

Shock Formation and Vorticity Creation for 3d Euler

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Abstract

We analyze the shock formation process for the 3D nonisentropic Euler equations with the ideal gas law, in which sound waves interact with entropy waves to produce vorticity. Building on our theory for isentropic flows in [3,4], we give a constructive proof of shock formation from smooth initial data. Specifically, we prove that there exist smooth solutions to the nonisentropic Euler equations which form a generic stable shock with explicitly computable blowup time, location, and direction. This is achieved by establishing the asymptotic stability of a generic shock profile in modulated self-similar variables, controlling the interaction of wave families via: (i) pointwise bounds along Lagrangian trajectories, (ii) geometric vorticity structure, and (iii) high-order energy estimates in Sobolev spaces. © 2022 Wiley Periodicals LLC.

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1 Introduction

The three-dimensional Euler equations of gas dynamics, introduced by Euler in [12], are a hyperbolic system of five coupled equations, and can be written as

$$(1.1a) \quad \rho(\partial_t u + (u \cdot \nabla_x)u) + \nabla_x p(\rho, k) = 0,$$

$$(1.1b) \quad \partial_t \rho + (u \cdot \nabla_x)\rho + \rho \operatorname{div}_x u = 0,$$

$$(1.1c) \quad \partial_t k + (u \cdot \nabla_x)k = 0,$$

for spacial variable $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, temporal variable $t \in \mathbb{R}$, velocity $u : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$, density $\rho : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}_+$, and entropy $k : \mathbb{R}^3 \times \mathbb{R}$. The pressure¹ $p = p(\rho, k) : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a function of both density and entropy, with equation-of-state given by the ideal gas law

$$p(\rho, k) = \frac{1}{\gamma} \rho^\gamma e^k,$$

where the adiabatic constant $\gamma > 1$. If smooth initial conditions are prescribed at an initial time t_0 , then a classical solution to (1.1) exists up to a finite time T_* , the lifespan, when a singularity or blowup develops [27]. The mechanism of blowup for smooth solutions to (1.1) as $t \rightarrow T_*$, including rate, direction, locus, and profile is heretofore unknown.

Our primary aim is the detailed analysis of the formation of the first shock or blowup for smooth solutions to (1.1). We prove that for an open set of initial conditions, smooth solutions to (1.1) evolve steepening wavefronts and form an asymptotically self-similar cusp-type first shock with explicit rate, location, and direction. The major difficulty in the analysis of the nonisentropic Euler dynamics stems from the interaction of sound waves, entropy waves, and vorticity waves. Nonisentropic flows can have a misalignment of density and entropy gradients, thus leading to dynamic vorticity creation, even from irrotational initial data.

To highlight the challenge created by the interaction of different wave families, we must examine the evolution of the vorticity vector, which we shall now derive. To do so, it is convenient to write the Euler equations using the sound speed. We introduce the adiabatic exponent

$$\alpha = \frac{\gamma-1}{2}$$

¹ The evolution equation for ρ can be replaced with the equation for pressure given by $\partial_t p + (u \cdot \nabla_x)p + \gamma p \operatorname{div}_x u = 0$.

so that the sound speed $c(\rho) = \sqrt{\partial p / \partial \rho}$ can be written as $c = e^{k/2} \rho^\alpha$ and $p = \frac{1}{\gamma} \rho c^2$. We define the scaled sound speed σ by

$$(1.2) \quad \sigma = \frac{1}{\alpha} c = \frac{1}{\alpha} e^{\frac{k}{2}} \rho^\alpha,$$

and write the Euler equations (1.1) as a system for (u, σ, k) as follows:

$$(1.3a) \quad \partial_t u + (u \cdot \nabla_x) u + \alpha \sigma \nabla_x \sigma = \frac{\alpha}{2\gamma} \sigma^2 \nabla_x k,$$

$$(1.3b) \quad \partial_t \sigma + (u \cdot \nabla_x) \sigma + \alpha \sigma \operatorname{div}_x u = 0,$$

$$(1.3c) \quad \partial_t k + (u \cdot \nabla_x) k = 0.$$

We let $\omega = \operatorname{curl}_x u$ denote the vorticity vector, and define the *specific vorticity vector* by $\zeta = \frac{\omega}{\rho}$. A straightforward computation shows that ζ is a solution to

$$(1.4) \quad \partial_t \zeta + (u \cdot \nabla_x) \zeta - (\zeta \cdot \nabla_x) u = \frac{\alpha}{\gamma} \frac{\sigma}{\rho} \nabla_x \sigma \times \nabla_x k.$$

The term $\frac{\alpha}{\gamma} \frac{\sigma}{\rho} \nabla_x \sigma \times \nabla_x k$ on the right side of (1.4) can also be written as $\rho^{-3} \nabla_x \rho \times \nabla_x p$ and is referred to as *baroclinic torque*. Clearly, the potential vorticity, the component of ζ in the direction of the density gradient, can only be generated by vortex stretching, whereas baroclinic vorticity modes are produced from the interaction of acoustic waves and entropy waves. This (baroclinic) vorticity production is the fundamental mechanism for the excitation and stabilization of both the Rayleigh-Taylor and Richtmyer-Meshkov instabilities, and plays a fundamental role in atmospheric science as well as numerous flows of engineering significance.

Of course, it is possible to simplify the Euler dynamics in a manner that still retains the steepening of sound waves, but removes complications associated to the interaction of different wave families. This can be achieved by considering the subclass of flows for which the entropy is a constant; such flows are called *isentropic*, and the pressure is a function of density alone: $p = \frac{1}{\gamma} \rho^\gamma$. Note that for isentropic flow, baroclinic torque vanishes, and thus the specific vorticity ζ is Lie-advected as a vector field. Acoustic modes can no longer interact with entropy waves to create vorticity; rather, vorticity is merely advected. As such, two further subclasses of flows exist: *irrotational* flow and flow with advected vorticity. For irrotational flow, only sound waves propagate, while for initial data with vorticity, there is an interaction between acoustic modes and vorticity modes that must be carefully analyzed, as controlling the growth of vorticity is essential to the study of shock formation. For nonisentropic dynamics, the presence of baroclinic torque creates a fundamentally new challenge in the estimation of the growth of vorticity. Why? Because as the first shock forms, the magnitude of baroclinic torque becomes infinite! Even though the baroclinic torque blows up, using geometric coordinates adapted to the steepening wave front we are able to obtain a number of cancellations in the vorticity equation, which allow us to prove that the vorticity remains bounded up to the time of shock formation. Furthermore, *irrotational* initial data can be chosen with non-zero baroclinic torque such that vorticity is

instantaneously produced and remains nontrivial throughout the shock formation process. By a significant extension of the methodology we developed in [3, 4], we shall prove the following:

THEOREM 1.1 (Rough statement of the main theorem). *For an open set of smooth initial data with a maximally negative gradient of size $\mathcal{O}(1/\varepsilon)$, for $\varepsilon > 0$ sufficiently small, there exist smooth solutions to the nonisentropic 3D Euler equations (1.1) that form a shock singularity at time $T_* = \mathcal{O}(\varepsilon)$. The first singularity occurs at a single point in space, whose location can be explicitly computed, along with the precise time at which it occurs. The blowup profile is shown to be a cusp with $C^{1/3}$ regularity, and the singularity is given by an asymptotically self-similar shock profile that is stable with respect to the $H^m(\mathbb{R}^3)$ topology for $m \geq 18$. If an irrotational initial velocity is prescribed, vorticity is instantaneously produced, and remains bounded and nontrivial up to the blowup time T_* .*

A precise statement of the main result will be given below as Theorem 3.2.

1.1 Prior results

In one space dimension, the theory of finite-time blowup of smooth solutions and shock formation to the Euler equations is well established. The literature is too vast to provide a review here. See, for example, [11, 13–17, 19, 26]. In contrast, in multiple space dimensions and with no symmetry assumptions, only the *isentropic* shock formation problem has been studied: shock formation was established for irrotational flows by [7, 9] (see also [8]), for 2D isentropic flows with vorticity by [3, 18], and for 3D isentropic flows with vorticity by [4]. Under a spherical symmetry assumption, which reduces the *nonisentropic* Euler equations to a 1D system, the shock formation process was studied in [30]. For *nonisentropic* flow in multiple space dimensions and without symmetry assumptions, prior to this paper it was *only* known that C^1 solutions have a finite lifespan [27].

As we noted above, one of the major difficulties in the analysis of nonisentropic flows is due to the interaction of multiple wave families: sound waves, vorticity waves, and entropy waves. Indeed, the analysis of quasilinear hyperbolic systems with multiple wave speeds is just emerging. As stated in [29], prior to the results in [3, 4, 18, 29], there have been no constructive proofs of shock formation for a quasilinear hyperbolic system in more than one spatial dimension, featuring multiple wave speeds. We note that the irrotational (isentropic) Euler equations can be written as a scalar quasilinear wave equation with only one wave speed; formation of shocks for systems with a single wave speed have been studied by [1, 2, 7, 9, 24, 25, 28].

Finally, we mention that there are other possible blowup mechanisms for the Euler equations; for example, a precise characterization of implosion for spherically symmetric isentropic flow has recently been given in [21, 22].

1.2 Main ideas in the proof

Because of the presence of multiple wave speeds, multiple wave families, and their nonlinear interactions, the Euler dynamics offer a rich tapestry of dynamic behavior, and yet when zooming in on the formation of the first shock, the Euler solution shares fundamental features with the wave-steepening blowup of the 3D Burgers solution. For this reason, our study of the mechanism of shock formation for smooth solutions of (1.3) as $t \rightarrow T_*$ makes use of a blowup profile $\bar{W}(y)$, one example of a stable stationary solution to the 3D self-similar Burgers equation

$$(1.5) \quad -\frac{1}{2}\bar{W} + \left(\frac{3}{2}y_1 + \bar{W}\right)\partial_{y_1}\bar{W} + \frac{1}{2}y_2\partial_{y_2}\bar{W} + \frac{1}{2}y_3\partial_{y_3}\bar{W} = 0,$$

which has an explicit representation. If we consider the 3D Burgers equation $\partial_t v + v \cdot \nabla v = 0$ in physical spacetime variables (x, t) , then a smooth solution $v = (v_1, v_2, v_3)$ that forms a first shock at $t = T_*$ is given by²

$$(1.6) \quad \begin{aligned} &v_1(x_1, x_2, x_3, t) \\ &= (T_* - t)^{1/2} \bar{W} \left(\frac{x_1}{(T_* - t)^{3/2}}, \frac{x_2}{(T_* - t)^{1/2}}, \frac{x_3}{(T_* - t)^{1/2}} \right) \end{aligned}$$

with $v_2 = 0$ and $v_3 = 0$. Explicit properties of the blowup profile $\bar{W}(y)$ together with the solution for $v_1(x, t)$ give precise information of the blowup mechanism as $t \rightarrow T_*$, including the blowup time T_* , the blowup location $x = 0$, and the blowup direction e_1 . We note that we have made a particular choice of direction for our Burgers solution v ; specifically, we have chosen to let the wave steepen along the e_1 blowup direction, whereas we could have used the profile \bar{W} to form a blowup in any direction.

Although the nonisentropic Euler system is significantly more complicated, we are nevertheless able to use the Burgers stationary solution \bar{W} to describe the blowup mechanism for smooth solutions of (1.3) as $t \rightarrow T_*$. This requires a number coordinate and variable transformations that are constructed upon two geometric principles: first, we build into our transformations a family of time-dependent modulation functions whose purpose is to fight against the destabilizing action of the finite-dimensional symmetry groups of the Euler equations, and second, we design a coordinate system which both follows and deforms with the steepening Euler solution.

Let us now elaborate on these ideas. The blowup profile $\bar{W}(y)$ has an explicit formula which shows that $y = 0$ is a global minimum for $\partial_{y_1}\bar{W}(y)$, and with the following properties verified: $\bar{W}(0) = 0$, $\partial_{y_1}\bar{W}(0) = -1$, $\partial_{y_2}\bar{W}(0) = \partial_{y_3}\bar{W}(0) = 0$, $\nabla_y^2\bar{W}(0) = 0$, and

$$(1.7) \quad \nabla^2\partial_{y_1}\bar{W}(0) > 0.$$

²In fact, as established in Appendix A.1 there are many closely related stable self-similar solutions to the Burgers equations which allow for a slight modification of v_1 .

Positive-definiteness of the Hessian of $\partial_1 W$ at $y = 0$ is a *genericity condition* for the blowup mechanism, and has been used in the study of blowup for quasilinear wave equations [1] and discussed in [5, 7] as an important selection criterion for stable shocks.

Returning now to the identity (1.6), if the initial time is fixed to be $t_0 = -\varepsilon$ for $\varepsilon > 0$, we can set $T_* = 0$; the initial condition for v_1 is then given by

$$v_1(x, -\varepsilon) = \varepsilon^{\frac{1}{2}} \bar{W}(\varepsilon^{-\frac{3}{2}} x_1, \varepsilon^{-\frac{1}{2}} x_2, \varepsilon^{-\frac{1}{2}} x_3).$$

With $(y_1, y_2, y_3) = ((-t)^{-3/2} x_1, (-t)^{-1/2} x_2, (-t)^{-1/2} x_3)$, we see that the properties of $\bar{W}(y)$ at $y = 0$ show that $v_1(0, -\varepsilon) = 0$, $\partial_{x_1} v_1(0, -\varepsilon) = \frac{1}{\varepsilon} \partial_{y_1} \bar{W}(0) = -\frac{1}{\varepsilon}$, $\partial_{x_2} v_1(0, -\varepsilon) = 0$, $\partial_{x_3} v_1(0, -\varepsilon) = 0$, $\nabla_x^2 v_1(0, -\varepsilon) = 0$, and the genericity condition (1.7) is also satisfied so that $\nabla_x^2 \partial_{x_1} v_1(0, -\varepsilon) > 0$. We see that for the 3D Burgers equation, if we start with a maximally negative slope equal to $-\frac{1}{\varepsilon}$ at time $t = -\varepsilon$ and $x = 0$, then the first shock occurs at time $T_* = 0$ and $x = 0$, and by virtue of (1.6), the blowup mechanism is self-similar

$$(1.8) \quad \partial_{x_1} v_1(0, t) = \frac{1}{T_* - t} \partial_1 \bar{W}(0) = -\frac{1}{T_* - t}.$$

Of course, no such formula as (1.6) exists for the Euler equations, but we can nevertheless use the properties of \bar{W} to develop a new type of stability theory for the Euler equations in self-similar variables.

Thus, the first step in our proof of shock formation for the nonisentropic Euler equations is the mapping of the physical space-time coordinates (x, t) to *modulated* self-similar space-time coordinates (y, s) , together with a succession of transformations that map the original variables (u, σ, k) into geometric Riemann-like variables (W, Z, A, K) , in which the dynamically dominant variable $W(y, s)$ mimics the properties of $\bar{W}(y)$ near the blowup location $y = 0$. The use of modulation functions for the analysis of self-similar dispersive equations was pioneered in [20, 23]. The initial data is prescribed at self-similar time $s_0 = -\log \varepsilon$, and we require $\partial^\gamma W(y, -\log \varepsilon)$ to verify the same conditions as $\partial^\gamma \bar{W}(y)$ at the point $y = 0$ for all multi-indices $|\gamma| \leq 2$. Just as we noted above, we are now making a choice of blowup direction; the initial data is chosen so that its maximal negative slope is in the e_1 -direction, but unlike the Burgers solution, the rotational symmetry of the Euler dynamics does not preserve this direction. In fact, the various symmetries of the Euler equations prevent these conditions on $\partial^\gamma W(0, s)$ to be maintained under the natural evolution, and for this reason, ten time-dependent modulation functions are used to ensure that $\partial^\gamma W(0, s) = \partial^\gamma \bar{W}(0)$ for $|\gamma| \leq 2$ and for all $s \geq -\log \varepsilon$. Of these ten modulation functions, seven of them are associated to symmetries of the Euler equations (see section 1.3 in [4]), and three of the modulation functions are associated to a spatially quadratic time-dependent parametrization $f(t, x_2, x_3) = \phi_{22}(t)x_2^2 + 2\phi_{23}(t)x_2x_3 + \phi_{33}(t)x_3^2$ of the steepening front, where the matrix $\phi_{\mu\nu}(t)$ modulates the curvature, and denotes the induced second-fundamental form. Associated to this parametric surface $f(x_2, x_3, t)$

is a time-dependent orthonormal basis (N, T^2, T^3) representing the normal and tangential directions. The steepening front moves in the N -direction and the dominant Riemann variable is defined as $w = u \cdot N + \sigma$. With respect to coordinates x , which themselves depend on f , the variable $w(x, t)$ is associated to the dominant self-similar variable $W(y, s)$ by a formula which is analogous to (1.6):

$$\begin{aligned} w(x_1, x_2, x_3, t) \\ &= (\tau(t) - t)^{\frac{1}{2}} W\left(\frac{x_1}{(\tau(t) - t)^{3/2}}, \frac{x_2}{(\tau(t) - t)^{1/2}}, \frac{x_3}{(\tau(t) - t)^{1/2}}, s\right), \\ -s &= \log(\tau(t) - t), \end{aligned}$$

where $\tau(t)$ modulates the blowup time and converges to T_* as $t \rightarrow T_*$. Differentiating w in the direction N of the steepening front, it can be shown that

$$(1.9) \quad \partial_N w(\xi(t), t) = e^s \partial_{y_1} W(0, s) = -\frac{1}{\tau(t) - t} \rightarrow -\infty \quad \text{as } t \rightarrow T_*,$$

where $\xi(t)$ modulates the blowup location. The blowup (1.9) is the geometric analogue of (1.8), and requires a well-defined limit as $t \rightarrow T_*$ which, in turn, requires that $W(y, s)$ remains well defined for all $-\log \varepsilon \leq s$.

It therefore becomes clear that in order to establish stable self-similar shock formation, we must prove global existence of solutions to the Euler equations in self-similar coordinates (y, s) , and the majority of our work is devoted to this end. The understanding of the damping/antidamping structure of the Euler equations in self-similar coordinates (y, s) along Lagrangian trajectories is key to our analysis; the undifferentiated Euler equations have antidamping terms, but upon spatial differentiation, damping emerges, and the more derivatives that are applied, the stronger the damping becomes. A consequence of this observation is that pointwise bounds for lower-order derivatives cannot rely on either damping or traditional Eulerian-type analysis, but rather on sharp (lower) bounds on the motion of the three families of trajectories associated to the three wave speeds present. In self-similar coordinates, almost all of the trajectories in these three wave families *escape to infinity* and having sharp rates-of-escape for each family can be combined with spatial decay properties of the Riemann-type function $W(y, s)$ to close a system of highly coupled bootstrap bounds for derivatives up to order 2.

On the other hand, it is not possible to close estimates for the Euler equations using only pointwise bounds due to inherent derivative loss, and higher-order energy estimates must therefore be employed. Modified energy estimates are performed for a system of variables comprised of U , $Se^{-K/2\gamma}$, and $e^{K/2\gamma}$, where U , S , and K are the self-similar versions of u , σ , and k , respectively. The use of these variables removes the hyperbolic degeneracy associated to vanishing density. Combined with the weighted pointwise bounds for lower-order derivatives, we prove global existence in a modified \dot{H}^m -norm, $m \geq 18$.

While for the subclass of irrotational flows the above two types of estimates suffice, for rotational flows it is essential to obtain uniform bounds for the vorticity all the way to the blowup time. Even for isentropic dynamics, in which the specific vorticity is Lie-advected, analysis in self-similar coordinates appears to create logarithmic losses in temporal decay (see [4]). Instead, the specific vorticity ζ is estimated in physical coordinates using geometric components $(\zeta \cdot \mathbf{N}, \zeta \cdot \mathbf{T}^2, \zeta \cdot \mathbf{T}^3)$, which yield a cancellation at highest order. For the nonisentropic dynamics, an additional difficulty arises because the vorticity equation (1.4) is forced by the baroclinic torque $\frac{\alpha}{\gamma} \frac{\sigma}{\rho} \nabla_x \sigma \times \nabla_x \mathbf{k}$, which blows up as $t \rightarrow T_*$. Indeed, from formula (2.25) below, and the bounds established in Sections 6 and 7 we may show that the tangential components of the baroclinic torque term satisfy

$$\left| \left(\frac{\sigma}{\rho} \mathbf{T} \cdot \nabla_x \sigma \times \nabla_x \mathbf{k} \right) (\xi(t), t) \right| \gtrsim \frac{1}{T_* - t}.$$

A main feature of our proof is to show that in spite of the fact that the Lie-advection for the specific vorticity is forced by a diverging term, ζ remains uniformly bounded up to T_* . This is achieved by noting that the divergence of the velocity gains a space derivative when integrated along trajectories with speed u , and by taking advantage of certain cancellations that arise due to our geometric framework.

