Editors' Suggestion

Three-Loop Gluon Scattering in QCD and the Gluon Regge Trajectory

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We compute the three-loop helicity amplitudes for the scattering of four gluons in QCD. We employ projectors in the 't Hooft-Veltman scheme and construct the amplitudes from a minimal set of physical building blocks, which allows us to keep the computational complexity under control. We obtain relatively compact results that can be expressed in terms of harmonic polylogarithms. In addition, we consider the Regge limit of our amplitude and extract the gluon Regge trajectory in full three-loop QCD. This is the last missing ingredient required for studying single-Reggeon exchanges at next-to-next-to-leading logarithmic accuracy.

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Introduction.—Scattering amplitudes in quantum chromodynamics (QCD) are one of the fundamental ingredients to describe the dynamics of the high energy collision events produced at the Large Hadron Collider (LHC) at CERN. As a matter of fact, such probability amplitudes for processes involving four or five elementary particles and up to two loops in perturbation theory, are routinely used to measure the properties of standard model particles as the Higgs boson and to study its interactions with fermions and electroweak bosons [1]. Moreover, by providing the building blocks for precise estimates of standard model processes, they also allow us to put stringent constraints on new physics signals predicted by various beyond the standard model scenarios.

In addition to their practical use for collider physics, the analytic calculation of scattering amplitudes in QCD provides an invaluable source of information to understand general properties of perturbative quantum field theory (QFT). In fact, with more loops and more external particles participating to the scattering process, the analytic structure of scattering amplitudes becomes increasingly rich, in particular due to the appearance of new classes of special functions, whose branch cut and analytical structure are to reproduce those dictated by causality and unitarity in QFT. In recent years, a considerable effort has been devoted to study the properties of these functions from first principles. The goal is to understand whether an upper bound can be established for the type of mathematical objects that can appear in the calculation of physically relevant scattering processes. While we are far from being able to provide a complete answer to this question, the multitude of data collected in the form of increasingly complicated amplitudes, have already revealed crucial to discover and classify ubiquitous classes of such functions, most notably the so-called multiple polylogarithms [2–4] and more recently their elliptic generalizations [5–10]. Most of these discoveries have been inspired by analytical results for scattering amplitudes up to two loops, both in massless and in massive theories, which have been an important focus of the efforts of the particle physics community in the last two decades. A natural step forward in these investigations is to push these calculations one loop higher to determine which degree of generalization is required. In combination with more general results on the simplified universal properties of QCD in special kinematical limits, perturbative calculations can also be used to have a glimpse of some all-order QCD structures, which only emerge summing infinite classes of diagrams. One of the classical examples of such kinematical configurations is the so-called Regge limit [11], where the energy of the colliding partons is assumed to be much larger than the typical transferred momentum. In this limit, the Balitskii-Fadin-Kuraev-Lipatov (BFKL) formalism [12,13] allows one to reformulate the calculation of scattering amplitudes in terms of the exchange of so-called reggeized gluons, which resum specific contribution to the strong interaction among elementary partons to all orders in the QCD coupling constants.

Motivated by these considerations, in this Letter we focus on the scattering of four gluons at three loops in QCD. This is the most complex of all scattering processes in QCD that involve four massless particles, both for the number of terms involved in its calculation, and also for its color and infrared structure. As of today, this process was known to three-loop order only in the simpler setting of $\mathcal{N}=4$ super-Yang-Mills (SYM) theory [14] and in the planar approximation for pure Yang-Mills theory [15]. The high-energy limit of these results have been studied in Refs. [16,17], respectively. In this Letter, we build upon the techniques that we have developed for the calculations of simpler four-particle scattering processes [18–20] and compute the three loop scattering amplitudes for gluon-gluon scattering in full, nonplanar QCD.

