

Transmission Problems for Parabolic Operators on Polygonal Domains and Applications to the Finite Element Method

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Abstract We study linear parabolic equations $\partial_t u + Lu = f$, where $L = -\operatorname{div}(A\nabla)$ is a second-order strongly elliptic operator, on non-smooth two-dimensional bounded domains. The domain is polygonal and not assumed to be convex. The coefficient matrix A is piecewise smooth and exhibits jump discontinuities across a finite number of piecewise smooth curves, collectively denoted the interface. We assume mixed Dirichlet-Neumann boundary conditions and standard transmission conditions at the interface. Under some additional assumptions, we establish well-posedness of the initial-value problem using suitable weighted Sobolev spaces. The solution admits a decomposition $u = u_{\text{reg}} + w_s$, into a function u_{reg} that belongs to the weighted Sobolev space and a function w_s that is locally constant near the vertices, thus proving well-posedness in an augmented space. We use the theoretical analysis to devise graded meshes that give quasi-optimal rates of convergence for a fully discrete scheme that utilizes finite elements on a space grid and finite differences in time.

Keywords Parabolic operators · transmission · mixed boundary-value problems · polygonal domains · finite element method

Mathematics Subject Classification (2010) 35K10 · 35K45 · 65M60

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Conflict of interest

The authors have no conflict of interest.

1 Introduction

We consider second-order, linear, strongly parabolic operators on non-smooth domains in two space dimensions. We restrict our attention to non-convex polygonal domains. The coefficients are assumed piecewise smooth and globally bounded, but may jump across a finite number of curves, which we collectively denote as the interface. We assume that standard transmission conditions hold at the interface and impose mixed Dirichlet-Neumann boundary conditions. We are interested in the well-posedness and regularity of the initial-boundary-value-transmission problem (IBVTP for short). Parabolic transmission problems arise in a variety of applications to composite materials, where properties vary from one material to the other, for instance heat conduction problems with or without heat loss across interfaces. They also arise in parabolic free-boundary problems, such as in the Stefan problem and in the Mullins-Sekerka model of solidification in binary alloys [43], in the context of non-linear equations.

Transmission problems for elliptic equations have been widely studied in both smooth and non-smooth settings. It is well known that elliptic regularity does not hold in general for non-smooth domains. The singular behavior has been well studied for conical or corner singularities starting with the seminal work of Kondratiev [30]. Among the extensive literature on the subject, we refer to [6, 15–19, 37, 38]. The singularities can be expressed in terms of singular exponents and singular functions, using a local coordinate representation. In particular, if the domain is convex, weak solutions with data in L^2 still satisfy Sobolev H^2 regularity. Points where the boundary conditions change behave similarly to geometric singularities at the boundary. In the case where interfaces are present, regularity holds on each side of the interface, when the interface is sufficiently regular and does not touch the boundary of the domain [13, 14, 23, 31, 35, 51, 52]. However, in many applications, the interface is itself singular and crosses the boundary. In the elliptic case the problem has been studied using various techniques, including layer potentials for Lipschitz domains and interfaces (we refer to [24–26, 29, 42] among several works on the subject), and functional analysis techniques for corner domains [6, 8, 21, 22, 39, 40, 44–46, 50].

There is significantly less work on regularity for parabolic problems in non-smooth domains and with mixed boundary conditions (see e.g. [20, 28] and references therein), and even fewer works for transmission problems (see e.g. [41] and references therein). If the contact angle is a right angle, then reflection techniques can be used [1, 48, 49]. Recently, maximal L^p -regularity was established for parabolic transmission problems on smooth domains with a general contact angle [5]. In our work, we investigate parabolic transmission problems in two-dimensional polygonal domains with general polygonal interfaces, using the framework of weighted Sobolev spaces. For elliptic equations, this framework lends itself to studying quasi-optimal convergence rates for the Finite Element Method (FEM for short) using graded meshes (see e.g. [2, 10–12, 9, 32–34]). Even when interfaces are absent, grading is necessary to achieve the optimal rate of convergence (up to constants) near vertices with aperture angle larger than π , so-called *re-entrant corners*.

For conforming schemes, such as the standard FEM, one typically exploits a variational formulation of the problem. For elliptic equations, this formulation leads to the invertibility of the forward operator, and hence solvability, via a coercivity property, in a natural

weighted space. Solvability and higher regularity for other weights can then be inferred by using Fredholm theory and the fact the index of the operator does not change under small perturbations of the weight. This approach has the advantage of not requiring a careful study of the singular functions and exponents, which can be difficult to carry out for variable coefficients operators, especially in higher dimensions. In order to calculate the index of the forward operator, one must account for certain special situations, in particular the presence of the vertices with Neumann boundary conditions on both sides, and singular points along interfaces. In fact, at these vertices the local operator pencil that describes the spectral properties of the forward operator has a larger kernel. These exceptional points can be treated one by one, by adding more singular functions to the regular part of the solution. By regular part, we mean here the part that belongs to the weighted Sobolev space where the solution lies in the case of homogeneous Dirichlet boundary conditions. In our case, because of the range of weights we consider the regular part vanishes at the vertices. In the parabolic case, a main difficulty arises in constructing a basis of functions on which to expand an approximate solution that lies in the same space as the true solution. For instance, one cannot use eigenfunctions of the Laplace operator as a Galerkin basis, as commonly done on smooth domains, since these eigenfunctions do not satisfy the transmission conditions at the interface.

In this work, we also utilize suitable weighted spaces to establish existence and uniqueness of weak solutions to the IBVTP for the operator L , and then to establish higher regularity on each side of the interface, provided the data have higher regularity and satisfy certain compatibility conditions to control time derivatives. We employ the regularity results to devise a grading scheme of an initially uniform triangular mesh of the domain Ω . The grading allows to prove quasi-optimal rates of convergence for the FEM, using polynomial elements. In fact, we show that we can use the grading scheme already introduced in [32] for elliptic problems. Our results could, in principle, be extended to 3D polyhedral domains, following in particular the approach in [8, 9] for elliptic problems. However, there are additional difficulties in the parabolic case, stemming from needed compatibility conditions between the data and at the interface.

We follow standard notation for function spaces, in particular $H^s(\Omega)$, $s > 0$, denote the L^2 -based Sobolev space of order s , while $H^{-s}(\Omega)$ denotes the dual space to $H_0^s(\Omega)$, which is the closure in $H^s(\Omega)$ of the space of smooth compactly supported functions in Ω . With $\langle \cdot, \cdot \rangle$ we denote the standard L^2 inner product. Throughout, at times we will use subscript to indicate derivatives, e.g. $u_t = \partial_t u$.

We conclude the introduction with an outline of the paper. In Section 2 we introduce the problem and the geometric set-up. Next, in Section 3 we discuss weighted Sobolev spaces and recall the well-posedness theory for elliptic problems. In Section 4 we state and prove our main results, which give well-posedness for the parabolic mixed-boundary-value-transmission problem in weighted spaces, under suitable conditions. Then in Section 5 we derive a formula to compute the singular behavior of solutions at the vertices. Lastly, in Section 6 we discuss a mixed fully-discrete scheme to numerically solve the parabolic problem and prove quasi-optimal rates of convergence for it, while in Section 7 we present some numerical tests to validate the theoretical analysis.

2 Preliminaries

We denote the domain by $\Omega \subset \mathbb{R}^2$ and assume that it is partitioned into a finite number of polygonal subdomains Ω_k , $k = 1, \dots, K$:

$$\overline{\Omega} = \bigcup_{k=1}^K \overline{\Omega_k}. \quad (1)$$

We denote $\Gamma := \bigcup_{k=1}^K \partial\Omega_k \setminus \partial\Omega$ and call it the *interface*. The interface is the union of finitely many piecewise-linear curves and may intersect the boundary of Ω . We denote by ν the unit outer normal vector to $\partial\Omega$, which is well defined except at vertices. We also give an orientation on each open sides of Γ that is compatible with a choice of orientation on each Ω_k , and denote by ν the unit normal induced by this orientation, again well defined except at vertices. We assume that the interior aperture angle α_Q at each vertex Q satisfies $0 < \alpha_Q < 2\pi$. Hence, re-entrant corners, where $\alpha_Q > \pi$, are allowed. We assume that Ω is a Lipschitz domain for simplicity, which excludes pathologies such as internal sides and vertices touching a side.

We consider a linear, strongly elliptic operator L in divergence form:

$$Lw := -\operatorname{div}(A\nabla w) = -\sum_{i,j=1}^2 \partial_j(a_{ij}\partial_i w), \quad (2)$$

for w a sufficiently regular, real-valued function on $\overline{\Omega}$. The coefficient matrix A is assumed positive definite and bounded on $\overline{\Omega}$, smooth on each subdomain $\overline{\Omega_k}$, $k = 1, \dots, K$, and it may have jump discontinuities across Γ . For simplicity, we assume also that A is symmetric, so L is formally self-adjoint. We restrict ourselves to divergence-form operators to which Theorem 1 directly applies. However, we expect that adding lower-order terms to L may be treated via a perturbation argument.

We decompose the boundary $\partial\Omega$ into two disjoint subsets, $\partial_D\Omega$ and $\partial_N\Omega$, with $\partial_D\Omega$ closed in $\partial\Omega$ with non-empty interior, so $\partial_N\Omega$ is open and possibly empty. We impose Dirichlet boundary conditions on $\partial_D\Omega$ and Neumann boundary conditions on $\partial_N\Omega$. We allow for a forcing term f , so we reduce to the case of homogeneous boundary conditions. We impose standard transmission conditions at the interface Γ . We therefore study the following problem for the parabolic operator $\partial_t + L$ on a space-time cylinder $\Omega \times (0, T)$, $0 < T \leq \infty$:

$$\begin{cases} Lu + u_t = f, & \text{in } \bigcup_{k=1}^K \Omega_k \times (0, T), \\ u = 0, & \text{on } \partial_D\Omega \times (0, T), \\ \nabla_{\nu}^A u = 0, & \text{on } \partial_N\Omega \times (0, T), \\ u = g, & \text{in } \Omega \times \{t = 0\}, \\ u^+ = u^-, \quad \nabla_{\nu}^A u^+ = \nabla_{\nu}^A u^- & \text{on } \Gamma \times (0, T), \end{cases} \quad (3)$$

where the unknown $u : \overline{\Omega} \times [0, T) \rightarrow \mathbb{R}$, ∇_{ν}^A denotes the conormal derivative associated with L , namely:

$$\nabla_{\nu}^A u := \nu \cdot A \nabla u,$$

and $\pm$ refers to non-tangential limits at the two sides of the interface away from singular points. We define the singular set associated to (3) as follows. $\mathcal{V}$ denotes the set of all geometric vertices of Ω_k , $k = 1, \dots, K$, and of all the points where the boundary conditions change. We note that $\mathcal{V}$ includes points, such as all non-smooth points of Γ and the points where Γ meets $\partial\Omega$, that are not determined by the geometry of the domain Ω . As in [39], we call such points *artificial vertices*. The reason for including the artificial vertices is that singularities in the solution at these points is of the same type as at true vertices. At artificial vertices, the interior angle is π by definition (see Fig. 1).

For reasons that will be clear later, we subdivide the set of vertices further. Any vertex on $\partial\Omega$ has two sides that lie also on the boundary, although it may also have sides that belong to the interface Γ . Then, we denote by $\mathcal{V}_1$ the set of all boundary vertices such that at least one boundary edge belongs to $\partial_D\Omega$ and we let $\mathcal{V}_2$ contain all the remaining vertices. In particular, $\mathcal{V}_1$ contains all the points where the boundary conditions change, while $\mathcal{V}_2$ contains all interior singular points of the interface Γ and all boundary points where both sides belong to $\partial_N\Omega$. We refer to the latter type of vertices as *Neumann-Neumann vertices* (see Fig. 2).

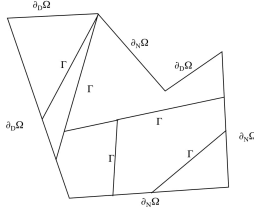


Fig. 1 The domain Ω and Interface Γ .

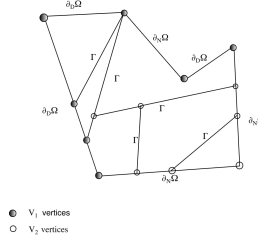


Fig. 2 Two Types of Vertices

Remark 1 The case of *cracks*, that is, side internal to Ω , where the aperture angle is 2π can also be included, introducing a geometric structure on Ω that unfolds the boundary. Then, a crack has two distinct sides in this structure. This construction allows also to handle vertices touching a side. We refer to [8, 32] for more details on this construction.

We next discuss the corresponding variational formulation of Problem (3). To this end, we introduce the following bilinear form:

$$B[u, w] = \int_{\Omega} \sum_{i,j=1}^2 a_{ij} u_{x_i} w_{x_j} dx,$$

defined on the Sobolev space $H^1(\Omega)$. We also set:

$$H_D^1(\Omega) := \{f \in H^1(\Omega) \mid u|_D = 0\},$$

where the restriction of u at the boundary is in trace sense. $(H_D^1(\Omega))'$ denotes the Banach space dual of $H_D^1(\Omega)$. This space can be canonically identified with the direct sum $H^{-1}(\Omega) \otimes H^{-1/2}(\partial_N\Omega)$, where $H^{-1/2}(\partial_N\Omega) = (H_{00}^{1/2}(\partial_N\Omega))'$ and $H_{00}^{1/2}$ is the set of all functions in $H^{1/2}(\partial_N\Omega)$ that can be extended by zero to $\partial\Omega$ (we refer the reader to [3, 36] for further details).

Definition 1 Let $f \in L^2(0, T; (H_D^1(\Omega))')$ and let $g \in L^2(\Omega)$. The function $u \in L^2(0, T; H_D^1(\Omega))$ is a weak solution of Problem (3), if $u_t \in L^2(0, T; (H_D^1(\Omega))')$ and for all $w \in L^2(0, T; H_D^1(\Omega))$:

$$B[u, w] + \langle u_t, w \rangle = \langle f, w \rangle, \quad \text{a.e. } 0 < t < T, \quad (4)$$

and $u(\cdot, 0) = g$ in trace sense.

