Quark and Gluon Form Factors in Four-Loop QCD

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We compute the photon-quark and Higgs-gluon form factors to four-loop order within massless perturbative quantum chromodynamics. Our results constitute ready-to-use building blocks for N^{4}LO cross sections for Drell-Yan processes and gluon-fusion Higgs boson production at the LHC. We present complete analytic expressions for both form factors and show several of the most complicated master integrals.

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Introduction.—A number of experimental results obtained at the Large Hadron Collider (LHC) at CERN have reached a precision below the percent level, often superseding the original expectations. A fundamental ingredient to the successful interpretation of precise data is the computation of higher-order quantum corrections, most importantly those stemming from the strong interaction. In many cases next-to-next-to-leading order (NNLO) corrections have become standard. In fact, nowadays $2 \to 2$ scattering processes are routinely computed at this order, in some cases even taking into account massive particles in the loops. Also for $2 \to 3$ processes more and more results become available (see, e.g., Refs. [1–7]).

There are a few processes which are known to third order, or N^{3}LO, in perturbative quantum chromodynamics (QCD). Among them are the Drell-Yan production of W and Z bosons [8,9] as well as Higgs boson production in gluon fusion in the infinite top-mass limit [10,11] at the LHC. In the latter case the high-order corrections are particularly important due to the slow convergence of the perturbative series. Similar observations have been made for the threshold production cross section of the top quark pairs in electron positron annihilation, where third-order corrections are necessary to obtain theory uncertainties of a few percent [12]. For more generic $2 \to 2$ processes like dijet production, virtual corrections at third-order QCD became available only recently (see, e.g., Refs. [13,14]), providing first ingredients to such N^{3}LO cross sections.

In the coming years the precise determination of the Higgs boson properties will be one of the central topics at the LHC. In this context it is important to improve the precision for the production cross section, both experimentally and from the theory side. First steps towards the N^{4}LO corrections of the Higgs boson production cross section have been undertaken in Ref. [15]. In this Letter, we provide the first ready-to-use ingredient to the N^{4}LO cross section for $gg \to H + X$ by presenting the virtual corrections to the effective Higgs-gluon vertex up to four-loop order. Similarly, we provide the four-loop corrections to the photon-quark vertex which are a building block of the N^{4}LO corrections to the process $q\bar{q} \to Z/W$. Historically, also at N^{3}LO the purely virtual corrections were the first building blocks to become available with the calculation of the three-loop form factors more than a decade ago [16–18]. Subsequently, the real-radiation contributions have been added step-by-step until first results for the Higgs production cross section became available [19–25].

The relevant form factors for the $q\bar{q}\gamma^*$ and ggH vertex functions Γ_q^μ and $\Gamma_g^{\mu\nu}$, respectively, are given by the projections

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}(\not{q}_2\Gamma_q^\mu\not{q}_1\gamma_\mu), \tag{1}$$

$$F_g(q^2) = \frac{(q_1 \cdot q_2 g_{\mu\nu} - q_{1,\mu}q_{2,\nu} - q_{1,\nu}q_{2,\mu})}{2(1-\epsilon)}\Gamma_g^{\mu\nu}. \tag{2}$$

Here, the overall normalization is chosen such that both form factors are one at leading order. We employ conventional dimensional regularization and use $\epsilon = (4-d)/2$, where d is the space-time dimension. The outgoing momentum of the photon (Higgs) is $q = q_1 + q_2$, where q_1 and q_2 are the incoming momenta of the quark and antiquark (gluons) for F_q (F_g), and we have $q_1^2 = q_2^2 = 0$ and $q^2 \neq 0$.

