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Proof of a three-loop relation between the Regge limits of four-point amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity

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Abstract: A previously proposed all-loop-orders relation between the Regge limits of four-point amplitudes of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and $\mathcal{N} = 8$ supergravity is established at the three-loop level. We show that the Regge limit of known expressions for the amplitudes obtained using generalized unitarity simplifies in both cases to a (modified) sum over three-loop ladder and crossed-ladder scalar diagrams. This in turn is consistent with the result obtained using the eikonal representation of the four-point gravity amplitude. A possible exact three-loop relation between four-point amplitudes is also considered.

Keywords: Extended Supersymmetry, Scattering Amplitudes, Supergravity Models, Supersymmetric Gauge Theory

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1 Introduction

In recent years, connections between the perturbative amplitudes of gauge theory and gravity have been explored intensively (see ref. [1] for a recent review). The first hints of such a connection came from string theory, when Kawai, Lewellen, and Tye obtained a relation between open and closed supersymmetric string amplitudes at tree level [2]. In the low energy limit, this implies relations between tree-level gauge-theory and gravity amplitudes.

The most comprehensive current understanding of such relations is through the double copy of Bern, Carrasco, and Johansson [3, 4]. When gauge-theory amplitudes are written

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in terms of color factors and kinematic numerators of graphs with trivalent vertices, with the kinematic numerators obeying the same algebraic relations as those satisfied by the color factors (color-kinematic duality), then gravity amplitudes can be obtained from the gauge-theory amplitudes by replacing the color factors with a second copy of the kinematic numerators. That this procedure gives correct gravity amplitudes was proven at tree level in ref. [5]. At higher loops, the gauge-theory amplitudes are written in terms of integrals over loop momenta of graphs with trivalent vertices, with the kinematic numerators of the integrands chosen to obey color-kinematic duality. That the double-copy procedure generates correct loop-level gravity amplitudes was demonstrated through four loops for four-point amplitudes of $\mathcal{N}=8$ supergravity [4, 6]. At five loops and above, it is difficult to find representations of the $\mathcal{N}=4$ SYM four-point amplitude with manifest color-kinematic duality, but double-copy representations of the $\mathcal{N}=8$ supergravity four-point amplitude have nonetheless been obtained [7, 8].

It is important to emphasize that, at loop level, the double-copy prescription applies at the level of *integrands*. The double copy does not imply direct relations between the *integrated amplitudes* of gauge theory and gravity except in those cases (e.g., one- and two-loop four-point amplitudes, or one-loop five-point amplitudes) in which kinematic numerators are independent of loop momenta and therefore can be pulled outside the integrals [9–13].

A higher-loop relation between gauge-theory and gravity amplitudes was recently conjectured to hold in the Regge limit by one of the authors [14]. In that paper, the Regge limit of (nonplanar) ℓ-loop $\mathcal{N}=4$ SYM four-point amplitudes was examined, and a basis of color factors suitable for that limit was presented. The coefficients of the four-point amplitude in that basis were calculated through three-loop order, using the Regge limit of the full amplitude previously obtained by Henn and Mistlberger [15]. One of those coefficients, denoted $B_{\ell\ell}^{(\ell)}$, whose Laurent expansion begins at $1/\epsilon^{\ell}$, was shown to be proportional to the Regge limit of the ℓ-loop $\mathcal{N}=8$ supergravity four-point amplitude, at least through the first three orders in the Laurent expansion, thus motivating the conjecture.

At one and two loops, the conjectured relation reads

$$\frac{1}{(\kappa_D/2)^2 stu}\frac{\mathcal{M}^{(1)}}{\mathcal{M}^{(0)}} \quad\longrightarrow\quad -\frac{\left[A_1^{(1)}+A_3^{(1)}\right]}{g_D^2 st A_1^{(0)}}\,, \tag{1.1}$$

$$\frac{1}{(\kappa_D/2)^4\ stu}\frac{\mathcal{M}^{(2)}}{\mathcal{M}^{(0)}} \quad\longrightarrow\quad -\frac{s}{6}\,\frac{\left[A_7^{(2)}-A_9^{(2)}\right]}{g_D^4 st A_1^{(0)}}\,. \tag{1.2}$$

These are in fact a specialization to the Regge limit of previously known [16, 17] exact relations (i.e. relations that hold in all kinematic regions), namely eqs. (3.10) and (4.13), as will be shown explicitly below. Here $\mathcal{M}^{(\ell)}$ denotes the ℓ-loop four-point amplitude of $\mathcal{N}=8$ supergravity, while $A_\lambda^{(\ell)}$ denote the color-ordered ℓ-loop four-point amplitudes of $\mathcal{N}=4$ SYM theory (i.e., the coefficients of the ℓ-loop amplitude in the $(3\ell+3)$-dimensional extended trace basis $t_\lambda^{(\ell)}$ defined in section 2). For each theory, the loop- and tree-level amplitudes carry the same helicity dependence, so the ratios $\mathcal{M}^{(\ell)}/\mathcal{M}^{(0)}$ and $A_\lambda^{(\ell)}/A_1^{(0)}$ are helicity-independent functions of the Mandelstam variables s, t, and u. Since loop-level amplitudes of massless particles are IR-divergent, we dimensionally regularize them in $D = 4 - 2\epsilon$ dimensions, with g_D and κ_D denoting the gauge and gravitational couplings in D dimensions,[1] and the amplitudes are expressed as Laurent expansions in ϵ. Here and throughout this paper, the long right arrow denotes a relation valid in the Regge limit $|t| \ll s$.

The conjectured relation at three loops

$$\frac{1}{(\kappa_D/2)^6 \; stu} \frac{\mathcal{M}^{(3)}}{\mathcal{M}^{(0)}} \quad \longrightarrow \quad \frac{s^2}{12} \frac{\left[4\left(A_1^{(3)} + A_3^{(3)}\right) - \left(A_4^{(3)} + A_6^{(3)} + A_7^{(3)} + A_9^{(3)}\right)\right]}{g_D^6 st A_1^{(0)}} \tag{1.3}$$

was shown [14] to hold through $\mathcal{O}(\epsilon^0)$ of the Laurent expansion (i.e., the first four non-vanishing terms) using the explicit calculations of Henn and Mistlberger [15, 18]. The two- and three-loop relations, eqs. (1.2) and (1.3), are special cases of the more general all-loop conjecture, namely

$$\frac{1}{(\kappa_D/2)^{2\ell} \; stu} \frac{\mathcal{M}^{(\ell)}}{\mathcal{M}^{(0)}} \quad \longrightarrow \quad -\frac{s^{\ell-1}}{2\cdot 3^{\ell/2}} \frac{\left[A_{3\ell+1}^{(\ell)} - A_{3\ell+3}^{(\ell)}\right]}{g_D^{2\ell} st A_1^{(0)}}, \qquad \text{even } \ell \geq 2\,, \tag{1.4}$$

$$\frac{1}{(\kappa_D/2)^{2\ell} \; stu} \frac{\mathcal{M}^{(\ell)}}{\mathcal{M}^{(0)}} \quad \longrightarrow \quad \frac{s^{\ell-1}}{4\cdot 3^{(\ell-1)/2}} \frac{\left[4\left(A_{3\ell-8}^{(\ell)} + A_{3\ell-6}^{(\ell)}\right) - \left(A_{3\ell-5}^{(\ell)} + A_{3\ell-3}^{(\ell)} + A_{3\ell-2}^{(\ell)} + A_{3\ell}^{(\ell)}\right)\right]}{g_D^{2\ell} st A_1^{(0)}}, \qquad \text{odd } \ell \geq 3\,. \tag{1.5}$$

In ref. [14], these relations were verified to hold at all loop orders for the first three (IR-divergent) terms in the Laurent expansion, using the known structure of IR divergences in both theories.

In ref. [14], it was suggested that the three-loop relation (1.3) could be established to all orders in the Laurent expansion by examining known expressions for the amplitudes in terms of scalar integrals obtained through generalized unitarity. That is the main aim of this paper. We will demonstrate that both sides of eq. (1.3) simplify in the Regge limit to the same (modified) sum over three-loop ladder and crossed-ladder scalar diagrams, thus proving the conjectured relation. The sum is modified in the sense that two of the crossed-ladder diagrams are multiplied by a factor of one-half relative to the remaining diagrams.

Alternatively, the eikonal approximation [19–21] may be used to obtain a representation (6.7) of the supergravity amplitude as an integral over impact-parameter space [22–32]. This may then be evaluated to give an explicit result (6.9) for the ℓ-loop supergravity amplitude in the Regge limit [31]. We show that this result is consistent with the representation obtained in the current paper of the Regge limit of the $\mathcal{N} = 8$ supergravity amplitude at one, two, and three loops as a sum of ladder and crossed-ladder scalar diagrams. The modification of the sum at three loops mentioned above is crucial for this consistency.

In this paper, we also observe that the three-loop relation (1.3) is the Regge limit of a certain exact relation (5.28) that would be valid if only a certain subset of the scalar diagrams were included in the evaluation of the three-loop amplitudes. Testing this exact

[1] Our convention is $(\kappa_D/2)^2 = 8\pi G_D$.

relation against the Laurent expansions of the full three-loop amplitudes, we find that it holds at $\mathcal{O}(1/\epsilon^3)$ and $\mathcal{O}(1/\epsilon^2)$, only breaking down at $\mathcal{O}(1/\epsilon)$, cf. eq. (5.29).

The outline of this paper is as follows: in section 2 we review the form of maximally supersymmetric four-point amplitudes obtained from generalized unitarity, and the definition of color-ordered amplitudes in the extended trace basis. In sections 3 and 4 we write down the one- and two-loop amplitudes for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity in terms of scalar integrals, the exact relations that hold among them, and the Regge limits of these relations. In section 5 we write down the three-loop $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity amplitudes in terms of scalar integrals, and then obtain their approximate form in the Regge limit, thus establishing the three-loop relation (1.3). We also present a putative exact three-loop relation, and show that it only breaks down at $\mathcal{O}(1/\epsilon)$. In section 6, we show that the expressions obtained in the previous three sections can be recast as a (modified) sum over ladder and crossed-ladder scalar diagrams, and show how this is related to the eikonal representation of gravity amplitudes in impact-parameter space. Section 7 summarizes the results of the paper.

2 Maximally supersymmetric four-point amplitudes

Generalized unitarity [33, 34] has been used to find representations of various loop-level amplitudes of maximally supersymmetric $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity theories in terms of planar and nonplanar scalar integrals [6–9, 35–40]. In this section, we review the results of this approach, and establish our conventions for integrals and color-ordered amplitudes.

2.1 $\mathcal{N} = 4$ SYM four-point amplitude

The ℓ-loop $\mathcal{N} = 4$ SYM four-point amplitude can be expressed as a linear combination of products of color factors $c^{(x)}$ and scalar integrals $I^{(x,1)}$ associated with a set of diagrams x with trivalent vertices. The color factor $c^{(x)}$ associated with each diagram is defined by decorating each vertex of the diagram with a structure constant $\tilde{f}^{abc}$, related to the conventionally defined structure constants f^{abc} by

$$\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} \tag{2.1}$$

and then contracting indices connected by internal lines. The scalar integral $I^{(x,1)}$ associated with the diagram x is defined (following the conventions of eq. (2.4) of ref. [39]) as

$$I^{(x,1)}(p_1, p_2, p_3, p_4) = (-i)^\ell \int \prod_{i=1}^{\ell} \frac{d^D \ell_i}{(2\pi)^D} \frac{N^{(x,1)}}{\prod_{j=1}^{3\ell+1} l_j^2} \tag{2.2}$$

where $N^{(x,1)}$ is a specified numerator factor for the diagram. It will later be convenient for us to define the related scalar integral $I^{(x,0)}$, without a numerator factor,

$$I^{(x,0)}(p_1, p_2, p_3, p_4) = (-i)^\ell \int \prod_{i=1}^{\ell} \frac{d^D \ell_i}{(2\pi)^D} \frac{1}{\prod_{j=1}^{3\ell+1} l_j^2} \,. \tag{2.3}$$

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The ℓ-loop amplitude is then given by a sum over all the diagrams and, to ensure Bose symmetry, a sum over all permutations of the external legs

$$\mathcal{A}^{(\ell)} \sim \sum_x \sum_{S_4} \lambda^{(x)} c^{(x)}_{ijkl} I^{(x,1)}_{ijkl} \tag{2.4}$$

where S_4 denotes that $ijkl$ runs over all permutations of 1234, the $\lambda^{(x)}$ are simple combinatorial factors, and we define

$$I^{(x)}_{ijkl} \equiv I^{(x)}(p_i, p_j, p_k, p_\ell)\,. \tag{2.5}$$

The integrals $I^{(x)}(p_1, p_2, p_3, p_4)$ (with or without numerator factors) are functions of the Mandelstam invariants

$$s = (p_1 + p_2)^2, \qquad t = (p_1 + p_4)^2, \qquad u = (p_1 + p_3)^2 \tag{2.6}$$

and hence are invariant under a four element normal subgroup of permutations isomorphic to the Klein four-group

$$I^{(x)}(p_1, p_2, p_3, p_4) = I^{(x)}(p_2, p_1, p_4, p_3) = I^{(x)}(p_3, p_4, p_1, p_2) = I^{(x)}(p_4, p_3, p_2, p_1) \tag{2.7}$$

or equivalently,

$$I^{(x)}_{1234} = I^{(x)}_{2143} = I^{(x)}_{3412} = I^{(x)}_{4312}\,. \tag{2.8}$$

In the following subsection, we will demonstrate that the SU(N) gauge group color factors are also invariant under the Klein four-group

$$c^{(x)}_{1234} = c^{(x)}_{2143} = c^{(x)}_{3412} = c^{(x)}_{4312}\,. \tag{2.9}$$

Hence, using eqs. (2.8) and (2.9), the sum over permutations in eq. (2.4) is reduced to

$$\sum_{S_4} c^{(x)}_{ijkl} I^{(x,1)}_{ijkl} = 4 \sum_{S_3} c^{(x)}_{1ijk} I^{(x,1)}_{1ijk} \tag{2.10}$$

where S_3 denotes that ijk runs over all permutations of 234.

