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Stabilization of Mechanical Systems on Semidirect Product Lie Groups with Broken Symmetry via Controlled Lagrangians ⋆

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Abstract: Motivated by the problem of stabilizing bottom-heavy underwater vehicles, we find the matching condition for controlled Lagrangians via the kinetic shaping for mechanical systems on a semidirect product Lie group with broken symmetry. Of particular interest is the problem of stabilizing a steady translational motion of an underwater vehicle. We show that the control by the kinetic shaping stabilizes the equilibrium, and also that an additional dissipative control renders it asymptotically stable.

Keywords: Stabilization; controlled Lagrangians; Euler–Poincaré mechanical systems; broken symmetry; semidirect product; underwater vehicle

1. INTRODUCTION

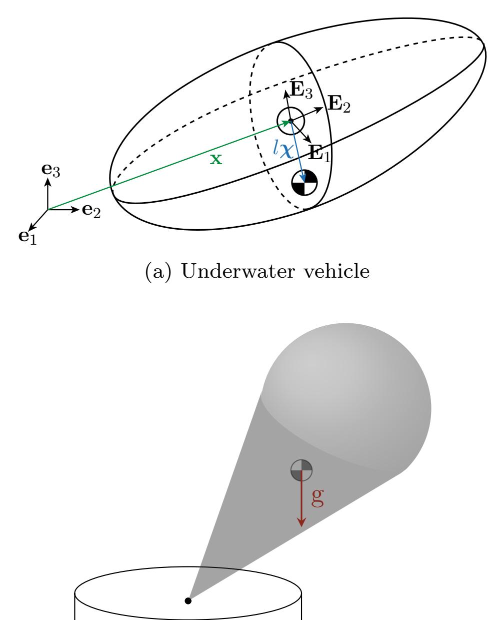
The goal of this paper is to stabilize a class of unstable equilibria of an ellipsoidal bottom-heavy underwater vehicle; see Fig. 1a. This system is very similar to the heavy top spinning on a movable base in Fig. 1b from our previous work Contreras and Ohsawa [2020] in the following sense:

(i) Their configuration space is the semidirect product Lie group $\mathsf{SE}(3) := \mathsf{SO}(3) \ltimes \mathbb{R}^3$.
(ii) One cannot decouple the dynamics into those in $\mathsf{SO}(3)$ and $\mathbb{R}^3$ as in the standard rigid body dynamics because of their interactions.
(iii) Their $\mathsf{SE}(3)$-symmetry is broken by the gravity.

In fact, we may write the equations of motion for both systems as Euler–Poincaré equations with advected parameters.

In Contreras and Ohsawa [2020], we found feedback controls that can be applied to the base to stabilize the upright spinning position of the heavy top by extending the method of controlled Lagrangians (see, e.g., Blankenstein et al. [2002], Bloch et al. [Oct 2001, 2001, Dec 2000], Chang et al. [2002], Chang and Marsden [2004], Hamberg [1999, 2000], Ortega et al. [1998, 2001]) to the Euler–Poincaré equations with advected parameters.

Despite the similarity to the heavy top on a movable base, the underwater vehicle poses its own challenge. Particularly, according to Kirchhoff's theory of rigid body interacting with potential flow, the body-fluid interactions result in the so-called added mass and added inertia; see Leonard [1997a], Leonard and Marsden [1997]. They depend on the shape of the vehicle, and in general the



(a) Underwater vehicle

(b) Heavy top spinning on movable base

Fig. 1. (a) Bottom-heavy underwater vehicle (see, e.g., Leonard [1997a,b], Leonard and Marsden [1997]); its center of mass (CM) is below the center of buoyancy (CB). (b) Heavy top spinning on a (point-mass) movable base Contreras and Ohsawa [2020]. In both systems, the configuration space is the semidirect product Lie group $\mathsf{SE}(3) := \mathsf{SO}(3) \ltimes \mathbb{R}^3$, and the gravity breaks the symmetry of the system.

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translational component of the mass matrix is not a multiple of the identity matrix as in the heavy top on a movable base.

We generalize our previous work to accommodate this difference, and show that the resulting control by the kinetic shaping stabilizes the class of unstable equilibria from Leonard [1997a] corresponding to steady translational motions of the vehicle. We also find an additional dissipative control that helps achieve the asymptotic stability.

2. SEMIDIRECT PRODUCT LIE GROUPS

We start with a brief summary of semidirect product Lie groups. This section is a condensed version of [Contreras and Ohsawa, 2021, Section 2].

2.1 Semidirect Product Lie Groups and Lie Algebras

Let G be a Lie group, V be a vector space, and $\mathsf{GL}(V)$ be the set of all invertible linear transformations on V. Let $\lambda\colon \mathsf{G} \to \mathsf{GL}(V)$ be a (left) representation of G on V, i.e., $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$ for any $g_1, g_2 \in \mathsf{G}$. We then define the semidirect product Lie group $\mathsf{S} := \mathsf{G} \ltimes V$ under the multiplication

$$s_1 \cdot s_2 = (g_1, x_1) \cdot (g_2, x_2) = (g_1 g_2, \lambda(g_1)x_2 + x_1).$$

Let $\mathfrak{g}$ be the Lie algebra of G. Then the representation λ induces the Lie algebra representation $\lambda'\colon \mathfrak{g} \to \mathfrak{gl}(V)$ as follows:

$$\begin{aligned}\lambda'(\xi)v &:= \left.\frac{d}{dt}\lambda(\exp(t\xi))v\right|_{t=0} \\ &= \lambda^i_{\alpha k}\xi^\alpha v^k \\ &= \xi_V(v),\end{aligned}$$

where ξ_V is the infinitesimal generator on V corresponding to ξ. Then we have the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} \ltimes V$ equipped with the commutator

$$\begin{aligned}\mathrm{ad}_{(\xi,v)}(\eta, w) &:= [(\xi, v), (\eta, w)] \\ &= (\mathrm{ad}_\xi \eta,\ \lambda'(\xi)w - \lambda'(\eta)v).\end{aligned}$$

Then one can find the momentum map $\mathbf{J}\colon T^*V \cong V \times V^* \to \mathfrak{g}^*$ associated with the cotangent lift of the G-action λ on V as follows:

$$\langle \mathbf{J}(v, a), \xi\rangle = \langle a, \xi_V(v)\rangle \iff J^k_{\alpha j} v^j a_k \xi^\alpha = a_k \lambda^k_{\alpha j}\xi^\alpha v^j,$$

which results in $J^k_{\alpha j} = \lambda^k_{\alpha j}$, i.e.,

$$\mathbf{J}(v, a) = \lambda^k_{\alpha j} v^j a_k. \tag{1}$$

This is nothing but the so-called diamond operator $\diamond\colon V \times V^* \to \mathfrak{g}^*$ (see Cendra et al. [1998], Holm et al. [1998] and [Holm et al., 2009, §7.5]), i.e., $v \diamond a = \mathbf{J}(v, a)$.

