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A Note on S. Weinberg, "Massless Particles in Higher Dimensions"

Jacques Distler

Theory Group
Department of Physics,
University of Texas at Austin,
Austin, TX 78712, USA

Email: `distler@golem.ph.utexas.edu`

Abstract

In [1], Weinberg made a conjecture about the little-group representations of massless particles that can be created out of the vacuum by the action of a local operator in d dimensions, generalizing his old result [2] in $d = 4$. In this note, I prove his conjecture and extend it to arbitrary irreps of $so(1, d-1)$.

In [1], Steven Weinberg posed the following question. Consider a local operator, $O(x)$, transforming in some irreducible (infinite-dimensional) representation of the Poincaré algebra $iso(1,d-1)$. Assume that $O(x)$ has a nonzero matrix element between a massless 1-particle state and the vacuum

$$0 \neq \langle 0|O(x)|k, R'\rangle$$

By translational invariance, this matrix element is nonvanishing if and only if

$$0 \neq \langle 0|O(0)|k, R'\rangle \tag{1}$$

We may therefore assume that $O(0)$ transforms as some (nonunitary) finite-dimensional irreducible representation, R, of the Lorentz algebra $so(1,d-1)$ which leaves the point $x=0$ fixed. Weinberg's question is:

- Given R, what representation, R', of $so(d-2) \subset iso(d-2)$ is compatible with a nonzero matrix element (1)?

Weinberg computed some examples, and conjectured an answer for (tensorial) representations R, given by Young Tableaux. Here I prove his conjecture and extend it to all finite-dimensional irreps, R.

The problem reduces to one in Lie theory. As an irrep of $so(1,d-1)$, R is *a fortiori* a representation of the little algebra, $iso(d-2) \subset so(1,d-1)$. As a representation of $iso(d-2)$, R is *reducible* but *indecomposable*.

We can write $iso(d-2) = so(d-2) \ltimes K$, where K is the $(d-2)$-dimensional abelian subalgebra of $so(1,d-1)$ which (along with $so(d-2)$) leaves fixed a particular null momentum. The irreducible subrepresentation, $R' \subset R$ is such that[1] K restricts to *zero* on R'. An alternative characterization of R' is that it is *simultaneously* an irrep of $iso(d-2)$ and of $so(1,1) \times so(d-2)$, where the $so(d-2)$ is the common subalgebra of these two maximal subalgebras of $so(1,d-1)$.

With this reformulation, we can ask, "If R' is an irrep of $so(1,1) \times so(d-2)$, which irrep is it?"

To answer that, we note that the highest-weight of R is contained in R'.

Proof: Since K raises[2] the $so(1,1)$ weight, it necessarily annihilates the highest weight of R. Acting with $so(d-2)$ does not change the $so(1,1)$ weight and the commutator of an element of $so(d-2)$ with K lies in K. Hence, acting on the highest weight of R with the generators of $iso(d-2)$, we get an irrep R' of $iso(d-2)$ with K represented by 0. By construction, R' is also an irrep of $so(1,1) \times so(d-2)$.

With that in mind, let us decompose R under $so(1,1) \times so(d-2)$

$$R = \bigoplus_i (\lambda_i) \otimes R_i \tag{2}$$

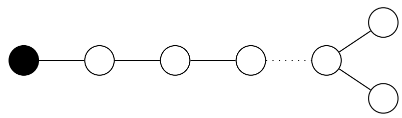
[1] As Wigner pointed out in 1939 [3], the restriction to $K=0$ is necessary to avoid a continuous infinity of particle species.

[2] For our conventions for the $so(1,1)$ weights, see (3).

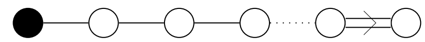
where R_i is an irrep of $so(d-2)$ and λ_i is the $so(1,1)$ weight labeling the corresponding 1-dimensional irrep of $so(1,1)$. Without loss of generality, we can order

$$\lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots$$

The embedding $so(1,1) \times so(d-2) \hookrightarrow so(1,d-1)$ is the one obtained by omitting the left-most node of the Dynkin diagram.



d even



d odd

The remaining nodes are the simple roots of $so(d-2)$. The highest weight of R under $so(1,d-1)$ is also the highest weight under the $so(1,1) \times so(d-2)$ subalgebra. That is, R' is the representation $(\lambda_1) \otimes R_1$ in (2).

