

Some Remarks on Non-Symmetric Interpolation Macdonald Polynomials

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We provide elementary identities relating the three known types of non-symmetric interpolation Macdonald polynomials. In addition we derive a duality for non-symmetric interpolation Macdonald polynomials. We consider some applications of these results, in particular to binomial formulas involving non-symmetric interpolation Macdonald polynomials.

1 Introduction

The symmetric interpolation Macdonald polynomials $R_\lambda(x; q, t) = R_\lambda(x_1, \dots, x_n; q, t)$ form a distinguished inhomogeneous basis for the algebra of symmetric polynomials in n variables over the field $\mathbb{F} := \mathbb{Q}(q, t)$. They were first introduced in [4, 13], building on joint work by one of the authors with Knop [5] and earlier work with Kostant [6, 7, 12]. These polynomials are indexed by the set of partitions with at most n parts

$$\mathcal{P}_n := \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0 \}.$$

For a partition $\mu \in \mathcal{P}_n$ we define $|\mu| = \mu_1 + \dots + \mu_n$ and write

$$\bar{\mu} = (q^{\mu_1} \tau_1, \dots, q^{\mu_n} \tau_n) \text{ where } \tau := (\tau_1, \dots, \tau_n) \text{ with } \tau_i := t^{1-i}.$$

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Then $R_\lambda(x) = R_\lambda(x; q, t)$ is, up to normalization, characterized as the unique nonzero symmetric polynomial of degree at most $|\lambda|$ satisfying the vanishing conditions

$$R_\lambda(\bar{\mu}) = 0 \text{ for } \mu \in \mathcal{P}_n \text{ such that } |\mu| \leq |\lambda|, \mu \neq \lambda.$$

The normalization is fixed by requiring that the coefficient of $x^\lambda := x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ in the monomial expansion of $R_\lambda(x)$ is 1. In spite of their deceptively simple definition, these polynomials possess some truly remarkable properties. For instance, as shown in [4, 13], the top homogeneous part of $R_\lambda(x)$ is the Macdonald polynomial $P_\lambda(x)$ [9] and $R_\lambda(x)$ satisfies the extra vanishing property $R_\lambda(\bar{\mu}) = 0$ unless $\lambda \subseteq \mu$ as Ferrer diagrams. Other key properties of $R_\lambda(x)$, which were proven by Okounkov [10], include the binomial theorem, which gives an explicit expansion of $R_\lambda(ax) = R_\lambda(ax_1, \dots, ax_n; q, t)$ in terms of the $R_\mu(x; q^{-1}, t^{-1})$'s over the field $\mathbb{K} := \mathbb{Q}(q, t, a)$, and the *duality* or *evaluation symmetry*, which involves the evaluation points

$$\tilde{\mu} = (q^{-\mu_n} \tau_1, \dots, q^{-\mu_1} \tau_n), \quad \mu \in \mathcal{P}_n$$

and takes the form

$$\frac{R_\lambda(a\tilde{\mu})}{R_\lambda(a\tau)} = \frac{R_\mu(a\tilde{\lambda})}{R_\mu(a\tau)}.$$

The interpolation polynomials have natural non-symmetric analogs $G_\alpha(x) = G_\alpha(x; q, t)$, which were also defined in [4, 13]. These are indexed by the set of compositions with at most n parts, $\mathcal{C}_n := (\mathbb{Z}_{\geq 0})^n$. For a composition $\beta \in \mathcal{C}_n$ we define

$$\bar{\beta} := w_\beta(\bar{\beta}_+),$$

where w_β is the shortest permutation such that $\beta_+ = w_\beta^{-1}(\beta)$ is a partition. Then $G_\alpha(x)$ is, up to normalization, characterized as the unique polynomial of degree at most $|\alpha| := \alpha_1 + \cdots + \alpha_n$ satisfying the vanishing conditions

$$G_\alpha(\bar{\beta}) = 0 \text{ for } \beta \in \mathcal{C}_n \text{ such that } |\beta| \leq |\alpha|, \beta \neq \alpha.$$

The normalization is fixed by requiring that the coefficient of $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in the monomial expansion of $G_\alpha(x)$ is 1.

Many properties of the symmetric interpolation polynomials $R_\lambda(x)$ admit non-symmetric counterparts for the $G_\alpha(x)$. For instance, the top homogeneous part of $G_\alpha(x)$

is the non-symmetric Macdonald polynomial $E_\alpha(x)$ and $G_\alpha(x)$ satisfies an extra vanishing property [4]. An analog of the binomial theorem, proved by one of us in [14, Thm. 1.1], gives an explicit expansion of $G_\alpha(ax; q, t)$ in terms of a 2nd family of interpolation polynomials $G'_\alpha(x) = G'_\alpha(x; q, t)$. These latter polynomials are characterized by having the same top homogeneous part as $G_\alpha(x)$, namely the non-symmetric polynomial $E_\alpha(x)$, and the following vanishing conditions at the evaluation points $\tilde{\beta} := \overline{(-w_0\beta)}$, with w_0 the longest element of the symmetric group S_n :

$$G'_\alpha(\tilde{\beta}) = 0 \text{ for } |\beta| < |\alpha|.$$

The 1st result of the present paper is a Demazure-type formula for the primed interpolation polynomials $G'_\alpha(x)$ in terms of $G_\alpha(x)$, which involves the symmetric group action on the algebra of polynomials in n variables over $\mathbb{F}$ by permuting the variables, as well as the associated Hecke algebra action in terms of Demazure-Lusztig operators H_w ($w \in S_n$) as described in the next section.

Theorem A. Write $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$. Then we have

$$G'_\alpha(t^{n-1}x; q^{-1}, t^{-1}) = t^{(n-1)|\alpha| - I(\alpha)} w_0 H_{w_0} G_\alpha(x; q, t).$$

This is restated and proved in Theorem 1 below.

The 2nd result is the following duality theorem for $G_\alpha(x)$, which is the non-symmetric analog of Okounkov's duality result.

Theorem B. For all compositions $\alpha, \beta \in \mathcal{C}_n$ we have

$$\frac{G_\alpha(a\tilde{\beta})}{G_\alpha(a\tau)} = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)}.$$

This is a special case of Theorem 17 below.

We now recall the interpolation O -polynomials introduced in [14, Thm. 1.1]. Write x^{-1} for $(x_1^{-1}, \dots, x_n^{-1})$. Then it was shown in [14, Thm. 1.1] that there exists a unique polynomial $O_\alpha(x) = O_\alpha(x; q, t; a)$ of degree at most $|\alpha|$ with coefficients in the field $\mathbb{K}$ such that

$$O_\alpha(\bar{\beta}^{-1}) = \frac{G_\beta(a\tilde{\alpha})}{G_\beta(a\tau)} \text{ for all } \beta.$$

Our 3rd result is a simple expression for the O -polynomials in terms of the interpolation polynomials $G_\alpha(x)$.

Theorem C. For all compositions $\alpha \in \mathcal{C}_n$ we have

$$O_\alpha(x) = \frac{G_\alpha(t^{1-n}aw_0x)}{G_\alpha(a\tau)}.$$

This is deduced in Proposition 22 below as a direct consequence of non-symmetric duality. We also obtain new proofs of Okounkov's [10] duality theorem, as well as the dual binomial theorem of Lascoux et al. [8], which gives an expansion of the primed-interpolation polynomials $G'_\alpha(x)$ in terms of the $G_\beta(ax)$'s.

2 Demazure-Lusztig Operators and the Primed Interpolation Polynomials

We use the notations from [14]. The correspondence with the notations from the other important references [4], [13] and [10] is listed in [14, Section 2] (directly after Lemma 2.8).

Let S_n be the symmetric group in n letters and $s_i \in S_n$ the permutation that swaps i and $i+1$. The s_i ($1 \leq i < n$) are Coxeter generators for S_n . Let $\ell : S_n \rightarrow \mathbb{Z}_{\geq 0}$ be the associated length function. Let S_n act on $\mathbb{Z}^n$ and $\mathbb{K}^n$ by $s_i v := (\dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots)$ for $v = (v_1, \dots, v_n)$. Write $w_0 \in S_n$ for the longest element, given explicitly by $i \rightarrow n+1-i$ for $i = 1, \dots, n$.

