Entanglement Dynamics From Random Product States: Deviation From Maximal Entanglement

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Abstract—We study the entanglement dynamics of quantum many-body systems and prove the following: (I) For any geometrically local Hamiltonian on a lattice, starting from a random product state the entanglement entropy is bounded away from the maximum entropy at all times with high probability. (II) In a spin-glass model with random all-to-all interactions, starting from any product state the average entanglement entropy is bounded away from the maximum entropy at all times. We also extend these results to any unitary evolution with charge conservation and to the Sachdev-Ye-Kitaev model. Our results highlight the difference between the entanglement generated by (chaotic) Hamiltonian dynamics and that of random states, for the latter is nearly maximal.

Index Terms—Chaos, dynamics, entropy, quantum entanglement, quantum mechanics.

I. INTRODUCTION

ENTANGLEMENT, a concept of quantum information theory, has been widely used in condensed matter and statistical physics to provide insights beyond those obtained via "conventional" quantities. A large body of literature is available on the static [2]–[15] and dynamical [16]–[22] behavior of entanglement in various systems. The scaling of entanglement [23] reflects the classical simulability of quantum many-body systems [24]–[40].

The dynamics of chaotic (not necessarily geometrically) local Hamiltonians is of high current interest. Since these models are almost by definition not exactly solvable, heuristic descriptions of the universal aspects of the dynamics have been developed. It is important to understand the extent to which the heuristic descriptions reflect reality.

In this paper, we study the entanglement dynamics [41], [42] starting from a random product state, which is typically a "massive" superposition of energy eigenstates [43]. The time evolution under a chaotic local Hamiltonian is so complex that heuristically, one might expect that the state at long times behaves like a random state. Therefore,

Conjecture 1: For chaotic (not necessarily geometrically) local Hamiltonians, starting from a random product state the

entanglement entropy approaches that of a random state at long times.

This conjecture is not mathematically precise because "chaotic" is not defined. We do not attempt to define it here, for there is no clear-cut definition of quantum chaos.

Recall that the entanglement entropy of a random state is nearly maximal [44]–[47]. However, we prove the following results.

- For any geometrically local Hamiltonian on a lattice, starting from a random product state the entanglement entropy is bounded away from the maximum entropy at all times with high probability.
- In a spin-glass model with random all-to-all interactions, starting from any product state the average entanglement entropy is bounded away from the maximum entropy at all times.

We also extend these results to any unitary evolution with charge conservation and to the Sachdev-Ye-Kitaev (SYK) model [48]–[50]. Our results highlight the difference between the entanglement generated by (chaotic) Hamiltonian dynamics and that of random states. The difference is a consequence of energy conservation, which prevents the time-evolved state from behaving like a completely random state [51]. For chaotic Hamiltonian dynamics at long times, if our upper bounds on the entanglement entropy are tight, then the difference is a subleading correction, and Conjecture 1 holds to leading order.

II. PRELIMINARIES

Throughout this paper, standard asymptotic notations are used extensively. Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be two functions. One writes $f(x) = O(g(x))$ if and only if there exist constants $M, x_0 > 0$ such that $f(x) \le Mg(x)$ for all $x > x_0$; $f(x) = \Omega(g(x))$ if and only if there exist constants $M, x_0 > 0$ such that $f(x) \ge Mg(x)$ for all $x > x_0$; $f(x) = \Theta(g(x))$ if and only if there exist constants $M_1, M_2, x_0 > 0$ such that $M_1 g(x) \le f(x) \le M_2 g(x)$ for all $x > x_0$.

Definition 1 (Entanglement Entropy): The entanglement entropy of a bipartite pure state ρ_{AB} is defined as the von Neumann entropy

$$S(\rho_A) := -\operatorname{tr}(\rho_A \ln \rho_A) \tag{1}$$

of the reduced density matrix $\rho_A = \operatorname{tr}_B \rho_{AB}$.

We briefly review the entanglement of random states.

Theorem 1 (Conjectured and Partially Proved by Page [44]; Proved in Refs. [45]–[47]): For a bipartite pure state ρ_{AB}

chosen uniformly at random with respect to the Haar measure,

$$\underset{\rho_{AB}}{\mathbb{E}}\, S(\rho_A) = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} - \frac{d_A-1}{2d_B} = \ln d_A - \frac{d_A}{2d_B} + \frac{O(1)}{d_A d_B},$$

(2)

where $d_A \le d_B$ are the local dimensions of subsystems A and B, respectively.

Let $\gamma \approx 0.577216$ be the Euler-Mascheroni constant. The second step of Eq. (2) uses the formula

$$\sum_{k=1}^{d_B} \frac{1}{k} = \ln d_B + \gamma + \frac{1}{2d_B} + O(1/d_B^2).$$

(3)

The distribution of $S(\rho_A)$ is highly concentrated around the mean $\mathbb{E}_{\rho_{AB}} S(\rho_A)$ [52]. This can be seen from the exact formula [53], [54] for the variance of $S(\rho_A)$.

Consider a system of N qubits labeled by $1, 2, \ldots, N$. Let

$$\sigma_j^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_j^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_j^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(4)

be the Pauli matrices for qubit j.

