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# On a partition identity of Lehmer

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ABSTRACT

Euler's identity equates the number of partitions of any non-negative integer $n$ into odd parts and the number of partitions of $n$ into distinct parts. Beck conjectured and Andrews proved the following companion to Euler's identity: the excess of the number of parts in all partitions of $n$ into odd parts over the number of parts in all partitions of $n$ into distinct parts equals the number of partitions of $n$ with exactly one even part (possibly repeated). Beck's original conjecture was followed by generalizations and so-called "Beck-type" companions to other identities.

In this paper, we establish a collection of Beck-type companion identities to the following result mentioned by Lehmer at the 1974 International Congress of Mathematicians: the excess of the number of partitions of $n$ with an even number of even parts over the number of partitions of $n$ with an odd number of even parts equals the number of partitions of $n$ into distinct, odd parts. We also establish various generalizations of Lehmer's identity, and prove related Beck-type companion identities. We use both analytic and combinatorial methods in our proofs.



## 1. Introduction and statement of results

Many results in the theory of partitions concern identities asserting that the set $\mathcal{P}_X(n)$ of partitions of $n$ satisfying condition $X$ and the set $\mathcal{P}_Y(n)$ of partitions of $n$ satisfying condition $Y$ are equinumerous. Likely the oldest such result is Euler's identity that the number of partitions of $n$ into odd parts is equal to the number of partitions of $n$ into distinct parts. In 2017, Beck made the following conjecture ([10], [2, Conjecture]):

**Conjecture 1** *(Beck). The excess of the number of parts in all partitions of n into odd parts over the number of parts in all partitions of n into distinct parts equals the number of partitions of n with exactly one even part (possibly repeated).*

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Beck's conjecture was quickly proved analytically by Andrews [2], who additionally showed that this excess also equals the number of partitions of $n$ with exactly one part repeated (and all other parts distinct). The conjecture was also proved combinatorially by Yang [11] and Ballantine–Bielak [4] independently. This work was followed by generalizations and Beck-type companions to other well known identities (e.g., [3], [5], [9], [11]). In general, a Beck-type companion identity to $|\mathcal{P}_X(n)| = |\mathcal{P}_Y(n)|$ is an identity that equates the excess of the number of parts in all partitions in $\mathcal{P}_X(n)$ over the number of parts in all partitions in $\mathcal{P}_Y(n)$ to the number of partitions of $n$ satisfying a condition closely related to $X$ (or $Y$).

In this article, we establish a number of Beck-type identities related to a result of Lehmer, which he informally mentioned at the 1974 International Congress of Mathematicians [8]: for every non-negative integer $n$, we have that

$$2p_e(n,2) = p(n) + q_o(n), \tag{1}$$

where

$$p_e(n,2) := p(n \mid \text{ the number of even parts is even})$$

and

$$q_o(n) := p(n \mid \text{ distinct, odd parts}).$$

Here and throughout we use the standard notations $p(n)$ and $p(n \mid X)$ to denote the number of partitions of $n$, and the number of partitions of $n$ satisfying condition $X$, respectively. If we also denote by

$$p_o(n,2) := p(n \mid \text{ the number of even parts is odd}),$$

identity (1) is equivalent to the following statement which we refer to as Lehmer's identity.

**Theorem 1.1.** *For any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we have*

$$p_e(n,2) = p_o(n,2) + q_o(n). \tag{2}$$

An analytic proof of Theorem 1.1 is immediate: The generating series for $p_e(n,2) - p_o(n,2)$ and $q_o(n)$ are given by $(q;q^2)_\infty^{-1}(-q^2;q^2)_\infty^{-1}$ and $(-q;q^2)_\infty$, respectively. Then Theorem 1.1 follows from the fact that

$$(-q;q^2)_\infty = \frac{(-q;q)_\infty}{(-q^2;q^2)_\infty}$$

and Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

Here and throughout, the $q$-Pochhammer symbol is given by

$$(a;q)_n := \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$
$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n.$$

In [8], Gupta provided a beautiful combinatorial proof of Theorem 1.1. We also note that (2) is equivalent to the following identity due to Glaisher ([6, p.129] [7, p.256])

$$p_e(n) - p_o(n) = (-1)^n q_o(n),$$

where

$$p_{e/o}(n) := p(n \mid \text{even/odd number of parts}).$$

Our first main result, Theorem 1.2 below, is a Beck-type companion identity to Lehmer's identity (2). To state it, we first set some additional notation. We begin by formally defining a *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ of *size* $n \in \mathbb{N}_0$ to be a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$ called *parts* that add up to $n$. For convenience, we abuse notation and use $\lambda$ to denote either the multiset of its parts or the non-increasing sequence of parts. We write $a \in \lambda$ to mean the positive integer $a$ is a part of $\lambda$. The empty partition is the only partition of size 0. Thus, $p(0) = 1$. We write $|\lambda|$ for the size of $\lambda$ and $\lambda \vdash n$ to mean that $\lambda$ is a partition of size $n$. For a pair of partitions $(\lambda, \mu)$ we also write $(\lambda, \mu) \vdash n$ to mean $|\lambda| + |\mu| = n$. We use the convention that $\lambda_k = 0$ for all $k$ greater than the number of parts. When convenient we will also use the exponential notation for parts in a partition: the exponent of a part is the multiplicity of the part in the partition, e.g., we write $(a^b)$ for the partition consisting of $b$ parts equal to $a$. Further, we denote by calligraphy style capital letters the set of partitions enumerated by the function denoted by the same letter. For example, $\mathcal{Q}_o(n)$ denotes the set of partitions of $n$ into distinct odd parts. We also define $\mathcal{Q}_o := \bigcup_{n\geq 0} \mathcal{Q}_o(n)$.

**Theorem 1.2.** *Let $n \in \mathbb{N}_0$. The excess of the number of parts in all partitions in $\mathcal{P}_e(n,2)$ over the number of parts in all partitions in $\mathcal{P}_o(n,2) \cup \mathcal{Q}_o(n)$ equals the number of partitions of $n$ with exactly one even part, possibly repeated, and all other parts odd and distinct.*

**Remark 1.** As proved in [3], the excess in Theorem 1.2 is almost always equal to the number of parts in all self-conjugate partitions of $n$. Hence, the excess in the number of parts in all partitions in $\mathcal{P}_e(n,2)$ over the number of parts in all partitions in $\mathcal{P}_o(n,2)$ is almost always equal to the total number of parts in all self-conjugate partitions of $n$ and in all partitions of $n$ into distinct odd parts. More precisely, if $N(x)$ is the number of times the above statement is true for $n \leq x$, then $\lim_{x\to\infty} N(x)/x = 1$.

We also establish a *restricted* Beck-type identity accompanying (2) in which we only count the number of even parts in partitions in $\mathcal{P}_e(n,2)$ and $\mathcal{P}_o(n,2)$; this result is given in Theorem 1.3 below. To ease notation in the statement of this result and other Beck-type identities that follow, we introduce the following definition. Let $n, r, a, b$ be non-negative integers such that $1 \leq ab \leq n$. We define

$$\mathcal{B}_r(n,a,b) := \left\{ \lambda \vdash n - rab \;\middle|\; \begin{array}{l} \lambda \neq (ra, r(a-2)), \text{ and} \\ r(a+b+1) \notin \lambda, \text{ and} \\ \lambda_1 - \lambda_2 \leq 2r(a+b+1) \text{ or } \lambda_1 = 3r(a+b+1) \end{array} \right\}.$$

We write $\mathcal{B}(n,a,b)$ for $\mathcal{B}_1(n,a,b)$.

**Theorem 1.3.** *Let $n \in \mathbb{N}_0$. The excess of the number of parts in all partitions in $\mathcal{Q}_o(n)$ plus the number of even parts in all partitions in $\mathcal{P}_o(n,2)$ over the number of even parts in all partitions in $\mathcal{P}_e(n,2)$ equals the number of pairs of partitions $(\lambda, (a^b))$ satisfying the following conditions:*

i. *$a, b$ are both odd,*
ii. *$\lambda \in \mathcal{Q}_o \cap \mathcal{B}(n,a,b)$, i.e., $\lambda$ has distinct odd parts, is not equal to $(a, a-2)$, does not have $a+b+1$ as a part, and satisfies $\lambda_1 - \lambda_2 \leq 2(a+b+1)$ or $\lambda_1 = 3(a+b+1)$.*

**Remark 2.** If $n$ is even, the condition $\lambda \neq (a, a-2)$ in ii. is vacuously true.

In general, whenever we refer to pairs of the form $(\lambda, (a^b))$, we require $(a^b)$ to be nonempty (i.e. $a, b > 0$), while $\lambda$ is allowed to be the empty partition.