Finally, we examine baroclinic vorticity production. We prove that even if the initial velocity is irrotational, vorticity is instantaneously produced due to the baroclinic torque, and our analysis shows that this created vorticity remains non-trivial in an open neighborhood of the steepening front all the way up to the first shock. We thus provide a constructive proof of shock formation for Euler in the regime in which vorticity is created, and not simply Lie advected.

1.3 Outline

In Section 2 we introduce a succession of variable changes and Riemann-type variables which allow us to write the Euler equations in modulated self-similar coordinates. A precise specification of the data and the statement of the main results is then given in Section 3. In Section 4 we introduce the bootstrap assumptions for the modulation functions as well as the primary variables solving the self-similar Euler equations; these bootstrap assumptions consist of carefully chosen weighted (in both space and time) bounds. A fundamental aspect of our proof requires a detailed estimates for the rates of escape of the trajectories corresponding to the different wave speeds, and Section 5 is devoted to this analysis. In Section 6 we establish pointwise bounds for the vorticity, and in Section 7 we show that there exists irrotational initial velocity fields from which vorticity is created and remains nontrivial at the first shock. Energy estimates in self-similar variables are established in Section 8, using the modified variables (2.41). In Section 9 we establish weighted (pointwise) estimates for functions appearing in the forcing, damping, and transport of the differentiated Euler system. In turn, these weighted bounds allow us to close the bootstrap assumptions for W , Z , A , K , and

their partial derivatives, and this is achieved in Sections [10-11](#) while in Section [12](#), we close the bootstrap bounds for the dynamic modulation functions. Finally, in Section [13](#), we explain how all of the obtained bounds are used to prove Theorem [3.2](#); in particular, we show that $\lim_{s \rightarrow \infty} W(y, s) = \bar{W}_{\mathcal{A}}(y)$ for any fixed $y \in \mathbb{R}^3$, where $\bar{W}_{\mathcal{A}}(y)$ is a stable stationary solution to the self-similar 3D Burgers equations. A family of such stationary solutions is constructed in Appendix [A](#), which also contains an interpolation inequality that is used throughout the paper, as well as some detailed computations leading to the evolution equations for the modulation functions.

2 Transforming the Euler Equations into Geometric Self-Similar Variables

We now make a succession of variable transformations for both dependent and independent variables. We begin by rescaling time as

$$(2.1) \quad t \mapsto \frac{1+\alpha}{2}t = t.$$

We next introduce ten modulation variables which satisfy a coupled system of ODEs that will be given in [\(12.12\)](#)–[\(12.13\)](#). For each time t , they are defined as follows:

- $R(t) \in \mathbb{S}^2$: rotation matrix from e_1 to the direction of steepening front $n(t)$,
- $\xi(t) \in \mathbb{R}^3$: translation vector used to fix the location of the developing shock,
- $\phi(t) \in \mathbb{R}^3$: 2x2 symmetric matrix giving the curvature of the shock front,
- $\tau(t) \in \mathbb{R}$: scalar used to track exact the blowup time,
- $\kappa(t) \in \mathbb{R}$: scalar used to fix the speed of the developing shock.

The matrix $R(t)$ is defined in terms of two time-dependent rotation angles $n_2(t)$ and $n_3(t)$ as follows. We define

$$n(t) = \left(\sqrt{1 - n_2^2(t) - n_3^2(t)}, n_2(t), n_3(t) \right)$$

and a skew-symmetric matrix \tilde{R} whose first row is the vector $(0, -n_2, -n_3)$, first column is $(0, n_2, n_3)$, and has 0 entries otherwise. In terms of \tilde{R} , we define the rotation matrix

$$(2.2) \quad R(t) = \text{Id} + \tilde{R}(t) + \frac{1 - e_1 \cdot n(t)}{|e_1 \times n(t)|^2} \tilde{R}^2(t).$$

It is the two angles $n_2(t)$ and $n_3(t)$ whose evolution is given in [\(12.12\)](#).

Using these modulation functions, we next proceed to make a succession of transformations of both the independent and dependent variables, finally arriving at a novel modulated self-similar form of the dynamics.

2.1 Rotating the direction and translating the location of the steepening wavefront

We introduce the new independent variable

$$(2.3) \quad \tilde{x} = R^T(t)(x - \xi(t))$$

and corresponding dependent variables as

$$(2.4) \quad \tilde{u}(\tilde{x}, t) = R^T(t)u(x, t), \quad \tilde{\sigma}(\tilde{x}, t) = \sigma(x, t), \quad \tilde{k}(\tilde{x}, t) = k(x, t).$$

It follows that (1.3) is transformed to

$$(2.5a) \quad \frac{1+\alpha}{2} \partial_t \tilde{u} - \dot{Q} \tilde{u} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{u} + \alpha \tilde{\sigma} \nabla_{\tilde{x}} \tilde{\sigma} = \frac{\alpha}{2\gamma} \tilde{\sigma}^2 \nabla_{\tilde{x}} \tilde{k},$$

$$(2.5b) \quad \frac{1+\alpha}{2} \partial_t \tilde{\sigma} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{\sigma} + \alpha \tilde{\sigma} \operatorname{div}_{\tilde{x}} \tilde{u} = 0,$$

$$(2.5c) \quad \frac{1+\alpha}{2} \partial_t \tilde{k} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{k} = 0,$$

where

$$(2.6) \quad \dot{Q} = \dot{R}^T R \quad \text{and} \quad \tilde{v}(\tilde{x}, t) := \dot{Q} \tilde{x} - R^T \dot{\xi}.$$

The density and pressure in this rotated and translated frame are given by

$$(2.7) \quad \tilde{\rho}(\tilde{x}, t) = \rho(x, t), \quad \tilde{p}(\tilde{x}, t) = p(x, t)$$

satisfy

$$(2.8a) \quad \frac{1+\alpha}{2} \partial_t \tilde{\rho} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{\rho} + \tilde{\rho} \operatorname{div}_{\tilde{x}} \tilde{u} = 0,$$

$$(2.8b) \quad \frac{1+\alpha}{2} \partial_t \tilde{p} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{p} + \gamma \tilde{p} \operatorname{div}_{\tilde{x}} \tilde{u} = 0,$$

and we also have the alternative form of the momentum equation

$$(2.9) \quad \frac{1+\alpha}{2} \partial_t \tilde{u} - \dot{Q} \tilde{u} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{u} + (\alpha \tilde{\sigma})^{-1/\alpha} e^{\tilde{k}/2\alpha} \nabla_{\tilde{x}} \tilde{p} = 0.$$

This follows from the form of the momentum equation given by

$$\partial_t u + (u \cdot \nabla_x) u + (\alpha \sigma)^{-1/\alpha} e^{k/2\alpha} \nabla_x p = 0$$

where, from (1.2), we have used that $\rho^{-1} = (\alpha \sigma)^{-1/\alpha} e^{k/2\alpha}$.

Similarly, defining the transformed *specific vorticity* vector $\tilde{\zeta}$ by

$$(2.10) \quad \tilde{\zeta}(\tilde{x}, t) = R^T(t) \zeta(x, t),$$

we have that $\tilde{\zeta}$ solves

$$(2.11) \quad \frac{1+\alpha}{2} \partial_t \tilde{\zeta} - \dot{Q} \tilde{\zeta} + ((\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}}) \tilde{\zeta} - (\tilde{\zeta} \cdot \nabla_{\tilde{x}}) \tilde{u} = \frac{\alpha}{\gamma} \frac{\tilde{\sigma}}{\tilde{\rho}} \nabla_{\tilde{x}} \tilde{\sigma} \times \nabla_{\tilde{x}} \tilde{k}.$$

Deriving (2.11) from (1.4) fundamentally uses that \dot{Q} is skew-symmetric, and the fact that the cross product is invariant to rotation.

2.2 Coordinates adapted to the shape of the steepening wavefront

We next define a quadratic surface over the $\tilde{x}_2\tilde{x}_3$ -plane given by the graph

$$(2.12) \quad (f(\tilde{x}_2, \tilde{x}_3, t), \tilde{x}_2, \tilde{x}_3),$$

which approximates the steepening shock, and where

$$(2.13) \quad f(\tilde{x}, t) = \frac{1}{2}\phi_{\nu\gamma}(t)\tilde{x}_\nu\tilde{x}_\gamma.$$

Associated to the parametrized surface (2.12), we define the unit-length normal and tangent vectors³

$$(2.14a) \quad \mathbf{N} = \mathbf{J}^{-1}(1, -f_{,2}, -f_{,3}),$$

$$(2.14b) \quad \mathbf{T}^2 = \left(\frac{f_{,2}}{J}, 1 - \frac{(f_{,2})^2}{J(J+1)}, \frac{-f_{,2}f_{,3}}{J(J+1)} \right),$$

$$(2.14c) \quad \mathbf{T}^3 = \left(\frac{f_{,3}}{J}, \frac{-f_{,2}f_{,3}}{J(J+1)}, 1 - \frac{(f_{,3})^2}{J(J+1)} \right),$$

where $J = (1 + |f_{,2}|^2 + |f_{,3}|^2)^{\frac{1}{2}}$ ⁴

In order to ‘flatten’ the developing shock front, we make one further transformation of the independent space variables⁵

$$(2.15) \quad x_1 = \tilde{x}_1 - f(\tilde{x}_2, \tilde{x}_3, t), \quad x_2 = \tilde{x}_2, \quad x_3 = \tilde{x}_3,$$

and define the transformed dependent variables by

$$(2.16a) \quad \overset{\circ}{u}(x, t) = \tilde{u}(\tilde{x}, t) = \tilde{u}(x_1 + f(x_2, x_3, t), x_2, x_3, t),$$

$$(2.16b) \quad \overset{\circ}{\sigma}(x, t) = \tilde{\sigma}(\tilde{x}, t) = \tilde{\sigma}(x_1 + f(x_2, x_3, t), x_2, x_3, t),$$

$$(2.16c) \quad \overset{\circ}{\rho}(x, t) = \tilde{\rho}(\tilde{x}, t) = \tilde{\rho}(x_1 + f(x_2, x_3, t), x_2, x_3, t),$$

$$(2.16d) \quad \overset{\circ}{k}(x, t) = \tilde{k}(\tilde{x}, t) = \tilde{k}(x_1 + f(x_2, x_3, t), x_2, x_3, t),$$

$$(2.16e) \quad \overset{\circ}{p}(x, t) = \tilde{p}(\tilde{x}, t) = \tilde{p}(x_1 + f(x_2, x_3, t), x_2, x_3, t).$$

We shall also make use of the α -dependent parameters

$$(2.17a) \quad \beta_1 = \beta_1(\alpha) = \frac{1}{1+\alpha}, \quad \beta_2 = \beta_2(\alpha) = \frac{1-\alpha}{1+\alpha},$$

$$(2.17b) \quad \beta_3 = \beta_3(\alpha) = \frac{\alpha}{1+\alpha}, \quad \beta_4 = \beta_4(\alpha) = \frac{\beta_3(\alpha)}{1+2\alpha},$$

where $0 \leq \beta_i = \beta_i(\alpha) < 1$.

³ As we noted in [4], $(\mathbf{N}, \mathbf{T}^2, \mathbf{T}^3)$ defines an orthonormal basis and $\mathbf{T}^2 \times \mathbf{T}^3 = \mathbf{N}$, $\mathbf{N} \times \mathbf{T}^2 = \mathbf{T}^3$, and $\mathbf{N} \times \mathbf{T}^3 = -\mathbf{T}^2$.

⁴ Here and throughout the paper we are using the notation $\varphi_{,\mu} = \partial_{x_\mu}\varphi$ and $\partial_\mu\varphi = \partial_{y_\mu}\varphi$.

⁵ Note that only the \tilde{x}_1 -coordinate is modified.

Using the time rescaling from (2.1), the system (2.5) can be written as (2.16) as

(2.18a)

$$\begin{aligned} \partial_t \mathring{u} - 2\beta_1 \mathring{Q} \mathring{u} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{u} + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{u} \\ + 2\beta_3 \mathring{\sigma} (\mathbf{J}\mathbf{N} \partial_1 \mathring{\sigma} + \delta^\nu \partial_\nu \mathring{\sigma}) = \beta_4 \mathring{\sigma}^2 (\mathbf{J}\mathbf{N} \partial_1 \mathring{k} + \delta^\nu \partial_\nu \mathring{k}), \end{aligned}$$

(2.18b)

$$\begin{aligned} \partial_t \mathring{\sigma} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{\sigma} + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{\sigma} \\ + 2\beta_3 \mathring{\sigma} (\partial_1 \mathring{u} \cdot \mathbf{N}\mathbf{J} + \partial_\nu \mathring{u}_\nu) = 0, \end{aligned}$$

(2.18c)

$$\partial_t \mathring{k} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{k} + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{k} = 0,$$

where in analogy to (2.16), we have denoted

$$(2.19) \quad v(x, t) = \tilde{v}(\tilde{x}, t) = \tilde{v}(x_1 + f(x_2, x_3, t), x_2, x_3, t).$$

Note in particular the identity $v_i(x, t) = \mathring{Q}_{i1}(x_1 + f(\tilde{x}, t)) + \mathring{Q}_{i\nu}x_\nu - R_{ji}\dot{\xi}_j$. The density equation (2.8a) becomes

$$(2.20) \quad \begin{aligned} \partial_t \mathring{\rho} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{\rho} \\ + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{\rho} + 2\beta_1 \mathring{\rho} (\partial_1 \mathring{u} \cdot \mathbf{N}\mathbf{J} + \partial_\nu \mathring{u}_\nu) = 0, \end{aligned}$$

the pressure equation (2.8b) is transformed to

$$(2.21) \quad \begin{aligned} \partial_t \mathring{p} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{p} \\ + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{p} + 2\beta_1 \gamma \mathring{p} (\partial_1 \mathring{u} \cdot \mathbf{N}\mathbf{J} + \partial_\nu \mathring{u}_\nu) = 0, \end{aligned}$$

and the alternative form of the momentum equation (2.9) is written as

$$(2.22) \quad \begin{aligned} \partial_t \mathring{u} - 2\beta_1 \mathring{Q} \mathring{u} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{u} + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{u} \\ + 2\beta_1 (\alpha \mathring{\sigma})^{-\frac{1}{\alpha}} e^{\frac{\mathring{k}}{2\alpha}} (\mathbf{J}\mathbf{N} \partial_1 \mathring{p} + \delta^\nu \partial_\nu \mathring{p}) = 0. \end{aligned}$$

Similarly, the transformed specific vorticity vector is

$$(2.23) \quad \mathring{\zeta}(x, t) = \tilde{\zeta}(\tilde{x}, t) = \tilde{\zeta}(x_1 + f(x_2, x_3, t), x_2, x_3, t),$$

so that the equation (2.11) becomes

$$(2.24) \quad \begin{aligned} \partial_t \mathring{\zeta} - 2\beta_1 \mathring{Q} \mathring{\zeta} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N} \right) \partial_1 \mathring{\zeta} + 2\beta_1 (v_\nu + \mathring{u}_\nu) \partial_\nu \mathring{\zeta} \\ - 2\beta_1 \mathbf{J}\mathbf{N} \cdot \mathring{\zeta} \partial_1 \mathring{u} - 2\beta_1 \mathring{\zeta}_\nu \partial_\nu \mathring{u} = \frac{\alpha}{\gamma} \frac{\mathring{\sigma}}{\mathring{\rho}} \nabla_{\tilde{x}} \mathring{\sigma} \times \nabla_{\tilde{x}} \mathring{k}. \end{aligned}$$

Note that the gradient appearing on the right side is with respect to \tilde{x} . We record for later use that

$$(2.25) \quad \begin{aligned} \nabla_{\tilde{x}} \circ \times \nabla_{\tilde{x}} \circ \mathbf{k} &= (\partial_{T_2} \circ \partial_{T_3} \circ \mathbf{k} - \partial_{T_3} \circ \partial_{T_2} \circ \mathbf{k}) \mathbf{N} \\ &+ (\partial_{T_3} \circ \partial_{N_1} \circ \mathbf{k} - \partial_{N_1} \circ \partial_{T_3} \circ \mathbf{k}) T^2 + (\partial_{N_1} \circ \partial_{T_2} \circ \mathbf{k} - \partial_{T_2} \circ \partial_{N_1} \circ \mathbf{k}) T^3, \end{aligned}$$

where

$$\partial_N = \mathbf{N} \cdot \nabla_{\tilde{x}} \quad \text{and} \quad \partial_{T^\nu} = T^\nu \cdot \nabla_{\tilde{x}}.$$

2.3 Riemann variables adapted to the shock geometry

Just as for the isentropic Euler equations that we analyzed in [4], the nonisentropic Euler system (2.18) has a rad geometric structure arising from the use of Riemann-type variables, defined by

$$(2.26) \quad w = \mathring{u} \cdot \mathbf{N} + \mathring{\sigma}, \quad z = \mathring{u} \cdot \mathbf{N} - \mathring{\sigma}, \quad a_\nu = \mathring{u} \cdot T^\nu,$$

so that

$$(2.27) \quad \mathring{u} \cdot \mathbf{N} = \frac{1}{2}(w + z), \quad \mathring{\sigma} = \frac{1}{2}(w - z).$$

The Euler system (2.18) can be written in terms of $(w, z, a_2, a_3, \mathbf{k})$ as⁶

(2.28a)

$$\begin{aligned} \partial_t w &+ \left(2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} \right) + \mathbf{J}w + \beta_2 \mathbf{J}z \right) \partial_1 w \\ &+ (2\beta_1 v_\mu + w N_\mu - \beta_2 z N_\mu + 2\beta_1 a_\nu T_\mu^\nu) \partial_\mu w \\ &= -2\beta_3 \mathring{\sigma} T_\mu^\nu \partial_\mu a_\nu + 2\beta_1 a_\nu T_i^\nu \dot{N}_i + 2\beta_1 \dot{Q}_{ij} a_\nu T_j^\nu N_i - 2\beta_3 \mathring{\sigma} (a_\nu T_{\mu,\mu}^\nu + \mathring{u} \cdot \mathbf{N} N_{\mu,\mu}) \\ (2.28b) \quad &+ 2\beta_1 (v_\mu + \mathring{u} \cdot \mathbf{N} N_\mu + a_\nu T_\mu^\nu) a_\gamma T_i^\gamma N_{i,\mu} + \beta_4 \mathring{\sigma}^2 (\mathbf{J} \partial_1 \mathring{\mathbf{k}} + N_\mu \partial_\mu \mathring{\mathbf{k}}), \end{aligned}$$

$$\begin{aligned} \partial_t z &+ \left(2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} \right) + \beta_2 \mathbf{J}w + \mathbf{J}z \right) \partial_1 z \\ &+ (2\beta_1 v_\mu + \beta_2 w N_\mu + z N_\mu + 2\beta_1 a_\nu T_\mu^\nu) \partial_\mu z \\ &= 2\beta_3 \mathring{\sigma} T_\mu^\nu \partial_\mu a_\nu + 2\beta_1 a_\nu T_i^\nu \dot{N}_i + 2\beta_1 \dot{Q}_{ij} a_\nu T_j^\nu N_i + 2\beta_3 \mathring{\sigma} (a_\nu T_{\mu,\mu}^\nu + \mathring{u} \cdot \mathbf{N} N_{\mu,\mu}) \\ (2.28c) \quad &+ 2\beta_1 (v_\mu + \mathring{u} \cdot \mathbf{N} N_\mu + a_\nu T_\mu^\nu) a_\gamma T_i^\gamma N_{i,\mu} + \beta_4 \mathring{\sigma}^2 (\mathbf{J} \partial_1 \mathring{\mathbf{k}} + N_\mu \partial_\mu \mathring{\mathbf{k}}), \end{aligned}$$

$$\begin{aligned} \partial_t a_\nu &+ \left(2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}v \cdot \mathbf{N} \right) + \beta_1 \mathbf{J}w + \beta_1 \mathbf{J}z \right) \partial_1 a_\nu \\ &+ 2\beta_1 (v_\mu + \frac{1}{2}(w + z) N_\mu + a_\gamma T_\mu^\gamma) \partial_\mu a_\nu \\ &= -2\beta_3 \mathring{\sigma} T_\mu^\nu \partial_\mu \mathring{\sigma} + 2\beta_1 (\mathring{u} \cdot \mathbf{N} N_i + a_\gamma T_i^\gamma) \dot{T}_i^\nu + 2\beta_1 \dot{Q}_{ij} \left((\mathring{u} \cdot \mathbf{N} N_j + a_\gamma T_j^\gamma) T_i^\nu \right. \\ (2.28d) \quad &\left. + \beta_1 (v_\mu + \mathring{u} \cdot \mathbf{N} N_\mu + 2a_\gamma T_\mu^\gamma) (\mathring{u} \cdot \mathbf{N} N_i + a_\gamma T_i^\gamma) T_{i,\mu}^\nu + \beta_4 \mathring{\sigma}^2 T_\mu^\nu \partial_\mu \mathring{\mathbf{k}}, \right. \end{aligned}$$

⁶The time rescaling (2.1) sets the coefficient of $w \partial_1 w$ in (2.28b) to 1, which provides a convenient framework to study the w -equation as a perturbation of Burgers-type evolution.