Let us also find an expression for the dual $\lambda'(\xi)^*$ of $\lambda'(\xi)$:

$$\begin{aligned}&\langle \lambda'(\xi)^* a, v\rangle = \langle a, \lambda'(\xi)v\rangle \\ &\qquad \iff (\lambda'(\xi)^* a)_k v^k = a_i \lambda^i_{\alpha k}\xi^\alpha v^k,\end{aligned}$$

which gives

$$(\lambda'(\xi)^* a)_k = \lambda^i_{\alpha k}\xi^\alpha a_i.$$

We may now write the coadjoint representation on the dual $\mathfrak{s}^*$ of $\mathfrak{s}$ as follows:

$$\begin{aligned}\mathrm{ad}^*_{(\xi,v)}(\mu, a) &= \left(\mathrm{ad}^*_\xi \mu - \mathbf{J}(v, a),\ \lambda'(\xi)^* a\right) \\ &= \left(c^\beta_{\gamma\alpha}\xi^\gamma \mu_\beta - \lambda^i_{\alpha j} v^j a_i,\ \lambda^j_{\beta i}\xi^\beta a_j\right).\end{aligned} \tag{2}$$

Example 1. ($\mathsf{S} = \mathsf{SE}(3) = \mathsf{SO}(3) \ltimes \mathbb{R}^3$). Consider the representation $\lambda\colon \mathsf{SO}(3) \to \mathsf{GL}(\mathbb{R}^3) = \mathsf{GL}(3, \mathbb{R})$ defined by the standard matrix-vector multiplication, i.e., $\lambda(R)\mathbf{x} = R\mathbf{x}$. Then, in terms of the hat map

$$\hat{(\,\cdot\,)}\colon \mathbb{R}^3 \to \mathfrak{so}(3); \qquad \mathbf{a} \mapsto \hat{a} := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix},$$

we have

$$\begin{aligned}\lambda'(\hat{\Omega})\mathbf{v} = \lambda'_{\mathbf{v}}(\hat{\Omega}) &= \left.\frac{d}{dt}\exp(t\hat{\Omega})\mathbf{v}\right|_{t=0} \\ &= \hat{\Omega}\mathbf{v} \\ &= \boldsymbol{\Omega} \times \mathbf{v} \\ &= \varepsilon^i{}_{\alpha k}\Omega^\alpha v^k.\end{aligned}$$

Therefore, we have $\lambda^i_{\alpha k} = \varepsilon^i{}_{\alpha k}$. Note that, using the above identification of $\mathbb{R}^3$ with $\mathfrak{so}(3)$, the structure constants satisfy $c^\alpha_{\beta\gamma} = \varepsilon^\alpha{}_{\beta\gamma}$ as well. As a result, we may write the coadjoint action as follows:

$$\begin{aligned}\mathrm{ad}^*_{(\boldsymbol{\Omega},\mathbf{v})}(\boldsymbol{\mu}, \mathbf{a}) &= \left(\varepsilon^\gamma{}_{\beta\alpha}\,\Omega^\beta \mu_\gamma - \varepsilon^k{}_{\alpha j}\, v^j a_k,\ \varepsilon^j{}_{\beta i}\,\Omega^\beta a_j\right) \\ &= (\boldsymbol{\mu} \times \boldsymbol{\Omega} + \mathbf{a} \times \mathbf{v},\ \mathbf{a} \times \boldsymbol{\Omega}).\end{aligned}$$

3. EULER–POINCARÉ EQUATION WITH ADVECTED PARAMETERS

3.1 Recovering Broken Symmetry of Lagrangian

Consider a mechanical system defined on a semidirect product Lie group $\mathsf{S} = \mathsf{G} \ltimes V$ with Lagrangian $L_{\Gamma_0}\colon T\mathsf{S} \to \mathbb{R}$ with parameters $\Gamma_0 \in X^*$, where X^* is the dual of a vector space X. Specifically, we consider the Lagrangian of the following form:

$$L_{\Gamma_0}(s, \dot{s}) = \frac{1}{2}\langle\!\langle \dot{s}, \dot{s}\rangle\!\rangle - U_{\Gamma_0}(s),$$

where $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ is a left-invariant metric on $T\mathsf{S}$, i.e., for any $s, s_0 \in \mathsf{S}$ and any $\dot{s} \in T_s\mathsf{S}$,

$$\langle\!\langle T_s\mathsf{L}_{s_0}(\dot{s}), T_s\mathsf{L}_{s_0}(\dot{s})\rangle\!\rangle = \langle\!\langle \dot{s}, \dot{s}\rangle\!\rangle,$$

where L stands for the left translation, i.e., $\mathsf{L}_{s_0}(s) = s_0 s$ for any $s_0, s \in \mathsf{S}$, and $T\mathsf{L}$ is its tangent lift. So the kinetic term is S-invariant.

Suppose however that the potential is *not* S-invariant, i.e., there exist $s_0, s \in \mathsf{S}$ such that $U_{\Gamma_0}(s_0 s) \neq U_{\Gamma_0}(s)$. This breaks the S-symmetry of the Lagrangian L_{Γ_0}.