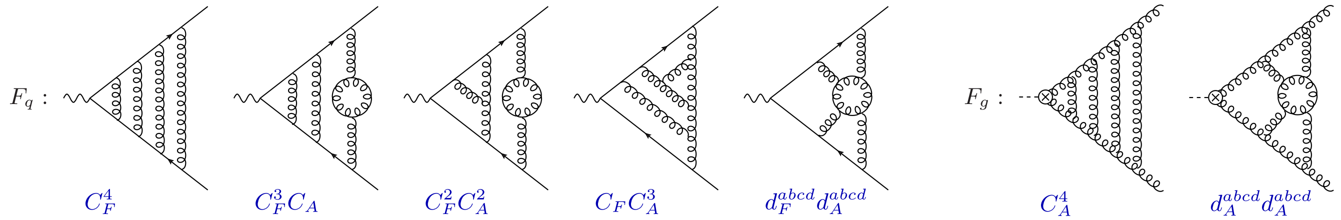


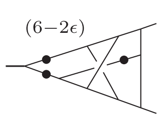
FIG. 1. Sample Feynman diagrams and color factors for the nonfermionic contributions to F_g and F_q at four-loop order. Straight and curly lines denote quarks and gluons, respectively. Both planar and nonplanar diagrams contribute.

The two- and three-loop QCD corrections to F_q and F_g are available from Refs. [16,18,26–32] and in Refs. [33,34] the three-loop results have been obtained up to order ϵ^2. At four loops, only partial results have been obtained so far. In Fig. 1 we show sample Feynman diagrams for the purely gluonic corrections to F_q and F_g in four-loop QCD; sample diagrams for the fermionic part can be found in Fig. 1 of Ref. [35]. Altogether 5728 and 43 220 Feynman diagrams contribute to the quark and gluon form factor at this perturbative order, respectively.

The results, which are presented in this Letter, finalize a long-running effort to compute QCD form factors to four loops. First results have been obtained in 2016 [36,37] where planar diagrams for F_q have been presented in the large-N_c limit. Fermionic corrections with two closed fermion bubbles are available from [38] and the complete contribution from color structure $(d_F^{abcd})^2$ has been computed in [39,40]. For F_q and F_g, all corrections with three or two closed fermion loops have been calculated in [41,42], respectively, including also the singlet contributions. The complete set of poles of F_q and F_g in the dimensional regulator has been obtained through direct diagrammatic evaluation in [43]. Finally, the complete fermionic corrections to F_q and F_g have been computed in Ref. [35].

Calculation.—The calculation of the four-loop form factors presents two major challenges. The first one is connected to a minimal representation of the form factors. After generating the Feynman diagrams with QGRAF [44], we apply the projectors and perform the numerator and color algebra with FORM4 [45] and COLOR.H [46]. In this way, we can write the form factors as a linear combination of a large number of scalar Feynman integrals, each belonging to one of 100 twelve-line top-level topologies or a subtopology thereof. Fixing the twelve propagators and six irreducible numerators of its top-level topology, a scalar integral can be described by eighteen integers indicating the exponents of the propagators and numerators. By choosing the irreducible numerators as suitably defined inverse propagators, all top-level topologies can be described in terms of the ten complete sets of denominators described in [47]. Integration-by-parts (IBP) reductions [48–50] systematically establish linear relations between the integrals, allowing us to express the form factors as a linear combination of a minimal set of so-called master integrals. For our calculation we use the setup described in [40] based on the program REDUZE2 [51] and the in-house code FINRED, employing techniques from [52–58].

The second challenge is the computation of the master integrals. Here we follow two complementary approaches. The first one is based on the construction of finite master integrals [34,59,60], in $d_0 - 2\epsilon$ dimensions where $d_0 = 4, 6, \ldots$. Provided a linearly reducible [61,62] Feynman parametric representation can be found, the ϵ expansions of such master integrals may be computed analytically using the program HYPERINT [63]. The dimensionally shifted integrals can be related to master integrals in $4 - 2\epsilon$ dimensions using IBP relations derived with first- and second-order annihilators in the Lee-Pomeransky representation [64]. We wish to point out that in this approach, the integration can be performed at the level of individual integrals. In practice, evaluating higher orders of the ϵ expansion gets ever more demanding due to the rise in algebraic complexity. To determine the form factors F_q and F_g, we computed a number of integrals to transcendental weight eight in this approach, including computationally demanding nonplanar integrals with twelve different propagators. For one such irreducible topology with a single twelve-line master integral we find



