



On computational capabilities of Ising machines based on nonlinear oscillators

Mikhail Erementchouk*, Aditya Shukla, Pinaki Mazumder

Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109, USA

ARTICLE INFO

Article history:

Received 16 February 2022

Accepted 28 April 2022

Available online 11 May 2022

Communicated by Liang Huang

Keywords:

Ising machines

Combinatorial optimization

Dynamical computations

Kuramoto model

ABSTRACT

Dynamical Ising machines are actively investigated from the perspective of finding efficient heuristics for NP-hard optimization problems. However, the existing data demonstrate super-polynomial scaling of the running time with the system size, which is incompatible with large NP-hard problems. We show that oscillator networks implementing the Kuramoto model of synchronization are capable of demonstrating polynomial scaling. The dynamics of these networks is related to the semidefinite programming relaxation of the Ising model ground state problem. Consequently, such networks, as we numerically demonstrate, are capable of producing the best possible approximation in polynomial time. To reach such performance, however, the reconstruction of the binary Ising state (rounding) must be specially addressed. We demonstrate that commonly implemented forced collapse to a close-to-Ising state may diminish the computational capabilities up to their complete invalidation. Therefore, consistent treatment of rounding may cardinally improve various operation metrics of already existing and upcoming dynamical Ising machines.

© 2022 Elsevier B.V. All rights reserved.

1. Introduction

Challenges set by large-scale NP-hard problems make unconventional models of computation of special interest and importance. One of such models is based on the Ising model describing a network of coupled classical spins. In 1970–1980-s, researchers realized that reaching the equilibrium of the Ising model is equivalent to solving certain optimization problems [1–3]. Furthermore, in [3], it was observed that the ground state of the Ising model on a graph delivers the maximal cut of the graph. This tied the Ising model with a series of other NP-hard problems as established in [4,5] and explicated in [6]. These observations exposed the Ising model as a special model of computation, which represents computing tasks in terms of set partitioning.

Recently, a significant research effort is put into development of continuous dynamical Ising machines [7–25]. Characteristically, these machines do not represent the classical spins by a binary (taking values ± 1) object. Instead, they leverage the emergent property of specially constructed continuous dynamical system to minimize the Ising model Hamiltonian.

To better understand challenges faced by Ising machines, it should be reminded that finding the maximum cut is an APX-hard problem [26,27]. If $P \neq NP$, an APX-hard problem cannot be solved with arbitrarily good approximation within time scaling

polynomially with the system size. For finding the maximal cut, this means that any Ising solver acting on a sufficiently rich set of large graphs can either guarantee, at best, $16/17 \approx 0.941$ [28] value of the maximal cut, or its running time will scale super-polynomially. The fact that an Ising machine is continuous does not change the implications of the APX-hardness. Moreover, finding the ground state of the Ising model admits various continuous (exact) representations. We consider an Ising machine based on one of such representation and show that its *performance* – the degree of approximation achievable in polynomial time – is the same as of a simple local search.

The questions of performance received little attention in the Ising machine literature. Virtually, the only adopted method of evaluating the accuracy is testing the machines against a selective set of benchmarks. While such tests are, of course, important, they provide little insight into how the machines would operate on a general class of problems, especially while dealing with graphs of gigantic size (say, with millions or billions of nodes) [29]. In turn, studies of the dependence of the running time on the problem size are also scarce. The existing data [30,31] demonstrate super-polynomial scaling: the best result reported in [31] is $O(e^{\sqrt{N}})$, where N is the number of graph nodes. Such scaling effectively puts large NP-hard problem out of the reach.

In the present paper, we show that Ising machines based on networks implementing the Kuramoto model of synchronization [32–35] are capable of demonstrating scaling compatible with large NP-hard problems. The ability of these machines [15,16,21–23,36,37] to find the ground state of the Ising model is of the same

* Corresponding author.

E-mail addresses: merement@gmail.com (M. Erementchouk), aditshuk@umich.edu (A. Shukla), pinakimazum@gmail.com (P. Mazumder).

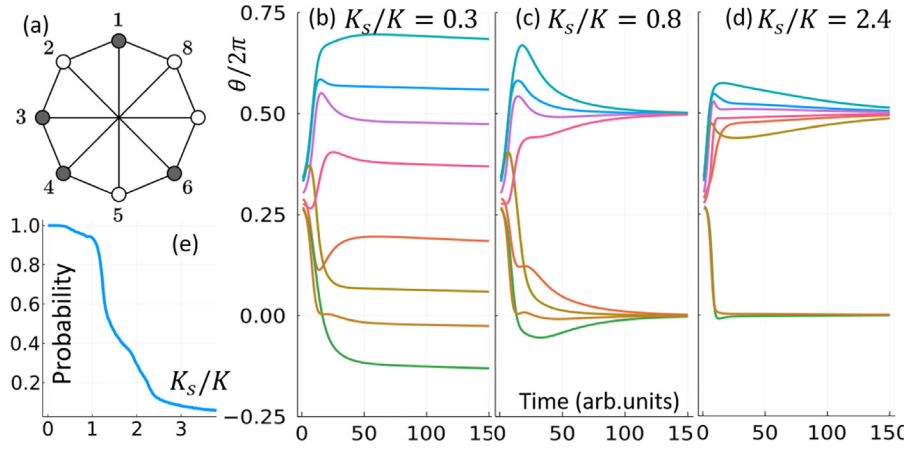


Fig. 1. (a) A graph with 8 nodes and one of its maximal cut partitions. (b)–(d) Representative dynamical regimes obtained for different K/K_s . The obtained cut values are (c) 10, (d) 6. (e) The probability of finding a max-cut configuration as a function of K/K_s .

origin as for the semidefinite programming (SDP) relaxation [38]. This suggests that such oscillatory Ising machines can reach the theoretical limits: the best classically possible quality of solutions [27,39] in time that scales almost linearly with the problem size [40,41].

The best theoretical performance, however, is not achieved automatically within the commonly adopted ways to implement machines based on synchronized networks. As the dynamical model describing these machines, we consider a network of oscillators with identical natural frequencies. In the rotating frame, the network is governed by [37,42]

$$\dot{\theta}_m = K \sum_n A_{m,n} \sin(\theta_n - \theta_m) + K_s \sin(2\theta_m), \quad (1)$$

where θ_m is the m th oscillator phase, $A_{m,n}$ is the network adjacency matrix, K is the coupling strength, and K_s is the strength of the phase injection [43–45], which facilitates aligning individual oscillator phases. To distinguish from the model with variable frequencies, we will refer to (1) as the quasistatic Kuramoto model (QKM).