We further suppose that we can fix this in the following way: Define an extended potential $U\colon \mathsf{S} \times X^* \to \mathbb{R}$ so that $U(s, \Gamma_0) = U_{\Gamma_0}(s)$ for any $s \in \mathsf{S}$, and let $\kappa\colon \mathsf{S} \to \mathsf{GL}(X)$ be a representation of S on X, and $\kappa^*\colon \mathsf{S} \to \mathsf{GL}(X^*)$ be the induced representation on the dual X^*. We assume that κ helps us recover the S-symmetry of the potential as follows: For any $s_0, s \in \mathsf{S}$ and any $\Gamma \in X^*$,

$$U(s_0 s, \kappa(s)^*\Gamma) = U(s_0 s, \Gamma).$$

Now let us define an extended Lagrangian $L\colon T\mathsf{S} \times X^* \to \mathbb{R}$ by setting

$$L(s, \dot{s}, \Gamma) := \frac{1}{2}\langle\!\langle \dot{s}, \dot{s}\rangle\!\rangle - U(s, \Gamma),$$

and also define the action

$$\begin{aligned}&\Psi\colon \mathsf{S} \times (T\mathsf{S} \times X^*) \to T\mathsf{S} \times X^*; \\ &\quad (s_0, (s, \dot{s}, \Gamma)) \mapsto \Psi_{s_0}(s, \dot{s}, \Gamma) := (s_0 s, T_s\mathsf{L}_{s_0}(\dot{s}), \kappa^*(s_0)\Gamma).\end{aligned}$$

Then we see that the extended Lagrangian now possesses the S-symmetry, i.e., $L \circ \Psi_{s_0} = L$ for any $s_0 \in \mathsf{S}$.

3.2 Euler–Poincaré Equation with Advected Parameters

Let us define (with an abuse of notation) the reduced potential

$$U\colon X^* \to \mathbb{R}; \qquad U(\Gamma) := U(e, \Gamma),$$

as well as the reduced Lagrangian $\ell\colon \mathfrak{s} \times X^* \to \mathbb{R}$ as

$$\ell(\xi, v, \Gamma) := L(e, (\xi, v), \Gamma) = K(\xi, v) - U(\Gamma)$$

with the kinetic energy term K defined as

$$\begin{aligned} K(\xi, v) &:= \frac{1}{2}\langle\!\langle (\xi, v), (\xi, v) \rangle\!\rangle \\ &= \frac{1}{2}\mathbb{G}_{\alpha\beta}\xi^\alpha\xi^\beta + \mathbb{G}_{\alpha j}\xi^\alpha v^j + \frac{1}{2}\mathbb{G}_{ij}v^i v^j. \end{aligned} \tag{3}$$

We also define $\mathbb{G}_{i\beta} := \mathbb{G}_{\beta i}$ component-by-component so that $\mathbb{G}_{\alpha j}\xi^\alpha v^j = \mathbb{G}_{i\beta}v^i\xi^\beta$.

The Euler–Poincaré equation with advected parameters (see Cendra et al. [1998], Holm et al. [1998] and [Holm et al., 2009, §7.5]) are then given by

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta \ell}{\delta(\xi, v)}\right) &= \operatorname{ad}^*_{(\xi,v)} \frac{\delta \ell}{\delta(\xi, v)} + \mathbf{K}\left(\frac{\delta \ell}{\delta \Gamma}, \Gamma\right), \\ \frac{d\Gamma}{dt} &= \kappa'(\xi, v)^*\Gamma, \end{aligned}$$

where we defined, for any smooth function $f\colon E \to \mathbb{R}$ on a real vector space E, its functional derivative $\delta f/\delta x \in E^*$ at $x \in E$ such that, for any $\delta x \in E$, under the natural dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle\colon E^* \times E \to \mathbb{R}$,

$$\left\langle \frac{\delta f}{\delta x}, \delta x \right\rangle = \left.\frac{d}{dt} f(x + t\,\delta x)\right|_{t=0}.$$

Note also that $\mathbf{K}\colon X \times X^* \to \mathfrak{s}^*$ is the momentum map associated with the above action κ defined in a similar manner to $\mathbf{J}$:

$$\begin{aligned} \langle \mathbf{K}(x, \Gamma), (\xi, v) \rangle &= \langle \mathbf{K}_{\mathfrak{g}^*}(x, \Gamma), \xi \rangle + \langle \mathbf{K}_{V^*}(x, \Gamma), v \rangle \\ &= \langle \Gamma, \kappa'(\xi, v)(x) \rangle, \end{aligned}$$

where we split the components of $\mathbf{K}$ into those in $\mathfrak{g}^*$ and V^* as $\mathbf{K}_{\mathfrak{g}^*}$ and $\mathbf{K}_{V^*}$. Then, using the formula (2) for the coadjoint action on $\mathfrak{s}^*$, we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta \ell}{\delta \xi}\right) &= \operatorname{ad}^*_\xi \frac{\delta \ell}{\delta \xi} - \mathbf{J}\left(v, \frac{\delta \ell}{\delta v}\right) + \mathbf{K}_{\mathfrak{g}^*}\left(\frac{\delta \ell}{\delta \Gamma}, \Gamma\right), \\ \frac{d}{dt}\left(\frac{\delta \ell}{\delta v}\right) &= \lambda'(\xi)^* \frac{\delta \ell}{\delta v} + \mathbf{K}_{V^*}\left(\frac{\delta \ell}{\delta \Gamma}, \Gamma\right), \\ \frac{d\Gamma}{dt} &= \kappa'(\xi, v)^*\Gamma. \end{aligned} \tag{4}$$

Example 2. (Underwater vehicle). Consider the underwater vehicle shown in Fig. 1a in ideal potential flow. Following Leonard [1997a,b], Leonard and Marsden [1997], the configuration space is $\mathsf{S} = \mathsf{SE}(3)$, i.e., rotations about the center of buoyancy and its translational positions. Let $\{\mathbf{e}_i\}_{i=1}^3$ and $\{\mathbf{E}_i\}_{i=1}^3$ be the orthonormal spatial/inertial and body frames, respectively; the origin of the body frame is at the center of buoyancy. The orientation $R \in \mathsf{SO}(3)$ of the vehicle is defined so that $\mathbf{E}_i = R\mathbf{e}_i$ for $i = 1, 2, 3$. Note that our definitions of $\mathbf{e}_3$ and $\mathbf{E}_3$ are the opposite of those in Leonard [1997a,b], Leonard and Marsden [1997]. Letting $\mathbf{x} \in \mathbb{R}^3$ be the position of the center of buoyancy in the spatial frame, we have an element $(R, \mathbf{x}) \in \mathsf{SE}(3)$ giving the orientation and the position of the vehicle.