To be more explicit, we need some notation. Highest weight representations, with highest weight Λ, will be denoted by their Dynkin labels,

$$n_i = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

where the α_i are the simple roots. Our convention for the $so(1,1)$ weights will be that the adjoint representation of $so(1,d-1)$ decomposes as

$$\begin{aligned} [0,1,0,0,\dots,0] = (2) \otimes [1,0,0,\dots,0] \ &\oplus \ (0) \otimes [0,1,0,\dots,0] \\ &\oplus \ (0) \otimes [0,0,0,\dots,0] \ \oplus \ (-2) \otimes [1,0,0,\dots,0] \end{aligned} \tag{3}$$

Here

- $K = (2) \otimes [1,0,0,\dots,0]$. *I.e.*, the generators of K transform as a vector of $so(d-2)$ and with weight $+2$ under $so(1,1)$.
- $(0) \otimes [0,1,0,\dots,0]$ is the adjoint of $so(d-2)$ and
- $(0) \otimes [0,0,0,\dots,0]$ is the generator of $so(1,1)$.

The normalization of the $so(1,1)$ weights (λ) is such that tensorial representations have λ even and spinorial representations have λ odd.

Let

$$R = [n, n_1, n_2, \dots, n_r] \tag{4}$$

be our chosen highest weight representation of $so(1,d-1)$. The simple roots of $so(d-2)$ were obtained by omitting the first simple root. The corresponding Dynkin labels are obtained by omitting the first Dynkin label of R. The highest-weight of the $so(1,d-1)$ irrep R, with Dynkin labels (4), is the highest weight of the $so(d-2)$ irrep with Dynkin labels $[n_1, n_2, \dots, n_r]$. That is, our sought-after representation of $so(d-2)$ is

$$R_1 = [n_1, n_2, \dots, n_r] \tag{5}$$

Though we don't need it, the $so(1,1)$ weight is also determined:

$$\lambda_1 = \begin{cases} 2n + n_r + 2\sum_{i=1}^{r-1} n_i & d = 2r+3 \\ 2n + n_r + n_{r-1} + 2\sum_{i=1}^{r-2} n_i & d = 2r+2 \end{cases} \tag{6}$$

Finally, let us translate Weinberg's Young diagrams into the corresponding Dynkin labels of irreps. Consider a Young diagram, whose rows have lengths $l_0 \geq l_1 \geq \cdots \geq l_r \geq 0$, where $r = (d-2)/2$ for d even and $(d-3)/2$ for d odd.

For d odd, the corresponding Dynkin labels for R are

$$\begin{aligned} n &= l_0 - l_1 \\ n_i &= l_i - l_{i+1}, \quad i = 1, \dots, r-1 \\ n_r &= 2l_r \end{aligned}$$

For d even,

$$\begin{aligned} n &= l_0 - l_1 \\ n_i &= l_i - l_{i+1}, \quad i = 1, \dots, r-2 \\ n_{r-1} + n_r &= 2l_{r-1} \\ |n_{r-1} - n_r| &= 2l_r \end{aligned}$$

Note that (of course) we only get tensorial representations this way ($n_r =$ even for d odd or $n_{r-1} + n_r =$ even for d even). Moreover, when d is even and $l_r > 0$, the Young diagram corresponds to a *reducible* representation, decomposing into *two* irreps whose Dynkin labels differ by exchanging $n_{r-1} \leftrightarrow n_r$.

For tensorial representations, dropping the first Dynkin label in passing from R to R_1 is precisely the same as Weinberg's conjectured "decapitation" procedure: removing the first row of the Young diagram. But it extends naturally to spinorial representations as well. And, for d even, it takes care of the reducibility of Young diagrams with $l_r > 0$. Finally, it gives an interpretation of the $so(1,1)$ weight in (6): $\lambda_1 = 2l_0$, where l_0 is the length of the row that he removes.

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