The relation with the Ising model is often suggested along the following lines. QKM can be considered as induced by the Lyapunov function (a function monotonously decreasing with time with evolution of the system) [42]

$$\mathcal{H}^{(QKM)} = \frac{K}{2} \sum_{m,n} A_{m,n} \cos(\theta_m - \theta_n) + \frac{K_s}{2} \sum_m \cos^2(\theta_m). \quad (2)$$

When the phase distribution is binary, $\theta_m = \sigma_m \pi/2$ with $\sigma_m = \pm 1$, $\mathcal{H}^{(QKM)}$ turns into an Ising Hamiltonian. Since $\mathcal{H}^{(QKM)}$ decreases with time, the arrival at a state with clustered phases can be expected to deliver an approximation to the ground state.

There are difficulties with such arguments: there is no guarantee that the system will converge to a binary-like state, nor that the resultant state will deliver the optimum solution. These difficulties manifest themselves even on small graphs, like shown in Fig. 1(a) (the same graph was considered in [16]). Depending on K_s/K , the phase evolution governed by Eqs. (1) can be in any of three regimes:

1. well-defined clusters do not form;
2. clusters form and yield the maximal cut;
3. clusters form but do not yield the maximal cut.

It may appear as if QKM-based machines are unreliable if K_s does not ensure operating in regime 2. Moreover, as we will show, regimes 1 and 3 can be regarded as generic.

However, this is regime 1, without well formed phase clusters, which delivers the solution. It suffices that the network *admits* the binary state without necessarily settling in it. Consequently, obtaining a steady state of a dynamic Ising machine must be followed by rounding: finding the best phase reference point. Forcing phases to form well-defined clusters does not achieve this. This is supported by Fig. 1(e) showing that when K_s is too large, the probability of success reduces to random guessing.

2. Ising model and the max-cut problem

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be an undirected graph on the sets of nodes $\mathcal{V}$ and edges $\mathcal{E}$. We will denote the number of nodes and edges by $N = |\mathcal{V}|$ and $M = |\mathcal{E}|$, respectively. The spin configurations are described by binary functions on the graph: $\sigma : \mathcal{V} \rightarrow \{-1, 1\}$. In other words, to each node, we assign a binary variable $\sigma_m \in \{-1, 1\}$.

With each configuration, an energy is associated

$$\mathcal{H}(\sigma) = \sum_{(m,n) \in \mathcal{E}} A_{m,n} \sigma_m \sigma_n = \frac{1}{2} \sigma \cdot \hat{A} \cdot \sigma = \frac{1}{2} \text{Tr}[\hat{A} \hat{\mathcal{E}}], \quad (3)$$

where $\hat{A}$ is the graph adjacency matrix, and $\hat{\mathcal{E}} = \sigma \otimes \sigma$, so that $\mathcal{E}_{m,n} = \sigma_m \sigma_n$. In general, a weight function $\hat{J} : \mathcal{E} \rightarrow \mathbb{R}$ can also be provided. We will focus on the simpler case described above and only briefly discuss what changes in the general case.

Each configuration σ naturally defines partitioning $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_-$, with $\mathcal{V}_+$ and $\mathcal{V}_-$ being sets, where σ takes values $+1$ and -1 , respectively. Conversely, any partitioning, $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_-$, defines a configuration and, thereby, can be characterized by the energy, which is related to the cut of $\mathcal{G}$ induced by the partitioning. A cut, $\mathcal{C}(\mathcal{V}_+, \mathcal{V}_-)$ is a set of edges with one of the ends in $\mathcal{V}_+$ and another in $\mathcal{V}_-$. The cut size, $C(\sigma) = |\mathcal{C}|$, can be found by summing over all edges a function equal to 1 on edges from $\mathcal{C}$, and to 0 elsewhere. In terms of σ , the value of this function on edge (m, n) can be written as $(1 - \sigma_m \sigma_n)/2$ leading to [3]

$$C(\sigma) = \frac{1}{2} \sum_{(m,n) \in \mathcal{E}} A_{m,n} (1 - \sigma_m \sigma_n) = \frac{M}{2} - \frac{1}{2} \mathcal{H}(\sigma). \quad (4)$$

Thus, finding the maximal cut, $\bar{C}_{\mathcal{G}} = \max_{\sigma \in \{-1, 1\}^N} C(\sigma)$ is equivalent to finding the ground state energy.

Owing to $A_{m,m} = 0$, finding the maximal cut amounts to finding a maximal value of a linear function on the vertices of hypercube $[-1, 1]^N$. In turn, any linear function on a convex polyhedron reaches its extrema on the vertices. This yields a

continuous representation of the max-cut problem [46], when instead of binary functions σ one considers $\xi: \mathcal{V} \rightarrow [-1, 1]^N$:

$$\bar{C}_{\mathcal{G}} = \max_{\xi \in [-1, 1]^N} C(\xi). \quad (5)$$

Let us briefly consider a dynamical Ising machine utilizing this representation. Away from the extreme points ($\xi_m = \sigma_m = \pm 1$), the machine's equations of motion are

$$\dot{\xi}_m = \frac{\partial C(\xi)}{\partial \xi_m} = - \sum_n A_{m,n} \xi_n. \quad (6)$$

Solutions obtained with the help of this machine are determined by the machine's final state. It is easy to show that these are binary configurations ($\xi_m = \sigma_m = \pm 1$) characterized by the majority rule: for each node at least half of the incident edges are cut. Or, equivalently, $F_m \geq 0$, where

$$F_m = - \sum_n A_{m,n} \sigma_m \sigma_n. \quad (7)$$

We limit ourselves to the case of graphs without nodes with even degrees (the number of incident edges). In this case, any state satisfying the majority rule is stable and has the basin of attraction of finite volume. Thus, such Ising machine finds the same configurations as a simple local search algorithm with the respective consequences for performance.

In view of this consideration, it must be emphasized that the dynamical Ising machines based on synchronized oscillator networks implement a different operational principle and are not bound by the performance limitations typical for the local search.