The precise form of the ℓ-loop four-point amplitude for $\mathcal{N} = 4$ SYM theory is then given by [39]

$$\mathcal{A}^{(\ell)} = 4K(-g_D^2)^{\ell+1} \sum_x \sum_{S_3} \lambda^{(x)} c^{(x)}_{1ijk} I^{(x,1)}_{1ijk} \tag{2.11}$$

where g_D is the coupling constant in $D = 4 - 2\epsilon$ dimensions, and K is a factor common to all loop orders depending on the helicities of the external states. For four external gluons, for example, K is given by eq. (7.4.42) of ref. [41].

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2.2 Decomposition in an SU(N) trace basis

An alternative representation of the ℓ-loop four-point amplitude $\mathcal{A}^{(\ell)}$ for an SU(N) gauge theory is the decomposition [42]

$$\mathcal{A}^{(\ell)} = \sum_{\lambda=1}^{6} A^{(\ell)}_{[\lambda]} c_{[\lambda]} \tag{2.12}$$

in terms of a six-dimensional basis $c_{[\lambda]}$ of single and double traces

$$\begin{aligned} c_{[1]} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2}), & c_{[4]} &= \mathrm{Tr}(T^{a_1}T^{a_3})\,\mathrm{Tr}(T^{a_2}T^{a_4}),\\ c_{[2]} &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) + \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2}), & c_{[5]} &= \mathrm{Tr}(T^{a_1}T^{a_4})\,\mathrm{Tr}(T^{a_2}T^{a_3}),\\ c_{[3]} &= \mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3}) + \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}), & c_{[6]} &= \mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4}) \end{aligned} \tag{2.13}$$

where a_i are the adjoint color indices of the external particles, and T^a are generators[2] in the fundamental representation of SU(N). All other possible trace terms vanish for SU(N) since $\mathrm{Tr}(T^a) = 0$. The coefficients $A^{(\ell)}_{[\lambda]}$, called color-ordered amplitudes, are gauge-invariant.

To obtain the color-ordered amplitudes from $\mathcal{A}^{(\ell)}$, one replaces the structure constants $\tilde{f}^{abc}$ of an arbitrary color factor c_{1ijk} appearing in eq. (2.11) with (recalling eq. (2.1))

$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c) \tag{2.14}$$

and then repeatedly utilizes

$$\begin{aligned} \mathrm{Tr}(AT^a)\,\mathrm{Tr}(BT^a) &= \mathrm{Tr}(AB) - \frac{1}{N}\,\mathrm{Tr}(A)\,\mathrm{Tr}(B)\,,\\ \mathrm{Tr}(AT^aBT^a) &= \mathrm{Tr}(A)\,\mathrm{Tr}(B) - \frac{1}{N}\,\mathrm{Tr}(AB) \end{aligned} \tag{2.15}$$

to reduce c_{1ijk} to a linear combination of traces (2.13), for example

$$c_{1234} = \sum_{\lambda=1}^{6} M_{[\lambda]} c_{[\lambda]} \qquad \text{or} \qquad c_{1234} = (M_{[1]}, M_{[2]}, M_{[3]};\; M_{[4]}, M_{[5]}, M_{[6]}) \tag{2.16}$$

where $M_{[\lambda]}$ are polynomials in N. By inspection the elements of the trace basis $c_{[\lambda]}$ are invariant under the Klein four-group, and therefore so are any of the color factors c_{1234}, as claimed above in eq. (2.9). Given eq. (2.16), one can write down the decomposition of color factors with permuted legs as

$$\begin{aligned} c_{1243} &= (M_{[2]}, M_{[1]}, M_{[3]};\; M_{[5]}, M_{[4]}, M_{[6]}),\\ c_{1342} &= (M_{[3]}, M_{[1]}, M_{[2]};\; M_{[6]}, M_{[4]}, M_{[5]}),\\ c_{1324} &= (M_{[3]}, M_{[2]}, M_{[1]};\; M_{[6]}, M_{[5]}, M_{[4]}),\\ c_{1423} &= (M_{[2]}, M_{[3]}, M_{[1]};\; M_{[5]}, M_{[6]}, M_{[4]}),\\ c_{1432} &= (M_{[1]}, M_{[3]}, M_{[2]};\; M_{[4]}, M_{[6]}, M_{[5]}). \end{aligned} \tag{2.17}$$

[2]Our conventions are $\mathrm{Tr}(T^aT^b) = \delta^{ab}$ so that $[T^a, T^b] = i\sqrt{2}f^{abc}T^c$ and $f^{abc} = (-i/\sqrt{2})\,\mathrm{Tr}([T^a, T^b]T^c)$.

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2.3 Extended trace basis

The ℓ-loop color-ordered amplitudes $A^{(\ell)}_{[\lambda]}$ can be further decomposed in powers of N as [35]

$$A^{(\ell)}_{[\lambda]} = \begin{cases} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} N^{\ell-2k} A^{(\ell,2k)}_{\lambda}, & \text{for } \lambda = 1,2,3, \\ \sum_{k=0}^{\lfloor \frac{\ell-1}{2} \rfloor} N^{\ell-2k-1} A^{(\ell,2k+1)}_{\lambda}, & \text{for } \lambda = 4,5,6 \end{cases} \tag{2.18}$$

where $A^{(\ell,0)}_{\lambda}$ are leading-order-in-N (planar) amplitudes, and $A^{(\ell,k)}_{\lambda}$, $k = 1, \cdots, \ell$, are subleading-order-in-N, yielding $(3\ell+3)$ color-ordered amplitudes at ℓ loops. This suggests an enlargement of the six-dimensional trace basis $c_{[\lambda]}$ to an extended $(3\ell+3)$-dimensional trace basis $t^{(\ell)}_{\lambda}$, defined by

$$\begin{aligned} t^{(\ell)}_{1+6k} &= N^{\ell-2k}\, c_{[1]}\,, & t^{(\ell)}_{4+6k} &= N^{\ell-2k-1}\, c_{[4]}\,, \\ t^{(\ell)}_{2+6k} &= N^{\ell-2k}\, c_{[2]}\,, & t^{(\ell)}_{5+6k} &= N^{\ell-2k-1}\, c_{[5]}\,, \\ t^{(\ell)}_{3+6k} &= N^{\ell-2k}\, c_{[3]}\,, & t^{(\ell)}_{6+6k} &= N^{\ell-2k-1}\, c_{[6]}\,. \end{aligned} \tag{2.19}$$

Then eq. (2.16) becomes

$$c_{1234} = \sum_{\lambda=1}^{3\ell+3} m_\lambda t^{(\ell)}_{\lambda} \quad \text{or} \quad c_{1234} = (m_1, m_2, m_3; \;\; m_4, m_5, m_6; \;\; m_7, m_8, m_9; \;\; \cdots) \tag{2.20}$$

where m_λ are integers, and eq. (2.12) becomes

$$\mathcal{A}^{(\ell)} = \sum_{\lambda=1}^{3\ell+3} A^{(\ell)}_{\lambda} t^{(\ell)}_{\lambda}, \qquad \text{where} \qquad A^{(\ell)}_{\lambda+6k} = \begin{cases} A^{(\ell,2k)}_{\lambda}, & \lambda = 1,2,3\,, \\ A^{(\ell,2k+1)}_{\lambda}, & \lambda = 4,5,6\,. \end{cases} \tag{2.21}$$

The color-ordered amplitudes $A^{(\ell)}_{\lambda}$ are not all independent but are related by various group-theory constraints [43], which were summarized in ref. [14].

Since the tree-level color-ordered amplitude is given by [36]

$$A^{(0)}_1 = (-g_D)^2 \frac{4K}{st} \tag{2.22}$$

we can rewrite eq. (2.11) in the form

$$\mathcal{A}^{(\ell)} = (-g_D^2)^{\ell} st A^{(0)}_1 \sum_x \sum_{S_3} \lambda^{(x)} c^{(x)}_{1ijk} I^{(x,1)}_{1ijk}\,. \tag{2.23}$$

In subsequent sections, explicit expressions for the one-, two-, and three-loop color factors will be used to derive the color-ordered amplitudes in terms of the scalar integrals $I^{(x,1)}$.

2.4 $\mathcal{N} = 8$ supergravity four-point amplitude

Using generalized unitarity, the ℓ-loop $\mathcal{N} = 8$ supergravity four-point amplitude can also be expressed as a linear combination of scalar integrals

$$I^{(x,2)}(p_1, p_2, p_3, p_4) = (-i)^{\ell} \int \prod_{i=1}^{\ell} \frac{d^D \ell_i}{(2\pi)^D} \frac{N^{(x,2)}}{\prod_{j=1}^{3\ell+1} l_j^2} \tag{2.24}$$

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which are analogous to the gauge-theory integrals (2.2), but with different numerator factors. The sum over all permutations of external legs can again be reduced, using the invariance (2.8) under the Klein four-group, to a sum over S_3.

The precise form of the ℓ-loop four-point amplitude for $\mathcal{N} = 8$ supergravity is then given by [36, 38]

$$\mathcal{M}^{(\ell)} = 16K\tilde{K}(-1)^{\ell+1}\left(\frac{\kappa_D}{2}\right)^{2\ell+2}\sum_x\sum_{S_3}\lambda^{(x)}I^{(x,2)}_{1ijk} \tag{2.25}$$

where the combinatorial factors $\lambda^{(x)}$ are the same as those for the gauge-theory amplitude (2.23). Since the tree-level four-point gravity amplitude is [36]

$$\mathcal{M}^{(0)} = \left(\frac{\kappa_D}{2}\right)^2\frac{16K\tilde{K}}{stu} \tag{2.26}$$

we may rewrite eq. (2.25) as

$$\mathcal{M}^{(\ell)} = (-1)^{\ell+1}\left(\frac{\kappa_D}{2}\right)^{2\ell} stu\mathcal{M}^{(0)}\sum_x\sum_{S_3}\lambda^{(x)}I^{(x,2)}_{1ijk}\,. \tag{2.27}$$

3 One-loop relation

In this section we review the one-loop $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity four-point amplitudes, and the exact relation between them. Finally we examine the limiting form of the one-loop relation in the Regge limit.

3.1 One-loop $\mathcal{N} = 4$ SYM amplitude

Only the box diagram contributes to the one-loop $\mathcal{N} = 4$ SYM four-point amplitude [16, 36]

$$\mathcal{A}^{(1)} = -g_D^2 stA_1^{(0)}\sum_{S_3}\tfrac{1}{2}c^{(\mathrm{box})}_{1ijk}I^{(\mathrm{box})}_{ijkl} \tag{3.1}$$

where the one-loop box color factor is

$$c^{(\mathrm{box})}_{1234} \equiv \tilde{f}^{ea_1b}\tilde{f}^{ba_2c}\tilde{f}^{ca_3d}\tilde{f}^{da_4e} \tag{3.2}$$

and the box scalar integral is (since the numerator factor $N^{(\mathrm{box},1)} = 1$)

$$I^{(\mathrm{box})}_{1ijk} \equiv I^{(\mathrm{box})}(p_1,p_i,p_j,p_k) = -i\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2(\ell-p_1)^2(\ell-p_1-p_i)^2(\ell+p_k)^2}\,. \tag{3.3}$$

One easily ascertains that, in addition to being invariant under the Klein four-group (2.8), the one-loop color factor and box integral satisfy the reflection symmetry

$$c^{(\mathrm{box})}_{1ijk} = c^{(\mathrm{box})}_{1kji}\,, \qquad I^{(\mathrm{box})}_{1ijk} = I^{(\mathrm{box})}_{1kji} \tag{3.4}$$

so eq. (3.1) reduces to

$$\mathcal{A}^{(1)} = -g_D^2 stA_1^{(0)}\left[c^{(\mathrm{box})}_{1234}I^{(\mathrm{box})}_{1234} + c^{(\mathrm{box})}_{1342}I^{(\mathrm{box})}_{1342} + c^{(\mathrm{box})}_{1423}I^{(\mathrm{box})}_{1423}\right]. \tag{3.5}$$

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The one-loop color factor can be decomposed into

$$c_{1234}^{(\text{box})} = t_1^{(1)} + 2(t_4^{(1)} + t_5^{(1)} + t_6^{(1)}) = (1,0,0;\ 2,2,2) \tag{3.6}$$

with other permutations satisfying eq. (2.17). Consequently, the one-loop amplitude can be decomposed in the extended trace basis (2.21) with color-ordered amplitudes given by

$$\begin{aligned} A_1^{(1)} &= -g_D^2 st A_1^{(0)} I_{1234}^{(\text{box})}\,, \\ A_2^{(1)} &= -g_D^2 st A_1^{(0)} I_{1342}^{(\text{box})}\,, \\ A_3^{(1)} &= -g_D^2 st A_1^{(0)} I_{1423}^{(\text{box})} \end{aligned} \tag{3.7}$$

and the other color-ordered amplitudes satisfying $A_4^{(1)} = A_5^{(1)} = A_6^{(1)} = 2(A_1^{(1)} + A_2^{(1)} + A_3^{(1)})$.