(2.28e)

$$\partial_t \overset{\circ}{k} + 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + Jv \cdot N + Ju^\circ \cdot N \right) \partial_1 \overset{\circ}{k} + 2\beta_1 (v_v + \overset{\circ}{u}_v) \partial_v \overset{\circ}{k} = 0.$$

2.4 Euler equations in modulated self-similar Riemann-type variables

Finally, to facilitate the analysis of shock formation, we introduce the (modulated) self-similar variables:

$$(2.29a) \quad s = s(t) = -\log(\tau(t) - t),$$

$$(2.29b) \quad y_1 = y_1(x_1, t) = \frac{x_1}{(\tau(t) - t)^{\frac{3}{2}}} = x_1 e^{3s/2},$$

$$(2.29c) \quad y_j = y_j(x_j, t) = \frac{x_j}{(\tau(t) - t)^{1/2}} = x_j e^{s/2} \quad \text{for } j \in \{2, 3\}.$$

Using the self-similar variables y and s , we rewrite the functions w , z , a_v , $\overset{\circ}{k}$, and v , defined in (2.26) and (2.19), as

$$(2.30a) \quad w(x, t) = e^{-s/2} W(y, s) + \kappa(t),$$

$$(2.30b) \quad z(x, t) = Z(y, s),$$

$$(2.30c) \quad a_v(x, t) = A_v(y, s),$$

$$(2.30d) \quad \overset{\circ}{k}(x, t) = K(y, s),$$

$$(2.30e) \quad v(x, t) = V(y, s),$$

so that

$$(2.31) \quad V_i(y, s) = \dot{Q}_{i1} \left(e^{-3s/2} y_1 + \frac{1}{2} e^{-s} \phi_{v\mu} y_v y_\mu \right) + e^{-s/2} \dot{Q}_{iv} y_v - R_{ji} \dot{\xi}_j.$$

Introducing the parameter

$$\beta_\tau = \beta_\tau(t) = \frac{1}{1-\tau(t)},$$

the Euler system (2.28) is written in self-similar coordinates as

(2.32a)

$$\left(\partial_s - \frac{1}{2} \right) W + \left(g_W + \frac{3}{2} y_1 \right) \partial_1 W + \left(h_W^\mu + \frac{1}{2} y_\mu \right) \partial_\mu W = F_W - e^{-\frac{s}{2}} \beta_\tau \dot{k},$$

$$(2.32b) \quad \partial_s Z + \left(g_Z + \frac{3}{2} y_1 \right) \partial_1 Z + \left(h_Z^\mu + \frac{1}{2} y_\mu \right) \partial_\mu Z = F_Z,$$

$$(2.32c) \quad \partial_s A_v + \left(g_U + \frac{3}{2} y_1 \right) \partial_1 A_v + \left(h_U^\mu + \frac{1}{2} y_\mu \right) \partial_\mu A_v = F_{A_v},$$

$$(2.32d) \quad \partial_s K + \left(g_U + \frac{3}{2} y_1 \right) \partial_1 K + \left(h_U^\nu + \frac{1}{2} y_\nu \right) \partial_\nu K = 0,$$

where the y_1 transport functions are defined by

(2.33a)

$$g_W = \beta_\tau J W + \beta_\tau e^{s/2} \left(-\dot{f} + J(\kappa + \beta_2 Z + 2\beta_1 V \cdot N) \right) = \beta_\tau J W + G_W,$$

(2.33b)

$$g_Z = \beta_2 \beta_\tau J W + \beta_\tau e^{\frac{s}{2}} \left(-\dot{f} + J(\beta_2 \kappa + Z + 2\beta_1 V \cdot N) \right) = \beta_2 \beta_\tau J W + G_Z$$

(2.33c)

$$g_U = \beta_1 \beta_\tau J W + \beta_\tau e^{\frac{s}{2}} (-\dot{f} + J(\beta_1 \kappa + \beta_1 Z + 2\beta_1 V \cdot N)) = \beta_1 \beta_\tau J W + G_U,$$

the y_v transport functions are given as

(2.34a)

$$h_W^\mu = \beta_\tau e^{-s} N_\mu W + \beta_\tau e^{-\frac{s}{2}} (2\beta_1 V_\mu + N_\mu \kappa - \beta_2 N_\mu Z + 2\beta_1 A_\gamma T_\mu^\gamma),$$

(2.34b)

$$h_Z^\mu = \beta_\tau \beta_2 e^{-s} N_\mu W + \beta_\tau e^{-\frac{s}{2}} (2\beta_1 V_\mu + \beta_2 N_\mu \kappa + N_\mu Z + 2\beta_1 A_\gamma T_\mu^\gamma),$$

(2.34c)

$$h_U^\mu = \beta_\tau \beta_1 e^{-s} N_\mu W + \beta_\tau e^{-\frac{s}{2}} (2\beta_1 V_\mu + \beta_1 N_\mu \kappa + \beta_1 N_\mu Z + 2\beta_1 A_\gamma T_\mu^\gamma),$$

and the forcing functions are

$$\begin{aligned} F_W = & -2\beta_3 \beta_\tau S T_\mu^\nu \partial_\mu A_\nu + 2\beta_1 \beta_\tau e^{-\frac{s}{2}} A_\nu T_i^\nu \dot{N}_i + 2\beta_1 \beta_\tau e^{-\frac{s}{2}} \dot{Q}_{ij} A_\nu T_j^\nu N_i \\ & + 2\beta_1 \beta_\tau e^{-\frac{s}{2}} (V_\mu + N_\mu U \cdot N + A_\nu T_\mu^\nu) A_\gamma T_i^\gamma N_{i,\mu} \end{aligned}$$

(2.35a)

$$-2\beta_3 \beta_\tau e^{-\frac{s}{2}} S (A_\nu T_{\mu,\mu}^\nu + U \cdot N N_{\mu,\mu}) + \beta_4 \beta_\tau S^2 (J e^{\frac{s}{2}} \partial_1 K + N_\mu \partial_\mu K)$$

$$\begin{aligned} F_Z = & 2\beta_3 \beta_\tau e^{-\frac{s}{2}} S T_\mu^\nu \partial_\mu A_\nu + 2\beta_1 \beta_\tau e^{-s} A_\nu T_i^\nu \dot{N}_i + 2\beta_1 \beta_\tau e^{-s} \dot{Q}_{ij} A_\nu T_j^\nu N_i \\ & + 2\beta_1 \beta_\tau e^{-s} (V_\mu + N_\mu U \cdot N + A_\nu T_\mu^\nu) A_\gamma T_i^\gamma N_{i,\mu} \end{aligned}$$

(2.35b)

$$+ 2\beta_3 \beta_\tau e^{-s} S (A_\nu T_{\mu,\mu}^\nu + U \cdot N N_{\mu,\mu}) + \beta_4 \beta_\tau S^2 (J e^{\frac{s}{2}} \partial_1 K + N_\mu e^{-\frac{s}{2}} \partial_\mu K)$$

$$\begin{aligned} F_{A_v} = & -2\beta_3 \beta_\tau e^{-\frac{s}{2}} S T_\mu^\nu \partial_\mu S + 2\beta_1 \beta_\tau e^{-s} (U \cdot N N_i + A_\gamma T_i^\gamma) \dot{T}_i^\nu \\ & + 2\beta_1 \beta_\tau e^{-s} \dot{Q}_{ij} (U \cdot N N_j + A_\gamma T_j^\gamma) T_i^\nu + \beta_4 \beta_\tau e^{-\frac{s}{2}} S^2 T_\mu^\nu \partial_\mu K \end{aligned}$$

(2.35c)

$$+ 2\beta_1 \beta_\tau e^{-s} (V_\mu + U \cdot N N_\mu + A_\gamma T_\mu^\gamma) (U \cdot N N_i + A_\gamma T_i^\gamma) T_{i,\mu}^\nu.$$

In (2.35) we have also used the self-similar variants of \hat{u} , $\hat{\sigma}$, and \hat{k} , which, together with the self-similar variant of \hat{p} , are given by

$$(2.36a) \quad \hat{u}(x, t) = U(y, s),$$

$$(2.36b) \quad \hat{\rho}(x, t) = R(y, s),$$

$$(2.36c) \quad \hat{\sigma}(x, t) = S(y, s),$$

$$(2.36d) \quad \hat{p}(x, t) = P(y, s),$$

so that

$$(2.37) \quad U \cdot N = \frac{1}{2}(\kappa + e^{-\frac{s}{2}} W + Z) \quad \text{and} \quad S = \frac{1}{2}(\kappa + e^{-\frac{s}{2}} W - Z).$$

The system (2.32) may be written as

$$\begin{aligned}\partial_s W - \frac{1}{2}W + (\mathcal{V}_W \cdot \nabla)W &= F_W, \\ \partial_s Z + (\mathcal{V}_Z \cdot \nabla)Z &= F_Z, \\ \partial_s A_v + (\mathcal{V}_U \cdot \nabla)A_v &= F_{A_v}, \\ \partial_s K + (\mathcal{V}_U \cdot \nabla)K &= 0,\end{aligned}$$

where the transport velocities are abbreviated as

$$(2.39a) \quad \mathcal{V}_W = (g_W + \frac{3}{2}y_1, h_W^2 + \frac{1}{2}y_2, h_W^3 + \frac{1}{2}y_3),$$

$$(2.39b) \quad \mathcal{V}_Z = (g_Z + \frac{3}{2}y_1, h_Z^2 + \frac{1}{2}y_2, h_Z^3 + \frac{1}{2}y_3),$$

$$(2.39c) \quad \mathcal{V}_U = (g_U + \frac{3}{2}y_1, h_U^2 + \frac{1}{2}y_2, h_U^3 + \frac{1}{2}y_3).$$

2.5 Self-similar Euler equations in terms of velocity, pressure, and entropy

From (2.18), (2.21), (2.22), (2.29), (2.36a), (2.36c) we deduce that (U, P, K) are solutions of

$$(2.40a) \quad \begin{aligned} &\partial_s U_i - 2\beta_1\beta_\tau e^{-s} \dot{Q}_{ij}U_j + (\mathcal{V}_U \cdot \nabla)U_i \\ &+ 2\beta_\tau\beta_1(\alpha S)^{-\frac{1}{\alpha}} e^{\frac{K}{2\alpha}} (\mathbf{JN}_i e^{\frac{s}{2}} \partial_1 P + \delta^{iv} e^{-\frac{s}{2}} \partial_v P) = 0, \end{aligned}$$

$$(2.40b) \quad \partial_s P + (\mathcal{V}_U \cdot \nabla)P + 2\beta_\tau\beta_1\gamma e^{\frac{s}{2}} P \partial_1 U \cdot \mathbf{N} \mathbf{J} + 2\beta_\tau\beta_1\gamma e^{-\frac{s}{2}} P \partial_v U_v = 0,$$

$$(2.40c) \quad \partial_s K + (\mathcal{V}_U \cdot \nabla)K = 0.$$

For the purpose of performing high-order energy estimates, it is convenient to introduce

$$(2.41) \quad \mathcal{P} = S e^{-\frac{K}{2\gamma}} = \frac{1}{\alpha} (\gamma P)^{\frac{\alpha}{\gamma}}, \quad \mathcal{H} = e^{\frac{K}{2\gamma}},$$

and re-express the system of equations (2.40) as the following $(U, \mathcal{P}, \mathcal{H})$ -system:

$$(2.42a) \quad \begin{aligned} &\partial_s U_i + (\mathcal{V}_U \cdot \nabla)U_i + 2\beta_\tau\beta_3\mathcal{H}^2 \mathcal{P} (\mathbf{JN}_i e^{\frac{s}{2}} \partial_1 \mathcal{P} + \delta^{iv} e^{-\frac{s}{2}} \partial_v \mathcal{P}) \\ &= 2\beta_\tau\beta_1 e^{-s} \dot{Q}_{ij}U_j, \end{aligned}$$

$$(2.42b) \quad \partial_s \mathcal{P} + (\mathcal{V}_U \cdot \nabla)\mathcal{P} + 2\beta_\tau\beta_3\mathcal{P} (e^{\frac{s}{2}} \mathbf{JN} \cdot \partial_1 U + e^{-\frac{s}{2}} \partial_v U_v) = 0,$$

$$(2.42c) \quad \partial_s \mathcal{H} + (\mathcal{V}_U \cdot \nabla)\mathcal{H} = 0.$$

Finally, we define the self-similar variant of the specific vorticity via

$$(2.43) \quad \overset{\circ}{\zeta}(x, t) = \Omega(y, s).$$

2.6 Evolution of higher-order derivatives

Higher-order derivatives for the (W, Z, A, K) -system

We shall also need the differentiated form of the system (2.32), which we record here for convenience. For a multi-index $\gamma \in \mathbb{N}_0^3$, we use the notation $\gamma =$

$(\gamma_1, \check{\gamma}) = (\gamma_1, \gamma_2, \gamma_3)$. We have that

(2.44a)

$$\left(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_\tau (1 + \gamma_1 \mathbf{1}_{\gamma_1 \geq 2}) \mathcal{J} \partial_1 W \right) \partial^\gamma W + (\mathcal{V}_W \cdot \nabla) \partial^\gamma W = F_W^{(\gamma)},$$

(2.44b)

$$\left(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_2 \beta_\tau \gamma_1 \mathcal{J} \partial_1 W \right) \partial^\gamma Z + (\mathcal{V}_Z \cdot \nabla) \partial^\gamma Z = F_Z^{(\gamma)},$$

(2.44c)

$$\left(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_1 \beta_\tau \gamma_1 \mathcal{J} \partial_1 W \right) \partial^\gamma A_v + (\mathcal{V}_U \cdot \nabla) \partial^\gamma A_v = F_{Av}^{(\gamma)},$$

(2.44d)

$$\left(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_1 \beta_\tau \gamma_1 \mathcal{J} \partial_1 W \right) \partial^\gamma K + (\mathcal{V}_U \cdot \nabla) \partial^\gamma K = F_K^{(\gamma)},$$

where $|\gamma| \geq 1$ and the forcing terms are

$$\begin{aligned} F_W^{(\gamma)} = & \partial^\gamma F_W - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} G_W \partial_1 \partial^\beta W + \partial^{\gamma-\beta} h_W^\mu \partial_\mu \partial^\beta W) \\ & - \beta_\tau \mathbf{1}_{|\gamma| \geq 3} \sum_{\substack{1 \leq |\beta| \leq |\gamma|-2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{J} W) \partial_1 \partial^\beta W \\ & - \beta_\tau \mathbf{1}_{|\gamma| \geq 2} \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1 = \gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{J} W) \partial_1 \partial^\beta W \end{aligned} \quad (2.45)$$

for the $\partial^\gamma W$ evolution, and

$$\begin{aligned} F_Z^{(\gamma)} = & \partial^\gamma F_Z - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} G_Z \partial_1 \partial^\beta Z + \partial^{\gamma-\beta} h_Z^\mu \partial_\mu \partial^\beta Z) \\ & - \beta_2 \beta_\tau \mathbf{1}_{|\gamma| \geq 2} \sum_{\substack{0 \leq |\beta| \leq |\gamma|-2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{J} W) \partial_1 \partial^\beta Z \\ & - \beta_2 \beta_\tau \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1 = \gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{J} W) \partial_1 \partial^\beta Z \end{aligned} \quad (2.46a)$$

$$\begin{aligned} F_{Av}^{(\gamma)} = & \partial^\gamma F_{Av} - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} G_U \partial_1 \partial^\beta A_v + \partial^{\gamma-\beta} h_U^\mu \partial_\mu \partial^\beta A_v) \\ & - \beta_1 \beta_\tau \mathbf{1}_{|\gamma| \geq 2} \sum_{\substack{0 \leq |\beta| \leq |\gamma|-2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{J} W) \partial_1 \partial^\beta A_v \\ & - \beta_1 \beta_\tau \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1 = \gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{J} W) \partial_1 \partial^\beta A_v \end{aligned} \quad (2.46b)$$

$$\begin{aligned}
F_K^{(\gamma)} = & - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} \left(\partial^{\gamma-\beta} G_U \partial_1 \partial^\beta K + \partial^{\gamma-\beta} h_U^\mu \partial_\mu \partial^\beta K \right) \\
& - \beta_1 \beta_\tau \mathbf{1}_{|\gamma| \geq 2} \sum_{\substack{0 \leq |\beta| \leq |\gamma|-2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta K \\
(2.46c) \quad & - \beta_1 \beta_\tau \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1=\gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta K
\end{aligned}$$

for the $\partial^\gamma Z$, $\partial^\gamma A_\nu$, and $\partial^\gamma K$ evolutions.

Higher-order derivatives for \tilde{W}

We let $\bar{W}(y)$ denote a particular self-similar, stable, stationary solution of the 3D Burgers equation, given by

$$(2.47) \quad \bar{W}(y) = \langle \check{y} \rangle W_{1D} \left(\frac{y_1}{\langle \check{y} \rangle^3} \right)$$

where $\langle \check{y} \rangle = 1 + y_2^2 + y_3^2$ is the Japanese bracket, and where $W_{1D}(y_1)$ is the stable globally self-similar solution of the 1D Burgers equation, i.e., $W_{1D}(y_1)$ is a solution to $W_{1D} + W_{1D}^3 = -y_1$. We refer the reader to [6], [10], and section 2.7 of [4] for the explicit form of $W_{1D}(y_1)$ and for properties of $\bar{W}(y)$. We note that \bar{W} is one example from the 10-dimensional family $\bar{W}_{\mathcal{A}}$ of stable stationary solutions to the self-similar 3D Burgers equation, which are given by Proposition A.1 in Appendix A.1. The symmetric 3-tensor \mathcal{A} represents $\partial^\gamma \bar{W}_{\mathcal{A}}(0)$ for $|\gamma| = 3$. The function \bar{W} is in fact equal to $\bar{W}_{\mathcal{A}}$ for the case that $\mathcal{A}_{111} = 6$, $\mathcal{A}_{122} = \mathcal{A}_{133} = 2$, and all other components vanish.