$$= \left(-\frac{119}{48}\zeta_7 - \frac{5}{6}\zeta_5\zeta_2 - \frac{53}{10}\zeta_3\zeta_2^2 + 3\zeta_3^2 + \frac{79}{42}\zeta_2^3 + \frac{25}{6}\zeta_5 - \frac{5}{3}\zeta_3\zeta_2 + \frac{1}{15}\zeta_2^2 + 2\zeta_3\right) + \epsilon\left(-\frac{991}{30}\zeta_{5,3} - \frac{323}{2}\zeta_5\zeta_3 - \frac{81}{2}\zeta_3^2\zeta_2 + \frac{127223}{31500}\zeta_2^4 - \frac{2827}{24}\zeta_7 + \frac{73}{6}\zeta_5\zeta_2 - 14\zeta_3\zeta_2^2 + \frac{41}{3}\zeta_3^2 + \frac{1696}{315}\zeta_2^3 + \frac{401}{3}\zeta_5 + \frac{206}{3}\zeta_3\zeta_2 + \frac{23}{15}\zeta_2^2 + 14\zeta_3\right) + \mathcal{O}(\epsilon^2) \quad (3)$$

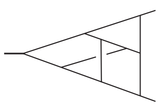
in the conventions of Ref. [65]. In particular, the integral is defined in $6-2\epsilon$ and each dot indicates a squared propagator. We would like to mention that no integral in this topology was needed for the calculation of the $\mathcal{N}=4$ Sudakov form factor [47]. Our result above is expressed in terms of regular zeta values, ζ_n $(n=2,\ldots,7)$, and

$$\zeta_{5,3}=\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\frac{1}{m^5n^3}\approx 0.037\,707\,672\,984\,8 \qquad (4)$$

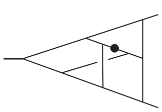
is the only irreducible multiple zeta value at weight eight.

Our second approach for computing master integrals is the method of differential equations [66–69] based on "canonical" bases [70]. Since our master integrals only depend on one kinematic parameter, q^2, we have to introduce a second mass scale such that nontrivial differential equations can be established. With canonical bases, this idea was first implemented in [71]. For our application it is advantageous to make one of the massless external legs massive. Choosing $q_1^2\neq 0$ has the advantage that the boundary conditions can be fixed for $q_1^2=q^2$ since then the vertex integrals turn into two-point integrals, which are well studied in the literature [72,73]. The differential equations are used to transport the information to $q_1^2=0$, which corresponds to the vertex diagrams we are interested in. To construct canonical bases we apply the algorithm of Ref. [74] implemented in [75]. Details of this approach can, e.g., be found in Ref. [39]. When constructing canonical bases, we also need IBP reduction to master integrals. Here, we apply FIRE [76] for this.

For one of the most complicated twelve-line topologies which did not enter the $\mathcal{N}=4$ Sudakov form factor we obtain for its two master integrals