The kinetic energy of the system is given in the form (3); see Section 5.1 below for more details. On the other hand, assuming the neutral buoyancy, the potential term is given as

$$U_{\mathbf{e}_3}(R, \mathbf{x}) = mgl\mathbf{e}_3 \cdot (R\boldsymbol{\chi}) = mgl\boldsymbol{\chi} \cdot (R^{-1}\mathbf{e}_3),$$

where $l\boldsymbol{\chi}$ is the position vector—l being its length and $\boldsymbol{\chi}$ being the unit vector for the direction—of the center of mass measured from the center of buoyancy; see Fig. 1a. Hence we define the extended potential $U\colon \mathsf{SE}(3) \times (\mathbb{R}^3)^* \to \mathbb{R}$ by setting

$$U((R, \mathbf{x}), \mathbf{\Gamma}) := mgl\boldsymbol{\chi} \cdot (R^{-1}\mathbf{\Gamma})$$

so that $U((R, \mathbf{x}), \mathbf{e}_3) = U_{\mathbf{e}_3}(R, \mathbf{x})$.

Also define the representation

$$\kappa\colon \mathsf{SE}(3) \to \mathsf{GL}(\mathbb{R}^3); \qquad \kappa(R, \mathbf{x})\mathbf{y} := R\mathbf{y}.$$

Then, identifying $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$ via the inner product, we have

$$\kappa^*(R, \mathbf{x})\mathbf{\Gamma} = R\mathbf{\Gamma}.$$

As a result we recovered the $\mathsf{SE}(3)$-symmetry: For any $(R_0, \mathbf{x}_0), (R, \mathbf{x}) \in \mathsf{SE}(3)$ and any $\mathbf{\Gamma} \in \mathbb{R}^3$,

$$U((R_0, \mathbf{x}_0) \cdot (R, \mathbf{x}), \kappa^*(R_0, \mathbf{x}_0)\mathbf{\Gamma}) = U((R, \mathbf{x}), \mathbf{\Gamma}).$$

Hence we may define the reduced potential $U\colon (\mathbb{R}^3)^* \to \mathbb{R}$ as

$$U(\mathbf{\Gamma}) := U((I, \mathbf{0}), \mathbf{\Gamma}) = mgl\boldsymbol{\chi} \cdot \mathbf{\Gamma},$$

and the reduced Lagrangian $\ell\colon \mathfrak{se}(3) \times (\mathbb{R}^3)^* \to \mathbb{R}$ as

$$\ell(\mathbf{\Omega}, \mathbf{v}, \mathbf{\Gamma}) = \frac{1}{2}\langle\!\langle (\mathbf{\Omega}, \mathbf{v}), (\mathbf{\Omega}, \mathbf{v}) \rangle\!\rangle - mgl\boldsymbol{\chi} \cdot \mathbf{\Gamma}.$$

The representation κ gives rise to the momentum map

$$\begin{aligned} \mathbf{K}(\mathbf{y}, \mathbf{\Gamma}) &= \left(\mathbf{K}_{\mathfrak{so}(3)^*}(\mathbf{y}, \mathbf{\Gamma}), \mathbf{K}_{(\mathbb{R}^3)^*}(\mathbf{y}, \mathbf{\Gamma})\right) \\ &= (\mathbf{y} \times \mathbf{\Gamma}, \mathbf{0}). \end{aligned}$$

As a result, the Euler–Poincaré equation (4) with advected parameters gives

$$\begin{aligned} \dot{\mathbf{\Pi}} &= \mathbf{\Pi} \times \mathbf{\Omega} + \mathbf{P} \times \mathbf{v} - mgl\boldsymbol{\chi} \times \mathbf{\Gamma}, \\ \dot{\mathbf{P}} &= \mathbf{P} \times \mathbf{\Omega}, \\ \dot{\mathbf{\Gamma}} &= \mathbf{\Gamma} \times \mathbf{\Omega} \end{aligned} \tag{5}$$

as in Leonard [1997a,b], Leonard and Marsden [1997], where we defined the the angular and linear impulses as

$$\Pi_\alpha := \frac{\partial \ell}{\partial \Omega^\alpha} = \mathbb{G}_{\alpha\beta}\Omega^\beta + \mathbb{G}_{\alpha j}v^j$$

and

$$P_i := \frac{\partial \ell}{\partial v^i} = \mathbb{G}_{i\beta}\Omega^\beta + \mathbb{G}_{ij}v^j,$$

respectively.

4. CONTROLLED LAGRANGIAN AND MATCHING

4.1 Controlled Euler–Poincaré Equation with Advected Parameters

Consider the controlled Euler–Poincaré equation with advected parameters:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta \ell}{\delta \xi}\right) &= \operatorname{ad}^*_\xi \frac{\delta \ell}{\delta \xi} - \mathbf{J}\left(v, \frac{\delta \ell}{\delta v}\right) + \mathbf{K}_{\mathfrak{g}^*}\left(\frac{\delta \ell}{\delta \Gamma}, \Gamma\right) + u^{\mathfrak{g}^*}, \\ \frac{d}{dt}\left(\frac{\delta \ell}{\delta v}\right) &= \lambda'(\xi)^* \frac{\delta \ell}{\delta v} + \mathbf{K}_{V^*}\left(\frac{\delta \ell}{\delta \Gamma}, \Gamma\right) + u^{V^*}, \\ \frac{d\Gamma}{dt} &= \kappa'(\xi, v)^*\Gamma. \end{aligned} \tag{6}$$

Note that, for our special case of interest with $\mathsf{S} = \mathsf{SE}(3)$, the control $u^{\mathfrak{g}^*} = u^{\mathfrak{so}(3)^*}$ is a torque whereas $u^{V^*} = u^{(\mathbb{R}^3)^*}$ is a linear force. Hence we refer to $u^{\mathfrak{g}^*}$ and u^{V^*} as a torque and a force, respectively, for the general case as well.