We consider the process

$$g(p_1)+g(p_2)+g(p_3)+g(p_4)\to 0, \tag{1}$$

where all momenta are taken to be incoming and massless

$$p_1^\mu+p_2^\mu+p_3^\mu+p_4^\mu=0, \qquad p_i^2=0. \tag{2}$$

The scattering process above can be parametrized in terms of the usual set of Mandelstam invariants

$$s=(p_1+p_2)^2, \quad t=(p_1+p_3)^2, \quad u=(p_2+p_3)^2, \tag{3}$$

which satisfy the relation $u=-t-s$. We work in *dimensional regularization* to regulate all ultraviolet and infrared divergences. More precisely, we adopt the 't Hooft-Veltman scheme [21], where loop momenta are taken to be $d=4-2\epsilon$ dimensional, while momenta and polarizations associated with external particles are kept in four dimensions. The physical scattering process $gg\to gg$ (relevant for dijet production) can be obtained from (1) by crossing $p_{3,4}\to -p_{3,4}$. In order to parametrize the kinematics for this process, it is useful to define the dimensionless ratio

$$x=-t/s, \tag{4}$$

so that in the physical region $p_1+p_2\to p_3+p_4$ we have

$$s>0, \qquad t<0, \qquad u<0; \qquad 0<x<1. \tag{5}$$

Color and Lorentz decomposition.—We write the scattering amplitude for $gg\to gg$ as

$$\mathcal{A}^{a_1a_2a_3a_4}=4\pi\alpha_{s,b}\sum_{i=1}^{6}\mathcal{A}^{[i]}\mathcal{C}_i, \tag{6}$$

where $\alpha_{s,b}$ is the bare strong coupling, $\mathcal{A}^{[i]}$ are color-ordered *partial amplitudes*, and the color basis $\{\mathcal{C}_i\}$ reads

$$\begin{aligned}
\mathcal{C}_1&=\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]+\mathrm{Tr}[T^{a_1}T^{a_4}T^{a_3}T^{a_2}],\\
\mathcal{C}_2&=\mathrm{Tr}[T^{a_1}T^{a_2}T^{a_4}T^{a_3}]+\mathrm{Tr}[T^{a_1}T^{a_3}T^{a_4}T^{a_2}],\\
\mathcal{C}_3&=\mathrm{Tr}[T^{a_1}T^{a_3}T^{a_2}T^{a_4}]+\mathrm{Tr}[T^{a_1}T^{a_4}T^{a_2}T^{a_3}],\\
\mathcal{C}_4&=\mathrm{Tr}[T^{a_1}T^{a_2}]\mathrm{Tr}[T^{a_3}T^{a_4}],\\
\mathcal{C}_5&=\mathrm{Tr}[T^{a_1}T^{a_3}]\mathrm{Tr}[T^{a_2}T^{a_4}],\\
\mathcal{C}_6&=\mathrm{Tr}[T^{a_1}T^{a_4}]\mathrm{Tr}[T^{a_2}T^{a_3}].
\end{aligned} \tag{7}$$

Here, the adjoint representation index a_i corresponds to the ith external gluon, while T_{ij}^a are the fundamental $\mathrm{SU}(N_c)$ generators normalized such that $\mathrm{Tr}[T^aT^b]=\frac{1}{2}\delta^{ab}$. As it is well known, the partial amplitudes $\mathcal{A}^{[i]}$ are independently gauge invariant. The advantage of using a color-ordered decomposition is that, by construction, the amplitudes $\mathcal{A}^{[i]}$ are not all independent under crossings of the external momenta. We can restrict ourselves to compute only two of the structures above and obtain all other partial amplitudes by crossing symmetry. For definiteness, we choose to focus on $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[4]}$.

In order to compute $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[4]}$, it is convenient to further decompose them with respect to a basis of Lorentz covariant tensor structures. In the following, we denote the polarization vector of the ith external gluon as $\epsilon(p_i)=\epsilon_i$, which satisfies the transversality condition $\epsilon_i\cdot p_i=0$. By making the cyclic gauge choice $\epsilon_i\cdot p_{i+1}=0$, with $p_5=p_1$, and restricting ourselves to physical four-dimensional external states, one finds [22,23] that each partial amplitude can be decomposed as

$$\mathcal{A}^{[j]}(s,t)=\sum_{i=1}^{8}\mathcal{F}_i^{[j]}T_i, \tag{8}$$

where the coefficient functions $\mathcal{F}_i^{[j]}$ are usually referred to as *form factors* and the tensors T_i are defined as

