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Remarks on anomalous dissipation for passive scalars

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We consider the problem of anomalous dissipation for passive scalars advected by an incompressible flow. We review known results on anomalous dissipation from the point of view of the analysis of partial differential equations, and present simple rigorous examples of scalars that admit a Batchelor-type energy spectrum and exhibit anomalous dissipation in the limit of zero scalar diffusivity.

To Uriel Frisch, on the occasion of his 80th birthday.

1. Introduction

We consider the motion of a passive scalar in an incompressible fluid. We confine ourselves to two space dimensions, which impose more restrictions on the motion of the fluid, but simplify the analysis somewhat. Our results can be generalized to three space dimensions.

We are interested in *anomalous dissipation*, that is, the scalar dissipation rate does not vanish in the limit of zero scalar diffusivity. Anomalous dissipation has been observed in experiments as well as in numerical simulations of turbulent mixing (we refer to [1] and references therein for a discussion of experimental and numerical evidence).

We suppose the scalar θ is advected by a given divergence-free vector field and, at the same time, diffuses. In turbulent mixing, it is commonly assumed that the feedback of the scalar field onto the flow is absent or negligible. The fluid flow is assumed turbulent, and the corresponding velocity field is often modeled as a random field satisfying certain prescribed statistics and having spectra of a certain form. For isotropic, homogeneous turbulence, we have the celebrated Kolmogorov's theory in 3 space dimensions, and the Kraichnan-Batchelor theory in 2 space dimensions (see [2] for a comprehensive treatment).

In turbulent mixing, several regimes can be identified, depending on the relative strength of the fluid viscosity ν and scalar diffusivity κ.

This relative strength is encoded in the *Schmidt number* $Sc := \frac{\nu}{\kappa}$, which can vary several orders of magnitude. We are interested here in the so-called *viscous–convective range*, where $Sc \gg 1$. In this range, a power law for the scalar spectrum $\bar{E}_\theta$ was derived by Bachelor from dynamical arguments, assuming a stationary rate of strain for the flow [3]:

$$\bar{E}_\theta(k) = C_B\, \chi \left(\frac{\nu}{\kappa}\right)^{1/2} k^{-1}, \tag{1.1}$$

where k is the wavenumber, C_B is the *Batchelor constant*, and χ is the mean scalar dissipation rate. Evidence of the validity of this power law is somewhat mixed, especially the value of the Batchelor constant (see again [1] and references therein), but we will assume it throughout.

There is an extensive literature on mixing in fluids. From a dynamical systems point of view, we mention only the comprehensive reference [4]. There is also an important connection with Rayleigh-Bénard convection, where the Schmidt number is replaced by the Prandtl number (see e.g. [5, 6]). Mixing has been recognized as an important mechanism for stabilization in fluids, with connections with phenomena such as inviscid damping for the Euler equations [7] and enhanced dissipation [8]. Recently, several works have addressed rigorous bounds on mixing rates, utilizing so-called mix-norms [9, 10, 11, 12], which allow for a quantitative approach [13, 14, 15, 16]. These bounds have been shown to be sharp under various constraints on the flow, such as finite energy or enstrophy, both in the deterministic setting [16, 17, 18, 19], as well as in the stochastic setting [20]. All these results concern pure mixing and transport of the scalar field and can be considered in the limit of infinite Schmidt number. There are significantly fewer works in pde/analysis addressing mixing with diffusion and, in particular, bounds on the energy dissipation rate or the spectrum for scalar turbulence at large, but finite, Schmidt number [21, 22, 23, 24]. In fact, transport can both enhance as well as balance diffusion.

Historically, the *Kraichnan model* [25] has played an important role in the study of anomalous diffusion and anomalous scaling (among the many references, we mention in particular [26, 27, 28, 29, 30], and the monograph [31]). In this model, the passive scalar is advected by a random flow, generated by a Gaussian-in-space, white-in-time vector field. The velocity is only Hölder continuous and hence the Lagrangian trajectories are not unique. In this model, anomalous dissipation can be established rigorously, using the results in [32, 33]. The non-uniqueness of Lagrangian trajectories is a more general mechanisms for *spontaneous stochasticity* (see [34, 35, 36, 37] and references therein).

The purpose of this work is to present simple deterministic examples of scalar fields and advecting flows, compatible with the Batchelor scaling (1.1), for which anomalous dissipation exists and it is a purely diffusive effect. The flows in these examples are exact solutions of the incompressible Euler or Navier-Stokes equations, with a certain symmetry compatible with that of the scalar. Their effect is purely kinematic and not dynamic.

2. Preliminaries

Throughout we denote the scalar field by θ^κ, where $\theta^\kappa(x, y, t)$ is the function of time t and space (x, y), a point in either $\mathbb{R}^2$ or the torus $\mathbb{T}^2$. A function on the torus $\mathbb{T}^2$ will be assumed to be 1-periodic for convenience, but any period can be considered. Our results can also be extended to 3 space dimensions, but 2 space dimensions is a natural setting for scalar mixing as it is the most constrained, non-trivial dimension. In what follows, we use the shorthand notation for functions of space and time

$$f(t)(x, y) := f(x, y, t).$$

The scalar θ^κ will be assumed to satisfy the advection-diffusion equation in weak sense:

$$\partial_t \theta^\kappa + \mathbf{u} \cdot \nabla \theta^\kappa = \kappa\, \Delta \theta^\kappa, \tag{2.1}$$