**Remark 3.** Beck's Conjecture 1 can also be formulated in the language of pairs as in Theorem 1.3:
The excess of the number of parts in all partitions of $n$ into odd parts over the number of parts in all partitions of $n$ into distinct parts equals the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ satisfying the following conditions:

i. $a$ is even,
ii. $\lambda$ is a partition into odd parts.

Next, we give a collection of Beck-type companion identities to the following generalization of Lehmer's identity (2), which we prove in Section 3. For the remainder of the paper, we let $r \in \mathbb{N}$.

**Theorem 1.4.** *For any $n \in \mathbb{N}_0$, we have*

$$p_e(n,2r) = p_o(n,2r) + q_o(n,r), \tag{3}$$

*where*

$$p_{e/o}(n,2r) := p(n \mid \text{all parts allowed, even/odd number of parts divisible by } 2r)$$
$$q_o(n,r) := p\left(n \;\middle|\; \begin{array}{l} \text{parts are not divisible by } 2r, \\ \text{parts divisible by } r \text{ are distinct} \end{array}\right)$$
$$= p\left(n \mid \text{all parts divisible by } r \text{ are distinct, odd multiples of } r\right).$$

Note that for $r = 1$, identity (3) reduces to identity (2).
Our first Beck-type companion identity to (3) is given by the next theorem which becomes Theorem 1.2 when $r = 1$.

**Theorem 1.5.** *Let $n \in \mathbb{N}_0$. The excess in the total number of parts in all partitions in $\mathcal{P}_e(n,2r)$ over the total number of parts in all partitions in $\mathcal{P}_o(n,2r) \cup \mathcal{Q}_o(n,r)$ equals the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ such that*

i. $2r \mid a$,
ii. $\lambda \in \mathcal{Q}_o(n-ab, r)$.

**Remark 4.** Equivalently, the excess of Theorem 1.5 equals the number of partitions of $n$ in which, among the parts divisible by $r$, there is a single even multiple of $r$ and this part is possibly repeated, while all other parts divisible by $r$ are odd multiples of $r$ and they are distinct.

Theorem 1.6 below is a restricted Beck-type companion identity to (3), in which we only count the number of parts divisible by $r$ in $\mathcal{Q}_o(n, r)$, and the number of parts divisible by $2r$ in $\mathcal{P}_e(n, 2r)$ and $\mathcal{P}_o(n, 2r)$. The theorem reduces to Theorem 1.3 when $r = 1$.

**Theorem 1.6.** *Let $n \in \mathbb{N}_0$. The excess of the number of parts divisible by $r$ in all partitions in $\mathcal{Q}_o(n, r)$ plus the number of parts divisible by $2r$ in all partitions in $\mathcal{P}_o(n, 2r)$ over the number of parts divisible by $2r$ in all partitions in $\mathcal{P}_e(n, 2r)$ equals the number of pairs of partitions $(\lambda, ((ar)^b))$ satisfying the following conditions:*

i. *$a, b$ are both odd,*
ii. *$\lambda \in \mathcal{Q}_o(n - rab, r)$ such that, if we write $\lambda = \lambda^{ndiv} \cup \lambda^{div}$ where $\lambda^{div}$ contains all parts of $\lambda$ that are divisible by $r$, then $\lambda^{div} \in \mathcal{B}_r(n - |\lambda^{ndiv}|, a, b)$.*

Recall that $\lambda \cup \mu$ is the partition whose parts are precisely the parts of $\lambda$ and $\mu$ (with multiplicities).

Next we give another generalization of Lehmer's identity (2). To describe this, we let $r \in \mathbb{N}$, and let $L_r \subseteq \{2, 4, 6, \ldots, 2r\}$, with $L_r \neq \varnothing$. We use the sets $L_r$ to restrict even parts of partitions to lie within certain arithmetic progressions. More precisely, we define

$$p_{e/o}(n, L_r, 2r) := p\left(n \;\middle|\; \begin{array}{l}\text{all parts allowed,}\\ \text{even parts} \equiv \ell \pmod{2r}, \ell \in L_r,\\ \text{even/odd number of even parts}\end{array}\right),$$

$$q(n, L_r, r) := p\left(n \;\middle|\; \begin{array}{l}\text{all parts distinct,}\\ \text{even parts} \not\equiv \ell \pmod{2r}, \ell \in L_r\end{array}\right).$$

**Theorem 1.7.** *For any $n \in \mathbb{N}_0$, we have*

$$p_e(n, L_r, 2r) = p_o(n, L_r, 2r) + q(n, L_r, r). \tag{4}$$

Note that in the case $L_r = \{2, 4, \ldots, 2r\}$, identity (4) is equivalent to identity (2).

The next theorem is a Beck-type companion identity to (4), which becomes Theorem 1.2 when $L_r = \{2, 4, \ldots, 2r\}$.

**Theorem 1.8.** *Let $n \in \mathbb{N}_0$. The excess in the total number of parts in all partitions in $\mathcal{P}_e(n, L_r, 2r)$ over the total number of parts in all partitions in $\mathcal{P}_o(n, L_r, 2r) \cup \mathcal{Q}(n, L_r, r)$ equals the number of pairs of partitions $(\lambda, (a^b))$ satisfying the following conditions:*

i. *$a$ is even,*
ii. *$\lambda \in \mathcal{Q}(n - ab, L_r, r)$.*

A restricted Beck-type companion identity to (4) is given by the next theorem, where we only count the number of even parts in $\mathcal{P}_e(n, L_r, 2r)$ and $\mathcal{P}_o(n, L_r, 2r)$. The theorem becomes Theorem 1.3 when $L_r = \{2, 4, \ldots, 2r\}$.

**Theorem 1.9.** *Let $n \in \mathbb{N}_0$. The excess of the number of parts in all partitions in $\mathcal{Q}(n, L_r, r)$ plus the number of even parts in all partitions in $\mathcal{P}_o(n, L_r, 2r)$ over the number of even parts in all partitions in $\mathcal{P}_e(n, L_r, 2r)$ equals the number of pairs of partitions $(\lambda, (a^b))$ satisfying the following conditions:*

i. *$a, b$ are both odd,*
ii. *$\lambda \in \mathcal{Q}(n - ab, L_r, r)$ such that, if we write $\lambda = \lambda^e \cup \lambda^o$, where $\lambda^e$ consists of all the even parts of $\lambda$ and $\lambda^o$ consists of all the odd parts of $\lambda$, then $\lambda^o \in \mathcal{B}(n - |\lambda^e|, a, b)$.*

The next result is a new restricted Beck-type companion identity to Lehmer's identity (2), different from Theorem 1.3. We only count the number of parts in certain arithmetic progressions in $\mathcal{Q}_o(n, 2)$, $\mathcal{P}_e(n, 2)$ and $\mathcal{P}_o(n, 2)$.

To describe it, for $r \in \mathbb{N}$, let

$$L_r \subseteq \{2, 4, 6, \ldots, 2r\}, \quad \text{and} \quad O_r \subseteq \{1, 3, 5, \ldots, 2r-1\}.$$

**Theorem 1.10.** *Let $n$ be a positive integer, and $L_r$ and $O_r$ as above such that if $n \equiv 0 \pmod 4$ then $2 \notin L_r$. The excess of the number of parts $\equiv \ell \pmod{2r}$, $\ell \in O_r$, in all partitions in $\mathcal{Q}_o(n)$ plus the number of parts $\equiv \ell \pmod{2r}$, $\ell \in L_r$, in all partitions in $\mathcal{P}_o(n,2)$ over the number of parts $\equiv \ell \pmod{2r}$, $\ell \in L_r$, in all partitions in $\mathcal{P}_e(n,2)$ equals the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ satisfying the following conditions:*

i. *$a \equiv \ell \pmod{2r}$ for some $\ell \in L_r \cup O_r$, and $b$ is odd. Moreover, if $a$ is odd, then $b = 1$,*
ii. *$\lambda \in \mathcal{Q}_o$. Moreover, if $a$ is odd, then $a \notin \lambda$; if $a$ is even, then $\lambda_1 - \lambda_2 \leq a$ and $\lambda \notin \left\{ \left(\frac{a}{2}+1, \frac{a}{2}-1\right), \left(\frac{a}{2}+2, \frac{a}{2}-2\right) \right\}$.*

*If $n \equiv 0 \pmod 4$ and $2 \in L_r$, the excess is one less than the number of pairs counted above. Moreover, if we additionally have that $n \notin \{4, 8, 12, 16, 20\}$, then the excess is equal to the number of pairs $(\lambda, (a^b))$ satisfying i. and ii. with the additional condition $(\lambda, (a^b)) \neq ((9,7,5,1),(2^b))$.*

**Remark 5.** If $n \not\equiv 0 \pmod 4$, then the condition $\lambda \notin \left\{ \left(\frac{a}{2}+1, \frac{a}{2}-1\right), \left(\frac{a}{2}+2, \frac{a}{2}-2\right) \right\}$ is vacuously true.