The weak formulation contains both the homogeneous Neumann boundary condition on $\partial_N \Omega$ as well as the standard transmission conditions on Γ as the natural conditions. A regular enough solution of Problem (3) is also a weak solution by an application of Green's identities. Since u is assumed in $H^1(0, T; (H_D^1(\Omega))' \cap L^2(0, T; H_D^1(\Omega)))$, by interpolation its trace at zero is well-defined as an L^2 function (see e.g. [27]).

We study existence and uniqueness of weak solutions and possible higher regularity away from the singular set in certain weighted Sobolev spaces, motivated by analogous results for elliptic equations.

3 Weighted Sobolev Spaces

In this section, we discuss Sobolev spaces weighted by powers of the distance to the singular set. These spaces were used by Kondratiev [30] and other authors (in particular, Mazya and Plamenevskii [38]) to study elliptic equations on domains with conical points.

We begin by introducing the *weight function*, which is essentially the distance to the vertices.

$$\rho(x) = \prod_{Q \in \mathcal{V}} d(x, Q). \quad (5)$$

where $d(x, Q)$ represents the (Euclidean) distance between the point x and the vertex Q .

Given $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$, we define the m th-order *weighted Sobolev space* with weight a :

$$\mathcal{K}_a^m(\Omega) = \{u \in L_{loc}^2(\Omega), \rho^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), |\alpha| \leq m\}. \quad (6)$$

$\mathcal{K}_a^m$ is a Banach space with norm

$$\|u\|_{\mathcal{K}_a^m(\Omega)}^2 = \sum_{|\alpha| \leq m} \left\| \rho^{|\alpha|-a} \partial^\alpha u \right\|_{L^2(\Omega)}^2. \quad (7)$$

By definition $\mathcal{K}_0^0(\Omega) = L^2(\Omega)$. Furthermore, the spaces $\mathcal{K}_a^m$ form a scale, in the sense that:

$$\mathcal{K}_a^m \subset \mathcal{K}_{a'}^{m'}, \quad m' \leq m, a' \leq a.$$

Whenever needed (e.g. when one must take derivatives of the weight function), we will implicitly replace ρ with a smooth function r_Ω , which is comparable to ρ and induces an equivalent norm.

We define $\mathcal{K}_a^{-m}(\Omega)$, $m \in \mathbb{Z}_+$ by duality. Let $\mathcal{K}^a(\Omega)$ denote the closure of $C_c^\infty(\Omega)$ in $\mathcal{K}_a^m(\Omega)$. Then, $\mathcal{K}_{-a}^{-m} = (\mathcal{K}_m^a(\Omega))'$. We can also define spaces on the boundary with good trace properties as follows. $\mathcal{K}_a^m(\partial\Omega)$ is the direct sum of the corresponding spaces on each of the edges with no compatibility at the vertices. The spaces $\mathcal{K}_a^s(\partial\Omega)$, $s > 0$, can then be defined as interpolation spaces. (We refer to [39] for a more in depth discussion of the spaces $\mathcal{K}_a^m$ and their properties.)

By a weighted form of Poincaré's inequality (see e.g. [32, Lemma 3.5] for a proof),

$$\|w\|_{\mathcal{K}_1^0(\Omega)}^2 \leq C_\Omega \int_\Omega |\nabla w(x)|^2 dx, \quad (8)$$

for any $w \in H_D^1(\Omega)$, if no Neumann-Neumann vertices exist. Hence in this case,

$$H_D^1(\Omega) \subset \mathcal{K}_1^1(\Omega).$$

We note that, if u is continuous and $u \in \mathcal{H}_1^1(\Omega)$, then u must vanish at the vertices of Ω . By analogy, for general weights we set:

$$\mathcal{H}_a := \{w \in \mathcal{H}_{1+a}^1(\Omega), u = 0 \text{ on } \partial_D \Omega\}.$$

We now introduce weighted spaces adapted to the transmission problem (3). Given the partition (1) of the domain Ω , we define *broken* weighted Sobolev spaces [2, 9] by

$$\hat{\mathcal{H}}_a^m(\Omega) := \{w \in L^2(\Omega), w \in \mathcal{H}_a^m(\Omega_k), k = 1, 2, \dots, K\}, \quad (9)$$

with the induced norm:

$$\|w\|_{\hat{\mathcal{H}}_a^m(\Omega)} = \|w\|_{L^2(\Omega)}^2 + \sum_{k=1}^K \|w\|_{\mathcal{H}_a^m(\Omega_k)}.$$

The L^2 norm is redundant for positive weights.

We will analyze the IBVTP (3) by relating it to the corresponding elliptic boundary-value-transmission problem (BVTP for short):

$$\begin{cases} Lu = f, & \text{on } \cup_{k=1}^K \Omega_k, \\ u = 0, & \text{on } \partial_D \Omega, \\ \nabla_v^A u = 0 & \text{on } \partial_N \Omega; \\ u^+ - u^- = 0, \quad \nabla_v^A u^+ - \nabla_v^A u^- = 0, & \text{on } \Gamma, \end{cases} \quad (10)$$

with the associated variational formulation:

$$B(u, w) = \langle f, w \rangle, \quad \forall w \in H_D^1(\Omega),$$

for $u \in H_D^1(\Omega)$. If $\mathcal{V}_2 = \emptyset$, it follows from (8) that the bilinear form B is strictly coercive on $\mathcal{H}_0$ and the solution operator of (10) is Fredholm of index zero. Using properties of the index, in particular, its behavior under conjugation with the weight, one readily obtains well-posedness of (10) in $\mathcal{H}_a$ for a small enough.

When $\mathcal{V}_2 \neq \emptyset$, the spectral properties of the solution operator change, since solutions are allowed to be locally constant near each vertex in $\mathcal{V}_2$. Solvability is restored by augmenting the space $\mathcal{H}_0$ with a finite dimensional space $\mathcal{W}_s$. A basis for this space consists of functions $\chi_Q \in C_0^\infty(\Omega)$, one for each $Q \in \mathcal{V}_2$, where χ_Q is constant equal to 1 in a neighborhood of Q . Higher regularity in $\hat{\mathcal{H}}_{1+a}^m$ can then be bootstrapped for more regular data.

To state the main well-posedness result for (10), we recall the following notation from [32]. For $m \in \mathbb{Z}_+$ and $a \geq 0$, we set:

$$\begin{aligned} V_{a+1}^{m+1}(\Omega) = \{ & u \mid u = u_{\text{reg}} + w_s, u_{\text{reg}} \in \hat{\mathcal{H}}_{a+1}^{m+1}(\Omega) \cap \mathcal{H}_{a+1}^1(\Omega), w_s \in \mathcal{W}_s \} \\ & \cap \{ u \mid u = 0 \text{ on } \partial_D \Omega, \nabla_v^A u = 0 \text{ on } \partial_N \Omega, \nabla_v^A u^+ = \nabla_v^A u^- \text{ on } \Gamma \}. \end{aligned} \quad (11)$$

We let $P : u \mapsto f$ denote the forward operator for Problem (10).

Theorem 1 ([32]) *Let $m \in \mathbb{N}_+$. There exists $\eta = \eta(\Omega, L) > 0$ such that, if $a \in (0, \eta)$ and $f \in \hat{\mathcal{H}}_{a-1}^{m-1}(\Omega)$, then the elliptic problem (10) has a unique weak solution $u \in H_D^1(\Omega)$, which can be uniquely decomposed as $u = u_{\text{reg}} + w_s$, where $u_{\text{reg}} \in \hat{\mathcal{H}}_{a+1}^{m+1}(\Omega) \cap \mathcal{H}_a^1(\Omega)$, and $w_s \in \mathcal{W}_s$.*

Furthermore, the operator $P : u \mapsto f$ gives an isomorphism between the space $V_{a+1}^{m+1}(\Omega)$ and $\hat{\mathcal{H}}_{a-1}^{m-1}(\Omega)$, and the following estimate holds:

$$\|u_{\text{reg}}\|_{\mathcal{H}_{a+1}^1(\Omega)} + \|u_{\text{reg}}\|_{\hat{\mathcal{H}}_{a+1}^{m+1}(\Omega)} + \|w_s\|_{L^2(\Omega)} \leq C \|f\|_{\hat{\mathcal{H}}_{a-1}^{m-1}(\Omega)}, \quad (12)$$

with C a constant that depends on L and Ω .

The constant η is related to the singular exponents for the problem. We will refer to it as the *singularity constant*. We call u_{reg} the *regular part* of u , as it belongs to the weighted Sobolev space $\mathcal{K}_{a+1}^{m+1}(\Omega)$ and vanishes at the vertices, while we refer to w_s as the *singular part* of u , because it does not. Since W_s is a finite-dimensional space, we can replace the L^2 norm with any finite norm. We observe that when $\mathcal{V}_2 = \emptyset$, i.e., all the vertices in $\mathcal{V}$ have at least one Dirichlet side and there is no interior vertex on the interface Γ , $w_s = 0$ and $u = u_{reg}$. The theorem holds also for $m = 0$, but the isomorphism is $P : V_{1+a}^1 \rightarrow (V_{1+a}^1)'$.

4 Well-posedness for the parabolic transmission problem: weak solutions

We begin our analysis of Problem (3) by temporarily assuming that $\mathcal{V}_2 = \emptyset$. Our first result is an existence and uniqueness result for weak solutions in weighted spaces.

Theorem 2 *Let η be the singularity constant in the Theorem 1 and let $\mathcal{V}_2 = \emptyset$. Then there exists a constant $0 < \delta \leq \eta$, such that if $f \in L^2(0, T; (V_{a+1}^1(\Omega))')$ and if $g \in \mathcal{K}_a^0(\Omega)$ for $a \in (0, \delta)$, then there is a unique weak solution to (3) $u \in L^2(0, T; \mathcal{K}_{a+1}^1(\Omega)) \cap H^1(0, T; (V_{a+1}^1(\Omega))')$ satisfying*

$$\|u\|_{L^2(0, T; \mathcal{K}_{a+1}^1(\Omega))}^2 + \|u_t\|_{L^2(0, T; (\mathcal{V}_{a+1}^1(\Omega))')}^2 \leq \|f\|_{L^2(0, T; (\mathcal{V}_{a+1}^1(\Omega))')}^2 + \|g\|_{\mathcal{K}_a^0(\Omega)}^2.$$

A more precise estimate of δ will be presented later.

The theorem is a regularity theorem for weak solutions. Hence, the gist of the proof consists in analyzing the behavior of the solution at the singular points. The other steps in the proof generally follow a well-known approach (see e.g. the monographs [27] and [36]).

Proof We have

$$f \in L^2(0, T; (V_{a+1}^1(\Omega))') \subset L^2(0, T; (H_D^1(\Omega))'), \quad g \in \mathcal{K}_a^0(\Omega) \subset L^2(\Omega).$$

Hence, a standard Galerkin approximation gives the existence and uniqueness of a weak solution in $u \in L^2(0, T; H_D^1(\Omega))$ to Problem (3) (see e.g. [27]).

To analyze the behavior in three main steps:

Step 1. We seek an adapted weight function $\vartheta(x) > 0$ on Ω , such that:

- (a) ϑ behaves like ρ pointwise, i.e., there exist two constants $0 < C_1 < C_2$ depending only on Ω and the coefficient matrix A , such that $C_1 \rho < \vartheta < C_2 \rho$ pointwise.
- (b) ϑ is sufficiently smooth (at least C^1) inside each subdomain, except near the corners;
- (c) ϑ is continuous across the interface, that is, $\vartheta^+ - \vartheta^- = 0$ on Γ ;
- (d) $\nabla_v^A(\vartheta^a) = 0$ on $\partial_N \Omega$ and $\nabla_v^A(\vartheta^a)^+ = \nabla_v^A(\vartheta^a)^- = 0$ on Γ .

We first construct ϑ in the case that the coefficient matrix $A = [a_{ij}]$ of the operator L is constant on each subdomain. We denote a point in $\mathbb{R}^2$ by (x, y) . For each vertex $Q_i = (x_i, y_i)$, $i = 1, \dots, N$ (where N is the cardinality of $\mathcal{V}$), we denote by Ω_{il} , $l = 1, \dots, K(i)$, the subdomains of Ω that contain the vertex Q_i , where $K(i) \leq K$. Then we let

$$\vartheta_i(x, y) = \vartheta_{il}(x, y) = C_{il} \{(x - x_i, y - y_i)A^{-1}(x - x_i, y - y_i)^T\}^{1/2}, \quad (13)$$

defined in a neighborhood of the vertex Q_i in each Ω_{il} . The constants C_{il} are chosen to make ϑ_i continuous across Γ , and we can set $C_{i1} = 1$ without loss of generality. We choose a constant $C_0 > 0$ small enough so that the sets

$$\mathcal{B}_{Q_i}(C_0) = \{(x, y) \mid \vartheta_i(x, y) \leq C_0\} \quad (14)$$

for each $Q_i \in \mathcal{V}$ are pairwise disjoint in Ω . We next pick ζ_i to be a continuous function defined on $\{(x, y) \mid C_0/2 \leq \vartheta_i(x, y) \leq C_0\}$ with the following properties:

- (i) $\zeta_i = \kappa(\vartheta_i)$ for some continuous function κ defined on the closed interval $[C_0/2, C_0]$;
- (ii) $\kappa = 1$ on $[C_0/2, 2C_0/3]$, $\kappa = 0$ on $[5C_0/6, C_0]$, and κ is regular (C^1 suffices) and non-increasing on $[C_0/2, C_0]$.

We can now construct ϑ as follows. For notational convenience, we denote with ϑ_0 the constant function 1 on $\bar{\Omega}$. Then, we let

$$\vartheta(x, y) = \vartheta_0(x, y) = 1 \quad (15a)$$

if $(x, y) \in \Omega \setminus \bigcup_{Q_i \in \mathcal{V}} \mathcal{B}_{Q_i}(C_0)$; and

$$\vartheta(x, y) = \vartheta_i(x, y) \quad (15b)$$

if $(x, y) \in \mathcal{B}_{Q_i}(C_0/2)$; finally

$$\vartheta(x, y) = (1 - \zeta_i(x, y))\vartheta_0(x, y) + \zeta_i(x, y)\vartheta_i(x, y) \quad (15c)$$

if $(x, y) \in \mathcal{B}_{Q_i}(C_0)$ but $(x, y) \notin \mathcal{B}_{Q_i}(C_0/2)$. It is straightforward to check that ϑ satisfied conditions (a)–(d) (note that since ϑ_0 is constant, it satisfies the transmission conditions automatically irrespective of the values of A).