on $\mathbb{R}^2 \times [0, \infty)$ or $\mathbb{T}^2 \times [0, \infty)$, where again κ is the diffusivity of the scalar and $\mathbf{u}$ is a given divergence-free vector field. The advecting field $\mathbf{u}$ will satisfy either the Navier-Stokes ($\nu > 0$) or the Euler equations ($\nu = 0$) for a certain pressure field p:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu\, \Delta \mathbf{u} + \nabla p, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{2.2}$$

where ν denotes the fluid viscosity and $\operatorname{div} = \nabla \cdot$ is the divergence operator. Since we will keep ν fixed, we do not explicitly indicate the dependence of $\mathbf{u}$ on ν. We give the initial condition for (2.1) as $\theta^\kappa(x, y, 0) = \theta_0(x, y)$ and we take θ_0 independent of diffusivity κ for reasons that will be clear later. On the full plane $\mathbb{R}^2$, we will also impose that θ_0 be compactly supported. These conditions on θ_0 can be relaxed.

In the viscous-convective regime at fixed viscosity $\nu > 0$, we can assume that the flow is regular, which makes it easier to treat pure transport. Formally as $\kappa \to 0$, θ^κ converges to a solution θ^0 of the linear transport equation:

$$\partial_t \theta^0 + \mathbf{u} \cdot \nabla \theta^0 = 0, \tag{2.3}$$

with initial condition $\theta^0(x, y, 0) = \theta_0(x, y)$. If $\mathbf{u}$ is sufficiently regular, at least Lipschitz continuous in space uniformly in time, and θ_0 is a continuous function, we can solve (2.3) by the method of characteristics:

$$\theta^0(x, y, t) = \theta_0(\boldsymbol{\Phi}_t^{-1}(x, y)),$$

where $\boldsymbol{\Phi}$ is the flow of the vector field $\mathbf{u}$. The formula above still holds for a.a points (x, y) if $\mathbf{u}$ has only Sobolev regularity and θ_0 is only essentially bounded. The Di Perna-Lions theory of *renormalized solutions* [38] guarantees the existence of a unique weak solution of (2.3) on $[0, T)$ for $\theta_0 \in L^r(\mathbb{R}^2)$, if $\mathbf{u}$ is bounded and $\mathbf{u} \in L^1((0, T); W^{1,p})$, where $1/p + 1/r = 1$, $p > 1$ (see [39] for an important extension to the case $p = 1$). We use standard notation for Sobolev spaces, that is, for $1 \le p \le \infty$, $k \in \mathbb{Z}_+$, we let:

$$W^{k,p}(\mathbb{R}^2) = \{ f \in L^p(\mathbb{R}^2) \mid \partial^\alpha f \in L^p(\mathbb{R}^2),\ |\alpha| \le k \},$$

and similarly on the torus. We also set $H^k = W^{k,2}$ as customary. In the setting of the Di Perna-Lions theory, the unique weak solution has a Lagrangian representation and can be obtained as a limit of regularized solutions. In particular, θ^0 can be obtained as a weak limit of solutions θ^κ of (2.1). Since the flow preserves the Lebesgue measure if $\mathbf{u}$ is divergence free, all L^p-norms of θ^0 are preserved.

3. The scalar spectrum and scalar dissipation

We next discuss the spectrum and the dissipation rate for solutions of (2.1). As long as $\mathbf{u}$ is uniformly bounded and divergence free and $\theta_0 \in L^2$, there exists a unique weak solution $\theta^\kappa \in C([0, \infty); L^2) \cap L^2((0, \infty); H^1)$ of (2.1).

Then a simple energy estimate shows that, for $0 \le s \le t$,

$$\|\theta^\kappa(t)\|_{L^2}^2 - \|\theta^\kappa(s)\|_{L^2}^2 = 2\kappa \int_s^t \|\nabla \theta^\kappa(\tau)\|_{L^2}^2 \, d\tau. \tag{3.1}$$

The term on the right-hand side of (3.1), starting at $s = 0$, is the *rate of scalar dissipation*

$$\chi^\kappa = \chi^\kappa(\theta^\kappa) := 2\kappa \int_0^t \|\nabla \theta^\kappa(\tau)\|_{L^2}^2 \, d\tau.$$

We say that there is *anomalous dissipation* if

$$\liminf_{\kappa \to 0_+} \chi^\kappa = \chi > 0.$$

This is the analog for scalar turbulence of anomalous energy dissipation in turbulence. Given that the mean is preserved under the time evolution for weak solutions of both (2.1) and (2.3), one can reduce to consider mean-free scalar fields for which the L^2-norm squared can be interpreted as the covariance in the tracer field and χ is a measure of decay of correlations.

A related concept is that of the *scalar spectrum* $E_{\theta^\kappa}(k,t)$, which is akin to the energy spectrum $E_{\mathbf{u}}(k,t)$ for turbulence. For a given $t \geq 0$, it is the density function in the wavenumber $k \in [0,\infty)$ with respect to the 1D Lebesgue measure such that:

$$\int_0^\infty E_{\theta^\kappa}(k,t)\,dk = \int_{\mathbb{R}^2} \left(\theta^\kappa(x,y,t)\right)^2\,dx\,dy. \tag{3.2}$$

On the torus $\mathbb{T}^2$, we replace the integral by a sum over all integer wavenumbers. We consider the spectrum as a function of time, as in decaying turbulence. In the turbulence literature, the average with respect to an invariant measure or the long-time average, assuming ergodicity, $\bar{E}_{\theta^\kappa}(k)$ is typically used.