$$\begin{aligned}=\;&\frac{1}{\varepsilon^8}\left(\frac{1}{72}\right)+\frac{1}{\varepsilon^7}\left(\frac{17}{96}\right)+\frac{1}{\varepsilon^6}\left(-\frac{19}{72}\zeta_2+\frac{5}{36}\right)+\frac{1}{\varepsilon^5}\left(-\frac{101}{36}\zeta_3-\frac{719}{288}\zeta_2-\frac{511}{288}\right)+\frac{1}{\varepsilon^4}\left(-\frac{1967}{360}\zeta_2^2-\frac{4123}{144}\zeta_3\right.\\&\left.+\frac{11}{96}\zeta_2+\frac{2113}{288}\right)+\frac{1}{\varepsilon^3}\left(-\frac{697}{8}\zeta_5+\frac{235}{9}\zeta_3\zeta_2-\frac{182903}{2880}\zeta_2^2-\frac{29}{24}\zeta_3+\frac{1637}{288}\zeta_2-\frac{1477}{72}\right)+\frac{1}{\varepsilon^2}\left(\frac{27083}{144}\zeta_3^2-\frac{130951}{2520}\zeta_2^3\right.\\&\left.-\frac{12907}{16}\zeta_5+\frac{16883}{144}\zeta_3\zeta_2+\frac{2101}{320}\zeta_2^2+\frac{5527}{36}\zeta_3+\frac{2899}{144}\zeta_2+\frac{4223}{144}\right)+\frac{1}{\varepsilon}\left(-\frac{916123}{384}\zeta_7+\frac{16565}{48}\zeta_5\zeta_2+\frac{727901}{720}\zeta_3\zeta_2^2\right.\\&\left.+\frac{26927}{18}\zeta_3^2-\frac{13321753}{20160}\zeta_2^3+\frac{2795}{12}\zeta_5+\frac{147}{4}\zeta_3\zeta_2+\frac{979013}{2880}\zeta_2^2-\frac{77021}{144}\zeta_3-\frac{5771}{16}\zeta_2+\frac{13315}{144}\right)+\left(\frac{110723}{120}\zeta_{5,3}\right.\\&+\frac{104859}{8}\zeta_5\zeta_3-\frac{1123}{9}\zeta_3^2\zeta_2-\frac{6055979}{21000}\zeta_2^4-\frac{5285507}{384}\zeta_7-\frac{1373}{4}\zeta_5\zeta_2+\frac{11089517}{1440}\zeta_3\zeta_2^2+\frac{88945}{96}\zeta_3^2+\frac{1391189}{2240}\zeta_2^3\\&\left.+\frac{20717}{6}\zeta_5-\frac{118961}{144}\zeta_3\zeta_2-\frac{1021111}{720}\zeta_2^2+\frac{1701}{4}\zeta_3+\frac{358103}{144}\zeta_2-\frac{128443}{144}\right)+\mathcal{O}(\varepsilon)\end{aligned}\qquad (5)$$



$$\begin{aligned}=\;&\frac{1}{\varepsilon^7}\left(-\frac{5}{216}\right)+\frac{1}{\varepsilon^6}\left(-\frac{13}{36}\zeta_2-\frac{6157}{2592}\right)+\frac{1}{\varepsilon^5}\left(-\frac{43}{36}\zeta_3-\frac{1661}{432}\zeta_2-\frac{267889}{15552}\right)+\frac{1}{\varepsilon^4}\left(\frac{2621}{360}\zeta_2^2-\frac{536}{27}\zeta_3\right.\\&\left.+\frac{77653}{2592}\zeta_2+\frac{3522299}{93312}\right)+\frac{1}{\varepsilon^3}\left(\frac{173}{3}\zeta_5+\frac{722}{9}\zeta_3\zeta_2+\frac{235093}{4320}\zeta_2^2+\frac{806659}{2592}\zeta_3+\frac{4152625}{15552}\zeta_2+\frac{1129069}{17496}\right)\\&+\frac{1}{\varepsilon^2}\left(\frac{12499}{36}\zeta_3^2+\frac{6801}{40}\zeta_2^3+\frac{34745}{72}\zeta_5+\frac{23542}{27}\zeta_3\zeta_2+\frac{287881}{405}\zeta_2^2+\frac{5714285}{1944}\zeta_3+\frac{805613}{46656}\zeta_2-\frac{2167037021}{1679616}\right)\\&+\frac{1}{\varepsilon}\left(\frac{137837}{96}\zeta_7+\frac{5739}{4}\zeta_5\zeta_2+\frac{26929}{180}\zeta_3\zeta_2^2+\frac{2608907}{432}\zeta_3^2+\frac{16974331}{10080}\zeta_2^3+\frac{1920841}{288}\zeta_5+\frac{579521}{1296}\zeta_3\zeta_2+\frac{151664975}{31104}\zeta_2^2\right.\\&\left.-\frac{210823813}{46656}\zeta_3-\frac{206363129}{139968}\zeta_2+\frac{84786801307}{10077696}\right)+\left(-\frac{221827}{60}\zeta_{5,3}-\frac{1225}{3}\zeta_5\zeta_3-\frac{40667}{9}\zeta_3^2\zeta_2+\frac{47962441}{28000}\zeta_2^4\right.\\&+\frac{1542237}{32}\zeta_7+\frac{114838}{9}\zeta_5\zeta_2+\frac{9353537}{2160}\zeta_3\zeta_2^2+\frac{7987499}{2592}\zeta_3^2+\frac{496790909}{90720}\zeta_2^3+\frac{4253033}{108}\zeta_5\\&\left.-\frac{127736653}{7776}\zeta_3\zeta_2-\frac{8249965}{11664}\zeta_2^2-\frac{82032149}{34992}\zeta_3+\frac{5255296693}{839808}\zeta_2-\frac{2529781809149}{60466176}\right)+\mathcal{O}(\varepsilon).\end{aligned}\qquad (6)$$