3. QKM, XY model, and rank-2 SDP

An extension of the Ising model, the XY model, is obtained by considering vector-valued functions on the graph: $\xi: \mathcal{V} \rightarrow \mathbb{S}^2$, where $\mathbb{S}^2$ is the set of unit vectors on a 2D Euclidean plane, $\mathbb{R}^2$. The configuration energy is given by

$$\mathcal{H}^{(XY)}(\xi) = \frac{1}{2} \text{Tr}[\hat{A} \hat{\mathcal{E}}^{(XY)}] - \frac{K_s}{2} \sum_{m \in \mathcal{V}} \tilde{\xi}_m \cdot \vec{l} \cdot \tilde{\xi}_m, \quad (8)$$

where $\mathcal{E}_{m,n}^{(XY)} = \tilde{\xi}_m \cdot \tilde{\xi}_n$, $\vec{l} = \vec{l} \otimes \vec{l}$ is the anisotropy tensor, $\vec{l}$ is the anisotropy axis, and K_s is the anisotropy constant. Depending on whether $K_s = 0$ or $K_s \neq 0$, the XY model is called isotropic or anisotropic. Since $\tilde{\xi}_m$ are 2D vectors (rather than, say, 3D), the sign of K_s plays no role. We assume that $K_s \geq 0$ so that the anisotropy aligns the spins along the line defined by $\vec{l}$.

Treating $\mathcal{H}^{(XY)}(\xi)$ as a Lyapunov function, the dynamics is defined by $\dot{\tilde{\xi}}_m = -\partial \mathcal{H}^{(XY)} / \partial \tilde{\xi}_m = \vec{g}_m(\xi) \vec{\gamma}_m$, where $\vec{\gamma}_m$ is any of the two unit vectors orthogonal to $\tilde{\xi}_m$, and

$$\vec{g}_m(\xi) = - \sum_n A_{m,n} \vec{\gamma}_m \cdot \tilde{\xi}_n + K_s \vec{\gamma}_m \cdot \vec{l} \cdot \tilde{\xi}_m. \quad (9)$$

Equations of motion in this form ensure that $|\tilde{\xi}_m|$ is an integral of motion. Alternatively, this can be enforced by representing $\tilde{\xi}_m = (\cos(\theta) \sin(\theta))^T$ in terms of the angle θ_m , say, with respect to $\vec{l}$. Using this in Eq. (8), we obtain Eq. (2).

Thus, QKM describes the dynamics of vector spins in the XY model [42]. This relationship provides a convenient phenomenology for discussing the dynamic Ising machines implementing QKM as it abstracts from the challenges of physical realizations of oscillator networks. Therefore, for brevity, we will refer to these networks as XY machines.

The equivalence QKM $\leftrightarrow$ XY implies that the network evolution realizes the gradient descent for the XY model. Generally, the outcome of the evolution is a configuration with an unconstrained mutual orientation of vector spins. This poses two questions:

1. What is the relation between the final state of the XY model and the ground state of the Ising model?
2. How to reconstruct a feasible binary distribution from the ensemble of arbitrarily oriented spins?

Finding the maximal cut (or the ground state of the Ising model) can be formulated as an integer program:

$$\bar{C}_{\mathcal{G}} = \frac{M}{2} - \min_{\hat{\mathcal{E}}} \frac{1}{4} \text{Tr}[\hat{A} \hat{\mathcal{E}}] \quad (10)$$

with constraints $\mathcal{E}_{m,m} = 1$, and $\text{rank}(\mathcal{E}) = 1$. This problem is APX-hard [26,27] meaning that, unless $P = NP$, there is no polynomial-time algorithm providing arbitrarily good approximation. For such problems, the algorithms are characterized by the approximation ratio: $\rho = C_{\mathcal{G}}(\xi_g) / \bar{C}_{\mathcal{G}}$, where $C_{\mathcal{G}}(\xi_g)$ is the best solution the algorithm is guaranteed to produce in polynomial time. For example, for local search algorithms one has $\rho \gtrsim 0.5$ [47]. In particular, this holds for the example of a dynamical machine driven by Eq. (6). Of course, due to the special form of the respective worst-case Hamiltonians (see, for instance, [48]), this does not preclude local search algorithms from performing very well on some classes of graphs, which may be of practical relevance.

Other approaches to solving (10) are based on simplifying the problem by relaxing constraints and, subsequently, reconstructing a feasible (satisfying the original constraints) configuration by rounding. Importantly, since the relaxation cannot change the problem complexity class, the complexity is delegated to the rounding stage, which therefore requires special attention (see e.g. [49,50]).

It was discovered in [51,52] that semidefinite programming (SDP) relaxation is uniquely efficient in solving the max-cut problem. This relaxation requires that $\hat{\mathcal{E}}$ is symmetric positive semidefinite with $\text{rank}[\mathcal{E}] = k > 1$ (rank- k relaxation). This is equivalent to considering configurations $\xi: \mathcal{V} \rightarrow \mathbb{S}^k$ and minimizing $\mathcal{H}^{(k)}(\xi) = \sum_{\mathcal{G}} A_{m,n} \tilde{\xi}_m \cdot \tilde{\xi}_n$. Given the solution of the relaxed problem, the feasible configuration is obtained as $\sigma_m = \text{sign}(\tilde{\xi}_m^{(m)} \cdot \vec{l})$, where $\vec{l} \in \mathbb{S}^k$. Averaging the obtained cuts over randomly chosen $\vec{l}$ results in [52,53]

$$\langle C_{\mathcal{G}}(\xi) \rangle_{\vec{l}} \geq \rho_{\text{GW}} \bar{C}_{\mathcal{G}}, \quad (11)$$

where $\rho_{\text{GW}} = \min_{\theta > 0} \theta / (\pi \sin^2(\theta/2)) \gtrsim 0.878$.

The significance of this result is two-fold. First of all, it establishes a rounding procedure recovering the solution in polynomial time. Second, Eq. (11) estimates (not necessarily tightly) the so-called integrality gap [38,54] and exhibits a relation between the solutions of the relaxed and integer problems. This result holds for a weighted Ising model with weights of the uniform sign. For models with variable signs, the standard argument fails, but a guaranteed approximation ratio can still be proven [55–57].

To put this estimate into perspective, if $P \neq NP$, the best classically possible approximation ratio for the max-cut problem is $16/17 \gtrsim 0.941$ [28]. However, assuming additionally the unique games conjecture [58,59], it can be shown that ρ_{GW} is the best approximation achievable in polynomial time [27,39–41].