Generally speaking, our proofs are both analytic and combinatorial in nature. In Sections 2 to 4, we prove Theorems 1.2 through 1.9. In Section 5, we prove Theorem 1.10 and give several important examples. In Section 6, we establish the non-negativity of the coefficients of some related $q$-series.

## 2. Proofs of Theorems 1.2 and 1.3

Consider the generating series

$$F(z;q) := \frac{1}{(zq;q^2)_\infty (-zq^2;q^2)_\infty} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{m} p(n \mid m \text{ parts, of which } s \text{ parts are even})(-1)^s z^m q^n,$$

$$E(z;q) := \frac{1}{(q;q^2)_\infty (-zq^2;q^2)_\infty} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(n \mid \text{the number of even parts is } m)(-z)^m q^n,$$

and

$$Q_o(z;q) := (-zq;q^2)_\infty = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(n \mid \text{parts must be odd and distinct, } m \text{ parts}) z^m q^n.$$

To prove Theorem 1.2, note that $\frac{\partial}{\partial z}\big|_{z=1}(F(z;q) - Q_o(z;q))$ gives the generating series for the excess of the number of parts in all partitions in $\mathcal{P}_e(n,2)$ over the number of parts in all partitions in $\mathcal{Q}_o(n) \cup \mathcal{P}_o(n,2)$. We have

$$\begin{aligned}
\frac{\partial}{\partial z}\Big|_{z=1}(F(z;q) - Q_o(z;q)) &= (-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1-q^{2k+1}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1+q^{2k}} - \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1+q^{2k+1}} \right) \\
&= (-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1-q^{2k+1}} - \sum_{k=1}^{\infty} \frac{q^{k}}{1+q^{k}} \right) \\
&= (-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1-q^{2k+1}} - \sum_{k=1}^{\infty} \frac{q^{k}}{1-q^{2k}} + \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \right) \\
&= (-q;q^2)_\infty \left( \sum_{k=1}^{\infty} \frac{q^{k}}{1-q^{k}} - \sum_{k=1}^{\infty} \frac{q^{k}}{1-q^{2k}} \right) \\
&= (-q;q^2)_\infty \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}}.
\end{aligned}$$

The last expression is the generating series for the number of partitions of $n$ with exactly one even part, possibly repeated, and all other parts odd and distinct. This proves Theorem 1.2.

To prove Theorem 1.3 we note that $\frac{\partial}{\partial z}\big|_{z=1}(Q_o(z;q) - E(z;q))$ is the generating series for the excess of the number of parts in all partitions in $\mathcal{Q}_o(n)$ plus the number of even parts in all partitions in $\mathcal{P}_o(n,2)$ over the number of even parts in all partitions in $\mathcal{P}_e(n,2)$. We compute

$$\frac{\partial}{\partial z}\Big|_{z=1}(Q_o(z;q) - E(z;q)) = (-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1+q^{2k+1}} + \sum_{k=1}^{\infty} \frac{q^{2k}}{1+q^{2k}} \right)$$

$$= (-q;q^2)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1+q^k}$$
$$= (-q;q^2)_\infty \left( \sum_{k=1}^{\infty} \frac{q^k}{1-q^{2k}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \right).$$

Let

$$p_{e^o}(n) := p(n \mid \text{odd number of identical even parts}),$$

i.e.,

$$p_{e^o}(n) := |\{\lambda \vdash n \mid \lambda = (a^b), a \text{ even and } b \text{ odd}\}|.$$

Define $p_{e^e}(n)$, $p_{o^e}(n)$, and $p_{o^o}(n)$ similarly.

Then

$$\sum_{k=1}^{\infty} \frac{q^k}{1-q^{2k}} = \sum_{n=1}^{\infty} (p_{o^o}(n) + p_{e^o}(n)) q^n,$$

and

$$\sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} = \sum_{n=1}^{\infty} (p_{o^e}(n) + p_{e^e}(n)) q^n.$$

Since conjugation gives a bijection between $\mathcal{P}_{o^e}(n)$ and $\mathcal{P}_{e^o}(n)$, we further have

$$\sum_{k=1}^{\infty} \frac{q^k}{1-q^{2k}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} = \sum_{n=1}^{\infty} (p_{o^o}(n) - p_{e^e}(n)) q^n.$$

Therefore

$$(-q;q^2)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1+q^k} = \left( \sum_{n=0}^{\infty} q_o(n) q^n \right) \left( \sum_{n=1}^{\infty} (p_{o^o}(n) - p_{e^e}(n)) q^n \right)$$
$$= \sum_{n=1}^{\infty} \left( \sum_{m=0}^{n-1} q_o(m) p_{o^o}(n-m) - q_o(m) p_{e^e}(n-m) \right) q^n,$$

and the excess in question is given by

$$\sum_{m=0}^{n-1} \left( q_o(m) p_{o^o}(n-m) - q_o(m) p_{e^e}(n-m) \right).$$

Equivalently, this is the excess of the number of elements in

$$B(n) := \{(\lambda, (a^b)) \vdash n \mid \lambda \in \mathcal{Q}_o, a, b \text{ odd}\}$$

over that in

$$A(n) := \{(\lambda, (a^b)) \vdash n \mid \lambda \in \mathcal{Q}_o, a, b \text{ even}\}.$$

To measure this excess, we construct an injection $T$ from $A(n)$ to $B(n)$ as follows. We partition the set $A(n)$ into three disjoint subsets:

$$A_1(n) := \{(\lambda, (a^b)) \in A(n) \mid a+b-1 \notin \lambda\};$$
$$A_2(n) := \{(\lambda, (a^b)) \in A(n) \mid a+b-1 \in \lambda \text{ and } \lambda \text{ has at least two parts}\};$$
$$A_3(n) := \{(\lambda, (a^b)) \in A(n) \mid \lambda = (a+b-1)\}.$$

We define $T$ on each $A_i(n)$ in the following way.

1. If $(\lambda, (a^b)) \in A_1(n)$ (including the case where $\lambda$ is empty), then

$$T(\lambda, (a^b)) := \left( \lambda \cup \{a+b-1\}, \left( (a-1)^{b-1} \right) \right).$$

2. If $(\lambda, (a^b)) \in A_2(n)$, then let $m$ denote the largest part of $\lambda$ that is not $a+b-1$ and define
$$T(\lambda, (a^b)) := \left((\lambda \setminus \{m, a+b-1\}) \cup \{2a+2b-2+m\}, \left((a-1)^{b-1}\right)\right),$$
where $\lambda \setminus \{m, a+b-1\}$ is the partition obtained by removing parts $a+b-1$ and $m$ from $\lambda$.
3. If $(\lambda, (a^b)) \in A_3(n)$, then $T(\lambda, (a^b)) := \left((a+1, a-1), \left((a+1)^{b-1}\right)\right)$.

The image sets are thus

$$T(A_1(n)) = \{(\mu, (c^d)) \in B(n) \mid c+d+1 \in \mu\};$$
$$T(A_2(n)) = \left\{(\mu, (c^d)) \in B(n) \,\middle|\, \begin{array}{l} c+d+1 \notin \mu,\ \mu_1 \neq 3(c+d+1), \\ \text{and } \mu_1 - \mu_2 > 2(c+d+1) \end{array}\right\};$$
$$T(A_3(n)) = \{(\mu, (c^d)) \in B(n) \mid \mu = (c, c-2)\}.$$

Note that $T(A_1(n))$, $T(A_2(n))$, and $T(A_3(n))$ are disjoint, and their union $T(A(n))$ is a subset of $B(n)$. Define the map $L$ from $T(A(n))$ to $A(n)$ as follows:

1. If $(\mu, (c^d)) \in T(A_1(n))$, then
$$L(\mu, (c^d)) := \left(\mu \setminus \{c+d+1\}, \left((c+1)^{d+1}\right)\right).$$
2. If $(\mu, (c^d)) \in T(A_2(n))$, then
$$L(\mu, (c^d)) := \left((\mu \setminus \{\mu_1\}) \cup \{c+d+1, \mu_1 - 2(c+d+1)\}, \left((c+1)^{d+1}\right)\right).$$
3. If $(\mu, (c^d)) \in T(A_3(n))$, then
$$L(\mu, (c^d)) := \left((c+d-1), \left((c-1)^{d+1}\right)\right).$$

Then $L$ and $T$ are inverses of each other. Since $T$ gives a bijection between $A(n)$ and $T(A(n)) \subseteq B(n)$, the excess in question is given by the number of elements in

$$\begin{aligned} B(n) \setminus T(A(n)) &= B(n) \setminus (T(A_1(n)) \cup T(A_2(n)) \cup T(A_3(n))) \\ &= \left\{(\mu, (c^d)) \in B(n) \,\middle|\, \begin{array}{l} c+d+1 \notin \mu, \mu \neq (c, c-2), \text{ and} \\ \mu_1 - \mu_2 \leq 2(c+d+1) \text{ or } \mu_1 = 3(c+d+1) \end{array}\right\}, \\ &= \left\{(\mu, (c^d)) \in B(n) \,\middle|\, \mu \in \mathcal{B}(n, c, d)\right\}. \end{aligned}$$

Theorem 1.3 now follows.