A feature of our second method is that it provides the results for all master integrals of a given family. Often the simpler integrals with less lines could also be computed with the first approach, which moreover gave results through to transcendental weight seven for almost all of the integrals. This provided us with plenty of analytical cross checks. For all integrals which were not checked by redundant analytical calculations, we employed FIESTA5 [77] to verify our analytical results within a typical relative precision of 10^{-4} using a basis of finite integrals.

Results.—Our calculation of the master integrals through to weight eight allows us to present complete analytic results for F_q and F_g. It is convenient to define their perturbative expansion in terms of the bare strong coupling constant α_s^0 as

$$F_x = 1 + \sum_{n\geq 1} \left(\frac{\alpha_s^0}{4\pi}\right)^n \left(\frac{4\pi}{e^{\gamma_E}}\right)^{n\epsilon} \left(\frac{\mu^2}{-q^2 - i0}\right)^{n\epsilon} F_x^{(n)}, \quad (7)$$

with $x \in \{q, g\}$. Here, γ_E denotes Euler's constant, and μ is the 't Hooft scale.

While the ϵ expansion of the fermionic corrections starts at order $1/\epsilon^7$, the purely gluonic corrections also have $1/\epsilon^8$ poles and, correspondingly, zeta values with transcendental weight up to eight in the finite part. Since all pole parts are known analytically from Ref. [43], see also [15,36,40,78–87], it is sufficient to consider the finite terms in the following. The complete expressions can be found in a computer-readable ancillary file attached to this Letter available on the arXiv. We obtain for the finite part of the quark form factor