$$\begin{aligned}
&T_1=\epsilon_1\cdot p_3\epsilon_2\cdot p_1\epsilon_3\cdot p_1\epsilon_4\cdot p_2,\\
&T_2=\epsilon_1\cdot p_3\epsilon_2\cdot p_1\epsilon_3\cdot\epsilon_4, \qquad T_3=\epsilon_1\cdot p_3\epsilon_3\cdot p_1\epsilon_2\cdot\epsilon_4,\\
&T_4=\epsilon_1\cdot p_3\epsilon_4\cdot p_2\epsilon_2\cdot\epsilon_3, \qquad T_5=\epsilon_2\cdot p_1\epsilon_3\cdot p_1\epsilon_1\cdot\epsilon_4,\\
&T_6=\epsilon_2\cdot p_1\epsilon_4\cdot p_2\epsilon_1\cdot\epsilon_3, \qquad T_7=\epsilon_3\cdot p_1\epsilon_4\cdot p_2\epsilon_1\cdot\epsilon_2,\\
&T_8=\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4+\epsilon_1\cdot\epsilon_4\epsilon_2\cdot\epsilon_3+\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4.
\end{aligned} \tag{9}$$

The form factors can be extracted by defining a set of eight projectors P_i which are in one to one correspondence with the tensors in Eq. (9), such that $P_i\cdot T_j=\sum_{pol}P_iT_j=\delta_{ij}$.

Helicity amplitudes.—In this Letter we are ultimately interested in the helicity amplitudes $\mathcal{A}_\lambda$, where

$\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and λ_i is the helicity of the ith external particle. In the four-gluon case we need to consider $2^4 = 16$ possible helicity choices. However, only eight helicity amplitudes are independent as the remaining ones can be obtained by parity conjugation, which effectively transforms the helicities as $\boldsymbol{\lambda} \to -\boldsymbol{\lambda}$. The independent helicity amplitudes are in one to one correspondence with the Lorentz tensors of Eq. (9) and their color stripped counterparts can in fact be written as a linear combination of the form factors $\mathcal{F}_i^{[j]}$. In order to make this relation explicit, we start from the tensor decomposition in Eq. (8) and employ the *spinor-helicity formalism* [24] to fix the helicities of the external gluons. We write the gluon polarization vectors for fixed $\pm$ helicity as

$$\epsilon^{\mu}_{i,+} = \frac{[i+1|\gamma^{\mu}|i\rangle}{\sqrt{2}[i|i+1]}, \qquad \epsilon^{\mu}_{i,-} = \frac{[i|\gamma^{\mu}|i+1\rangle}{\sqrt{2}\langle i+1|i\rangle}, \quad (10)$$

where we used the cyclic gauge choice introduced above, identifying $|5] \equiv |1]$ and $|5\rangle \equiv |1\rangle$. By inserting the specific representation of Eq. (10) in Eq. (8), we can write the color-ordered partial amplitudes as

$$\mathcal{A}_{\boldsymbol{\lambda}}^{[i]} = \mathcal{H}_{\boldsymbol{\lambda}}^{[i]} s_{\boldsymbol{\lambda}}, \quad (11)$$

where $s_{\boldsymbol{\lambda}}$ is a phase that carries all the spinor weight. The decomposition (11) is not unique. Here, we follow [25] and choose

$$\begin{aligned}
s_{++++} &= \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}, \qquad & s_{-+++} &= \frac{\langle 12\rangle\langle 14\rangle[24]}{\langle 34\rangle\langle 23\rangle\langle 24\rangle}, \\
s_{+-++} &= \frac{\langle 21\rangle\langle 24\rangle[14]}{\langle 34\rangle\langle 13\rangle\langle 14\rangle}, \qquad & s_{++-+} &= \frac{\langle 32\rangle\langle 34\rangle[24]}{\langle 14\rangle\langle 21\rangle\langle 24\rangle}, \\
s_{+++-} &= \frac{\langle 42\rangle\langle 14\rangle[12]}{\langle 13\rangle\langle 23\rangle\langle 12\rangle}, \qquad & s_{++--} &= \frac{\langle 12\rangle[34]}{[12]\langle 34\rangle}, \\
s_{+-+-} &= \frac{\langle 13\rangle[24]}{[13]\langle 24\rangle}, \qquad & s_{+--+} &= \frac{\langle 14\rangle[23]}{[14]\langle 23\rangle}.
\end{aligned} \quad (12)$$