Definition 2 (Haar-Random Product State): Let $|\Psi\rangle = \bigotimes_{j=1}^{N} |\Psi_j\rangle$ be a Haar-random product state, where each $|\Psi_j\rangle$ is chosen independently and uniformly at random with respect to the Haar measure.

III. RESULTS

This section consists of four independent subsections, which can be read without consulting each other.

A. Geometrically Local Hamiltonians

For notational simplicity and without loss of generality, we present the results for geometrically local Hamiltonians in one spatial dimension. (It is easy to see that the same result holds in higher dimensions.) Consider a chain of N qubits governed by a local Hamiltonian

$$H^{\text{lat}} = \sum_{j=1}^{N} H_j,$$

(5)

where H_j represents the nearest-neighbor interaction between qubits at positions j and $j+1$. For concreteness, we use periodic boundary conditions, but our argument also applies to other boundary conditions. Assume without loss of generality that $\operatorname{tr} H_j = 0$ (traceless) so that the mean energy of H^{lat} is 0. We do not assume translational invariance. In particular, $\|H_j\|$ may be site dependent but should be $\Theta(1)$ for all j.

Let A be a contiguous subsystem of n qubits and $\bar{A}$ be the rest of the system. Assume without loss of generality that $n \le N/2$. Let $\mathbb{E}_{|A|=n}$ denote averaging over all contiguous subsystems of size n. There are N such subsystems.

Theorem 2: Initialize the system in a Haar-random product state $|\Psi\rangle$ (Definition 2). Let

$$\rho_A(t) = \operatorname{tr}_{\bar{A}}(e^{-iH^{\text{lat}}t} |\Psi\rangle\langle\Psi| e^{iH^{\text{lat}}t})$$

(6)

be the reduced density matrix of subsystem A at time t. For $n > 1$,

$$\Pr_{\Psi} \left(\sup_{t \in \mathbb{R}} \underset{|A|=n}{\mathbb{E}}\, S(\rho_A(t)) = n \ln 2 - \Omega(n/N) \right) \ge 1 - \delta,$$

(7)

where $\delta > 0$ is an arbitrarily small constant.

Corollary 1: Using the notation of Theorem 2, if H^{lat} is translationally invariant, then for $n > 1$,

$$\sup_{t \in \mathbb{R}} \mathbb{E}_{\Psi}\, S(\rho_A(t)) = n \ln 2 - \Omega(n/N).$$

(8)

Proof: Since the ensemble of Haar-random product states is translationally invariant, averaging over subsystems is not necessary if we average over this ensemble. ∎

For $1 < n = O(1)$, the bound (8) is saturated by any translationally invariant H^{lat} whose spectrum has non-degenerate gaps.

Definition 3 (Non-Degenerate Gap): The spectrum $\{E_j\}$ of a Hamiltonian has non-degenerate gaps if the differences $\{E_j - E_k\}_{j \ne k}$ are all distinct, i.e., for any $j \ne k$,

$$E_j - E_k = E_{j'} - E_{k'} \implies (j = j') \text{ and } (k = k').$$

(9)

Theorem 3: Using the notation of Theorem 2, if H^{lat} is translationally invariant and if the spectrum has non-degenerate gaps, then for $1 \le n = O(1)$ and sufficiently large τ,

$$\Pr_{t \in [0,\tau]} \left(\mathbb{E}_{\Psi} S(\rho_A(t)) = n \ln 2 - O(1/N) \right) = 1 - e^{-\Omega(N)},$$

(10)

where t is uniformly distributed in the interval $[0, \tau]$.

B. Unitary Evolution With Charge Conservation

Consider a system of N qubits without an underlying lattice structure (of course, Theorem 4 below remains valid in the presence of a lattice).

Let m, n be positive integers such that $n \le N/2$ and that mn is a multiple of N. Let $A_1, A_2, \ldots, A_m$ be m possibly overlapping subsystems, each of which has exactly n qubits. Suppose that each qubit in the system is in exactly mn/N out of these m subsystems. For each j, let $\bar{A}_j$ be the complement of A_j so that $A_j \otimes \bar{A}_j$ defines a bipartition of the system.

Let $\sigma^z := \sum_{j=1}^{N} \sigma_j^z$ be the total charge operator and $U(t)$ be a unitary operator such that $[U(t), \sigma^z] = 0$. Note that $U(t)$ need not be generated by a time-independent Hamiltonian. It can be the time evolution operator of a quantum circuit with charge conservation [55]–[58].

Theorem 4: Initialize the system in a Haar-random product state $|\Psi\rangle$ (Definition 2). Let

$$\rho_{A_j}(t) = \operatorname{tr}_{\bar{A}_j}(U(t)|\Psi\rangle\langle\Psi|U^\dagger(t))$$

(11)

be the reduced density matrix of subsystem A_j at time t. Then,

$$\Pr_{\Psi} \left(\sup_{t \in \mathbb{R}} \frac{1}{m} \sum_{j=1}^{m} S(\rho_{A_j}(t)) = n \ln 2 - \Omega(n/N) \right) \ge 1 - \delta,$$

(12)

where $\delta > 0$ is an arbitrarily small constant.