$$\begin{aligned}
F_q^{(4)}|_{\epsilon^0} = {}& C_F^4\Big(-\frac{32\,384}{15}\zeta_{5,3} + \frac{739\,328}{45}\zeta_5\zeta_3 - \frac{66\,392}{27}\zeta_3^2\zeta_2 + \frac{7\,486\,576}{7875}\zeta_2^4 - \frac{538\,913}{14}\zeta_7 + \frac{20\,464}{5}\zeta_5\zeta_2 + \frac{1276}{15}\zeta_3\zeta_2^2 \\
& - \frac{192\,866}{27}\zeta_3^2 - \frac{737\,186}{63}\zeta_2^3 - \frac{468\,509}{5}\zeta_5 - \frac{269\,026}{9}\zeta_3\zeta_2 + \frac{304\,331}{30}\zeta_2^2 + \frac{1\,823\,767}{48}\zeta_2 + \frac{384\,343}{6}\zeta_3 + \frac{6\,579\,473}{128}\Big) \\
& + C_F^3 C_A\Big(\frac{20\,948}{15}\zeta_{5,3} - \frac{18\,364}{5}\zeta_5\zeta_3 + \frac{3296}{9}\zeta_3^2\zeta_2 + \frac{5\,472\,536}{2625}\zeta_2^4 + \frac{847\,073}{56}\zeta_7 - \frac{445\,861}{45}\zeta_5\zeta_2 - \frac{1\,686\,022}{135}\zeta_3\zeta_2^2 \\
& - \frac{7\,935\,031}{162}\zeta_3^2 + \frac{73\,358\,647}{5670}\zeta_2^3 + \frac{73\,074\,974}{405}\zeta_5 + \frac{12\,562\,265}{162}\zeta_3\zeta_2 + \frac{11\,388\,811}{9720}\zeta_2^2 - \frac{1\,176\,750\,697}{7776}\zeta_2 \\
& + \frac{1\,623\,827\,681}{11\,664}\zeta_3 - \frac{163\,952\,683\,523}{839\,808}\Big) + C_F^2 C_A^2\Big(\frac{54\,254}{45}\zeta_{5,3} - \frac{45\,281}{9}\zeta_5\zeta_3 + \frac{38\,482}{27}\zeta_3^2\zeta_2 - \frac{21\,476\,059}{15\,750}\zeta_2^4 \\
& - \frac{4\,032\,413}{144}\zeta_7 + \frac{3\,166\,909}{270}\zeta_5\zeta_2 + \frac{4\,135\,486}{405}\zeta_3\zeta_2^2 + \frac{114\,329\,701}{1458}\zeta_3^2 - \frac{2\,153\,423}{420}\zeta_2^3 - \frac{1\,049\,804\,117}{9720}\zeta_5 \\
& - \frac{179\,904\,821}{2916}\zeta_3\zeta_2 - \frac{506\,374\,351}{29\,160}\zeta_2^2 + \frac{38\,578\,372\,511}{209\,952}\zeta_2 - \frac{33\,927\,798\,065}{104\,976}\zeta_3 + \frac{1\,181\,337\,259\,783}{3\,779\,136}\Big) \\
& + C_F C_A^3\Big(-\frac{14\,161}{30}\zeta_{5,3} + \frac{21\,577}{6}\zeta_5\zeta_3 - \frac{1963}{3}\zeta_3^2\zeta_2 + \frac{10\,233\,079}{15\,750}\zeta_2^4 + \frac{199\,283}{48}\zeta_7 - \frac{20\,821}{9}\zeta_5\zeta_2 \\
& - \frac{68\,311}{45}\zeta_3\zeta_2^2 - \frac{3\,175\,501}{108}\zeta_3^2 - \frac{565\,843}{1620}\zeta_2^3 + \frac{75\,664\,147}{3240}\zeta_5 + \frac{4\,325\,527}{324}\zeta_3\zeta_2 + \frac{1\,709\,477}{180}\zeta_2^2 - \frac{842\,991\,647}{11\,664}\zeta_2 \\
& + \frac{47\,856\,067}{324}\zeta_3 - \frac{5\,465\,282\,473}{34\,992}\Big) + \frac{d_F^{abcd}d_A^{abcd}}{N_F}\Big(260\zeta_{5,3} - 5092\zeta_5\zeta_3 - 16\zeta_3^2\zeta_2 - \frac{496\,766}{525}\zeta_2^4 + 3518\zeta_7 \\
& - \frac{4744}{3}\zeta_5\zeta_2 + \frac{6584}{15}\zeta_3\zeta_2^2 + \frac{39\,986}{9}\zeta_3^2 + \frac{526\,496}{945}\zeta_2^3 - \frac{180\,566}{27}\zeta_5 + \frac{3020}{3}\zeta_3\zeta_2 \\
& + \frac{1220}{9}\zeta_2^2 + \frac{10\,570}{9}\zeta_2 + \frac{169\,532}{27}\zeta_3 - \frac{1580}{3}\Big) + \text{contributions with closed fermion loop from Ref. [35]} \qquad (8)
\end{aligned}$$