Of paramount importance to our analysis is the evolution of the perturbation

$$(2.48) \quad \tilde{W}(y, s) = W(y, s) - \bar{W}(y),$$

which satisfies

$$\begin{aligned}
(2.49) \quad & \partial_s \tilde{W} + (\beta_\tau J \partial_1 \bar{W} - \tfrac{1}{2}) \tilde{W} + (\mathcal{V}_W \cdot \nabla) \tilde{W} \\
& = F_W - e^{-\frac{s}{2}} \beta_\tau \dot{K} + ((\beta_\tau J - 1) \bar{W} - G_W) \partial_1 \bar{W} - h_W^\mu \partial_\mu \bar{W} =: \tilde{F}_W.
\end{aligned}$$

Applying ∂^γ to (2.49), we obtain that $\partial^\gamma \tilde{W}$ obeys

$$\begin{aligned}
(2.50) \quad & \left(\partial_s + \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_\tau J (\partial_1 \bar{W} + \gamma_1 \partial_1 W) \right) \partial^\gamma \tilde{W} \\
& + (\mathcal{V}_W \cdot \nabla) \partial^\gamma \tilde{W} = \tilde{F}_W^{(\gamma)}
\end{aligned}$$

for $|\gamma| \geq 1$, where the forcing term $\tilde{F}_W^{(\gamma)}$ is given by

$$\begin{aligned}
 \tilde{F}_W^{(\gamma)} &= \partial^\gamma \tilde{F}_W \\
 &\quad - \sum_{0 \leq \beta < \gamma} \binom{\gamma}{\beta} \left(\partial^{\gamma-\beta} G_W \partial_1 \partial^\beta \tilde{W} + \partial^{\gamma-\beta} h_W^\mu \partial_\mu \partial^\beta \tilde{W} + \beta_\tau \partial^{\gamma-\beta} (J \partial_1 \tilde{W}) \partial^\beta \tilde{W} \right) \\
 (2.51) \quad &\quad - \beta_\tau \mathbf{1}_{|\gamma| \geq 2} \sum_{\substack{1 \leq |\beta| \leq |\gamma|-2 \\ \beta \leq \gamma}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta \tilde{W} \\
 &\quad - \beta_\tau \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1=\gamma_1}} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta \tilde{W}.
 \end{aligned}$$

2.7 Constraints and the evolution of dynamic modulation variables

The use of modulated self-similar variables allows us to ensure that the evolution of W in (2.32a) maintains the constraints

$$(2.52) \quad W(0, s) = 0, \quad \partial_1 W(0, s) = -1, \quad \tilde{\nabla} W(0, s) = 0, \quad \nabla^2 W(0, s) = 0,$$

for all $s \geq -\log \varepsilon$. This is achieved by choosing our 10 time-dependent dynamic modulation parameters $\{n_\nu\}_{\nu=2}^3, \{\xi_i\}_{i=1}^3, \kappa, \tau, \{\phi_{\nu\mu}\}_{\nu,\mu=2}^3$ to satisfy a 10-by-10 coupled system of ODEs, which we describe next.

At time $t = -\varepsilon$ the modulation parameters are defined as

$$(2.53) \quad \kappa(-\varepsilon) = \kappa_0, \quad \tau(-\varepsilon) = \xi(-\varepsilon) = n_\mu(-\varepsilon) = 0, \quad \phi_{\nu\mu}(-\varepsilon) = \phi_{0,\nu\mu},$$

where κ_0 is defined in (3.29) and ϕ_0 is defined by (3.8). In order to determine the time derivatives of our 10 modulation parameters, we use the explicit form of the evolution equations for W , ∇W , and $\nabla^2 W$ (cf. (2.32a) and (2.44a)), which are evaluated at $y = 0$ and take into account the constraints in (2.52). Note that in this subsection we only collect the equations which *implicitly* define the evolution of the modulation parameters; only in Section 12 do we untangle the coupled nature of these implicitly defined ODEs to actually define the evolution of the constraints (cf. (12.12) and (12.13)), and prove that the resulting ODEs are globally well-posed.

Throughout the paper, for a function $\varphi(y, s)$, we shall denote $\varphi(0, s)$ by $\varphi^0(s)$. We make a preliminary observation regarding the value at $y = 0$ for the forcing terms $F_W^{(\gamma)}$ that appear in the evolution (2.44a) for $\partial^\gamma W$. Using (2.52) it is not hard to check that for any $\gamma \in \mathbb{N}_0^3$ with $|\gamma| = 1$ or $|\gamma| = 2$ we have that

$$(2.54) \quad F_W^{(\gamma),0} = \partial^\gamma F_W^0 + \partial^\gamma G_W^0.$$

Therefore, it is sufficient to know the derivatives up to order 2 of F_W and G_W at $y = 0$; these derivatives may be computed explicitly, and for convenience of the reader we have listed them in Appendix (A.3); see equations (A.7), (A.8), (A.9), (A.10), (A.10), (A.11), and (A.11). Next, we turn to the evolution equations for the modulation parameters.

First, we evaluate the equation for W in (2.32a) at $y = 0$ to obtain a definition for $\dot{\kappa}$. Using (2.32a) and (2.52) we obtain that

$$(2.55) \quad -G_W^0 = F_W^0 - e^{-\frac{s}{2}} \beta_\tau \dot{\kappa} \quad \Rightarrow \quad \dot{\kappa} = \frac{1}{\beta_\tau} e^{\frac{s}{2}} (F_W^0 + G_W^0).$$

Second, we evaluate the equation for $\partial_1 W$ at $y = 0$ and obtain a formula for $\dot{\tau}$. Indeed, using that $-1 + \beta_\tau = \frac{\dot{\tau}}{1-\dot{\tau}} = \dot{\tau} \beta_\tau$, we obtain from (2.44a) with $\gamma = e_1$ that

$$(2.56) \quad -(1 - \beta_\tau) = \partial_1 F_W^0 + \partial_1 G_W^0 \quad \Rightarrow \quad \dot{\tau} = \frac{1}{\beta_\tau} (\partial_1 F_W^0 + \partial_1 G_W^0).$$

Third, we turn to the evolution equation for $\check{\nabla} W$ at $y = 0$, which allows us to compute \dot{Q}_{1j} . Evaluating (2.44a) with $\gamma = e_v$ at $y = 0$ and using (2.54) we obtain for $v \in \{2, 3\}$ that

$$(2.57) \quad F_W^{0,(0,1,0)} = F_W^{0,(0,0,1)} = 0 \quad \Rightarrow \quad \partial_v F_W^0 + \partial_v G_W^0 = 0.$$

It is not immediately apparent that (2.57) determines \dot{Q}_{1j} . In order to see this one has to inspect the explicit formula for $\partial_v G_W^0$ in (A.7c), and to note that $\partial_v G_W^0 = 2\beta_1 \dot{Q}_{1v} +$ terms which are all small (bounded by ε to a positive power). This is explained in (12.3) below. Note that once \dot{Q}_{1j} is known, we can determine \dot{n} thorough an algebraic computation; this will be achieved in (12.5) below.

Fourth, we analyze the evolution of $\partial_1 \nabla W$ at $y = 0$. This constraint allows us to compute G_W^0 and $h_W^{\mu,0}$, which will in turn allow us to express $\dot{\xi}_i$; we initially focus on computing G_W^0 and $h_W^{\mu,0}$. Evaluating (2.44a) with $\gamma = e_1 + e_i$ at $y = 0$ for $i \in \{1, 2, 3\}$, and using (2.54), we obtain

$$(2.58) \quad G_W^0 \partial_{1i1} W^0 + h_W^{\mu,0} \partial_{1i\mu} W^0 = \partial_{1i} F_W^0 + \partial_{1i} G_W^0.$$

On the left side of the above identity we recognize the matrix

$$(2.59) \quad \mathcal{H}^0(s) := (\partial_1 \nabla^2 W)^0(s)$$

acting on the vector with components G_W^0 , $h_W^{2,0}$, and $h_W^{3,0}$. We will show (see (12.14) below) that the matrix \mathcal{H}^0 remains very close to the matrix $\text{diag}(6, 2, 2)$ for all $s \geq -\log \varepsilon$, and thus it is invertible. Therefore, we can express

$$(2.60a) \quad G_W^0 = (\mathcal{H}^0)^{-1}_{1i} (\partial_{1i} F_W^0 + \partial_{1i} G_W^0)$$

$$(2.60b) \quad h_W^{\mu,0} = (\mathcal{H}^0)^{-1}_{\mu i} (\partial_{1i} F_W^0 + \partial_{1i} G_W^0).$$

Once (2.60) is obtained, we may derive the evolution for $\dot{\xi}_i$. Indeed, from (2.34a), (2.13) evaluated at $\tilde{x} = 0$, the definition of V in (2.31), the constraints in (2.52), and the identities $N_\mu^0 = 0$, $T_\mu^{\gamma,0} = \delta_{\gamma\mu}$ we have that

$$(2.61) \quad \frac{1}{\beta_\tau} h_W^{\mu,0} = 2\beta_1 e^{-\frac{s}{2}} (A_\mu^0 - R_{j\mu} \dot{\xi}_j),$$

Similarly, from the definition of G_W in (2.33a), (2.13), and the constraints in (2.52), we deduce that

$$(2.62) \quad \frac{1}{\beta_\tau} G_W^0 = e^{\frac{s}{2}} \left(\kappa + \beta_2 Z^0 - 2\beta_1 R_{j1} \dot{\xi}_j \right).$$

Since the matrix R is orthogonal (hence invertible), it is clear that (2.60), (2.61), and (A.7a) determine $\dot{\xi}_j$.

Lastly, we use the evolution of $\widetilde{\nabla}^2 W$ at $y = 0$ in order to determine $\dot{\phi}_{v\gamma}$. Evaluating (2.44a) with $\gamma = e_v + e_\gamma$ at $y = 0$ and using (2.54), we obtain

$$(2.63) \quad G_W^0 \partial_{1v\gamma} W^0 + h_W^{\mu,0} \partial_{\mu v\gamma} W^0 = \partial_{v\gamma} F_W^0 + \partial_{v\gamma} G_W^0$$

for $v, \gamma \in \{2, 3\}$. Using (2.60a) and (2.60b) we rewrite the above identity as

$$(2.64) \quad \begin{aligned} \partial_{v\gamma} G_W^0 &= (\mathcal{H}^0)^{-1}_{1i} (\partial_{1i} F_W^0 + \partial_{1i} G_W^0) \partial_{1v\gamma} W^0 \\ &+ (\mathcal{H}^0)^{-1}_{\mu i} (\partial_{1i} F_W^0 + \partial_{1i} G_W^0) \partial_{\mu v\gamma} W^0 - \partial_{v\gamma} F_W^0. \end{aligned}$$

As with (2.57) earlier, it is not immediately clear that (2.64) determines the evolution of $\dot{\phi}_{v\gamma}$. In order to see this, we need to inspect the precise definition of $\partial_{v\gamma} G_W^0$ (cf. (A.7e) below), which yields that $\dot{\phi}_{v\gamma} = -e^{s/2} \frac{1}{\beta_\tau} \partial_{v\gamma} G_W^0 + \text{terms which are smaller (by a positive power of } \varepsilon \text{)}$. Details are given in (12.10) below.

The computations in this subsection derive implicit definitions for the time derivatives of our ten modulation parameters. In Section 12 we will show that the resulting system of ODEs for the modulation parameters is in fact solvable globally in time.

3 Main Results

3.1 Data in physical variables (x, t)

It is convenient to set $t_0 = -\varepsilon$. This corresponds to $t_0 = -\frac{2}{1+\alpha}\varepsilon$. We define initial conditions for the modulation variables as follows:

$$(3.1) \quad \begin{aligned} \kappa_0 &:= \kappa(-\varepsilon), \quad \tau_0 := \tau(-\varepsilon) = 0, \quad \xi_0 := \xi(-\varepsilon) = 0, \\ \check{n}_0 &:= \check{n}(-\varepsilon) = 0, \quad \phi_0 := \phi(-\varepsilon), \end{aligned}$$

where

$$(3.2) \quad \kappa_0 > 1, \quad |\phi_0| \leq \varepsilon.$$

Next, we define the initial value for the parametrization f of the front by

$$f_0(\check{x}) = \frac{1}{2} \phi_{0\nu\mu} x_\nu x_\mu,$$

and according to (2.14), we define the orthonormal basis (N_0, T_0^2, T_0^3) by

$$(3.3a) \quad N_0 = J_0^{-1}(1, -f_{0,2}, -f_{0,3}) \quad \text{where } J_0 = (1 + |f_{0,2}|^2 + |f_{0,3}|^2)^{\frac{1}{2}},$$

$$(3.3b) \quad \begin{aligned} T_0^2 &= \left(\frac{f_{0,2}}{J_0}, 1 - \frac{(f_{0,2})^2}{J_0(J_0+1)}, \frac{-f_{0,2}f_{0,3}}{J_0(J_0+1)} \right), \\ T_0^3 &= \left(\frac{f_{0,3}}{J_0}, \frac{-f_{0,2}f_{0,3}}{J_0(J_0+1)}, 1 - \frac{(f_{0,3})^2}{J_0(J_0+1)} \right). \end{aligned}$$

From (3.2) and (3.3) we deduce

$$(3.4) \quad |N_0 - e_1| \leq \varepsilon, \quad |T_0^\nu - e_\nu| \leq \varepsilon.$$

At $t = -\varepsilon$, the variable x is given by

$$(3.5) \quad x_1 = x_1 - f_0(\check{x}), \quad x_2 = x_2, \quad x_3 = x_3,$$

which is a consequence of (3.1), (2.3), and (2.15).

The remaining initial conditions are for the velocity field, density, and entropy which then provides us with the rescaled sound speed:

$$u_0(x) := u(x, -\varepsilon), \quad \rho_0(x) := \rho(x, -\varepsilon), \quad k_0(x) := k(x, -\varepsilon), \quad \sigma_0(x) := \frac{\rho_0^\alpha}{\alpha} e^{\frac{k_0}{2}}.$$

Following (2.16) and (2.26), we introduce the Riemann-type variables at initial time $t = -\varepsilon$ as

$$(3.6a) \quad \tilde{w}_0(x) := u_0(x) \cdot N_0(\check{x}) + \sigma_0(x),$$

$$(3.6b) \quad \tilde{z}_0(x) := u_0(x) \cdot N_0(\check{x}) - \sigma_0(x),$$

$$(3.6c) \quad \tilde{a}_{0\nu}(x) := u_0(x) \cdot T^\nu(\check{x}).$$

Using (3.5) and the fact that $\tilde{w}_0(x) = w(x, -\varepsilon)$ and that $\check{\nabla} f_0(0) = 0$, it follows that

$$(3.7) \quad \partial_{x_\nu} \partial_{x_\mu} w_0(0) = \partial_{x_\nu} \partial_{x_\mu} \tilde{w}_0(0) + \partial_{x_1} w_0(0) \phi_{0\nu\mu}.$$

As we will explain below, we will require that $\partial_{x_1} \tilde{w}_0(0) = -\frac{1}{\varepsilon}$, $\check{\nabla}_x \tilde{w}_0(0) = 0$, $\check{\nabla}_x^2 w_0(0) = 0$, and that $|\check{\nabla}_x^2 \tilde{w}_0(0)| \leq 1$, and thus from (3.7), we find that

$$(3.8) \quad \phi_{0\nu\mu} = \varepsilon \partial_{x_\nu} \partial_{x_\mu} \tilde{w}_0(0),$$

which shows that (3.2) holds.

In order to establish the formation of a stable self-similar shock, we shall stipulate conditions on the initial data. It is convenient to first explain these conditions in self-similar variables, and we now proceed to do so.

3.2 Data in self-similar variables (y, s)

At $s = -\log \varepsilon$ we have that $\tau_0 = 0$, and thus the self-similar variables y are given by

$$(3.9) \quad y_1 = \varepsilon^{-\frac{3}{2}} x_1 = \varepsilon^{-\frac{3}{2}} (x_1 - f_0(\check{x})) \quad \text{and} \quad \check{y} = \varepsilon^{-\frac{1}{2}} \check{x} = \varepsilon^{-\frac{1}{2}} \check{x}.$$

Second, we use (2.30), (3.1), and (3.6), to define

$$\begin{aligned} W(y, -\log \varepsilon) &= \varepsilon^{-\frac{1}{2}} (\tilde{w}_0(x) - \kappa_0), & Z(y, -\log \varepsilon) &= \tilde{z}_0(x), \\ A_v(y, -\log \varepsilon) &= \tilde{a}_{0v}(x), & K(y, -\log \varepsilon) &= \tilde{k}_0(x). \end{aligned}$$

This initial data is supported in the set \mathcal{X}_0 , given by

$$(3.10) \quad \mathcal{X}_0 = \{|y_1| \leq \varepsilon^{-1}, |\check{y}| \leq \varepsilon^{-\frac{1}{3}}\}.$$

At $y = 0$, we shall mimic the behavior of $\bar{W}(0)$ and assume that at initial time $s = -\log \varepsilon$,

$$(3.11a) \quad W(0, -\log \varepsilon) = 0, \quad \partial_1 W(0, -\log \varepsilon) = -1,$$

$$(3.11b) \quad \check{\nabla} W(0, -\log \varepsilon) = 0, \quad \nabla^2 W(0, -\log \varepsilon) = 0.$$

We define a sufficiently large parameter $M = M(\alpha, \kappa_0) \geq 1$ (which is in particular independent of ε), a small length scale ℓ , and a large length scale \mathcal{L} by

$$(3.12a) \quad \ell = (\log M)^{-5},$$

$$(3.12b) \quad \mathcal{L} = \varepsilon^{-\frac{1}{10}}.$$

For $|y| \leq \ell$ we shall prove that W is well approximated by its series expansion at $y = 0$, while for $\ell \leq |y| \leq \mathcal{L}$ we show that W and ∇W track \bar{W} and $\nabla \bar{W}$, respectively.

For the initial datum of $\tilde{W} = W - \bar{W}$ given by

$$\tilde{W}(y, -\log \varepsilon) = W(y, -\log \varepsilon) - \bar{W}(y),$$

we suppose that for $|y| \leq \mathcal{L}$,

$$(3.13a) \quad \eta^{-\frac{1}{6}}(y) |\tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{10}}$$

$$(3.13b) \quad \eta^{\frac{1}{3}}(y) |\partial_1 \tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{11}}$$

$$(3.13c) \quad |\check{\nabla} \tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{12}},$$

where $\eta(y) = 1 + y_1^2 + |\check{y}|^6$. In the smaller region $|y| \leq \ell$, we assume that

$$(3.14) \quad |\partial^\gamma \tilde{W}(y, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{8}} \quad \text{for } |\gamma| = 4,$$

and at $y = 0$, we have that

$$(3.15) \quad |\partial^\gamma \tilde{W}(0, -\log \varepsilon)| \leq \varepsilon^{\frac{1}{2} - \frac{4}{2m-7}} \quad \text{for } |\gamma| = 3.$$

For y in the region $\{|y| \geq \mathcal{L}\} \cap \mathcal{X}_0$, we suppose that

$$(3.16a) \quad \eta^{-\frac{1}{6}}(y)|W(y, -\log \varepsilon)| \leq 1 + \varepsilon^{\frac{1}{11}},$$

$$(3.16b) \quad \eta^{\frac{1}{3}}(y)|\partial_1 W(y, -\log \varepsilon)| \leq 1 + \varepsilon^{\frac{1}{12}},$$

$$(3.16c) \quad |\widetilde{\nabla} W(y, -\log \varepsilon)| \leq \frac{3}{4},$$

while for the second derivatives of W , globally for all $y \in \mathcal{X}_0$, we shall assume that

$$(3.17a) \quad \eta^{\frac{1}{3}}(y)|\partial^\gamma W(y, -\log \varepsilon)| \leq 1 \quad \text{for } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1,$$

$$(3.17b) \quad \eta^{\frac{1}{3}}(y)\psi^{-\frac{1}{4}}(y, -\log \varepsilon)|\partial^\gamma W(y, -\log \varepsilon)| \leq 1 \quad \text{for } \gamma = (2, 0, 0),$$

$$(3.17c) \quad \eta^{\frac{1}{6}}(y)|\widetilde{\nabla}^2 W(y, -\log \varepsilon)| \leq 1,$$

where $\psi(y, -\log \varepsilon) = \eta^{-1}(y) + \varepsilon^3 \eta(y)$.