C. Spin-Glass Model

Consider a system of N qubits. Let $J := \{J_{jklm}\}_{1 \le j < k \le N}^{l,m \in \{x,y,z\}}$ be a collection of $d_N := 9N(N-1)/2$ independent real Gaussian random variables with zero mean $\overline{J_{jklm}} = 0$ and unit variance $\overline{J_{jklm}^2} = 1$. The Hamiltonian of the spin-glass model is [59]

$$H_J^{\text{sg}} = \frac{1}{\sqrt{d_N}} \sum_{1 \le j < k \le N} \sum_{l,m \in \{x,y,z\}} J_{jklm} \sigma_j^l \sigma_k^m. \qquad (13)$$

Let $A \subset \{1, 2, \ldots, N\}$ so that $A \sqcup \bar{A}$ defines a bipartition of the system. Assume without loss of generality that $|A| \le N/2$. Let $\mathbb{E}_{|A|=n}$ denote averaging over all subsystems of size n. There are $\binom{N}{n}$ such subsystems.

Theorem 5: Initialize the system in an arbitrary (deterministic) product state $|\psi\rangle = \bigotimes_{j=1}^{N} |\psi_j\rangle$. Let

$$\rho_{J,A}(t_J) = \text{tr}_{\bar{A}}(e^{-iH_J^{\text{sg}}t_J} |\psi\rangle \langle\psi| e^{iH_J^{\text{sg}}t_J}) \qquad (14)$$

be the reduced density matrix of subsystem A at time t_J. For $n > 1$,

$$\mathbb{E}_{J} \sup_{t_J \in \mathbb{R}} \mathbb{E}_{|A|=n} S(\rho_{J,A}(t_J)) = n \ln 2 - \Omega(n^2/N^2). \qquad (15)$$

D. SYK Model

Consider a system of N Majorana fermions $\chi_1, \chi_2, \ldots, \chi_N$ with $\{\chi_j, \chi_k\} = 2\delta_{jk}$, where N is an even number. Let $K := \{K_{jklm}\}_{1 \le j < k < l < m \le N}$ be a collection of $\binom{N}{4}$ independent real Gaussian random variables with zero mean $\overline{K_{jklm}} = 0$ and unit variance $\overline{K_{jklm}^2} = 1$. The Hamiltonian of the SYK model is [48]–[50]

$$H_K^{\text{SYK}} = \frac{1}{\sqrt{\binom{N}{4}}} \sum_{1 \le j < k < l < m \le N} K_{jklm} \chi_j \chi_k \chi_l \chi_m. \qquad (16)$$

Let $A \subset \{1, 2, \ldots, N\}$ with $|A|$ even so that $A \sqcup \bar{A}$ defines a bipartition of the system. Assume without loss of generality that $|A| \le N/2$. Let $\mathbb{E}_{|A|=n}$ denote averaging over all subsystems of size n. There are $\binom{N}{n}$ such subsystems.

Theorem 6: Initialize the system in a state $|\psi\rangle$ such that a constant fraction of the expectation values $\{\langle\psi|\chi_j\chi_k\chi_l\chi_m|\psi\rangle\}_{1 \le j < k < l < m \le N}$ are non-vanishing, i.e.,

$$|\{(j,k,l,m) : |\langle\psi|\chi_j\chi_k\chi_l\chi_m|\psi\rangle| = \Theta(1)\}| = \Theta(N^4). \quad (17)$$

Let

$$\rho_{K,A}(t_K) = \text{tr}_{\bar{A}}(e^{-iH_K^{\text{SYK}}t_K} |\psi\rangle \langle\psi| e^{iH_K^{\text{SYK}}t_K}) \qquad (18)$$

be the reduced density matrix of subsystem A at time t_K. For $n \ge 4$,

$$\mathbb{E}_{K} \sup_{t_K \in \mathbb{R}} \mathbb{E}_{|A|=n} S(\rho_{K,A}(t_K)) = \frac{n \ln 2}{2} - \Omega(n^4/N^4). \qquad (19)$$

Unfortunately, not all product states satisfy Eq. (17). It is not difficult to see that the product states defined in Ref. [60] are counterexamples. One might expect that a Haar-random product state, if properly defined, satisfies Eq. (17) with overwhelming probability.

In fermionic systems, defining a Haar-random product state is tricky. Since the Hamiltonian (16) conserves fermion parity, the Hilbert space is split into an even sector and an odd sector, which do not interact with each other. It is controversial whether to allow the superposition of states from both sectors. While being compatible with the axioms of quantum mechanics, such a superposition is widely believed to be unphysical. On the other hand, it is not clear how to define a Haar-random product state with definite fermion parity. The statement of Theorem 6 avoids the controversy and related technical difficulties by introducing the condition (17) instead of claiming $|\psi\rangle$ to be a Haar-random product state.

IV. PROOFS

This section consists of four subsections. Subsections IV-A, IV-B, IV-C, IV-D use the notations of Subsections III-A, III-B, III-C, III-D, respectively.