From now on we will focus on the calculation of $\mathcal{H}_{\boldsymbol{\lambda}}^{[j]}$, which we will refer to as helicity amplitudes, with a slight abuse of notation. The $\mathcal{H}_{\boldsymbol{\lambda}}^{[j]}$ can be expanded in terms of the bare QCD coupling in the usual way:

$$\mathcal{H}_{\boldsymbol{\lambda}} = \sum_{k=0}^{3} \bar{\alpha}_{s,b}^k S_\epsilon^k \mathcal{H}_{\boldsymbol{\lambda}}^{(k)} + \mathcal{O}(\bar{\alpha}_{s,b}^4), \quad (13)$$

where we have omitted the color structure index $[j]$ for ease of reading and defined $\bar{\alpha}_{s,b} = \alpha_{s,b}/(4\pi)$ and $S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma_E}$. Here, we focus on the computation of the three-loop amplitude $\mathcal{H}_{\boldsymbol{\lambda}}^{(3)}$. As a byproduct we also recomputed the tree level, one- and two-loop amplitudes as a check of our framework and found prefect agreement with previous results in the literature [26,27].

We use QGRAF [28] to produce the relevant Feynman diagrams: there are four different diagrams at tree level, 81 at one loop, 1771 at two loops and 48723 at three loops. We then use FORM [29] to apply the projection operators P_i to suitable combinations of the Feynman diagrams and in this way write the helicity amplitudes $\mathcal{H}_{\boldsymbol{\lambda}}^{[1]}$, $\mathcal{H}_{\boldsymbol{\lambda}}^{[4]}$ as a linear combination of scalar Feynman integrals. The integrals appearing in the computation of these amplitudes can be written as

$$\mathcal{I}^{\text{top}}_{n_1,\ldots,n_N} = \mu_0^{2L\epsilon} e^{L\epsilon\gamma_E} \int \prod_{i=1}^{L} \left(\frac{d^d k_i}{i\pi^{\frac{d}{2}}}\right) \frac{1}{D_1^{n_1}\cdots D_N^{n_N}}, \quad (14)$$

where L stands for the number of loops, k_i are the loop momenta, $\gamma_E \approx 0.5772$ is the Euler constant, μ_0 is the dimensional regularization scale, and $\epsilon = (4-d)/2$ is the dimensional regulator. Here, "top" can be any of the planar or nonplanar integral families which are given explicitly in Ref. [19]. At three loops we find that a staggering number of $\sim\mathcal{O}(10^7)$ scalar integrals contribute to the amplitude. However, these integrals are not linearly independent and can be related using symmetry relations and integration by parts identities [30,31]. We performed this reduction with REDUZE2 [32,33] and FINRED, an in-house implementation based on Laporta's algorithm, finite field techniques [34–37] and syzygy algorithms [38–43]. In this way we were able to express the helicity amplitudes in terms of the 486 *master integrals* (MIs), which were first computed in Ref. [44] and more recently in Ref. [20] in terms of simple harmonic polylogarithms (HPLs) [2]. After inserting the analytic expressions for the master integrals, we obtain the *bare* helicity amplitudes $\mathcal{H}_{\boldsymbol{\lambda}}^{(j)}$ as a Laurent series in ϵ up to $\mathcal{O}(\epsilon^0)$ in terms of HPLs up to transcendental weight six.