We now modify this construction for the case that A is variable and regular on each subdomain. We just sketch the construction for brevity. In the constant-coefficient case, ϑ is given by (15), that is, near each vertex Q_i it is defined on sectors emanating from the vertex Q_i determined by the values of A . When A is not constant on each subdomains, these sectors become deformed. However, sufficiently close to the vertex, they are still approximately straight sectors so that we can follow the same construction as in the constant case for a small neighborhood of each vertex Q_i , and then using again the functions ζ_i and ϑ_0 construct ϑ globally in Ω .

Step 2. We conjugate the operator L with ϑ^a . We let $\bar{u} = u/\vartheta^a$. If u solves (3), then $\bar{u}$ solves the following problem:

$$\begin{cases} \vartheta^{-a} L \vartheta^a \bar{u} + \bar{u}_t = \bar{f} & \text{in } \bigcup_{k=1}^K \Omega_k \times (0, T), \\ \bar{u} = 0 & \text{on } \partial_D \Omega \times (0, T), \\ \nabla_v^A \bar{u} = 0 & \text{on } \partial_N \Omega \times (0, T), \\ \bar{u} = \bar{g} & \text{in } \Omega \times \{t = 0\}, \\ \bar{u}^+ = \bar{u}^-, \quad \nabla_v^A \bar{u}^+ = \nabla_v^A \bar{u}^- & \text{on } \Gamma \times (0, T), \end{cases} \quad (16)$$

taking into accounts the conditions (a)–(d) satisfied by ϑ , where $\bar{f} = f/\vartheta^a \in L^2(0, T; \mathcal{H}_{-1}^{-1}(\Omega)) \subset L^2(0, T; (H_D^1(\Omega))')$, $\bar{g} = g/\vartheta^a \in L^2(\Omega)$. We solve Problem (16) via a Galerkin approximation. We observe that the transmission conditions and the Neumann condition are not those naturally associated to the conjugated operator, so we will need to construct an adapted Galerkin basis in **Step 3**.

We begin by noticing that a weak solution $\bar{u} \in L^2(0, T; H_D^1(\Omega))$ must satisfy for all $v \in H_D^1(\Omega)$:

$$\begin{aligned} \langle f - \bar{u}_t, v \rangle &= \langle \vartheta^{-a} L \vartheta^a \bar{u}, v \rangle \\ &= (\nabla \bar{u}, \nabla v)_A - a^2 \left(\frac{\bar{u} v}{\vartheta^2} \nabla \vartheta, \nabla \vartheta \right)_A + a \left(\frac{\bar{u}}{\vartheta} \nabla v, \nabla \vartheta \right)_A - a \left(\frac{v}{\vartheta} \nabla \bar{u}, \nabla \vartheta \right)_A =: B_a[\bar{u}, v] \end{aligned} \quad (17)$$

where we used the notation $(X, Y)_A := \int_{\Omega} X^T A Y \, dx$. We view (17) as an elliptic problem.

We show next that B_a is continuous and coercive on $H_D^1(\Omega)$. Continuity follows from the weighted form of Poincaré's inequality (8) and Condition (a) for ϑ :

$$\begin{aligned} B_a[\bar{u}, v] &\leq \alpha_1 \|\nabla \bar{u}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \alpha_2 \|\bar{u}/\vartheta\|_{L^2(\Omega)} \|v/\vartheta\|_{L^2(\Omega)} + \\ &\quad \alpha_3 \|\bar{u}/\vartheta\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \alpha_4 \|\nabla \bar{u}\|_{L^2(\Omega)} \|v/\vartheta\|_{L^2(\Omega)} \\ &\leq \alpha \|\nabla \bar{u}\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \quad (18) \end{aligned}$$

for some positive constant $\alpha = \alpha(a) > 0$. To ensure coercivity of the bilinear form, we need to restrict the range of the weight a . Since A is bounded above and below, it is enough to find a lower bound for B_A , assuming A is constant. In this case, we have an explicit formula for ϑ , formula (15). We then have:

$$\begin{aligned} B_a[\bar{u}, \bar{u}] &= (\nabla \bar{u}, \nabla \bar{u})_A - a^2 \left(\frac{\bar{u}^2}{\vartheta^2} \nabla \vartheta, \nabla \vartheta \right)_A \\ &\geq \bigcup_{Q_i \in \mathcal{V}} \left(\int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A (\nabla \bar{u}) dS - a^2 \int_{\mathcal{B}_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)^T A^{-1} (x-x_i, y-y_i)} dS \right) \\ &\quad + \beta_0 \|\nabla \bar{u}\|_{L^2(\Omega \setminus \bigcup_{Q_i \in \mathcal{V}} \mathcal{B}_{Q_i}(C_0/2))}^2 - \gamma a^2 \|\bar{u}\|_{L^2(\Omega \setminus \bigcup_{Q_i \in \mathcal{V}} \mathcal{B}_{Q_i}(C_0/2))}^2 \quad (19) \end{aligned}$$

for some $\beta_0, \gamma_0 > 0$ that do not depend on $\bar{u}$. We focus on the first term of (19). For each vertex Q_i we define $r_i = \inf_{\vartheta(X)=C_0/2} d(X, Q_i)$ so that the part of the disk centered at Q_i in Ω with radius r_i lies inside $\mathcal{B}_{Q_i}(C_0/2)$. We denote this sector by D_i and its aperture angle by ω_i . Since A is symmetric positive definite, by the weighted Poincaré's inequality (8) it follows that

$$\begin{aligned} \int_{\mathcal{B}_{Q_i}(C_0/2)} (\bar{u}_x, \bar{u}_y)^T A (\bar{u}_x, \bar{u}_y) dS &\geq \int_{D_i} (\bar{u}_x, \bar{u}_y)^T A (\bar{u}_x, \bar{u}_y) dS \geq \beta_1 \|\nabla \bar{u}\|_{L^2(D_i)}^2 \\ &\geq \beta_1 \left(\frac{\pi}{2\omega_i} \right)^2 \int_{D_i} \left(\frac{\bar{u}}{d((x,y), Q_i)} \right)^2 dS \geq \frac{\beta_2}{\omega_i^2} \int_{D_i} \frac{\bar{u}^2}{(x-x_i, y-y_i)^T A^{-1} (x-x_i, y-y_i)} dS \\ &\geq \frac{\beta_2}{\omega_i^2} \int_{\mathcal{B}_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)^T A^{-1} (x-x_i, y-y_i)} dS - \frac{\beta_3}{\omega_i^2 r_i^2} \|\bar{u}\|_{L^2(\mathcal{B}_{Q_i}(C_0/2))}^2. \quad (20) \end{aligned}$$

for some constant $\beta_2, \beta_3 > 0$ depending only on A . We then let $\omega = \max_{Q_i \in \mathcal{V}} \{\omega_i\}$ and choose

$\delta = \min\{\frac{\sqrt{\beta_2}}{\omega}, \eta\}$. By combining the inequalities (19) and (20) we can conclude that $a^2 < \beta_2/\omega_i^2$ for each vertex Q_i if $0 < a < \delta$. Consequently, if a lies in this range, then there exist constants $\beta, \gamma > 0$ such that

$$B_a[\bar{u}, \bar{u}] \geq \beta \|\nabla \bar{u}\|_{L^2(\Omega)}^2 - \gamma \|\bar{u}\|_{L^2(\Omega)}^2. \quad (21)$$

for all $\bar{u} \in H_D^1(\Omega)$.

Step 3. We construct an orthonormal basis of $L^2(\Omega)$, which is also orthogonal in $H_D^1(\Omega)$, adapted to the boundary and transmission conditions for Problem 16. Recall that we assume here that $\mathcal{V}_2 = \emptyset$. Then, $H_D^1(\Omega) \subset \mathcal{H}_1^1(\Omega)$. According to Theorem 1, the solution operator of the BVTP (10) maps $L^2(\Omega) \subset \mathcal{H}_{a-1}^0(\Omega)$ into $V_{1+a}^2 \subset L^2(\Omega)$ for $0 < a > \eta$ boundedly.

Hence, by Rellich Theorem it is a compact operator on $L^2(\Omega)$ and self adjoint, since A is symmetric. Consequently, there exists a countable orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions of L , $\{w_i\}_{i=1}^\infty$. But $w_i \in V_{1+a}^2$, so it satisfies the boundary and transmission conditions in (16). Hence by Green's formulas:

$$B[w_i, w_j] = \lambda_i \langle w_i, w_j \rangle = \lambda_j \langle w_i, w_j \rangle = 0$$

as $\lambda_i \neq \lambda_j$ (if $\lambda_i = \lambda_j$, we use that the basis is orthogonal with respect to the L^2 inner product); that is, $\{w_i\}_{i=1}^\infty$ is also a complete B-orthogonal basis in $H_D^1(\Omega)$. We take $\{w_i\}_{i=1}^\infty$ as the basis of the Galerkin approximation for the parabolic problem (16). For $m \in \mathbb{N}$, we define

$$\bar{u}_m(t, \cdot) = \sum_{i=1}^m d_m^i(t) w_i, \quad i = 1, 2, \dots, m.$$

and denote $\bar{u}_m(t) := \bar{u}_m(t, \cdot)$. The function $\bar{u}_m$ is an exact solution of the variational problem (6) in the span of $w_1, \dots, w_m$ if $\{d_m^i(t)\}_{i=1}^m$ satisfies the following system of ODEs:

$$B_a[\bar{u}_m(t), w_i] + (\bar{u}_m'(t), w_i) = \langle \bar{f}, w_i \rangle, \quad i = 1, 2, \dots, m, \quad (22)$$

with initial conditions

$$d_m^i(0) = (\bar{g}, w_i), \quad i = 1, 2, \dots, m,$$

where $\bar{f} = f/\vartheta^a$, $\bar{g} = g/\vartheta^a$. Given the properties of B_a , there exists a unique solution for each $m \in \mathbb{N}$. In addition, a simple energy estimate, using the coercivity and boundedness of B_a , gives

$$\begin{aligned} \|\bar{u}_m\|_{L^2(0,T;H_D^1(\Omega))}^2 + \|\bar{u}_m'\|_{L^2(0,T;H^{-1}(\Omega))}^2 + \|\bar{u}_m\|_{L^\infty(0,T;L^2(\Omega))}^2 \\ \leq C(\|\bar{f}\|_{L^2(0,T;(H_D^1(\Omega))')}^2 + \|\bar{g}\|_{L^2(\Omega)}^2), \end{aligned} \quad (23)$$

with the constant C depending only on Ω and the coefficient matrix A . Let $u_m = \vartheta^a \bar{u}_m$. For each $t > 0$, we have from the weighted Poincaré's estimate (8) that

$$\left\| \frac{u_m}{\vartheta^{a+1}} \right\|_{L^2(\Omega)} = \left\| \frac{\bar{u}_m}{\vartheta} \right\|_{L^2(\Omega)} \leq C \|\nabla \bar{u}_m\|_{L^2(\Omega)}, \quad (24a)$$

and

$$\left\| \frac{\nabla u_m}{\vartheta^a} \right\|_{L^2(\Omega)} \leq C \left(\left\| \nabla \frac{u_m}{\vartheta^a} \right\|_{L^2(\Omega)} + \left\| \frac{u_m}{\vartheta^{a+1}} \right\|_{L^2(\Omega)} \right) \leq C \|\nabla \bar{u}_m\|_{L^2(\Omega)}. \quad (24b)$$

It follows from (23)-(24) that u_m is uniformly bounded $L^2(0, T; \mathcal{K}_{a+1}^1(\Omega))$. Consequently, there exists a subsequence (not relabeled) that converges weakly to some element U of $L^2(0, T; \mathcal{K}_{a+1}^1(\Omega))$. However, since $\bar{u}_m$ satisfies (22) and the Galerkin basis is complete in $H_D^1(\Omega)$, it is straightforward to check that $\bar{U} = \vartheta^{-a} U$ satisfies (17), from which it follows immediately that U satisfies (4) and, hence, must agree with u . Then by (24) u satisfies

$$\|u\|_{L^2(0,T;\mathcal{K}_{a+1}^1(\Omega))}^2 \leq \|f\|_{L^2(0,T;(V_{a+1}^1(\Omega))')}^2 + \|g\|_{\mathcal{K}_a^0(\Omega)}^2.$$

Similarly, passing to the limit $m \rightarrow \infty$ in (23) gives

$$\|u\|_{L^\infty(0,T;\mathcal{K}_a^0(\Omega))}^2 + \|u_t\|_{L^2(0,T;(V_{a+1}^1(\Omega))')}^2 \leq \|f\|_{L^2(0,T;(V_{a+1}^1(\Omega))')}^2 + \|g\|_{\mathcal{K}_a^0(\Omega)}^2.$$

The proof of the theorem is now complete. $\square$

We next improve the regularity in time of the weak solution, if the data f and g are more regular. With Theorem 2 at hand, bootstrap of regularity follows in a manner similar to that for smooth domains.