For technical reasons, we will utilize the *Littlewood-Paley (LP) spectrum* instead [40], which is defined as:

$$E^{LP}_{\theta^\kappa}(\zeta) := \frac{1}{\zeta}\|\Delta_j\theta_\kappa\|^2_{L^2(\mathbb{R}^2)}, \qquad 2^{j-1} < \zeta < 2^{j+1}, \tag{3.3}$$

where $j \in \mathbb{Z}$ and Δ_j is a Littlewood-Paley operator at frequency 2^j, which is defined as follows. We use $\hat{}$ to denote the Fourier Transform and $\check{}$ to denote its inverse as customary. Let ψ_j be a smooth bump function supported on the dyadic shell $D_j = \{\boldsymbol{\xi};\, 2^j < |\boldsymbol{\xi}| < 2^{j+1}\}$, $j \in \mathbb{Z}_+$, in frequency space, and let ψ_{-1} be a smooth bump function supported on the unit ball. We can choose $\psi_j(\boldsymbol{\xi}) = \psi_0(2^j\,\boldsymbol{\xi})$, radially symmetric, positive, and such that $\{\psi_j\}_{j=-1}^\infty$ form a dyadic partition of unity (this is standard construction, see e.g. [41]). Then,

$$\sum_{j=-1}^{\infty} \psi_j(\xi)\,f(\boldsymbol{\xi}) = f(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} = (\xi_1,\xi_2),$$

in the sense of distributions, for any locally integrable function f on $\mathbb{R}^2$. Finally, we let

$$\Delta_j h(x,y) = \check{\psi}_j * h(x,y).$$

and we set $\zeta = |\boldsymbol{\xi}|$. It can be shown that the spectrum and the Littlewood-Paley spectrum are comparable, but the Littlewood-Paley spectrum can be defined for functions that are not necessarily square integrable. For functions on the torus, we can define Δ_j in a similar fashion by restricting frequencies to $D_j \cap \mathbb{Z}^2$ (see [42]). We also recall that, if a function h is compactly supported on the unit cube, it can be viewed as a function on the torus $\mathbb{T}^2$ via its one-periodic extension $\tilde{h}$, and then the Fourier coefficients of $\tilde{h}$ are given by $\hat{h}(\mathbf{k})$, $\mathbf{k} \in \mathbb{Z}^2$.

As a consequence of the Di Perna-Lions theory, no anomalous dissipation exists if $\theta_0 \in L^2$ and $\mathbf{u}$ is sufficiently regular. Therefore, mathematically there are two possible scenarios for scalar anomalous dissipation in the regime we consider:

(1) the advecting field $\mathbf{u}$ is not regular enough for the Di Perna-Lions theory to apply;
(2) the initial data θ_0 is not square integrable, but $\theta^\kappa(t) \in L^2$ for $t > 0$.

Case (1) may arise if the advecting flow is a turbulent flow at very high Reynolds number. In the so-called *inertial–convective range*, the Schmidt number is taken to be of order one, so this regime covers the simultaneous limit $\nu \to 0$ and $\kappa \to 0$. In the inertial-convective range, the scalar spectrum and dissipation were investigated by Obukhov [43] and Corrsin [44], who derived the power law for the averaged spectrum by dimensional considerations:

$$\bar{E}_{\theta^\kappa}(k) = C_{OB}\,\bar{\chi}\,\bar{\epsilon}\,k^{-5/3},$$

where $\bar{\epsilon}$ is the (averaged) energy dissipation rate in the fluid and C_{OB} is the Obukhov-Corrsin constant. In this derivation, both the (averaged) spectrum dissipation rate $\bar{\chi}$ and $\bar{\epsilon}$ are assumed independent of κ and ν respectively. By Onsager's conjecture [45], which has been rigorously established now (we refer to [46, 47, 48] and references therein), the flow $\mathbf{u}$ can only be Hölder continuous of exponent $1/3$ for anomalous energy dissipation to occur. The Obukhov-Corrsin theory also predicts the scaling of structure functions, if the fluid flow obeys Kolmogorov's scaling of fully developed isotropic, homogeneous turbulence.

Anomalous scalar dissipation for fluid velocities with Hölder regularity was recently investigated in [24]. They gave sufficient conditions for anomalous dissipation in terms of the mixing rates of the advecting flow. They showed that for any initial data $\theta_0 \in H^2(\mathbb{T}^2)$, given $T > 0$, $\alpha \in [0, 1)$, there exists a velocity field $\mathbf{u} \in C^\infty([0, T) \times \mathbb{T}^2) \cap L^1([0, T]; C^\alpha(\mathbb{T}^2))$ and a constant $\chi_\alpha > 0$ such that

$$\chi^\kappa \geq \chi_\alpha \, \|\theta_0\|^2_{L^2}.$$

In their example, $\mathbf{u}$ and χ_α depends on T, α and θ_0, but are independent of κ. The velocity is constructed out of alternating shear flows with a smooth, approximately linear profile. We will also utilize shear flows for our example on the torus. The authors of [24] observe that the Obukhov-Corrsin scaling argument can be generalized to predict that, in the limit $\nu, \kappa \to 0$ with $\kappa = O(\nu)$, if $\mathbf{u}$ is Hölder continuous of exponent at most $\alpha \in (0, 1)$, then θ^κ can be Hölder continuous of exponent at most $\beta = (1 - \alpha/2)$. They show that this result is sharp in the sense that no anomalous scalar dissipation can occur if $\beta > (1 - \alpha)/2$ or if $\beta = (1 - \alpha)/2$ and, in addition, $\mathbf{u} \in L^1_{\text{loc}}([0, T); W^{1,\infty}(\mathbb{T}^2))$. Their example of anomalous dissipation corresponds to the borderline case $\alpha = 1_-$. We also mention the recent preprint [49], where the authors also consider the advection-diffusion equation (2.1) with rough advecting field and study in particular the vanishing viscosity limit. They prove that, if $\theta_0 \in L^2(\mathbb{T}^2)$, as long as $\mathbf{u} \in L^1((0, T); W^{1,1}(\mathbb{T}^2))$, then no anomalous dissipation is possible, because the limit solutions are Lagrangian.

