopic classification of representations of quasi-split unitary groups, *Mem. Amer. Math. Soc.* **235** no. 1108 (2015), vi+248. MR 3338302. Zbl 1316.22018. https://doi.org/10.1090/memo/1108.

[Nek07a] J. Nekovář, The Euler system method for CM points on Shimura curves, in *L-Functions and Galois Representations*, *London Math. Soc. Lecture Note Ser.* **320**, Cambridge Univ. Press, Cambridge, 2007, pp. 471–547. MR 2392363. Zbl 1152.11023. https://doi.org/10.1017/CBO9780511721267.014.

[Nek07b] J. Nekovář, On the parity of ranks of Selmer groups. III, *Doc. Math.* **12** (2007), 243–274. MR 2350290. Zbl 1201.11067. Available at https://elibm.org/article/10000094.

[Ral82] S. Rallis, Langlands' functoriality and the Weil representation, *Amer. J. Math.* **104** no. 3 (1982), 469–515. MR 0658543. Zbl 0532.22016. https://doi.org/10.2307/2374151.

[Ral84] S. Rallis, Injectivity properties of liftings associated to Weil representations, *Compositio Math.* **52** no. 2 (1984), 139–169. MR 0750352. Zbl 0624.22012. Available at http://www.numdam.org/item?id=CM_1984__52_2_139_0.

[Ram18] D. Ramakrishnan, A theorem on GL(n) à là Tchebotarev, 2018. arXiv 1806.08429.

[RSZ20] M. Rapoport, B. Smithling, and W. Zhang, Arithmetic diagonal cycles on unitary Shimura varieties, *Compos. Math.* **156** no. 9 (2020), 1745–1824. MR 4167594. Zbl 1456.14031. https://doi.org/10.1112/s0010437x20007289.

[Sch75] W. Schmid, Some properties of square-integrable representations of semisimple Lie groups, *Ann. of Math.* (2) **102** no. 3 (1975), 535–564. MR 0579165. Zbl 0347.22011. https://doi.org/10.2307/1971043.

[Shi] S. W. Shin, On the cohomological base change for unitary similitude groups, an appendix to *Galois Representations Associated to Holomorphic limits of Discrete Series* by W. Goldring, available on author's website.

[Tan99] V. Tan, Poles of Siegel Eisenstein series on U(n,n), *Canad. J. Math.* **51** no. 1 (1999), 164–175. MR 1692899. Zbl 0963.11028. https://doi.org/10.4153/CJM-1999-010-4.

[Xue19] H. Xue, Arithmetic theta lifts and the arithmetic Gan-Gross-Prasad conjecture for unitary groups, *Duke Math. J.* **168** no. 1 (2019), 127–185. MR 3909895. Zbl 1446.11117. https://doi.org/10.1215/00127094-2018-0039.

[Yam14] S. Yamana, L-functions and theta correspondence for classical groups, *Invent. Math.* **196** no. 3 (2014), 651–732. MR 3211043. Zbl 1303.11054. https://doi.org/10.1007/s00222-013-0476-x.

[YZZ09] X. Yuan, S.-W. Zhang, and W. Zhang, The Gross-Kohnen-Zagier theorem over totally real fields, *Compos. Math.* **145** no. 5 (2009), 1147–1162. MR 2551992. Zbl 1235.11056. https://doi.org/10.1112/S0010437X08003734.

[YZZ13] X. Yuan, S.-W. Zhang, and W. Zhang, *The Gross-Zagier Formula on Shimura curves*, *Ann. of Math. Stud.* **184**, Princeton Univ. Press, Princeton, NJ, 2013. MR 3237437. Zbl 1272.11082. https://doi.org/10.1515/9781400845644.

[Zha19] S. Zhang, Linear forms, algebraic cycles, and derivatives of L-series, *Sci. China Math.* **62** no. 11 (2019), 2401–2408. MR 4028281. Zbl 1443.11077. https://doi.org/10.1007/s11425-019-1589-7.

[Zha09] W. Zhang, *Modularity of Generating Functions of Special Cycles on Shimura Varieties*, ProQuest LLC, Ann Arbor, MI, 2009, Thesis (Ph.D.)–Columbia Univ. MR 2717745. Available at http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:3373585.

[Zha12] W. Zhang, Gross-Zagier formula and arithmetic fundamental lemma, in *Fifth International Congress of Chinese Mathematicians. Part* 1, 2, *AMS/IP Stud. Adv. Math.*, 51, *pt.* 1 **2**, Amer. Math. Soc., Providence, RI, 2012, pp. 447–459. MR 2908086. Zbl 1252.11052.

[Zha21] W. Zhang, Weil representation and arithmetic fundamental lemma, *Ann. of Math.* (2) **193** no. 3 (2021), 863–978. MR 4250392. Zbl 07353244. https://doi.org/10.4007/annals.2021.193.3.5.

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