3.2 One-loop $\mathcal{N} = 8$ supergravity amplitude and exact relation

The one-loop $\mathcal{N} = 8$ supergravity four-point amplitude is [16, 36]

$$\mathcal{M}^{(1)} = \left(\frac{\kappa_D}{2}\right)^2 stu\mathcal{M}^{(0)} \sum_{S_3} \tfrac{1}{2} I_{1ijk}^{(\text{box})} \tag{3.8}$$

where again the numerator factor $N^{(\text{box},2)} = 1$. Using the symmetry (3.4), the amplitude reduces to

$$\mathcal{M}^{(1)} = \left(\frac{\kappa_D}{2}\right)^2 stu\mathcal{M}^{(0)} \left[I_{1234}^{(\text{box})} + I_{1342}^{(\text{box})} + I_{1423}^{(\text{box})}\right]. \tag{3.9}$$

One can see the double-copy prescription at work in eq. (3.9); one simply replaces the color factors $c_{ijkl}^{(\text{box})}$ in eq. (3.5) with (constant) kinematic numerators to obtain the gravity amplitude.

Comparing eq. (3.9) with eq. (3.7), one establishes the exact one-loop relation [16]

$$\frac{1}{(\kappa_D/2)^2 stu} \frac{\mathcal{M}^{(1)}}{\mathcal{M}^{(0)}} = \frac{-\left[A_1^{(1)} + A_2^{(1)} + A_3^{(1)}\right]}{g_D^2 st A_1^{(0)}}\,. \tag{3.10}$$

3.3 One-loop Regge limit

To determine the behavior of the four-point amplitudes in the Regge limit ($|t| \ll s$), we need to examine the individual integrals that contribute. The known exact expression for the one-loop box integral [44] has the kinematic prefactor

$$I_{1ijk}^{(\text{box})} \sim \frac{1}{s_{1i}s_{1k}} \tag{3.11}$$

where

$$s_{ij} \equiv (p_i + p_j)^2. \tag{3.12}$$

In the Regge limit $|t| \ll s$ (so that $u = -s - t \sim -s$), one thus has $I_{1234}^{(\text{box})} \sim 1/(st)$ and $I_{1423}^{(\text{box})} \sim 1/(ut) \sim -1/(st)$ whereas $I_{1342}^{(\text{box})} \sim 1/(su) \sim -1/s^2$, so that

$$I_{1234}^{(\text{box})}, \quad I_{1423}^{(\text{box})} \gg I_{1342}^{(\text{box})}\,. \tag{3.13}$$

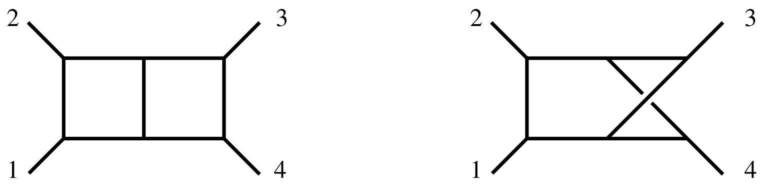

Figure 1. Two-loop diagrams P and NP.

From eqs. (3.7) and (3.13) one then has that the $\mathcal{N}=4$ SYM color-ordered amplitudes obey

$$A_1^{(1)}, \quad A_3^{(1)} \gg A_2^{(1)} \tag{3.14}$$

so the exact one-loop SYM/supergravity relation (3.10) reduces in the Regge limit to

$$\frac{1}{(\kappa_D/2)^2 stu} \frac{\mathcal{M}^{(1)}}{\mathcal{M}^{(0)}} \quad \longrightarrow \quad \frac{-\left[A_1^{(1)}+A_3^{(1)}\right]}{g_D^2 st A_1^{(0)}} \tag{3.15}$$

in agreement with eq. (1.1). Moreover, eq. (3.13) implies that the one-loop supergravity amplitude (3.9) reduces in the Regge limit to

$$\mathcal{M}^{(1)} \quad \longrightarrow \quad \left(\frac{\kappa_D}{2}\right)^2 stu\mathcal{M}^{(0)} \left[I_{1234}^{(\text{box})} + I_{1423}^{(\text{box})}\right] \tag{3.16}$$

the sum of a box and a "crossed-box" diagram. We will revisit this in section 6.

4 Two-loop relation

In this section we review the two-loop $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity four-point amplitudes [36]. We will then re-establish the known exact relation [17] between them, as well as its limiting form in the Regge limit.

4.1 Two-loop $\mathcal{N}=4$ SYM amplitude

The two-loop $\mathcal{N}=4$ SYM four-point amplitude is given by [35, 36]

$$\mathcal{A}^{(2)} = g_D^4 \; stA_1^{(0)} \sum_{S_3} \left[c_{1ijk}^{(P)} I_{1ijk}^{(P,1)} + c_{1ijk}^{(NP)} I_{1ijk}^{(NP,1)}\right] \tag{4.1}$$

where P and NP denote the planar double box and non-planar diagrams shown in figure 1, and the associated two-loop color factors are

$$\begin{aligned} c_{1234}^{(P)} &= \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{cgd} \tilde{f}^{dfe} \tilde{f}^{ga_3h} \tilde{f}^{ha_4f} \,, \\ c_{1234}^{(NP)} &= \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{cgd} \tilde{f}^{hfe} \tilde{f}^{ga_3h} \tilde{f}^{da_4f} \,. \end{aligned} \tag{4.2}$$

The two-loop integrals *without numerator factors* are

$$I_{1ijk}^{(P,0)} = -\int \frac{d^D\ell_1}{(2\pi)^D} \frac{d^D\ell_2}{(2\pi)^D} \frac{1}{\ell_1^2 (\ell_1+\ell_2)^2 \ell_2^2 (\ell_1-p_1)^2 (\ell_1-p_1-p_i)^2 (\ell_2-p_k)^2 (\ell_2-p_j-p_k)^2} \,,$$

$$I_{1ijk}^{(NP,0)} = -\int \frac{d^D\ell_1}{(2\pi)^D} \frac{d^D\ell_2}{(2\pi)^D} \frac{1}{\ell_1^2 (\ell_1+\ell_2)^2 \ell_2^2 (\ell_1-p_i)^2 (\ell_1+\ell_2+p_1)^2 (\ell_2-p_j)^2 (\ell_2-p_j-p_k)^2} \,.$$

Including the two-loop gauge-theory numerator factors, we have

$$I^{(P,1)}_{1ijk} = s_{1i} I^{(P,0)}_{1ijk} \,, \qquad\qquad I^{(NP,1)}_{1ijk} = s_{1i} I^{(NP,0)}_{1ijk} \,. \tag{4.3}$$

The non-planar color factor and integral have the additional symmetries

$$c^{(NP)}_{1ijk} = c^{(NP)}_{1ikj} \,, \qquad\qquad I^{(NP,1)}_{1ijk} = I^{(NP,1)}_{1ikj} \tag{4.4}$$

so eq. (4.1) reduces to

$$\begin{aligned}\mathcal{A}^{(2)} = g_D^4 \; stA_1^{(0)} \Big[& c^{(P)}_{1234} I^{(P,1)}_{1234} + c^{(P)}_{1243} I^{(P,1)}_{1243} + c^{(P)}_{1342} I^{(P,1)}_{1342} + c^{(P)}_{1324} I^{(P,1)}_{1324} + c^{(P)}_{1423} I^{(P,1)}_{1423} + c^{(P)}_{1432} I^{(P,1)}_{1432} \\ & + 2c^{(NP)}_{1234} I^{(NP,1)}_{1234} + 2c^{(NP)}_{1342} I^{(NP,1)}_{1342} + 2c^{(NP)}_{1423} I^{(NP,1)}_{1423} \Big] \,.\end{aligned} \tag{4.5}$$

The two-loop color factors may be decomposed in the extended trace basis as

$$\begin{aligned} c^{(P)}_{1234} &= (1,\,0,\,0;\quad 0,\quad 0,\,6;\,2,\,2,\,-4)\,, \\ c^{(NP)}_{1234} &= (0,\,0,\,0;\,-2,\,-2,\,4;\,2,\,2,\,-4)\,. \end{aligned} \tag{4.6}$$

Plugging eq. (4.6) together with the other permutations obtained via eq. (2.17) into eq. (4.5), we compute the color-ordered amplitudes [35]

$$\begin{aligned}
A_1^{(2)} &= g_D^4 \; stA_1^{(0)} \Big[I^{(P,1)}_{1234} + I^{(P,1)}_{1432} \Big] \,, \\
A_2^{(2)} &= g_D^4 \; stA_1^{(0)} \Big[I^{(P,1)}_{1243} + I^{(P,1)}_{1342} \Big] \,, \\
A_3^{(2)} &= g_D^4 \; stA_1^{(0)} \Big[I^{(P,1)}_{1324} + I^{(P,1)}_{1423} \Big] \,, \\
A_4^{(2)} &= 2g_D^4 \; stA_1^{(0)} \Big[3I^{(P,1)}_{1342} + 3I^{(P,1)}_{1324} - 2I^{(NP,1)}_{1234} + 4I^{(NP,1)}_{1342} - 2I^{(NP,1)}_{1423} \Big] \,, \\
A_5^{(2)} &= 2g_D^4 \; stA_1^{(0)} \Big[3I^{(P,1)}_{1423} + 3I^{(P,1)}_{1432} - 2I^{(NP,1)}_{1234} - 2I^{(NP,1)}_{1342} + 4I^{(NP,1)}_{1423} \Big] \,, \\
A_6^{(2)} &= 2g_D^4 \; stA_1^{(0)} \Big[3I^{(P,1)}_{1234} + 3I^{(P,1)}_{1243} + 4I^{(NP,1)}_{1234} - 2I^{(NP,1)}_{1342} - 2I^{(NP,1)}_{1423} \Big] \,, \\
A_7^{(2)} &= 2g_D^4 \; stA_1^{(0)} \Big[I^{(P,1)}_{1234} + I^{(P,1)}_{1243} - 2I^{(P,1)}_{1342} - 2I^{(P,1)}_{1324} + I^{(P,1)}_{1423} + I^{(P,1)}_{1432} + 2I^{(NP,1)}_{1234} - 4I^{(NP,1)}_{1342} + 2I^{(NP,1)}_{1423} \Big] \,, \\
A_8^{(2)} &= 2g_D^4 \; stA_1^{(0)} \Big[I^{(P,1)}_{1234} + I^{(P,1)}_{1243} + I^{(P,1)}_{1342} + I^{(P,1)}_{1324} - 2I^{(P,1)}_{1423} - 2I^{(P,1)}_{1432} + 2I^{(NP,1)}_{1234} + 2I^{(NP,1)}_{1342} - 4I^{(NP,1)}_{1423} \Big] \,, \\
A_9^{(2)} &= 2g_D^4 \; stA_1^{(0)} \Big[-2I^{(P,1)}_{1234} - 2I^{(P,1)}_{1243} + I^{(P,1)}_{1342} + I^{(P,1)}_{1324} + I^{(P,1)}_{1423} + I^{(P,1)}_{1432} - 4I^{(NP,1)}_{1234} + 2I^{(NP,1)}_{1342} + 2I^{(NP,1)}_{1423} \Big] \,.
\end{aligned} \tag{4.7}$$

From these, using $s+t+u=0$, we easily obtain the linear combination

$$\begin{aligned}\frac{uA_7^{(2)} + tA_8^{(2)} + sA_9^{(2)}}{g_D^4 stA_1^{(0)}} = -6\Big[& s(I^{(P,1)}_{1234} + I^{(P,1)}_{1243}) + u(I^{(P,1)}_{1342} + I^{(P,1)}_{1324}) + t(I^{(P,1)}_{1423} + I^{(P,1)}_{1432}) \\ & + 2sI^{(NP,1)}_{1234} + 2uI^{(NP,1)}_{1342} + 2tI^{(NP,1)}_{1423} \Big]\end{aligned} \tag{4.8}$$

which will be used below.