A. Proof of Theorem 2

Lemma 1: For a (possibly mixed) density matrix ρ, let $\rho_A = \text{tr}_{\bar{A}}\rho$ be the reduced density matrix of subsystem A. For $n > 1$,

$$\mathbb{E}_{|A|=n} S(\rho_A) \le \frac{n}{2} \mathbb{E}_{|A|=2} S(\rho_A). \qquad (20)$$

Proof: Using the subadditivity [61] of the von Neumann entropy,

$$\mathbb{E}_{|A|=n} S(\rho_A) \le \mathbb{E}_{|A|=n-2} S(\rho_A) + \mathbb{E}_{|A|=2} S(\rho_A), \qquad (21)$$

$$\mathbb{E}_{|A|=1} S(\rho_A) \ge \frac{1}{2} \mathbb{E}_{|A|=2} S(\rho_A). \qquad (22)$$

Using the strong subadditivity [62],

$$\mathbb{E}_{|A|=3} S(\rho_A) \le 2 \mathbb{E}_{|A|=2} S(\rho_A) - \mathbb{E}_{|A|=1} S(\rho_A). \qquad (23)$$

Combining these inequalities, we obtain (20). ∎

Lemma 2 [63]: Let ρ_j be a density matrix of qubits at positions j and $j+1$ such that

$$|\text{tr}(\rho_j H_j)| \ge \epsilon_j \|H_j\| \qquad (24)$$

for some $\epsilon_j \ge 0$. Then,

$$S(\rho_j) \le 2\ln 2 - \epsilon_j^2/2. \qquad (25)$$

Proof: We include the proof of this lemma for completeness. Let I_4 be the identity matrix of order 4. Let $\|X\|_1 := \text{tr}\sqrt{X^\dagger X}$ denote the trace norm. Since H_j is traceless, ϵ_j provides a lower bound on the deviation of ρ_j from the maximally mixed state:

$$\epsilon_j \le |\text{tr}(\rho_j H_j)|/\|H_j\| = |\text{tr}((\rho_j - I_4/4)H_j)|/\|H_j\|$$

$$\le \|\rho_j - I_4/4\|_1 = \sum_{i=1}^{4} |\lambda_i - 1/4|, \quad (26)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of ρ_j. An upper bound on $S(\rho_j)$ is $\max\{-\sum_{i=1}^{4} p_i \ln p_i\}$ subject to the constraints

$$\sum_{i=1}^{4} p_i = 1, \quad \sum_{i=1}^{4} |p_i - 1/4| \ge \epsilon_j. \qquad (27)$$

Since the Shannon entropy is Schur concave, it suffices to consider the following three cases:

- $p_1 = p_2 = 1/4 + \epsilon_j/4$, $p_3 = p_4 = 1/4 - \epsilon_j/4$;
- $p_1 = 1/4 + \epsilon_j/2$, $p_2 = p_3 = p_4 = 1/4 - \epsilon_j/6$;
- (if $\epsilon_j \leq 1/2$) $p_1 = 1/4 - \epsilon_j/2$, $p_2 = p_3 = p_4 = 1/4 + \epsilon_j/6$.

In all these cases, by Taylor expansion we can prove

$$-\sum_{i=1}^{4} p_i \ln p_i \leq 2\ln 2 - \epsilon_j^2/2 \qquad (28)$$

for $\epsilon_j \ll 1$. We have checked numerically that this inequality remains valid for any $\epsilon_j \leq 1$. ∎

Lemma 3: For a Haar-random product state $|\Psi\rangle$ (Definition 2),

$$\Pr_{\Psi}(|\langle\Psi|H^{\text{lat}}|\Psi\rangle| = \Omega(\sqrt{N})) \geq 1 - \delta. \qquad (29)$$

Proof: For $j = 2, 3, \ldots, N$, we assume that the expansion of H_j in the Pauli basis does not contain any terms acting only on the qubit at position j (this is without loss of generality since such terms can be included in H_{j-1}). Under this assumption, it is easy to see that

$$\mathop{\mathbb{E}}_{\Psi_{j+1}} \langle\Psi_j\Psi_{j+1}|H_j|\Psi_j\Psi_{j+1}\rangle = 0 \qquad (30)$$

for any $|\Psi_j\rangle$. Thus, $\{\langle\Psi_j\Psi_{j+1}|H_j|\Psi_j\Psi_{j+1}\rangle\}_{j=1}^{N-1}$ is a martingale difference sequence, and (29) follows from the martingale central limit theorem. ∎

We are ready to prove Theorem 2. Let

$$\rho = e^{-iH^{\text{lat}}t}|\Psi\rangle\langle\Psi|e^{iH^{\text{lat}}t}, \quad \epsilon_j = |\text{tr}(\rho H_j)|/\|H_j\| \qquad (31)$$

so that

$$\sum_{j=1}^{N} \epsilon_j = \sum_{j=1}^{N} \Theta(|\text{tr}(\rho H_j)|) = \Omega(1)\left|\sum_{j=1}^{N} \text{tr}(\rho H_j)\right|$$
$$= \Omega(|\text{tr}(\rho H^{\text{lat}})|) = \Omega(|\langle\Psi|H^{\text{lat}}|\Psi\rangle|). \qquad (32)$$

Note that ρ, ϵ_j are functions of time and should carry t as an argument, which is omitted for notational simplicity. Using Lemmas 1, 2, the RMS-AM inequality, and Eq. (32) sequentially,

$$\mathop{\mathbb{E}}_{|A|=n} S(\rho_A) \leq \frac{n}{2} \mathop{\mathbb{E}}_{|A|=2} S(\rho_A) \leq \frac{n}{2N} \sum_{j=1}^{N} (2\ln 2 - \epsilon_j^2/2)$$

$$\leq n\ln 2 - \frac{n}{4N^2}\left(\sum_{j=1}^{N} \epsilon_j\right)^2 = n\ln 2 - \frac{n\Omega(\langle\Psi|H^{\text{lat}}|\Psi\rangle^2)}{N^2}. \qquad (33)$$

We complete the proof of Theorem 2 by combining this inequality with Lemma 3.