We would like to match this control system with the Euler–Poincaré equation with advected parameters for a different reduced Lagrangian $\ell_{\tau,\sigma,\rho} \colon \mathfrak{s} \times X^* \to \mathbb{R}$:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta \ell_{\tau,\sigma,\rho}}{\delta \xi}\right) &= \mathrm{ad}_\xi^* \frac{\delta \ell_{\tau,\sigma,\rho}}{\delta \xi} - \mathbf{J}\left(v, \frac{\delta \ell_{\tau,\sigma,\rho}}{\delta v}\right) \\ &\quad + \mathbf{K}_{\mathfrak{g}^*}\left(\frac{\delta \ell_{\tau,\sigma,\rho}}{\delta \Gamma}, \Gamma\right), \\ \frac{d}{dt}\left(\frac{\delta \ell_{\tau,\sigma,\rho}}{\delta v}\right) &= \lambda'(\xi)^* \frac{\delta \ell_{\tau,\sigma,\rho}}{\delta v} + \mathbf{K}_{V^*}\left(\frac{\delta \ell_{\tau,\sigma,\rho}}{\delta \Gamma}, \Gamma\right), \\ \frac{d\Gamma}{dt} &= \kappa'(\xi, v)^* \Gamma. \end{aligned} \tag{7}$$

4.2 Controlled Lagrangian

Now we would like to find the controlled Lagrangian $\ell_{\tau,\sigma,\rho}$ such that (7) gives (6). Then we determine the controls $(u^{\mathfrak{g}^*}, u^{V^*})$ such that (6) and (7) become equivalent. As a result, the dynamics of the controlled system (6) is described by the "free" system (7) with the new Lagrangian $\ell_{\tau,\sigma,\rho}$.

Specifically, we would like to seek the controlled Lagrangian of the form

$$\ell_{\tau,\sigma,\rho}(\xi, v, \Gamma) := K_{\tau,\sigma,\rho}(\xi, v) - U(\Gamma),$$

where $K_{\tau,\sigma,\rho}$ is the modified kinetic energy as in Bloch et al. [2001]: Using the kinetic energy K and the metric tensor $\mathbb{G}$ from (3), in index notation,

$$K_{\tau,\sigma,\rho}(\xi, v) := K(\xi, v) + \frac{1}{2}\Delta_{\alpha\beta}\xi^\alpha\xi^\beta + \Delta_{i\beta}v^i\xi^\beta + \frac{1}{2}\Delta_{ij}v^iv^j$$

with

$$\begin{aligned} \Delta_{\alpha\beta} &:= \left(\mathbb{G}_{i\beta} + \sigma_{ij}\tau^j_\beta\right)\tau^i_\alpha + \Delta_{i\beta}\left(\mathbb{G}^{ik}\mathbb{G}_{k\alpha} + \tau^i_\alpha\right), \\ \Delta_{i\beta} &:= \rho_{ij}\left(\mathbb{G}^{jk}\mathbb{G}_{k\beta} + \tau^j_\beta\right) - \mathbb{G}_{i\beta}, \\ \Delta_{ij} &:= \rho_{ij} - \mathbb{G}_{ij}, \end{aligned}$$

where $\mathbb{G}^{ij}$ stands for the *inverse* of the matrix $\mathbb{G}_{ij}$ and we use the same convention for other matrices too; σ, ρ, and τ are constant matrices—σ and ρ being symmetric—to be determined below.

4.3 Matching Condition for Control by Linear Forces

Suppose that we would like to stabilize the system by applying external (linear) forces to the system. Practically speaking, the system is either pushed by some external means or controlled by jets attached to the body; the latter is more amenable to our formulation because our equations are written in the body frame.

Assuming that there is no torque applied to the system as a control, we have $u^{\mathfrak{g}^*} = 0$ (see Contreras and Ohsawa [2021] for matching with $u^{\mathfrak{g}^*} \neq 0$ and $u^{V^*} \neq 0$ with a potential shaping). In order to satisfy this condition, it is sufficient to impose

$$\frac{\delta \ell_{\tau,\sigma,\rho}}{\delta \xi} = \frac{\delta \ell}{\delta \xi}, \qquad \mathbf{J}\left(v, \frac{\delta \ell}{\delta v} - \frac{\delta \ell_{\tau,\sigma,\rho}}{\delta v}\right) = 0. \tag{8}$$

The first condition is equivalent to $\Delta_{\alpha\beta}\xi^\beta + \Delta_{\alpha j}v^j = 0$ for any $\xi \in \mathfrak{g}$ and any $v \in V$. Hence this reduces to $\Delta_{\alpha\beta} = 0$ and $\Delta_{\alpha j} = 0$. Then $\Delta_{i\beta} = 0$ as well, but then this gives

$$\tau^i_\beta = \left(\rho^{ij} - \mathbb{G}^{ij}\right)\mathbb{G}_{j\beta}, \tag{9}$$

whereas substituting $\Delta_{i\beta} = 0$ into $\Delta_{\alpha\beta} = 0$, we obtain $\left(\mathbb{G}_{i\beta} + \sigma_{ij}\tau^j_\beta\right)\tau^i_\alpha = 0$. We see that this is satisfied if $\mathbb{G}_{i\beta} + \sigma_{ij}\tau^j_\beta = 0$, but then this in turn is satisfied if

$$\sigma^{ij} = \mathbb{G}^{ij} - \rho^{ij}. \tag{10}$$

On the other hand, the second condition in (8) is written as, using (1), $\lambda^k_{\alpha j}v^j(\Delta_{k\beta}\xi^\beta + \Delta_{kl}v^l) = 0$. Taking $\Delta_{i\beta} = 0$ and the expression for Δ_{kl} into account, we have $\lambda^k_{\alpha j}(\rho_{kl} - \mathbb{G}_{kl})v^jv^l = 0$. Since this holds for any $v \in V$, it implies that $\lambda^k_{\alpha j}(\rho_{kl} - \mathbb{G}_{kl})$ is skew-symmetric with respect to the indices (j, l), i.e.,

$$\lambda^k_{\alpha l}(\rho_{kj} - \mathbb{G}_{kj}) = -\lambda^k_{\alpha j}(\rho_{kl} - \mathbb{G}_{kl}). \tag{11}$$

As a result, we have the following: Under the matching conditions (9)—(11) and the feedback control

$$u^{\mathfrak{g}^*} = 0, \qquad u^{V^*}_i = (\mathbb{G}_{ij} - \rho_{ij})\dot{v}^j - \lambda^j_{\beta i}(\mathbb{G}_{jk} - \rho_{jk})\xi^\beta v^k,$$

the systems (6) and (7) are equivalent. This result is a slight generalization of the main result from Contreras and Ohsawa [2020], in which we assumed that $\mathsf{S} = \mathsf{SE}(3)$.