## 3. Proofs of Theorems 1.4, 1.5, and 1.6

For $r \in \mathbb{N}$, we define

$$\begin{aligned} F_r(z; q) &:= \frac{1}{(zq; q^{2r})_\infty (zq^2; q^{2r})_\infty \cdots (zq^{2r-1}; q^{2r})_\infty \cdot (-zq^{2r}; q^{2r})_\infty} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{m} p\left(n \,\middle|\, \begin{array}{l} \text{all parts allowed,} \\ m \text{ parts,} \\ s \text{ parts divisible by } 2r \end{array}\right) (-1)^s z^m q^n, \\ R_r(z; q) &:= \frac{(-zq^r; q^{2r})_\infty}{(zq; q^r)_\infty (zq^2; q^r)_\infty \cdots (zq^{r-1}; q^r)_\infty} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p\left(n \,\middle|\, \begin{array}{l} \text{parts are not divisible by } 2r, \\ m \text{ parts,} \\ \text{parts divisible by } r \text{ are distinct} \end{array}\right) z^m q^n. \end{aligned}$$

Hence, the generating series for $p_e(n, 2r) - p_o(n, 2r)$ and $q_o(n, r)$ are $F_r(1; q)$ and $R_r(1; q)$, respectively. We have

$$\begin{aligned} F_r(1;q) &= \frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty}\cdot\frac{1}{(-q^{2r};q^{2r})_\infty} = \frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty}\cdot\frac{(-q^r;q^{2r})_\infty}{(-q^r;q^r)_\infty} \\ &= \frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty}\cdot(q^r;q^{2r})_\infty\cdot(-q^r;q^{2r})_\infty = \frac{(q^r;q^r)_\infty}{(q;q)_\infty}\cdot(-q^r;q^{2r})_\infty \\ &= R_r(1;q). \end{aligned}$$

Here we used the facts

$$(-q^{2r};q^{2r})_\infty(-q^r;q^{2r})_\infty = (-q^r;q^r)_\infty \quad \text{and} \quad (q^{2r};q^{2r})_\infty(q^r;q^{2r})_\infty = (q^r;q^r)_\infty$$

in the second and fourth equality respectively, and used Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}$$

(by replacing $q$ by $q^r$) in the third equality. Theorem 1.4 now follows.

To prove Theorem 1.5, we have that $\frac{\partial}{\partial z}\big|_{z=1}(F_r(z;q)-R_r(z;q))$ is the generating series for the excess of the total number of parts in all partitions in $\mathcal{P}_e(n,2r)$ over the total number of parts in all partitions in $\mathcal{P}_o(n,2r)\cup\mathcal{Q}_o(n,r)$. We have

$$\begin{aligned} \frac{\partial}{\partial z}\Big|_{z=1}(F_r(z;q)-R_r(z;q)) &= R_r(1;q)\left(\sum_{\ell=1}^{2r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+2kr}}{1-q^{\ell+2kr}} - \sum_{k=1}^{\infty}\frac{q^{2kr}}{1+q^{2kr}} - \sum_{\ell=1}^{r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+kr}}{1-q^{\ell+kr}} - \sum_{k=0}^{\infty}\frac{q^{r+2kr}}{1+q^{r+2kr}}\right) \\ &= R_r(1;q)\left(\sum_{\ell=1}^{2r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+2kr}}{1-q^{\ell+2kr}} - \sum_{\ell=1}^{r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+kr}}{1-q^{\ell+kr}} - \sum_{k=1}^{\infty}\frac{q^{kr}}{1+q^{kr}}\right) \\ &= R_r(1;q)\left(\sum_{\ell=1}^{2r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+2kr}}{1-q^{\ell+2kr}} - \sum_{\ell=1}^{r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+kr}}{1-q^{\ell+kr}} - \sum_{k=1}^{\infty}\frac{q^{kr}}{1-q^{2kr}} + \sum_{k=1}^{\infty}\frac{q^{2kr}}{1-q^{2kr}}\right) \\ &= R_r(1;q)\left(\sum_{k=1}^{\infty}\frac{q^{k}}{1-q^{k}} - \sum_{\ell=1}^{r-1}\sum_{k=0}^{\infty}\frac{q^{\ell+kr}}{1-q^{\ell+kr}} - \sum_{k=1}^{\infty}\frac{q^{kr}}{1-q^{2kr}}\right) \\ &= R_r(1;q)\left(\sum_{k=1}^{\infty}\frac{q^{kr}}{1-q^{kr}} - \sum_{k=1}^{\infty}\frac{q^{kr}}{1-q^{2kr}}\right) \\ &= R_r(1;q)\sum_{k=1}^{\infty}\frac{q^{2kr}}{1-q^{2kr}}. \end{aligned} \tag{5}$$

This is the generating series for the number of pairs of partitions $(\lambda,(a^b))\vdash n$ so that

i. $2r \mid a$,
ii. $\lambda\in\mathcal{Q}_o(n-ab,r)$.

Equivalently, (5) is the generating series for the number of partitions of $n$ in which among the parts divisible by $r$ there is exactly one even multiple of $r$, possibly repeated, and all other parts divisible by $r$ are odd multiples of $r$ and are distinct. This proves Theorem 1.5.

To prove Theorem 1.6, we define

$$\begin{aligned} E_r(z;q) &:= \frac{1}{(q;q^{2r})_\infty(q^2;q^{2r})_\infty\cdots(q^{2r-1};q^{2r})_\infty\cdot(-zq^{2r};q^{2r})_\infty} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(n \mid \text{all parts allowed, } m \text{ parts divisible by } 2r)(-z)^m q^n, \\ Q_r(z;q) &:= \frac{(-zq^r;q^{2r})_\infty}{(q;q^r)_\infty(q^2;q^r)_\infty\ldots(q^{r-1};q^r)_\infty} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p\left(n \;\middle|\; \begin{array}{l}\text{parts are not divisible by } 2r,\\ \text{parts divisible by } r \text{ are distinct,}\\ m \text{ parts divisible by } r\end{array}\right) z^m q^n. \end{aligned}$$

As in the proof of Theorem 1.5, we compute

$$\frac{\partial}{\partial z}\Big|_{z=1}(Q_r(z;q)-E_r(z;q))=\frac{(-q^r;q^{2r})_\infty}{(q;q^r)_\infty(q^2;q^r)_\infty\cdots(q^{r-1};q^r)_\infty}\sum_{k=1}^{\infty}\frac{q^{kr}}{1+q^{kr}}. \tag{6}$$

In the proof of Theorem 1.3, we have shown that

$$(-q;q^2)_\infty\sum_{k=1}^{\infty}\frac{q^k}{1+q^k} \tag{7}$$

is the generating series for the number of pairs of partitions $(\lambda,(a^b))\vdash n$ satisfying the following conditions:

i. $a,b$ are both odd,
ii. $\lambda\in\mathcal{Q}_o\cap\mathcal{B}(n,a,b)$.

For each $r\in\mathbb{N}$, replacing $q$ by $q^r$ in (7) implies that

$$(-q^r;q^{2r})_\infty\sum_{k=1}^{\infty}\frac{q^{kr}}{1+q^{kr}}$$

is the generating series for the number of pairs $(\lambda^{div},((ar)^b))\vdash n$ satisfying the following conditions:

i. $a,b$ are both odd,
ii. $\lambda^{div}\in\mathcal{Q}_o(n-rab,r)\cap\mathcal{B}_r(n,a,b)$ and every part of $\lambda^{div}$ is divisible by $r$.

Theorem 1.6 follows from equation (6).