4.2 Two-loop $\mathcal{N}=8$ supergravity amplitude and exact relation

The two-loop $\mathcal{N}=8$ supergravity four-point amplitude is given by [36]

$$\mathcal{M}^{(2)} = -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)} \sum_{S_3} \left[I^{(P,2)}_{1ijk} + I^{(NP,2)}_{1ijk}\right] \tag{4.9}$$

where the two-loop gravity integrals include the numerator factors

$$I^{(P,2)}_{1ijk} = s_{1i}^2 I^{(P,0)}_{1ijk}, \qquad I^{(NP,2)}_{1ijk} = s_{1i}^2 I^{(NP,0)}_{1ijk}. \tag{4.10}$$

Using the symmetry $I^{(NP,2)}_{1ijk} = I^{(NP,2)}_{1ikj}$, we may write eq. (4.9) as

$$\begin{aligned}\mathcal{M}^{(2)} = -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)}\Big[&I^{(P,2)}_{1234} + I^{(P,2)}_{1243} + I^{(P,2)}_{1342} + I^{(P,2)}_{1324} + I^{(P,2)}_{1423} + I^{(P,2)}_{1432}\\ &+ 2I^{(NP,2)}_{1234} + 2I^{(NP,2)}_{1342} + 2I^{(NP,2)}_{1423}\Big].\end{aligned} \tag{4.11}$$

From eqs. (4.3) and (4.10) we have $I^{(P,2)}_{1ijk} = s_{1i}I^{(P,1)}_{1ijk}$ and $I^{(NP,2)}_{1ijk} = s_{1i}I^{(NP,1)}_{1ijk}$ so that

$$\begin{aligned}\mathcal{M}^{(2)} = -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)}\Big[&s(I^{(P,1)}_{1234} + I^{(P,1)}_{1243}) + u(I^{(P,1)}_{1342} + I^{(P,1)}_{1324}) + t(I^{(P,1)}_{1423} + I^{(P,1)}_{1432})\\ &+ 2sI^{(NP,1)}_{1234} + 2uI^{(NP,1)}_{1342} + 2tI^{(NP,1)}_{1423}\Big].\end{aligned} \tag{4.12}$$

This expression can be understood in terms of the double copy by replacing the color factors of eq. (4.5) with kinematic numerators; see ref. [12] for details.

Comparing eqs. (4.8) and (4.12), one obtains the exact two-loop relation [17]

$$\frac{1}{(\kappa_D/2)^4\ stu}\frac{\mathcal{M}^{(2)}}{\mathcal{M}^{(0)}} = \frac{\frac{1}{6}\left[uA^{(2)}_7 + tA^{(2)}_8 + sA^{(2)}_9\right]}{g_D^4 stA^{(0)}_1}. \tag{4.13}$$

4.3 Two-loop Regge limit

To determine the Regge limit of the two-loop amplitude, we examine the kinematic prefactors of the contributing integrals. The known explicit expression for the planar double box integral [45] has prefactor

$$I^{(P,0)}_{1ijk} \sim \frac{1}{s_{1i}^2 s_{1k}}. \tag{4.14}$$

The known explicit expression for the nonplanar integral [46] has separate contributions with prefactors

$$I^{(NP,0)}_{1ijk} \sim \frac{1}{s_{1i}^2 s_{1j}}, \ \frac{1}{s_{1i}^2 s_{1k}} \tag{4.15}$$

which ensures that the symmetry $I^{(NP,0)}_{1ijk} = I^{(NP,0)}_{1ikj}$ is respected. The gauge-theory integrals satisfy eq. (4.3) so

$$I^{(P,1)}_{1ijk} \sim \frac{1}{s_{1i}s_{1k}}, \qquad I^{(NP,1)}_{1ijk} \sim \frac{1}{s_{1i}s_{1j}}, \ \frac{1}{s_{1i}s_{1k}}. \tag{4.16}$$

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Thus in the Regge limit, one has $I^{(P,1)}_{1234}, I^{(P,1)}_{1324}, I^{(P,1)}_{1432}, I^{(P,1)}_{1423} \sim 1/(st)$ and $I^{(P,1)}_{1243}, I^{(P,1)}_{1342} \sim 1/s^2$, similar to the one-loop box integral, while all permutations of the nonplanar integral $I^{(NP,1)}$ go as $1/(st)$. All in all, we have

$$I^{(P,1)}_{1234}, I^{(P,1)}_{1324}, I^{(P,1)}_{1432}, I^{(P,1)}_{1423}, I^{(NP,1)}_{1234}, I^{(NP,1)}_{1342}, I^{(NP,1)}_{1423} \gg I^{(P,1)}_{1243}, I^{(P,1)}_{1342}\,. \tag{4.17}$$

Given eq. (4.17), we see from eq. (4.7) that $A^{(2)}_7$, $A^{(2)}_8$, and $A^{(2)}_9$ are all comparable in the Regge limit, so that the exact relation (4.13) reduces in this limit to

$$\frac{1}{(\kappa_D/2)^4\; stu}\frac{\mathcal{M}^{(2)}}{\mathcal{M}^{(0)}} \quad\longrightarrow\quad \frac{-\frac{1}{6}s\left[A^{(2)}_7 - A^{(2)}_9\right]}{g_D^4 st A^{(0)}_1} \tag{4.18}$$

in agreement with eq. (1.2).

It is instructive to explicitly express eq. (4.18) in terms of the contributing integrals. Using eq. (4.7) together with eq. (4.17) we have in the Regge limit

$$\begin{aligned} A^{(2)}_7 - A^{(2)}_9 \quad &\longrightarrow\quad 6g_D^4\; stA^{(0)}_1\left[I^{(P,1)}_{1234} - I^{(P,1)}_{1324} + 2I^{(NP,1)}_{1234} - 2I^{(NP,1)}_{1342}\right] \\ &\longrightarrow\quad g_D^4\; stA^{(0)}_1(6s)\left[I^{(P,0)}_{1234} + I^{(P,0)}_{1324} + 2I^{(NP,0)}_{1234} + 2I^{(NP,0)}_{1342}\right]. \end{aligned} \tag{4.19}$$

The gravity integrals satisfy eq. (4.10) so eqs. (4.14) and (4.15) imply

$$I^{(P,2)}_{1ijk} \sim \frac{1}{s_{1k}}, \qquad\qquad I^{(NP,2)}_{1ijk} \sim \frac{1}{s_{1j}}, \frac{1}{s_{1k}} \tag{4.20}$$

and thus $I^{(P,2)}_{1234}, I^{(P,2)}_{1324}, I^{(NP,2)}_{1234}, I^{(NP,2)}_{1342}$ go as $1/t$, while $I^{(P,2)}_{1243}, I^{(P,2)}_{1342}, I^{(P,2)}_{1432}, I^{(P,2)}_{1423}, I^{(NP,2)}_{1234}$ go as $1/s$ in the Regge limit. All in all,

$$I^{(P,2)}_{1234}, I^{(P,2)}_{1324}, I^{(NP,2)}_{1234}, I^{(NP,2)}_{1342} \gg I^{(P,2)}_{1243}, I^{(P,2)}_{1342}, I^{(P,2)}_{1432}, I^{(P,2)}_{1423}, I^{(NP,2)}_{1423}\,. \tag{4.21}$$

Thus the two-loop supergravity amplitude (4.11) reduces in the Regge limit to

$$\begin{aligned} \mathcal{M}^{(2)} \quad &\longrightarrow\quad -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)}\left[I^{(P,2)}_{1234} + I^{(P,2)}_{1324} + 2I^{(NP,2)}_{1234} + 2I^{(NP,2)}_{1342}\right] \\ &\longrightarrow\quad -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)}s^2\left[I^{(P,0)}_{1234} + I^{(P,0)}_{1324} + 2I^{(NP,0)}_{1234} + 2I^{(NP,0)}_{1342}\right]. \end{aligned} \tag{4.22}$$

In section 6, we will see that this is just the sum over all ladder and crossed-ladder diagrams.

Comparing eqs. (4.19) and (4.22), we can see explicitly that eq. (4.18) is satisfied.

5 Three-loop relation

In this section we present the expressions obtained in refs. [37, 38] for the three-loop $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity four-point amplitudes in terms of a sum of scalar integrals. Unlike the one- and two-loop cases, we will not be able to establish an exact relation between the three-loop amplitudes (but see subsection 5.4). However, by examining the Regge limits of the relevant integrals we will verify the Regge limit relation

$$\frac{1}{(\kappa_D/2)^6\; stu}\frac{\mathcal{M}^{(3)}}{\mathcal{M}^{(0)}} \quad\longrightarrow\quad \frac{s^2}{12}\,\frac{\left[4\left(A^{(3)}_1 + A^{(3)}_3\right) - \left(A^{(3)}_4 + A^{(3)}_6 + A^{(3)}_7 + A^{(3)}_9\right)\right]}{g_D^6 st A^{(0)}_1} \tag{5.1}$$

that was conjectured in ref. [14].

$I^{(x,1)}$	$N^{(x,1)}$
(a)–(d)	s_{12}^2
(e)–(g)	$s_{12}\, s_{46}$
(h)	$s_{12}(\tau_{26}+\tau_{36})+s_{14}(\tau_{15}+\tau_{25})+s_{12}s_{14}$
(i)	$s_{12}s_{45}-s_{14}s_{46}-\frac{1}{3}(s_{12}-s_{14})l_7^2$

Table 1. $\mathcal{N}=4$ SYM numerator factors (from ref. [38]).

5.1 Three-loop $\mathcal{N}=4$ SYM amplitude

The three-loop $\mathcal{N}=4$ SYM four-point amplitude is given by [37, 38]

$$\mathcal{A}^{(3)} = -g_D^6 st A_1^{(0)} \sum_{S_3} \Big[c_{1ijk}^{(a)} I_{1ijk}^{(a,1)} + c_{1ijk}^{(b)} I_{1ijk}^{(b,1)} + \tfrac{1}{2} c_{1ijk}^{(c)} I_{1ijk}^{(c,1)} + \tfrac{1}{4} c_{1ijk}^{(d)} I_{1ijk}^{(d,1)} \\ + 2c_{1ijk}^{(e)} I_{1ijk}^{(e,1)} + 2c_{1ijk}^{(f)} I_{1ijk}^{(f,1)} + 4c_{1ijk}^{(g)} I_{1ijk}^{(g,1)} + \tfrac{1}{2} c_{1ijk}^{(h)} I_{1ijk}^{(h,1)} + 2c_{1ijk}^{(i)} I_{1ijk}^{(i,1)} \Big] \tag{5.2}$$

where $c^{(x)}$ and $I^{(x,1)}$ are color factors and scalar integrals associated with the nine diagrams shown in figure 2. The $\mathcal{N}=4$ SYM numerator factors $N^{(x,1)}$ appearing in the integrals $I^{(x,1)}$ are given in ref. [38] and reproduced in table 1, with the invariants appearing in the table defined as

$$\tau_{ij} = 2p_i\cdot l_j\,, \qquad s_{ij} = \begin{cases} (p_i+p_j)^2, & i,j\le 4; \\ (p_i+l_j)^2, & i\le 4<j; \\ (l_i+l_j)^2, & 4<i,j. \end{cases} \tag{5.3}$$

The three-loop color factors may be decomposed in the extended trace basis as

$$\begin{aligned}
c_{1234}^{(a)} &= (1,\,0,\,0;\ \ 0,\ \ 0,\ 14;\ \ 2,\ \ 2,\ \ 0;\ \ 8,\ \ 8,\ \ 8),\\
c_{1234}^{(b)} &= (0,\,0,\,0;\ \ 0,\ \ 0,\ \ 8;\ \ 0,\ \ 0,\ \ 4;\ \ 8,\ \ 8,\ \ 8),\\
c_{1234}^{(c)} &= (0,\,0,\,0;\ \ 0,\ \ 0,\ \ 8;\ \ 0,\ \ 0,\ \ 4;\ \ 8,\ \ 8,\ \ 8),\\
c_{1234}^{(d)} &= (0,\,0,\,0;\ \ 2,\ \ 2,\ \ 4;\ -2,\ -2,\ \ 8;\ \ 8,\ \ 8,\ \ 8),\\
c_{1234}^{(e)} &= (1,\,0,\,0;\ \ 0,\ \ 0,\ \ 2;\ \ 8,\ -4,\ -6;\ -4,\ -4,\ -4),\\
c_{1234}^{(f)} &= (0,\,0,\,0;\ -2,\ -2,\ \ 0;\ \ 8,\ -4,\ -6;\ -4,\ -4,\ -4),\\
c_{1234}^{(g)} &= (0,\,0,\,0;\ \ 0,\ \ 0,\ -4;\ \ 6,\ -6,\ -2;\ -4,\ -4,\ -4),\\
c_{1234}^{(h)} &= (0,\,0,\,0;\ \ 0,\ \ 2,\ \ 2;\ -6,\ \ 4,\ \ 4;\ \ 4,\ \ 4,\ \ 4),\\
c_{1234}^{(i)} &= (0,\,0,\,0;\ \ 0,\ -2,\ \ 2;\ \ 0,\ \ 2,\ -2;\ \ 0,\ \ 0,\ \ 0).
\end{aligned} \tag{5.4}$$