For the initial conditions of Z , A , and K , we require that

$$(3.18) \quad |\partial^\gamma Z(y, -\log \varepsilon)| \leq \begin{cases} \varepsilon^{\frac{3}{2}}, & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 1, 2, \\ \varepsilon, & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 0, 1, 2, \end{cases}$$

$$(3.19) \quad |\partial^\gamma A(y, -\log \varepsilon)| \leq \begin{cases} \varepsilon^{\frac{3}{2}}, & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ \varepsilon, & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 0, 1, 2, \end{cases}$$

$$(3.20) \quad |\partial^\gamma K(y, -\log \varepsilon)| \leq \begin{cases} \varepsilon^2, & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, 1, \\ \varepsilon^{\frac{9}{4}} \eta^{-\frac{1}{15}}(y) & \text{if } \gamma_1 = 2 \text{ and } |\check{\gamma}| = 0, \\ \varepsilon, & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 0, 1, 2. \end{cases}$$

Consequently, the initial specific vorticity in self-similar variables satisfies

$$(3.21) \quad \|\Omega(\cdot, -\log \varepsilon) \cdot N_0\|_{L^\infty} \leq \varepsilon^{\frac{1}{4}} \quad \text{and} \quad \|\Omega(\cdot, -\log \varepsilon) \cdot T_0^v\|_{L^\infty} \leq 1,$$

and the initial scaled sound speed satisfies

$$(3.22) \quad \|S(y, -\log \varepsilon) - \frac{\kappa_0}{2}\|_{L^\infty} \leq \varepsilon^{\frac{1}{7}}.$$

Lastly, for the Sobolev norm of the initial condition, we suppose that for all $m \geq 18$,

$$(3.23) \quad \varepsilon \|W(\cdot, -\log \varepsilon)\|_{\dot{H}^m}^2 + \|Z(\cdot, -\log \varepsilon)\|_{\dot{H}^m}^2 + \|A(\cdot, -\log \varepsilon)\|_{\dot{H}^m}^2 + \|K(\cdot, -\log \varepsilon)\|_{\dot{H}^m}^2 \leq \varepsilon.$$

LEMMA 3.1 (Initial datum suitable for vorticity creation). *There exists initial datum $W(y, -\log \varepsilon)$ with support in the set \mathcal{X}_0 defined in (3.10), which satisfies the bounds (3.13)–(3.17), and which additionally can be chosen to satisfy*

$$(3.24) \quad -\frac{1}{2}|y_1|^{-\frac{2}{3}} \leq \partial_1 W(y, -\log \varepsilon) \leq -\frac{1}{4}|y_1|^{-\frac{2}{3}} \\ \text{for } \varepsilon^{-\frac{1}{10}} \leq |y_1| \leq 2\kappa_0 \varepsilon^{-\frac{1}{2}}, \quad |\check{y}| \leq \varepsilon^{\frac{1}{3}}.$$

Moreover, associated to this choice of $W(y, -\log \varepsilon)$, letting $Z(y, -\log \varepsilon) = 0$ and $\phi_0 = 0$, there exists an $A(y, -\log \varepsilon)$, such that

$$\begin{aligned}
 (3.25) \quad & \tilde{u}(\tilde{x}, -\varepsilon) \\
 &= U(y, -\log \varepsilon) \\
 &= \left(\frac{1}{2}(\varepsilon^{\frac{1}{2}} W(y, -\log \varepsilon) + \kappa_0), A_2(y, -\log \varepsilon), A_3(y, -\log \varepsilon)\right)
 \end{aligned}$$

is irrotational with respect to the physical space variable \tilde{x} .

PROOF. The proof of (3.24) is based on the introduction of cutoff functions in both the y_1 -direction and in the \check{y} -direction, and the multiplication of the globally self-similar profile \bar{W} by these cutoffs. The only nontrivial part of this argument is to choose the dependence of the aforementioned cutoffs on ε^{-1} .

We start by defining a cutoff function with two parameters. For $b \geq 2a > 0$ we let $\eta[a, b](r)$ be a smooth nonincreasing function which is identically equal to 1 for $r \in [0, a]$, and vanishes identically for $r \in [a + b, \infty)$. For the purposes of this lemma we may take the piecewise linear cutoff function and mollify it with a compactly supported mollifier with characteristic length which is ε -dependent. For example, we may mollify with a mollifier of compact support at scale $\varepsilon^{1/10}$ the function which equals 1 for $r \leq a\varepsilon^{1/10}$, equals 0 for $r \geq a + b - \varepsilon^{1/10}$, and is given by $1 - (r - a - \varepsilon^{1/10})(b - 2\varepsilon^{1/10})^{-1}$ for $a < r < a + b$. In particular, we may ensure that up to a constant factor of $\varepsilon^{1/10}$ the derivative of $\eta[a, b](r)$ is given by $-b^{-1}$ on the region $r \in (a, a + b)$, and vanishes outside of this region. Similarly, the second derivative of this cutoff function is bounded by a constant multiple of $b^{-1}\varepsilon^{-1/10}$ on the region where it does not vanish.

Finally, we define the initial datum $W(y, -\log \varepsilon)$ to be a cutoff version of \bar{W} , according to

$$(3.26) \quad W(y, -\log \varepsilon) = \bar{W}(y) \eta[\varepsilon^{-\frac{1}{2} - \frac{1}{16}}, \varepsilon^{-\frac{3}{4}}](|y_1|) \eta[\varepsilon^{-\frac{1}{4}}, 100\varepsilon^{-\frac{1}{4}}](|\check{y}|).$$

A lengthy but routine computation which uses properties of the explicit function \bar{W} (see, e.g., [4, equation (2.48) and remark 3.3]) shows that the function $W(y, -\log \varepsilon)$ satisfies the conditions (3.13)–(3.17). We omit these details, but give the proof of condition 3.24, which is essential for the vorticity creation argument. We note that for $|y_1| \leq 2\kappa_0\varepsilon^{-1/2}$ we have that $\eta[\varepsilon^{-(1/2)-(1/16)}, \varepsilon^{-3/4}](|y_1|) = 1$, and for $|\check{y}| \leq \varepsilon^{1/3}$ we have $\eta[\varepsilon^{-1/4}, 100\varepsilon^{-1/4}](|\check{y}|) = 1$. Thus, in the region relevant for (3.24), by using (2.47) we have

$$(3.27) \quad \partial_1 W(y, -\log \varepsilon) = \partial_1 \bar{W}(y) = \frac{1}{1+|\check{y}|^2} W'_{\text{ID}}\left(\frac{y_1}{(1+|\check{y}|^2)^{3/2}}\right).$$

The function W_{ID} is explicit, and the Taylor series of its derivative around infinity is given by $W'_{\text{ID}}(r) = -\frac{1}{3}r^{-2/3} - \frac{1}{9}r^{-4/3} + \mathcal{O}(r^{-8/3})$. Using that we are interested in a region where $|\check{y}| \leq \varepsilon^{1/3}$, and $\varepsilon^{-1/10} \leq |y_1| \leq 2\kappa_0\varepsilon^{-1/2}$, upon choosing ε sufficiently small (so that the Taylor series expansion around infinity is the relevant

one), we immediately deduce that from (3.27) that

$$-\frac{1+\varepsilon^{10}}{3}|y_1|^{-\frac{2}{3}} \leq \frac{1}{1+|\check{y}|^2} W'_{\text{ID}} \left(\frac{y_1}{(1+|\check{y}|^2)^{\frac{2}{3}}} \right) \leq -\frac{1-\varepsilon^{10}}{3}|y_1|^{-\frac{2}{3}}$$

in the region of relevance to (3.24). This establishes the existence of W satisfying (3.24) as well as the bounds (3.13)–(3.17).

Next, for $W(y, -\log \varepsilon)$ given by (3.26) and with $Z(y, -\log \varepsilon) = 0$, we shall now prove the existence of an irrotational initial velocity field $\tilde{u}(\tilde{x}, \varepsilon)$ satisfying (3.25).

We first set $\phi_0 = 0$ so that $N_0 = e_1$, $T_0^\nu = e_\nu$, and $J_0 = 1$, and $(\tilde{x}_1, \tilde{x}_\nu) = (\varepsilon^{-3/2} y_1, \varepsilon^{-1/2} y_\nu)$. We have that $\tilde{w}_0(\tilde{x}, -\varepsilon) = \varepsilon^{1/2} W(y, -\log \varepsilon) + \kappa_0$, and from (3.25), we see that

$$\tilde{u}(\tilde{x}, \varepsilon)_1 = \frac{1}{2} \tilde{w}_0.$$

In order to ensure that $\tilde{u}_1 = \partial_{\tilde{x}_1} \Psi$, we define

$$\Psi(\tilde{x}) = \frac{1}{2} \int_0^{\tilde{x}_1} \tilde{w}_0(\tilde{x}'_1, \tilde{x}) d\tilde{x}'_1 - \frac{1}{2} \int_0^\infty \tilde{w}_0(\tilde{x}'_1, \tilde{x}) d\tilde{x}'_1$$

for $\tilde{x}_1 > 0$ and then extend $\Psi(\tilde{x})$ as an even function in \tilde{x}_1 . We now define

$$(3.28) \quad \tilde{a}_\nu(\tilde{x}, -\varepsilon) = \partial_{\tilde{x}_\nu} \Psi(\tilde{x}),$$

so that $\tilde{u}(\tilde{x}, -\varepsilon) = \nabla_{\tilde{x}} \Psi(\tilde{x})$, which implies that $\text{curl}_{\tilde{x}} \tilde{u}(x, -\varepsilon) = 0$. We write (3.28) in self-similar coordinates as

$$A_\nu(y, -\log \varepsilon) = -\frac{1}{2} \varepsilon^{\frac{3}{2}} \int_{y_1}^\infty \partial_\nu W(y'_1, \check{y}, -\log \varepsilon) dy'_1.$$

Using the definition of $W(y, -\log \varepsilon)$ given in (3.26), a lengthy computation shows that $A(y, -\log \varepsilon)$ satisfies the bounds (3.19) and (3.23). \square

3.3 Statement of the main theorem in self-similar variables and asymptotic stability

THEOREM 3.2 (Stability and shock formation via self-similar variables). *For $\alpha = \frac{\gamma-1}{2}$ and $\gamma > 1$, let $\kappa_0 = \kappa_0(\alpha) > 1$ be chosen sufficiently large. Suppose that at initial time $s = -\log \varepsilon$, the initial data $(W_0, Z_0, A_0, K_0) = (W, Z, A, K)|_{s=-\log \varepsilon}$ are supported in the set \mathcal{X}_0 from (3.10), and obey conditions (3.11)–(3.23). Assume that the modulation functions have initial conditions compatible with (3.1)–(3.2).*

There exist $M = M(\alpha, \kappa_0) \geq 1$ sufficiently large, $\varepsilon = \varepsilon(\alpha, \kappa_0, M) \in (0, 1)$ sufficiently small, unique smooth modulation functions $(\kappa, \tau, \xi, n, \phi)$ which solve the system (12.12) and (12.13), and unique global-in-time solutions (W, Z, A, K) to (2.32) with the following properties. The functions (W, Z, A, K) are supported in the time-dependent cylinder $\mathcal{X}(s)$ defined in (4.4),

$(W, Z, A, K) \in C([-\log \varepsilon, +\infty); H^m) \cap C^1([-\log \varepsilon, +\infty); H^{m-1})$ for $m \geq 18$,

and

$$\begin{aligned} & \|W(\cdot, s)\|_{\dot{H}^m}^2 + e^s \|Z(\cdot, s)\|_{\dot{H}^m}^2 + e^s \|A(\cdot, s)\|_{\dot{H}^m}^2 + e^s \|K(\cdot, s)\|_{\dot{H}^m}^2 \\ & \leq 16\kappa_0^2 \lambda^{-m} e^{-s-\log \varepsilon} + (1 - e^{-s} \varepsilon^{-1}) M^{4m} \end{aligned}$$

for a constant $\lambda = \lambda(m) \in (0, 1)$. The modulation functions $(\kappa, \tau, \xi, n, \phi)$ lie in $C^1([- \varepsilon, T_*))$ and satisfy the bounds (4.1). The Riemann function $W(y, s)$ remains close to the generic and stable self-similar blowup profile $\bar{W}\tilde{u}$; upon defining the weight function $\eta(y) = 1 + y_1^2 + |\check{y}|^6$, the perturbation $\tilde{W} = W - \bar{W}$ satisfies

$$|\tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{11}} \eta^{\frac{1}{6}}(y), \quad |\partial_1 \tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{12}} \eta^{-\frac{1}{3}}(y), \quad |\check{\nabla} \tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{13}},$$

for all $|y| \leq \varepsilon^{-\frac{1}{10}}$ and $s \geq -\log \varepsilon$. Furthermore, $\partial^\gamma \tilde{W}(0, s) = 0$ for all $|\gamma| \leq 2$, and the bounds (4.9) and (4.10) hold. Additionally, $W(y, s)$ satisfies the bounds given in (4.7) and (4.19).

As $s \rightarrow \infty$, $W(y, s)$ converges to an asymptotic profile $\bar{W}_{\mathcal{A}}(y)$ which satisfies:

- $\bar{W}_{\mathcal{A}}$ is a C^∞ smooth solution to the self-similar 3D Burgers equation (1.5).
- $\bar{W}_{\mathcal{A}}(y)$ obeys the genericity condition (1.7).
- $\bar{W}_{\mathcal{A}}$ is uniquely determined by the 10 parameters $\mathcal{A}_\alpha = \lim_{s \rightarrow \infty} \partial^\alpha W(0, s)$ for $|\alpha| = 3$.

The amplitude of the functions Z , A , and K remains $\mathcal{O}(\varepsilon)$ for all $s \geq -\log \varepsilon$, while for each $|\gamma| \leq m$, $\partial^\gamma Z(\cdot, s) \rightarrow 0$, $\partial^\gamma A(\cdot, s) \rightarrow 0$, and $\partial^\gamma K(\cdot, s) \rightarrow 0$ as $s \rightarrow +\infty$, and Z and A satisfy the bounds (4.12), (4.13), (4.14).

The scaled sound speed $S(y, s)$ satisfies

$$\|S(\cdot, s) - \frac{\kappa_0}{2}\|_{L^\infty} \leq \varepsilon^{\frac{1}{8}} \quad \text{for all } s \geq -\log \varepsilon.$$

The specific vorticity $\Omega(y, s) = \overset{\circ}{\zeta}(x, t)$ satisfies for all $s \geq -\log \varepsilon$,

$$\|\Omega \circ \Phi_U^{y_0}(\cdot, s) - \Omega(\cdot, -\log \varepsilon)\|_{L^\infty} \leq \varepsilon^{\frac{1}{20}}$$

where $\Phi_U^{y_0}$ is defined in (5.11). Furthermore, there exists irrotational initial data from which vorticity is instantaneously created and remains nonzero in a neighborhood of the shock location $(0, T_*)$: see Theorem 7.4 for details.

For concision, the initial data was assumed to have the support property (3.10) and satisfy the conditions (3.11). By using the symmetries of the Euler equations, we can generalize these conditions to allow for data in a non-trivial open set in the H^m topology.

THEOREM 3.3 (Open set of initial conditions). *Let $\tilde{\mathcal{F}}$ denote the set of initial data satisfying the hypothesis of Theorem 3.2. There exists an open neighborhood of $\tilde{\mathcal{F}}$ in the H^m topology, denoted by \mathcal{F} , such that for any initial data to the Euler equations taken from \mathcal{F} , the conclusions of Theorem 3.2 hold.*

3.4 Shock formation in physical variables (x, t)

We shall now interpret the assumptions and results of Theorem 3.2 in the context of physical variables (x, t) . The function

$$\tilde{w}_0(x) = w(x, -\varepsilon) = \varepsilon^{\frac{1}{2}} W(y, -\log \varepsilon) + \kappa_0$$

is chosen such that the minimum (negative) slope of \tilde{w}_0 occurs in the e_1 direction, and $\partial_{x_1} \tilde{w}_0$ attains its global minimum at $x = 0$, and from (3.11), satisfies

$$(3.29) \quad \tilde{w}_0(0) = \kappa_0, \quad \partial_{x_1} \tilde{w}_0(0) = -\frac{1}{\varepsilon}, \quad \tilde{\nabla}_x \tilde{w}_0(0) = 0, \quad \nabla_x \partial_{x_1} \tilde{w}_0(0) = 0.$$

Of course, there are a number of additional conditions on $\tilde{w}_0(x)$ and its partial derivatives which exactly correspond to conditions (3.13)–(3.17) by the change of variables (2.29), but the conditions (3.29) are fundamental to the stable self-similar point shock formation process.

We shall assume that the support of the initial data $(\tilde{w}_0 - \kappa_0, \tilde{z}_0, \tilde{a}_0)$, is contained in the set $\mathcal{X}_0 = \{|x_1| \leq \frac{1}{2}\varepsilon^{\frac{1}{2}}, |\check{x}| \leq \varepsilon^{\frac{1}{6}}\}$, which in turn shows that $u_0 \cdot N_0 - \frac{\kappa_0}{2}$, $\sigma_0 - \frac{\kappa_0}{2}$, and $u_0 \cdot T^\nu$ are compactly supported in \mathcal{X}_0 . In view of the coordinate transformation (3.5) and the bound (3.2), the functions of x defined in (3.6), namely (w_0, z_0, a_0, k_0) , have spatial support contained in the set $\{|x_1| \leq \frac{1}{2}\varepsilon^{1/2} + \varepsilon, |\check{x}| \leq \varepsilon^{1/6}\} \subset \{|x_1| \leq \varepsilon^{1/2}, |\check{x}| \leq \varepsilon^{1/6}\}$. This larger set corresponds to the support condition (3.10) under the transformation (2.29).

For the initial conditions of \tilde{z}_0, \tilde{a}_0 , and k_0 , from (3.18)–(3.20), we have that⁷

$$\begin{aligned} |\tilde{z}_0(x)| &\leq \varepsilon, & |\partial_{x_1} \tilde{z}_0(x)| &\leq 1, & |\tilde{\nabla}_x \tilde{z}_0(x)| &\leq \varepsilon^{\frac{1}{2}}, \\ |\tilde{a}_0(x)| &\leq \varepsilon, & |\partial_{x_1} \tilde{a}_0(x)| &\leq 1, & |\tilde{\nabla}_x \tilde{a}_0(x)| &\leq \varepsilon^{\frac{1}{2}}, \\ |\tilde{k}_0(x)| &\leq \varepsilon, & |\partial_{x_1} \tilde{k}_0(x)| &\leq \varepsilon^{\frac{1}{2}}, & |\tilde{\nabla}_x \tilde{k}_0(x)| &\leq \varepsilon^{\frac{1}{2}}, \end{aligned}$$

together with conditions on higher-order derivatives⁸ that follow (3.18)–(3.20) and (3.23).

The initial specific vorticity

$$\tilde{\zeta}(\tilde{x}, -\varepsilon) = \overset{\circ}{\zeta}(x, -\varepsilon) = \Omega(y, -\log \varepsilon)$$

satisfies condition (3.21), and the initial scale sound speed

$$\tilde{\sigma}(\tilde{x}, -\varepsilon) = \overset{\circ}{\sigma}(x, -\varepsilon) = S(y, -\log \varepsilon)$$

satisfies (3.22).

⁷ The bound for $\partial_{x_1} a_0$ can be replaced by a bound that depends on κ_0 , thus permitting arbitrarily large initial vorticity.

⁸ We deduce from (3.23) that at $t = -\varepsilon$, the Sobolev norm must satisfy

$$\sum_{|\nu|=m} \varepsilon^2 \|\partial_x^\nu w_0\|_{L^2}^2 + \|\partial_x^\nu z_0\|_{L^2}^2 + \|\partial_x^\nu a_0\|_{L^2}^2 + \|\partial_x^\nu k_0\|_{L^2}^2 \leq \varepsilon^{\frac{7}{2} - (3\gamma_1 + |\check{\nu}|)}.$$

See (3.21)–(3.22) in [4] for details.