## 4. Proofs of Theorems 1.7, 1.8, and 1.9

For $r\in\mathbb{N}$, $L_r\subseteq\{2,4,\dots,2r\}$ as in Section 1, we define

$$\begin{aligned}
E_{r,L_r}(z;q)&:=\frac{1}{(q;q^2)_\infty\prod_{\ell\in L_r}(-zq^\ell;q^{2r})_\infty}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}p\left(n\;\middle|\;\begin{array}{l}\text{all odd parts allowed,}\\ \text{even parts}\equiv\ell\pmod{2r},\ \ell\in L_r,\\ m\text{ even parts}\end{array}\right)(-z)^mq^n,\\
F_{r,L_r}(z;q)&:=\frac{1}{(zq;q^2)_\infty\prod_{\ell\in L_r}(-zq^\ell;q^{2r})_\infty}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s=0}^{m}p\left(n\;\middle|\;\begin{array}{l}\text{all odd parts allowed,}\\ \text{even parts}\equiv\ell\pmod{2r},\ \ell\in L_r,\\ m\text{ parts, }s\text{ even parts}\end{array}\right)(-1)^sz^mq^n,\\
Q_{r,L_r}(z;q)&:=\prod_{\substack{j=1\\ j\notin L_r}}^{2r}(-zq^j;q^{2r})_\infty\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}p\left(n\;\middle|\;\begin{array}{l}\text{all odd parts allowed,}\\ \text{even parts}\not\equiv\ell\pmod{2r},\ \ell\in L_r,\\ m\text{ parts, all distinct,}\end{array}\right)z^mq^n.
\end{aligned}$$

Theorem 1.7 now follows from the fact that $E_{r,L_r}(1;q)=Q_{r,L_r}(1;q)$, which is not difficult to obtain after a short calculation using Euler's identity.

The proof of Theorem 1.8 is similar to the proofs of Theorems 1.2 and 1.5, and can be seen from

$$\frac{\partial}{\partial z}\Big|_{z=1}(Q_{r,L_r}(z;q)-F_{r,L_r}(z;q))=Q_{r,L_r}(1;q)\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}.$$

To prove Theorem 1.9, we compute

$$\frac{\partial}{\partial z}\Big|_{z=1}(Q_{r,L_r}(z;q)-E_{r,L_r}(z;q)) = \prod_{\substack{j=1\\ j\notin L_r}}^{2r}(-q^j;q^{2r})_\infty\left(\sum_{k=1}^{\infty}\frac{q^k}{1+q^k}\right)$$
$$= \left(\prod_{\substack{j=1\\ j \text{ even},\ j\notin L_r}}^{2r}(-q^j;q^{2r})_\infty\right)(-q;q^2)_\infty\sum_{k=1}^{\infty}\frac{q^k}{1+q^k}. \tag{8}$$

Using the combinatorial interpretation of (7) in the proof of Theorem 1.3, Theorem 1.9 follows from (8).

## 5. Proof of Theorem 1.10

Let $r\in\mathbb{N}$, $L_r\subseteq\{2,4,\dots,2r\}$ and $O_r\subseteq\{1,3,\dots,2r-1\}$ as in Section 1. Also let $L_r^c=\{2,4,\dots,2r\}\setminus L_r$ and $O_r^c=\{1,3,\dots,2r-1\}\setminus O_r$. Define

$$\widetilde{E}_{r,L_r}(z;q) := \frac{1}{(q;q^2)_\infty\prod_{j\in L_r^c}(-q^j;q^{2r})_\infty\prod_{\ell\in L_r}(-zq^\ell;q^{2r})_\infty}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(p_e(n,m;L_r)-p_o(n,m;L_r))z^mq^n,$$
$$\widetilde{Q}_{r,O_r}(z;q) := \prod_{j\in O_r^c}(-q^j;q^{2r})_\infty\prod_{\ell\in O_r}(-zq^\ell;q^{2r})_\infty$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_o(n,m;O_r)z^mq^n,$$

where

$$p_{e/o}(n,m;L_r) := p\left(n\;\middle|\;\begin{array}{l}\text{all parts allowed,}\\ \text{even/odd no. of even parts,}\\ m\text{ (even) parts}\equiv\ell\pmod{2r},\ \ell\in L_r\end{array}\right),$$
$$q_o(n,m;O_r) := p\left(n\;\middle|\;\begin{array}{l}\text{all parts odd and distinct,}\\ m\text{ (odd) parts}\equiv\ell\pmod{2r},\ \ell\in O_r\end{array}\right).$$

When $z=1$, $\widetilde{E}_{r,L_r}(1;q)=\widetilde{Q}_{r,O_r}(1;q)$ recovers Lehmer's identity (2) in Theorem 1.1.

We compute that

$$\frac{\partial}{\partial z}\Big|_{z=1}(\widetilde{Q}_{r,O_r}(z;q)-\widetilde{E}_{r,L_r}(z;q)) = (-q;q^2)_\infty\sum_{\ell\in L_r\cup O_r}\sum_{k=0}^{\infty}\frac{q^{2kr+\ell}}{1+q^{2kr+\ell}}. \tag{9}$$

To prove Theorem 1.10, it suffices to prove the case where $L_r\cup O_r=\{\ell\}$ for each positive integer $\ell\le 2r$. In Section 5.1, we state and prove Proposition 5.1, which establishes the non-negativity of the $q$-series coefficients of the series in (9) (noting that special case $\ell=2$ is more delicate). We then prove Theorem 1.10, making use of Proposition 5.1 and its proof.

### 5.1. Non-negativity of $q$-series coefficients

We use the notation $F(q)\succeq 0$ to mean that the coefficients of $F(q)$ when expanded as a $q$-series are all non-negative.

**Proposition 5.1.** *Let $r\in\mathbb{N}$, and $\ell$ a positive integer such that $\ell\le 2r$.*
*If $\ell\neq 2$, then*

$$(-q;q^2)_\infty\sum_{k=0}^{\infty}\frac{q^{2kr+\ell}}{1+q^{2kr+\ell}}\succeq 0.$$

*If $\ell = 2$, then the only possible negative coefficients of*

$$(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2kr+2}}{1+q^{2kr+2}} \tag{10}$$

*(when expanded as a q-series) are the coefficients of $q^4, q^8, q^{12}, q^{16}$, and $q^{20}$, and any such negative coefficient is equal to $-1$. Precisely, the set of all n such that the coefficient of $q^n$ in (10) is negative (and thus equal to $-1$) is given as a function of r in the following table:*

| $r$ | $\{n\}$ |
|---|---|
| 1 | $\{\,\}$ |
| 2, 4, 7 | $\{4, 8, 12\}$ |
| 3 | $\{4\}$ |
| 5 | $\{4, 8\}$ |
| 6, 9 | $\{4, 8, 12, 16\}$ |
| 8, *or* $\geq 10$ | $\{4, 8, 12, 16, 20\}$ |

**Proof of Proposition 5.1.** We divide our proof into three cases: $\ell$ odd, $\ell$ even but $\ell \neq 2$, and $\ell = 2$.

For $0 < \ell \leq 2r$ odd, we have that

$$(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1+q^{2kr+\ell}} = \sum_{k=0}^{\infty} \prod_{\substack{m=0 \\ 2m+1 \neq 2kr+\ell}}^{\infty} q^{2kr+\ell}(1+q^{2m+1}) \succeq 0. \tag{11}$$

For $0 < \ell \leq 2r$ even, we note that

$$(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1+q^{2kr+\ell}} = \sum_{k=0}^{\infty} (-q;q^2)_\infty (1-q^{2kr+\ell}) \frac{q^{2kr+\ell}}{1-q^{2(2kr+\ell)}}. \tag{12}$$

We first assume that $\ell \neq 2$. Using (12), it suffices to show that $(-q;q^2)_\infty(1-q^{2a}) \succeq 0$ for any integer $a \geq 2$. We apply the well-known identity (see, e.g., [1, (2.2.6) with $q \mapsto q^2, t \mapsto q$])

$$(-q;q^2)_\infty = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}$$

to re-write

$$\begin{aligned}(-q;q^2)_\infty(1-q^{2a}) &= (1-q^{2a}) \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n} \\ &= (1-q^{2a}) + \frac{(1-q^{2a})}{(1-q^2)} \left( \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^4;q^2)_{n-1}} \right) \\ &= 1 - q^{2a} + \left( \sum_{t=0}^{a-1} q^{2t} \right) \left( \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^4;q^2)_{n-1}} \right). \end{aligned} \tag{13}$$

Thus, it suffices to show that the coefficient of $q^{2a}$ in the $q$-series expansion of

$$\left( \sum_{t=0}^{a-1} q^{2t} \right) \left( \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^4;q^2)_{n-1}} \right) \tag{14}$$

is strictly positive. We re-write $2a = u^2 + v$, where $u^2$ is the largest even perfect square at most equal to $2a$ (with $u$ a non-negative even integer), and $v$ is a non-negative integer. Note that $u^2 \geq 4$ (so $u \geq 2$), since $2a \geq 4$. Since $u$ is even, $v$ is even, and since $u^2 \geq 4$, we have that $0 \leq v \leq 2(a-2)$. That is, $v = 2t$ for some $0 \leq t \leq a-2$. For this $t$, we consider

$$q^{2t} \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^4;q^2)_{n-1}}, \tag{15}$$

which appears in (14). We extract the $n = u$ term $q^{u^2}/(q^4;q^2)_{u-1}$ from the sum in (15), noting that $u \geq 2$. Expanding this as a $q$-series, we obtain

$$\frac{q^{u^2}}{(q^4;q^2)_{u-1}} = q^{u^2} + \Sigma_u(q), \tag{16}$$

where $\Sigma_u(q) \succeq 0$. Multiplying (16) by $q^{2t}$, we find the term $q^{2t+u^2} = q^{2a}$ in the $q$-expansion of (14); moreover, we have explained that $\Sigma_u(q) \succeq 0$, and it is clear that the remaining coefficients in the $q$-expansion of (14) are non-negative. This completes the proof of non-negativity in the case of even $\ell \neq 2$.