Corollary 1 *Let u be the weak solution of (3) given in Theorem 2. If $f \in L^2(0, T; \mathcal{K}_a^0(\Omega))$ and $g \in \mathcal{K}_{a+1}^1(\Omega) \cap H_D^1(\Omega)$ with $a \in (0, \delta)$, then $u \in L^\infty(0, T; \mathcal{K}_{a+1}^1(\Omega)) \cap H^1(0, T; \mathcal{K}_a^0(\Omega))$.*

Proof We utilize the notation introduced in the proof of Theorem 2. From the proof of Theorem 2, $\bar{u}$ is a limit of approximate solutions $\bar{u}_m$, where $\bar{u}_m$ solves Problem (16), and estimate (23) holds. Then $\bar{f} \in L^2(0, T; L^2(\Omega))$, and $\bar{g} \in L^2(0, T; H_D^1(\Omega))$. By the coercivity and boundedness of the bilinear form B on $H_D^1(\Omega)$, it follows that for each $0 < t < T$,

$$\begin{aligned} \|\bar{u}_m(t)\|_{H_D^1(\Omega)}^2 &\leq \beta_4 B[\bar{u}_m(t), \bar{u}_m(t)] \leq \beta_4 (B[\bar{u}_m(0), \bar{u}_m(0)] + \int_0^t B[\bar{u}_m(\tau), \bar{u}_m'(\tau)] d\tau) \\ &\leq \beta_4 B[\bar{u}_m(0), \bar{u}_m(0)] + \beta_4 \int_0^t -(\bar{u}_m'(\tau), \bar{u}_m'(\tau)) + (\bar{u}_m'(\tau), \bar{f}(\tau)) d\tau \\ &\leq \beta_5 \|\bar{g}\|_{H_D^1(\Omega)}^2 + \beta_6 \|\bar{f}\|_{L^2(0, T; L^2(\Omega))}^2 - \frac{\beta_4}{2} \|\bar{u}_m'\|_{L^2(0, T; L^2(\Omega))}^2, \end{aligned}$$

where we used (22) in the integral. Recalling that $u_m = \vartheta^a \bar{u}^m$, we have that

$$\|u_m\|_{L^\infty(0, T; \mathcal{K}_{a+1}^1(\Omega))}^2 + \|u_m'\|_{L^2(0, T; \mathcal{K}_a^0(\Omega))}^2 \leq C(\|g\|_{\mathcal{K}_{a+1}^1(\Omega)}^2 + \|\bar{f}\|_{L^2(0, T; \mathcal{K}_a^0(\Omega))}^2). \quad (25)$$

Passing to the limit $m \rightarrow \infty$ and using the weak lower semicontinuity of the norms then yields $u \in L^\infty(0, T; \mathcal{K}_{a+1}^1(\Omega)) \cap L^\infty(0, T; H_D^1(\Omega))$ and $u_t \in L^2(0, T; \mathcal{K}_a^0(\Omega))$, with the same bound as in (25). $\square$

Remark 2 Using the above corollary, we can bootstrap space regularity once. First, we note that u solves the parabolic problem (3) strongly, that is, $Lu = f - u_t$ a.e. in $\cup_{k=1}^K \Omega_k \times T$. Since $f - u_t \in L^2(0, T; \mathcal{K}_a^0(\Omega))$, from elliptic regularity (Theorem 1) we deduce $u = P^{-1}(f - u_t) \in L^2(0, T; V_{b+1}^2(\Omega))$ for any $0 < b < \min\{\eta, a+1\}$, where η is the singularity constant in Theorem 1.

To bootstrap regularity in the space variables further, we need to impose compatibility conditions on the data. We measure higher space regularity in broken weighted spaces $\mathcal{K}_{1+a}^m$.

Lemma 1 *Let $\mathcal{V}_2 = \emptyset$. Assume that $g \in V_{a+1}^2(\Omega)$, $f_t \in L^2(0, T; (V_{a+1}^1(\Omega))')$, $f(0) - Pg \in \mathcal{K}_a^0(\Omega)$, for $a \in (0, \delta)$. Let u be the weak solution of IBVTP (3). Then, u_t belongs to $L^2(0, T; \mathcal{K}_{a+1}^1(\Omega)) \cap L^2(0, T; H_D^1(\Omega))$ and is the unique weak solution of the IBVTP*

$$\begin{cases} Lw + w_t = f_t, & \text{in } \cap_{k=1}^K \Omega_k \times (0, T), \\ w = 0, & \text{on } \partial_D \Omega \times (0, T), \\ \nabla_v^A w = 0, & \text{on } \partial_N \Omega \times (0, T), \\ w^+ = w^-, \quad \nabla_v^A w^+ = \nabla_v^A w^-, & \text{on } \Gamma \times (0, T), \\ w(0) = f(0) - Pg, & \text{on } \Omega \times \{t = 0\}. \end{cases} \quad (26)$$

Proof By classical results, there exists a unique weak solution of (26), satisfying

$$B[w, v] + \langle w_t, v \rangle = \langle f_t, v \rangle \quad \forall v \in H_D^1(\Omega) \text{ and a.e. } 0 \leq t \leq T, \quad (27)$$

along with the initial condition $w(0) = f(0) - Pg \in \mathcal{K}_a^0(\Omega)$ in trace sense. By Theorem 2 $w \in L^2(0, T; \mathcal{K}_{a+1}^1(\Omega)) \cap L^2(0, T; H_D^1(\Omega)) \cap L^\infty(0, T; \mathcal{K}_a^0(\Omega))$. Let

$$\tilde{u}(t) = \int_0^t w(\tau) d\tau + g \quad (28)$$

for each $t \in [0, T]$. By interpolation, $w \in C([0, T]; \mathcal{K}_a^0(\Omega))$, hence the function $\tilde{u}(t)$ is well defined and is Lipschitz continuous in the $\mathcal{K}_a^0(\Omega)$ norm. In particular, $(\tilde{u}_t, v) = (w, v)$ for $t \in [0, T]$ pointwise. Integrating (27) gives

$$\int_0^t B[w(\tau), v] d\tau + \int_0^t \langle w_t(\tau), v \rangle = \int_0^t \langle f_t(\tau), v \rangle d\tau.$$

Recall that v is constant in time. Given that $w \in L^2(0, T; \mathcal{K}_{a+1}^1(\Omega))$, $w_t \in L^2(0, T; (V_{a+1}^1(\Omega))')$, $f_t \in L^2(0, T; (V_{a+1}^1(\Omega))')$, we can conclude that

$$B[\tilde{u}(t) - \tilde{u}(0), v] + \langle w(t) - w(0), v \rangle = \langle f(t) - f(0), v \rangle. \quad (29)$$

As $g = \tilde{u}(0) \in V_{a+1}^2(\Omega)$, we can apply Green's formulas and get

$$B[\tilde{u}(0), v] + \langle w(0) - \langle f(0), v \rangle = \langle Lg + f(0) - Lg - f(0), v \rangle = 0, \quad (30)$$

so that (29) becomes

$$B[\tilde{u}(t), v] + \langle \tilde{u}_t(t), v \rangle = \langle f(t), v \rangle. \quad (31)$$

By uniqueness of weak solutions, $\tilde{u} = u$. Hence, $\tilde{u}_t = w = u_t$ solves (26) and has the stated regularity. $\square$

As expected, we can achieve higher regularity by imposing more compatibility conditions on the data.

Theorem 3 *Let $\delta > 0$ be as in Theorem 2 and assume $\mathcal{V}_2 = \emptyset$. Given $m \in \mathbb{N}$ and $a \in (0, \delta)$, assume $g \in \hat{\mathcal{K}}_{a+1}^{2m+1}(\Omega)$ and*

$$\begin{aligned} \frac{d^i f}{dt^i} &\in C([0, T]; \hat{\mathcal{K}}_a^{2m-1-2i}(\Omega)), & i = 0, 1, \dots, m-1, \\ \frac{d^i f}{dt^i} &\in L^2(0, T; \hat{\mathcal{K}}_a^{2m-2i}(\Omega)), & i = 0, 1, \dots, m. \end{aligned}$$

Suppose also the following m^{th} - order compatibility conditions hold:

$$\begin{aligned} g_0 &:= g \in V_{a+1}^2(\Omega) \cap \hat{\mathcal{K}}_{a+1}^{2m+1}(\Omega), \\ g_1 &:= f(0) - Pg_0 \in V_{a+1}^2(\Omega) \cap \hat{\mathcal{K}}_{a+1}^{2m-1}(\Omega), \\ &\dots \\ g_{m-1} &:= \frac{d^{m-2} f}{dt^{m-2}}(0) - Pg_{m-2} \in V_{a+1}^2(\Omega) \cap \hat{\mathcal{K}}_{a+1}^3(\Omega), \\ g_m &:= \frac{d^{m-1} f}{dt^{m-1}}(0) - Pg_{m-1} \in \mathcal{K}_{a+1}^1(\Omega) \cap H_D^1(\Omega) \end{aligned} \quad (32)$$

If u is the weak solution of Problem (3) on $[0, T]$, then

$$\frac{d^i u}{dt^i} \in L^\infty(0, T; \hat{\mathcal{K}}_{a+1}^{2m+1-2i}(\Omega)) \cap L^\infty(0, T; \mathcal{K}_{a+1}^1(\Omega)) \quad i = 0, 1, \dots, m;$$

and the following estimate holds

$$\begin{aligned} \sum_{i=0}^m \left\| \frac{d^i u}{dt^i} \right\|_{L^\infty(0,T; \hat{\mathcal{K}}_{a+1}^{2m+1-2i}(\Omega))} &\leq C \left(\sum_{i=0}^m \left\| \frac{d^i f}{dt^i} \right\|_{L^2(0,T; \mathcal{K}_a^{2m-2i}(\Omega))} \right. \\ &\quad \left. + \sum_{i=0}^{m-1} \left\| \frac{d^i f}{dt^i} \right\|_{L^\infty(0,T; \hat{\mathcal{K}}_a^{2m-1-2i}(\Omega))} + \sum_{i=0}^m \|g_i\|_{\hat{\mathcal{K}}_{a+1}^{2m+1-2i}(\Omega)} \right), \end{aligned}$$

with C depending on m , Ω , T and the coefficients of L .

The proof uses induction on m and Lemma 1. For brevity's sake, we omit the proof and refer to [53].

Remark 3 We give an example of data that satisfy the compatibility conditions (32). We recall that w_j is an eigenfunction of the operator L satisfying the boundary and transmission conditions. Then, Theorem 1 allows to bootstrap the regularity of the eigenfunctions and guarantees that $w_j \in V_{a+1}^2 \cap \hat{\mathcal{K}}_{a+1}^{2m+1}(\Omega)$ for all $m \in \mathbb{N}$. Next, we pick f and g to be finite linear combinations of the eigenfunctions:

$$g = c_1 w_1 + c_2 w_2 + \dots + c_N w_N,$$

and

$$f = d_1(t)w_1 + d_2(t)w_2 + \dots + d_N(t)w_N,$$

where $\{c_l\}_{l=1}^N \in \mathbb{R}$, $\{d_l(t)\}_{l=1}^N \in C^m[0, T]$. Then the compatibility conditions automatically hold.

We now turn to the case when $\mathcal{V}_2 \neq \emptyset$, which means that Neumann-Neumann vertices are present and the interface Γ has singular points. We study well-posedness of the parabolic problem again by viewing it as an elliptic problem with time-dependent forcing and exploiting Theorem 1. Higher regularity can be obtained by imposing more compatibility conditions on the data in the spirit of Theorem 3.

Theorem 4 *Let u be the unique weak solution of the parabolic problem (3) on $[0, T]$. Let η be the singularity constant in Theorem 1 and assume that*

$$\begin{aligned} f_{tt} &\in L^2(0, T; (H_D^1(\Omega))'), \quad f_t \in C(0, T; L^2(\Omega)), \quad f \in C(0, T; \hat{\mathcal{K}}_a^2(\Omega)), \\ g &\in V_{a+1}^2(\Omega), \quad g_1 = f(0) - Pg \in V_{a+1}^2(\Omega), \quad g_2 = f_t(0) - Pg_1 \in L^2(\Omega). \end{aligned}$$

Then, the weak solution u of Problem 3 can be decomposed as $u = u_{reg} + w_s$, where

$$u_{reg} \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; \hat{\mathcal{K}}_{a+1}^2(\Omega)) \cap L^\infty(0, T; \hat{\mathcal{K}}_{a+1}^4(\Omega));$$

and $w_s \in W_s$. In addition, the following bound holds:

$$\begin{aligned} &\|w_s\|_{L^\infty(0,T; \hat{H}^2(\Omega))} + \|u_{reg}\|_{W^{2,\infty}(0,T; L^2(\Omega))} + \|u_{reg}\|_{W^{1,\infty}(0,T; \hat{\mathcal{K}}_{a+1}^2(\Omega))} + \|u_{reg}\|_{L^\infty(0,T; \hat{\mathcal{K}}_{a+1}^4(\Omega))} \\ &\leq C(\|f_{tt}\|_{L^2(0,T; (H_D^1(\Omega))')} + \|f_t\|_{C(0,T; L^2(\Omega))} + \|f\|_{C(0,T; \hat{\mathcal{K}}_a^2(\Omega))} + \|g\|_{V_{a+1}^2(\Omega)} \\ &\quad + \|f(0) - Pg\|_{V_{a+1}^2(\Omega)}), \end{aligned}$$

with C depending only on Ω , a , and L .

Proof Existence and uniqueness of the weak solution follows exactly as in the case $\mathcal{V}_2 = \emptyset$. Differentiating (3) twice gives that $u_{tt} \in L^\infty(0, T; L^2(\Omega))$ by the Galerkin approximation and energy estimates. Since $Pu_t = f_t - u_{tt}$ for a.e. $t > 0$, where P is the solution operator for the elliptic problem (10), and P gives an isomorphism from $V_{a+1}^2(\Omega)$ to $\mathcal{H}_{a-1}^0(\Omega) \supset L^2(\Omega)$, we obtain from (12) that $u_t = P^{-1}(f_t - u_{tt}) \in L^\infty(0, T; V_{a+1}^2(\Omega))$. Similarly, $u = P^{-1}(f - u_t)$ implies a bound on the $L^\infty(0, T; \mathcal{H}_{a+1}^4)$ norm of u . $\square$

5 Vertex Singularities

In this section, we study more closely the singularities in the solution of Problem 10 at the vertices, in the case that the operator L has piecewise-constant coefficients in each subdomain Ω_k , $k = 1, \dots, K$, generalizing the results of Kondratiev [30] for the Laplace operator and of Nicaise [45]-[46] for L a scalar multiple of the Laplacian in each Ω_k . We assume $\mathcal{V}_2 = \emptyset$ in order to provide an explicit characterization of the first singular exponent, which is determined by the value of η . Being able to compute or estimate η will be important for applications to the FEM in Section 6.

Theorem 5 *Let $\eta > 0$ be the singularity constant in Theorem 1, and let $\delta > 0$ be the constant in Theorem 2. Assume $\mathcal{V}_2 = \emptyset$. Assume also the coefficient matrix A of the operator L is constant on each subdomain Ω_k , $k = 1, 2, \dots, K$. Then*

$$\max \delta^2 = \eta^2 = \inf_{Q_i \in \mathcal{V}_1, \bar{u} \in H_D^1(\Omega), \bar{u} \neq 0} \frac{\int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A (\nabla \bar{u}) dS}{\int_{\mathcal{B}_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i) A^{-1} (x-x_i, y-y_i)^T} dS}, \quad (33)$$

where the sets $\mathcal{B}_{Q_i}$ are defined in (14) and $Q_i = (x_i, y_i)$.