Case (2) is compatible with the Batchelor scaling of the spectrum (1.1). In [22], a modified LP spectrum, which uses the L^1 norm of $\Delta_j \theta^\kappa$, instead of the L^2 norm was studied. An upper bound of order k^{-1} for its decay in a suitable weighted long-time average was obtained, assuming the velocity field $\mathbf{u}$ is in the Sobolev space H^s, $s \in [0, 1]$, uniformly in time. The case $s = 1$ is within the Di Perna-Lions theory. The Batchelor scaling was recently investigated in the stochastic setting in [23]. This setting is a mathematical model for the viscous-convective range in scalar turbulence. The scalar θ^κ is assumed to satisfy (2.1) with a source term, which is smooth in space and white noise in time, while $\mathbf{u}$ is assumed to satisfy the 2D Navier-Stokes equations at fixed viscosity with a non-degenerate stochastic forcing that is sufficiently regular in space (the case of the 3D hyperviscous Navier-Stokes equations was also considered) . The authors establish rigorously that the averaged cumulative spectrum up to wavenumber k_0 decays like $\log k_0$ with constants uniform in κ. This result is slightly weaker than (1.1), and follows from results on the mixing properties of the flow of $\mathbf{u}$, which was shown to be almost surely exponential mixing by the same authors in [20], using some deep dynamical systems tools.

If one *assumes* that the energy spectrum or the LP spectrum decay like k^{-1}, then one can conclude that θ^κ belongs to the Besov space $B^0_{2,\infty}$. For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B^s_{p,q}$ is defined as follows:

$$B^s_{p,q}(\mathbb{R}^2) := \{f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_{B^s_{p,q}} := \|2^{js} \|\Delta_j f\|_{L^p(\mathbb{R}^2)}\|_{\ell^r} < \infty\}, \tag{3.4}$$

with a similar definition on the torus $\mathbb{T}^2$ [42]. These are Banach spaces and the dual $(B^s_{p,q})' = B^{-s}_{p',q'}$ for $1 \leq p, q < \infty$, where p', q' are the conjugate exponents to p, q. Above, $\mathcal{S}'(\mathbb{R}^2)$ denotes the space of tempered distributions, the dual to the space $\mathcal{S}(\mathbb{R}^2)$ of smooth, rapidly decreasing functions.

The significance of the space $B^0_{2,\infty}$ in turbulence was already noted by Eyink [50] in the context of the Kraichnan-Batchelor theory of 2D turbulence. In fact, if $\mathbf{u}$ solves the two-dimensional Navier-Stokes (respectively Euler) equations, then its (scalar) vorticity $\omega = \operatorname{curl} \mathbf{u}$ satisfies (2.1) (respectively (2.3)). The Navier-Stokes and Euler equations in vorticity-velocity formulation have *active transport*, since $\mathbf{u}$ depends on ω via the Biot-Savart law (for recent results on renomalizing certain active transport equations we refer to [51]). We recall that the enstrophy is defined as the L^2-norm squared of vorticity, so formally the problem of anomalous enstrophy dissipation is equivalent to the problem of anomalous scalar dissipation. Assuming the Kraichnan scaling $O(k^{-3})$ for the energy spectrum of $\mathbf{u}$, associated to the forward enstrophy cascade, implies that $\omega \in B^0_{2,\infty}$.

We next address the well-posedness of (2.1)-(2.3) in the space $B^0_{2,\infty}$. We discuss only the case of the whole space $\mathbb{R}^2$, the case of the torus $\mathbb{T}^2$ is similar. Uniqueness of solutions in the periodic case holds for mean-free initial data. We recall that we are interested in the regime where the fluid viscosity is fixed and the flow can be assumed to be regular. We hence impose that $\mathbf{u}$ is bounded in space and time, and $\mathbf{u} \in L^1((0,T); W^{1,\infty} \cap H^2)$.

The existence of a weak solution for both (2.1) and (2.3) follows via a suitable regularization by standard methods [52]. The key point is whether solutions remain in $B^0_{2,\infty}$ for $t > 0$ and estimates on derivatives in the case $\kappa > 0$. Obtaining bounds on the Besov norm is rather delicate, as the norm in $B^0_{2,\infty}$ is not rearrangement invariant. Indeed, the solutions constructed in [23] are not known to be in $B^0_{2,\infty}$ uniformly in κ, even though they satisfy a uniform estimate on the cumulative spectrum.

We introduce certain spaces of functions of time close to L^q spaces [53]. Given $0 < T < \infty$, for $s \in \mathbb{R}$, $1 \leq q \leq \infty$, we let:

$$\widetilde{L}^q((0,T); B^s_{2,\infty}) = \{ f \in \mathcal{S}' \mid \|u\|_{\widetilde{L}^q((0,T);B^s_{2,\infty})} := \sup_{j \geq -1} \|\Delta_j \theta\|_{L^q((0,T);L^p)} < \infty \}. \tag{3.5}$$

One has that

$$\widetilde{L}^r((0,T); B^s_{2,\infty}) \subset L^q((0,T); B^s_{2,\infty}) \subset \widetilde{L}^m((0,T); B^s_{2,\infty}), \quad m < q < r. \tag{3.6}$$

Then the following bounds hold.