When $\ell = 2$, we begin with the identity in (13), which also holds for $a = 1$. In this case, the $q$-series expansion for the expression in (14) is $q + q^4 + O(q^8)$ (and has non-negative coefficients); that is, the coefficient of $q^2$ is 0. Thus, the $q$-expansion of (13) in this case is $1 + q - q^2 + q^4 + O(q^8)$, and has non-negative coefficients for all powers of $q$ greater than 4. Referring to (12), as above, we have that

$$\sum_{k=1}^{\infty} (-q;q^2)_\infty (1 - q^{2kr+\ell}) \frac{q^{2kr+\ell}}{1 - q^{2(2kr+\ell)}} \succeq 0 \tag{17}$$

for any even $\ell \in L_r$, including $\ell = 2$. We also have from the above that the $k = 0$ term from (12) satisfies

$$(-q;q^2)_\infty (1 - q^{\ell}) \frac{q^{\ell}}{1 - q^{2\ell}} \succeq 0 \tag{18}$$

for even $\ell \geq 4$. For $\ell = 2$, we have shown that the only negative term appearing in the $q$-expansion

$$(-q;q^2)_\infty (1 - q^2) = 1 + q - q^2 + q^4 + q^8 + q^9 + q^{12} + q^{13} + q^{15} + 2q^{16} + O(q^{17}) \tag{19}$$

is $-q^2$. We multiply (19) by

$$\frac{q^2}{1 - q^4} = q^2 + q^6 + q^{10} + q^{14} + q^{18} + q^{22} + O(q^{26}) \tag{20}$$

(which clearly has non-negative $q$-series coefficients) to obtain

$$q^2 + q^3 - q^4 + 2q^6 + q^7 - q^8 + 3q^{10} + 2q^{11} - q^{12} + 4q^{14} + 3q^{15} - q^{16} + q^{17} + 6q^{18} + 4q^{19} - q^{20} + \Sigma(q), \tag{21}$$

where $\Sigma(q) = O(q^{21})$. We now argue that $\Sigma(q) \succeq 0$. It is not difficult to see that the only powers of $q$ in the expansion for $\Sigma(q)$ which may possibly have negative coefficients are those $q^m$ such that $m \equiv 0 \pmod 4$, $m \geq 24$ (where we have also used that $\Sigma(q) = O(q^{21})$). Now, any $m \equiv 0 \pmod 4$ such that $m \geq 24$ can also be written as $m = (3+1)^2 + 6 + (2+4c)$ for some integer $c \geq 0$. Thus, we also obtain the term $+q^m$ after multiplying (19) and (20) as follows. We use the expression in (13) (with $a = 1$ and $t = 0$) for (19), and take the numerator $q^{(3+1)^2}$ of the $n = 3$ term and also $q^6$ from the expansion of the denominator $(q^4;q^2)_3$ of that same term. This yields a term $q^{(3+1)^2+6}$ after multiplying. We now multiply by the term $q^{2+4c}$ from the expansion of $q^2/(1-q^4)$ in (20). Overall, this yields after multiplication the term $q^{(3+1)^2+6+(2+4c)} = q^m$, which cancels with the earlier $-q^m$. This shows that $\Sigma(q) \succeq 0$.

Thus, the only negative coefficients of the series in (21) are $q^4, q^8, q^{12}, q^{16}$, and $q^{20}$, and these coefficients are all equal to $-1$. When added to the rest of the sum in (17) (which has non-negative coefficients), this argument shows that the only powers of $q$ in the expansion of (10) (equivalently, (12) with $\ell = 2$) with potentially negative coefficients are $q^4, q^8, q^{12}, q^{16}, q^{20}$, and that any such negative coefficient must be $-1$, as claimed. Moreover, for $r \geq 10$, and any $k \geq 1$, we have that

$$\frac{q^{2kr+2}}{1 - q^{2(2kr+2)}} = O(q^{22}),$$

which, when combined with the above argument, proves that the coefficients of $q^4, q^8, q^{12}, q^{16}$, and $q^{20}$ are all equal to $-1$. The remaining negative coefficients as given in the table in Proposition 5.1 for $1 \leq r \leq 9$ are easily calculated directly. □

### 5.2. Combinatorial interpretation of Proposition 5.1

In this section, we give a combinatorial interpretation of the coefficient of $q^n$ in the $q$-series of Proposition 5.1 in terms of the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ satisfying certain conditions. This will complete the proof of Theorem 1.10.

Let $\ell$ be a positive integer such that $\ell \leq 2r$.

If $\ell$ is odd, then the coefficient of $q^n$ in (11) is the number of pairs of partitions $(\lambda, (a)) \vdash n$ satisfying $a \equiv \ell \pmod{2r}$, $\lambda \in \mathcal{Q}_o$, and $a \notin \lambda$.

If $\ell$ is even, $\ell \neq 2$, we substitute (13) in (12) to obtain

$$(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1+q^{2kr+\ell}} = \sum_{\substack{a \equiv \ell \pmod{2r} \\ a>0}} \left(1 + \left(\sum_{t=0}^{\frac{a}{2}-1} q^{2t}\right)\left(\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^4;q^2)_n}\right)\right) \frac{q^a}{1-q^{2a}} \tag{22}$$

$$- \sum_{\substack{a \equiv \ell \pmod{2r} \\ a>0}} q^a \frac{q^a}{1-q^{2a}}. \tag{23}$$

The $q$-series

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^4;q^2)_n}$$

is the generating series for the number of self-conjugate partitions of $n \geq 1$ which are either (1) or whose smallest part is at least 2. Thus, this is also the generating series of the number of partitions of $n \geq 1$ into distinct odd parts which are either (1) or partitions whose first two parts differ by exactly 2. Moreover,

$$\sum_{t=0}^{\frac{a}{2}-1} q^{2t}$$

is the generating series for partitions of $n$ into a single even part no larger than $a-2$.

By adding a non-negative even integer no larger than $a-2$ to (1) or the first part of a partition into distinct odd parts whose first two parts differ by exactly 2, we see that

$$\left(\sum_{t=0}^{\frac{a}{2}-1} q^{2t}\right)\left(\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^4;q^2)_n}\right)$$

is the generating series for the number of partitions $\lambda$ of $n \geq 1$ into distinct odd parts with $\lambda_1 - \lambda_2 \leq a$. Note that $\lambda_2$ can be 0.

Thus, (22) is the generating series for the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ with $b$ odd, $a \equiv \ell \pmod{2r}$, and $\lambda \in \mathcal{Q}_o$ satisfying $\lambda_1 - \lambda_2 \leq a$. Note that $\lambda$ may be empty.

To interpret (23), for $a \geq 4$ and even, define $\mu(a) \in \mathcal{Q}_o(a)$ to be the partition

$$\mu(a) := \begin{cases} \left(\frac{a}{2}+1, \frac{a}{2}-1\right) & \text{if } \frac{a}{2} \text{ is even} \\ \left(\frac{a}{2}+2, \frac{a}{2}-2\right) & \text{if } \frac{a}{2} \text{ is odd.} \end{cases} \tag{24}$$

Thus, (23) is the generating series for the number of pairs of partitions $(\mu(a), (a^b)) \vdash n$ with $b$ odd, $a \equiv \ell \pmod{2r}$.

Therefore, the coefficient of $q^n$ in

$$(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1+q^{2kr+\ell}}$$

is equal to the number of the pairs of partitions $(\lambda, (a^b)) \vdash n$ with $b$ odd, $a \equiv \ell \pmod{2r}$, and $\lambda \in \mathcal{Q}_o$ satisfying $\lambda_1 - \lambda_2 \leq a$ and $\lambda \neq \mu(a)$.