For $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$ define $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n) \in \mathbb{F}^n$ by $\bar{v}_i := q^{v_i} t^{-k_i(v)}$ with

$$k_i(v) := \#\{k < i \mid v_k \geq v_i\} + \#\{k > i \mid v_k > v_i\}.$$

If $v \in \mathbb{Z}^n$ has non-increasing entries $v_1 \geq v_2 \geq \dots \geq v_n$, then $\bar{v} = (q^{v_1} \tau_1, \dots, q^{v_n} \tau_n)$. For arbitrary $v \in \mathbb{Z}^n$ we have $\bar{v} = w_v(\bar{v}_+)$ with $w_v \in S_n$ the shortest permutation such that $v_+ := w_v^{-1}(v)$ has non-increasing entries, see [4, Section 2]. We write $\tilde{v} := \overline{-w_0 v}$ for $v \in \mathbb{Z}^n$.

Note that $\bar{\alpha}_n = t^{1-n}$ if $\alpha \in \mathcal{C}_n$ with $\alpha_n = 0$.

For a field F we write $F[x] := F[x_1, \dots, x_n]$, $F[x^{\pm 1}] := F[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $F(x)$ for the quotient field of $F[x]$. The symmetric group acts by algebra automorphisms on $\mathbb{F}[x]$ and $\mathbb{F}(x)$, with the action of s_i by interchanging x_i and x_{i+1} for $1 \leq i < n$. Consider the $\mathbb{F}$ -linear operators

$$H_i = ts_i - \frac{(1-t)x_i}{x_i - x_{i+1}}(1-s_i) = t + \frac{x_i - tx_{i+1}}{x_i - x_{i+1}}(s_i - 1)$$

on $\mathbb{F}(x)$ ($1 \leq i < n$) called Demazure-Lusztig operators, and the automorphism Δ of $\mathbb{F}(x)$ defined by

$$\Delta f(x_1, \dots, x_n) = f(q^{-1}x_n, x_1, \dots, x_{n-1}).$$

Note that H_i ($1 \leq i < n$) and Δ preserve $\mathbb{F}[x^{\pm 1}]$ and $\mathbb{F}[x]$. Cherednik [1, 2] showed that the operators H_i ($1 \leq i < n$) and Δ satisfy the defining relations of the type A extended affine Hecke algebra,

$$\begin{aligned} (H_i - t)(H_i + 1) &= 0, \\ H_i H_j &= H_j H_i, \quad |i - j| > 1, \\ H_i H_{i+1} H_i &= H_{i+1} H_i H_{i+1}, \\ \Delta H_{i+1} &= H_i \Delta, \\ \Delta^2 H_1 &= H_{n-1} \Delta^2 \end{aligned}$$

for all the indices such that both sides of the equation make sense (see also [4, Section 3]). For $w \in S_n$ we write $H_w := H_{i_1} H_{i_2} \cdots H_{i_\ell}$ with $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ a reduced expression for $w \in S_n$. It is well defined because of the braid relations for the H_i 's. Write $\overline{H}_i := H_i + 1 - t = tH_i^{-1}$ and set

$$\xi_i := t^{1-n} \overline{H}_{i-1} \cdots \overline{H}_1 \Delta^{-1} H_{n-1} \cdots H_i, \quad 1 \leq i \leq n. \quad (1)$$

The operators ξ_i 's are pairwise commuting invertible operators, with inverses

$$\xi_i^{-1} = \overline{H}_i \cdots \overline{H}_{n-1} \Delta H_1 \cdots H_{i-1}.$$

The ξ_i^{-1} ($1 \leq i \leq n$) are the Cherednik operators [2, 4].

The monic non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ of degree $\alpha \in \mathcal{C}_n$ is the unique polynomial satisfying

$$\xi_i^{-1} E_\alpha = \overline{\alpha}_i E_\alpha, \quad i = 1, \dots, n$$

and normalized such that the coefficient of x^α in E_α is 1.

Let ι be the field automorphism of $\mathbb{K}$ inverting q , t and a . It restricts to a field automorphism of $\mathbb{F}$, inverting q and t . We extend ι to a $\mathbb{Q}$ -algebra automorphism of $\mathbb{K}[x]$

and $\mathbb{F}[x]$ by letting ι act on the coefficients of the polynomial. Write

$$G_\alpha^\circ := \iota(G_\alpha), \quad E_\alpha^\circ := \iota(E_\alpha)$$

for $\alpha \in \mathcal{C}_n$. Note that $\bar{v}^{-1} = (\iota(\bar{v}_1), \dots, \iota(\bar{v}_n))$.

Put $H_i^\circ, H_w^\circ, \bar{H}_i^\circ, \Delta^\circ$ and ξ_i° for the operators $H_i, H_w, \bar{H}_i, \Delta$ and ξ_i with q, t replaced by their inverses. For instance,

$$H_i^\circ = t^{-1}s_i - \frac{(1-t^{-1})x_i}{x_i - x_{i+1}}(1-s_i),$$

$$\Delta^\circ f(x_1, \dots, x_n) = f(qx_n, x_1, \dots, x_{n-1}).$$

We then have $\xi_i^\circ E_\alpha^\circ = \bar{\alpha}_i E_\alpha^\circ$ for $i = 1, \dots, n$, which characterizes E_α° up to a scalar factor.

Theorem 1. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(x) = t^{(1-n)|\alpha|+I(\alpha)} w_0 H_{w_0}^\circ G_\alpha^\circ(t^{n-1}x) \quad (2)$$

with $I(\alpha) := \#\{i < j \mid \alpha_i \geq \alpha_j\}$.

Remark. Formally set $t = q^r$, replace x by $1 + (q-1)x$, divide both sides of (2) by $(q-1)^{|\alpha|}$ and take the limit $q \rightarrow 1$. Then

$$G'_\alpha(x; r) = (-1)^{|\alpha|} \sigma(w_0) w_0 G_\alpha(-x - (n-1)r; r) \quad (3)$$

for the non-symmetric interpolation Jack polynomial $G_\alpha(\cdot; r)$ and its primed version (see [14]). Here σ denotes the action of the symmetric group with $\sigma(s_i)$ the rational degeneration of the Demazure-Lusztig operators H_i , given explicitly by

$$\sigma(s_i) = s_i + \frac{r}{x_i - x_{i+1}}(1 - s_i),$$

see [14, Section 1]. To establish the formal limit (3) one uses that $\sigma(w_0)w_0 = w_0\sigma^\circ(w_0)$ with σ° the action of the symmetric group defined in terms of the rational degeneration

$$\sigma^\circ(s_i) = s_i - \frac{r}{x_i - x_{i+1}}(1 - s_i)$$

of H_i° . Formula (3) was obtained before in [14, Thm. 1.10].

Proof. We show that the right-hand side of (2) satisfies the defining properties of G'_α . For the vanishing property, note that

$$t^{n-1}w_0\tilde{\beta} = \bar{\beta}^{-1} \quad (4)$$

(this is the q -analog of [14, Lem. 6.1(2)]); hence,

$$(w_0H_{w_0}^\circ G_\alpha^\circ(t^{n-1}x))|_{x=\tilde{\beta}} = (H_{w_0}^\circ G_\alpha^\circ(x))|_{x=\bar{\beta}^{-1}}.$$

This expression is zero for $|\beta| < |\alpha|$ since it is a linear combination of the evaluated interpolation polynomials $G_\alpha^\circ(\overline{w\beta}^{-1})$ ($w \in S_n$) by [14, Lem. 2.1(2)].

It remains to show that the top homogeneous terms of both sides of (2) are the same, that is, that

$$E_\alpha = t^{I(\alpha)}w_0H_{w_0}^\circ E_\alpha^\circ. \quad (5)$$

Note that $\Psi := w_0H_{w_0}^\circ$ satisfies the intertwining properties

$$\begin{aligned} H_i\Psi &= t\Psi\bar{H}_i^\circ, \\ \Delta\Psi &= t^{n-1}\Psi\bar{H}_{n-1}^\circ \cdots \bar{H}_1^\circ(\Delta^\circ)^{-1}H_{n-1}^\circ \cdots H_1^\circ \end{aligned} \quad (6)$$

for $1 \leq i < n$ (use e.g., [2, Prop. 3.2.2]). It follows that $\xi_i^{-1}\Psi = \Psi\xi_i^\circ$ for $i = 1, \dots, n$. Therefore,

$$E_\alpha(x) = c_\alpha\Psi E_\alpha^\circ(x)$$

for some constant $c_\alpha \in \mathbb{F}$. But the coefficient of x^α in Ψx^α is $t^{-I(\alpha)}$; hence, $c_\alpha = t^{I(\alpha)}$. ■