Theorem 3.1. *Let $\mathbf{u} \in L^1((0,T); W^{1,\infty} \cap H^2)$. Let θ^κ be a weak solution of (2.1) on $[0,T)$ with initial data $\theta_0 \in B^0_{2,\infty}$. Then, there exists a universal constant $\alpha > 0$ such that*

$$\|\theta^\kappa\|_{\widetilde{L}^\infty((0,T);B^0_{2,\infty})} + \alpha\kappa^{1/4} \|\theta^\kappa - \Delta_{-1}\theta^\kappa\|_{\widetilde{L}^1((0,T);B^2_{2,\infty})} \leq C \|\theta_0\|_{B^0_{2,\infty}}, \tag{3.7}$$

where C depends on $\mathbf{u}$ and T, but is independent of κ and θ_0. In particular the weak solution is unique.

For a proof of this result we refer to Proposition 2.1 in [54].

Since applying the operator Δ_j destroys the identity

$$\int_{\mathbb{R}^2} (\mathbf{u} \cdot \nabla)\theta^\kappa\, \theta^\kappa \, dx\, dy = 0$$

that gives the energy estimate (3.1), it is not clear that energy dissipation can be defined for such solutions. We note, however, that by (3.7), $\theta^\kappa(t) \in B^2_{2,\infty} \subset L^2$ for a. a. $t > 0$. Hence, by uniqueness of the weak solution, θ^κ has finite energy for positive times.

Theorem 3.2. *Let $\mathbf{u} \in L^1((0,T); W^{1,\infty} \cap H^2)$ and bounded. Let θ^0 be a weak solution of (2.3) on $[0,T)$ with initial data $\theta_0 \in B^0_{2,\infty}$. Then*

$$\|\theta^0\|_{\widetilde{L}^\infty((0,T);B^0_{2,\infty})} \leq \widetilde{C} \|\theta_0\|_{B^0_{2,\infty}}, \tag{3.8}$$

where $\widetilde{C}$ depends on $\mathbf{u}$ and T, but is independent of θ_0. In particular, the weak solution is unique.

For a proof of this result we refer to Theorem 3.18 and Theorem 3.19 in [52].

In both cases, the solutions belongs to $C_w([0,T]; B^0_{2,\infty})$, which is the space of continuous functions in time into $B^0_{2,\infty}$, endowed with the weak$*$ topology.

We next discuss whether the weak solution of (2.3) with initial data in $B^0_{2,\infty}$ can be obtained in the limit of vanishing diffusivity. We follow the approach in [50] for the Euler equations. From (3.6) and (3.7), we have a uniform bound

$$\sup_{0<\kappa} \|\theta^\kappa(t)\|_{L^r((0,T);B^0_{2,\infty})} \leq \delta, \tag{3.9}$$

for some constant $\delta > 0$, which may depend on the initial data, on $1 \leq r < \infty$, and $T > 0$, but is independent of κ.

Proposition 3.1. *Let θ_0 and $\mathbf{u}$ satisfies the hypotheses of Theorems 3.1 and 3.2. Let θ^κ and θ^0 be the weak solutions of (2.1) and (2.3), respectively, with initial data θ_0. Then*

$$\theta^\kappa \rightharpoonup \theta^0, \qquad \textit{as } \kappa \to 0,$$

weakly* in $L^2((0,T);B^0_{2,\infty})$.

Proof. From (3.9), the family $\{\theta^\kappa\}_{0<\kappa<\kappa_0}$ is uniformly bounded in κ in $L^2((0,T);B^0_{2,\infty})$. Hence there is a uniform bound in κ on the time derivative $\partial_t\theta^\kappa$ in $H^{-1}((0,T);B^0_{2,\infty})$, and in $L^2((0,T);B^{-1}_{2,1})$ by duality. Consequently, there exists a sequence θ^{κ_n} converging weakly* to some $\vartheta \in L^2((0,T);B^0_{2,\infty})$, such that $\partial_t\theta^{\kappa_n}$ converges weakly* to $\partial_t\vartheta$ in $L^2((0,T);B^{-1}_{2,1})$ by weak continuity of the derivatives. By linearity, pairing (2.1) with a suitable test function and passing to a weak limit gives that ϑ is a weak solution of (2.3). Weak convergence of the time derivative implies convergence in $C([0,T];B^{-\epsilon}_{2,\infty})$, $\epsilon>0$, so that $\vartheta(0)=\theta_0$. By uniqueness of weak solutions, $\vartheta=\theta^0$. Hence any weakly convergent sequence in the family $\{\theta^\kappa\}$ converges to θ^0 as $\kappa\to 0$. □

4. Examples of anomalous dissipation

In this section, we present two simple examples of anomalous scalar dissipation, one on the whole plane $\mathbb{R}^2$, the second one on the torus $\mathbb{T}^2$.

The first example is a simple adaptation of an example of enstrophy dissipation for solutions of the 2D Navier-Stokes equations in vorticity-velocity formulation, obtained by the author, M. C. Lopes, and H. J. Nussenzveig-Lopes in [55].