To obtain the three-loop color-ordered amplitudes $A_\lambda^{(3)}$, one simply substitutes eq. (5.4), together with other permutations obtained via eq. (2.17), into eq. (5.2) and reads off the coefficient of each $t_\lambda^{(3)}$ as a linear combination of the integrals $I^{(x,1)}$. From these expressions, which we will not reproduce here, one can compute the particular linear combination

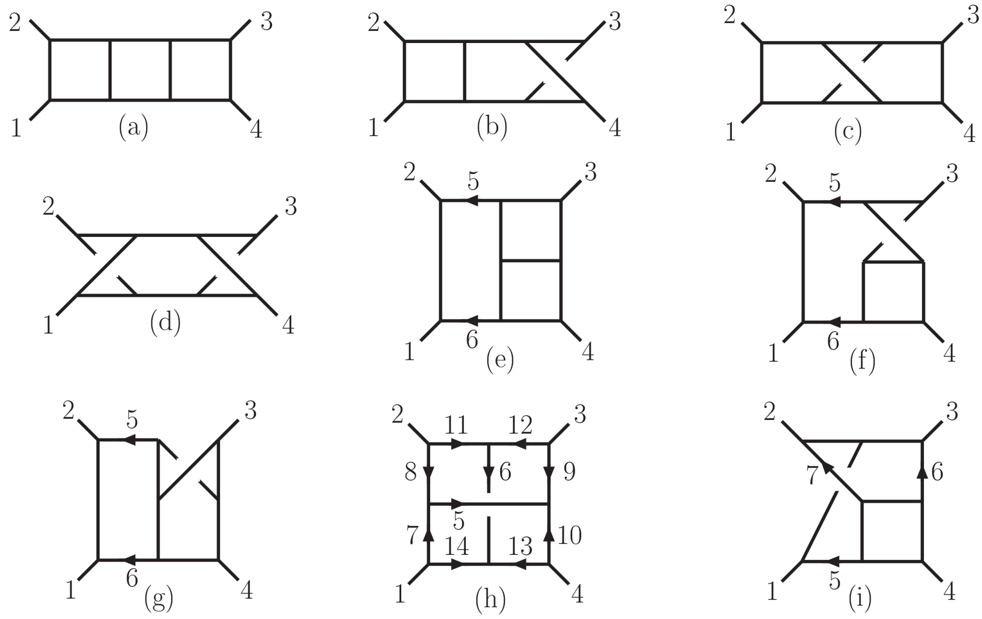

Figure 2. Three-loop diagrams (courtesy of ref. [38]).

appearing on the r.h.s. of the relation (5.1)

$$
\begin{aligned}
&\frac{4\left(A_1^{(3)}+A_3^{(3)}\right)-\left(A_4^{(3)}+A_6^{(3)}+A_7^{(3)}+A_9^{(3)}\right)}{g_D^6 st A_1^{(0)}}\\
&\quad=12\Big[I_{1234}^{(a,1)}+\tfrac{4}{3}I_{1243}^{(a,1)}+I_{1324}^{(a,1)}+\tfrac{4}{3}I_{1342}^{(a,1)}+2I_{1234}^{(b,1)}+2I_{1342}^{(b,1)}+\tfrac{1}{2}\left(2I_{1234}^{(c,1)}+2I_{1342}^{(c,1)}\right)\\
&\qquad+\tfrac{1}{4}\left(2I_{1234}^{(d,1)}+2I_{1342}^{(d,1)}\right)+2\left(-\tfrac{2}{3}I_{1243}^{(e,1)}-\tfrac{2}{3}I_{1342}^{(e,1)}\right)+2\left(-I_{1243}^{(f,1)}-I_{1342}^{(f,1)}\right)\\
&\qquad+4\left(-I_{1243}^{(g,1)}-I_{1342}^{(g,1)}\right)+\tfrac{1}{2}\left(I_{1243}^{(h,1)}+I_{1342}^{(h,1)}\right)\Big]
\end{aligned}
\tag{5.5}
$$

where we have used

$$
I_{1ijk}^{(b,1)}=I_{1ikj}^{(b,1)}, \qquad I_{1ijk}^{(c,1)}=I_{1ikj}^{(c,1)}, \qquad I_{1ijk}^{(d,1)}=I_{1ikj}^{(d,1)}
\tag{5.6}
$$

which are manifestly satisfied for the nonplanar diagrams b, c, and d.

5.2 Three-loop $\mathcal{N}=8$ supergravity amplitude

The three-loop $\mathcal{N}=8$ supergravity four-point amplitude is given by [37, 38]

$$
\begin{aligned}
\mathcal{M}^{(3)}=\left(\frac{\kappa_D}{2}\right)^6 stu\mathcal{M}^{(0)}\sum_{S_3}\Big[&I_{1ijk}^{(a,2)}+I_{1ijk}^{(b,2)}+\tfrac{1}{2}I_{1ijk}^{(c,2)}+\tfrac{1}{4}I_{1ijk}^{(d,2)}\\
&+2I_{1ijk}^{(e,2)}+2I_{1ijk}^{(f,2)}+4I_{1ijk}^{(g,2)}+\tfrac{1}{2}I_{1ijk}^{(h,2)}+2I_{1ijk}^{(i,2)}\Big]
\end{aligned}
\tag{5.7}
$$

where $I^{(x,2)}$ are scalar integrals associated with the nine diagrams shown in figure 2. The $\mathcal{N}=8$ SYM numerator factors $N^{(x,2)}$ appearing in the integrals $I^{(x,2)}$ are given in ref. [38]

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$I^{(x,2)}$	$N^{(x,2)}$
(a)–(d)	$[s_{12}^2]^2$
(e)–(g)	$[s_{12}\, s_{46}]^2$
(h)	$(s_{12}s_{89} + s_{14}s_{11,14} - s_{12}s_{14})^2 - s_{12}^2(2(s_{89} - s_{14}) + l_6^2)l_6^2 - s_{14}^2(2(s_{11,14} - s_{12}) + l_5^2)l_5^2$ $- s_{12}^2(2l_8^2 l_{10}^2 + 2l_7^2 l_9^2 + l_8^2 l_7^2 + l_9^2 l_{10}^2) - s_{14}^2(2l_{11}^2 l_{13}^2 + 2l_{12}^2 l_{14}^2 + l_{11}^2 l_{12}^2 + l_{13}^2 l_{14}^2) + 2s_{12}s_{14}l_5^2 l_6^2$
(i)	$(s_{12}s_{45} - s_{14}s_{46})^2 - (s_{12}^2 s_{45} + s_{14}^2 s_{46} + \frac{1}{3}s_{12}s_{13}s_{14})l_7^2$

Table 2. $\mathcal{N} = 8$ supergravity numerator factors (from ref. [38]).

and reproduced in table 2. In general there are no additional symmetries among the integrals except for the three nonplanar diagrams b, c, and d, which manifestly satisfy

$$I^{(b,2)}_{1ijk} = I^{(b,2)}_{1ikj}\,, \qquad I^{(c,2)}_{1ijk} = I^{(c,2)}_{1ikj}\,, \qquad I^{(d,2)}_{1ijk} = I^{(d,2)}_{1ikj}\,. \tag{5.8}$$

5.3 Three-loop Regge limit

We now examine the Regge limit of the integrals appearing in eqs. (5.5) and (5.7). In particular, we will find that

- the leading integrals in the SYM amplitude go as $1/(st)$,
- the leading integrals in the supergravity amplitude go as s/t, and
- the other integrals are suppressed by one or more factors of t/s.

This will lead to simplified expressions for the four-point amplitudes in the Regge limit.

We begin with the two planar diagrams, a and e, for which explicit expressions for the gauge-theory integrals are known [47], from which we see that they have kinematic prefactors

$$I^{(a,1)}_{1ijk} \sim \frac{1}{s_{1i}s_{1k}}\,, \qquad I^{(e,1)}_{1ijk} \sim \frac{1}{s_{1i}s_{1k}}\,. \tag{5.9}$$

This implies that, in the Regge limit,

$$I^{(a,1)}_{1234}\,, I^{(a,1)}_{1324} \gg I^{(a,1)}_{1243}\,, I^{(a,1)}_{1342}\,, I^{(e,1)}_{1243}\,, I^{(e,1)}_{1342} \tag{5.10}$$

so that the four gauge-theory integrals on the r.h.s. of eq. (5.10), which go as $1/s^2$, can be neglected relative to the two on the l.h.s., which go as $1/(st)$. (We only consider those integrals appearing in eq. (5.5).)

The numerator factor for the gravity integral $I^{(a,2)}_{1234}$ is s_{12}^4, compared to s_{12}^2 for gauge theory, so that $I^{(a,2)}_{1ijk} = s_{1i}^2 I^{(a,1)}_{1ijk}$. Hence from eq. (5.9), we have

$$I^{(a,2)}_{1ijk} \sim \frac{s_{1i}}{s_{1k}}\,. \tag{5.11}$$

Therefore, in the Regge limit

$$I^{(a,2)}_{1234}\,, I^{(a,2)}_{1324} \gg I^{(a,2)}_{1243}\,, I^{(a,2)}_{1342} \gg I^{(a,2)}_{1432}\,, I^{(a,2)}_{1423} \tag{5.12}$$

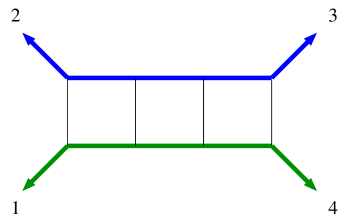

Figure 3. Momentum routing for $I^{(a)}_{1234}$.

so only the two integrals on the l.h.s. of eq. (5.12), which go as s/t, contribute to eq. (5.7) for $\mathcal{M}^{(3)}$ to leading order in the Regge limit.

Next, we expect that, like the two-loop nonplanar integrals (4.16), the three-loop nonplanar gauge-theory integrals corresponding to b, c, d have separate contributions with kinematic prefactors

$$I^{(b,1)}_{1ijk}, I^{(c,1)}_{1ijk}, I^{(d,1)}_{1ijk} \sim \frac{1}{s_{1i}s_{1j}}, \frac{1}{s_{1i}s_{1k}} \tag{5.13}$$

to ensure the symmetry (5.6). Hence the contributions from these integrals appearing in eq. (5.5) all go as $1/(st)$ in the Regge limit. The corresponding gravity integrals satisfy $I^{(x,2)}_{1ijk} = s^2_{1i} I^{(x,1)}_{1ijk}$, so from eq. (5.13), we have

$$I^{(b,2)}_{1ijk}, I^{(c,2)}_{1ijk}, I^{(d,2)}_{1ijk} \sim \frac{s_{1i}}{s_{1j}}, \frac{s_{1i}}{s_{1k}}. \tag{5.14}$$

Hence in the Regge limit the gravity integrals satisfy

$$I^{(x,2)}_{1234}, I^{(x,2)}_{1342} \gg I^{(x,2)}_{1423} \qquad \text{for } x = b, c, d \tag{5.15}$$

so only the two integrals on the l.h.s. of eq. (5.15), which go as s/t, contribute to $\mathcal{M}^{(3)}$ to leading order in the Regge limit.

Since the numerator factors for diagrams a, b, c, and d are independent of loop momenta for both gauge theory and gravity, the corresponding integrals are simply proportional to the same integrals without any numerator factors

$$I^{(x,1)}_{12jk} = s^2 I^{(x,0)}_{12jk}, \qquad I^{(x,1)}_{13jk} = u^2 I^{(x,0)}_{13jk} \longrightarrow s^2 I^{(x,0)}_{13jk}, \qquad \text{for } x = a, b, c, d, \tag{5.16}$$

$$I^{(x,2)}_{12jk} = s^4 I^{(x,0)}_{12jk}, \qquad I^{(x,2)}_{13jk} = u^4 I^{(x,0)}_{13jk} \longrightarrow s^4 I^{(x,0)}_{13jk}, \qquad \text{for } x = a, b, c, d \tag{5.17}$$

which will be used below.