If $\ell = 2$, the argument above fails only in the interpretation of $q^a \frac{q^a}{1-q^{2a}}$ when $a = 2$. However, this $q$-series is just $\sum_{k=1}^{\infty} q^{4k}$. Thus, if $\ell = 2$ and $n \equiv 0 \pmod 4$, the coefficient of $q^n$ in (12) is one less than the number of pairs of partitions described above. If $n \geq 24$, $n \equiv 0 \pmod 4$, then $((9,7,5,1), (2^{(n-22)/2}))$ is a pair counted by the sequence whose generating series is (22). Thus, if $n \notin \{4, 8, 12, 16, 20\}$, the coefficient of $q^n$ in (12) is non-negative and equal to the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ with $b$ odd, $a \equiv \ell \pmod{2r}$, and $\lambda \in \mathcal{Q}_o$ satisfying $\lambda_1 - \lambda_2 \leq a$, $\lambda \neq \mu(a)$ and, if $n \equiv 0 \pmod 4$, also $(\lambda, (a^b)) \neq ((9,7,5,1), (2^{(n-22)/2}))$.

If $\ell = 2$, $r \geq 10$, and $n \in \{4, 8, 12, 16, 20\}$, then $q^n$ appears only when $a = 2$ in (22). If $b$ is odd, $n - 2b \equiv 2 \pmod 4$ and $n - 2b \leq 18$. Thus, a partition $\lambda \vdash n - 2b$ into distinct parts would have two parts. However, since $n - 2b \equiv 2 \pmod 4$, it follows that $\lambda_1 - \lambda_2 \geq 4$. Therefore, there are no pairs of partitions $(\lambda, (a^b))$ counted by the sequence whose generating series is (22) and the coefficient of $q^n$ in (12) is $-1$. For $r < 10$, one can easily verify that the values of $n$ giving negative coefficients are as in the statement of the theorem.

### 5.3. Examples of Theorem 1.10

In this section, we give some examples of Theorem 1.10 for specific choices of $L_r$ and $O_r$ in which the excess is non-negative for all $n$. Example 1 gives a new interpretation for the excess studied in Theorem 1.3. Examples 2 and 3, when specialized to $r = 1$, are also related to the excess studied in Theorem 1.3.

#### 5.3.1. Example 1: $L_r \cup O_r = \{1, \dots, 2r\}$

The excess in this case is the same as that in Theorem 1.3, but the combinatorial description given by Theorem 1.3 and Theorem 1.10 are different.

When $n \not\equiv 0 \pmod 4$, Theorem 1.10 becomes: The excess of the number of parts in all partitions in $\mathcal{Q}_o(n)$ plus the number of even parts in all partitions of $\mathcal{P}_e(n, 2)$ over the number of even parts in all partitions in $\mathcal{P}_o(n, 2)$ equals the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ satisfying the following conditions:

i. $b$ is odd. Moreover, if $a$ is odd, then $b = 1$,
ii. $\lambda \in \mathcal{Q}_o$. Moreover, if $a$ is odd, then $a \notin \lambda$; if $a$ is even, then $\lambda_1 - \lambda_2 \leq a$.

When $n \in \{4, 8, 12, 16, 20\}$, since $2 \in L_r \cup O_r$, Theorem 1.10 does not guarantee the non-negativity of the excess. However, since $L_r \supseteq \{2, 4, \dots, 2r\}$, when $n \equiv 0 \pmod 4$, we can exclude $(\varnothing, (n))$ instead of $((9, 7, 5, 1), (2^b))$, proving non-negativity of the excess. Specifically, the excess is now equal to the number of pairs of partitions $(\lambda, (a^b)) \vdash n$ satisfying the following conditions:

i. $b$ is odd. Moreover, if $a$ is odd, then $b = 1$,
ii. $\lambda \in \mathcal{Q}_o$. Moreover, if $a$ is odd, then $a \notin \lambda$; if $a$ is even, then $\lambda_1 - \lambda_2 \leq a$ and $\lambda \neq \mu(a)\}$; if $a \equiv 0 \pmod 4$ and $b = 1$, then $\lambda \neq \varnothing$.

With the modification explained above we have the following corollary of Theorem 1.10:

**Corollary 5.2.** *The excess of the number of even parts in all partitions of $\mathcal{P}_o(n, 2)$ over the number of even parts in all partitions of $\mathcal{P}_e(n, 2)$ equals the number of partitions $\lambda$ of $n$ such that exactly one part is even, all other parts are odd and distinct, the even part may be repeated an odd number of times and, if we write $\lambda = (\lambda^o \cup ((2k)^b))$ with $\lambda^o \in \mathcal{Q}_o$, $k \geq 1$, then $\lambda^o_1 - \lambda^o_2 \leq 2k$, $\lambda^o \neq \mu(2k)$, and if $k$ is even and $b = 1$ then $\lambda^o \neq \varnothing$.*

**Proof.** Corollary 5.2 follows from the fact that counting $(\lambda, (a)) \vdash n$ with $a$ odd and $a \notin \lambda$ is the same as counting parts in $\mathcal{Q}_o(n)$. When $a = 2k$, we insert the even (possibly repeated) part into the partition into distinct odd parts to obtain a statement similar to the original Beck conjecture. □

Combining Theorem 1.2 and Corollary 5.2, we arrive at the following corollary.

**Corollary 5.3.** *The excess of the number of odd parts in all partitions in $\mathcal{P}_e(n, 2)$ over the number of odd parts in $\mathcal{P}_o(n, 2) \cup \mathcal{Q}_o(n)$ equals the number of partitions $\lambda$ of $n$ satisfying either*

i. *$\lambda$ has exactly one even part, possibly repeated, and all other parts are odd and distinct, or*
ii. *all parts of $\lambda$ are odd and exactly one part $b$ is repeated. Moreover, let $\lambda^o = \lambda \setminus (b^{2k})$ be the partition obtained by removing from $\lambda$ the largest even number of parts equal to $b$, then $\lambda^o_1 - \lambda^o_2 \leq 2k$, $\lambda^o \neq \mu(2k)$, and if $k$ is even and $b = 1$ then $\lambda^o \neq \varnothing$.*

**Proof.** The excess in Corollary 5.3 is the sum of the excess in Theorem 1.2 and Corollary 5.2. The partitions described in ii. are disjoint from the partitions described in *i.* They are in one-to-one correspondence with the partitions described in Corollary 5.2. To see this correspondence, consider a partition $\mu = (\mu^o \cup ((2k)^b))$ as in Corollary 5.2. Then define $\lambda = \mu^o \cup (b^{2k})$. Now $b$, which is odd, is a repeated part. The part $b$ may have already existed in $\mu^o$, so its multiplicity in $\lambda$ can be even or odd. □

#### 5.3.2. Example 2: $L_r \cup O_r = \{r, 2r\}$

In this case,

$$\frac{\partial}{\partial z}\Big|_{z=1} (\widetilde{Q}_{r,O_r}(z; q) - \widetilde{E}_{r,L_r}(z; q)) = (-q; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{kr}}{1 + q^{kr}}.$$

Theorem 1.10 implies that the series has non-negative coefficients when $r \geq 3$, because $2 \notin L_r \cup O_r$. When $r = 1$, both Example 1 and Proposition 5.1 show that the series has non-negative coefficients. When $r = 2$, the series also has non-negative coefficients. As in Example 1, when $n \equiv 0 \pmod 4$, we can exclude $(\varnothing, (n))$ instead of $((9, 7, 5, 1), (2^b))$. Alternatively, one

can see that in Proposition 5.1, the coefficient of $q^n$ in $(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{4k+2}}{1+q^{4k+2}}$ is negative (and equal to $-1$) only for $n = 4, 8, 12$, while it can be computed that the coefficient of $q^4, q^8, q^{12}$ in $(-q;q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{4k+4}}{1+q^{4k+4}}$ are $1, 1, 4$, respectively.

*5.3.3. Example 3: $L_r \cup O_r = \{1, 2r\}$*

In this case,

$$\frac{\partial}{\partial z}\Big|_{z=1} (\widetilde{Q}_{r,O_r}(z;q) - \widetilde{E}_{r,L_r}(z;q)) = (-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2kr+1}}{1+q^{2kr+1}} + \sum_{k=1}^{\infty} \frac{q^{2kr}}{1+q^{2kr}} \right).$$

Theorem 1.10 (for $r \geq 2$) and Example 1 (for $r = 1$) imply that the series has non-negative coefficients.

**Remark 6.** The derivative differences given in Examples 2 and 3 also have interpretations as generating series for the number of pairs of partitions satisfying certain conditions as in Theorem 1.10.