We now summarize the statement of Theorem 3.2 in the physical variables. Suppose that the initial data $\tilde{w}_0, \tilde{z}_0, \tilde{a}_0$, and k_0 satisfy the conditions stated above and that $\alpha = \frac{\gamma-1}{2} > 0$ is fixed. There exist a sufficiently large $\kappa_0 = \kappa_0(\alpha) > 1$ and a sufficiently small $\varepsilon = \varepsilon(\alpha, \kappa_0) \in (0, 1)$ such that there exists a time $T_* = \mathcal{O}(\varepsilon^2)$, unique modulation functions $(\kappa, \tau, \xi, n, \phi) \in C^1([- \varepsilon, T_*])$, which solve the system (12.12) and (12.13), and a unique solution

$$(u, \rho, k) \in C([- \varepsilon, T_*]; H^m) \cap C^1([- \varepsilon, T_*]; H^{m-1})$$

to (1.1) which blows up in an asymptotically self-similar fashion at time T_* , at a single point $\xi_* \in \mathbb{R}^3$. In particular, the following results hold:

- (i) The blowup time $T_* = \mathcal{O}(\varepsilon^2)$ and the blowup location $\xi_* = \mathcal{O}(\varepsilon)$ are explicitly computable, with T_* defined by the condition $\int_{-\varepsilon}^{T_*} (1 - \dot{\tau}(t)) dt = \varepsilon$ and with the blowup location given by $\xi_* = \lim_{t \rightarrow T_*} \xi(t)$. The amplitude modulation function satisfies $|\kappa_* - \kappa_0| = \mathcal{O}(\varepsilon^{\frac{3}{2}})$ where $\kappa_* = \lim_{t \rightarrow T_*} \kappa(t)$.
- (ii) For each $t \in [- \varepsilon, T_*)$, we have $|\mathbf{N}(\tilde{x}, t) - \mathbf{N}_0(\tilde{x})| + |\mathbf{T}^\nu(\tilde{x}, t) - \mathbf{T}_0^\nu(\tilde{x})| = \mathcal{O}(\varepsilon)$.
- (iii) We have

$$\sup_{t \in [- \varepsilon, T_*)} \|\tilde{u} \cdot \mathbf{N} - \frac{\kappa_0}{2}\|_{L^\infty} + \|\tilde{u} \cdot \mathbf{T}^\nu\|_{L^\infty} + \|\tilde{\sigma} - \frac{\kappa_0}{2}\|_{L^\infty} + \|\zeta\|_{L^\infty} \lesssim 1.$$

- (iv) There holds

$$\lim_{t \rightarrow T_*} \mathbf{N} \cdot \nabla_{\tilde{x}} \tilde{w}(\xi(t), t) = -\infty$$

and

$$\frac{1}{2(T_* - t)} \leq \|\mathbf{N} \cdot \nabla_{\tilde{x}} \tilde{w}(\cdot, t)\|_{L^\infty} \leq \frac{2}{T_* - t}$$

as $t \rightarrow T_*$.

- (v) At the time of blowup, $\tilde{w}(\cdot, T_*)$ has a cusp-type singularity with $C^{1/3}$ Hölder regularity.
- (vi) Only the $\partial_{\mathbf{N}}$ derivative of $\tilde{u} \cdot \mathbf{N}$ and $\tilde{\rho}$ blow up, while the other first-order derivatives remain bounded:

$$(3.30a) \quad \lim_{t \rightarrow T_*} \mathbf{N} \cdot \nabla_{\tilde{x}} (\tilde{u} \cdot \mathbf{N})(\xi(t), t) = \lim_{t \rightarrow T_*} \mathbf{N} \cdot \nabla_{\tilde{x}} \tilde{\rho}(\xi(t), t) = -\infty,$$

$$(3.30b) \quad \sup_{t \in [- \varepsilon, T_*)} \|\mathbf{T}^\nu \cdot \nabla_{\tilde{x}} \tilde{\rho}(\cdot, t)\|_{L^\infty} + \|\mathbf{T}^\nu \cdot \nabla_{\tilde{x}} \tilde{u}(\cdot, t)\|_{L^\infty} + \|\mathbf{N} \cdot \nabla_{\tilde{x}} (\tilde{u} \cdot \mathbf{T}^\nu)(\cdot, t)\|_{L^\infty} \lesssim 1.$$

- (vii) Both \tilde{k} and $\nabla_{\tilde{x}} \tilde{k}$ remain bounded:

$$(3.31) \quad \sup_{t \in [- \varepsilon, T_*)} \|\tilde{k}(\cdot, t)\|_{L^\infty} + \|\nabla_{\tilde{x}} \tilde{k}(\cdot, t)\|_{L^\infty} \lesssim \varepsilon^{\frac{1}{8}}.$$

- (viii) Let $\partial_t X(\mathbf{x}, t) = u(X(\mathbf{x}, t), t)$ with $X(\mathbf{x}, -\varepsilon) = \mathbf{x}$ so that $X(\mathbf{x}, t)$ is the Lagrangian flow. Then there exists constants c_1, c_2 such that $c_1 \leq |\nabla_{\mathbf{x}} X(\mathbf{x}, t)| \leq c_2$ for all $t \in [- \varepsilon, T_*)$.

(ix) The scaled sound $\tilde{\sigma}$ remains uniformly bounded from below and satisfies

$$\|\tilde{\sigma}(\cdot, t) - \frac{\kappa_0}{2}\|_{L^\infty} \leq \varepsilon^{1/8} \text{ for all } t \in [-\varepsilon, T_*].$$

(x) The vorticity satisfies $\|\omega(\cdot, t)\|_{L^\infty} \leq C_0 \|\omega(\cdot, -\varepsilon)\|_{L^\infty}$ for all $t \in [-\varepsilon, T_*]$ for a universal constant C_0 , and if $|\omega(\cdot, -\varepsilon)| \geq c_0 > 0$ on the set $B(0, 2\varepsilon^{3/4})$, then at the blowup location ξ_* there is nontrivial vorticity, and moreover

$$|\omega(\cdot, T_*)| \geq \frac{c_0}{C_0} \quad \text{on the set } B(0, \varepsilon^{3/4}).$$

4 Bootstrap Assumptions

As discussed above, the proof of Theorem 3.2 consists of a bootstrap argument, which we make precise in this section. For M sufficiently large, depending on κ_0 and on α , and for ε sufficiently small, depending on M , κ_0 , and α , we postulate that the modulation functions are bounded as in (4.1), that (W, Z, A, K) are supported in the set given by (4.4), that W satisfies (4.7), \tilde{W} obeys (4.8)–(4.10), and that Z , A , and K are bounded as in (4.12)–(4.14). All these bounds have explicit constants in them. In the subsequent sections of the paper, we prove that these estimates in fact hold with strictly better prefactors, which in view of a continuation argument yields the proof of Theorem 3.2.

4.1 Dynamic variables

For the dynamic modulation variables, we assume that

$$(4.1a) \quad \begin{aligned} \frac{1}{2}\kappa_0 \leq \kappa(t) \leq 2\kappa_0, \quad |\tau(t)| \leq M\varepsilon^2, \quad |\xi(t)| \leq M^{\frac{1}{4}}\varepsilon, \\ |\check{n}(t)| \leq M^2\varepsilon^{\frac{3}{2}}, \quad |\phi(t)| \leq M^2\varepsilon, \end{aligned}$$

$$(4.1b) \quad \begin{aligned} |\dot{\kappa}(t)| \leq e^{-\frac{3s}{10}}, \quad |\dot{\tau}(t)| \leq Me^{-s}, \quad |\dot{\xi}(t)| \leq M^{\frac{1}{4}}, \\ |\dot{\check{n}}(t)| \leq M^2\varepsilon^{\frac{1}{2}}, \end{aligned}$$

for all $-\varepsilon \leq t < T_*$.

From (2.6) and (A.4)–(A.5) in [4], and the bootstrap assumptions (4.1), we obtain that

$$(4.2) \quad |\dot{Q}(t)| \leq 2M^2\varepsilon^{\frac{1}{2}}.$$

Also, from the $\dot{\tau}$ estimate in (4.1b), we obtain

$$(4.3) \quad |1 - \beta_\tau| = \frac{|\dot{\tau}|}{1 - \dot{\tau}} \leq 2Me^{-s} \leq 2M\varepsilon$$

upon taking ε sufficiently small.

4.2 Spatial support bootstrap

We shall assume that (W, Z, A) have support in the set

$$(4.4) \quad \mathcal{X}(s) := \left\{ |y_1| \leq 2\varepsilon^{\frac{1}{2}} e^{\frac{3}{2}s}, |\check{y}| \leq 2\varepsilon^{\frac{1}{6}} e^{\frac{s}{2}} \right\} \quad \text{for all } s \geq -\log \varepsilon.$$

We introduce the weights

$$\eta(y) = 1 + y_1^2 + |\check{y}|^6 \quad \text{and} \quad \tilde{\eta}(y) = \eta(y) + |\check{y}|^2,$$

as well as the s -dependent weight function

$$\psi(y, s) = \frac{1}{\eta(y)} + e^{-3s} \eta(y).$$

For $y \in \mathcal{X}(s)$, we note that

$$(4.5) \quad \eta(y) \leq 40\varepsilon e^{3s} \quad \Leftrightarrow \quad \eta^{\frac{1}{3}}(y) \leq 4\varepsilon^{\frac{1}{3}} e^s$$

for all $y \in \mathbb{R}^3$. Since $\eta\psi = 1 + e^{-3s}\eta^2$, we have $e^{-3s}\eta^2 \leq \eta\psi$, and thus

$$(4.6) \quad e^{-s} \lesssim \psi^{\frac{q}{3}} \eta^{-\frac{1}{3}(2-q)}$$

holds for $1 < q \leq 2$.

4.3 W bootstrap

The bootstrap assumptions on W and its derivatives are

$$(4.7) \quad |\partial^\gamma W(y, s)| \leq \begin{cases} (1 + \varepsilon^{\frac{1}{20}}) \eta^{\frac{1}{6}}, & \text{if } |\gamma| = 0, \\ \tilde{\eta}^{-\frac{1}{3}} \left(\frac{\gamma}{2}\right) \mathbf{1}_{|y| \leq \mathcal{L}} + 2\eta^{-\frac{1}{3}}(y) \mathbf{1}_{|y| \geq \mathcal{L}}, & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ 1, & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ M^{\frac{2}{3}} \eta^{-\frac{1}{3}}, & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ M^{\frac{1}{3}} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}}, & \text{if } \gamma_1 = 2 \text{ and } |\check{\gamma}| = 0, \\ M\eta^{-\frac{1}{6}}, & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2. \end{cases}$$

Next, for $|y| \leq \mathcal{L}$, we assume that⁹

$$(4.8a) \quad |\tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{11}} \eta^{\frac{1}{6}}(y),$$

$$(4.8b) \quad |\partial_1 \tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{12}} \eta^{-\frac{1}{3}}(y),$$

$$(4.8c) \quad |\check{\nabla} \tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{13}},$$

where \mathcal{L} is defined as in (3.12b). Furthermore, for $|y| \leq \ell$ (as defined in (3.12a)) we assume that

$$(4.9a) \quad \begin{aligned} |\partial^\gamma \tilde{W}(y, s)| &\leq (\log M)^4 \varepsilon^{\frac{1}{10}} |y|^{4-|\gamma|} + M \varepsilon^{\frac{1}{4}} |y|^{3-|\gamma|} \\ &\leq 2(\log M)^4 \varepsilon^{\frac{1}{10}} \ell^{4-|\gamma|}, \end{aligned} \quad |\gamma| \leq 3,$$

$$(4.9b) \quad |\partial^\gamma \tilde{W}(y, s)| \leq \varepsilon^{\frac{1}{10}} (\log M)^{|\check{\gamma}|}, \quad |\gamma| = 4,$$

⁹ While the first three bounds stated in (4.7) follow directly from the properties of \tilde{W} stated in (2.48) of [4], and those of \tilde{W} in (4.8), the estimate for $\partial_1 W$ makes use of the fact that $\tilde{\eta}^{-1/3}(y) + \varepsilon^{1/12} \eta^{-1/3}(y) \leq \tilde{\eta}^{-1/3}(y/2)$.

while at $y = 0$, we assume that

$$(4.10) \quad |\partial^\gamma \tilde{W}(0, s)| \leq \varepsilon^{\frac{1}{4}} \quad \text{for all } |\gamma| = 3,$$

for all $s \geq -\log \varepsilon$.

LEMMA 4.1 (Lower bound for $J\partial_1 W$).

$$(4.11) \quad \begin{aligned} J\partial_1 W(y, s) &\geq -1 \quad \text{and} \quad J\partial_1 \bar{W}(y, s) \geq -1 \\ &\text{for all } y \in \mathbb{R}^3, s \geq -\log \varepsilon. \end{aligned}$$

The proof of this lemma is given in the proof of Lemma 4.2 in [4].

4.4 Z and A bootstrap

The bootstrap assumptions on Z , A , K , and their derivatives are:

$$(4.12) \quad |\partial^\gamma Z(y, s)| \leq \begin{cases} M^{\frac{1+|\gamma|}{2}} e^{-\frac{3}{2}s}, & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 1, 2 \\ M \varepsilon^{\frac{2-|\gamma|}{2}} e^{-\frac{|\gamma|}{2}s}, & \text{if } \gamma_1 = 0 \text{ and } |\gamma| = 0, 1, 2, \end{cases}$$

$$(4.13) \quad |\partial^\gamma A(y, s)| \leq \begin{cases} M e^{-\frac{3}{2}s}, & \text{if } \gamma_1 = 1 \text{ and } |\gamma| = 0 \\ M \varepsilon^{\frac{2-|\gamma|}{2}} e^{-\frac{|\gamma|}{2}s}, & \text{if } \gamma_1 = 0 \text{ and } |\gamma| = 0, 1, 2, \end{cases}$$

$$(4.14) \quad |\partial^\gamma K(y, s)| \leq \begin{cases} \varepsilon^{\frac{1}{4}} e^{-\frac{3}{2}s}, & \text{if } \gamma_1 = 1 \text{ and } |\gamma| = 0 \\ \varepsilon^{\frac{1}{8}} e^{-\frac{13}{8}s}, & \text{if } \gamma_1 = 1 \text{ and } |\gamma| = 1 \\ \varepsilon^{\frac{1}{8}} e^{-2s} \eta^{-\frac{1}{15}}(y), & \text{if } \gamma_1 = 2 \text{ and } |\gamma| = 0 \\ \varepsilon^{\frac{1}{8}} e^{-\frac{|\gamma|}{2}s}, & \text{if } \gamma_1 = 0 \text{ and } |\gamma| = 1, 2. \end{cases}$$

Remark 4.2. Since K satisfies a transport equation, the pointwise bound

$$(4.15) \quad |K(y, s)| \leq \varepsilon$$

follows directly from the initial datum assumption (3.20).

4.5 Further consequences of the bootstrap assumptions

The bootstrap bounds (4.1), (4.5), (4.7)–(4.10), (4.12), and (4.13) have a number of consequences, which we collect here for future reference. The first is a global-in-time L^2 -based Sobolev estimate:

PROPOSITION 4.3 (\dot{H}^m estimate for W , Z , and A). *For integers $m \geq 18$ and for a constant $\lambda = \lambda(m)$,*

$$(4.16a) \quad \begin{aligned} &\|Z(\cdot, s)\|_{\dot{H}^m}^2 + \|A(\cdot, s)\|_{\dot{H}^m}^2 + \|K(\cdot, s)\|_{\dot{H}^m}^2 \\ &\leq 16\kappa_0^2 \lambda^{-m} \varepsilon^{-1} e^{-2s} + (e^{-s} - e^{-2s} \varepsilon^{-1}) M^{4m} \end{aligned}$$

$$(4.16b) \quad \begin{aligned} &\|W(\cdot, s)\|_{\dot{H}^m}^2 \\ &\leq 16\kappa_0^2 \lambda^{-m} \varepsilon^{-1} e^{-s} + (1 - e^{-s} \varepsilon^{-1}) M^{4m} \end{aligned}$$

for all $s \geq -\log \varepsilon$.

The proof of Proposition 4.3, which will be given at the end of Section 8, relies only upon the initial data assumption (3.23), on the support bound (4.5), on L^∞ estimates for $\partial^\gamma W$, $\partial^\gamma Z$, and $\partial^\gamma K$ when $|\gamma| \leq 2$, on $\partial^\gamma A$ pointwise bounds for $|\gamma| \leq 1$, and on $\tilde{V}^2 A$ bounds. That is, Proposition 4.3 follows directly from (3.23) and the bootstrap assumptions (4.1), (4.5), (4.7), (4.12), and (4.13).

The reason we state Proposition 4.3 at this stage of the analysis is that the \dot{H}^m estimates and linear interpolation yield useful information for higher-order derivatives of (W, Z, A, K) , which are needed in order to close the bootstrap assumptions for high-order derivatives. These bounds are summarized as follows:

LEMMA 4.4. *For integers $m \geq 18$, we have that*

$$(4.17) \quad |\partial^\gamma A(y, s)| \lesssim \begin{cases} e^{-(\frac{3}{2} - \frac{2|\gamma|-1}{2m-5})s} & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 2, 3, \\ e^{-(1 - \frac{|\gamma|-1}{2m-7})s} & \text{if } |\gamma| = 3, 4, 5, \end{cases}$$

$$(4.18) \quad |\partial^\gamma Z(y, s)| \lesssim \begin{cases} e^{-(\frac{3}{2} - \frac{3}{2m-7})s} & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 3, \\ e^{-(1 - \frac{|\gamma|-1}{2m-7})s} & \text{if } |\gamma| = 3, 4, 5, \end{cases}$$

$$(4.19) \quad |\partial^\gamma W(y, s)| \lesssim \begin{cases} e^{\frac{2s}{2m-7}} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 2, \\ e^{\frac{s}{2m-7}} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 3, \\ e^{\frac{3s}{2m-7}} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} & \text{if } \gamma_1 \geq 2 \text{ and } |\gamma| = 3, \end{cases}$$

$$(4.20) \quad |\partial^\gamma K(y, s)| \lesssim \begin{cases} e^{-(\frac{13}{8} - \frac{9}{4(2m-7)})s} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 2, \\ e^{-(2 - \frac{4}{2m-7})s} \eta^{-\frac{1}{15}} & \text{if } \gamma_1 \geq 2 \text{ and } |\gamma| = 3, \\ e^{-(1 - \frac{|\gamma|-2}{2m-7})s} & \text{if } |\gamma| = 3, 4, 5. \end{cases}$$

PROOF. The bounds for (4.17) and (4.18), as well as the first two estimates in (4.19), are proven in lemma 4.4 in [4].