Now we turn to the more difficult integrals, whose numerator factors are dependent on the loop momenta. To determine the Regge limits of these integrals, we consider the routing of hard momenta through each of the integrals. Consider diagram a shown in figure 3, where we have adopted the convention that all external momenta are outgoing. In the Regge limit $p_4 \to -p_1$ and $p_3 \to -p_2$, and the hard momentum associated with each of these external legs flows through the thick (green and blue) lines of the diagram. The thin (black) lines of the diagram carry the much softer exchanged momenta, which we generically denote by

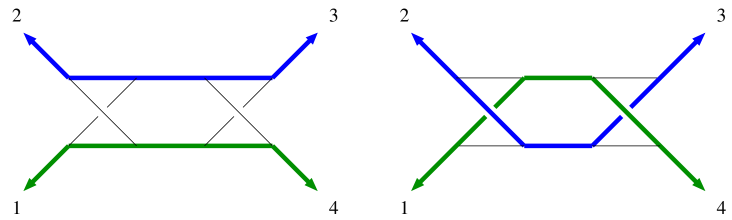

Figure 4. Two possible momentum routings for $I_{1234}^{(d)}$.

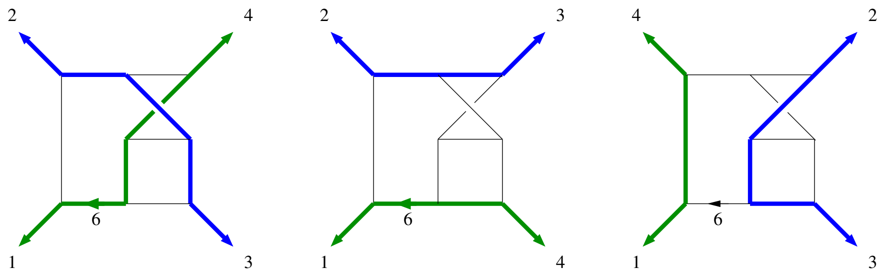

Figure 5. Possible routings for $I_{1243}^{(f)}$, $I_{1234}^{(f)}$, and $I_{1423}^{(f)}$ respectively.

q in this paper. In the Regge limit, q^2 is of order t. Sometimes more than one routing of hard momenta through the diagram is possible, as shown in figure 4 for the "double-cross" diagram d. This fact will play a key role in the discussion of ladder diagrams in section 6.

Now consider the routing of hard momenta through the gauge-theory integral $I_{1243}^{(f,1)}$, shown in the first diagram of figure 5. The numerator factor for this integral is $s_{12}s_{36}$, where $s_{36} = (p_3 + l_6)^2$. The momentum l_6 flowing through the thick (green) line in the direction indicated by the arrow is approximately equal to p_1 (that is, $l_6 = p_1 + q$, where q is the soft momentum flowing through the line connecting legs 1 and 2). Hence in the Regge limit the numerator factor $s_{12}s_{36} \to s_{12}s_{13} \to -s^2$, which we can then pull out of the integral. The same reasoning obtains for $I_{1342}^{(f,1)}$, and also for $I_{1243}^{(g,1)}$ and $I_{1342}^{(g,1)}$, so that

$$\begin{aligned} I_{1243}^{(f,1)} &\longrightarrow -s^2 I_{1243}^{(f,0)}, & I_{1342}^{(f,1)} &\longrightarrow -s^2 I_{1342}^{(f,0)}, \\ I_{1243}^{(g,1)} &\longrightarrow -s^2 I_{1243}^{(g,0)}, & I_{1342}^{(g,1)} &\longrightarrow -s^2 I_{1342}^{(g,0)}. \end{aligned} \tag{5.18}$$

Now consider the corresponding gravity integral $I_{1243}^{(f,2)}$. The numerator factor for this integral is $(s_{12}s_{63})^2$, which by the reasoning above goes to s^4 in the Regge limit. Using similar reasoning with the other integrals we obtain

$$\begin{aligned} I_{1243}^{(f,2)} &\longrightarrow s^4 I_{1243}^{(f,0)}, & I_{1342}^{(f,2)} &\longrightarrow s^4 I_{1342}^{(f,0)}, \\ I_{1243}^{(g,2)} &\longrightarrow s^4 I_{1243}^{(g,0)}, & I_{1342}^{(g,2)} &\longrightarrow s^4 I_{1342}^{(g,0)}. \end{aligned} \tag{5.19}$$

For the gravity integral $I_{1234}^{(f,2)}$, one possible routing of hard momentum is shown in the second diagram of figure 5. The hard momentum continues to flow through 6, so $l_6 \to p_1$,

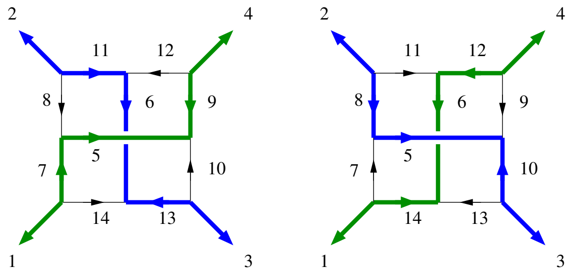

Figure 6. Possible momentum routings for $I^{(h)}_{1243}$.

but the numerator factor in this case is $(s_{12}s_{46})^2 \to (s_{12}s_{14})^2 = s^2t^2$, so this integral is suppressed relative to $I^{(f,2)}_{1243}$, and similar reasoning holds $I^{(f,2)}_{1324}$, and also for $I^{(g,2)}_{1234}$ and $I^{(g,2)}_{1324}$.

For the gravity integral $I^{(f,2)}_{1423}$, one possible routing of hard momenta is shown in the third diagram of figure 5, so that l_6 is soft. Consequently, the numerator factor $(s_{14}s_{36})^2 \to t^4$, suppressing this integral even further in the Regge limit. Similar reasoning holds for $I^{(f,2)}_{1432}$, and also for $I^{(g,2)}_{1423}$ and $I^{(g,2)}_{1432}$. Thus, the only permutations of the f and g integrals that contribute to $\mathcal{M}^{(3)}$ in the Regge limit are those shown in eq. (5.19).

Next, we consider diagram e. The gravity integral $I^{(e,2)}_{1ijk}$ has numerator factor $(s_{1i}s_{k6})^2$, compared to the gauge-theory numerator factor $s_{1i}s_{k6}$. If $i \neq 4$, the routing of hard momentum goes through l_6, so $l_6 \to p_1$ in which case $s_{1i}s_{k6} \to s_{1i}s_{1k}$. Thus in the Regge limit

$$I^{(e,2)}_{1ijk} \quad \longrightarrow \quad s_{1i}s_{1k}I^{(e,1)}_{1ijk}\,, \qquad \text{for } i \neq 4\,. \tag{5.20}$$

The kinematic prefactor for the gauge-theory integral $I^{(e,1)}_{1ijk}$ is given by eq. (5.9), so we have

$$I^{(e,2)}_{1ijk} \quad \longrightarrow \quad \frac{s_{1i}s_{1k}}{s_{1i}s_{1k}} = 1 \ll \frac{s}{t}\,, \qquad \text{for } i \neq 4 \tag{5.21}$$

which is much less than the contribution of the other gravity diagrams. The Regge limit of $I^{(e,1)}_{1ijk}$ with $i = 4$ is even more highly suppressed (since l_6 is soft), therefore diagram e does not contribute to $\mathcal{M}^{(3)}$ at all in the Regge limit.

Finally we turn to the formidable integral $I^{(h,1)}_{1243}$, through which there are two possible routings of the hard momentum, as shown in figure 6. For the first routing, one has $l_5 \to p_4$ and $l_6 \to p_3$. The numerator factor for this gauge-theory integral (cf. table 1) simplifies in the Regge limit to

$$s_{12}(\tau_{26}+\tau_{46})+s_{13}(\tau_{15}+\tau_{25})+s_{12}s_{13} \;\to\; s_{12}(s_{23}+s_{34})+s_{13}(s_{14}+s_{24})+s_{12}s_{13} \;\to\; s^2\,. \tag{5.22}$$

In the second routing shown in figure 6, one has $l_5 \to p_3$ and $l_6 \to p_1$, and the numerator factor again approaches s^2 in the Regge limit. Similar reasoning holds for $I^{(h,1)}_{1342}$. Consequently, we see that the gauge-theory integrals corresponding to diagram h go as

$$I^{(h,1)}_{1243} \quad \longrightarrow \quad s^2 I^{(h,0)}_{1243}\,, \qquad\qquad I^{(h,1)}_{1342} \quad \longrightarrow \quad s^2 I^{(h,0)}_{1342}\,. \tag{5.23}$$

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For the gravity integrals corresponding to diagram h, the numerator factor (given in table 2) is more complicated. In the Regge limit, however, most of the terms are subleading. Any term l_j^2 goes either as q^2 (if the momentum flowing through leg is soft, $l_j \sim q$) or as $q \cdot p_i$ (if the momentum flowing through leg is hard, $l_j \sim p_i + q$, where p_i is the momentum of one of the external legs), both of which are $\ll s$. Thus the gravity numerator factor for $I_{1243}^{(h,2)}$ simplifies to

$$(s_{12}s_{89} + s_{13}s_{11,14} - s_{12}s_{13})^2 \;\rightarrow\; s^4 \tag{5.24}$$

because s_{89} and $s_{11,14}$ are both $\sim q \cdot p_i$. Consequently we may write

$$I_{1243}^{(h,2)} \;\longrightarrow\; s^4 I_{1243}^{(h,0)}, \qquad\qquad I_{1342}^{(h,2)} \;\longrightarrow\; s^4 I_{1342}^{(h,0)}. \tag{5.25}$$

Straightforward consideration of all other permutations of the external legs for this diagram shows that the corresponding gravity integrals are subleading in the Regge limit. The same is true for the gravity integrals corresponding to all permutations of external legs for diagram i. (Note that diagram i does not contribute at all to the gauge theory expression (5.5).)

We are now in a position to combine all of the results obtained above to evaluate both eqs. (5.5) and (5.7) in the Regge limit. Taking into account eq. (5.10), and using eqs. (5.16), (5.18), and (5.23), we find that the linear combination of gauge-theory amplitudes (5.5) evaluates in the Regge limit to

$$\begin{aligned}
&\frac{4\left(A_1^{(3)} + A_3^{(3)}\right) - \left(A_4^{(3)} + A_6^{(3)} + A_7^{(3)} + A_9^{(3)}\right)}{g_D^6 st A_1^{(0)}}\\
&\quad\longrightarrow\; 12s^2\Big[I_{1234}^{(a,0)} + I_{1324}^{(a,0)} + 2I_{1234}^{(b,0)} + 2I_{1342}^{(b,0)} + \tfrac{1}{2}\left(2I_{1234}^{(c,0)} + 2I_{1342}^{(c,0)}\right) + \tfrac{1}{4}\left(2I_{1234}^{(d,0)} + 2I_{1342}^{(d,0)}\right)\\
&\qquad\qquad + 2\left(I_{1243}^{(f,0)} + I_{1342}^{(f,0)}\right) + 4\left(I_{1243}^{(g,0)} + I_{1342}^{(g,0)}\right) + \tfrac{1}{2}\left(I_{1243}^{(h,0)} + I_{1342}^{(h,0)}\right)\Big].
\end{aligned} \tag{5.26}$$

Similarly, we can evaluate the gravity amplitude (5.7) by omitting all the integrals suppressed in the Regge limit, and using eqs. (5.17), (5.19), and (5.25) to obtain

$$\begin{aligned}
&\frac{1}{(\kappa_D/2)^6 \; stu}\frac{\mathcal{M}^{(3)}}{\mathcal{M}^{(0)}}\\
&\quad\longrightarrow\; s^4\Big[I_{1234}^{(a,0)} + I_{1324}^{(a,0)} + 2I_{1234}^{(b,0)} + 2I_{1342}^{(b,0)} + \tfrac{1}{2}\left(2I_{1234}^{(c,0)} + 2I_{1342}^{(c,0)}\right) + \tfrac{1}{4}\left(2I_{1234}^{(d,0)} + 2I_{1342}^{(d,0)}\right)\\
&\qquad\qquad + 2\left(I_{1243}^{(f,0)} + I_{1342}^{(f,0)}\right) + 4\left(I_{1243}^{(g,0)} + I_{1342}^{(g,0)}\right) + \tfrac{1}{2}\left(I_{1243}^{(h,0)} + I_{1342}^{(h,0)}\right)\Big].
\end{aligned} \tag{5.27}$$

Comparing eqs. (5.26) and (5.27), we immediately see that the conjectured three-loop SYM/supergravity relation (5.1) is confirmed in the Regge limit.

5.4 An exact three-loop relation?

It might be asked whether the three-loop relation (5.1) proved in the previous subsection is a specialization to the Regge limit of some exact relation, as was the case at one and two loops. Because the three-loop kinematic numerators for both supergravity and SYM theory are dependent on the loop momenta for diagrams (e) through (i) of figure 2 (see tables 1 and 2), the double-copy procedure, which relates the integrands of these amplitudes, does not give a straightforward relation between the integrated amplitudes (unlike the one- and two-loop cases).