Importantly, we have the equivalence XY $\leftrightarrow$ SDP₂, where SDP₂ is the rank-2 SDP relaxation [60]. Indeed, for such relaxation, one has $\hat{\mathcal{E}} = \mathbf{s}^{(1)} \otimes \mathbf{s}^{(1)} + \mathbf{s}^{(2)} \otimes \mathbf{s}^{(2)}$ and hence $\mathcal{E}_{m,n} = s_n^{(1)} s_m^{(1)} + s_n^{(2)} s_m^{(2)} = \tilde{\xi}_n \cdot \tilde{\xi}_m$, where $\tilde{\xi}_m = (s_m^{(1)}, s_m^{(2)})^T$ are unit (because of the constraint $\mathcal{E}_{m,m} = 1$) vectors.

This identifies the origin of the computational capabilities of XY machines and, hence, Ising machines based on synchronizing oscillator networks. Their dynamics implements the gradient descent minimization of a rank-2 relaxation of the Ising ground state problem. Instead of exploring the configuration space, as, say, is done in the Ising machine described by Eq. (6), the XY machines solve a different but tightly related problem.

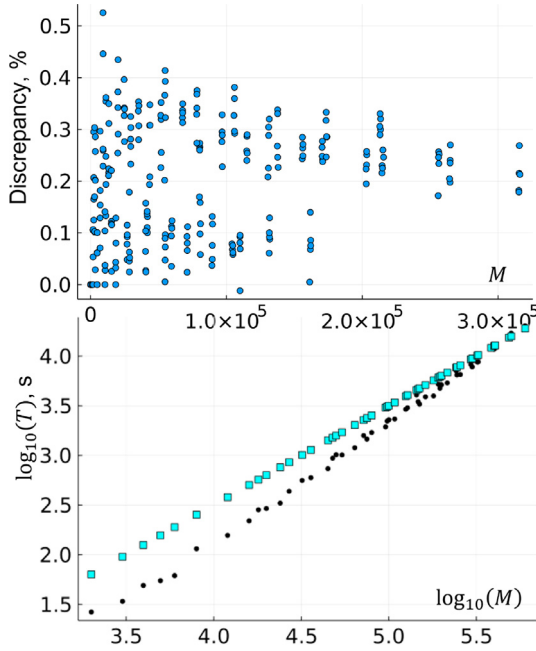


Fig. 2. Performance of a QKM-based dynamical Ising machine. (a) Discrepancy between the Ising machine and Circut results. (b) Scaling of the running wall-time with the number of edges in the graph (black circles) vs. the true linear scaling (cyan squares).

Fig. 2 illustrates the performance of the dynamical Ising machine implementing QKM. The dynamics of the machine was taken to be governed by Eq. (1) with $K_s = 0$ and solved using the first order Euler approximation. The machine ran on a series of Erdős-Rényi graphs $\mathcal{G}_{N,p}$, where p is the probability for an edge to present, with $50 \leq N \leq 2000$ and $0.1 \leq p \leq 0.3$. For each graph, the machine ran for 350 time-steps each $20K/N$ long. After each run, the final configuration was rounded using the same algorithm as in Circut [60] and post-processed as described below. This procedure was repeated 300 times from independently chosen random configurations, and the best value of cut was recorded. The obtained cut values were compared with Circut results, which remains one of the best max-cut heuristic solvers [61]. Except for a single instance, the results obtained by the Ising machine were within 0.5 percent of Circut's.

As an estimate of running time, the wall-time, T , was measured (Fig. 2). It must be noted that, by design, the number of elementary operations in simulating the machine dynamics is $\mathcal{O}(M)$. The observed deviation from this scaling is due to the rounding procedure, which, as implemented, scales at worst as $\mathcal{O}(NM)$.

The post-rounding processing consisted of two steps based on the observation that rounding does not necessarily respect the majority rule. The first step implemented the local search ensuring that all nodes obey $F_m \geq 0$. The second step, relevant for nodes with $F_m = 0$, ensured that for each cut edge at least half adjacent edges should be cut (otherwise, the cut can be increased by reverting spins at the incident nodes).

4. The detrimental effect of forced binarization

It must be emphasized that the chain of equivalences

$$\text{QKM} \leftrightarrow \text{XY} \leftrightarrow \text{SDP}_2$$

concerns only the dynamics of the Ising machines. The principal part of accessing the computational resource associated with SDP

is to properly recover a feasible binary state from the configuration of unit vectors of the XY model or relative oscillator phases in a synchronized network. Rounding the state of the Ising machines remains an under-explored problem since the existing implementations pursue dynamics collapsing the machine to a binary state. This can be achieved when the anisotropy is sufficiently strong. At the same time, as demonstrated by Fig. 1e, strong anisotropy may disrupt finding the configuration delivering the maximal cut. Here, we show that such detrimental effect of strong anisotropy is generic.

The equilibrium configurations of the oscillator networks implementing QKM are determined by $g_m(\xi^{(0)}) = 0$. The dynamics of weak excitations is obtained by representing $\tilde{\xi}_m = \xi_m^{(0)} + x_m \tilde{\gamma}_m^{(0)}$, with $x_m \ll 1$, so that

$$\begin{aligned} \dot{x}_m = & \sum_n A_{m,n} \tilde{\xi}_m^{(0)} \cdot \tilde{\xi}_n^{(0)} (x_m - x_n) \\ & - K_s \left(2 \tilde{\xi}_m^{(0)} \cdot \vec{l} \cdot \tilde{\xi}_m^{(0)} - 1 \right) x_n, \end{aligned} \quad (12)$$

or $\dot{\mathbf{x}} = [\hat{L}(\xi^{(0)}) - \hat{K}(\xi^{(0)})] \mathbf{x}$.

The dynamics is attracted to (Lyapunov) stable equilibria, that is with negative semidefinite $\hat{L}(\xi^{(0)}) - \hat{K}(\xi^{(0)})$. We note that the Laplacian structure of $\hat{L}(\xi^{(0)})$ implies that it always has a zero eigenvalue. As a result, in the isotropic case, weak perturbations of stable configurations exponentially converge to their projection on the homogeneous displacement, $x_m \equiv x$, which reflects the rotational symmetry of the XY model.