Consider the Demazure operators H_i and the Cherednik operators ξ_j^{-1} as operators on the space $\mathbb{F}[x^{\pm 1}]$ of Laurent polynomials. For an integral vector $u \in \mathbb{Z}^n$, let $E_u \in \mathbb{F}[x^{\pm 1}]$ be the common eigenfunction of the Cherednik operators ξ_j^{-1} with eigenvalues $\bar{u}_j$ ($1 \leq j \leq n$), normalized such that the coefficient of $x^u := x_1^{u_1} \cdots x_n^{u_n}$ in E_u is 1. For $u = \alpha \in \mathcal{C}_n$ this definition reproduces the non-symmetric Macdonald polynomial $E_\alpha \in \mathbb{F}[x]$ as defined before. Note that

$$E_{u+(1^n)} = x_1 \cdots x_n E_u(x).$$

It is now easy to check that formula (5) is valid with α replaced by an arbitrary integral vector u ,

$$E_u = t^{I(u)} w_0 H_{w_0}^\circ E_u^\circ \quad (7)$$

with $E_u^\circ := \iota(E_u)$. Furthermore, one can show in the same vein as the proof of (5) that

$$w_0 E_{-w_0 u}(x^{-1}) = E_u(x)$$

for an integral vector u , where $p(x^{-1})$ stands for inverting all the parameters $x_1, \dots, x_n$ in the Laurent polynomial $p(x) \in \mathbb{F}[x^{\pm 1}]$. Combining this equality with (7) yields

$$E_{-w_0 u}(x^{-1}) = t^{I(u)} H_{w_0}^\circ E_u^\circ(x),$$

which is a special case of a known identity for non-symmetric Macdonald polynomials (see [2, Prop. 3.3.3]).

3 Evaluation Formulas

In [14, Thm. 1.1] the following combinatorial evaluation formula

$$G_\alpha(a\tau) = \prod_{s \in \alpha} \left(\frac{t^{1-n} - q^{a'(s)+1} t^{1-l'(s)}}{1 - q^{a(s)+1} t^{l(s)+1}} \right) \prod_{s \in \alpha} (at^{l'(s)} - q^{a'(s)}) \quad (8)$$

was obtained, with $a(s)$, $l(s)$, $a'(s)$ and $l'(s)$ the arm, leg, coarm and coleg of $s = (i, j) \in \alpha$, defined by

$$\begin{aligned} a(s) &:= \alpha_i - j, & l(s) &:= \#\{k > i \mid j \leq \alpha_k \leq \alpha_i\} + \#\{k < i \mid j \leq \alpha_k + 1 \leq \alpha_i\}, \\ a'(s) &:= j - 1, & l'(s) &:= \#\{k > i \mid \alpha_k > \alpha_i\} + \#\{k < i \mid \alpha_k \geq \alpha_i\}. \end{aligned}$$

By (8) we have

$$E_\alpha(\tau) = \lim_{a \rightarrow \infty} a^{-|\alpha|} G_\alpha(a\tau) = \prod_{s \in \alpha} \left(\frac{t^{1-n+l'(s)} - q^{a'(s)+1} t}{1 - q^{a(s)+1} t^{l(s)+1}} \right),$$

which is the well-known evaluation formula [1, 2] for the non-symmetric Macdonald polynomials. Note that for $\alpha \in \mathcal{C}_n$,

$$\ell(w_0) - I(\alpha) = \#\{i < j \mid \alpha_i < \alpha_j\}.$$

Lemma 2. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(a\tau) = t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G_\alpha^\circ(a\tau^{-1}).$$

Proof. Since $t^{n-1}w_0\tau = \tau^{-1} = \bar{0}^{-1}$ we have by Theorem 1,

$$\begin{aligned} G'_\alpha(a\tau) &= t^{(1-n)|\alpha|+I(\alpha)} (H_{w_0}^\circ G_\alpha^\circ)(a\bar{0}^{-1}) \\ &= t^{(1-n)|\alpha|+I(\alpha)-\ell(w_0)} G_\alpha^\circ(a\bar{0}^{-1}), \end{aligned}$$

where we have used [14, Lem. 2.1(2)] for the 2nd equality. ■

We now derive a relation between the evaluation formulas for $G_\alpha(x)$ and $G_\alpha^\circ(x)$. To formulate this we write, following [8],

$$n(\alpha) := \sum_{s \in \alpha} l(s), \quad n'(\alpha) := \sum_{s \in \alpha} a(s).$$

Note that $n'(\alpha) = \sum_{i=1}^n \binom{\alpha_i}{2}$; hence, it only depends on the S_n -orbit of α , while

$$n(\alpha) = n(\alpha^+) + \ell(w_0) - I(\alpha). \quad (9)$$

The following lemma is a non-symmetric version of the 1st displayed formula on [10, page 537].

Lemma 3. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(a\tau) = (-a)^{|\alpha|} t^{(1-n)|\alpha|-n(\alpha)} q^{n'(\alpha)} G_\alpha^\circ(a^{-1}\tau^{-1}).$$

Proof. This follows from the explicit evaluation formula (8) for the non-symmetric interpolation Macdonald polynomial G_α . ■

Following [8, (3.9)] we define $\tau_\alpha \in \mathbb{F}$ ($\alpha \in \mathcal{C}_n$) by

$$\tau_\alpha := (-1)^{|\alpha|} q^{n'(\alpha)} t^{-n(\alpha^+)}. \quad (10)$$

It only depends on the S_n -orbit of α .

Corollary 4. For $\alpha \in \mathcal{C}_n$ we have

$$G'_\alpha(a^{-1}\tau) = \tau_\alpha^{-1} a^{-|\alpha|} G_\alpha(a\tau).$$

Proof. Use Lemmas 2 and 3 and (9). ■

4 Normalized Interpolation Macdonald Polynomials

We need the basic representation of the (double) affine Hecke algebra on the space of $\mathbb{K}$ -valued functions on $\mathbb{Z}^n$, which is constructed as follows.

For $v \in \mathbb{Z}^n$ and $y \in \mathbb{K}^n$ write $v^\natural := (v_2, \dots, v_n, v_1 + 1)$ and $y^\natural := (y_2, \dots, y_n, qy_1)$. Denote the inverse of $^\natural$ by $^\sharp$, so $v^\sharp = (v_n - 1, v_1, \dots, v_{n-1})$ and $y^\sharp = (y_n/q, y_1, \dots, y_{n-1})$. We have the following lemma (cf. [4, 13, 14]).

Lemma 5. Let $v \in \mathbb{Z}^n$ and $1 \leq i < n$. Then we have

1. $s_i(\bar{v}) = \bar{s}_i \bar{v}$ if $v_i \neq v_{i+1}$.
2. $\bar{v}_i = t\bar{v}_{i+1}$ if $v_i = v_{i+1}$.
3. $\bar{v}^\natural = \overline{v^\sharp}$.

Let $\mathbb{H}$ be the double affine Hecke algebra over $\mathbb{K}$. It is isomorphic to the subalgebra of $\text{End}(\mathbb{K}[x^{\pm 1}])$ generated by the operators H_i ($1 \leq i < n$), $\Delta^{\pm 1}$, and the multiplication operators $x_j^{\pm 1}$ ($1 \leq j \leq n$).

For a unital $\mathbb{K}$ -algebra A we write $\mathcal{F}_A$ for the space of A -valued functions $f : \mathbb{Z}^n \rightarrow A$ on $\mathbb{Z}^n$.

Corollary 6. Let A be a unital $\mathbb{K}$ -algebra. Consider the A -linear operators $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}$ and $\widehat{x}_j$ ($1 \leq j \leq n$) on $\mathcal{F}_A$ defined by

$$\begin{aligned} (\widehat{H}_i f)(v) &:= tf(v) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}} (f(s_i v) - f(v)), \\ (\widehat{\Delta} f)(v) &:= f(v^\sharp), \quad (\widehat{\Delta}^{-1} f)(v) := f(v^\natural), \\ (\widehat{x}_j f)(v) &:= a\bar{v}_j f(v) \end{aligned} \tag{11}$$

for $f \in \mathcal{F}_A$ and $v \in \mathbb{Z}^n$. Then $H_i \mapsto \widehat{H}_i$ ($1 \leq i < n$), $\Delta \mapsto \widehat{\Delta}$ and $x_j \mapsto \widehat{x}_j$ ($1 \leq j \leq n$) defines a representation $\mathbb{H} \rightarrow \text{End}_A(\mathcal{F}_A)$, $X \mapsto \widehat{X}$ ($X \in \mathbb{H}$) of the double affine Hecke algebra $\mathbb{H}$ on $\mathcal{F}_A$.