and for the finite part of the gluon form factor

$$
\begin{aligned}
F_g^{(4)}|_{\epsilon^0} = C_A^4\Big(&-\frac{2591}{90}\zeta_{5,3}+\frac{1018949}{90}\zeta_5\zeta_3-\frac{35689}{27}\zeta_3^2\zeta_2+\frac{18282694}{7875}\zeta_2^4-\frac{27705161}{504}\zeta_7+\frac{1160731}{270}\zeta_5\zeta_2\\
&-\frac{1928564}{405}\zeta_3\zeta_2^2-\frac{1296845}{1458}\zeta_3^2-\frac{727183}{1134}\zeta_2^3+\frac{6161623}{243}\zeta_5-\frac{3233651}{729}\zeta_3\zeta_2+\frac{54443689}{14580}\zeta_2^2+\frac{839716507}{104976}\zeta_2\\
&-\frac{84995881}{52488}\zeta_3+\frac{96887974603}{3779136}\Big)+\frac{d_A^{abcd}d_A^{abcd}}{N_A}\Big(260\zeta_{5,3}-5092\zeta_5\zeta_3-16\zeta_3^2\zeta_2-\frac{496766}{525}\zeta_2^4-\frac{6776}{3}\zeta_7\\
&-5016\zeta_5\zeta_2+\frac{2992}{3}\zeta_3\zeta_2^2+\frac{31588}{3}\zeta_3^2+\frac{1073972}{945}\zeta_2^3-6460\zeta_5+\frac{6752}{9}\zeta_3\zeta_2+\frac{24616}{45}\zeta_2^2-\frac{4682}{27}\zeta_2-\frac{1310}{9}+\frac{68410}{9}\zeta_3\Big)\\
&+\text{contributions with closed fermion loop from Ref. [35].} \qquad (9)
\end{aligned}
$$

We expressed our results in terms of invariants of a general Lie algebra, where C_R denotes the quadratic Casimir operator, d_R^{abcd} the fully symmetrical tensor originating from the trace over generators, and N_R the dimension of the fundamental and adjoint representation, $R = F$, A, respectively. For a $SU(N_c)$ gauge group the invariants or color factors are obtained as

$$
\begin{aligned}
C_F &= (N_c^2-1)/(2N_c),\\
C_A &= N_c,\\
d_F^{abcd}d_A^{abcd}/N_F &= (N_c^2-1)(N_c^2+6)/48,\\
d_A^{abcd}d_A^{abcd}/N_A &= N_c^2(N_c^2+36)/24. \qquad (10)
\end{aligned}
$$

All terms shown in Eqs. (8) and (9) are new. The complete four-loop results for F_q and F_g are obtained after adding the fermionic contributions given in Eqs. (10) and (11) of Ref. [35].

We performed several checks of our results, which we describe in the following. First, the leading-color limit of Eq. (8) agrees with the result of Ref. [37]. While all color structures of Eq. (8) contribute in this limit, it can be derived from just planar loop integrals, see also Ref. [65] for an independent calculation. Second, we observe that our weight-eight result for $F_g^{(4)}/N_c^4$ agrees with the corresponding expression of the four-loop Sudakov form factor in $\mathcal{N}=4$ supersymmetric Yang Mills theory, see Eq. (4.1) of Ref. [47], after expressing the color factors in terms of N_c using Eqs. (10). Furthermore, after adjusting the QCD color factors such that the bosonic and fermionic degrees of freedom are in the same color representation, i.e., $C_F \to C_A$, $N_F \to N_A$ and $d_A^{abcd}d_F^{abcd} \to d_A^{abcd}d_A^{abcd}$, we obtain identical results for the weight-eight coefficients of $F_q^{(4)}$ and $F_g^{(4)}$. These relations between the maximal transcendental parts involve all nonfermionic color coefficients of $F_q^{(4)}$ and $F_g^{(4)}$ and test both leading and subleading color contributions.

Conclusions.—In this Letter, we provide the perturbative corrections to the photon-quark and Higgs-gluon form factors at relative order α_s^4. This is the first complete calculation of vertex functions in four-loop massless QCD. Our analytical results have been obtained by combining two powerful multiloop techniques: the direct integration of finite master integrals and the method of differential equations. The final expressions are presented in terms of zeta values with transcendental weight up to eight, allowing for a straightforward numerical evaluation. Our results constitute the virtual contributions to a number of cross sections and decay rates at N^4LO, including Drell-Yan processes and gluon-fusion Higgs boson production at the LHC.

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