It must be emphasized, that because of the nonlinear coupling between oscillators in QKM, matrix $\hat{L}(\xi^{(0)})$ coincides with the graph Laplacian *only* when all $\xi^{(0)}$ have the same orientation. For other configurations, the spectral properties of the graph Laplacian and $\hat{L}(\xi^{(0)})$ are drastically different.

It follows from (9), that $g_m(\xi)$ vanishes on Ising-like configurations $\xi^{(i)}$ with $\tilde{\xi}_m^{(i)} = \sigma_m \vec{l}$. It is straightforward to show that in the isotropic case, all configurations σ that do not produce the maximal cut are unstable [60]. In turn, maximal cut configurations are stable only on bipartite graphs and selected families of non-bipartite graphs. As a result, anisotropy plays the major role in the emergence of Ising-like configurations in the dynamics of XY machines.

A complete framework describing the effect of anisotropy on the convergence properties of XY machines is yet to be developed. In the present paper, we limit ourselves to an analysis of the structure of the binary configurations enabled by anisotropy.

Since the effect of anisotropy on binary configurations reduces to a simple displacing the spectrum of $\hat{L}(\sigma)$ by $-K_s$, any such configuration can be stable provided

$$K_s \geq \kappa(\sigma) = \lambda_1(\hat{L}(\sigma)), \quad (13)$$

where we have introduced $\kappa(\sigma)$, the *instability* of σ , which is defined by $\lambda_1(\hat{L}(\sigma))$, the maximal eigenvalue of $\hat{L}(\sigma)$. Thus, except for special graphs, the instability of binary configurations of the XY model is positive.

Based on this, we can identify characteristic values of anisotropy, when the significant impact on the computational capabilities can be expected. The condition $\hat{L}(\sigma) - \hat{K}(\sigma) \preceq 0$ implies that for any unit $\mathbf{u} \in \mathbb{R}^N$, one has $K_s - \mathbf{u} \cdot \hat{L}(\sigma) \cdot \mathbf{u} \geq 0$, or that

$$K_s + \sum_m u_m^2 F_m + \sum_{m,n} A_{m,n} u_m u_n \geq 0, \quad (14)$$

where F_m are given by Eq. (7).

This yields the first characteristic value of anisotropy, $K_s^{(1)} = |\mu_N(\hat{A})|$, where $\mu_N(\hat{A})$ is the smallest eigenvalue of the graph adjacency matrix. When K_s reaches $K_s^{(1)}$, any binary states satisfying the majority rule $F_m \geq 0$ become stable. Using a simple bound

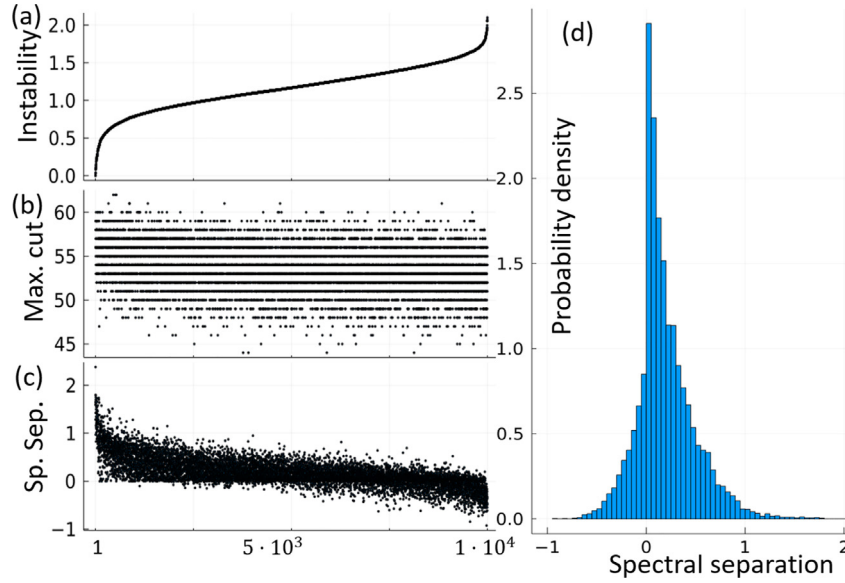


Fig. 3. (a) The instability of the ground (max-cut) state, (b) the maximal cut, and (c) the spectral separation $\delta_{\mathcal{G}}$ for 10^4 connected Erdős-Rényi graphs $\mathcal{G}_{17,0.7}$. The samples are arranged according to the instability. (d) The empirical probability distribution function of $\delta_{\mathcal{G}}$.

$|\mu_N(\hat{A})| \leq \Delta_{\mathcal{G}}$, where $\Delta_{\mathcal{G}}$ is the graph maximal degree, the following rule can be formulated. When $K_s = \Delta_{\mathcal{G}}$, the structure of binary configurations produced by the anisotropic XY machine is, at best, the same as obtained by simple local search.

Clearly, when anisotropy increases further, binary configurations that do not satisfy the majority rule (and, hence, cannot maximize cut) also belong to the set of stable configurations. In other words, the quality of the solutions may become worse than that of the local search. Finally, when anisotropy is too strong, $K_s \geq K_s^{(2)} = \lambda_1(\hat{L})$, where $\lambda_1(\hat{L})$ is the maximal eigenvalue of the graph Laplacian, all binary configurations are stable. For instance, this is the case when $K_s \geq 2\Delta_{\mathcal{G}}$.

While identifying “dangerous” values of the anisotropy, this consideration leaves open the question whether moderate anisotropy $K_s < \Delta_{\mathcal{G}}$ can be used to arrive at a maximal-cut binary configuration.

A parameter characterizing how widely one can vary K_s without introducing sub-optimal configurations is the difference between instabilities of the least unstable maximal cut and non-max-cut configurations

$$\delta_{\mathcal{G}} = \min_{\sigma \in \mathcal{M}_{\mathcal{G}}} \kappa(\sigma) - \min_{\xi \notin \mathcal{M}_{\mathcal{G}}} \kappa(\sigma), \quad (15)$$

where $\mathcal{M}_{\mathcal{G}}$ is the set of max-cut configurations. Importantly, neither the magnitude of $\delta_{\mathcal{G}}$ nor even its sign are bounded on a sufficiently rich set of graphs. As an example of such set we have considered 10^4 connected random (Erdős-Rényi) graphs $\mathcal{G}_{17,0.7}$. Fig. 3 shows the instability of the max-cut configuration together with the max-cut values (Fig. 3b) and spectral separations $\delta_{\mathcal{G}}$ (Fig. 3c). It reveals that there is only a weak correlation between the instabilities of the max-cut and non-max-cut configurations. This is summarized in Fig. 3d depicting the distribution function of $\delta_{\mathcal{G}}$. It shows that the probability that in a randomly chosen graph the max-cut-state will become stable at the lowest value of anisotropy is rather moderate (0.2 for the considered set).