Remark 4 It is known that the singular exponents can be computed by solving suitable eigenvalue problems after blowing up the vertex singularities. Then, formula (33) can be thought of as a Rayleigh quotient for these eigenvalue problems. In particular, if $L = -\Delta$ on Ω , then we recover the well-known formula

$$\max \delta = \eta = \min_{Q_i \in \mathcal{V}} \left\{ \frac{\pi}{\sigma_i \omega_i} \right\},$$

where ω_i is the interior angle of Ω in the vertex Q_i , $\sigma_i = 1$ if Q_i has two Dirichlet sides, $\sigma_i = 2$ if Q_i has one Neumann side and one Dirichlet side. (We refer to [53] for a proof of this fact.)

Proof We first prove that $\max \delta = \eta$ and then establish formula (33) by using polar coordinates near each vertex Q_i and by solving a one-dimensional eigenvalue problem. In the course of the proof, we denote

$$F_i(\bar{u})^2 = \frac{\int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A (\nabla \bar{u}) dS}{\int_{\mathcal{B}_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i) A^{-1} (x-x_i, y-y_i)^T} dS},$$

for $\bar{u} \in H_D^1(\Omega)$, $\bar{u} \neq 0$, and take the positive root. By convention, we set $F_i(\bar{u}) = \infty$ if $\bar{u}$ is identically zero in $\mathcal{B}_{Q_i}(C_0/2)$. Estimate (20) shows that $\inf_{\bar{u} \in H_D^1(\Omega)} F_i(\bar{u}) > 0$ for each $Q_i \in \mathcal{V}$.

We recall the weight function ϑ is given in (15). If $0 < a < \inf_{\bar{u} \in H_D^1(\Omega)} F_i(\bar{u})$ for each i , then

$$\begin{aligned} & \int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A(\nabla \bar{u}) dS - a^2 \int_{\mathcal{B}_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)A^{-1}(x-x_i, y-y_i)^T} dS \\ & \geq (1 - \frac{a^2}{\inf_{\bar{u} \in H_D^1(\Omega)} F_i^2(\bar{u})}) \int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A(\nabla \bar{u}) dS \geq C \|\nabla \bar{u}\|_{L^2(\mathcal{B}_{Q_i}(C_0/2))}^2 \end{aligned} \quad (34)$$

for some constant $C > 0$ depending on Ω , a , and A , so that for all non-zero $\bar{u} \in H_D^1(\Omega)$ it follows from (19) that

$$B_a[\bar{u}, \bar{u}] \geq \beta \|\nabla \bar{u}\|_{L^2(\Omega)}^2 - \gamma \|\bar{u}\|_{L^2(\Omega)}^2 \quad (35)$$

for some $\beta, \gamma > 0$. We choose $\bar{u} = u\vartheta^a$, where u is the weak solution of the elliptic problem (10). Then,

$$B_a[\bar{u}, v] = \langle \bar{f}, v \rangle \quad \forall v \in H_D^1(\Omega),$$

with $\bar{f} = f/\vartheta^a$. Next, we prove that $\|\bar{u}\|_{L^2(\Omega)} \leq C \|\bar{f}\|_{(H_D^1(\Omega))'}$ for some constant $C > 0$. In fact, by contradiction suppose there exist sequences $\{\bar{f}_k\}_{k=1}^\infty \subset (H_D^1(\Omega))'$ and $\{\bar{u}_k\}_{k=1}^\infty \subset H_D^1(\Omega)$ such that

$$\|\bar{u}_k\|_{L^2(\Omega)} > k \|\bar{f}_k\|_{(H_D^1(\Omega))'}, \quad B_a[\bar{u}_k, v] = \langle \bar{f}_k, v \rangle \quad \forall v \in H_D^1(\Omega).$$

Without loss of generality, we may assume $\|\bar{u}_k\|_{L^2(\Omega)} = 1$, so that $\bar{f}_k \rightarrow 0$ in $(H_D^1(\Omega))'$. By (35) and Cauchy's inequality,

$$\beta \|\nabla \bar{u}_k\|_{L^2(\Omega)}^2 - \gamma \|\bar{u}_k\|_{L^2(\Omega)}^2 \leq B_a[\bar{u}_k, \bar{u}_k] = \langle \bar{f}_k, \bar{u}_k \rangle \leq C \left(\|\bar{u}_k\|_{H_D^1(\Omega)}^2 + \|\bar{f}_k\|_{(H_D^1(\Omega))'}^2 \right). \quad (36)$$

The sequence $\{\bar{u}_k\}_{k=1}^\infty$ is therefore bounded in $H_D^1(\Omega)$ and there exists a subsequence satisfying

$$\bar{u}_{k_j} \rightharpoonup \bar{u}_0 \quad \text{in } H_D^1(\Omega), \quad \bar{u}_{k_j} \rightarrow \bar{u}_0 \quad \text{in } L^2(\Omega).$$

In particular, $\|\bar{u}_0\|_{L^2(\Omega)} = 1$. Since $\bar{f}_k \rightarrow 0$ in $(H_D^1(\Omega))'$, we deduce

$$B_a[\bar{u}_0, v] = 0 \quad \forall v \in H_D^1(\Omega),$$

which implies $\bar{u}_0 = 0$, a contradiction. From (36) it follows that $\|\bar{u}\|_{H_D^1(\Omega)} \leq C \|\bar{f}\|_{(H_D^1(\Omega))'}$, or equivalently

$$\|u\|_{\mathcal{H}_{a+1}^1(\Omega)} \leq \|f\|_{(\mathcal{H}_{a+1}^1(\Omega))'}, \quad (37)$$

Since by Theorem 1, we have solvability in $\mathcal{H}_{1+a}^1$ for $0 < a < \eta$, it follows that

$$\eta \geq \inf_{Q_i \in \mathcal{I}_1, \bar{u} \in H_D^1(\Omega), \bar{u} \neq 0} \left(\frac{\int_{\mathcal{B}_{Q_i}(C_0/2)} (\nabla \bar{u})^T A(\nabla \bar{u}) dS}{\int_{\mathcal{B}_{Q_i}(C_0/2)} \frac{\bar{u}^2}{(x-x_i, y-y_i)A^{-1}(x-x_i, y-y_i)^T} dS} \right)^{1/2} =: \bar{F}.$$

From the proof of Theorem 2 we have that δ is chosen to satisfy (21). Therefore, $\max \delta = \eta$ by (35) above.

Next, we are going to show that $\bar{F}$ is the optimal value for η , that is, $\eta \leq \bar{F}$. To establish this upper bound, we will explicitly compute $\bar{F}$, using the special form of the operator L .

First, we can reduce to the case when L is a multiple of the Laplacian in each subdomain Ω_k , $k = 1, \dots, K$ by essentially diagonalizing the matrix A . We fix a vertex $Q_i \in \mathcal{V}$. Since formula (33) is local near each vertex, we are only interested in the part of the interface that joins Q_i (see Fig. 2, top vertex). As in the proof of Theorem 2, we denote the subdomains adjacent to the vertex Q_i by Ω_{il} , $l = 1, \dots, K(i)$. Let ω_i denote the aperture angle of the corner with vertex Q_i in Ω , and let (r, ϕ) denote polar coordinates centered at Q_i with main axis along one of the Dirichlet sides of the corner. Because we assume that $\mathcal{V}_2 = \emptyset$, there is always at least one side of the corner with vertex Q_i with Dirichlet boundary conditions. Then sufficiently close to the vertex Q_i , each subdomain Ω_{il} is a sector centered at Q_i with angle $\omega_{l+1} - \omega_l$, $l = 0, \dots, K(i)$, where $\omega_{i,0} = 0$ and $\omega_{i,K(i)} = \omega_i$. Without loss of generality, we can label the vectors in such a way that $0 = \omega_{i,0} < \omega_{i,1} < \dots < \omega_{i,K(i)} = \omega_i$. We recall the definition of $\mathcal{B}_{Q_i}(C_0/2)$:

$$\mathcal{B}_{Q_i}(C_0/2) = \{ (x, y) \mid \vartheta_i(x, y) = C_{il} \{ (x - x_i, y - y_i) A^{-1} (x - x_i, y - y_i)^T \}^{1/2} \leq C_0/2 \}.$$

We next diagonalize the matrix $A = A_{il}$ in each Ω_{il} (recall A is symmetric and definite positive) via an orthogonal matrix R_{il} and then apply a dilation matrix Λ_{il} to rescale the eigenvalues to be all equal to a positive constant d_{il} . We set $\Phi_{il} = \Lambda_{il} R_{il}$. We introduce a new coordinate system (r', ϕ') obtained by applying Φ_{il} in each sector $\Omega_{il} \cap \mathcal{B}_{Q_i}(C_0/2)$:

$$(r' \cos(\phi'), r' \sin(\phi')) = \Phi_{il} (r \cos(\phi), r \sin(\phi)), \quad \phi \in (\omega_l, \omega_{l+1}), \quad 0 < r < C_0/2.$$

We choose the constants d_{il} so that this transformation is continuous across the interface between different sectors and fixes the Dirichlet side corresponding to $\phi = 0$. That is, we take the matrix Φ satisfying:

- (i) $\text{Det}(\Phi_{il}) > 0$;
- (ii) $\Phi_{il}(r, 0) = (r, 0)$;
- (iii) $\Phi_{il}(r \cos(\omega_l), r \sin(\omega_l)) = P_{i, l+1}(r \cos(\omega_l), r \sin(\omega_l))$, $l = 1, 2, \dots, L-1$;
- (iv) $\Phi_{il} A_{il} \Phi_{il}^T = d_{il} I$ for some constant $d_{il} > 0$.

These requirements uniquely determine Φ . Under the transformation Φ , each sector $\Omega_{il} \cap \mathcal{B}_{Q_i}(C_0/2)$ is mapped onto another sector centered at Q_i :

$$\mathcal{B}_{Q_i}(r'_{i,l}/2) = \{ (r', \phi') \mid \frac{C_{il}}{\sqrt{d_{il}}} r' \leq C_0/2, \omega'_{i, l-1} \leq \phi' \leq \omega'_{i,l} \},$$

with $0 = \omega'_0 < \omega'_{i,1} < \omega'_{i,2} < \dots < \omega'_{i,L} = \omega'_i < 2\pi$, and $r'_{i,l} = \frac{C_0 \sqrt{d_{il}}}{C_{il}}$ for each l . Because of Condition (iii) on Φ , $r'_{i,l} = r_{i, l+1}$ and we denote this common value simply as r'_0 , suppressing the dependence of i .

Let now $\bar{u} \in H_B^1(\Omega)$. With slight abuse of notation, we write $\bar{u}(r, \phi)$ for $\bar{u}(r \cos(\phi), r \sin(\phi))$ in $\mathcal{B}_{Q_i}(C_0/2)$. We let $\hat{u}(r', \phi')$ be the image of $\bar{u}(r, \phi)$. Then $\hat{u}$ is well defined on the sector $\mathcal{B}'_{Q_i}(r'_0)$ and $(\nabla \hat{u}) = (\Phi^{-1})^T (\nabla \bar{u})$. A simple change of variables show that $F_i^2(\bar{u})$ is transformed into:

$$\hat{F}_i^2(\hat{u}) = \frac{\sum_{l=1}^{K(i)} (\text{Det}(A_{il}))^{1/2} (\|\hat{u}_{r'}\|_{L^2(\mathcal{B}'_{Q_i}(r'_0/2) \cap \Omega_{il})}^2 + \|\hat{u}_{r'}/r'\|_{L^2(\mathcal{B}'_{Q_i}(r'_0/2) \cap \Omega_{il})}^2)}{\sum_{l=1}^{K(i)} (\text{Det}(A_{il}))^{1/2} \|\hat{u}/r'\|_{L^2(\mathcal{B}'_{Q_i}(r'_0/2) \cap \Omega_{il})}^2}. \quad (38)$$

It is also straightforward to show that, if $\bar{u}$ satisfies (16), then locally near the vertex Q_i , $\hat{u}$ is harmonic in each sector $\omega'_{i,l} < \phi' < \omega'_{i, l+1}$, $l = 0, \dots, K(i) - 1$, satisfies homogeneous

Dirichlet or homogeneous Neumann conditions on the sides $\phi' = \omega'_0 = 0$ and $\phi' = \omega'_i$, depending on whether the vertex Q_i was a Dirichlet-Dirichlet or a Dirichlet-Neumann vertex, and satisfies standard transmission conditions (continuity of $\hat{u}$ and its conormal derivative) at $\phi' = \omega'_{il}$, $l = 1, \dots, K(i) - 1$.

Our task is therefore reduced to computing $F_i(\bar{u})$ for each $Q_i \in \mathcal{V}$, in the case that the operator L is a constant multiple of the Laplacian, $L = -C_k \Delta$, $C_k > 0$, in each subdomain Ω_k , $k = 1, \dots, K$. Passing to polar coordinates (r, ϕ) again near the vertex Q_i gives:

$$F_i^2(\bar{u}) = \frac{\sum_{l=1}^{K(i)} C_{il} (\|\bar{u}_r\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2 + \|\bar{u}_\phi/r\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2)}{\sum_{l=1}^{K(i)} C_{il} \|\bar{u}/r\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2} \quad (39)$$

$$\geq \frac{\sum_{l=1}^{K(i)} C_{il} \|\bar{u}_\phi\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2}{\sum_{l=1}^{K(i)} C_{il} \|\bar{u}\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2}. \quad (40)$$

To minimize the right-hand side of (40), we consider a minimization problem for functions of the angle $0 < \phi < \omega_i$. Specifically, we characterize

$$\min_{\bar{u} \in H^1[0, \omega], \bar{u}(0)=0} \frac{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}_\phi^2 d\phi}{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}^2 d\phi} \quad (41)$$

when Q_i is a Neumann-Dirichlet vertex, or

$$\min_{\bar{u} \in H^1[0, \omega], \bar{u}(0)=\bar{u}(\omega)=0} \frac{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}_\phi^2 d\phi}{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}^2 d\phi} \quad (42)$$

when Q_i is a Dirichlet-Dirichlet vertex. The 1D Poincaré's inequality guarantees that the infimum is strictly positive in both cases. We denote it by λ_{i0}^2 . We will prove that the minimum is achieved.