Proof. Let $\mathcal{O} \subset \mathbb{K}^n$ be the smallest S_n -invariant and $\mathfrak{h}$ -invariant subset that contains $\{a\bar{v} \mid v \in \mathbb{Z}^n\}$. Note that $\mathcal{O}$ is contained in $\{y \in \mathbb{K}^n \mid y_i \neq y_j \text{ if } i \neq j\}$. The Demazure–Lusztig operators H_i ($1 \leq i < n$), $\Delta^{\pm 1}$ and the coordinate multiplication operators x_j ($1 \leq j \leq n$) act A -linearly on the space $F_A^{\mathcal{O}}$ of A -valued functions on $\mathcal{O}$, and hence turns $F_A^{\mathcal{O}}$ into an $\mathbb{H}$ -module. Define the surjective A -linear map

$$\text{pr} : F_A^{\mathcal{O}} \rightarrow \mathcal{F}_A$$

by $\text{pr}(g)(v) := g(a\bar{v})$ ($v \in \mathbb{Z}^n$).

We claim that $\text{Ker}(\text{pr})$ is an $\mathbb{H}$ -submodule of $F_A^{\mathcal{O}}$. Clearly $\text{Ker}(\text{pr})$ is x_j -invariant for $j = 1, \dots, n$. Let $g \in \text{Ker}(\text{pr})$. Part 3 of Lemma 5 implies that $\Delta g \in \text{Ker}(\text{pr})$. To show that $H_i g \in \text{Ker}(\text{pr})$ we consider two cases. If $v_i \neq v_{i+1}$ then $s_i \bar{v} = \bar{s}_i \bar{v}$ by part 1 of Lemma 5. Hence,

$$(H_i g)(a\bar{v}) = tg(a\bar{v}) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}}(g(a\bar{s}_i \bar{v}) - g(a\bar{v})) = 0.$$

If $v_i = v_{i+1}$ then $\bar{v}_i = t\bar{v}_{i+1}$ by part 2 of Lemma 5. Hence,

$$(H_i g)(\bar{v}) = tg(a\bar{v}) + \frac{\bar{v}_i - t\bar{v}_{i+1}}{\bar{v}_i - \bar{v}_{i+1}}(g(as_i \bar{v}) - g(a\bar{v})) = tg(a\bar{v}) = 0.$$

Hence, $\mathcal{F}_A$ inherits the $\mathbb{H}$ -module structure of $F_A^{\mathcal{O}}/\text{Ker}(\text{pr})$. It is a straightforward computation, using Lemma 5 again, to show that the resulting action of H_i ($1 \leq i < n$), Δ and x_j ($1 \leq j \leq n$) on $\mathcal{F}_A$ is by the operators $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}$ and $\widehat{x}_j$ ($1 \leq j \leq n$). ■

Remark 7. With the notations from (the proof of) Corollary 6, let $\tilde{g} \in F_A^{\mathcal{O}}$ and set $g := \text{pr}(\tilde{g}) \in \mathcal{F}_A$. In other words, $g(v) := \tilde{g}(a\bar{v})$ for all $v \in \mathbb{Z}^n$. Then

$$(\widehat{X}g)(v) = (X\tilde{g})(a\bar{v}), \quad v \in \mathbb{Z}^n$$

for $X = H_i, \Delta^{\pm 1}, x_j$.

Remark 8. Let $\mathcal{F}_A^+$ be the space of A -valued functions on $\mathcal{C}_n$. We sometimes will consider $\widehat{H}_i$ ($1 \leq i < n$), $\widehat{\Delta}^{-1}$ and $\widehat{x}_j$ ($1 \leq j \leq n$), defined by the formulas (11), as linear operators on $\mathcal{F}_A^+$.

Definition 9. We call

$$K_\alpha(x; q, t; a) := \frac{G_\alpha(x; q, t)}{G_\alpha(a\tau; q, t)} \in \mathbb{K}[x] \quad (12)$$

the normalized non-symmetric interpolation Macdonald polynomial of degree α .

We frequently use the shorthand notation $K_\alpha(x) := K_\alpha(x; q, t; a)$. We will see in a moment that formulas for non-symmetric interpolation Macdonald polynomials take the nicest form in this particular normalization.

Note that a cannot be specialized to 1 in (12) since $G_\alpha(\tau) = G_\alpha(\bar{0}) = 0$ if $\alpha \in \mathcal{C}_n$ is nonzero. Note furthermore that

$$\lim_{a \rightarrow \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)} \quad (13)$$

since $\lim_{a \rightarrow \infty} a^{-|\alpha|} G_\alpha(ax) = E_\alpha(x)$.

Recall from [4] the operator $\Phi = (x_n - t^{1-n})\Delta \in \mathbb{H}$ and the inhomogeneous Cherednik operators

$$\Xi_j = \frac{1}{x_j} + \frac{1}{x_j} H_j \cdots H_{n-1} \Phi H_1 \cdots H_{j-1} \in \mathbb{H}, \quad 1 \leq j \leq n.$$

The operators H_i , Ξ_j and Φ preserve $\mathbb{K}[x]$ (see [4]); hence, they give rise to $\mathbb{K}$ -linear operators on $\mathcal{F}_{\mathbb{K}[x]}^+$ (e.g., $(H_i f)(\alpha) := H_i(f(\alpha))$ for $\alpha \in \mathcal{C}_n$). Note that the operators H_i , Ξ_j and Φ on $\mathcal{F}_{\mathbb{K}[x]}^+$ commute with the hat-operators $\widehat{H}_i$, $\widehat{x}_j$ and $\widehat{\Delta}^{-1}$ on $\mathcal{F}_{\mathbb{K}[x]}^+$ (cf. Remark 8). The same remarks hold true for the space $\mathcal{F}_{\mathbb{K}(x)}$ of $\mathbb{K}(x)$ -valued functions on $\mathbb{Z}^n$ (in fact, in this case the hat-operators define a $\mathbb{H}$ -action on $\mathcal{F}_{\mathbb{K}(x)}$).

Let $K \in \mathcal{F}_{\mathbb{K}[x]}^+$ be the map $\alpha \mapsto K_\alpha(\cdot)$ ($\alpha \in \mathcal{C}_n$).

Lemma 10. For $1 \leq i < n$ and $1 \leq j \leq n$ we have in $\mathcal{F}_{\mathbb{K}[x]}^+$,

1. $H_i K = \widehat{H}_i K$.
2. $\Xi_j K = a \widehat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n} (a^2 \widehat{x}_1^{-1} - 1) \widehat{\Delta}^{-1} K$.

Proof. 1. To derive the formula we need to expand $H_i K_\alpha$ as a linear combination of the K_β 's. As a 1st step we expand $H_i G_\alpha$ as linear combination of the G_β 's.

If $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i < \alpha_{i+1}$ then

$$H_i G_\alpha(x) = \frac{(t-1)\bar{\alpha}_i}{\bar{\alpha}_i - \bar{\alpha}_{i+1}} G_\alpha(x) + G_{s_i \alpha}(x)$$

by [14, Lem. 2.2]. Using part 1 of Lemma 5 and the fact that H_i satisfies the quadratic relation $(H_i - t)(H_i + 1) = 0$, it follows that

$$H_i G_\alpha(x) = \frac{(t-1)\bar{\alpha}_i}{\bar{\alpha}_i - \bar{\alpha}_{i+1}} G_\alpha(x) + \frac{t(\bar{\alpha}_{i+1} - t\bar{\alpha}_i)(\bar{\alpha}_{i+1} - t^{-1}\bar{\alpha}_i)}{(\bar{\alpha}_{i+1} - \bar{\alpha}_i)^2} G_{s_i \alpha}(x)$$

if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i > \alpha_{i+1}$. Finally, $H_i G_\alpha(x) = t G_\alpha(x)$ if $\alpha \in \mathcal{C}_n$ satisfies $\alpha_i = \alpha_{i+1}$ by [4, Cor. 3.4].

An explicit expansion of $H_i K_\alpha$ as linear combination of the K_β 's can now be obtained using the formula

$$G_\alpha(a\tau) = \frac{\bar{\alpha}_{i+1} - t\bar{\alpha}_i}{\bar{\alpha}_{i+1} - \bar{\alpha}_i} G_{s_i\alpha}(a\tau)$$

for $\alpha \in \mathcal{C}_n$ satisfying $\alpha_i > \alpha_{i+1}$, cf. the proof of [14, Lem 3.1]. By a direct computation the resulting expansion formula can be written as $H_i K = \hat{H}_i K$.

2. See [4, Thm. 2.6].

3. Let $\alpha \in \mathcal{C}_n$. By [14, Lem. 2.2 (1)],

$$\Phi G_\alpha(x) = q^{-\alpha_1} G_{\alpha^\natural}(x).$$

By the evaluation formula (8) we have

$$\frac{G_{\alpha^\natural}(a\tau)}{G_\alpha(a\tau)} = at^{1-n+k_1(\alpha)} - q^{\alpha_1} t^{1-n}.$$

Hence,

$$\Phi K_\alpha(x) = t^{1-n} (a\bar{\alpha}_1^{-1} - 1) K_{\alpha^\natural}(x). \quad \blacksquare$$

Remark 11. Note that

$$\Phi K_\alpha(x) = (a\tilde{\alpha}_n - t^{1-n}) K_{\alpha^\natural}(x)$$

for $\alpha \in \mathcal{C}_n$ since $\bar{\alpha}^{-1} = t^{n-1} w_0 \tilde{\alpha}$.