uv renormalization and ir behavior.—The bare helicity amplitudes contain both ultraviolet (uv) and infrared (ir) divergencies that manifest as poles in the series expansions of the dimensional regulator ϵ. uv divergences can be removed by expressing the amplitudes in terms of the $\overline{\text{MS}}$ renormalized strong coupling $\alpha_s(\mu)$ using

$$\bar{\alpha}_{s,b}\mu_0^{2\epsilon} S_\epsilon = \bar{\alpha}_s(\mu)\mu^{2\epsilon} Z[\bar{\alpha}_s(\mu)], \quad (15)$$

where $\bar{\alpha}_s(\mu) = \alpha_s(\mu)/(4\pi)$, μ is the renormalization scale, and

$$Z[\bar{\alpha}_s] = 1 - \bar{\alpha}_s \frac{\beta_0}{\epsilon} + \bar{\alpha}_s^2\left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right) - \bar{\alpha}_s^3\left(\frac{\beta_0^3}{\epsilon^3} - \frac{7}{6}\frac{\beta_0\beta_1}{\epsilon^2} + \frac{\beta_2}{3\epsilon}\right) + \mathcal{O}(\bar{\alpha}_s^4). \quad (16)$$

The explicit form of the β-function coefficients β_i is immaterial for our discussion; for the reader's convenience, we provide them in Supplemental Material [45]. The uv-renormalized helicity amplitudes $\mathcal{H}_{\lambda,\mathrm{ren}}$ are obtained by expanding Eq. (6) in $\bar{\alpha}_s(\mu)$. In particular, $\mathcal{H}^{(k)}_{\lambda,\mathrm{ren}}$ is the (color- and helicity-stripped) coefficient of the $\bar{\alpha}_s^k$ term.

The renormalized amplitudes still contain poles of ir origin, whose structure is universal. The infrared structure of QCD scattering amplitudes was first studied at two loops in [46] and later extended to general processes and three loops in [47–55]. Up to three loop order, one can write [50,51]

$$\mathcal{H}_{\lambda,\mathrm{ren}} = \mathcal{Z}_{\mathrm{ir}} \mathcal{H}_{\lambda,\mathrm{fin}}, \tag{17}$$

where $\mathcal{H}_{\lambda,\mathrm{fin}}$ are *finite remainders* and $\mathcal{Z}_{\mathrm{ir}}$ is a color matrix that acts on the $\{\mathcal{C}_i\}$ basis (7). It can be written in terms of the so-called soft anomalous dimension $\mathbf{\Gamma}$ as

$$\mathcal{Z}_{\mathrm{ir}} = \mathbb{P} \exp\left[\int_\mu^\infty \frac{d\mu'}{\mu'} \mathbf{\Gamma}(\{p\},\mu')\right], \tag{18}$$

where the *path ordering* operator $\mathbb{P}$ reorganizes color operators in increasing values of μ' from left to right and is immaterial up to three loops since to this order $[\mathbf{\Gamma}(\mu),\mathbf{\Gamma}(\mu')] = 0$. The soft anomalous dimension can be written as

$$\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{dip}} + \mathbf{\Delta}_4. \tag{19}$$

The *dipole* term $\mathbf{\Gamma}_{\mathrm{dip}}$ is due to the pairwise exchange of color charge between external legs and reads

$$\mathbf{\Gamma}_{\mathrm{dip}} = \sum_{1\le i<j\le 4} \mathbf{T}_i^a \mathbf{T}_j^a \gamma^K \ln\left(\frac{\mu^2}{-s_{ij} - i\delta}\right) + 4\gamma^g, \tag{20}$$

where $s_{ij} = 2p_i \cdot p_j$, γ^K is the *cusp anomalous dimension* [56–62] and γ^g is the *gluon anomalous dimension* [63–66]. Their explicit form up to the order $\bar{\alpha}_s^3$ required here is reproduced in Supplemental Material [45] for convenience. In Eq. (20) we have also introduced the standard color insertion operators $\mathbf{T}_i^a$, which only act on the ith external color index. In particular, in our case their action on $\{\mathcal{C}_i\}$ is defined as $\mathbf{T}_i^a T^{b_i} = -i f^{a b_i c_i} T^{c_i} = [T^{b_i}, T^a]$.