We assume first that Q_i is a Neumann-Dirichlet vertex. Suppose that $\bar{u} \in H^1[0, \omega]$ is the minimizer of (41), then it is a stationary point of the functional in (41), which gives:

$$\frac{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}_\phi^2 d\phi}{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}^2 d\phi} = \frac{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}_\phi v_\phi d\phi}{\sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u} v d\phi} = \lambda_{i0}^2 \quad (43)$$

for all $v \in H^1[0, \omega]$ with $v(0) = 0$. Integrating by parts,

$$\begin{aligned} \sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}_\phi v_\phi d\phi &= - \sum_{l=1}^{K(i)} \int_{\omega_{i,l-1}}^{\omega_{il}} C_{il} \bar{u}_{\phi\phi} v d\phi \\ &\quad + \sum_{l=1}^{L-1} (C_{i,l+1} \bar{u}_\phi^+(\omega_l) - C_{il} \bar{u}_\phi^-(\omega_l)) v(\omega_l) + C_{iL} \bar{u}_\phi(\omega_i) v(\omega_i). \end{aligned}$$

As v is arbitrary, $\tilde{u}$ is the unique weak solution to the following eigenvalue problem

$$\begin{cases} -\tilde{u}_{\phi\phi} = \lambda_{i0}^2 \tilde{u}, & \text{in } (\omega_{i\ l-1}, \omega_{il}), \ l = 1, 2, \dots, K(i), \\ \tilde{u}(0) = 0, \\ \tilde{u}_{\phi}(\omega_i) = 0, \\ \tilde{u}^+(\omega_{il}) - \tilde{u}^-(\omega_{il}) = 0, & l = 1, 2, \dots, K(i) - 1, \\ C_{i\ l+1} \tilde{u}_{\phi}^+(\omega_{il}) - C_{il} \tilde{u}_{\phi}^-(\omega_{il}) = 0, & l = 1, 2, \dots, K(i) - 1, \end{cases} \quad (44)$$

with λ_{i0}^2 the smallest eigenvalue. We can solve explicitly this 1D eigenvalue problem to show the existence of a unique minimizer.

Similarly, for a Dirichlet-Dirichlet vertex Q_i , the minimizer $\bar{u}$ satisfies the following eigenvalue problem for the minimal eigenvalue:

$$\begin{cases} -\tilde{u}_{\phi\phi} = \lambda_{i0}^2 \tilde{u}, & \text{in } (\omega_{i\ l-1}, \omega_{il}), \ l = 1, 2, \dots, K(i), \\ \tilde{u}(0) = \tilde{u}(\phi) = 0, \\ \tilde{u}^+(\omega_{il}) - \tilde{u}^-(\omega_{il}) = 0, & l = 1, 2, \dots, K(i) - 1, \\ C_{i\ l+1} \tilde{u}_{\phi}^+(\omega_{il}) - C_{il} \tilde{u}_{\phi}^-(\omega_{il}) = 0 & l = 1, 2, \dots, K(i) - 1. \end{cases} \quad (45)$$

Again, the existence of a unique minimizer can be established by ODE methods.

Now, a direct calculation shows that λ_{i0}^2 is a lower bound for $F_i^2(\bar{u})$

$$F_i^2(\bar{u}) \geq \frac{\sum_{l=1}^{K(i)} C_{il} \|\bar{u}_{\phi}\|_{\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il}}^2}{\sum_{l=1}^{K(i)} C_{il} \|\bar{u}\|_{\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il}}^2} \geq \lambda_{i0}^2,$$

We prove next that it is actually the infimum over all restrictions to the sector $\mathcal{B}_{Q_i}(C_0/2)$ of functions $\bar{u} \in H_D^1(\Omega)$. To this end, for each $\varepsilon > 0$ we let

$$\bar{u}(r, \phi) = \chi(r) r^{\varepsilon} \tilde{u}(\phi) \in H_D^1(\Omega),$$

where $\chi(r)$ is a smooth cut-off function that it equals 1 on $[0, C_0/2]$, and equals 0 on $[C_0, \infty)$. We have then:

$$\begin{aligned} \lambda_{i0}^2 &\leq \inf_{\bar{u} \in H_D^1(\Omega)} F_i^2(\bar{u}) = \inf_{\bar{u} \in H_D^1(\Omega)} \frac{\sum_{l=1}^{K(i)} C_{il} (\|\bar{u}_r\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2 + \|\bar{u}_{\phi}/r\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2)}{\sum_{l=1}^L C_{il} \|\bar{u}/r\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2} \\ &\leq \lim_{\varepsilon \rightarrow 0} \frac{\sum_{l=1}^L C_{il} (\|\varepsilon r^{\varepsilon-1} \tilde{u}(\phi)\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2 + \|r^{\varepsilon-1} \tilde{u}_{\phi}(\phi)\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2)}{\sum_{l=1}^L C_{il} \|r^{\varepsilon-1} \tilde{u}(\phi)\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2} \\ &\leq \lim_{\varepsilon \rightarrow 0} \frac{\sum_{l=1}^L C_{il} \|r^{\varepsilon-1} \tilde{u}_{\phi}(\phi)\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2}{\sum_{l=1}^L C_{il} \|r^{\varepsilon-1} \tilde{u}(\phi)\|_{L^2(\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{il})}^2} = \frac{\sum_{l=1}^L \int_{\omega_{i\ l-1}}^{\omega_{il}} C_{il} \tilde{u}_{\phi}^2 d\phi}{\sum_{l=1}^L \int_{\omega_{i\ l-1}}^{\omega_{il}} C_{il} \tilde{u}^2 d\phi} = \lambda_{i0}^2. \end{aligned}$$

Therefore

$$\inf_{\bar{u} \in H_D^1(\Omega), \bar{u} \neq 0} F_i(\bar{u}) = \min_{Q_i \in \mathcal{V}} \lambda_{i0}, \quad (46)$$

It remains to show that $\eta \leq \lambda_{i0}$ for all $Q_i \in \mathcal{V}$. We construct a weak solution of Problem 10 for a suitable f , for which the regularity result in Theorem 1 does not hold if $\eta > \lambda_{i0}$ for some $Q_i \in \mathcal{V}$. In fact, we pick $f = 0$ in (10) and choose $a = \lambda_{i0}$. We let

$$\bar{u}(r, \phi) = \chi(r) r^{\lambda_{i0}} \tilde{u}_\phi(\phi),$$

where $\chi(r)$ is a smooth cut-off function that equals 1 on the interval $[0, C_0/2]$ and vanishes on $[C_0, \infty)$. Then $\bar{u}$ is supported on the sector $\mathcal{B}_{Q_i}(C_0)$, and

- (1) $\bar{u}$ satisfies the boundary conditions and the transmission conditions on the interface.
- (2) $\Delta \bar{u} = 0$ on the sectors $\mathcal{B}_{Q_i}(C_0/2) \cap \Omega_{i\ell}$, which implies $\Delta \bar{u} = 0$ in $\mathcal{K}_{\lambda_{i0}}^0(\Omega)$.
- (3) $\bar{u} \notin \mathcal{K}_{\lambda_{i0}+1}^1(\Omega)$, since $1/r \notin L^2(\Omega)$.

This example can be modified to show that, for the parabolic problem (3), necessarily $\delta \leq \lambda_{i0}$ for all $Q_i \in \mathcal{V}$. We omit the details. $\square$

Example 1 Assume Ω has interior aperture angle $\pi/4$ at one of its vertices Q , and the domain is split into two sub-polygons Ω_1 and Ω_2 . Also, we assume the coefficient matrix A takes values

$$A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 100 & 100 \\ 100 & 200 \end{pmatrix},$$

in Ω_1 and Ω_2 respectively. Then, the transformation matrix Φ takes the form

$$\Phi_1 = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} 2 & -1/2 \\ -1 & 3/2 \end{pmatrix},$$

with corresponding values $d_1 = 1/2$, $d_2 = 250$. Moreover,

$$\Phi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Phi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \Phi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}, \quad \Phi_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 5/2 \end{pmatrix},$$

so that $\omega'_1 = \arctan(1/3)$ and $\omega'_2 = 3\pi/4$. The value of λ_0 is determined by solving the following eigenvalue problem

$$\begin{cases} -\tilde{u}_{\phi'\phi'} = \lambda_0^2 \tilde{u}, & \text{in } (0, \omega'_1) \cup (\omega'_1, 3\pi/4), \\ \tilde{u}(0) = 0, \quad \tilde{u}_{\phi'}(3\pi/4) = 0, \\ \tilde{u}^+(\omega'_1) - \tilde{u}^-(\omega'_1) = 0, \quad \tilde{u}_{\phi'}^+(\omega'_1) - 100 \tilde{u}_{\phi'}^-(\omega'_1) = 0. \end{cases} \quad (47)$$

which gives $\lambda_0 \approx 0.1332$.

Remark 5 One common approach to constructing solutions of parabolic problems, even for semi-linear and some quasi-linear problem is by semi-group methods (see e.g. [47]). It is a natural question whether the operator L generates a C^0 semigroup on the weighted space $\mathcal{K}_a^0(\Omega)$. There are indications that this is not the case (we refer to [53] for further discussion on this point). We reserve to study this problem further in future work.

6 Applications to the FEM for parabolic problems

In two final sections, we discuss applications of the regularity results in weighted spaces for the IBVTP (3) to the implementation of the FEM. We use the FEM only for the space discretization, while we implement the backward Euler method in time. We do not assume $\mathcal{V}_2 = \emptyset$. In particular, we present numerical tests where Neumann-Neumann vertices and singular interfaces are present. The numerical scheme is not optimized — for instance, we do not make our method adaptive, though adaptivity is possible — as our examples have only pedagogical purpose.

We utilize a known open-source package, FEM2d_HEAT by J. Albery, C. Carstensen and S. A. Funken [4], implemented in MATLAB[®], but adapted to the operators and the geometric set-up considered in this work. In this package, the mass matrix is inverted via an LU decomposition. We choose to employ a backward first-order scheme in time for numerical efficiency and stability.

Our point of departure is a particular implementation of the FEM on polygonal domains for mixed-boundary-value-transmission elliptic problems that uses graded meshes and yields quasi-optimal rates of convergence [32]. Here we extend this scheme to the parabolic case and prove quasi-optimal rates of convergence for a mixed fully discrete scheme. There is an extensive literature on implementing the FEM on polygonal domains with focus on rates of convergence. We refer to the discussion in the Introduction. The graded mesh is obtained from a sequence of triangulations $\{\mathcal{T}_n\}_{n \geq 0}$ of Ω with the property that $\mathcal{T}_n$ consists of uniform triangles away from the vertices, but the triangles are graded near the vertices. The grading depends on the order m of the polynomials used for the FEM approximation and on the singularity constant η in Theorem 1. The grading is obtained as follows. Let $\mathcal{T}_0$ be a given uniform triangulation of Ω such that each vertex in $\mathcal{V}$ is also a vertex in $\mathcal{T}_0$, each triangle in $\mathcal{T}_0$ contains only one vertex in $\mathcal{V}$ and the sides of the interface Γ are also sides of triangles in $\mathcal{T}_0$. We also choose the size of each initial triangle small enough. Then $\mathcal{T}_n$ is constructed from $\mathcal{T}_{n-1}$ inductively. Let T be a triangle in $\mathcal{T}_{n-1}$ and let A be a vertex of T . We divide the side AB (and similarly for the other side of the triangle with vertex at A) according to the following rule. If $A = Q \in \mathcal{V}$ then we divide AB into AC and CB such that $|AC| = \kappa_Q |AB|$ for a certain constant κ_Q , otherwise we divide the side AB into two equal parts (see Fig. 3-4). The positive constant κ_Q cannot exceed $2^{-m} \eta_Q$, where η_Q is the singularity constant near the vertex Q , which can be computed by using Theorem 5. In fact, $\eta_Q = \lambda_{i0}$ in the notation of that proof of Theorem 5, where $Q = Q_i$. So, the grading is adapted to each vertex and there might not be any special grading at some vertex, as exemplified in our numerical tests. The refinement procedure and the choice of initial mesh guarantees that at each step in the refinement each vertex in $\mathcal{V}$ belongs to a triangle in the mesh and the interface is aligned with sides of elements of the mesh. So, no triangles in $\mathcal{T}_n$ can cross the interface. This refinement also preserves the quality of the mesh.

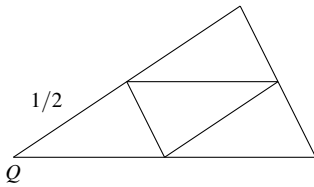


Fig. 3 Uniform Triangular Mesh

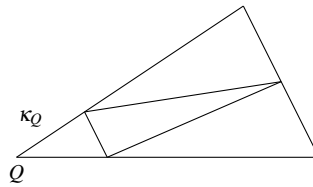


Fig. 4 Mesh Refinement with Grading

We let $S_n := S(\mathcal{T}_n, m) \subset H_D^1(\Omega)$ denote the finite element space of continuous functions on Ω that restrict to a polynomial of degree $m \geq 1$ on each triangle of $\mathcal{T}_n$. We recall the treatment of the elliptic problem (10) from [32]. We let $u_n \in S_n$ be the finite element approximation of the weak solution u , which is the exact solution of the variational problem

$$B[u_n, v_n] := \int_{\Omega} A_{ij} \partial_i u_n \partial_j v_n dx = (f, v_n) \quad \forall v_n \in S(\mathcal{T}_n, m). \quad (48)$$

Then the following quasi-optimal rates of convergence hold for the FEM approximation:

$$\|u - u_n\|_{H^1(\Omega)} \leq C' \|u - u_n\|_{\mathcal{H}_1^1(\Omega)} \leq C 2^{-nm} \|f\|_{\mathcal{H}_{a-1}^{m-1}(\Omega)}, \quad (49)$$

and

$$\|u - u_n\|_{L^2(\Omega)} \leq C 2^{-n(m+1)} \|f\|_{\mathcal{H}_{a-1}^{m-1}(\Omega)}, \quad (50)$$

for some constant $C > 0$. The rates in (49)-(50) are optimal in the sense that they are the rates valid on smooth domains, where the right hand side can be replaced by $\|u\|_{H^{m+1}(\Omega)}$ in that case. The quasi-optimality stems from the fact that we do not optimize constants.