5 Interpolation Macdonald Polynomials with Negative Degrees

In this section we give the natural extension of the interpolation Macdonald polynomials $G_\alpha(x)$ and $K_\alpha(x)$ to $\alpha \in \mathbb{Z}^n$. It will be the unique extension of $K \in \mathcal{F}_{\mathbb{K}[x]}^+$ to a map $K \in \mathcal{F}_{\mathbb{K}(x)}$ such that Lemma 10 remains valid.

Lemma 12. For $\alpha \in \mathcal{C}_n$ we have

$$G_\alpha(x) = q^{-|\alpha|} \frac{G_{\alpha+(1^n)}(qx)}{\prod_{i=1}^n (qx_i - t^{1-n})},$$

$$K_\alpha(x) = \left(\prod_{i=1}^n \frac{(1 - a\bar{\alpha}_i^{-1})}{(1 - qt^{n-1}x_i)} \right) K_{\alpha+(1^n)}(qx).$$

Proof. Note that for $f \in \mathbb{K}[x]$,

$$\Phi^n f(x) = \left(\prod_{i=1}^n (x_i - t^{1-n}) \right) f(q^{-1}x).$$

The 1st formula then follows by iteration of [14, Lem. 2.2(1)] and the 2nd formula from part 3 of Lemma 10. ■

For $m \in \mathbb{Z}_{\geq 0}$ we define $A_m(x; v) \in \mathbb{K}(x)$ by

$$A_m(x; v) := \prod_{i=1}^n \frac{(q^{1-m} a \bar{v}_i^{-1}; q)_m}{(qt^{n-1} x_i; q)_m} \quad \forall v \in \mathbb{Z}^n, \quad (14)$$

with $(y; q)_m := \prod_{j=0}^{m-1} (1 - q^j y)$ the q -shifted factorial.

Definition 13. Let $v \in \mathbb{Z}^n$ and write $|v| := v_1 + \dots + v_n$. Define $G_v(x) = G_v(x; q, t) \in \mathbb{F}(x)$ and $K_v(x) = K_v(x; q, t; a) \in \mathbb{K}(x)$ by

$$G_v(x) := q^{-m|v| - m^2 n} \frac{G_{v+(m^n)}(q^m x)}{\prod_{i=1}^n x_i^m (q^{-m} t^{1-n} x_i^{-1}; q)_m},$$

$$K_v(x) := A_m(x; v) K_{v+(m^n)}(q^m x),$$

where m is a nonnegative integer such that $v + (m^n) \in \mathcal{C}_n$ (note that G_v and K_v are well defined by Lemma 12).

Example 14. If $n = 1$ then for $m \in \mathbb{Z}_{\geq 0}$,

$$K_{-m}(x) = \frac{(qa; q)_m}{(qx; q)_m}, \quad K_m(x) = \left(\frac{x}{a} \right)^m \frac{(x^{-1}; q)_m}{(a^{-1}; q)_m}.$$

Lemma 15. For all $v \in \mathbb{Z}^n$,

$$K_v(x) = \frac{G_v(x)}{G_v(a\tau)}.$$

Proof. Let $v \in \mathbb{Z}^n$. Clearly $G_v(x)$ and $K_v(x)$ only differ by a multiplicative constant, so it suffices to show that $K_v(a\tau) = 1$. Fix $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n$. Then

$$K_v(a\tau) = A_m(a\tau; v) K_{v+(m^n)}(q^m a\tau) = A_m(a\tau; v) \frac{G_{v+(m^n)}(q^m a\tau)}{G_{v+(m^n)}(a\tau)} = 1,$$

where the last formula follows from a direct computation using the evaluation formula (8). ■

We extend the map $K : \mathcal{C}_n \rightarrow \mathbb{K}[x]$ to a map

$$K : \mathbb{Z}^n \rightarrow \mathbb{K}(x)$$

by setting $v \mapsto K_v(x)$ for all $v \in \mathbb{Z}^n$. Lemma 10 now extends as follows.

Proposition 16. We have, as identities in $\mathcal{F}_{\mathbb{K}(x)}$,

1. $H_i K = \widehat{H}_i K$.
2. $\Xi_j K = a \widehat{x}_j^{-1} K$.
3. $\Phi K = t^{1-n} (a^2 \widehat{x}_1^{-1} - 1) \widehat{\Delta}^{-1} K$.

Proof. Write $A_m \in \mathcal{F}_{\mathbb{K}(x)}$ for the map $v \mapsto A_m(x; v)$ for $v \in \mathbb{Z}^n$. Consider the linear operator on $\mathcal{F}_{\mathbb{K}(x)}$ defined by $(A_m f)(v) := A_m(x; v) f(v)$ for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$. For $1 \leq i < n$ we have $[H_i, A_m] = 0$ as linear operators on $\mathcal{F}_{\mathbb{K}(x)}$, since $A_m(x; v)$ is a symmetric rational function in $x_1, \dots, x_n$. Furthermore, for $v \in \mathbb{Z}^n$ and $f \in \mathcal{F}_{\mathbb{K}(x)}$,

$$(\widehat{H}_i \circ A_m) f(v) = (A_m \circ \widehat{H}_i) f(v) \quad \text{if } v_i \neq v_{i+1} \quad (15)$$

by part 2 of Lemma 5 and the fact that $A_m(x; v)$ is symmetric in $\bar{v}_1, \dots, \bar{v}_n$. Fix $v \in \mathbb{Z}^n$ and choose $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n$. Since

$$K_v(x) = A_m(x; v) K_{v+(m^n)}(q^m x)$$

we obtain from $[H_i, A_m] = 0$ and (15) that $(H_i K)(v) = (\widehat{H}_i K)(v)$ if $v_i \neq v_{i+1}$. This also holds true if $v_i = v_{i+1}$ since then $(\widehat{H}_i K)(v) = t K_v$ and $H_i K_{v+(m^n)}(q^m x) = t K_{v+(m^n)}(q^m x)$. This proves part 1 of the proposition.

Note that $\Phi K_v(x) = t^{1-n} (a \bar{v}_1^{-1} - 1) K_{v^\natural}(x)$ for arbitrary $v \in \mathbb{Z}^n$ by Lemma 10 and the commutation relation

$$\Phi \circ A_m = A_m \circ \Phi^{(q^m)}, \quad (16)$$

where $\Phi^{(q^m)} := (q^m x_n - t^{1-n}) \Delta$. This proves part 3 of the proposition.

Finally we have $\Xi_j K_v(x) = \bar{v}_j^{-1} K_v(x)$ for all $v \in \mathbb{Z}^n$ by $[H_i, A_m] = 0$, (16) and Lemma 10. This proves part 2 of the proposition. ■

6 Duality of the Non-Symmetric Interpolation Macdonald Polynomials

Recall the notation $\tilde{v} = \overline{-w_0 v}$ for $v \in \mathbb{Z}^n$.

Theorem 17. (Duality). For all $u, v \in \mathbb{Z}^n$ we have

$$K_u(a\tilde{v}) = K_v(a\tilde{u}). \quad (17)$$

Example 18. If $n = 1$ and $m, r \in \mathbb{Z}_{\geq 0}$ then

$$K_m(aq^{-r}) = q^{-mr} \frac{(a^{-1}; q)_{m+r}}{(a^{-1}; q)_m (a^{-1}; q)_r} \quad (18)$$

by the explicit expression for $K_m(x)$ from Example 14. The right-hand side of (18) is manifestly invariant under the interchange of m and r .

Proof. We divide the proof of the theorem in several steps. ■

Step 1. If $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $v \in \mathbb{Z}^n$ then $K_{s_i u}(a\tilde{v}) = K_v(a\widetilde{s_i u})$ for $v \in \mathbb{Z}^n$ and $1 \leq i < n$.