The *quadrupole* contribution $\mathbf{\Delta}_4$ in Eq. (19) accounts instead for the exchange of color charge among (up to) four external legs. It becomes relevant for the first time at three loops, $\mathbf{\Delta}_4 = \sum_{n=3}^\infty \bar{\alpha}_s^n \mathbf{\Delta}_4^{(n)}$, where it reads [55]

$$\mathbf{\Delta}_4^{(3)} = f_{abe} f_{cde}\Bigg[-16C \sum_{i=1}^4 \sum_{\substack{1\le j<k\le 4\\ j,k\neq i}} \{\mathbf{T}_i^a, \mathbf{T}_i^d\} \mathbf{T}_j^b \mathbf{T}_k^c + 128[\mathbf{T}_1^a \mathbf{T}_2^c \mathbf{T}_3^b \mathbf{T}_4^d D_1(x) - \mathbf{T}_4^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_3^d D_2(x)]\Bigg], \tag{21}$$

with $C = \zeta_5 + 2\zeta_2\zeta_3$. The functions $D_1(x)$ and $D_2(x)$ in our notation are reported in Supplemental Material [45].

We verified that the ir singularities of our three-loop amplitudes match perfectly those generated by Eqs. (17)–(21), which provides a highly nontrivial check of our results. Our results for the finite remainder $\mathcal{H}_{\lambda,\mathrm{fin}}$ are relatively compact, but still too long to be presented here. They are included in computer-readable format in Supplemental Material [67]. In Fig. 1, we plot our results for the interference with the tree level, defined as

$$\langle \mathcal{H}^{(0)} | \mathcal{H}^{(L)} \rangle \equiv \mathcal{N} \sum_{i,j=1}^{6} \mathcal{C}_i^\dagger \mathcal{C}_j \sum_\lambda \mathcal{H}_\lambda^{[i],(0)^*} \mathcal{H}_{\lambda,\mathrm{fin}}^{[j],(L)}, \tag{22}$$

where $\mathcal{N} = 1/[2(N_c^2-1)]^2$ is the initial-state color and helicity averaging factor and the polarization sum runs over all the 16 helicity configurations. Further, we have set $\mu^2 = s$, $\alpha_s = 0.118$, $N_c = 3$, and $n_f = 5$.

High energy limit and the gluon Regge trajectory.—QFT scattering amplitudes exhibit interesting factorization properties in the high energy (Regge) limit. In terms of the variables introduced in this Letter, this limit corresponds to $|s| \approx |u| \gg |t|$, or equivalently $x \to 0$. For studying this region it is convenient to split scattering amplitudes into parts of definite signature under the $s \leftrightarrow u$ exchange:

$$\mathcal{H}_{\mathrm{ren},\pm} = \frac{1}{2}[\mathcal{H}_{\mathrm{ren}}(s,u) \pm \mathcal{H}_{\mathrm{ren}}(u,s)]. \tag{23}$$

It is then useful to define the signature-even combination

$$L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2}\left[\ln\left(\frac{-s-i\delta}{-t}\right) + \ln\left(\frac{-u-i\delta}{-t}\right)\right] \tag{24}$$

and the color operators [68,69]

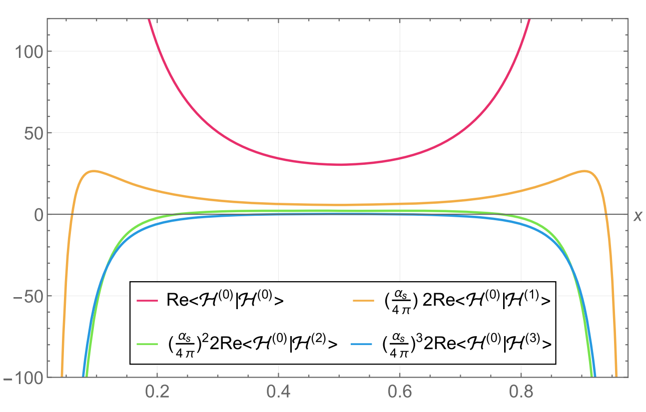

FIG. 1. Tree level amplitude squared and interferences of tree level with $L = 1, 2, 3$ loop amplitudes in dependence of $x = -t/s$.