We next tackle the numerical approximation of the parabolic problem (3). We are going to implement a mixed fully-discrete scheme, consisting in a FEM approximation in space and a finite-difference approximation in time, using backward Euler's method. We confine ourselves to a first-order method, though unconditionally stable, to reduce the computational cost. The numerical tests we perform have the sole purpose of validating the theoretical result.

Let u denote the weak solution of the IVBPT (3) on the time interval $[0, T]$. After N refinement of the mesh, we shall denote the maximum diameter of triangles in $\mathcal{T}_n$ by h , the space step size. Similarly, we shall denote by k the size of the time step after M steps, i.e., $k = T/M$. The numerical approximation $u_n^j \in S_n$, $n = 0, \dots, N$, $j = 0, \dots, M$, of u is the solution of the following coupled system:

$$\left(\frac{u_n^{j+1} - u_n^j}{k}, v_n \right) + B[u_n^{j+1}, v_n] = (f^{j+1}, v_n) \quad \forall v_n \in S_n, \quad (51)$$

$$(u_n^0, v_n) = (g, v_n) \quad \forall v_n \in S_n, \quad (52)$$

where $g \in L^2(\Omega)$ is the initial condition for the exact problem (3) and $f^j(\cdot) = f(\cdot, jk)$ with $f \in C([0, T], H_D^1(\Omega)')$.

Our main results in this section are an extension of the quasi-optimal rates of convergence to the parabolic setting, using the mixed, fully discrete scheme. One could also study convergence for a semi-discrete scheme, which is FEM in space and continuous in time. We consider only the case of piecewise-linear or quadratic polynomials, as higher-order polynomials require higher regularity for the solution in weighted spaces, which in turn requires higher compatibility conditions on the data.

Theorem 6 *Let u be the weak solution of the IBPTP (3) on $[0, T]$. Assume $0 < a < \min_{Q \in \mathcal{V}} \{\eta_Q\}$ and*

$$\begin{aligned} f_u &\in L^2(0, T; (H_D^1(\Omega))'), f_t \in C(0, T; L^2(\Omega)), f \in C(0, T; \mathcal{H}_a^2(\Omega)); \\ g &\in V_{a+1}^2(\Omega), g_1 = f(0) - Pg \in V_{a+1}^2(\Omega), g_2 = f_t(0) - Pg_1 \in L^2(\Omega); \end{aligned} \quad (53)$$

Let u_n^j , $j = 0, \dots, M$, $n = 0, \dots, N$ the solution of the system (51)-(52) with $m = 1$. Then, the following error estimates hold:

$$\begin{aligned} \max_{0 \leq j \leq M} \|u(jk) - u_n^j\|_{L^2(\Omega)} &= O(h^2 + k), \\ \max_{0 \leq j \leq M} \|u(jk) - u_n^j\|_{H_D^1(\Omega)} &= O(h + \frac{h^2 + k}{\sqrt{k}}). \end{aligned} \quad (54)$$

where $h = h_n$ is the size of the elements of the triangulation $\mathcal{T}_n$ and $k = k_j$ is the size of the time step.

Proof From Theorem 4 we can decompose u as $u = u_{reg} + w_s$, where

$$u_{reg} \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; \mathcal{K}_{a+1}^2(\Omega)) \cap L^\infty(0, T; \mathcal{K}_{a+1}^4(\Omega));$$

and $w_s \in W_s$. The rest of the proof follows a standard approach for FEM (see e.g. [7]). Since u is the exact solution of (3), and $f - u_t \in W^{1,\infty}(0, T; L^2(\Omega))$, there exists a unique solution $w_n(t) \in S_n$, $t \in [0, T]$ of the problem

$$B[w_n, v_n] = (f - u_t, v_n), \quad \forall v_n \in S_n, \quad 0 < t < T,$$

such that

$$B[\partial_t w_n, v_n] = (f_t - u_{tt}, v_n), \quad \forall v_n \in S_n, \quad 0 < t < T.$$

Since u solves (3) so that $B[u, v] = (f - u_t, v)$ for all $v \in H_D^1(\Omega)$ and $0 < t < T$, we also have:

$$B[w_n, v_n] = B[u, v_n], \quad \forall v_n \in S_n, \quad 0 < t < T. \quad (55)$$

Differentiating the above identity in time, we conclude that $\partial_t w_n \in S_n$ is the FEM approximation of $P^{-1}(f_t - u_{tt})$, where we recall P is the forward operator for the elliptic problem (10). From estimates (49), (50) with $m = 1$, we deduce that

$$\|u_t - \partial_t w_n\|_{L^2(\Omega)} \leq Ch^2 \|f_t - u_{tt}\|_{\mathcal{K}_{a-1}^0(\Omega)}, \quad (56a)$$

$$\|u_t - \partial_t w_n\|_{H_D^1(\Omega)} \leq Ch \|f_t - u_{tt}\|_{\mathcal{K}_{a-1}^0(\Omega)} \quad (56b)$$

for some constant $C > 0$ and $0 < t < T$.

From Taylor's formula, given the regularity of u , it follows that

$$\left\| \frac{u((j+1)k) - u(jk)}{k} - u_t((j+1)k) \right\|_{L^2(\Omega)} = \frac{k}{2} \|u_{tt}(jk + \varepsilon)\|_{L^2(\Omega)} \leq \frac{k}{2} \|u_{tt}\|_{L^\infty(0, T; L^2(\Omega))}, \quad (57)$$

while (56a) gives

$$\begin{aligned} \left\| \frac{w_n((j+1)k) - w_n(jk)}{k} - u_t((j+1)k) \right\|_{L^2(\Omega)} &= \left\| \frac{1}{k} \int_{jk}^{(j+1)k} [\partial_t w_n(s) - u_t(s)] ds \right\|_{L^2(\Omega)} \\ &\leq \|\partial_t w_n - u_t\|_{L^\infty(0, T; L^2(\Omega))} \leq Ch^2 \|f_t - u_{tt}\|_{L^\infty(0, T; \mathcal{K}_{a-1}^0(\Omega))} \leq Ch^2 \|f_t - u_{tt}\|_{L^\infty(0, T; L^2(\Omega))}. \end{aligned} \quad (58)$$

Now, (55) implies that for $j = 0, \dots, M$,

$$\begin{aligned} & \left(\frac{w_n((j+1)k) - w_n(jk)}{k}, v_n \right) + B[w_n((j+1)k), v_n] = B[u((j+1)k), v_n] + \\ & \left(\frac{u((j+1)k) - u(jk)}{k}, v_n \right) + \left(\frac{w_n((j+1)k) - u((j+1)k) - [w_n(jk) - u(jk)]}{k}, v_n \right). \end{aligned} \quad (59)$$

We rewrite this identity as:

$$\begin{aligned} & \left(\frac{w_n((j+1)k) - w_n(jk)}{k}, v_n \right) + B[w_n((j+1)k), v_n] = B[u((j+1)k), v_n] + \\ & (u_t((j+1)k), v_n) + (z_n^{j+1}, v_n) = (f((j+1)k), v_n) + (z_n^{j+1}, v_n) = \\ & (f^{j+1}, v_n) + (z_n^{j+1}, v_n) \quad \forall v_n \in S_n, j = 0, 1, \dots \end{aligned} \quad (60)$$

where, by exploiting estimates (57)-(58) above in this identity,

$$\|z_n^j\|_{L^2(\Omega)} \leq Ch^2 \|f_t\|_{L^\infty(0,T;L^2(\Omega))} + (Ch^2 + k) \|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))} \quad j = 0, 1, \dots \quad (61)$$

Next we compare (51) with (60), and let $y_n^j = u_n^j - w_n(jk)$, $j = 0, 1, \dots, M$, so that

$$\left(\frac{y_n^{j+1} - y_n^j}{k}, v_n \right) + B[y_n^{j+1}, v_n] = (z_n^{j+1}, v_n) \quad \forall v_n \in S_n, j = 0, 1, \dots$$

For each $j = 0, \dots, M$ we choose $v_n = y_n^{j+1} \in S_n$ in the identity above, obtaining

$$(y_n^{j+1}, y_n^{j+1}) + kB[y_n^{j+1}, y_n^{j+1}] = (y_n^j + kz_n^{j+1}, y_n^{j+1}), \quad (62)$$

Since the bilinear form $B[\cdot, \cdot]$ is positive definite, we deduce that

$$\max_{0 \leq j \leq M} \|y_n^j\|_{L^2(\Omega)} \leq \|y_n^0\|_{L^2(\Omega)} + k \sum_{s=1}^M \|z_n^s\|_{L^2(\Omega)} = k \sum_{s=1}^M \|z_n^s\|_{L^2(\Omega)},$$

where we used that $y_n^0 = 0$ as $B[u_n^0, v_n] = B[u(0), v_n] = B[w_n(0), v_n]$ for all $v_n \in S_n$. From (61) we then have

$$\begin{aligned} \max_{0 \leq j \leq M} \|y_n^j\|_{L^2(\Omega)} & \leq CT h^2 \|f_t\|_{L^\infty(0,T;L^2(\Omega))} + (Ch^2 + k)T \|u_{tt}\|_{L^\infty(0,T;L^2(\Omega))} \\ & \leq C^*(h^2 + k) (\|f'(0) - Pg\|_{\mathcal{H}_{a+1}^2(\Omega)} + \|f_t\|_{L^\infty(0,T;L^2(\Omega))} \\ & \quad + \|f_{tt}\|_{L^2(0,T;(H_D^1(\Omega))^*)} + \|g\|_{\mathcal{H}_{a+1}^2(\Omega)}). \end{aligned} \quad (63)$$

This bound implies

$$\max_{0 \leq j \leq M} \|u_n^j - w_n(kj)\|_{L^2(\Omega)} = O(h^2 + k).$$

Integrating equation (56a) from 0 to kj and combining it with the above bound, via the triangle inequality, then gives the desired L^2 estimate on the FEM approximation:

$$\max_{0 \leq j \leq M} \|u_n^j - u(kj)\|_{L^2(\Omega)} = O(h^2 + k).$$

Next we turn to the H^1 estimate. First, from (62) we have

$$Ck \|y_n^{j+1}\|_{H_D^1(\Omega)}^2 \leq kB[y_n^{j+1}, y_n^{j+1}] = (y_n^j + kz_n^{j+1} - y_n^{j+1}, y_n^{j+1}).$$

We estimate the right-hand side using (63), so that

$$\max_{0 \leq j \leq M} \|y_n^j\|_{H_D^1(\Omega)} \leq \frac{C}{\sqrt{k}} \max_{0 \leq j \leq M} \|y_n^j\|_{L^2(\Omega)} + C \max_{0 \leq j \leq M} \|y_n^j\|_{L^2(\Omega)}^{1/2} \|z_n^j\|_{L^2(\Omega)}^{1/2} = O\left(\frac{h^2 + k}{\sqrt{k}}\right).$$

We integrate (56b) from 0 to kj to obtain $\|u(jk) - w_n(jk)\|_{H_D^1(\Omega)} = O(h)$. Finally, recall that $y_n^j = u_n^j - w_n(jk)$ and combine the two previous estimates using the triangle inequality again to conclude that

$$\max_{0 \leq j \leq M} \|u_n^j - u(kj)\|_{H_D^1(\Omega)} = O(h + \frac{h^2 + k}{\sqrt{k}}).$$

□

Remark 6 If we use quadratic elements, that is, we fix $m = 2$, we obtain the improved rate:

$$\begin{aligned} \max_{0 \leq j \leq M} \|u_n^j - u(kj)\|_{L^2(\Omega)} &= O(h^3 + k), \\ \max_{0 \leq j \leq M} \|u_n^j - u(kj)\|_{H_D^1(\Omega)} &= O(h^2 + \frac{h^3 + k}{\sqrt{k}}). \end{aligned} \quad (64)$$

The proof is similar to the case $m = 1$, we only need to exploit higher regularity in (58).

7 Numerical tests

We implement the scheme presented in the Section 6 with three numerical tests. In these tests, we employ quadratic elements and use graded meshes, using the refinement strategy described in Section 6, taken from [32].

In the examples we present, the exact solution u is unknown, so we compute the convergence rates empirically *a posteriori* by comparing the approximate solutions for two consecutive refinements in space and/or time. For each example below, we let the mesh size of our initial triangulation $\mathcal{T}_0$ be $h = 2^{-2}$, so $\mathcal{T}_n$ will have maximal mesh size $h = 2^{-n-2}$. In addition, for each triangulation, we let the time step size $k = 4h^3$, so that we can expect an optimal error of order $O(h^{\frac{3}{2}})$ in $L^\infty H^1$ from the second equation in (64). Since the time and mesh sizes are related, we denote the approximate solution simply by u_n^h . Then theoretically

$$\lim_{n \rightarrow \infty} \log_2 e_n = \lim_{n \rightarrow \infty} \frac{\|u_{n-1}^h - u_n^h\|_{H_D^1(\Omega)}}{\|u_n^h - u_{n+1}^h\|_{H_D^1(\Omega)}} \geq 2\sqrt{2} \approx 2.828, \quad (65)$$

where the lower bound may not be optimal. This choice of time stepping is convenient for computing the convergence rate, however it is computationally intensive and we are able to refine the mesh only a few times (all calculations were done on a laptop computer). The numerical implementation is not the main focus of our work.

7.1 Domain with Non-smooth Interface

In the first test, we let the domain be a square that is divided into two polygons by the interface. Specifically, let $\Omega = (0, 1) \times (0, 1)$, $T = [0, 1]$, $\partial\Omega = \partial_D\Omega$, with the interface $\Gamma = \{ (1-y, y) \mid y \in (\frac{1}{2}, 1) \} \cup \{ (\frac{1}{2}, y) \mid y \in (0, \frac{1}{2}] \}$. We remark that Γ has an interior vertex at $(\frac{1}{2}, \frac{1}{2})$. Then the two subdomains are given by

$$\Omega_1 = \{ (x, y) \mid x \in (0, \frac{1}{2}), y \in (0, 1-x) \}, \quad \Omega_2 = \Omega \setminus \Omega_1,$$

while we choose the operator

$$L = \begin{cases} -\Delta & \text{on } \Omega_1, \\ -10 \Delta & \text{on } \Omega_2, \end{cases} \quad (66)$$

with data $g(x, y) \equiv 0$ and $f(x, y, t) = \sin(\pi x) \sin(\pi y) t$ for all $(x, y, t) \in (0, 1) \times (0, 1) \times [0, 1]$, so we solve for times $0 \leq t \leq T = 1$.