Proof of Step 1. Writing out the formula from part 1 of Proposition 16 gives

$$\begin{aligned} \frac{(t-1)\tilde{v}_i}{(\tilde{v}_i - \tilde{v}_{i+1})} K_u(a\tilde{v}) + \left(\frac{\tilde{v}_i - t\tilde{v}_{i+1}}{\tilde{v}_i - \tilde{v}_{i+1}} \right) K_u(a\widetilde{s_{n-i} v}) \\ = \frac{(t-1)\bar{u}_i}{(\bar{u}_i - \bar{u}_{i+1})} K_u(a\tilde{v}) + \left(\frac{\bar{u}_i - t\bar{u}_{i+1}}{\bar{u}_i - \bar{u}_{i+1}} \right) K_{s_i u}(a\tilde{v}). \end{aligned} \quad (19)$$

Replacing in (19) the role of u and v and replacing i by $n-i$ we get

$$\begin{aligned} \frac{(t-1)\tilde{u}_{n-i}}{(\tilde{u}_{n-i} - \tilde{u}_{n+1-i})} K_v(a\tilde{u}) + \left(\frac{\tilde{u}_{n-i} - t\tilde{u}_{n+1-i}}{\tilde{u}_{n-i} - \tilde{u}_{n+1-i}} \right) K_v(a\widetilde{s_i u}) \\ = \frac{(t-1)\bar{v}_{n-i}}{(\bar{v}_{n-i} - \bar{v}_{n+1-i})} K_v(a\tilde{u}) + \left(\frac{\bar{v}_{n-i} - t\bar{v}_{n+1-i}}{\bar{v}_{n-i} - \bar{v}_{n+1-i}} \right) K_{s_{n-i} v}(a\tilde{u}). \end{aligned} \quad (20)$$

Suppose that $s_{n-i} v = v$. Then $\bar{v}_{n-i} = t\bar{v}_{n+1-i}$ by the 2nd part of Lemma 5. Since $\tilde{v} = t^{1-n} w_0 \bar{v}^{-1}$, that is, $\tilde{v}_i = t^{1-n} \bar{v}_{n+1-i}^{-1}$, we then also have $\tilde{v}_i = t\tilde{v}_{i+1}$. It then follows by a direct computation that (19) reduces to $K_{s_i u}(a\tilde{v}) = K_u(a\tilde{v})$ and (20) to $K_v(a\widetilde{s_i u}) = K_v(a\tilde{u})$ if $s_{n-i} v = v$.

We now use these observations to prove Step 1. Assume that $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all v . We have to show that $K_{s_i u}(a\tilde{v}) = K_v(a\widetilde{s_i u})$ for all v . It is trivially true if $s_i u = u$, so we may assume that $s_i u \neq u$. Suppose that v satisfies $s_{n-i} v = v$. Then it follows from the previous paragraph that

$$K_{s_i u}(a\tilde{v}) = K_u(a\tilde{v}) = K_v(a\tilde{u}) = K_v(a\widetilde{s_i u}).$$

If $s_{n-i} v \neq v$ then (19) and the induction hypothesis can be used to write $K_{s_i u}(a\tilde{v})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_{s_{n-i} v}(a\tilde{u})$. Then (20) can be used to rewrite the term involving $K_{s_{n-i} v}(a\tilde{u})$ as an explicit linear combination of $K_v(a\tilde{u})$ and $K_v(a\widetilde{s_i u})$. Hence, we obtain an explicit expression of $K_{s_i u}(a\tilde{v})$ as linear combination of $K_v(a\tilde{u})$ and $K_v(a\widetilde{s_i u})$, which turns out to reduce to $K_{s_i u}(a\tilde{v}) = K_v(a\widetilde{s_i u})$ after a direct computation. ■

Step 2. $K_0(a\tilde{v}) = 1 = K_v(a\tilde{0})$ for all $v \in \mathbb{Z}^n$.

Proof of Step 2. Clearly $K_0(x) = 1$ and $K_v(a\tilde{0}) = K_v(a\tau) = 1$ for $v \in \mathbb{Z}^n$ by Lemma 15. ■

Step 3. $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for $v \in \mathbb{Z}^n$ and $\alpha \in \mathcal{C}_n$.

Proof of Step 3. We prove it by induction. It is true for $\alpha = 0$ by Step 2. Let $m \in \mathbb{Z}_{>0}$ and suppose that $K_\gamma(a\tilde{v}) = K_v(a\tilde{\gamma})$ for $v \in \mathbb{Z}^n$ and $\gamma \in \mathcal{C}_n$ with $|\gamma| < m$. Let $\alpha \in \mathcal{C}_n$ with $|\alpha| = m$.

We need to show that $K_\alpha(a\tilde{v}) = K_v(a\tilde{\alpha})$ for all $v \in \mathbb{Z}^n$. By Step 1 we may assume without loss of generality that $\alpha_n > 0$. Then $\gamma := \alpha^\sharp \in \mathcal{C}_n$ satisfies $|\gamma| = m - 1$, and $\alpha = \gamma^\natural$. Furthermore, note that we have the formula

$$(a\bar{v}_1^{-1} - 1)K_u(a\tilde{v}^\natural) = (a\bar{u}_1^{-1} - 1)K_{u^\natural}(a\tilde{v}) \quad (21)$$

for all $u, v \in \mathbb{Z}^n$, which follows by writing out the formula from part 3 of Lemma 16. Hence, we obtain

$$\begin{aligned} K_\alpha(a\tilde{v}) &= K_{\gamma^\natural}(a\tilde{v}) = \frac{(a\bar{v}_1^{-1} - 1)}{(a\bar{\gamma}_1^{-1} - 1)} K_\gamma(a\tilde{v}^\natural) \\ &= \frac{(a\bar{v}_1^{-1} - 1)}{(a\bar{\gamma}_1^{-1} - 1)} K_{v^\natural}(a\tilde{\gamma}) = K_v(a\tilde{\gamma}^\natural) = K_v(a\tilde{\alpha}), \end{aligned}$$

where we used the induction hypothesis for the 3rd equality and (21) for the 2nd and 4th equality. This proves the induction step. ■

Step 4. $K_u(a\tilde{v}) = K_v(a\tilde{u})$ for all $u, v \in \mathbb{Z}^n$.

Proof of Step 4. Fix $u, v \in \mathbb{Z}^n$. Let $m \in \mathbb{Z}_{\geq 0}$ such that $u + (m^n) \in \mathcal{C}_n$. Note that $q^m \tilde{v} = \widetilde{v - (m^n)}$ and $q^{-m} \tilde{u} = \widetilde{u + (m^n)}$. Then

$$\begin{aligned} K_u(a\tilde{v}) &= A_m(a\tilde{v}; u) K_{u+(m^n)}(q^m a\tilde{v}) \\ &= A_m(a\tilde{v}; u) K_{u+(m^n)}(a \widetilde{v - (m^n)}) \\ &= A_m(a\tilde{v}; u) K_{v-(m^n)}(a \widetilde{u + (m^n)}) \\ &= A_m(a\tilde{v}; u) K_{v-(m^n)}(q^{-m} a\tilde{u}) = A_m(a\tilde{v}; u) A_m(q^{-m} a\tilde{u}; v - (m^n)) K_v(a\tilde{u}), \end{aligned}$$

where we used Step 3 in the 3rd equality. The result now follows from the fact that

$$A_m(a\tilde{v}; u) A_m(q^{-m} a\tilde{u}; v - (m^n)) = 1,$$

which follows by a straightforward computation using (4). ■

7 Some Applications of Duality

7.1 Non-symmetric Macdonald polynomials

Recall that the (monic) non-symmetric Macdonald polynomial $E_\alpha(x)$ of degree α is the top homogeneous component of $G_\alpha(x)$, i.e.,

$$E_\alpha(x) = \lim_{a \rightarrow \infty} a^{-|\alpha|} G_\alpha(ax), \quad \alpha \in \mathcal{C}_n.$$

The normalized non-symmetric Macdonald polynomials are

$$\overline{K}_\alpha(x) := \lim_{a \rightarrow \infty} K_\alpha(ax) = \frac{E_\alpha(x)}{E_\alpha(\tau)}, \quad \alpha \in \mathcal{C}_n.$$

We write $\overline{K} \in \mathcal{F}_{\mathbb{F}[x]}^+$ for the resulting map $\alpha \mapsto \overline{K}_\alpha$. Taking limits in Lemma 10 we get the following.

Lemma 19. We have for $1 \leq i < n$ and $1 \leq j \leq n$,

1. $H_i \overline{K} = \widehat{H}_i \overline{K}$.
2. $\xi_j \overline{K} = \widehat{x}_j^{-1} \overline{K}$.
3. $x_n \Delta \overline{K} = t^{1-n} \widehat{x}_1^{-1} \widehat{\Delta}^{-1} \overline{K}$.