$$\mathbf{T}_s^2 = (\mathbf{T}_1+\mathbf{T}_2)^a(\mathbf{T}_1+\mathbf{T}_2)^a, \quad \mathbf{T}_t^2 = (\mathbf{T}_1+\mathbf{T}_3)^a(\mathbf{T}_1+\mathbf{T}_3)^a,$$
$$\mathbf{T}_u^2 = (\mathbf{T}_1+\mathbf{T}_4)^a(\mathbf{T}_1+\mathbf{T}_4)^a, \quad \mathbf{T}_{s-u}^2 = \frac{1}{2}(\mathbf{T}_s^2-\mathbf{T}_u^2). \tag{25}$$

At leading power in x and up to the next-to-leading logarithmic (NLL) accuracy, i.e., up to terms of the form $\bar{\alpha}_s^i L^{i-1}$, the odd amplitude has a simple factorized structure. Indeed, to all orders in the strong coupling, $\mathcal{H}_{\mathrm{ren},-}$ can be thought of as the amplitude for the exchange of a single "reggeized" t-channel gluon, whose interaction with the external high-energy gluons is described by so-called *impact factors* [12,70–73]. In the language of complex angular momentum [74], this single-particle exchange is usually referred to as the "Regge-pole" contribution.

Starting from next-to-next-to-leading logarithmic (NNLL) accuracy (i.e., from terms of the form $\bar{\alpha}_s^i L^{i-2}$), this simple factorization is broken and one needs to account for multiple Reggeon exchanges [16,73,75–80]. These are usually referred to as the "Regge-cut" contributions. For the signature-even amplitude, the Regge-cut contribution already enters at the first nontrivial logarithmic order (NLL). The presence of Regge cuts greatly increases the complexity of an all-order analysis. However, if one restricts oneself to fixed order and only considers the first nontrivial cut contribution (i.e., one works at NLL or NNLL for the even or odd amplitude), the problem simplifies dramatically. Indeed, this case can be dealt with using LO BFKL theory [73,77–80].

The only missing ingredient to fully characterize the signature even or odd amplitudes at NLL or NNLL and test Regge factorization to this accuracy is the three-loop gluon Regge trajectory. Currently, it is only known in $\mathcal{N}=4$ SYM [14,16], and in pure gluodynamics under some assumptions on the trajectory itself [15,17]. The three-loop calculation presented in this Letter allows us to extract the trajectory in full QCD, closing this gap.

Before presenting our results, we note that the definition itself of a Regge trajectory is subtle at NNLL [16,78–80]. In this Letter, for definiteness we follow the Regge-cut scheme of Ref. [16]. In particular, we write (in this section, we set the renormalization scale to $\mu^2=-t$)

$$\mathcal{H}_{\mathrm{ren},\pm} = Z_g^2 e^{L\mathbf{T}_t^2\tau_g}\sum_{n=0}^{3}\bar{\alpha}_s^n\sum_{k=0}^{n}L^k\mathcal{O}_k^{\pm,(n)}\mathcal{H}_{\mathrm{ren}}^{(0)}, \tag{26}$$

where $\tau_g=\sum_{n=1}\bar{\alpha}_s^n\tau_n$ is the gluon Regge trajectory and $Z_g=\sum_{n=0}\bar{\alpha}_s^n Z_g^{(n)}$ is a scalar factor accounting for collinear singularities [78] whose explicit value is given in Supplemental Material [45]. The nonvanishing odd signature color operators $\mathcal{O}_k^{-,(n)}$ read up to NNLL [78]

$$\mathcal{O}_0^{-,(0)}=1, \quad \mathcal{O}_0^{-,(1)}=2\mathcal{I}_1^g,$$
$$\mathcal{O}_0^{-,(2)}=[2\mathcal{I}_2^g+(\mathcal{I}_1^g)^2]+\mathcal{B}^{-,(2)}\left[(\mathbf{T}_{s-u}^2)^2-\frac{N_c^2}{4}\right],$$
$$\mathcal{O}_1^{-,(3)}=\mathcal{B}_1^{-,(3)}\mathbf{T}_{s-u}^2[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]+\mathcal{B}_2^{-,(3)}[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\mathbf{T}_{s-u}^2, \tag{27}$$

while the even signature ones are up to NLL [78]

$$\mathcal{O}_0^{+,(1)}=i\pi\mathcal{B}^{+,(1)}\mathbf{T}_{s-u}^2, \qquad \mathcal{O}_1^{+,(2)}=i\pi\mathcal{B}^{+,(2)}[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2],$$
$$\mathcal{O}_2^{+,(3)}=i\pi\mathcal{B}^{+,(3)}[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]]. \tag{28}$$