These simulation results show that the probability to have a graph admitting anisotropy governed selection of the ground state is small, which supports our statement that regimes 1 and 3 in Fig. 1 should be regarded as generic.

5. Conclusion

We have shown that the computational resource of dynamic Ising machines based on synchronizing networks of nonlinear oscillators originates from the factual realization of rank-2 semidefinite programming relaxation of the max-cut problem. In contrast to approaches aiming at direct exploration of the Ising model space state, these relaxations deliver the best (if $P \neq NP$) approximation achievable in polynomial time. This shows that Ising machines based on synchronizing networks are capable of providing good heuristics for a wide class of NP-hard problems.

At the same time, this relation shows that to reach theoretically possible performance, a rounding procedure must be supplied. An attempt to force the system to evolve towards an Ising-like state may disrupt the computational capabilities up to their complete invalidation when the dynamic Ising machine effectively acts as a random generator of configurations. For the Ising machines utilizing an effective anisotropy for ensuring final binary states, we have estimated the critical values of anisotropy corresponding to the loss of quality of solutions obtained by the collapsed state.

Finally, our consideration demonstrates that a quantitative evaluation of dynamic Ising machines requires an accurate description of their dynamics. The ability to yield the Ising Hamiltonian is not enough because of the wide variability of the approximation ratios: from the best classically possible to that of a random generator.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The work was supported by the US National Science Foundation under Grant No. 1710940 and by the Air Force Office of Scientific Research (AFOSR) under Grant No. FA9550-16-1-0363.