Let $\mathbf{u}$ be any *circularly symmetric* velocity field:

$$\mathbf{u}(x,y,t) = \frac{u(r,t)}{r}\,(-y,x), \qquad r=\sqrt{x^2+y^2}, \tag{4.1}$$

where $u(r,t)=\dfrac{1}{r}\displaystyle\int_0^r \sigma\,\omega(\sigma,t)\,d\sigma$ and ω is a given function. We take ω regular enough so that $\mathbf{u}$ satisfies the hypotheses on the flow in Section 3. Then, if ω is independent of time, $\mathbf{u}$ is an exact steady solution of the 2D Euler equations with radial scalar vorticity ω, and if ω solves the 2D heat equation, then $\mathbf{u}$ is an exact solution of the 2D Navier-Stokes equations with radial scalar vorticity ω. A circularly symmetric flow models a coherent vortex, such as a Rankine vortex, a hallmark of 2D turbulence.

We make the simple observation that, if θ_0 is any radial function in $\mathbb{R}^2$, then $\mathbf{u}\cdot\nabla\theta_0=0$. Hence, θ_0 is a steady solution of (2.3), while if θ^κ solves the 2D heat equation with initial data θ_0 and diffusion coefficient κ, then θ^κ is the unique solution of (2.1) with initial data θ_0.

Let now $\phi\in C^\infty_c(\mathbb{R}^2)$ be a radial, smooth bump function supported on the unit disk in $\mathbb{R}^2$ and set:

$$\theta_0(x,y)=\phi(x,y)\,\frac{1}{\sqrt{x^2+y^2}}, \qquad (x,y)\neq(0,0). \tag{4.2}$$

We have that $\theta_0\in L^p(\mathbb{R}^2)\cap B^0_{2,\infty}(\mathbb{R}^2)$ for all $1\le p<2$, that θ^κ, $0<\kappa\le\kappa_0$ for some fixed κ_0, satisfies (3.9) and that $\theta^\kappa\rightharpoonup\theta_0$ weakly* in $L^\infty((0,T);B^0_{2,\infty})$ for all $T>0$ [55, Proposition 4]. In fact, since the heat semigroup is strongly continuous in $B^0_{2,\infty}$ and the limit $\kappa\to 0_+$ for fixed $t>0$ is equivalent to the limit $t\to 0_+$ for fixed κ in this case, θ^κ converges strongly to θ_0 in $C([0,T);B^0_{2,\infty})$.

From [55, Theorem 5], we have the following result.

Theorem 4.1. *Let θ_0 be as in (4.2) and let $\mathbf{u}$ be as in (4.1). Let θ^κ be the solution of (2.1) with initial data θ_0. Then, for each $t>0$,*

$$\lim_{\kappa\to 0_+}\kappa\int_{\mathbb{R}^2}|\nabla\theta^\kappa(x,y,t)|^2\,dxdy=\frac{4\pi^3}{t}.$$

In particular,

$$\liminf_{\kappa\to 0_+} \kappa \int_{\mathbb{R}^2} |\nabla\theta^\kappa(x,y,t)|^2\, dxdy > 0.$$

In fact, Theorem 5 in [55] establishes a stronger result, namely, that

$$\kappa\, |\nabla\theta^\kappa(t)| \rightharpoonup \frac{4\pi^3}{t}\,\delta_0, \qquad \text{as } \kappa\to 0,$$

in the sense of distributions, for $t>0$ fixed , where δ_0 is the Dirac measure at the origin.

We modify this construction for the torus $\mathbb{T}^2$. We let $\mathbf{u}$ be any *shear flow*, without loss of generality we take a horizontal shear flow:

$$\mathbf{u}(x,y,t) = (u(y,t),0), \tag{4.3}$$

with u a 1-periodic function in y. If u is time independent, $\mathbf{u}$ is a steady solution of the Euler equations in $\mathbb{T}^2$ with constant pressure, while if u solves the 1D heat equation on $[-1/2,1/2]$ with periodic boundary conditions, $\mathbf{u}$ is an exact solution of the 2D Navier-Stokes equation with constant pressure (we exclude pressure-driven shear flows, such as Poiseuille flow, for which the pressure is not a periodic function necessarily). Again, we take u sufficiently regular so that $\mathbf{u}$ satisfies the regularity assumptions in Section 3.

We take the initial data for the scalar field in the form:

$$\theta_0(x,y) = \phi(y)\,\frac{1}{|y|^{1/2}}, \tag{4.4}$$

where $\phi\in C_c^\infty(-1/2,1/2)$ is a bump function supported on $(-1/4,1/4)$. It is immediate to see that $\theta_0\in L^p(\mathbb{T}^2)$, $1\le p<2$. We can choose ϕ even and so that θ_0 has average zero.

Let θ be any function on $\mathbb{T}^2$ of the form $\theta(x,y,t)=\vartheta(y,t)$, with ϑ a 1-periodic function in y. We observe that, if $\mathbf{u}$ is the shear flow in (4.3), then $\mathbf{u}\cdot\nabla\theta = u(y)\partial_x\vartheta = 0$, and that the 2D heat equation on $\mathbb{T}^2$ preserves the symmetry of θ. Hence, if $\theta_0(x,y)=\vartheta_0(y)$ and θ^κ solves the 2D heat equation in $\mathbb{T}^2$ with diffusion coefficient κ and initial data θ_0, then θ^κ solves (2.1), while θ_0 is a steady solution of (2.3), and $\theta^\kappa(x,y,t)=\vartheta^\kappa(y,t)$ with ϑ^κ solution of the 1D heat equation on $[-1/2,1/2]$ with initial data ϑ_0 and periodic boundary conditions.