In these equations, the coefficients $\mathcal{B}^{\pm,(L)}$ describe the Regge-cut contribution and are known [77,78]. $\mathcal{I}_j^g$ are the perturbative expansion coefficients of the gluon impact factor and can be extracted from a one- and two-loop calculation [27]. For convenience, we report both $\mathcal{B}^{\pm,(L)}$ and $\mathcal{I}_{1,2}^g$ in Supplemental Material [45]. As we noted earlier, the NNLL Regge trajectory instead requires a full three-loop calculation. To present our result for it, we define

$$K[\alpha_s(\mu)] = -\frac{1}{4}\int_{\infty}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma^K[\alpha_s(\lambda^2)], \tag{29}$$

together with its perturbative expansion $K=\sum_{n=1}K_i\bar{\alpha}_s^i$ whose coefficients are given in Supplemental Material [45]. The expansion coefficients of the gluon Regge trajectory τ_i can then be written as

$$\tau_1 = K_1+\mathcal{O}(\epsilon),$$
$$\tau_2 = K_2-\frac{56n_f}{27}+N_c\left(\frac{404}{27}-2\zeta_3\right)+\mathcal{O}(\epsilon),$$
$$\tau_3 = K_3+N_c^2\left(16\zeta_5+\frac{40\zeta_2\zeta_3}{3}-\frac{77\zeta_4}{3}-\frac{6664\zeta_3}{27}-\frac{3196\zeta_2}{81}+\frac{297029}{1458}\right)+\frac{n_f}{N_c}\left(-4\zeta_4-\frac{76\zeta_3}{9}+\frac{1711}{108}\right)$$
$$+N_c n_f\left(\frac{412\zeta_2}{81}+\frac{2\zeta_4}{3}+\frac{632\zeta_3}{9}-\frac{171449}{2916}\right)+n_f^2\left(\frac{928}{729}-\frac{128\zeta_3}{27}\right)+\mathcal{O}(\epsilon), \tag{30}$$

where the higher orders in ϵ for τ_1 and τ_2 can be found in Supplemental Material [45]. As expected, our lower-loop results are consistent with Ref. [81], see also [82]. For τ_3, the n_f-independent part of our result agrees with Ref. [17]. Furthermore, the highest transcendental-weight terms of the trajectory agree with the $\mathcal{N} = 4$ SYM result [14,16], as predicted by the maximal transcendentality principle [83–86]. On its own, the result (30) is not particularly illuminating. However, we have found the *same* trajectory using both the calculation outlined in this Letter and our previous $qq' \to qq'$ three-loop calculation [19]. This provides an important test of QCD Regge factorization at the three-loop level. We also stress that now *all* the ingredients for a NLL and NNLL analysis of the signature-even and signature-odd elastic amplitudes are known. In particular, we can now fully predict the yet unknown $qg \to qg$ three-loop amplitude to NNLL accuracy. Explicitly checking these predictions against a full calculation will provide a highly nontrivial test of the universality of Regge factorization in QCD.

Conclusion.—In this Letter, we have presented the first computation of the helicity amplitudes for the scattering of four gluons up to three loops in full QCD. We obtained compact results for the finite part of all independent helicity configurations in terms of harmonic polylogarithms up to weight six and we verified that the IR poles of our analytic amplitudes follow the predicted universal pattern up to three loops, which includes dipole and quadruple correlations. We also considered the high-energy (Regge) limit of our amplitudes, and extracted the full three-loop QCD gluon Regge trajectory. This was the last missing building block to describe single-Reggeon exchanges at NNLL accuracy.

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