References

- [1] S. Kirkpatrick, C.D. Gelatt, M.P. Vecchi, Optimization by simulated annealing, *Science* 220 (4598) (1983) 671–680, <http://dx.doi.org/10.1126/science.220.4598.671>.
- [2] Y. Fu, P.W. Anderson, Application of statistical mechanics to NP-complete problems in combinatorial optimisation, *J. Phys. A: Math. Gen.* 19 (9) (1986) 1605–1620, <http://dx.doi.org/10.1088/0305-4470/19/9/033>.
- [3] F. Barahona, M. Grötschel, M. Jünger, G. Reinelt, An application of combinatorial optimization to statistical physics and circuit layout design, *Oper. Res.* 36 (3) (1988) 493–513, <http://dx.doi.org/10.1287/opre.36.3.493>.
- [4] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher, J.D. Bohlinger (Eds.), *Complexity of Computer Computations*, Springer US, Boston, MA, 1972, pp. 85–103, http://dx.doi.org/10.1007/978-1-4684-2001-2_9.
- [5] M. Garey, D. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, *Theoret. Comput. Sci.* 1 (3) (1976) 237–267, [http://dx.doi.org/10.1016/0304-3975\(76\)90059-1](http://dx.doi.org/10.1016/0304-3975(76)90059-1).
- [6] A. Lucas, Ising formulations of many NP problems, *Front. Phys.* 2 (2014) 5, <http://dx.doi.org/10.3389/fphy.2014.00005>.
- [7] Z. Wang, A. Marandi, K. Wen, R.L. Byer, Y. Yamamoto, Coherent Ising machine based on degenerate optical parametric oscillators, *Phys. Rev. A* 88 (6) (2013) 063853, <http://dx.doi.org/10.1103/PhysRevA.88.063853>.
- [8] A.V. Novikov, E.N. Benderskaya, Oscillatory neural networks based on the Kuramoto model for cluster analysis, *Pattern Recognit. Image Anal.* 24 (3) (2014) 365–371, <http://dx.doi.org/10.1134/S1054661814030146>.
- [9] A. Marandi, Z. Wang, K. Takata, R.L. Byer, Y. Yamamoto, Network of time-multiplexed optical parametric oscillators as a coherent Ising machine, *Nat. Photonics* 8 (12) (2014) 937–942, <http://dx.doi.org/10.1038/nphoton.2014.249>.
- [10] I. Mahboob, H. Okamoto, H. Yamaguchi, An electromechanical Ising Hamiltonian, *Sci. Adv.* 2 (6) (2016) e1600236, <http://dx.doi.org/10.1126/sciadv.1600236>.
- [11] T. Inagaki, Y. Haribara, K. Igarashi, T. Sonobe, S. Tamate, T. Honjo, A. Marandi, P.L. McMahon, T. Umeki, K. Enbutsu, O. Tadanaga, H. Takenouchi, K. Aihara, K.-i. Kawarabayashi, K. Inoue, S. Utsunomiya, H. Takesue, A coherent Ising machine for 2000-node optimization problems, *Science* 354 (6312) (2016) 603–606, <http://dx.doi.org/10.1126/science.aah4243>.
- [12] A. Parihar, N. Shukla, M. Jerry, S. Datta, A. Raychowdhury, Vertex coloring of graphs via phase dynamics of coupled oscillatory networks, *Sci. Rep.* 7 (1) (2017) 911, <http://dx.doi.org/10.1038/s41598-017-00825-1>.
- [13] K.P. Kalinin, N.G. Berloff, Networks of non-equilibrium condensates for global optimization, *New J. Phys.* 20 (11) (2018) 113023, <http://dx.doi.org/10.1088/1367-2630/aae8ae>.
- [14] D. Pierangeli, G. Marcucci, C. Conti, Large-scale photonic Ising machine by spatial light modulation, *Phys. Rev. Lett.* 122 (21) (2019) 213902, <http://dx.doi.org/10.1103/PhysRevLett.122.213902>.
- [15] J. Chou, S. Brahamavar, S. Ghosh, W. Herzog, Analog coupled oscillator based weighted Ising machine, *Sci. Rep.* 9 (1) (2019) 14786, <http://dx.doi.org/10.1038/s41598-019-49699-5>.
- [16] T. Wang, J. Roychowdhury, OIM: Oscillator-based Ising machines for solving combinatorial optimisation problems, in: I. McQuillan, S. Seki (Eds.), *Unconventional Computation and Natural Computation*, Vol. 11493, Springer International Publishing, Cham, 2019, pp. 232–256, http://dx.doi.org/10.1007/978-3-030-19311-9_19.
- [17] T. Wang, L. Wu, J. Roychowdhury, New computational results and hardware prototypes for oscillator-based Ising machines, in: *Proceedings of the 56th Annual Design Automation Conference 2019*, ACM, Las Vegas NV USA, 2019, pp. 1–2, <http://dx.doi.org/10.1145/3316781.3322473>.
- [18] H. Goto, K. Tatsumura, A.R. Dixon, Combinatorial optimization by simulating adiabatic bifurcations in nonlinear Hamiltonian systems, *Sci. Adv.* 5 (4) (2019) eaav2372, <http://dx.doi.org/10.1126/sciadv.aav2372>.
- [19] K.P. Kalinin, A. Amo, J. Bloch, N.G. Berloff, Polaritonic XY-Ising machine, *Nanophotonics* 9 (13) (2020) 4127–4138, <http://dx.doi.org/10.1515/nanoph-2020-0162>.
- [20] Y. Yamamoto, T. Leleu, S. Ganguli, H. Mabuchi, Coherent Ising machines—Quantum optics and neural network perspectives, *Appl. Phys. Lett.* 117 (16) (2020) 160501, <http://dx.doi.org/10.1063/5.0016140>.
- [21] I. Ahmed, P.-W. Chiu, C.H. Kim, A probabilistic self-annealing compute fabric based on 560 hexagonally coupled ring oscillators for solving combinatorial optimization problems, in: *2020 IEEE Symposium on VLSI Circuits*, IEEE, Honolulu, HI, USA, 2020, pp. 1–2, <http://dx.doi.org/10.1109/VLSICircuits18222.2020.9162869>.
- [22] A. Mallick, M.K. Bashar, D.S. Truesdell, B.H. Calhoun, S. Joshi, N. Shukla, Using synchronized oscillators to compute the maximum independent set, *Nature Commun.* 11 (1) (2020) 4689, <http://dx.doi.org/10.1038/s41467-020-18445-1>.
- [23] S. Dutta, A. Khanna, S. Datta, Understanding the continuous-time dynamics of phase-transition nano-oscillator-based Ising Hamiltonian solver, *IEEE J. Explor. Solid-State Comput. Devices Circuits* 6 (2) (2020) 155–163, <http://dx.doi.org/10.1109/JXDCD.2020.3045074>.
- [24] R. Afoakwa, Y. Zhang, U.K.R. Vengalam, Z. Ignjatovic, M. Huang, BRIM: Bistable resistively-coupled Ising machine, in: *2021 IEEE International Symposium on High-Performance Computer Architecture, HPCA*, IEEE, Seoul, Korea (South), 2021, pp. 749–760, <http://dx.doi.org/10.1109/HPCA51647.2021.00068>.
- [25] H. Goto, K. Endo, M. Suzuki, Y. Sakai, T. Kanao, Y. Hamakawa, R. Hidaka, M. Yamasaki, K. Tatsumura, High-performance combinatorial optimization based on classical mechanics, *Sci. Adv.* 7 (6) (2021) eabe7953, <http://dx.doi.org/10.1126/sciadv.abe7953>.
- [26] C.H. Papadimitriou, M. Yannakakis, Optimization, approximation, and complexity classes, *J. Comput. System Sci.* 43 (3) (1991) 425–440, [http://dx.doi.org/10.1016/0022-0000\(91\)90023-X](http://dx.doi.org/10.1016/0022-0000(91)90023-X).
- [27] S. Khot, G. Kindler, E. Mossel, R. O'Donnell, Optimal inapproximability results for max-cut and other 2-variable CSPs? in: *45th Annual IEEE Symposium on Foundations of Computer Science*, IEEE, Rome, Italy, 2004, pp. 146–154, <http://dx.doi.org/10.1109/FOCS.2004.49>.
- [28] J. Håstad, Some optimal inapproximability results, *J. ACM* 48 (4) (2001) 798–859, <http://dx.doi.org/10.1145/502090.502098>.
- [29] A. Yurtsever, J.A. Tropp, O. Fercoq, M. Udell, V. Cevher, Scalable semidefinite programming, *SIAM J. Math. Data Sci.* 3 (1) (2021) 171–200, <http://dx.doi.org/10.1137/19M1305045>.
- [30] Experimental investigation of performance differences between coherent Ising machines and a quantum annealer, *Sci. Adv.* 5 (5) (2019) eaau0823, <http://dx.doi.org/10.1126/sciadv.aau0823>.
- [31] T. Leleu, F. Khoyrater, T. Levi, R. Hamerly, T. Kohno, K. Aihara, Scaling advantage of chaotic amplitude control for high-performance combinatorial optimization, *Commun. Phys.* 4 (1) (2021) 266, <http://dx.doi.org/10.1038/s42005-021-00768-0>.
- [32] Y. Kuramoto, Phase- and center-manifold reductions for large populations of coupled oscillators with application to non-locally coupled systems, *Int. J. Bifurcation Chaos* 07 (04) (1997) 789–805, <http://dx.doi.org/10.1142/S0218127497000595>.
- [33] H. Mori, Y. Kuramoto, *Dissipative Structures and Chaos*, Springer, Berlin ; New York, 1998.
- [34] J.A. Acebrón, L.L. Bonilla, C.J. Pérez Vicente, F. Ritort, R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Modern Phys.* 77 (1) (2005) 137–185, <http://dx.doi.org/10.1103/RevModPhys.77.137>.
- [35] B. Pietras, A. Daffertshofer, Network dynamics of coupled oscillators and phase reduction techniques, *Phys. Rep.* 819 (2019) 1–105, <http://dx.doi.org/10.1016/j.physrep.2019.06.001>.
- [36] M.K. Bashar, A. Mallick, D.S. Truesdell, B.H. Calhoun, S. Joshi, N. Shukla, Experimental demonstration of a reconfigurable coupled oscillator platform to solve the max-cut problem, *IEEE J. Explor. Solid-State Comput. Devices Circuits* 6 (2) (2020) 116–121, <http://dx.doi.org/10.1109/JXDCD.2020.3025994>.
- [37] T. Wang, L. Wu, P. Nobel, J. Roychowdhury, Solving combinatorial optimization problems using oscillator based Ising machines, *Nat. Comput.* (2021) <http://dx.doi.org/10.1007/s11047-021-09845-3>.
- [38] B. Gärtner, J. Matousek, *Approximation Algorithms and Semidefinite Programming*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2012, <http://dx.doi.org/10.1007/978-3-642-22015-9>.
- [39] P. Raghavendra, Optimal algorithms and inapproximability results for every CSP? in: *Proceedings of the 40th Annual ACM Symposium on Theory of Computing*, in: *STOC '08*, ACM, New York, NY, USA, 2008, pp. 245–254, <http://dx.doi.org/10.1145/1374376.1374414>.
- [40] L. Trevisan, Max cut and the smallest eigenvalue, *SIAM J. Comput.* 41 (6) (2012) 1769–1786, <http://dx.doi.org/10.1137/090773714>.
- [41] J.A. Soto, Improved analysis of a max-cut algorithm based on spectral partitioning, *SIAM J. Discrete Math.* 29 (1) (2015) 259–268, <http://dx.doi.org/10.1137/14099098X>.
- [42] S. Shinomoto, Y. Kuramoto, Phase transitions in active rotator systems, *Progr. Theoret. Phys.* 75 (5) (1986) 1105–1110, <http://dx.doi.org/10.1143/PTP.75.1105>.
- [43] R. Adler, A study of locking phenomena in oscillators, *Proc. IRE* 34 (6) (1946) 351–357, <http://dx.doi.org/10.1109/JRPROC.1946.229930>.
- [44] X. Zhang, X. Zhou, B. Aliener, A. Daryoush, A study of subharmonic injection locking for local oscillators, *IEEE Microw. Guid. Wave Lett.* 2 (3) (1992) 97–99, <http://dx.doi.org/10.1109/75.124911>.
- [45] P. Bhansali, J. Roychowdhury, Gen-adler: The generalized adler's equation for injection locking analysis in oscillators, in: *2009 Asia and South Pacific Design Automation Conference*, IEEE, Yokohama, Japan, 2009, pp. 522–527, <http://dx.doi.org/10.1109/ASPAC.2009.4796533>.
- [46] M.M. Deza, M. Laurent, *Geometry of Cuts and Metrics*, in: *Algorithms and Combinatorics*, vol. 15, Springer Berlin Heidelberg, Berlin, Heidelberg, 1997, <http://dx.doi.org/10.1007/978-3-642-04295-9>.
- [47] B. Korte, J. Vygen, *Combinatorial Optimization*, in: *Algorithms and Combinatorics*, vol. 21, Springer Berlin Heidelberg, Berlin, Heidelberg, 2018, <http://dx.doi.org/10.1007/978-3-662-56039-6>.