Remark 3. We may get rid of the acceleration term $\dot{v}$ from the above feedback control law because we can rewrite (7) so that $(\dot{\xi}, \dot{v})$ is given in terms of functions of (ξ, v, Γ).

Example 4. ($\mathsf{S} = \mathsf{SE}(3)$). As seen in Example 1, $\lambda^i_{\alpha k} = \varepsilon^i{}_{\alpha k}$ in this case, and so the third matching condition (11) becomes $\varepsilon^k{}_{\alpha l}(\rho_{kj} - \mathbb{G}_{kj}) = -\varepsilon^k{}_{\alpha j}(\rho_{kl} - \mathbb{G}_{kl})$. One may select ρ so that $\rho_{ij} - \mathbb{G}_{ij}$ becomes a non-zero constant multiple of the identity matrix, i.e.,

$$\rho_{ij} = \mathbb{G}_{ij} - \mathcal{K}\,\delta_{ij} \text{ for some } \mathcal{K} \in \mathbb{R}\backslash\{0\}.$$

Then the above condition becomes $\varepsilon^j{}_{\alpha l} = -\varepsilon^l{}_{\alpha j}$, which is trivially satisfied. The feedback control then becomes

$$\mathbf{u}^{\mathfrak{so}(3)^*} = 0, \qquad \mathbf{u}^{(\mathbb{R}^3)^*} = \mathcal{K}(\dot{\mathbf{v}} + \mathbf{\Omega} \times \mathbf{v}). \tag{12}$$

Note that we do not impose the assumption that $\mathbb{G}_{ij}$ is a constant multiple of the identity here as we did in Contreras and Ohsawa [2020]; this is important in the application to underwater vehicles as we shall see below.

5. APPLICATION

5.1 Underwater Vehicle

Consider the underwater vehicle from Example 2; see also Leonard [1997a,b], Leonard and Marsden [1997]. In addition to the assumptions mentioned in Example 2, we assume that the shape of vehicle is ellipsoidal, and the body frame introduced in Example 2 is aligned with the principal axes of the body, and also that the center of mass is aligned with the third principal axis $\mathbf{E}_3$ and is bottom heavy.

Then we have

$$\mathbb{G}_{\alpha\beta} = \mathrm{diag}(I_1, I_2, I_3), \qquad \mathbb{G}_{\alpha j} = ml\hat{\chi}, \\ \mathbb{G}_{ij} = \mathrm{diag}(m_1, m_2, m_3)$$

with $\boldsymbol{\chi} = (0, 0, -1)$. We are further assuming that the semi-axis along the principal axis $\mathbf{E}_2$ is longer than others as shown in Fig. 1a; this results in $m_1 > m_2$ and $m_3 > m_2$ as shown in [Leonard, 1997a, Appendix B].

The equilibrium of our interest is the steady translational motion along $\mathbf{E}_2$, i.e.,

$$\zeta_{\rm e} := (\mathbf{\Omega}_{\rm e}, \mathbf{v}_{\rm e}, \mathbf{\Gamma}_{\rm e}) = (\mathbf{0}, (0, v_0, 0), (0, 0, 1)). \tag{13}$$

As shown in [Leonard, 1997a, Theorem 2], this equilibrium is unstable under our assumption that the vehicle is bottom-heavy with $m_1 > m_2$.

5.2 Stabilization

Our goal is to stabilize the equilibrium by applying an external force control $\mathbf{u}^{(\mathbb{R}^3)^*}$ to (5):

$$\begin{cases} \dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega} + \mathbf{P} \times \mathbf{v} - mgl\boldsymbol{\chi} \times \mathbf{\Gamma}, \\ \dot{\mathbf{P}} = \mathbf{P} \times \mathbf{\Omega} + \mathbf{u}^{(\mathbb{R}^3)^*}, \\ \dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega} \end{cases}$$
$$\iff \begin{cases} \dot{\mathbf{\Pi}}_{\rm c} = \mathbf{\Pi}_{\rm c} \times \mathbf{\Omega} + \mathbf{P}_{\rm c} \times \mathbf{v} - mgl\boldsymbol{\chi} \times \mathbf{\Gamma}, \\ \dot{\mathbf{P}}_{\rm c} = \mathbf{P}_{\rm c} \times \mathbf{\Omega}, \\ \dot{\mathbf{\Gamma}} = \mathbf{\Gamma} \times \mathbf{\Omega}, \end{cases} \tag{14}$$

where we defined

$$\mathbf{\Pi}_{\rm c} := \frac{\partial \ell_{\tau,\sigma,\rho}}{\partial \mathbf{\Omega}}, \qquad \mathbf{P}_{\rm c} := \frac{\partial \ell_{\tau,\sigma,\rho}}{\partial \mathbf{v}}.$$

Hence the corresponding energy

$$E_{\tau,\sigma,\rho}(\mathbf{\Omega}, \mathbf{v}, \mathbf{\Gamma}) := K_{\tau,\sigma,\rho}(\mathbf{\Omega}, \mathbf{v}) + U(\mathbf{\Gamma})$$

is an invariant of the system. Additionally, the controlled system has the following invariants called Casimirs:

$$C_1 := \mathbf{P}_{\rm c} \cdot \mathbf{P}_{\rm c}, \qquad C_2 := \mathbf{P}_{\rm c} \cdot \mathbf{\Gamma}, \qquad C_3 := \mathbf{\Gamma} \cdot \mathbf{\Gamma}.$$