## 6. Further non-negativity results

The derivative difference in Example 3 can be expressed as

$$(-q;q^2)_\infty \left( \sum_{k=1}^{\infty} \frac{q^{2kr}}{1-q^{4kr}} - \sum_{k=0}^{\infty} \frac{q^{2(2kr+1)}}{1-q^{2(2kr+1)}} \right) \tag{25}$$

$$+ (-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2kr+1}}{1-q^{2(2kr+1)}} - \sum_{k=1}^{\infty} \frac{q^{4kr}}{1-q^{4kr}} \right). \tag{26}$$

In Theorem 6.1, we show that (25) has non-positive coefficients, and, in Theorem 6.2, we show that (26) has non-negative coefficients.

**Theorem 6.1.** *For $r \in \mathbb{N}$, we have that*

$$(-q;q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2(2kr+1)}}{1-q^{2(2kr+1)}} - \sum_{k=1}^{\infty} \frac{q^{2kr}}{1-q^{4kr}} \right) \succeq 0.$$

**Proof.** It suffices to construct an injection $T$ from

$$A(n) := \{(\lambda, (a^b)) \vdash n \mid \lambda \in \mathcal{Q}_o, a \equiv 0 \pmod{2r}, b \text{ odd}\}$$

to

$$B(n) := \{(\lambda, (a^b)) \vdash n \mid \lambda \in \mathcal{Q}_o, a \equiv 1 \pmod{2r}, b \text{ even}\}.$$

Let $(\lambda, (a^b)) \in A(n)$. Let $0 \leq c < 2r$ be the remainder of $b-1$ when divided by $2r$. Note that $c$ is even. We partition the set $A(n)$ into two disjoint subsets:

$$A_1(n) := \{(\lambda, (a^b)) \in A(n) \mid \lambda \neq \varnothing\};$$
$$A_2(n) := \{(\lambda, (a^b)) \in A(n) \mid \lambda = \varnothing\}.$$

We define $T$ on each $A_i(n)$ in the following way.

1. If $\lambda \neq \varnothing$, then
$$T(\lambda, (a^b)) = (\lambda \setminus \{\lambda_1\} \cup \{\lambda_1 + ab - (a-c)(b-c)\}, ((b-c)^{a-c})).$$
2. If $\lambda = \varnothing$, then
$$T(\varnothing, (a^b)) = (\mu(ab - (a-c)(b-c)), ((b-c)^{a-c})).$$

The image sets are thus

$$T(A_1(n)) = \{(\mu, (x^y)) \in B(n) \mid \mu_1 - \mu_2 > (y+z)(x+z) - xy\},$$
$$T(A_2(n)) = \{(\mu, (x^y)) \in B(n) \mid \mu = \mu((x+z)(y+z) - xy)\},$$

where $z$ is the remainder of $-y$ when divided by $2r$.

Note that $T$ maps $(\lambda, (a^b)) \in A(n)$ with $b \equiv \ell \pmod{2r}$ to $(\mu, (x^y)) \in B(n)$ with $y \equiv -\ell + 1 \pmod{2r}$. When $b \equiv 1 \pmod{2r}$, $T(\varnothing, (a^b)) = (\varnothing, (b^a)) \notin T(A_1(n))$. When $b \equiv \ell \not\equiv 1 \pmod{2r}$, because $(y+z)(x+z) - xy \geq 2(x+y) + 4 \geq 4$, $T(A_1(n))$ and $T(A_1(n))$ are disjoint.

Define the map $L$ from $T(A(n))$ to $A(n)$ as follows:

1. If $(\mu, (x^y)) \in T(A_1)$, then
$$L(\mu, (x^y)) = (\mu \setminus \{\mu_1\} \cup \{\mu_1 - (y+z)(x+z) + xy\}, ((y+z)^{x+z})).$$
2. If $(\mu, (x^y)) \in T(A_2)$, then
$$L(\mu, (x^y)) = (\varnothing, ((y+z)^{x+z})).$$

Then $L$ and $T$ are inverse to each other. Hence $T$ is an injection and Theorem 6.1 follows. □

**Remark 7.** When $r = 1$, the injection $T$ is the *bijection* $(\lambda, (a^b)) \mapsto (\lambda, (b^a))$, where the conjugation $(a^b) \mapsto (b^a)$ was used in the proof of Theorem 1.3.

**Theorem 6.2.** *For $r \in \mathbb{N}$, we have that*
$$(-q; q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2kr+1}}{1 - q^{2(2kr+1)}} - \sum_{k=1}^{\infty} \frac{q^{4kr}}{1 - q^{4kr}} \right) \succeq 0.$$

**First Proof.** Because the derivative difference in Example 3 has non-negative coefficients, Theorem 6.2 follows from Theorem 6.1. □

**Second Proof.** Alternatively, we can also prove Theorem 6.2 directly. It suffices to construct an injection $T$ from
$$A(n) := \{(\lambda, (a^b)) \vdash n \mid \lambda \in \mathcal{Q}_o, a \equiv 0 \pmod{2r}, b \text{ even}\}$$
to
$$B(n) := \{(\lambda, (a^b)) \vdash n \mid \lambda \in \mathcal{Q}_o, a \equiv 1 \pmod{2r}, b \text{ odd}\}.$$

We partition the set $A(n)$ into three disjoint subsets:
$$A_1(n) := \{(\lambda, (a^b)) \in A(n) \mid a + (2r-1)(b-1) \notin \lambda\};$$
$$A_2(n) := \{(\lambda, (a^b)) \in A(n) \mid a + (2r-1)(b-1) \in \lambda \text{ and } \lambda \text{ has at least two parts}\};$$
$$A_3(n) := \{(\lambda, (a^b)) \in A(n) \mid \lambda = (a + (2r-1)(b-1))\}.$$

We define $T$ on each $A_i(n)$ in the following way.

1. If $(\lambda, (a^b)) \in A_1(n)$ (including the case where $\lambda$ is empty), then
$$T(\lambda, (a^b)) := (\lambda \cup \{a + (2r-1)(b-1)\}, ((a+1-2r)^{b-1})).$$
2. If $(\lambda, (a^b)) \in A_2(n)$, let $m$ denote the largest part of $\lambda$ that is not $(a + (2r-1)(b-1))$. Then $T(\lambda, (a^b))$ equals
$$((\lambda \setminus \{a + (2r-1)(b-1), m\}) \cup \{2(a + (2r-1)(b-1)) + m\}, ((a+1-2r)^{b-1})).$$
3. If $(\lambda, (a^b)) \in A_3(n)$, then
$$T(\lambda, (a^b)) := ((a+1, a + (2r-2)b - (2r-1)), ((a+1)^{b-1})).$$

The image sets are thus
$$T(A_1(n)) = \{(\mu, (c^d)) \in B(n) \mid c + (2r-1)(d+1) \in \mu\},$$
$$T(A_2(n)) = \left\{ (\mu, (c^d)) \in B(n) \;\middle|\; \begin{array}{l} c + (2r-1)(d+1) \notin \mu, \\ \mu_1 - \mu_2 > 2(c + (2r-1)(d+1)), \\ \mu_1 \neq 3(c + (2r-1)(d+1)) \end{array} \right\},$$
$$T(A_3(n)) = \{(\mu, (c^d)) \in B(n) \mid \mu = (c, c + (2r-2)d - 2)\}.$$

Note that when $r = 1$, $(2r-2)d - 2 = -2 < 0$, and $2 \leq 2(c + (2r-1)(d+1))$. When $r > 1$, $(2r-2)d - 2 > 0$ and $(2r-2)d - 2 \leq 2(c + (2r-1)(d+1))$. Hence $T(A_1(n)), T(A_2(n)), T(A_3(n))$ are pairwise disjoint.

Define the map $L$ from $T(A(n))$ to $A(n)$ as follows:

1. If $(\mu, (c^d)) \in T(A_1(n))$, then
$$L(\mu, (c^d)) := (\mu \setminus \{c + (2r-1)(d+1)\}, ((c + 2r - 1)^{d+1})).$$
2. If $(\mu, (c^d)) \in T(A_2(n))$, then we define $L(\mu, (c^d))$ by
$$((\mu \setminus \{\mu_1\}) \cup \{c + (2r-1)(d+1), \mu_1 - 2(c + (2r-1)(d+1))\}, ((c + 2r - 1)^{d+1})).$$
3. If $(\mu, (c^d)) \in T(A_3(n))$, then
$$L(\mu, (c^d)) := ((c - 1 + (2r-1)d), ((c-1)^{d+1})).$$

Then $L$ and $T$ are inverses of each other. Hence, $T$ is an injection and Theorem 6.2 follows. □

**Remark 8.** When $r = 1$, the second proof of Theorem 6.2 recovers the proof of Theorem 1.3.

## Declaration of competing interest

No conflicts of interest.

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