Note that

$$(x_n \Delta)^n f(x) = \left(\prod_{i=1}^n x_i \right) f(q^{-1}x).$$

Then repeated application of part 3 of Lemma 19 shows that for $\alpha \in \mathcal{C}_n$,

$$\begin{aligned} E_\alpha(x) &= \frac{E_{\alpha+(1^n)}(x)}{x_1 \cdots x_n}, \\ \bar{K}_\alpha(x) &= q^{|\alpha|} t^{(1-n)n} \left(\prod_{i=1}^n (\bar{\alpha}_i x_i)^{-1} \right) \bar{K}_{\alpha+(1^n)}(x). \end{aligned} \quad (22)$$

As is well known and already noted in Section 2, the 1st equality allows to relate the non-symmetric Macdonald polynomials $E_v(x) := E_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]$ for arbitrary $v \in \mathbb{Z}^n$ to those labeled by compositions through the formula

$$E_v(x) = \frac{E_{v+(m^n)}(x)}{(x_1 \cdots x_n)^m}.$$

The 2nd formula of (22) can now be used to explicitly define the normalized non-symmetric Macdonald polynomials for degrees $v \in \mathbb{Z}^n$.

Definition 20. Let $v \in \mathbb{Z}^n$ and $m \in \mathbb{Z}_{\geq 0}$ such that $v + (m^n) \in \mathcal{C}_n$. Then $\bar{K}_v(x) := \bar{K}_v(x; q, t) \in \mathbb{F}[x^{\pm 1}]$ is defined by

$$\bar{K}_v(x) := q^{m|v|} t^{(1-n)nm} \left(\prod_{i=1}^n (\bar{v}_i x_i)^{-m} \right) \bar{K}_{v+(m^n)}(x).$$

Using

$$\lim_{a \rightarrow \infty} A_m(ax; v) = q^{-m^2 n} t^{(1-n)nm} \prod_{i=1}^n (\bar{v}_i x_i)^{-m}$$

and the definitions of $G_v(x)$ and $K_v(x)$ it follows that

$$\begin{aligned} \lim_{a \rightarrow \infty} a^{-|v|} G_v(ax) &= E_v(x), \\ \lim_{a \rightarrow \infty} K_v(ax) &= \bar{K}_v(x) \end{aligned}$$

for all $v \in \mathbb{Z}^n$, so in particular

$$\bar{K}_v(x) = \frac{E_v(x)}{E_v(\tau)} \quad \forall v \in \mathbb{Z}^n.$$

Lemma 19 holds true for the extension of $\bar{K}$ to the map $\bar{K} \in \mathcal{F}_{\mathbb{F}[x^{\pm 1}]}$ defined by $v \mapsto \bar{K}_v$ ($v \in \mathbb{Z}^n$). Taking the limit in Theorem 17 we obtain the well-known duality [1] of the Laurent polynomial versions of the normalized non-symmetric Macdonald polynomials.

Corollary 21. For all $u, v \in \mathbb{Z}^n$,

$$\bar{K}_u(\tilde{v}) = \bar{K}_v(\tilde{u}).$$

7.2 O -polynomials

We now show that the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17) directly implies the existence of the O -polynomials O_α (which is the nontrivial part of the proof of [14, Thm. 1.2]), and that it provides an explicit expression for O_α in terms of the non-symmetric interpolation Macdonald polynomial K_α .

Proposition 22. For all $\alpha \in C_n$ we have

$$O_\alpha(x) = K_\alpha(t^{1-n}aw_0x).$$

Proof. The polynomial $\tilde{O}_\alpha(x) := K_\alpha(t^{1-n}aw_0x)$ is of degree at most $|\alpha|$ and

$$\tilde{O}_\alpha(\bar{\beta}^{-1}) = K_\alpha(t^{1-n}aw_0\bar{\beta}^{-1}) = K_\alpha(a\tilde{\beta}) = K_\beta(a\tilde{\alpha})$$

for all $\beta \in C_n$ by (4) and Theorem 17. Hence, $\tilde{O}_\alpha = O_\alpha$. ■

7.3 Okounkov's duality

Write $F[x]^{S_n}$ for the symmetric polynomials in $x_1, \dots, x_n$ with coefficients in a field F . Write $C_+ := \sum_{w \in S_n} H_w$. The symmetric interpolation Macdonald polynomial $R_\lambda(x) \in \mathbb{F}[x]^{S_n}$ is the multiple of C_+G_λ such that the coefficient of x^λ is one (see, e.g., [13]). We write

$$K_\lambda^+(x) := \frac{R_\lambda(x)}{R_\lambda(a\tau)} \in \mathbb{K}[x]^{S_n}$$

for the normalized symmetric interpolation Macdonald polynomial. Then

$$C_+K_\alpha(x) = \left(\sum_{w \in S_n} t^{\ell(w)} \right) K_{\alpha_+}^+(x) \quad (23)$$

for $\alpha \in C_n$. Okounkov's [10, Section 2] duality result now reads as follows.

Theorem 23. For partitions $\lambda, \mu \in \mathcal{P}_n$ we have

$$K_\lambda^+(a\bar{\mu}^{-1}) = K_\mu^+(a\bar{\lambda}^{-1}).$$

Let us derive Theorem 23 as consequence of Theorem 17. Write $\widehat{C}_+ = \sum_{w \in S_n} \widehat{H}_w$, with $\widehat{H}_w := \widehat{H}_{i_1} \cdots \widehat{H}_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Write $f_\mu \in \mathcal{F}_{\mathbb{K}}$ for the function $f_\mu(u) := K_u(a\tilde{\mu})$ ($u \in \mathbb{Z}^n$). Then

$$\left(\sum_{w \in S_n} t^{\ell(w)} \right) K_\lambda^+(a\tilde{\mu}) = (C_+ K_\lambda)(a\tilde{\mu}) = (\widehat{C}_+ f_\mu)(\lambda) \quad (24)$$

by part 1 of Proposition 16. The duality (17) of K_u and (4) imply that

$$f_\mu(u) = K_\mu(a\tilde{u}) = (Jw_0 K_\mu(t^{1-n}x))|_{x=a^{-1}\tilde{u}} \quad (25)$$

with $(Jf)(x) := f(x_1^{-1}, \dots, x_n^{-1})$ for $f \in \mathbb{K}(x)$. A direct computation shows that

$$JH_i J = (H_i^\circ)^{-1}, \quad w_0 H_i w_0 = (H_{n-i}^\circ)^{-1} \quad (26)$$

for $1 \leq i < n$. In particular, $Jw_0 C_+ = C_+ Jw_0$. Combined with Remark 7 we conclude that

$$(\widehat{C}_+ f_\mu)(\lambda) = (Jw_0 C_+ K_\mu(t^{1-n}x))|_{x=a^{-1}\bar{\lambda}}.$$

By (23) and (4) this simplifies to

$$(\widehat{C}_+ f_\mu)(\lambda) = \left(\sum_{w \in S_n} t^{\ell(w)} \right) K_\mu^+(a\bar{\lambda}).$$

Returning to (24) we conclude that $K_\lambda^+(a\tilde{\mu}) = K_\mu^+(a\bar{\lambda})$. Since K_λ^+ is symmetric we obtain from (4) that

$$K_\lambda^+(a\bar{\mu}^{-1}) = K_\mu^+(a\bar{\lambda}^{-1}),$$

which is Okounkov's duality result.

7.4 A primed version of duality

We first derive the following twisted version of the duality of the non-symmetric interpolation Macdonald polynomials (Theorem 17).

Lemma 24. For $u, v \in \mathbb{Z}^n$ we have

$$(H_{w_0}K_u)(a\tilde{v}) = (H_{w_0}K_v)(a\tilde{u}). \quad (27)$$

Proof. We proceed as in the previous subsection. Set $f_v(u) := K_u(a\tilde{v})$ for $u, v \in \mathbb{Z}^n$. By part 1 of Proposition 16,

$$(H_{w_0}K_u)(a\tilde{v}) = (\widehat{H}_{w_0}f_v)(u).$$

Since $f_v(u) = (Iw_0K_v)(a^{-1}t^{n-1}\bar{u})$ by (4), Remark 7 implies that

$$(\widehat{H}_{w_0}f_v)(u) = (H_{w_0}Jw_0K_v)(a^{-1}t^{n-1}\bar{u}).$$

Now $H_{w_0}Jw_0 = Jw_0H_{w_0}$ by (26); hence,

$$(\widehat{H}_{w_0}f_v)(u) = (Jw_0H_{w_0}K_v)(a^{-1}t^{n-1}\bar{u}) = (H_{w_0}K_v)(a\tilde{u}),$$

which completes the proof. ■

Recall from Theorem 1 that

$$G'_\beta(x) = t^{(1-n)|\beta|+I(\beta)}\Psi G^\circ_\beta(t^{n-1}x)$$

with $\Psi := w_0H^\circ_{w_0}$. We define normalized versions by

$$K'_\beta(x) := \frac{G'_\beta(x)}{G'_\beta(a^{-1}\tau)} = t^{\ell(w_0)}\Psi K^\circ_\beta(t^{n-1}x), \quad \beta \in \mathcal{C}_n,$$