Lemma 4.1. *Let θ_0 be as in (4.4), then $\theta_0\in B^0_{2,\infty}(\mathbb{T}^2)\cap L^p(\mathbb{T}^2)$, $0\le p<2$. Let θ^κ be the solution of (2.1) with $\mathbf{u}$ as in (4.3) and initial data θ_0. Then, for any $T>0$, θ^κ satisfies the uniform bound (3.9) and θ^κ converges to the unique solution of (2.3) with initial data θ_0 in $C([0,T);B^0_{2,\infty}\cap L^p)$.*

Proof. We compute the Fourier coefficients of θ_0, $\widehat{\theta_0}(\mathbf{k})$, $\mathbf{k}=(k_1,k_2)\in\mathbb{Z}^2$. We note that $\widehat{\theta_0}(k_1,k_2)=0$ for $k_1\neq 0$ and $\widehat{\theta_0}(0,k_2)=\widehat{\vartheta_0}(k_2)$. Since ϑ is compactly supported on $(-1/2,1/2)$, we can compute its Fourier Transform $\widehat{\vartheta_0}(\xi)$, $\xi\in\mathbb{R}$, and evaluate it at integers $\xi=k_2$ to get the Fourier coefficients. Since ϑ_0 is the product of ϕ with $|y|^{-1/2}$, which has Fourier Transform $|\xi|^{-1/2}$, and ϕ is even, $\widehat{\vartheta_0}(\xi)=(\widehat{\phi}*|\cdot|^{1/2})(-\xi)$ in the sense of distributions. But $\widehat{\phi}\in\mathcal{S}(\mathbb{R})$, so the convolution can be written as an integral:

$$\widehat{\vartheta_0}(\xi) = \int_{\mathbb{R}} \widehat{\phi}(\xi-\zeta)\,|\zeta|^{-1/2}\,d\zeta.$$

As in [55], we define an auxiliary function $e(\xi) := |\xi|^{1/2}\,|\vartheta_0(\xi)| - 1$, and study its behavior under dilations. Let $s>0$ and write $\xi = r\,\sigma$, with $\sigma=\pm 1$ and $r=|\xi|$. Then

$$\begin{aligned} e(s\xi) &= |s|^{1/2} r^{1/2} \left| \int_{\mathbb{R}} \frac{1}{|\zeta|^{1/2}} \widehat{\phi}(-sr\sigma-\zeta)\,d\zeta \right| - 1 \\ &= \left| \int_{\mathbb{R}} \frac{1}{|\zeta'|^{1/2}}\, sr\,\widehat{\phi}(-sr(-\sigma-\zeta'))\,d\zeta' \right| - 1 = \left| \int_{\mathbb{R}} \frac{1}{|\zeta'|^{1/2}} \widehat{\phi}_{sr}(-\sigma-\zeta')\,d\zeta' \right| - 1, \end{aligned}$$

where we made the change of variables $\zeta = sr\,\zeta'$. Above, $\widehat{\phi}_t(y) = t\,\widehat{\phi}(t\,y)$, which is uniformly bounded in t and rapidly decreasing in y and t. So, the function e is bounded uniformly in both s and ξ. We show next that $e(s\xi) \to 0$ as $s \to \infty$. Let η be a smooth cut-off function supported in the interval $[-2,2]$ and equal to 1 on the interval $[-1/2,1/2]$. Next,

$$(|\zeta|^{-1/2} * \widehat{\phi}_{sr})(-\sigma) = \int_{\mathbb{R}} \eta(\zeta') \frac{1}{|-\sigma-\zeta'|^{1/2}}\,\widehat{\phi}_{sr}(\zeta')\,d\zeta'$$
$$+ \int_{\mathbb{R}} (1-\eta(\zeta') \frac{1}{|-\sigma-\zeta'|^{1/2}}\,\widehat{\phi}_{sr}(\zeta')\,d\zeta' =: I_1 + I_2.$$

Since $\widehat{\phi}$ is rapidly decreasing, for any fixed y and for any $N \in \mathbb{N}$ $\quad |\widehat{\phi}_{sr}(y)| \leq |sr|^{-N+1}|y|^{-N} \to 0$ as $s \to \infty$. The rest of the integrand in I_1 is in $L^1(\mathbb{R})$, so by the Dominated Convergence Theorem $I_1 \to 0$ as $s \to \infty$. We turn to I_2. Using the support properties of the cut-off function, we have

$$I_2 \leq \|\widehat{\phi}_{sr}\|_{L^\infty([-2,2]^c)} \int_{1/2<\zeta'<2} \frac{2}{|\zeta'|^{1/2}}\,d\zeta' + \int_{\zeta'>2} |\widehat{\phi}_{sr}(\zeta')|\,d\zeta',$$

where in the second integral we used that $|-\sigma-\zeta'|^{-1/2} \leq 1$ if $|\zeta'| > 2$, as $\sigma = \pm 1$.. Again. the rapid decrease of $\widehat{\phi}$ gives that $\|\widehat{\phi}_{sr}\|_{L^\infty([-2,2]^c)} \to 0$ as $s \to \infty$, and for a similar reason the second integral vanishes as well. We can therefore conclude that $|\widehat{\vartheta_0}(\xi) = |\xi|^{-1/2}(1+e(\xi))$, where $e(\xi) \to 0$ as $\xi \to \infty$, and consequently:

$$\widehat{\theta_0}(0,k_2)^2 = \frac{1}{|k_2|}(1+g(k_2)), \quad g(k_2) \to 0,\ k_2 \to \infty. \tag{4.5}$$