B. Proof of Theorem 4

The following lemmas are analogues of Lemmas 1, 2, 3, respectively.

Lemma 4: For a (possibly mixed) density matrix ρ, let $\rho_{A_j} = \text{tr}_{\bar{A}_j} \rho$ be the reduced density matrix of subsystem A_j, and ρ_k be that of qubit k. Then,

$$\frac{1}{m} \sum_{j=1}^{m} S(\rho_{A_j}) \leq \frac{n}{N} \sum_{k=1}^{N} S(\rho_k). \qquad (34)$$

Lemma 5: Let ρ_j be a density matrix of qubit j. Then,

$$S(\rho_j) \leq \ln 2 - \text{tr}^2(\rho_j\sigma_j^z)/2. \qquad (35)$$

Lemma 6: For a Haar-random product state $|\Psi\rangle$,

$$\Pr_{\Psi}(|\langle\Psi|\sigma^z|\Psi\rangle| = \Omega(\sqrt{N})) \geq 1 - \delta. \qquad (36)$$

Let $\rho = U(t)|\Psi\rangle\langle\Psi|U^\dagger(t)$. Theorem 4 can be proved in almost the same way as Theorem 2 by replacing H_j, H^{lat} with σ_j^z, σ^z, respectively.

C. Proof of Theorem 5

1) Proof Overview: We observe that all product states satisfy the energy condition (50), which is preserved under time evolution. To obtain an upper bound on the left-hand side of Eq. (15), we maximize the average subsystem entropy subject to the energy constraint (50). Since the thermal state maximizes the von Neumann entropy for a given energy, we assign a temperature to each subsystem for each disorder realization of the Hamiltonian (13). Lemma 11 implies that in order to maximize the average subsystem entropy, all these temperatures must have the same absolute value. Finally, we upper bound the average subsystem entropy using the thermodynamic relation (Lemma 9) between energy and entropy.

2) Complete Proof: We start with the spectral and thermodynamic properties of the spin-glass model (13).

Lemma 7: For any positive integer k,

$$\frac{1}{2^{2N}} \mathop{\mathbb{E}}_{J} \text{tr}^2((H_J^{\text{sg}})^k) \leq \frac{1}{2^N} \mathop{\mathbb{E}}_{J} \text{tr}((H_J^{\text{sg}})^{2k}) \leq (2k-1)!!. \qquad (37)$$

Proof: The first step follows from the RMS-AM inequality. The second step can be proved in the same way as (35) of Ref. [64]. ∎

Let

$$\varrho_J(\beta) := e^{-\beta H_J^{\text{sg}}} / \text{tr}\, e^{-\beta H_J^{\text{sg}}} \qquad (38)$$

be the thermal state of H_J^{sg} at inverse temperature β. Define a measure on $\mathbb{R}^{d_N}$ such that

$$\int_{\mathcal{J}} dJ = \Pr(J \in \mathcal{J}), \quad \forall \mathcal{J} \subseteq \mathbb{R}^{d_N}. \qquad (39)$$

For an arbitrary bipartition of $\mathbb{R}^{d_N} = \mathcal{J}^+ \sqcup \mathcal{J}^-$, let

$$\mathcal{E}(\beta) := \int_{\mathcal{J}^+} \text{tr}(\varrho_J(\beta)H_J^{\text{sg}})\, dJ - \int_{\mathcal{J}^-} \text{tr}(\varrho_J(-\beta)H_J^{\text{sg}})\, dJ \qquad (40)$$

so that $\mathcal{E}(0) = 0$ and that $\mathcal{E}$ is strictly monotonically decreasing.

Lemma 8: For $-c \leq \beta \leq 0$ with a small constant $c = \Theta(1)$,

$$\mathcal{E}(\beta) \leq -\beta + O(\beta^2). \qquad (41)$$

Proof: Since H_J^{sg} is traceless,

$$\operatorname{tr} e^{-\beta H_J^{\text{sg}}} \geq 2^N, \quad \forall J. \tag{42}$$