Interestingly, if one were to include *only* diagrams (a) through (d) of figure 2 (namely, those with numerator factors that are independent of loop momenta) in the expressions for the three-loop amplitudes (5.2) and (5.7) then one can derive a unique exact (i.e. not only in the Regge limit) relation between them, namely[3]

$$\frac{1}{(\kappa_D/2)^6\ stu}\frac{\mathcal{M}^{(3)}}{\mathcal{M}^{(0)}} \overset{?!}{=} \Big[2(s^2+u^2-2t^2)\left(A_1^{(3)}+A_2^{(3)}+A_3^{(3)}\right)-(u^2-t^2)A_4^{(3)}-(s^2-t^2)A_6^{(3)} \\ -(u^2+2t^2)A_7^{(3)}-3t^2A_8^{(3)}-(s^2+2t^2)A_9^{(3)}\Big]\Big/\left(12g_D^6 st A_1^{(0)}\right). \tag{5.28}$$

Moreover this relation indeed reduces to eq. (5.1) in the Regge limit.

One might then ask whether the relation (5.28) could be valid for the full amplitudes, despite the fact that it was obtained using only some of the contributing diagrams. Henn and Mistlberger have computed the Laurent expansions (through $\mathcal{O}(\epsilon^0)$) of the three-loop $\mathcal{N}=8$ supergravity amplitude, which starts at $\mathcal{O}(1/\epsilon^3)$, in ref. [18] and the color-ordered amplitudes of the three-loop $\mathcal{N}=4$ SYM amplitude, which start at $\mathcal{O}(1/\epsilon^6)$, in ref. [15]. Using their results, one finds, somewhat remarkably, that the relation (5.28) is satisfied through $\mathcal{O}(1/\epsilon^2)$, with the difference between the two sides given by the rather simple expression

$$(\text{l.h.s.}-\text{r.h.s})_{eq.\ (5.28)} = \frac{1}{(8\pi^2)^3}\frac{\zeta_5+2\zeta_2\zeta_3}{\epsilon}+\mathcal{O}(\epsilon^0)\,. \tag{5.29}$$

Thus, while the relation (5.28) is not generally valid (except in the Regge limit), it holds better than one might have anticipated. Note that the leading kinematical dependence on both sides of eq. (5.28) goes as $\sim s/t$ in the Regge limit so that the discrepancy (5.29) is subleading in an expansion in t/s. It would be interesting to understand the reason for this discrepancy.

6 All-loop order Regge limit of gravity amplitudes

In this section, we show that $\mathcal{N}=8$ supergravity four-point amplitudes at one, two, and three loops reduce in the Regge limit to a (modified) sum of ladder and crossed-ladder scalar diagrams, and explain how this is related to the eikonal representation of gravity amplitudes.

6.1 Ladder and crossed-ladder diagrams

The expressions for the Regge limits of the ℓ-loop $\mathcal{N}=8$ supergravity amplitudes (and therefore the equivalent linear combinations of color-ordered ℓ-loop $\mathcal{N}=4$ SYM amplitudes) obtained in the previous three sections take a very specific form: they are (almost) the sum over all ladder and crossed-ladder scalar diagrams at that loop order. To show this, we

[3]Note that $A_\lambda^{(3)}$ with $\lambda=5,10,11$, and 12 do not appear in this relation because they have been eliminated using the three-loop group theory relations; see eq. (2.13) of ref. [14].

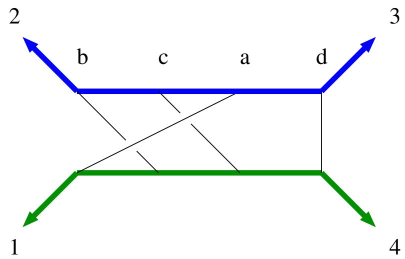

Figure 7. Crossed-ladder diagram corresponding to the integral $I^{(\mathrm{lad})}_{[abcd]}$.

introduce a special notation for (crossed-)ladder diagrams (see figure 7). The ℓ-loop crossed-ladder integral $I^{(\mathrm{lad})}_{[abc\ldots]}$ is defined with precisely the same prefactors (and no numerator factors) as $I^{(x,0)}$ (cf. eq. (2.3)), with the subscript in brackets describing how the rungs are connected between the rails of the ladder: the first vertex in the thick (green) line running from 1 to 4 is attached to the ath vertex in the thick (blue) line running from 2 to 3, the second vertex to the bth vertex, etc. The specific diagram shown in figure 7 corresponds to $I^{(\mathrm{lad})}_{[3124]}$.

The one-loop supergravity amplitude in the Regge limit, eq. (3.16), is easily seen to be the sum of a ladder and a crossed ladder diagram

$$\mathcal{M}^{(1)} \quad \longrightarrow \quad \left(\frac{\kappa_D}{2}\right)^2 stu\mathcal{M}^{(0)} \left[I^{(\mathrm{lad})}_{[12]} + I^{(\mathrm{lad})}_{[21]}\right] . \tag{6.1}$$

The two-loop supergravity amplitude in the Regge limit, eq. (4.22), which is equivalent by eqs. (2.8) and (4.4) to

$$\mathcal{M}^{(2)} \quad \longrightarrow \quad -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)} s^2 \left[I^{(P,0)}_{1234} + I^{(P,0)}_{1324} + I^{(NP,0)}_{1234} + I^{(NP,0)}_{4321} + I^{(NP,0)}_{1324} + I^{(NP,0)}_{4231}\right] \tag{6.2}$$

is written in the ladder notation as

$$\mathcal{M}^{(2)} \quad \longrightarrow \quad -\left(\frac{\kappa_D}{2}\right)^4 stu\mathcal{M}^{(0)} s^2 \left[I^{(\mathrm{lad})}_{[123]} + I^{(\mathrm{lad})}_{[321]} + I^{(\mathrm{lad})}_{[132]} + I^{(\mathrm{lad})}_{[213]} + I^{(\mathrm{lad})}_{[312]} + I^{(\mathrm{lad})}_{[231]}\right] \tag{6.3}$$

in which all six permutations of the three rungs are present. (This was observed in ref. [27].)

Finally, we recall from eq. (5.27) the three-loop supergravity amplitude in the Regge limit

$$\begin{aligned}\mathcal{M}^{(3)} \quad \longrightarrow \quad & \left(\frac{\kappa_D}{2}\right)^6 stu\mathcal{M}^{(0)} s^4 \Big[I^{(a,0)}_{1234} + I^{(a,0)}_{1324} + 2I^{(b,0)}_{1234} + 2I^{(b,0)}_{1342} \\ & + \tfrac{1}{2}\left(2I^{(c,0)}_{1234} + 2I^{(c,0)}_{1342}\right) + \tfrac{1}{4}\left(2I^{(d,0)}_{1234} + 2I^{(d,0)}_{1342}\right) + 2\left(I^{(f,0)}_{1243} + I^{(f,0)}_{1342}\right) \\ & + 4\left(I^{(g,0)}_{1243} + I^{(g,0)}_{1342}\right) + \tfrac{1}{2}\left(I^{(h,0)}_{1243} + I^{(h,0)}_{1342}\right)\Big].\end{aligned} \tag{6.4}$$

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Using the invariance of the integrals under the Klein four-group (2.8), this may be recast as

$$\begin{aligned}\mathcal{M}^{(3)} \quad &\longrightarrow \quad \left(\frac{\kappa_D}{2}\right)^6 \; stu\mathcal{M}^{(0)} s^4 \Big[I^{(a,0)}_{1234} + I^{(a,0)}_{1324} + I^{(b,0)}_{1234} + I^{(b,0)}_{4321} + I^{(b,0)}_{1324} + I^{(b,0)}_{4231} \\ &+ I^{(c,0)}_{1234} + I^{(c,0)}_{1324} + \tfrac{1}{2}\left(I^{(d,0)}_{1234} + I^{(d,0)}_{1324}\right) + I^{(f,0)}_{1243} + I^{(f,0)}_{4312} + I^{(f,0)}_{1342} + I^{(f,0)}_{4213} \\ &+ I^{(g,0)}_{1243} + I^{(g,0)}_{2134} + I^{(g,0)}_{3421} + I^{(g,0)}_{4312} + I^{(g,0)}_{1342} + I^{(g,0)}_{2431} + I^{(g,0)}_{3124} + I^{(g,0)}_{4213} + \tfrac{1}{2}\left(I^{(h,0)}_{1243} + I^{(h,0)}_{1342}\right) \Big] .\end{aligned} \tag{6.5}$$

We recognize all of these integrals as corresponding to ladders and crossed ladders

$$\begin{aligned}\mathcal{M}^{(3)} \quad &\longrightarrow \quad \left(\frac{\kappa_D}{2}\right)^6 \; stu\mathcal{M}^{(0)} s^4 \Big[I^{(\text{lad})}_{[1234]} + I^{(\text{lad})}_{[4321]} + I^{(\text{lad})}_{[1243]} + I^{(\text{lad})}_{[2134]} + I^{(\text{lad})}_{[4312]} + I^{(\text{lad})}_{[3421]} \\ &+ I^{(\text{lad})}_{[1324]} + I^{(\text{lad})}_{[4231]} + \tfrac{1}{2}\left(I^{(\text{lad})}_{[2143]} + I^{(\text{lad})}_{[3412]}\right) + I^{(\text{lad})}_{[1432]} + I^{(\text{lad})}_{[3214]} + I^{(\text{lad})}_{[4123]} + I^{(\text{lad})}_{[2341]} \\ &+ I^{(\text{lad})}_{[1423]} + I^{(\text{lad})}_{[1342]} + I^{(\text{lad})}_{[3124]} + I^{(\text{lad})}_{[2314]} + I^{(\text{lad})}_{[4132]} + I^{(\text{lad})}_{[2431]} + I^{(\text{lad})}_{[4213]} + I^{(\text{lad})}_{[3241]} + I^{(\text{lad})}_{[3142]} + I^{(\text{lad})}_{[2413]} \Big] .\end{aligned} \tag{6.6}$$

One slightly subtle point deserves to be noted in going from eq. (6.5) to eq. (6.6). Each of the integrals $I^{(h,0)}_{1243}$ and $I^{(h,0)}_{1342}$ has two possible momentum routings, as shown in figure 6. Each routing contributes to separate crossed-ladder integrals, $I^{(\text{lad})}_{[3142]}$ and $I^{(\text{lad})}_{[2413]}$, which accounts for the disappearance of the factor of 1/2 multiplying the last pair of terms in eq. (6.5).

Examination of eq. (6.6) shows that it contains all 24 permutations of the rungs of the crossed ladders, *except* that two of them include a factor of 1/2. This is a blessing in disguise, as we will now see.

6.2 Eikonal representation

An alternative approach to evaluating the Regge limit of supergravity uses the eikonal approximation [19–21] to write the gravitational amplitude (to all loop orders) in impact-parameter space [22–32]

$$\mathcal{M} \sim \int d^{D-2}\mathbf{x}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} \left(e^{i\chi(\mathbf{x}_\perp)} - 1\right) \tag{6.7}$$

where $\mathbf{x}_\perp$ is a $(D-2)$-dimensional vector transverse to the incoming particle direction, and $\mathbf{q}_\perp$ is the $(D-2)$-dimensional momentum transfer that is Fourier-conjugate to $\mathbf{x}_\perp$ (where $t \simeq -|\mathbf{q}_\perp|^2$ in the leading Regge limit). The quantity $i\chi(\mathbf{x}_\perp)$ is known as the eikonal phase, and is given in $D = 4 - 2\epsilon$ dimensions by

$$i\chi(\mathbf{x}_\perp) = \frac{-iG_D s}{\epsilon}\Gamma(1-\epsilon)(\pi \mathbf{x}_\perp^2)^\epsilon . \tag{6.8}$$

By expanding the exponential in eq. (6.7) in a Taylor series in G_D and Fourier transforming one obtains [31]

$$\mathcal{M}^{(\ell)} \quad \longrightarrow \quad \mathcal{M}^{(0)} \frac{1}{\ell!} \left[\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi}{-t}\right)^\epsilon \left(\frac{-iG_D s}{\epsilon}\right)\right]^\ell G^{(\ell)}(\epsilon) \tag{6.9}$$

where

$$G^{(\ell)}(\epsilon) = \frac{\Gamma^{\ell}(1-2\epsilon)\Gamma(1+\ell\epsilon)}{\Gamma^{\ell-1}(1-\epsilon)\Gamma^{\ell}(1+\epsilon)\Gamma(1-(\ell+1)\epsilon)}\,. \tag{6.10}$$

As explained in ref. [31], this result is consistent with the Regge limits of the Laurent expansions of the one-, two-, and three-loop $\mathcal{N}=8$ supergravity amplitudes obtained in ref. [18]. (A proposed extension [32] of the eikonal representation (6.7) to include subleading-level, i.e. $\mathcal{O}(-t/s)$, corrections also agrees with the results of ref. [18], up to a small discrepancy at the three-loop level that has still not been fully accounted for.)