We then consider the third estimate in (4.19) and hence estimate $\partial^\gamma W(y, s)$ for the case $\gamma_1 \geq 2$ and $|\gamma| = 3$. We write

$$\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \nabla \partial_{11} W = \underbrace{\nabla(\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W)}_{=:\text{I}} - \underbrace{\nabla(\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}}) \partial_{11} W}_{=:\text{II}}.$$

Since $|\nabla(\eta^{1/3} \psi^{-1/4})| \lesssim \eta^{1/3}$, it follows from (4.7) that

$$|\text{II}| \lesssim M^{\frac{1}{3}} \psi^{\frac{1}{4}} \lesssim M.$$

Now we apply Lemma A.2 to the function $\eta^{1/3} \psi^{-1/4} \partial_{11} W$, appeal to the estimate (4.7), and to the Leibniz rule to obtain that

$$|\text{I}| \lesssim \|\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W\|_{\dot{H}^{m-2}}^{\frac{2}{2m-7}} \|\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W\|_{L^\infty}^{\frac{2m-9}{2m-7}} \lesssim M \|\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W\|_{\dot{H}^{m-2}}^{\frac{2}{2m-7}},$$

where we have used that $m \geq 18$ for the last inequality as is required by Proposition 4.3. We next estimate the \dot{H}^{m-2} norm of $\eta^{1/3}\psi^{-1/4}\partial_{11}W$. To do so, we shall use the fact that $W(\cdot, s)$ has support in the set $\mathcal{X}(s)$ defined in (4.4). We find that

$$\begin{aligned}
 & \|\eta^{1/3}\psi^{-1/4}\partial_{11}W\|_{\dot{H}^{k-2}} \\
 & \lesssim \sum_{m'=0}^{m-2} \|D^{m-m'-2}(\eta^{1/3}\psi^{-1/4})D^{m'}\partial^{\nu''}W\|_{L^2} \\
 & \lesssim \sum_{m'=0}^{m-2} \|D^{m-m'-2}(\eta^{1/3}\psi^{-1/4})\|_{L^{\frac{2(m-1)}{m-2-m'}}(\mathcal{X}(s))} \|D^{m'}\partial^{\nu''}W\|_{L^{\frac{2(m-1)}{m'+1}}} \\
 (4.21) \quad & \lesssim \sum_{m'=0}^{m-2} \|D^{m-m'-2}(\eta^{1/3}\psi^{-1/4})\|_{L^{\frac{2(m-1)}{m-2-m'}}(\mathcal{X}(s))} \|\nabla W\|_{L^\infty}^{1-\frac{m'+1}{m-1}} \|W\|_{\dot{H}^m}^{\frac{m'+1}{m-1}}.
 \end{aligned}$$

Using (4.7) and Proposition 4.3, the W terms are bounded as

$$\|\nabla W\|_{L^\infty}^{1-\frac{m'+1}{m-1}} \|W\|_{\dot{H}^m}^{\frac{m'+1}{m-1}} \lesssim M^{2m}$$

for all $m \in \{0, \dots, m-2\}$. Moreover, using that $|D^{m-m'-2}(\eta^{1/3}\psi^{-1/4})| \lesssim \eta^{1/3}$ together with (4.5), we have that

$$(4.22) \quad \|D^{m-m'-2}(\eta^{1/3}\psi^{-1/4})\|_{L^{\frac{2(m-1)}{m-m'-2}}(\mathcal{X}(s))} \lesssim \varepsilon^{1/3} e^s,$$

with the usual abuse of notation $L^{\frac{2(m-1)}{m-m'-2}} = L^\infty$ for $m' = m-2$. Combining the above estimates, we obtain the inequality

$$(4.23) \quad |I| \lesssim M^{2m} (\varepsilon^{1/3} e^{11/8 s})^{\frac{2}{2m-7}} \lesssim e^{\frac{2s}{2(2m-7)}}$$

for ε sufficiently small. From the above estimate, we obtain the third inequality in (4.19).

We next consider the bounds (4.20), and we begin with the case that $\gamma_1 \geq 1$ and $|\tilde{\gamma}| = 2$. Applying Lemma A.2 to the function $\partial_1 \tilde{\nabla} K$, and using (4.14) and Proposition 4.3, we have that

$$\begin{aligned}
 \|\partial^\nu K\|_{L^\infty} & \lesssim \|K\|_{\dot{H}^m}^{\frac{2}{2m-7}} \|\partial_1 \tilde{\nabla} K\|_{L^\infty}^{\frac{2m-9}{2m-7}} \lesssim (M^{2m} e^{-\frac{s}{2}})^{\frac{2}{2m-7}} (\varepsilon^{1/8} e^{-13/8 s})^{\frac{2m-9}{2m-7}} \\
 & \lesssim e^{-\frac{109-26m}{8(2m-7)} s}.
 \end{aligned}$$

We next consider the second inequality in (4.20). In order to estimate the term $|\eta^{1/15} \nabla \partial_{11} K|$ we write

$$\eta^{1/15} \nabla \partial_{11} K = \underbrace{\nabla(\eta^{1/15} \partial_{11} K)}_{=:I} - \underbrace{\nabla \eta^{1/15} \partial_{11} K}_{=:II}.$$

Since $|\nabla \eta^{1/15}| \leq 1$, it follows from (4.14) that

$$|II| \lesssim e^{-2s}.$$

By Lemma A.2 and (4.14),

$$|I| \lesssim \|\eta^{\frac{1}{15}} \partial_{11} K\|_{\dot{H}^{m-2}}^{\frac{2}{2m-7}} \|\eta^{\frac{1}{15}} \partial_{11} K\|_{L^\infty}^{\frac{2m-9}{2m-7}} \lesssim M e^{-(2-\frac{4}{2m-7})s} \|\eta^{\frac{1}{15}} \partial_{11} K\|_{\dot{H}^{m-2}}^{\frac{2}{2m-7}},$$

Following the calculation (4.21), we have that

$$\|\eta^{\frac{1}{15}} \partial_{11} K\|_{\dot{H}^{m-2}} \lesssim \sum_{m'=0}^{m-2} \|D^{m-m'-2} \eta^{\frac{1}{15}}\|_{L^{\frac{2(m-1)}{m-2-m'}}(\mathcal{X}(s))} \|\nabla K\|_{L^\infty}^{1-\frac{m'+1}{m-1}} \|K\|_{\dot{H}^m}^{\frac{m'+1}{m-1}}.$$

Applying (4.5), we obtain that

$$\|D^{m-m'-2} \eta^{\frac{1}{15}}\|_{L^{\frac{2(m-1)}{m-2-m'}}(\mathcal{X}(s))} \lesssim \varepsilon^{\frac{1}{15}} e^{\frac{1}{5}s}.$$

From (4.14) and Proposition 4.3

$$\|\nabla K\|_{L^\infty}^{1-\frac{m'+1}{m-1}} \|K\|_{\dot{H}^m}^{\frac{m'+1}{m-1}} \leq e^{-\frac{s}{2}}.$$

From the above estimates, together with (4.14), we determine that

$$|I| \lesssim M e^{-(2-\frac{4}{2m-7})s} (\varepsilon^{\frac{1}{15}} e^{-\frac{3}{10}s})^{\frac{2}{2m-7}} \lesssim e^{-(2-\frac{4}{2m-7})s}.$$

This estimate establishes the second bound in (4.20). For $|\gamma| \in \{3, 4, 5\}$ we apply Lemma A.2 to $\nabla^2 K$, and together with (4.14) and Proposition 4.3, we find that

$$\begin{aligned} \|\partial^\gamma K\|_{L^\infty} &\lesssim \|K\|_{\dot{H}^m}^{\frac{2|\gamma|-4}{2m-7}} \|\nabla^2 K\|_{L^\infty}^{\frac{2m-3-2|\gamma|}{2m-7}} \\ &\lesssim (M^{2m} e^{-\frac{s}{2}})^{\frac{2|\gamma|-4}{2m-7}} (\varepsilon^{\frac{1}{8}} e^{-s})^{\frac{2m-3-2|\gamma|}{2m-7}} \lesssim e^{-(1-\frac{|\gamma|-2}{2m-7})s}, \end{aligned}$$

where we have assumed that ε is taken sufficiently small. \square

4.6 Bounds for $U \cdot N$ and S

Finally, we note that as a consequence of the definitions (2.37), we have the following estimates on $U \cdot N$ and S .

LEMMA 4.5. *For $y \in \mathcal{X}(s)$ we have*

$$(4.24) \quad |\partial^\gamma U \cdot N| + |\partial^\gamma S| \lesssim \begin{cases} M^{\frac{1}{4}} & \text{if } |\gamma| = 0, \\ M^{\frac{1}{3}} e^{-\frac{s}{2}} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ e^{-\frac{s}{2}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ M^{\frac{2}{3}} e^{-\frac{s}{2}} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ M^{\frac{2}{3}} e^{-\frac{s}{2}} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} & \text{if } \gamma_1 = 2 \text{ and } |\check{\gamma}| = 0, \\ M e^{-\frac{s}{2}} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \\ e^{(-\frac{1}{2} + \frac{3}{2m-7})s} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 2, \\ e^{(-\frac{1}{2} + \frac{1}{2m-7})s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 3, \\ e^{(-\frac{1}{2} + \frac{3}{2m-7})s} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} & \text{if } \gamma_1 \geq 2 \text{ and } |\check{\gamma}| = 3. \end{cases}$$

Additionally, for $|y| \leq \ell$ and $|\gamma| = 4$ we have the bound

$$|\partial^\gamma U \cdot \mathbf{N}| + |\partial^\gamma S| \lesssim e^{-\frac{s}{2}}.$$

PROOF. We shall only establish the bounds for $\partial^\gamma U \cdot \mathbf{N}$ as the estimates for $\partial^\gamma S$ are obtained in the identical fashion. Since $|\kappa| \leq M^{1/4}$, it follows from (2.37) that $|\partial^\gamma U \cdot \mathbf{N}| \lesssim |\kappa| \mathbf{1}_{|\gamma|=0} + e^{-s/2} |\partial^\gamma W| + |\partial^\gamma Z|$. The desired bounds are obtained by an application of (4.7), (4.9b), (4.12), Lemma 4.4, and (4.5). \square

PROPOSITION 4.6 (L^∞ bound for the sound speed). *We have that*

$$(4.25) \quad \|S(\cdot, s) - \frac{\kappa_0}{2}\|_{L^\infty} \leq \varepsilon^{\frac{1}{8}} \text{ for all } s \geq -\log \varepsilon.$$

PROOF. By (2.37), we have that

$$S(\cdot, s) - \frac{\kappa_0}{2} = \frac{\kappa - \kappa_0}{2} + \frac{1}{2}(e^{-\frac{s}{2}} W - Z).$$

By (4.1), (4.5), (4.7), and (4.12), and the triangle inequality,

$$\|S(\cdot, s) - \frac{\kappa_0}{2}\|_{L^\infty} \lesssim \varepsilon^{\frac{1}{6}},$$

which concludes the proof. \square

4.7 The blowup time and location

The blowup time T_* is defined uniquely by the condition $\tau(T_*) = T_*$, which by (2.53) is equivalent to

$$(4.26) \quad \int_{-\varepsilon}^{T_*} (1 - \dot{\tau}(t)) dt = \varepsilon.$$

The estimate for $\dot{\tau}$ in (4.1b) shows that for ε taken sufficiently small,

$$(4.27) \quad |T_*| \leq 2M^2 \varepsilon^2.$$

We also note here that the bootstrap assumption (4.1b) and the definition of T_* ensures that $\tau(t) > t$ for all $t \in [-\varepsilon, T_*)$. Indeed, when $t = -\varepsilon$, we have that $\tau(-\varepsilon) = 0 > -\varepsilon$, and the function $t \mapsto \int_{-\varepsilon}^t (1 - \dot{\tau}) dt' - \varepsilon = t - \tau(t)$ is strictly increasing.

The blowup location is determined by $\xi_* = \xi(T_*)$, which by (2.53) is

$$\xi_* = \int_{-\varepsilon}^{T_*} \dot{\xi}(t) dt.$$

In view of (4.1b), for ε small enough, find that

$$(4.28) \quad |\xi_*| \leq M\varepsilon,$$

so that the blowup location is $\mathcal{O}(\varepsilon)$ close to the origin.

4.8 Hölder bound for w

As we proved in [4], the self-similar scaling (2.29) and decay rate (4.7) for $W(y, s)$ show that

$$w \in L^\infty([-\varepsilon, T_*]; C^{1/3}),$$

and the C^α Hölder norms of w , with $\alpha > 1/3$, blowup as $t \rightarrow T_*$ with a rate proportional to $(T_* - t)^{(1-3\alpha)/2}$.

5 Bounds on Lagrangian Trajectories

5.1 The Lagrangian flows in self-similar variables

In self-similar variables (y, s) , we define Lagrangian flows associated to the transport velocities in (2.39) by

$$(5.1a) \quad \partial_s \Phi_W(y, s) = \mathcal{V}_W(\Phi_W(y, s), s), \quad \Phi_W(y, s_0) = y,$$

$$(5.1b) \quad \partial_s \Phi_Z(y, s) = \mathcal{V}_Z(\Phi_Z(y, s), s), \quad \Phi_Z(y, s_0) = y,$$

$$(5.1c) \quad \partial_s \Phi_U(y, s) = \mathcal{V}_U(\Phi_U(y, s), s), \quad \Phi_U(y, s_0) = y,$$

for $s_0 \geq -\log \varepsilon$. With Φ denoting either Φ_W , Φ_Z , or Φ_U , we shall denote trajectories emanating from a point y_0 at time s_0 by

$$(5.2) \quad \Phi^{y_0}(s) = \Phi(y_0, s) \quad \text{with } \Phi(y_0, s_0) = y_0.$$

Estimates for the support and a lower bound for Φ_W

Since the bounds for $|G_W|$, $|h_W|$, and $|W|$ are the same as in [4], the proofs of the following two lemmas are the same as lemmas 8.1 and 8.2 in [4].

The bootstrap assumption (4.4) on the size of the support is closed as follows:

LEMMA 5.1 (Estimates on the support). *Let Φ denote either $\Phi_W^{y_0}$, $\Phi_Z^{y_0}$, or $\Phi_U^{y_0}$. For any $y_0 \in \mathcal{X}_0$ defined in (3.10), we have that*

$$(5.3) \quad |\Phi_1(s)| \leq \frac{3}{2}\varepsilon^{\frac{1}{2}}e^{\frac{3}{2}s}, \quad |\check{\Phi}(s)| \leq \frac{3}{2}\varepsilon^{\frac{1}{6}}e^{\frac{s}{2}},$$

for all $s \geq -\log \varepsilon$.

We shall also make use of the lower bound given by

LEMMA 5.2. *Let $y_0 \in \mathbb{R}^3$ be such that $|y_0| \geq \ell$. Let $s_0 \geq -\log \varepsilon$. Then, the trajectory $\Phi_W^{y_0}$ moves away from the origin at an exponential rate, and we have the lower bound*

$$(5.4) \quad |\Phi_W^{y_0}(s)| \geq |y_0|e^{\frac{s-s_0}{5}}$$

for all $s \geq s_0$.

LEMMA 5.3. *Given $s_0 \geq -\log \varepsilon$ and $s > s_0$, let $y_0 \in \mathbb{R}^3$ be such that $|y_0| \geq \mathcal{L}$ and $|\check{\Phi}_W^{y_0}(s)| \leq M\varepsilon^{1/2}$. Then, we have that*

$$(5.5) \quad |(\Phi_W^{y_0})_1(s')| \geq \frac{3}{4}|(y_0)_1|e^{\frac{3(s'-s_0)}{2}} \quad \text{and} \quad |\check{\Phi}_W^{y_0}(s')| \leq M\varepsilon^{1/2}$$

for all $s_0 \leq s' \leq s$.

PROOF. Fix (y_0, s_0) and denote $(\Phi_W^{y_0})_1(s) = \Phi_1(s)$ and $\check{\Phi}_W^{y_0}(s) = \check{\Phi}(s)$.

According to (5.1) and (2.39), we have that $\partial_s \Phi_v = \frac{1}{2} \Phi_v + h_W^v \circ \Phi$. Solving this ODE on the interval $[s', s]$, with arbitrary $s' \in [s_0, s]$, we obtain that

$$\Phi_v(s') = \Phi_v(s) e^{-\frac{s-s'}{2}} - \int_{s'}^s e^{-\frac{s''-s'}{2}} h_W^v \circ \Phi(s'') ds''.$$

Using that by (9.5) we have $|h_W(\cdot, s)| \leq M^{1/2} e^{-\frac{s}{2}}$, and appealing to the assumption $|\Phi_v(s)| \leq M \varepsilon^{1/2}$, we obtain that

$$\begin{aligned} |\Phi_v(s')| &\leq |\Phi_v(s)| e^{-\frac{s-s'}{2}} + M^{1/2} \int_{s'}^s e^{-\frac{s''-s'}{2}} e^{-\frac{s''}{2}} ds'' \\ &\leq M \varepsilon^{1/2} e^{-\frac{s-s'}{2}} + M^{1/2} e^{-\frac{s'}{2}} (1 - e^{-(s-s')}) \leq M \varepsilon^{1/2}. \end{aligned}$$

where in the last inequality we have used that $s' \geq s_0 \geq -\log \varepsilon$, so that $e^{-\frac{s'}{2}} \leq \varepsilon^{1/2} e^{-\frac{(s'-s_0)}{2}}$. This proves the second claim in (5.5).

In order to prove the first claim in (5.5), we again recall (5.1) and (2.39), which gives $\partial_s \Phi_1 = \frac{3}{2} \Phi_1 + \beta_\tau W \circ \Phi + G_W \circ \Phi$. In view of the bound established for $\check{\Phi}$ and of the information we have from Lemma 5.2 we already know that $|y_0| \geq \mathcal{L}$ implies that $|\Phi_1(s')| \geq \frac{\mathcal{L}}{2} e^{(s'-s_0)/5}$ for all $s' \in [s_0, s]$, so that $\Phi_1(s')$ is much larger than 1. Thus, from (4.3) and the first bound in (4.7), we have

$$\begin{aligned} |\beta_\tau W \circ \Phi(s')| &\leq (1 + 2M\varepsilon)(1 + \varepsilon^{\frac{1}{20}})(1 + |\Phi_1(s')|^2 + (M\varepsilon^{1/2})^6)^{\frac{1}{6}} \\ &\leq 2|\Phi_1(s')|^{\frac{1}{3}}. \end{aligned}$$

Similarly, the first estimate in Lemma 9.2 in which we use an extra factor of M to absorb the implicit constant in the \lesssim symbol, and the previously established bound (5.3) imply that

$$\begin{aligned} |G_W \circ \Phi(s')| &\leq M^2 e^{-\frac{s'}{2}} + M^{\frac{3}{2}} e^{-s'} |\Phi_1(s')| + M^2 \varepsilon^{\frac{5}{6}} \leq M^2 e^{-s'} |\Phi_1(s')| \\ &\leq 2M^2 \varepsilon^{\frac{1}{3}} |\Phi_1(s')|^{\frac{1}{3}}. \end{aligned}$$

Combining the above two estimates with the ODE satisfied by Φ_1 , we derive that

$$\frac{1}{2} \frac{d}{ds} |\Phi_1(s')|^2 \geq \frac{3}{2} |\Phi_1(s')|^2 - 3 |\Phi_1(s')|^{\frac{4}{3}}.$$

By explicitly integrating the above ODE, and using our earlier observation that $|(y_1)_0| \geq 1/2 \varepsilon^{-\frac{1}{10}}$ for all $s' \in [s_0, s]$, we derive that

$$|\Phi_1(s')| \geq (|(y_0)_1|^{\frac{2}{3}} - 2)^{\frac{3}{2}} e^{\frac{3(s'-s_0)}{2}} \geq \frac{3}{4} |(y_0)_1| e^{\frac{3(s'-s_0)}{2}},$$

which completes the proof. \square

Lower bounds for Φ_Z and Φ_U

We now establish important lower bounds for $\Phi_Z^{y_0}(s)$ or $\Phi_U^{y_0}(s) = \Phi_Z^{y_0}(s)$.

LEMMA 5.4. *Let $\Phi(s)$ denote either $\Phi_Z^{y_0}(s)$ or $\Phi_U^{y_0}(s)$. If*

$$(5.6) \quad \kappa_0 \geq \frac{3}{1 - \max(\beta_1, \beta_2)},$$

then for any $y_0 \in \mathcal{X}_0$ defined in (3.10), there exists an $s_ \geq -\log \varepsilon$ such that*

$$(5.7) \quad |\Phi_1(s)| \geq \min(|e^{\frac{s}{2}} - e^{\frac{s_*}{2}}|, e^{\frac{s}{2}}).$$

In particular, we have the following inequalities:

$$(5.8) \quad \int_{-\log \varepsilon}^{\infty} e^{\sigma_1 s'} (1 + |\Phi_1(s')|)^{-\sigma_2} ds' \leq C,$$

for $0 \leq \sigma_1 \leq 1/2$ and $2\sigma_1 < \sigma_2$, where the constant C depends only on the choice of σ_1 and σ_2 .

This is a slight generalization of lemma 8.3 in [4], where we now allow the value $\sigma_1 = 1/2$. The only addition to the proof requires an estimate for the integral \mathcal{J} in the proof of lemma 8.3 in [4]. In particular, for $\sigma_1 = 1/2$, we see that

$$\mathcal{J} = 2 \int_{\varepsilon^{-1/2}}^{\infty} (1 + |r - e^{\frac{s_*}{2}}|)^{-\sigma_2} dr \lesssim 1.$$

The implicit constant only depends on σ_1 and σ_2 .