Using (42), Lemma 7, and the RMS-AM inequality,

$$
\begin{aligned}
&\int_{\mathcal{J}+} \operatorname{tr}(\varrho_J(\beta) H_J^{\text{sg}}) \, \mathrm{d}J - \int_{\mathcal{J}-} \operatorname{tr}(\varrho_J(-\beta) H_J^{\text{sg}}) \, \mathrm{d}J \\
&\leq \int_{\mathcal{J}+} \frac{\operatorname{tr}(e^{-\beta H_J^{\text{sg}}} H_J^{\text{sg}})}{2^N} \, \mathrm{d}J - \int_{\mathcal{J}-} \frac{\operatorname{tr}(e^{\beta H_J^{\text{sg}}} H_J^{\text{sg}})}{2^N} \, \mathrm{d}J \\
&= \sum_{k=0}^{+\infty} \int_{\mathcal{J}+} \frac{(-\beta)^k \operatorname{tr}((H_J^{\text{sg}})^{k+1})}{k! 2^N} \, \mathrm{d}J \\
&\quad - \sum_{k=0}^{+\infty} \int_{\mathcal{J}-} \frac{\beta^k \operatorname{tr}((H_J^{\text{sg}})^{k+1})}{k! 2^N} \, \mathrm{d}J \\
&= -\beta \sum_{k=0}^{+\infty} \frac{\beta^{2k} \, \mathbb{E}_J \operatorname{tr}((H_J^{\text{sg}})^{2k+2})}{(2k+1)! 2^N} \\
&\quad + \sum_{k=1}^{+\infty} \frac{\beta^{2k}}{(2k)! 2^N} \left(\int_{\mathcal{J}+} - \int_{\mathcal{J}-} \right) \operatorname{tr}((H_J^{\text{sg}})^{2k+1}) \, \mathrm{d}J \\
&\leq -\beta e^{\beta^2/2} + \sum_{k=1}^{+\infty} \frac{\beta^{2k}}{(2k)! 2^N} \sqrt{\mathbb{E}_J \operatorname{tr}^2((H_J^{\text{sg}})^{2k+1})} \\
&\leq -\beta e^{\beta^2/2} + \sum_{k=1}^{+\infty} \frac{\beta^{2k} \sqrt{(4k+1)!!}}{(2k)!} = -\beta + O(\beta^2). \tag{43}
\end{aligned}
$$

Let

$$\mathcal{S}(\beta) := \int_{\mathcal{J}+} S(\varrho_J(\beta)) \, \mathrm{d}J + \int_{\mathcal{J}-} S(\varrho_J(-\beta)) \, \mathrm{d}J \tag{44}$$

so that $\mathcal{S}(0) = N \ln 2$ and that $\mathcal{S}$ is strictly monotonically increasing (decreasing) for negative (positive) β.

Lemma 9: For β such that $0 \leq \mathcal{E}(\beta) = O(1)$,

$$\mathcal{S}(\beta) = N \ln 2 - \Omega((\mathcal{E}(\beta))^2). \tag{45}$$

Proof: Lemma 8 implies that

$$\beta = -\Omega(\mathcal{E}(\beta)). \tag{46}$$

Combining this with the thermodynamic relation

$$\mathrm{d}\mathcal{S}(\beta)/\mathrm{d}\beta = \beta \mathrm{d}\mathcal{E}(\beta)/\mathrm{d}\beta \implies \mathrm{d}\mathcal{S}(\beta)/\mathrm{d}\mathcal{E}(\beta) = \beta, \tag{47}$$

we obtain Eq. (45). ∎

We are ready to prove Theorem 5. Recall that $|\psi\rangle = \bigotimes_{j=1}^N |\psi_j\rangle$ is an arbitrary (deterministic) product state. Let

$$\mathcal{J}_+ := \{J : \langle\psi|H_J^{\text{sg}}|\psi\rangle > 0\}, \quad \mathcal{J}_- := \{J : \langle\psi|H_J^{\text{sg}}|\psi\rangle < 0\}. \tag{48}$$

$\mathcal{J}_+$ and $\mathcal{J}_-$ have the same volume as $J \in \mathcal{J}_+$ if and only if $-J \in \mathcal{J}_-$. Moreover, the complement of $\mathcal{J}_+ \sqcup \mathcal{J}_-$ has measure zero. Hence,

$$\mathbb{E}_J = \frac{1}{2} \mathbb{E}_{J \in \mathcal{J}_+} + \frac{1}{2} \mathbb{E}_{J \in \mathcal{J}_-}. \tag{49}$$

Lemma 10:

$$\mathbb{E}_J |\langle\psi|H_J^{\text{sg}}|\psi\rangle| = \Theta(1). \tag{50}$$

Proof: It follows from observation that

$$
\begin{aligned}
\langle\psi|H_J^{\text{sg}}|\psi\rangle &= 1/\sqrt{d_N} \\
&\times \sum_{1 \leq j < k \leq N} \sum_{l,m \in \{x,y,z\}} J_{jklm} \langle\psi_j|\sigma_j^l|\psi_j\rangle \langle\psi_k|\sigma_k^m|\psi_k\rangle \tag{51}
\end{aligned}
$$

is the sum of $\Theta(N^2)$ independent Gaussian random variables divided by $\Theta(N)$. ∎

Let

$$H_{J,A}^{\text{sg}} = \frac{1}{\sqrt{d_{|A|}}} \sum_{j,k \in A; j < k} \sum_{l,m \in \{x,y,z\}} J_{jklm} \sigma_j^l \sigma_k^m. \tag{52}$$