- [48] B. Kalantari, Quadratic functions with exponential number of local maxima, *Oper. Res. Lett.* 5 (1) (1986) 47–49, [http://dx.doi.org/10.1016/0167-6377\(86\)90100-8](http://dx.doi.org/10.1016/0167-6377(86)90100-8).
- [49] M. Laurent, S. Poljak, On a positive semidefinite relaxation of the cut polytope, *Linear Algebra Appl.* 223–224 (1995) 439–461, [http://dx.doi.org/10.1016/0024-3795\(95\)00271-R](http://dx.doi.org/10.1016/0024-3795(95)00271-R).
- [50] C. Delorme, S. Poljak, Combinatorial properties and the complexity of a max-cut approximation, *European J. Combin.* 14 (4) (1993) 313–333, <http://dx.doi.org/10.1006/eujc.1993.1035>.
- [51] M.X. Goemans, D.P. Williamson, .879-Approximation algorithms for MAX CUT and MAX 2SAT, in: *Proceedings of the 26th Annual ACM Symposium on Theory of Computing - STOC '94*, ACM Press, Montreal, Quebec, Canada, 1994, pp. 422–431, <http://dx.doi.org/10.1145/195058.195216>.
- [52] M.X. Goemans, D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *J. ACM* 42 (6) (1995) 1115–1145, <http://dx.doi.org/10.1145/227683.227684>.
- [53] N. Alon, B. Sudakov, Bipartite subgraphs and the smallest eigenvalue, *Combin. Probab. Comput.* 9 (1) (2000) 1–12, <http://dx.doi.org/10.1017/S0963548399004071>.
- [54] P. Raghavan, C.D. Tompson, Randomized rounding: A technique for provably good algorithms and algorithmic proofs, *Combinatorica* 7 (4) (1987) 365–374, <http://dx.doi.org/10.1007/BF02579324>.
- [55] M. Charikar, A. Wirth, Maximizing quadratic programs: Extending Grothendieck's inequality, in: *45th Annual IEEE Symposium on Foundations of Computer Science, IEEE, Rome, Italy, 2004*, pp. 54–60, <http://dx.doi.org/10.1109/FOCS.2004.39>.
- [56] N. Alon, A. Naor, Approximating the cut-norm via Grothendieck's inequality, in: *Proceedings of the 36th Annual ACM Symposium on Theory of Computing - STOC '04*, ACM Press, Chicago, IL, USA, 2004, p. 72, <http://dx.doi.org/10.1145/1007352.1007371>.
- [57] M.F. Anjos, H. Wolkowicz, Strengthened semidefinite relaxations via a second lifting for the Max-Cut problem, *Discrete Appl. Math.* 119 (1–2) (2002) 79–106, [http://dx.doi.org/10.1016/S0166-218X\(01\)00266-9](http://dx.doi.org/10.1016/S0166-218X(01)00266-9).
- [58] L. Trevisan, On Khot's unique games conjecture, *Bull. Amer. Math. Soc.* 49 (1) (2012) 91–111.
- [59] S. Khot, On the unique games conjecture (invited survey), in: *2010 IEEE 25th Annual Conference on Computational Complexity, 2010*, pp. 99–121, <http://dx.doi.org/10.1109/CCC.2010.19>.
- [60] S. Burer, R.D.C. Monteiro, Y. Zhang, Rank-two relaxation heuristics for MAX-CUT and other binary quadratic programs, *SIAM J. Optim.* 12 (2) (2002) 503–521, <http://dx.doi.org/10.1137/S1052623400382467>.
- [61] I. Dunning, S. Gupta, J. Silberholz, What works best when? A systematic evaluation of Heuristics for Max-Cut and QUBO, *INFORMS J. Comput.* 30 (3) (2018) 608–624, <http://dx.doi.org/10.1287/ijoc.2017.0798>.