What do eqs. (6.7) and (6.9) have to do with the expressions for $\mathcal{N}=8$ supergravity amplitudes in terms of scalar integrals (without numerator factors) obtained in eqs. (6.1), (6.3), and (6.6) above? Starting in the 1960's, efforts were made to derive the eikonal representation (6.7) from the sum of ladder and crossed-ladder scalar diagrams [21, 48]. (See appendix C of ref. [49] for a detailed accounting.) While successful at one and two loops, this project was discovered to fail at three loops and above [49–51]. At three loops, the obstacle is the fact that there are two possible routings of hard momentum through the "double-cross" ladder diagram d (as shown in figure 4) so that the contribution from this integral in the Regge limit is twice what it needs to be to give the eikonal result (6.9).

We found above, however, that the double-cross ladder diagrams that appear in the Regge limit of the three-loop $\mathcal{N}=8$ supergravity amplitude (6.6) obtained using generalized unitarity in refs. [37, 38] come equipped with a factor of $1/2$, which precisely corrects for this overcounting. Thus, in the Regge limit, the (modified) sum of ladders and crossed ladders given in eq. (6.6) yields the (correct) eikonal three-loop result (6.9).

7 Conclusions

In a previous paper, one of the authors analyzed the structure of the Regge limit of the (nonplanar) $\mathcal{N}=4$ SYM four-point amplitude [14], and based on those results, conjectured an all-loop-orders relation between the Regge limits of the four-point amplitudes of $\mathcal{N}=4$ SYM theory and $\mathcal{N}=8$ supergravity, viz. eqs. (1.4) and (1.5). The one- and two-loop Regge limit relations, eqs. (1.1) and (1.2), are consequences of known exact relations, eqs. (3.10) and (4.13), between $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ supergravity amplitudes.

In this paper, we established the conjectured (Regge limit) relation at the three-loop level, viz. eq. (1.3). We showed that the Regge limit of exact expressions for the amplitudes, obtained using generalized unitarity, simplifies in both cases to the same (modified) sum over three-loop ladder and crossed-ladder scalar diagrams, thus proving the conjectured relation. The sum is modified in the sense that two of the crossed-ladder diagrams are multiplied by a factor of one-half relative to the remaining diagrams.

We also presented an exact three-loop relation (5.28) that would be valid if only a certain subset of the scalar diagrams were included in the evaluation of the three-loop amplitudes, and which reduces to eq. (1.3) in the Regge limit. We tested this putative relation against the Laurent expansions of the full three-loop amplitudes, and found rather remarkably that it holds at $\mathcal{O}(1/\epsilon^3)$ and $\mathcal{O}(1/\epsilon^2)$, and only breaks down at $\mathcal{O}(1/\epsilon)$.

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The supergravity four-point amplitude can alternatively be evaluated in the Regge limit using the eikonal approximation to give a representation in impact-parameter space (6.7). This may in turn be evaluated [31] to give the expression (6.9), and shown to agree with the known Regge limit of the $\mathcal{N} = 8$ supergravity amplitude through three loops.

In this paper, we showed that the modification of the sum over crossed-ladder scalar diagrams described above is precisely what is required for the Regge limit of the sum to agree with correct three-loop result, whereas it is known that the unmodified sum fails to do so [49–51].

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References

[1] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, arXiv:1909.01358 [INSPIRE].

[2] H. Kawai, D.C. Lewellen and S.H.H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1 [INSPIRE].

[3] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [arXiv:0805.3993] [INSPIRE].

[4] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [arXiv:1004.0476] [INSPIRE].

[5] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, *Phys. Rev. D* **82** (2010) 065003 [arXiv:1004.0693] [INSPIRE].

[6] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes*, *Phys. Rev. D* **85** (2012) 105014 [arXiv:1201.5366] [INSPIRE].

[7] Z. Bern, J.J.M. Carrasco, W.-M. Chen, H. Johansson, R. Roiban and M. Zeng, *Five-loop four-point integrand of $N = 8$ supergravity as a generalized double copy*, *Phys. Rev. D* **96** (2017) 126012 [arXiv:1708.06807] [INSPIRE].

[8] Z. Bern et al., *Ultraviolet Properties of $\mathcal{N} = 8$ Supergravity at Five Loops*, *Phys. Rev. D* **98** (2018) 086021 [arXiv:1804.09311] [INSPIRE].

[9] J.J.M. Carrasco and H. Johansson, *Five-Point Amplitudes in $N = 4$ Super-Yang-Mills Theory and $N = 8$ Supergravity*, *Phys. Rev. D* **85** (2012) 025006 [arXiv:1106.4711] [INSPIRE].

[10] Z. Bern, C. Boucher-Veronneau and H. Johansson, *$N \geq 4$ Supergravity Amplitudes from Gauge Theory at One Loop*, *Phys. Rev. D* **84** (2011) 105035 [arXiv:1107.1935] [INSPIRE].

[11] S.G. Naculich and H.J. Schnitzer, *One-loop SYM-supergravity relation for five-point amplitudes*, *JHEP* **11** (2011) 001 [arXiv:1108.6326] [INSPIRE].

[12] C. Boucher-Veronneau and L.J. Dixon, *$N \geq 4$ Supergravity Amplitudes from Gauge Theory at Two Loops*, *JHEP* **12** (2011) 046 [arXiv:1110.1132] [INSPIRE].

[13] S.G. Naculich, H. Nastase and H.J. Schnitzer, *Linear relations between $N \geq 4$ supergravity and subleading-color SYM amplitudes*, *JHEP* **01** (2012) 041 [arXiv:1111.1675] [INSPIRE].

[14] S.G. Naculich, *All-loop-orders relation between Regge limits of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity four-point amplitudes*, *JHEP* **02** (2021) 044 [arXiv:2012.00030] [INSPIRE].

[15] J.M. Henn and B. Mistlberger, *Four-Gluon Scattering at Three Loops, Infrared Structure, and the Regge Limit*, *Phys. Rev. Lett.* **117** (2016) 171601 [arXiv:1608.00850] [INSPIRE].

[16] M.B. Green, J.H. Schwarz and L. Brink, *$N = 4$ Yang-Mills and $N = 8$ Supergravity as Limits of String Theories*, *Nucl. Phys. B* **198** (1982) 474 [INSPIRE].

[17] S.G. Naculich, H. Nastase and H.J. Schnitzer, *Subleading-color contributions to gluon-gluon scattering in $N = 4$ SYM theory and relations to $N = 8$ supergravity*, *JHEP* **11** (2008) 018 [arXiv:0809.0376] [INSPIRE].

[18] J.M. Henn and B. Mistlberger, *Four-graviton scattering to three loops in $\mathcal{N} = 8$ supergravity*, *JHEP* **05** (2019) 023 [arXiv:1902.07221] [INSPIRE].

[19] H. Cheng and T.T. Wu, *High-energy elastic scattering in quantum electrodynamics*, *Phys. Rev. Lett.* **22** (1969) 666 [INSPIRE].

[20] H.D.I. Abarbanel and C. Itzykson, *Relativistic eikonal expansion*, *Phys. Rev. Lett.* **23** (1969) 53 [INSPIRE].

[21] M. Levy and J. Sucher, *Eikonal approximation in quantum field theory*, *Phys. Rev.* **186** (1969) 1656 [INSPIRE].

[22] G. 't Hooft, *Graviton Dominance in Ultrahigh-Energy Scattering*, *Phys. Lett. B* **198** (1987) 61 [INSPIRE].

[23] D. Amati, M. Ciafaloni and G. Veneziano, *Superstring Collisions at Planckian Energies*, *Phys. Lett. B* **197** (1987) 81 [INSPIRE].

[24] I.J. Muzinich and M. Soldate, *High-Energy Unitarity of Gravitation and Strings*, *Phys. Rev. D* **37** (1988) 359 [INSPIRE].

[25] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, *Int. J. Mod. Phys. A* **3** (1988) 1615 [INSPIRE].

[26] D.N. Kabat and M. Ortiz, *Eikonal quantum gravity and Planckian scattering*, *Nucl. Phys. B* **388** (1992) 570 [hep-th/9203082] [INSPIRE].

[27] S.B. Giddings, M. Schmidt-Sommerfeld and J.R. Andersen, *High energy scattering in gravity and supergravity*, *Phys. Rev. D* **82** (2010) 104022 [arXiv:1005.5408] [INSPIRE].

[28] S. Melville, S.G. Naculich, H.J. Schnitzer and C.D. White, *Wilson line approach to gravity in the high energy limit*, *Phys. Rev. D* **89** (2014) 025009 [arXiv:1306.6019] [INSPIRE].

[29] R. Akhoury, R. Saotome and G. Sterman, *High Energy Scattering in Perturbative Quantum Gravity at Next to Leading Power*, *Phys. Rev. D* **103** (2021) 064036 [arXiv:1308.5204] [INSPIRE].

JHEP07(2022)043

[30] A. Luna, R. Monteiro, D. O'Connell and C.D. White, *The classical double copy for Taub-NUT spacetime*, *Phys. Lett. B* **750** (2015) 272 [arXiv:1507.01869] [INSPIRE].

[31] P. Di Vecchia, A. Luna, S.G. Naculich, R. Russo, G. Veneziano and C.D. White, *"A tale of two exponentiations in $\mathcal{N} = 8$ supergravity"*, *Phys. Lett. B* **798** (2019) 134927 [arXiv:1908.05603].

[32] P. Di Vecchia, S.G. Naculich, R. Russo, G. Veneziano and C.D. White, *A tale of two exponentiations in $\mathcal{N} = 8$ supergravity at subleading level*, *JHEP* **03** (2020) 173 [arXiv:1911.11716] [INSPIRE].

[33] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys. B* **425** (1994) 217 [hep-ph/9403226] [INSPIRE].

[34] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys. B* **435** (1995) 59 [hep-ph/9409265] [INSPIRE].

[35] Z. Bern, J.S. Rozowsky and B. Yan, *Two loop four gluon amplitudes in $N = 4$ superYang-Mills*, *Phys. Lett. B* **401** (1997) 273 [hep-ph/9702424] [INSPIRE].

[36] Z. Bern, L.J. Dixon, D.C. Dunbar, M. Perelstein and J.S. Rozowsky, *On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences*, *Nucl. Phys. B* **530** (1998) 401 [hep-th/9802162] [INSPIRE].

[37] Z. Bern, J.J. Carrasco, L.J. Dixon, H. Johansson, D.A. Kosower and R. Roiban, *Three-Loop Superfiniteness of $N = 8$ Supergravity*, *Phys. Rev. Lett.* **98** (2007) 161303 [hep-th/0702112] [INSPIRE].

[38] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *Manifest Ultraviolet Behavior for the Three-Loop Four-Point Amplitude of $N = 8$ Supergravity*, *Phys. Rev. D* **78** (2008) 105019 [arXiv:0808.4112] [INSPIRE].

[39] Z. Bern, J.J.M. Carrasco, L.J. Dixon, H. Johansson and R. Roiban, *The Complete Four-Loop Four-Point Amplitude in $N = 4$ Super-Yang-Mills Theory*, *Phys. Rev. D* **82** (2010) 125040 [arXiv:1008.3327] [INSPIRE].

[40] Z. Bern, J.J.M. Carrasco, H. Johansson and R. Roiban, *The Five-Loop Four-Point Amplitude of $N = 4$ super-Yang-Mills Theory*, *Phys. Rev. Lett.* **109** (2012) 241602 [arXiv:1207.6666] [INSPIRE].

[41] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory Vol. 1: 25th Anniversary Edition*, Cambridge University Press, Cambridge, U.K. (2012).

[42] Z. Bern and D.A. Kosower, *Color decomposition of one loop amplitudes in gauge theories*, *Nucl. Phys. B* **362** (1991) 389 [INSPIRE].

[43] S.G. Naculich, *All-loop group-theory constraints for color-ordered SU(N) gauge-theory amplitudes*, *Phys. Lett. B* **707** (2012) 191 [arXiv:1110.1859] [INSPIRE].

[44] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys. B* **412** (1994) 751 [hep-ph/9306240] [INSPIRE].

[45] V.A. Smirnov, *Analytical result for dimensionally regularized massless on shell double box*, *Phys. Lett. B* **460** (1999) 397 [hep-ph/9905323] [INSPIRE].

[46] J.B. Tausk, *Nonplanar massless two loop Feynman diagrams with four on-shell legs*, *Phys. Lett. B* **469** (1999) 225 [hep-ph/9909506] [INSPIRE].

[47] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev. D* **72** (2005) 085001 [hep-th/0505205] [INSPIRE].

[48] G. Tiktopoulos and S.B. Treiman, *Validity of the relativistic eikonal approximation*, *Phys. Rev. D* **2** (1970) 805 [INSPIRE].

[49] H. Cheng and T.T. Wu, *Expanding Protons: Scattering at High Energies*, MIT Press, Cambridge, U.S.A. (1987).

[50] G. Tiktopoulos and S.B. Treiman, *Relativistic eikonal approximation*, *Phys. Rev. D* **3** (1971) 1037 [INSPIRE].

[51] D.N. Kabat, *Validity of the Eikonal approximation*, *Comments Nucl. Part. Phys.* **20** (1992) 325 [hep-th/9204103] [INSPIRE].