Since $\sqrt{d_{|A|}/d_N} H_{J,A}^{\text{sg}}$ is the restriction of H_J^{sg} to subsystem A,

$$H_J^{\text{sg}} = \sqrt{d_N/d_n} \, \mathbb{E}_{|A|=n} H_{J,A}^{\text{sg}} \otimes I_{\bar{A}}, \tag{53}$$

where $I_{\bar{A}}$ is the identity operator on $\bar{A}$. Combining Eq. (53) with Eq. (49) and Lemma 10,

$$
\begin{aligned}
&\mathbb{E}_{J \in \mathcal{J}_+} \mathbb{E}_{|A|=n} \operatorname{tr}(\rho_{J,A}(t_J) H_{J,A}^{\text{sg}}) \\
&- \mathbb{E}_{J \in \mathcal{J}_-} \mathbb{E}_{|A|=n} \operatorname{tr}(\rho_{J,A}(t_J) H_{J,A}^{\text{sg}}) = \Theta(n/N). \tag{54}
\end{aligned}
$$

An upper bound on the left-hand side of Eq. (15) can be obtained as follows. For each tuple (J, A), we introduce a density matrix $\varrho_{J,A}$ supported on A. Since $\rho_{J,A}(t_J)$ and $\varrho_{J,A}$ are not related to each other, we use different fonts for rho to avoid confusion. We maximize $\mathbb{E}_J \mathbb{E}_{|A|=n} S(\varrho_{J,A})$ subject to the constraint

$$
\begin{aligned}
&\mathbb{E}_{J \in \mathcal{J}_+} \mathbb{E}_{|A|=n} \operatorname{tr}(\varrho_{J,A} H_{J,A}^{\text{sg}}) - \mathbb{E}_{J \in \mathcal{J}_-} \mathbb{E}_{|A|=n} \operatorname{tr}(\varrho_{J,A} H_{J,A}^{\text{sg}}) \\
&= \Theta(n/N). \tag{55}
\end{aligned}
$$

Lemma 11 below implies that the maximum is achieved when

$$\varrho_{J,A} = e^{\mp \beta H_{J,A}^{\text{sg}}} / \operatorname{tr} e^{\mp \beta H_{J,A}^{\text{sg}}} \tag{56}$$

is the thermal state of $H_{J,A}^{\text{sg}}$ at inverse temperature $\pm\beta$ for $J \in \mathcal{J}_\pm$, respectively.

Lemma 11: Let M be a positive integer and E be a real number. For $i = 1, 2, \ldots, M$, let G_i be a Hamiltonian on the Hilbert space $\mathcal{H}_i$, and ϱ_i be a density matrix on $\mathcal{H}_i$. The maximum average entropy $\sum_{i=1}^M S(\varrho_i)/M$ subject to the constraint

$$\frac{1}{M} \sum_{i=1}^M \operatorname{tr}(\varrho_i G_i) = E \tag{57}$$

is achieved when every $\varrho_i = e^{-\beta G_i} / \operatorname{tr} e^{-\beta G_i}$ is a thermal state at the same temperature, and the inverse temperature β can be obtained by solving the constraint (57).

Proof: Let $\varrho := \bigotimes_{i=1}^M \varrho_i$ be a density matrix on the Hilbert space $\mathcal{H} := \bigotimes_{i=1}^M \mathcal{H}_i$, and

$$G := \sum_{i=1}^M I^{\otimes(i-1)} \otimes G_i \otimes I^{\otimes(M-i)} \tag{58}$$

be a Hamiltonian on $\mathcal{H}$ so that

$$\operatorname{tr}(\varrho G) = \sum_{i=1}^M \operatorname{tr}(\varrho_i G_i) = ME. \tag{59}$$

The von Neumann entropy is additive: $S(\varrho) = \sum_{i=1}^{M} S(\varrho_i)$. To maximize $S(\varrho)$, ϱ must be a thermal state of G [65]:

$$\varrho = e^{-\beta G} / \operatorname{tr} e^{-\beta G} = \bigotimes_{i=1}^{M} e^{-\beta G_i} / \operatorname{tr} e^{-\beta G_i}. \quad (60)$$

Thus, each ϱ_i is a thermal state of G_i at the same inverse temperature β. ∎

Since $H_{J,A}^{\mathrm{sg}}$ is traceless, $\operatorname{tr}(e^{-\beta H_{J,A}^{\mathrm{sg}}} H_{J,A}^{\mathrm{sg}})$ is positive (negative) for negative (positive) β. Substituting Eq. (56) into Eq. (55), we see that the solution β is negative. Since $H_{J,A}^{\mathrm{sg}}$ is a spin-glass Hamiltonian for a system of n spins, Lemma 9 implies that

$$\mathbb{E}_{J} \mathbb{E}_{|A|=n} S(\varrho_{J,A}) = n \ln 2 - \Omega((n/N)^2). \quad (61)$$

We complete the proof of Theorem 5 by noting that the left-hand side of Eq. (61) is an upper bound on $\mathbb{E}_J \mathbb{E}_{|A|=n} S(\rho_{J,A}(t_J))$ for any $\{t_J \in \mathbb{R}\}_J$.

D. Proof of Theorem 6

Theorem 6 can be proved in almost the same way as Theorem 5. As an analogue of Lemma 10,

$$\mathbb{E}_{K} |\langle \psi | H_K^{\mathrm{SYK}} | \psi \rangle| = \Theta(1) \quad (62)$$

follows from Eq. (17). Moreover, "n/N" in Eqs. (54), (55), (61) and "$n \ln 2$" in Eq. (61) should be modified to n^2/N^2 and $n(\ln 2)/2$, respectively.