with $K^\circ_v := \iota(K_v)$ for $v \in \mathbb{Z}^n$ (the 2nd formula follows from Lemma 2). More generally, we define for $v \in \mathbb{Z}^n$,

$$K'_v(x) := t^{\ell(w_0)}\Psi K^\circ_v(t^{n-1}x). \quad (28)$$

We write $K' : \mathbb{Z}^n \rightarrow \mathbb{K}(x)$ for the map $v \mapsto K'_v$ ($v \in \mathbb{Z}^n$). Since $H_i\Psi = \Psi H^\circ_i$, part 1 of Proposition 16 gives $H_iK' = \widehat{H}_iK'$. Considering the action of $((x_n - 1)\Delta^\circ)^n$ on $K'_\beta(x)$ we get, using the fact that $((x_n - 1)\Delta^\circ)^n$ commutes with Ψ and part 3 of Proposition 16,

$$K'_v(x) = \left(\prod_{i=1}^n \frac{(1 - a^{-1}\bar{v}_i)}{(1 - q^{-1}x_i)} \right) K'_{v+(1^n)}(q^{-1}x),$$

in particular

$$K'_v(x) = \left(\prod_{i=1}^n \frac{(a^{-1}\bar{v}_i; q)_m}{(q^{-m}x_i; q)_m} \right) K'_{v+(m^n)}(q^{-m}x).$$

Example 25. For $n = 1$ we have $K'_v(x) = K_v^\circ(x)$ for $v \in \mathbb{Z}$; hence,

$$\begin{aligned} K'_{-m}(x) &= \frac{(q^{-1}a^{-1}; q^{-1})_m}{(q^{-1}x; q^{-1})_m} = (ax)^{-m} \frac{(qa; q)_m}{(qx^{-1}; q)_m}, \\ K'_m(x) &= (ax)^m \frac{(x^{-1}; q^{-1})_m}{(a; q^{-1})_m} = \frac{(x; q)_m}{(a^{-1}; q)_m} \end{aligned}$$

for $m \in \mathbb{Z}_{\geq 0}$ by Example 14.

Proposition 26. For all $u, v \in \mathbb{Z}^n$ we have

$$K'_v(a^{-1}\bar{u}) = K'_u(a^{-1}\bar{v}).$$

Proof. Note that

$$K'_v(a^{-1}\bar{u}) = t^{\ell(w_0)} \Psi K_v^\circ(t^{n-1}x)|_{x=a^{-1}\bar{u}} = t^{\ell(w_0)} (H_{w_0}^\circ K_v^\circ)(a^{-1}\tilde{u}^{-1})$$

by (4). By (27) the right-hand side is invariant under the interchange of u and v . ■

7.5 Binomial formula and dual binomial formula

In [14] the existence and uniqueness of O_α was used to prove the following binomial theorem [14, Thm. 1.3]. Define for $\alpha, \beta \in \mathbb{C}_n$ the generalized binomial coefficient by

$$\left[\begin{matrix} \alpha \\ \beta \end{matrix} \right]_{q,t} := \frac{G_\beta(\bar{\alpha})}{G_\beta(\bar{\beta})}. \quad (29)$$

Applying the automorphism ι of $\mathbb{F}$ to (29) we get

$$\left[\begin{matrix} \alpha \\ \beta \end{matrix} \right]_{q^{-1}, t^{-1}} = \frac{G_\beta^\circ(\bar{\alpha}^{-1})}{G_\beta^\circ(\bar{\beta}^{-1})}.$$

Theorem 27. For $\alpha, \beta \in \mathcal{C}_n$ we have the binomial formula

$$K_\alpha(ax) = \sum_{\beta \in \mathcal{C}_n} a^{|\beta|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} \frac{G'_\beta(x)}{G_\beta(a\tau)}. \quad (30)$$

Remark 28. 1. Note that the sum in (30) is finite, since the generalized binomial coefficient (29) is zero unless $\beta \subseteq \alpha$, with $\beta \subseteq \alpha$ meaning $\beta_i \leq \alpha_i$ for $i = 1, \dots, n$.

2. By Corollary 4 and (28) the binomial formula (30) can be alternatively written as

$$\begin{aligned} K_\alpha(ax) &= \sum_{\beta \in \mathcal{C}_n} \tau_\beta^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q^{-1}, t^{-1}} K'_\beta(x) \\ &= \sum_{\beta \in \mathcal{C}_n} \frac{K_\beta^\circ(\bar{\alpha}^{-1}) K'_\beta(x)}{\tau_\beta K_\beta^\circ(\bar{\beta}^{-1})} \\ &= t^{\ell(w_0)} \sum_{\beta \in \mathcal{C}_n} \frac{K_\beta^\circ(\bar{\alpha}^{-1}) \Psi K_\beta^\circ(t^{n-1}x)}{\tau_\beta K_\beta^\circ(\bar{\beta}^{-1})} \end{aligned} \quad (31)$$

with $\Psi = w_0 H_{w_0}^\circ$ (note that the dependence on a in the right-hand side of (31) is through the normalization factors of the interpolation polynomials $K_\beta^\circ(x)$ and $K'_\beta(x)$).

3. The binomial formula (30) and Theorem 1 imply the twisted duality (27) of K_α as follows. By the identity $H_{w_0} \Psi = w_0$ the binomial formula (31) implies the finite expansion

$$(H_{w_0} K_\alpha)(ax) = t^{\ell(w_0)} \sum_{\beta} \frac{K_\beta^\circ(\bar{\alpha}^{-1}) K_\beta^\circ(t^{n-1} w_0 x)}{\tau_\beta K_\beta^\circ(\bar{\beta}^{-1})}.$$

Substituting $x = \tilde{\gamma}$ and using (4) we obtain

$$(H_{w_0} K_\alpha)(a\tilde{\gamma}) = \sum_{\beta \in \mathcal{C}_n} \frac{K_\beta^\circ(\bar{\alpha}^{-1}) K_\beta^\circ(\bar{\gamma}^{-1})}{\tau_\beta K_\beta^\circ(\bar{\beta}^{-1})}.$$

The right-hand side is manifestly invariant under interchanging α and γ , which is equivalent to twisted duality (27).

In [8, Section 4] it is remarked that an explicit identity relating G'_α and G_α is needed to provide a proof of the dual binomial formula [8, Thm. 4.4] as a direct consequence of the binomial formula (30). We show here that Theorem 1 is providing the required identity. Instead of Theorem 1 we use its normalized version, encoded by (28).

The dual binomial formula [8, Thm. 4.4] in our notations reads as follows.

Theorem 29. For all $\alpha \in C_n$ we have

$$K'_\alpha(x) = \sum_{\beta \in C_n} \tau_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_\beta(ax). \quad (32)$$

The starting point of the alternative proof of (32) is the binomial formula in the form

$$K_\alpha(ax) = t^{\ell(w_0)} \sum_{\beta \in C_n} \frac{G_\beta^\circ(\bar{\alpha}^{-1}) \Psi K_\beta^\circ(t^{n-1}x)}{\tau_\beta G_\beta^\circ(\bar{\beta}^{-1})},$$

see (31). Replace (a, x, q, t) by $(a^{-1}, at^{n-1}x, q^{-1}, t^{-1})$ and act by $w_0 H_{w_0}$ on both sides. Since $w_0 H_{w_0} \Psi = \text{Id}$ we obtain

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \tau_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} K_\beta(ax).$$

Now use (28) to complete the proof of (32).

Remark 30. It follows from this proof of (32) that the dual binomial formula (32) can be rewritten as

$$\Psi K_\alpha^\circ(t^{n-1}x) = t^{-\ell(w_0)} \sum_{\beta} \frac{\tau_\beta K_\beta(\bar{\alpha}) K_\beta(ax)}{K_\beta(\bar{\beta})}. \quad (33)$$

As observed in [8, (4.11)], the binomial and dual binomial formula directly imply the orthogonality relations

$$\sum_{\beta \in C_n} \frac{\tau_\beta}{\tau_\alpha} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{q,t} \begin{bmatrix} \beta \\ \gamma \end{bmatrix}_{q^{-1}, t^{-1}} = \delta_{\alpha, \gamma}.$$

Since $\begin{bmatrix} \delta \\ \epsilon \end{bmatrix}_{q,t} = 0$ unless $\delta \supseteq \epsilon$, the terms in the sum are zero unless $\gamma \subseteq \beta \subseteq \alpha$.

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