We now have:

$$\|\theta_0\|^2_{B^0_{2,\infty}} \lesssim \|\theta_0\|^2_{L^1(\mathbb{T}^2)} + \sup_{j \geq -1} \left(\sum_{2^{j-1}<|k_i|<2^{j+1}} |\widehat{\theta_0}(k_1,k_2)|^2 \right)$$
$$\lesssim \|\theta_0\|^2_{L^1(\mathbb{T}^2)} + \sup_{j \geq -1} \left(\sum_{2^{j-1}<|k_2|<2^{j+1}} |\widehat{\theta_0}(0,k_2)|^2 \right) \lesssim \|\theta_0\|^2_{L^1(\mathbb{T}^2)} + \sup_{j \geq -1} 4 < \infty,$$

since there are $O(2^j)$ frequencies k_2 in each dyadic shells and the largest term in the sum if of order 2^{1-j} by (4.5).

The uniform bound (3.9) follows from the fact that the symbol of $e^{\kappa t\,\Delta}$, $e^{-\kappa t\,|\mathbf{k}|^2}$, is uniformly bounded in κ and t in L^∞. Lastly, the convergence of θ^κ to θ_0, which is the unique solution of the transport equation with data θ_0 is a direct consequence of the strong continuity of the heat semigroup in $L^p \cap B^0_{2,\infty}$. □

We turn to showing that anomalous dissipation holds for θ^κ.

Theorem 4.2. *In the hypotheses of Lemma 4.1, for each $t > 0$, there exists a constant $\overline{\gamma} > 0$ such that*

$$\lim_{\kappa \to 0_+} \kappa \int_{\mathbb{T}^2} |\nabla\theta^\kappa(x,y,t)|^2\,dxdy = \frac{\overline{\gamma}}{t}.$$

In particular,

$$\liminf_{\kappa \to 0_+} \kappa \int_{\mathbb{T}^2} |\nabla\theta^\kappa(x,y,t)|^2\,dxdy > 0.$$

Proof. By Plancherel's identity we can write:

$$\kappa \int_{\mathbb{T}^2} |\nabla\theta^\kappa(x,y,t)|^2\, dxdy = \kappa \sum_{\mathbf{k}\in\mathbb{Z}^2} |\mathbf{k}|^2\, e^{-2\kappa t\,|\mathbf{k}|^2}\, |\widehat{\theta_0}(\mathbf{k})|^2 = \kappa \sum_{k_2\in\mathbb{Z}} |k_2|^2\, e^{-2\kappa t\,|k_2|^2}\, |\widehat{\theta_0}(0,k_2)|^2.$$

From (4.5), $|\widehat{\theta_0}(0,k_2)|^2|k_2| = 1 + g(k_2)$, with $g(k_2) \to 0$ as $k_2 \to \infty$. Hence,

$$\sum_{k_2\in\mathbb{Z}} \kappa |k_2|^2\, e^{-2\kappa t\,|k_2|^2}\, |\widehat{\theta_0}(0,k_2)|^2 = \frac{1}{t}\sum_{k_2\in\mathbb{Z}} t\kappa\, |k_2|\; e^{-2\kappa t\,|k_2|^2} + \frac{1}{t}\sum_{k_2\in\mathbb{Z}} t\kappa\, |k_2|\, g(k_2)\, e^{-2\kappa t\,|k_2|^2}$$
$$:= S_1 + S_2,$$

where we have multiplied and divided each term in both sums by $t>0$. For S_1, we estimate the sum by the Euler-McLaurin's formula and conclude that

$$S_1 \sim \frac{\overline{\gamma}}{t}, \qquad \text{as } \kappa \to 0_+,$$

for some constant $\overline{\gamma} > 0$, since by making the change of variables $w = \sqrt{t\kappa}\, z$,

$$\int_{\mathbb{R}} t\kappa\, |z|\, e^{-2t\kappa\, z^2}\, dz = 2\int_0^\infty w\, e^{-2w^2}\, dw = \frac{1}{2}.$$

We estimate S_2 in a similar way and obtain that

$$S_2 \to 0, \qquad \text{as } \kappa \to 0_+,$$

given that

$$\int_{\mathbb{R}} t\kappa\, |z|\, g(z)\, e^{-2t\kappa\, z^2}\, dz = 2\int_0^\infty w\, g\Big(\frac{w}{\sqrt{t\kappa}}\Big) e^{-2w^2}\, dw \underset{\kappa\to 0_+}{\longrightarrow} 0,$$

which follows from the Dominated Convergence Theorem and the pointwise convergence $g(\frac{w}{\sqrt{t\kappa}}) \to 0$ as $\kappa \to 0_+$. □

5. Conclusion

In this note, we discuss anomalous dissipation in the limit of zero diffusivity from an analytic point of view, motivated by the viscous-convective range in scalar turbulence. After a brief review of the literature, we discuss the well-posedness of the advection-diffusion and the transport equations assuming only the Batchelor scaling for the scalar spectrum. Our main contribution is to present two simple examples of scalar fields, satisfying the Batchelor scaling for the spectrum, which exhibit anomalous dissipation as a purely diffusive effect. As already observed in [55], this scaling for the spectrum is consistent with an infinite reservoir of covariance for the scalar field, if all scales are taken into consideration. In this situation, diffusion alone can sustain anomalous dissipation. In scalar turbulence, a minimal characteristic length scale, the Batchelor scale, is observed, which is determined by a balance between stirring by the fluid and diffusion.

Data Accessibility. No data sets were used or produced in the course of this work.

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