We would like to establish the stability of the equilibrium $\zeta_{\rm e}$ using the energy-Casimir method. Specifically, we would like to find an energy-like invariant

$$\mathcal{E}(\mathbf{\Omega}, \mathbf{v}, \mathbf{\Gamma}) := E_{\tau,\sigma,\rho}(\mathbf{\Omega}, \mathbf{v}, \mathbf{\Gamma}) + \Phi(C_1, C_2, C_3) \tag{15}$$

with some smooth function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ so that the gradient $D\mathcal{E}(\zeta_{\rm e})$ vanishes and also the Hessian $D^2\mathcal{E}(\zeta_{\rm e})$ is positive-definite. Then $\mathcal{E}$ gives a control Lyapunov function to establish the stability of the equilibrium $\zeta_{\rm e}$.

One can show that the gradient $D\mathcal{E}(\zeta_{\rm e})$ vanishes if

$$D_1\Phi|_{\rm e} = \frac{1}{2(\mathcal{K} - m_2)}, \qquad D_2\Phi|_{\rm e} = 0, \qquad D_3\Phi|_{\rm e} = \frac{mgl}{2},$$

where D_i stands for the derivative with respect to the variable in the i-th variable, and $(\,\cdot\,)|_{\rm e}$ signifies that the function is evaluated at the equilibrium $\zeta_{\rm e}$.

On the other hand, by evaluating the leading principal minors of the Hessian $D^2\mathcal{E}|_{\rm e}$, one can show that it is positive-definite if all the components of the Hessian of Φ at $\zeta_{\rm e}$ vanish except

$$D_{11}^2\Phi|_{\rm e} = \frac{1}{(\mathcal{K} - m_2)^3 v_0^2}$$

and also the parameter $\mathcal{K}$ in (12) satisfies

$$m_2 < \mathcal{K} < \min\left\{ m_3, m_1 - \frac{m^2 l^2}{I_2} \right\}. \tag{16}$$

This implies that one may take

$$\Phi(C_1, C_2, C_3) = \frac{1}{2}\left(\frac{(C_1 - C_1|_{\rm e})^2}{(\mathcal{K} - m_2)^3 v_0^2} + \frac{C_1 - C_1|_{\rm e}}{\mathcal{K} - m_2} + mgl(C_3 - C_3|_{\rm e}) \right)$$

to satisfy the above conditions.

To summarize, we have the following: The unstable equilibrium (13) of the underwater vehicle is stabilized by the control (12) with any constant $\mathcal{K}$ satisfying the condition (16).

5.3 Asymptotic Stabilization by Dissipative Control

In order to achieve asymptotic stability, we would like to apply dissipative control in addition to (12), as is done in, e.g., Bloch et al. [Oct 2001,D, 2000]. Specifically, let us add a control $\mathbf{u}^{\rm d}$ to (14) as follows:

$$\begin{aligned} \dot{\mathbf{\Pi}}_{\rm c} &= \mathbf{\Pi}_{\rm c} \times \mathbf{\Omega} + \mathbf{P}_{\rm c} \times \mathbf{v} - mgl\boldsymbol{\chi} \times \mathbf{\Gamma}, \\ \dot{\mathbf{P}}_{\rm c} &= \mathbf{P}_{\rm c} \times \mathbf{\Omega} + \mathbf{u}^{\rm d}, \\ \dot{\mathbf{\Gamma}} &= \mathbf{\Gamma} \times \mathbf{\Omega}. \end{aligned} \tag{17}$$

This is equivalent to replacing $\mathbf{u}^{(\mathbb{R}^3)^*}$ by $\mathbf{u}^{(\mathbb{R}^3)^*} + \mathbf{u}^{\rm d}$ in (14).

Then, along the solutions of (17), the control Lyapunov function $\mathcal{E}$ from (15) is not an invariant any more. In fact,

$$\dot{\mathcal{E}} = (2D_1\Phi(C_1, C_2, C_3)\mathbf{P}_{\rm c} + \mathbf{v}) \cdot \mathbf{u}^{\rm d}.$$

Hence if we set, with a negative-definite 3×3 matrix $\mathcal{N}$,

$$\mathbf{u}^{\rm d} = \mathcal{N}(2D_1\Phi(C_1, C_2, C_3)\mathbf{P}_{\rm c} + \mathbf{v}), \tag{18}$$

then we have $\dot{\mathcal{E}} \le 0$.

One then needs to employ LaSalle's invariance principle to prove that this indeed gives the asymptotic stability of the equilibrium. This requires a detailed analysis of the invariant set defined by $\dot{\mathcal{E}} = 0$. We do not present this analysis here because it is lengthy and involved, and will present it in our future work. Instead, we will show the numerical evidence for it in the subsection to follow.

5.4 Numerical Results

Consider an underwater vehicle whose hull is an ellipsoidal shell with the outer semi-major axes $(a_1, a_2, a_3) = (5, 10, 4)$ [m] and the inner semi-major axes $(a_1 - h, a_2 - h, a_3 - h)$ with $h \simeq 0.1666$ [m] made of steel with density 8000 [kg/m^3].

For simplicity, we assume that all extra weight is concentrated at the point 1 meter below the center of the ellipsoids as a point mass with 40% of the weight of the shell; hence the center of mass is at $l\boldsymbol{\chi}$ with $l = 2/7$ [m] and $\boldsymbol{\chi} = (0, 0, -1) = -\mathbf{E}_3$.

Then the total mass of the vehicle is $m = 835{,}245$ [kg], and it is neutrally buoyant assuming that the mass density of the water is 997 [kg/m^3]—the "thickness" h of the hull is determined that way. Using formulas from [Leonard, 1997a, Appendix B], one obtains

$$(m_1, m_2, m_3) \simeq (1.330, 0.9860, 1.592) \times 10^6 \text{ [kg]}$$

and

$$(I_1, I_2, I_3) \simeq (2.787, 0.9020, 2.527) \times 10^7 \text{ [kg} \cdot \text{m}^2\text{]}.$$

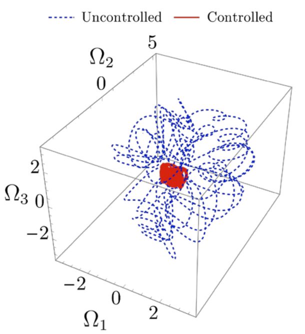
We set $\mathcal{K} \simeq 1.239 \times 10^6$ so that (16) is satisfied.