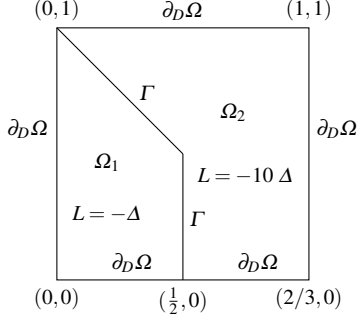


Fig. 5 Example: Non-smooth interface

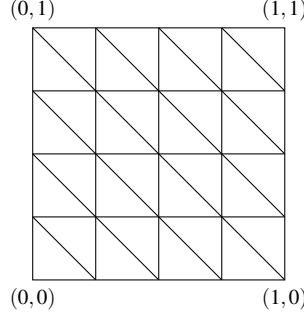


Fig. 6 Initial Triangulation $\mathcal{T}_0$

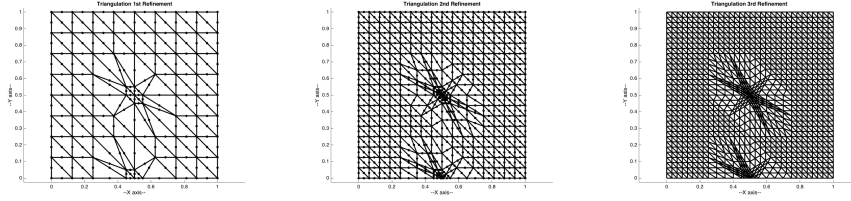


Fig. 7 First 3 mesh refinements $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ after the initial uniform mesh

The initial mesh $\mathcal{T}_0$ is uniform with mesh size $h = 0.25$. We note that there are 6 vertices on the boundary of Ω and the interface Γ , where the solution u is singular. By Theorem 5, at each vertex of the unit square, the singularity constant $\eta_Q = 2$, while at the vertices $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, η_Q equals 1 and $0.839 < 1$, respectively. Consequently at each corner of the unit square, the refinement ratio $\kappa_Q = 2^{-m} \eta_Q = 1/2$, and no grading is needed as expected. At the vertices $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$, $\kappa_Q = 1/4 = 0.25$ and $\kappa_Q = 0.839/4 = 0.210$, respectively, which gives the graded mesh in Fig. 7.

We tabulate the errors for the first three refinements (again we take the mesh size $h = 2^{-n-2}$ and time step size $k = 4h^3 = 2^{-3n-4}$ for $n = 1, 2, 3$) in Table 1. From Table 1, we

check that $e_1 = 2.191e^{-4}/1.295e^{-5} \approx 17$, while $e_2 = 1.295e^{-5}/2.037e^{-6} \approx 6$ at $t = 1$. The computational costs increase significantly for $n > 3$.

	$\ u_0^h - u_1^h\ _{H_D^1(\Omega)}$	$\ u_1^h - u_2^h\ _{H_D^1(\Omega)}$	$\ u_2^h - u_3^h\ _{H_D^1(\Omega)}$	$\ u_3^h\ _{H_D^1(\Omega)}$
t=0.0	0	0	0	0
t=0.25	1.202 e-5	7.166e-7	1.189e-7	7.261e-5
t=0.5	5.248e-5	3.113e-6	5.076e-7	3.137e-4
t=1.0	2.191e-4	1.295e-5	2.037 e-6	1.305e-3

Table 1 Errors: Domain with Non-smooth Interface

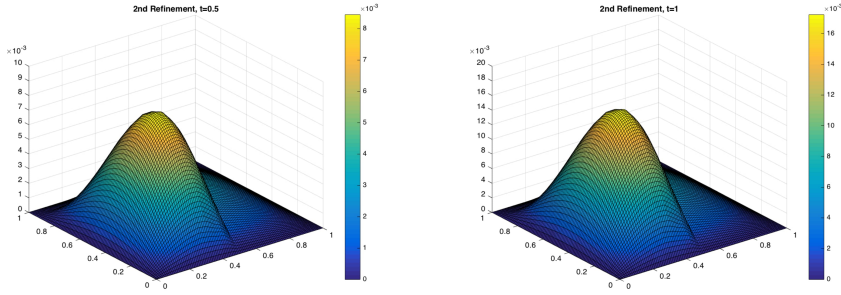
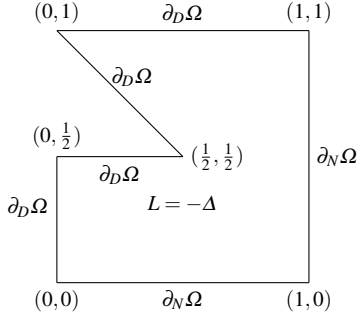
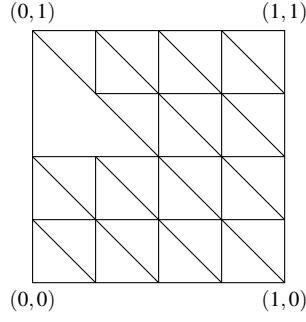


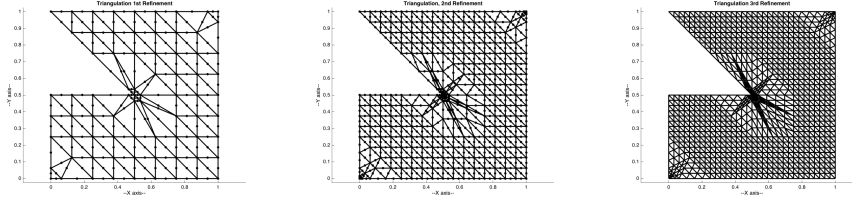
Fig. 8 Numerical solution $u_2^h \in S_2 : S(\mathcal{T}_2, 2)$ at time $t = 0.5$ and $t = 1$

7.2 Domain with Neumann-Neumann Vertices

We turn to our second example, in which the domain is non-convex and contains a Neumann-Neumann vertex. We let the domain Ω be the unit square $(0, 1) \times (0, 1)$ minus the triangle with vertices $(0, 1)$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. Moreover, we choose $L = -\Delta$ on Ω so that there is no interface. On the boundary, we impose Neumann condition on the sides from $(0, 0)$ to $(1, 0)$ and from $(1, 0)$ to $(1, 1)$, and impose Dirichlet condition on the other sides. As in the previous example, we take $T = 1$ and the data $g(x, y) \equiv 0$ and $f(x, y, t) = \sin(\pi x) \sin(\pi y) t$ for all $(x, y, t) \in (0, 1) \times (0, 1) \times [0, 1]$.

**Fig. 9** Example: Neumann-Neumann Vertices**Fig. 10** Initial Triangulation $\mathcal{T}_0$

As in the previous case, we calculate η_Q using Theorem 5. As a result, we can refine uniformly at the vertices $(0,1)$, $(1,0)$ and $(0, \frac{1}{2})$. At the vertices $(0,0)$ and $(1,1)$ where the boundary condition changes from Dirichlet to Neumann, the refinement ratio is $\kappa_Q = 2^{-2}\eta = 0.25$, while at the vertex $(\frac{1}{2}, \frac{1}{2})$, $\kappa_Q = 2^{-2} \times \frac{7}{4} = 0.143$ (see Fig. 11).

**Fig. 11** First 3 mesh refinements $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ after the initial uniform mesh

We tabulate the error again for the first three refinements in Table 2, and compute $e_1 = 3.100e^{-3}/7.263e^{-4} \approx 4.27$, and $e_2 = 7.263e^{-4}/1.883e^{-4} \approx 3.86$ at time $t = 1$. The error is stabilizing at the theoretical rate faster, as no interface is present.

	$\ u_0^h - u_1^h\ _{H_D^1(\Omega)}$	$\ u_1^h - u_2^h\ _{H_D^1(\Omega)}$	$\ u_2^h - u_3^h\ _{H_D^1(\Omega)}$	$\ u_3^h\ _{H_D^1(\Omega)}$
t=0.0	0	0	0	0
t=0.25	9.012e-5	2.447e-5	7.520e-6	3.028e-4
t=0.5	5.834e-4	1.448e-4	4.011e-5	1.878e-3
t=1.0	3.100e-3	7.263e-4	1.883e-4	9.737e-3

Table 2 Errors: Domain with Neumann-Neumann Vertices

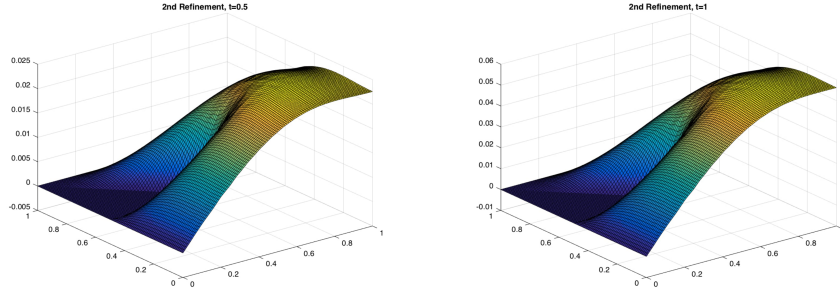


Fig. 12 Numerical solution $u_2^h \in S_2 : S(\mathcal{T}_2, 2)$ at time $t = 0.5$ and $t = 1$

7.3 Problem with High Contrast across Interface

In the last test, we let the domain Ω be the unit square partitioned equally into two subrectangles by an interface. We impose Dirichlet conditions on the entire boundary of Ω and solve again for $0 \leq t \leq T = 1$. The two subdomains are given by:

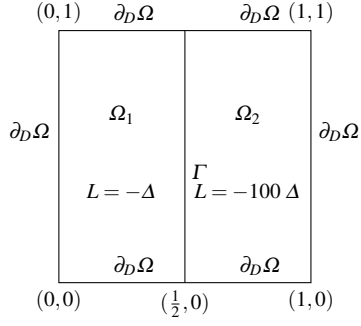
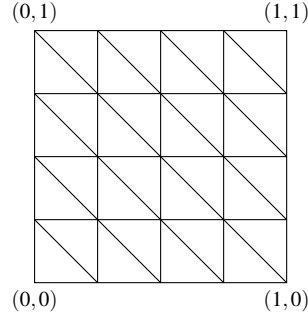
$$\Omega_1 = (0, \frac{1}{2}) \times (0, 1), \quad \Omega_2 = (\frac{1}{2}, 1) \times (0, 1)$$

The operator L is chosen to be

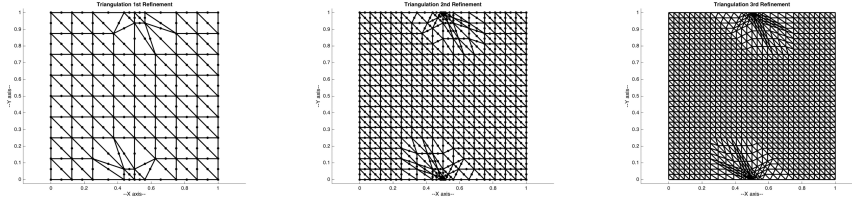
$$L = \begin{cases} -\Delta & \text{on } \Omega_1, \\ -100 \Delta & \text{on } \Omega_2. \end{cases} \quad (67)$$

Hence a high contrast across the interface Γ . To meet the compatibility conditions across the interface, we let the data $f \equiv 0$, and

$$g(x, y) = \begin{cases} y(1-y) \left(\frac{31300}{808}x - 200x^3 + 100x^4 \right) & \text{on } \Omega_1, \\ y(1-y) \left(\frac{1303}{808}(1-x) - 2(1-x)^3 + (1-x)^4 \right) & \text{on } \Omega_2. \end{cases} \quad (68)$$

**Fig. 13** Example: High Contrast Operator**Fig. 14** Initial Triangulation $\mathcal{T}_0$

We check again using Theorem 5 that at the four vertices of the unit square, $\kappa_Q = 0.5$ so no special grading is needed, while where the interface meets the boundary, the points $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$, $\eta_Q = 1$, so the refinement ratio is $\kappa_Q = 2^{-2} = 0.25$. The first three mesh refinements are shown in Fig. 15.

**Fig. 15** First 3 mesh refinements $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ after the initial uniform mesh

The corresponding error in the numerical solution is given in Table 3.

	$\ u_0^h - u_1^h\ _{H_b^1(\Omega)}$	$\ u_1^h - u_2^h\ _{H_b^1(\Omega)}$	$\ u_2^h - u_3^h\ _{H_b^1(\Omega)}$	$\ u_3^h\ _{H_b^1(\Omega)}$
t=0.0	9.401e-1	1.250e-2	2.381e-03	20.31
t=0.25	3.060e-4	1.491e-8	9.717e-10	1.736e-10
t=0.5	4.588e-9	2.410e-17	4.505e-20	1.425e-22
t=1.0	9.972e-19	3.268e-35	9.681e-41	8.524e-43

Table 3 Table: Problem with High Contrast Operator

From this table, we compute $e_1 = 9.972e^{-19}/3.268e^{-35} \approx 3.05e^{16}$ and $e_2 = 3.268e^{-35}/9.681e^{-41} \approx 3.38e^5$ at $t = 1$. So, the relative error is large. Significantly more refinements are needed to achieve the theoretical rate. The large numerical error stems from

the high contrast across the interface Γ , which has an effect on the time derivative as well. Further numerical tests have shown that a mesh size $h < \frac{1}{200}$ is needed to have an error of $< 10\%$.

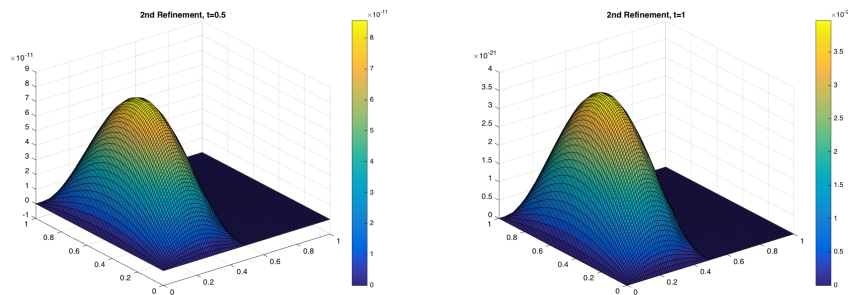


Fig. 16 Numerical solution $u_2^h \in S_2 : S(\mathcal{T}_2, 2)$ at time $t = 0.5$ and $t = 1$

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