APPENDIX
PROOF OF THEOREM 3

Let $\{|j\rangle\}_{j=1}^{2^N}$ be a complete set of eigenstates of H^{lat} and $\varrho_{j,A} := \operatorname{tr}_{\bar{A}} |j\rangle\langle j|$ be the reduced density matrix of subsystem A. The energy basis $\{|j\rangle\}$ is unambiguously defined. This is because the non-degenerate gap condition (9) implies that all eigenvalues of H^{lat} are distinct. Recall that n is the number of qubits in A.

Lemma 12: For $n = O(1)$,

$$\frac{1}{2^N} \sum_{j=1}^{2^N} S(\varrho_{j,A}) = n \ln 2 - O(1/N). \quad (63)$$

Proof: Using the monotonicity of the Rényi entropy and Theorem 1 in Ref. [7],

$$\frac{1}{2^N} \sum_{j=1}^{2^N} S(\varrho_{j,A}) \geq -\frac{1}{2^N} \sum_{j=1}^{2^N} \ln \operatorname{tr}(\varrho_{j,A}^2)$$

$$\geq -\ln\left(\frac{1}{2^N} \sum_{j=1}^{2^N} \operatorname{tr}(\varrho_{j,A}^2)\right) \geq -\ln\left(\frac{1}{2^n} + \frac{2^n}{N}\right)$$

$$= n \ln 2 - O(1/N). \quad (64)$$

∎

The effective dimension of $|\Psi\rangle$ is defined as

$$1/D_\Psi^{\mathrm{eff}} = \sum_{j=1}^{2^N} |\langle j | \Psi \rangle|^4. \quad (65)$$

Lemma 13 [43]:

$$\Pr_{\Psi}(D_\Psi^{\mathrm{eff}} = e^{\Omega(N)}) = 1 - e^{-\Omega(N)}. \quad (66)$$

Let

$$\rho^\infty := \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \rho(t)\, dt, \quad \rho(t) := e^{-iH^{\mathrm{lat}}t} |\Psi\rangle\langle\Psi| e^{iH^{\mathrm{lat}}t} \quad (67)$$

be the infinite time average and $\rho_A^\infty := \operatorname{tr}_{\bar{A}} \rho^\infty$ be the reduced density matrix of subsystem A. Expanding $|\Psi\rangle$ in the energy basis, it is easy to see that

$$\rho^\infty = \sum_{j=1}^{2^N} p_j |j\rangle\langle j|, \quad p_j := |\langle j | \Psi \rangle|^2 \quad (68)$$

is the so-called diagonal ensemble. Since the spectrum of H^{lat} has non-degenerate gaps,

Lemma 14 ([66], [67]):

$$\lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \|\rho_A(t) - \rho_A^\infty\|_1\, dt \leq 2^n / \sqrt{D_\Psi^{\mathrm{eff}}}. \quad (69)$$

Lemma 15 (Continuity of the Von Neumann Entropy [68], [69]): Let $T := \|\rho - \rho'\|_1/2$ be the trace distance between two density matrices ρ, ρ' on the Hilbert space $\mathbb{C}^D$. Then,

$$|S(\rho) - S(\rho')| \leq T \ln(D-1) - T \ln T - (1-T) \ln(1-T). \quad (70)$$

Since by definition $0 \leq T \leq 1$, the right-hand side of this inequality is well defined.

We are ready to prove Theorem 3. Lemmas 13, 14 imply that

$$\lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau \mathbb{E}_\Psi \|\rho_A(t) - \rho_A^\infty\|_1\, dt = e^{-\Omega(N)}. \quad (71)$$

Markov's inequality implies that

$$\Pr_{t \in [0,\tau]} \left(\mathbb{E}_\Psi \|\rho_A(t) - \rho_A^\infty\|_1 = e^{-\Omega(N)}\right) = 1 - e^{-\Omega(N)} \quad (72)$$

for sufficiently large τ. Due to the continuity of the von Neumann entropy (Lemma 15),

$$\mathbb{E}_\Psi \|\rho_A(t) - \rho_A^\infty\|_1 = e^{-\Omega(N)}$$

$$\implies \left|\mathbb{E}_\Psi S(\rho_A(t)) - \mathbb{E}_\Psi S(\rho_A^\infty)\right| \leq \mathbb{E}_\Psi |S(\rho_A(t)) - S(\rho_A^\infty)|$$

$$= e^{-\Omega(N)}. \quad (73)$$

Using the concavity of the von Neumann entropy and Lemma 12,

$$\mathbb{E}_\Psi S(\rho_A^\infty) = \mathbb{E}_\Psi S\left(\sum_{j=1}^{2^N} p_j \varrho_{j,A}\right) \geq \sum_{j=1}^{2^N} \mathbb{E}_\Psi p_j S(\varrho_{j,A})$$

$$= \frac{1}{2^N} \sum_{j=1}^{2^N} S(\varrho_{j,A}) = n \ln 2 - O(1/N). \quad (74)$$

Equation (10) follows from (72), (73), and (74).

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