We select an initial condition with a small perturbation to the equilibrium (13) with $v_0 = 30\,[\mathrm{m/s}]$ as follows:

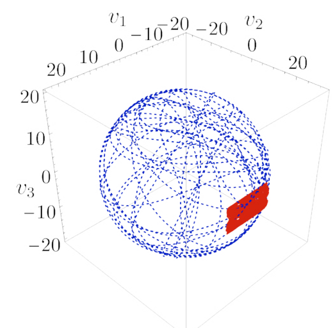
$$\boldsymbol{\Omega}(0) = (0.5,\, 0.25,\, 0.5), \qquad \mathbf{v}(0) = (1.5,\, 30,\, 1.5),$$
$$\boldsymbol{\Gamma}(0) = (\cos\theta_0 \sin\varphi_0, \sin\theta_0 \sin\varphi_0, \cos\varphi_0)$$

with $\theta_0 = \pi/3$ and $\varphi_0 = \pi/40$.

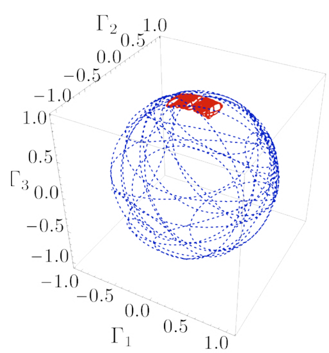
Figure 2 shows the trajectories of $\boldsymbol{\Omega}$, $\mathbf{v}$, and $\boldsymbol{\Gamma}$ for the uncontrolled and the non-dissipative controlled systems. The solution of the uncontrolled system (5) clearly shows that the equilibrium is unstable, whereas that of the controlled system (14) stays close to the equilibrium, indicating that the equilibrium is stabilized.



(a) Body angular velocity $\boldsymbol{\Omega}$



(b) Base velocity $\mathbf{v}$ in body frame



(c) Vertical upward direction $\boldsymbol{\Gamma}$ seen from body frame

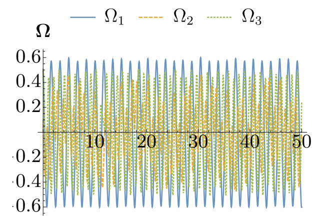
Fig. 2. Simulation results comparing the uncontrolled system (5) (dashed blue) and controlled system (14) (solid red) for the time interval $0 \le t \le 50$.

Figures 3 and 4 show the time evolutions of $(\boldsymbol{\Omega}, \mathbf{v}, \boldsymbol{\Gamma})$ for the non-dissipative controlled and dissipative controlled systems, (14) and (17), respectively; the negative-definite matrix $\mathcal{N}$ in (18) is chosen as $\mathcal{N} = -10^6 \operatorname{diag}(2, 1, 2)$. One sees that, without the dissipation, the solution oscillates near the equilibrium, whereas the dissipation damps the solution towards the equilibrium, although rather slowly for $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ compared to $\mathbf{v}$.

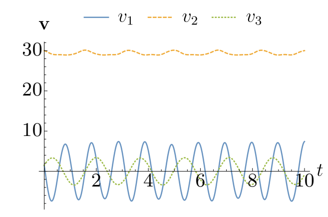
6. CONCLUSION

We extended the method of controlled Lagrangians with kinetic shaping to those mechanical systems defined on semidirect produce Lie groups with broken symmetry.

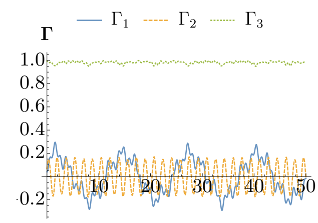
Our motivating example was an underwater vehicle with non-coincident centers of buoyancy and gravity.



(a) Body angular velocity $\boldsymbol{\Omega}$

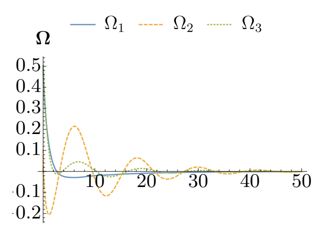


(b) Base velocity $\mathbf{v}$ in body frame

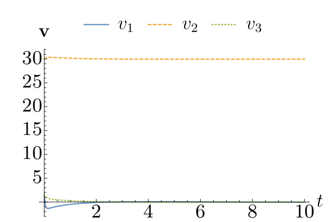


(c) Vertical upward direction $\boldsymbol{\Gamma}$ seen from body frame

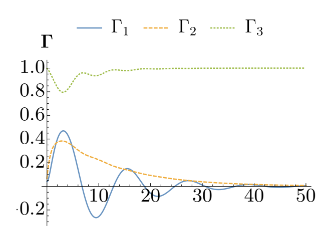
Fig. 3. Time evolution of controlled system (14) (without dissipation). Note that $\mathbf{v}$ is plotted for a shorter time interval to be compared with Fig. 4 below.



(a) Body angular velocity $\boldsymbol{\Omega}$



(b) Base velocity $\mathbf{v}$ in body frame



(c) Vertical upward direction $\boldsymbol{\Gamma}$ seen from body frame

Fig. 4. Time evolution of dissipative controlled system (17). Note that $\mathbf{v}$ is plotted for a shorter time interval because it is damped much faster than the other two quantities.

We found a matching condition for a class of Euler–Poincaré equations with advected parameters on $(\mathfrak{g} \ltimes V) \times X^*$.

We combined the resulting control with the energy–Casimir method to find a control stabilizing one of the classes of the unstable equilibria of ellipsoidal underwater vehicles found in Leonard [1997a].

This result also helped us find an additional dissipative control that renders the equilibria asymptotically stable.

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