The time integral of $|\partial_1 W|$ along $\Phi_Z^{y_0}$

An immediate consequence of (5.8) is the following:

COROLLARY 5.5. *For all $s \geq -\log \varepsilon$,*

$$(5.9) \quad \sup_{y_0 \in \mathcal{X}_0} \int_{-\log \varepsilon}^s |\partial_1 W| \circ \Phi_Z^{y_0}(s') ds' \lesssim 1.$$

PROOF OF COROLLARY 5.5. The bound (5.9) follows using the second estimate in (4.7) together with (5.8) with $\sigma_1 = 0$ and $\sigma_2 = \frac{2}{3}$. \square

5.2 The Lagrangian flow $\varphi(x, t)$

With respect to the independent variables (x, t) , the transport velocity for \mathring{u} in (2.22) is given by

$$(5.10) \quad \mathbf{v} = (v_1, v_2, v_3) = 2\beta_1 \left(-\frac{\dot{f}}{2\beta_1} + \mathbf{J}\mathbf{v} \cdot \mathbf{N} + \mathbf{J}\mathring{u} \cdot \mathbf{N}, v_2 + \mathring{u}_2, v_3 + \mathring{u}_3 \right).$$

We let $\varphi(x, t)$ denote the flow of \mathbf{v} so that

$$(5.11a) \quad \partial_t \varphi(x, t) = \mathbf{v}(\varphi(x, t), t), \quad t > -\varepsilon,$$

$$(5.11b) \quad \varphi(x, -\varepsilon) = x,$$

and we denote by $\varphi_{x_0}(t)$ the trajectory emanating from x_0 .

Asymptotic nonpositivity for $\partial_1 W$

LEMMA 5.6. *For all $y \in \mathbb{R}^3$ and $s \geq -\log \varepsilon$, we have*

$$(5.12) \quad \max \{ \partial_1 W(y, s), 0 \} \leq 4e^{-\frac{s}{15}}.$$

PROOF OF LEMMA 5.6. We start with the region $|y| \leq \mathcal{L} = \varepsilon^{-1/10}$. Here, due to the bootstrap (4.8b) for $\partial_1 \widetilde{W}$ and the fact that $\partial_1 \overline{W} \leq -\widetilde{\eta}^{-1/3}$ (see (2.48) in [4]), we deduce that

$$(5.13) \quad \begin{aligned} \partial_1 W(y, s) &= \partial_1 \overline{W}(y) + \partial_1 \widetilde{W}(y, s) \\ &\leq -\widetilde{\eta}^{-\frac{1}{3}}(y) + \varepsilon^{\frac{1}{12}} \eta^{-\frac{1}{3}}(y) < 0, \quad |y| \leq \varepsilon^{-\frac{1}{10}}, \end{aligned}$$

upon taking ε sufficiently small, and using that $\widetilde{\eta}(y) \leq 2\eta(y)$. Thus, for $|y| \leq \mathcal{L}$ the bound (5.12) holds.

Next, let us consider the region $|y| \geq e^{s/10}$. Here we have that $\eta(y) \geq 1/2 e^{s/5}$. Combining this bound with the second line of (4.7), we arrive at

$$|\partial_1 W(y, s)| \leq 2\eta^{-\frac{1}{3}}(y) \leq 4e^{-\frac{s}{15}}.$$

Thus, (5.12) also holds in the region $|y| \geq e^{s/10}$.

It remains to consider the region $\mathcal{L} < |y| < e^{s/10}$. Notice that by the definition of $\mathcal{L} = e^{-\log \varepsilon / 10}$, in this case we have that $s > -\log \varepsilon$. For such a fixed (y, s) we trace the particle trajectory of the flow \mathcal{V}_W backwards in time, and write $\Phi_W^{y_0}(s) = y$, where the initial datum $\Phi_W^{y_0}(s_0) = y_0$ is given by the property that $|y_0| = \mathcal{L}$ if $s_0 > -\log \varepsilon$, and $|y_0| > \mathcal{L}$ if $s_0 = -\log \varepsilon$. We claim that the second option is not possible, so that we must have $s_0 > -\log \varepsilon$ and $|y_0| = \mathcal{L}$. To see this, we appeal to Lemma 5.2, which is applicable since $|y_0| \geq \mathcal{L} \geq \ell$, and which gives the bound $|\Phi_W^{y_0}(s)| \geq |y_0| e^{(s-s_0)/5}$. Thus, in the case that $s_0 = -\log \varepsilon$ and $|y_0| > \mathcal{L}$, this implies

$$\begin{aligned} e^{\frac{s}{10}} > |y| &= |\Phi_W^{y_0}(s)| \geq |y_0| e^{\frac{s-s_0}{5}} > \mathcal{L} e^{\frac{s+\log \varepsilon}{5}} \\ &= \varepsilon^{-\frac{1}{10}} e^{\frac{s+\log \varepsilon}{5}} = e^{\frac{s}{10}} e^{\frac{s+\log \varepsilon}{10}} > e^{\frac{s}{10}}, \end{aligned}$$

since $s > -\log \varepsilon$. This yields the desired contradiction, which guarantees that $|y_0| = \mathcal{L}$ and $s_0 > -\log \varepsilon$. At this stage we appeal to the evolution of $\partial_1 W$ given in (2.44a) with $\gamma = (1, 0, 0)$, and deduce that $e^{\frac{s}{2}} \partial_1 W$ satisfies the equation

$$\partial_s(e^{\frac{s}{2}} \partial_1 W) + \left(\frac{1}{2} + \beta_\tau \mathcal{J} \partial_1 W\right)(e^{\frac{s}{2}} \partial_1 W) + (\mathcal{V}_W \cdot \nabla)(e^{\frac{s}{2}} \partial_1 W) = e^{\frac{s}{2}} F_W^{(1,0,0)}.$$

Composing with $\Phi_W^{y_0}$ and appealing to Grönwall's inequality on the interval $[s_0, s]$, we obtain that

$$\begin{aligned}
 (5.14) \quad & e^{\frac{s}{2}} \partial_1 W(y, s) \\
 &= e^{\frac{s_0}{2}} \partial_1 W(y_0, s_0) \exp\left(-\int_{s_0}^s \frac{1}{2} + \beta_\tau(J\partial_1 W) \circ \Phi_W^{y_0}(s') ds'\right) \\
 &+ \int_{s_0}^s e^{\frac{s'}{2}} F_W^{(1,0,0)} \circ \Phi_W^{y_0}(s') \\
 &\quad \cdot \exp\left(-\int_{s'}^s \frac{1}{2} + \beta_\tau(J\partial_1 W) \circ \Phi_W^{y_0}(s'') ds''\right) ds'.
 \end{aligned}$$

We now use the information that $|y_0| = \mathcal{L}$, and thus $\partial_1 W(y_0, s_0) < 0$. Hence, the first term on the right side of (5.14) is strictly negative (as the exponential is positive), so that it does not contribute to the positive part of $\partial_1 W$. We deduce, by also appealing to the $F_W^{(1,0,0)}$ estimate in (9.19) and the $\partial_1 W$ bootstrap in (4.7), that

$$\begin{aligned}
 & e^{\frac{s}{2}} \max\{\partial_1 W(y, s), 0\} \\
 & \leq \int_{s_0}^s e^{\frac{s'}{2}} |F_W^{(1,0,0)} \circ \Phi_W^{y_0}(s')| \exp\left(-\int_{s'}^s \frac{1}{2} + \beta_\tau(J\partial_1 W) \circ \Phi_W^{y_0}(s'') ds''\right) ds' \\
 & \lesssim M \int_{s_0}^s \eta^{-\frac{1}{3}} \circ \Phi_W^{y_0}(s') \exp\left(4 \int_{s'}^s \eta^{-\frac{1}{3}} \circ \Phi_W^{y_0}(s'') ds''\right) ds'.
 \end{aligned}$$

The proof is completed by appealing to the bound established in (11.32), namely $\int_{s_0}^s \eta^{-1/3} \circ \Phi_W^{y_0}(s') ds' \leq \varepsilon^{1/16}$, which holds for $|y_0| \geq \mathcal{L}$, and which implies

$$e^{\frac{s}{2}} \max\{\partial_1 W(y, s), 0\} \lesssim M \varepsilon^{\frac{1}{16}} \exp(4\varepsilon^{\frac{1}{16}}) \leq 1. \quad \square$$

From Lemma 5.6, we immediately deduce the following:

COROLLARY 5.7. *For any $t \in [-\varepsilon, T_*)$ we have*

$$(5.15) \quad \int_{-\varepsilon}^t \max\{\partial_{x_1} \tilde{u} \cdot \mathbf{N}, 0\} dt' \leq \varepsilon^{\frac{1}{16}}$$

uniformly pointwise in space.

PROOF. Recall that (cf. (2.27) and (2.30a)–(2.30b)) that

$$\partial_{x_1} \tilde{u} \cdot \mathbf{N} = \frac{1}{2}(\partial_{x_1} w + \partial_{x_1} z) = \frac{1}{2}e^s \partial_1 W + \frac{1}{2}e^{\frac{3s}{2}} \partial_1 Z.$$

From (4.12) we know that $e^{3s/2} |\partial_1 Z| \leq M^{1/2}$, and since the function $\max\{\cdot, 0\}$ is convex and in fact subadditive, we deduce from Lemma 5.6 that

$$\max\{\partial_{x_1} \tilde{u} \cdot \mathbf{N}, 0\} \leq \frac{1}{2}e^s \max\{\partial_1 W, 0\} + \frac{1}{2}e^{\frac{3s}{2}} \max\{\partial_1 Z, 0\} \leq 2e^{\frac{14s}{15}} + \frac{1}{2}M^{\frac{1}{2}}.$$

Writing $dt' = \beta_\tau e^{-s'} ds'$, the desired bound follows from

$$\int_{-\log \varepsilon}^{\infty} (2e^{\frac{14}{15}s'} + \frac{1}{2}M^{\frac{1}{2}})\beta_\tau e^{-s'} ds' \leq 60\varepsilon^{\frac{1}{15}} + M^{\frac{1}{2}}\varepsilon,$$

concluding the proof. \square

The time integral of $|\partial_1 W|$ along $\Phi_U^{y_0}$

We next establish the following:

LEMMA 5.8. *For all $s \geq -\log \varepsilon$,*

$$(5.16) \quad \sup_{y_0 \in \mathcal{Y}_0} \int_{-\log \varepsilon}^s |\partial_1 W| \circ \Phi_U^{y_0}(s') ds' \lesssim \varepsilon^{\frac{1}{18}}.$$

PROOF. From the definition of the transport velocity v in (5.10), observe that

$$(5.17) \quad \operatorname{div}_x v = \operatorname{div}_{\tilde{x}} \dot{u} = 2\beta_1(\partial_{x_1} \dot{u} \cdot N J + \partial_{x_\nu} \dot{u}_\nu)$$

where we have used the fact that

$$\operatorname{div}_x v = \partial_{x_j} v_j = \partial_{x_1} J N \cdot v + \partial_{x_\mu} v_\mu = \operatorname{div}_{\tilde{x}} v$$

and that from (2.19), $\operatorname{div}_{\tilde{x}} v = \dot{Q}_{ii} = 0$, and that $\operatorname{div}_x(-\dot{f}, 0, 0) = 0$. Hence, the conservation of mass equation (2.20) can be written as

$$(5.18) \quad \partial_t \dot{\rho} + v \cdot \nabla_x \dot{\rho} + \dot{\rho} \operatorname{div}_x v = 0,$$

and composing (5.18) with the flow φ given by (5.11), we see that

$$(5.19) \quad \partial_t(\dot{\rho} \circ \varphi) = (\dot{\rho} \circ \varphi)(\operatorname{div}_x v) \circ \varphi.$$

Since

$$(5.20) \quad \partial_t(\det \nabla_x \varphi) = \det \nabla_x \varphi (\operatorname{div}_x v) \circ \varphi,$$

and $\det \nabla_x \varphi(x, -\varepsilon) = 1$, it follows that

$$\dot{\rho} \circ \varphi = (\det \nabla_x \varphi)^{-1} \dot{\rho}_0.$$

Note that using (1.2), (4.25), and (4.15) yields

$$(5.21) \quad \begin{aligned} |\rho - (\frac{\alpha \kappa_0}{2})^{\frac{1}{\alpha}}| &= |(\alpha e^{-\frac{k}{2}} \sigma)^{\frac{1}{\alpha}} - (\frac{\alpha \kappa_0}{2})^{\frac{1}{\alpha}}| \\ &\lesssim |(\alpha e^{-\frac{k}{2}} \sigma)^{\frac{1}{\alpha}} - (\frac{\alpha e^{-\frac{k}{2}} \kappa_0}{2})^{\frac{1}{\alpha}}| + |(\frac{\alpha e^{-\frac{k}{2}} \kappa_0}{2})^{\frac{1}{\alpha}} - (\frac{\alpha \kappa_0}{2})^{\frac{1}{\alpha}}| \\ &\lesssim \varepsilon^{\frac{1}{8}} (\frac{\alpha \kappa_0}{2})^{\frac{1}{\alpha}-1} \lesssim \varepsilon^{\frac{1}{9}}. \end{aligned}$$

Therefore, by (5.21) and (5.21), we have that

(5.22)

$$|\det(\nabla_x \varphi(x, t)) - 1| \leq \left| \frac{\dot{\rho}_0}{\dot{\rho}} - 1 \right| \leq \left| \frac{\dot{\rho}_0}{\dot{\rho}} - \frac{(\frac{\alpha \kappa_0}{2})^{\frac{1}{\alpha}}}{\dot{\rho}} \right| + \left| \frac{(\frac{\alpha \kappa_0}{2})^{\frac{1}{\alpha}}}{\dot{\rho}} - 1 \right| \lesssim \varepsilon^{\frac{1}{9}}.$$

From (5.19) and (5.20), we have that

$$\frac{d}{dt} \det \nabla_x \varphi = \det \nabla_x \varphi (\operatorname{div}_x v) \circ \varphi = \det \nabla_x \varphi (\operatorname{div}_{\tilde{x}} \dot{u}) \circ \varphi$$

leads to

$$(5.23) \quad \det \nabla_x \varphi(x, t) = \exp \int_{-\varepsilon}^t (\operatorname{div}_{\tilde{x}} \mathring{u} \circ \varphi)(x, t') dt'.$$

Hence,

$$(5.24) \quad -\varepsilon^{\frac{1}{9}} \lesssim \int_{-\varepsilon}^t (\operatorname{div}_{\tilde{x}} \mathring{u} \circ \varphi)(x, t') dt' \lesssim \varepsilon^{\frac{1}{9}} \quad \text{for all } x \in \mathbb{R}^3.$$

From (2.30c), (2.36a), (4.13), and (4.24)

$$(5.25) \quad \|\partial_{x_v} \mathring{u}_v(\cdot, t)\|_{L^\infty} \lesssim 1.$$

It follows from (4.1a) and (5.25) that

$$(5.26) \quad \int_{-\varepsilon}^t \|\partial_{x_v} \mathring{u}_v(\cdot, t')\|_{L^\infty} dt' \lesssim \tau(t) + \varepsilon \lesssim M\varepsilon^2 + \varepsilon \leq \varepsilon^{\frac{1}{2}}.$$

Thus, with (9.1a), (5.17), (5.22), and (5.26), we have that

$$(5.27) \quad \begin{aligned} & \left| \int_{-\varepsilon}^t \partial_{x_1} \mathring{u} \cdot \mathbf{N} \circ \varphi dt' \right| \\ & \leq \left| \int_{-\varepsilon}^t \left(\frac{1}{j} \partial_{x_v} \mathring{u}_v \right) \circ \varphi dt' \right| + \frac{1}{2\beta_1} \left| \int (\operatorname{div}_{\tilde{x}} \mathring{u}) \circ \varphi dt' \right| \lesssim \varepsilon^{\frac{1}{9}}. \end{aligned}$$

By Corollary 5.7, the integral of the positive part of $\partial_{x_1} \mathring{u} \cdot \mathbf{N}$ is small. Therefore, the above estimate gives a bound on the negative part of $\partial_{x_1} \mathring{u} \cdot \mathbf{N}$ as well. In summary, by (5.27) and Corollary 5.7, we then have that

$$(5.28) \quad \int_{-\varepsilon}^t |\partial_{x_1} \mathring{u} \cdot \mathbf{N} \circ \varphi| dt' \leq \varepsilon^{\frac{1}{18}}.$$

Then, from (2.27) and the bootstrap assumptions (4.1a) and (4.12), we see that $\int_{-\varepsilon}^t |\partial_{x_1} w \circ \varphi| dt' \leq \varepsilon^{\frac{1}{19}}$, and in particular, for any $x_0 \in \mathcal{X}_0$, we have that

$$(5.29) \quad \sup_{x_0 \in \mathcal{X}_0} \int_{-\varepsilon}^t |\partial_{x_1} w \circ \varphi_{x_0}| dt' \leq \varepsilon^{\frac{1}{19}}.$$

Since the flow $\Phi(y, s)$ is related to the flow $\varphi(x, t)$ via

$$\Phi_1(y, s) = e^{\frac{3}{2}s} \varphi_1(x, t), \quad \Phi_v(y, s) = e^{\frac{s}{2}} \varphi_v(x, t),$$

and since $\partial_{x_1} w = e^s \partial_1 W$, using (2.29a), the estimate (5.16) follows. \square

The Lagrangian flow $X(\tilde{x}, t)$

We next introduce the Lagrangian flow X associated to the transport velocity in (2.11), namely, $2\beta_1(\tilde{v} + \tilde{u})$, as the solution to

$$(5.30a) \quad \partial_t X(\tilde{x}, t) = 2\beta_1(\tilde{v} + \tilde{u})(X(\tilde{x}, t), t), \quad t \in [-\varepsilon, T_*],$$

$$(5.30b) \quad X(\tilde{x}, -\varepsilon) = \tilde{x}.$$

Note that the flow $X(\tilde{x}, t)$ is related to the flow $\varphi(x, t)$ given in (5.11) via the transformation

$$(5.31) \quad \varphi_1(x, t) = X_1(\tilde{x}, t) - f(\tilde{X}(\tilde{x}, t), t), \quad \varphi_v(x, t) = X_v(\tilde{x}, t),$$

and that $X(\tilde{x}, t)$ is related to the flow $\Phi(y, s) := \Phi_U(y, s)$ by

$$(5.32) \quad \Phi_1(y, s) = e^{\frac{3}{2}s}(X_1(\tilde{x}, t) - f(\tilde{X}(\tilde{x}, t), t)), \quad \Phi_v(y, s) = e^{\frac{s}{2}}X_v(\tilde{x}, t).$$

In this subsection we obtain three results, which play an important role in the proof of vorticity creation: the first is an estimate on $|\nabla_{\tilde{x}} X(\cdot, t) - \text{Id}|$, cf. (5.39); the second is a precise bound on the label \tilde{x}_0 such that $X(\tilde{x}_0, t) \rightarrow 0$ as $t \rightarrow T_*$ (recall that 0 is the location at which the first singularity occurs), cf. Lemma 7.1; the third result is a precise lower bound on $-\int_{-\varepsilon}^{T_*} \partial_{\tilde{x}_1} \tilde{w} \circ X$, cf. Lemma 7.3.

First, we estimate the deformation rate of the flow X on the time interval $[-\varepsilon, T_*]$. The evolution of $\nabla_{\tilde{x}} X$ is given by

$$(5.33) \quad \frac{d}{dt} \partial_{\tilde{x}_j} X_i = 2\beta_1 (\partial_{\tilde{x}_k} (\tilde{v}_i + \tilde{u}_i) \circ X) \partial_{\tilde{x}_j} X_k.$$

We note that using the bounds (9.2), the argument given in (5.22)–(5.27), together with the identical argument given in section 13 of [4], we may show that there exists a universal constant $C \geq 1$ (in particular, ε -independent) such that

$$(5.34) \quad \frac{1}{C} \leq |\nabla_{\tilde{x}} X| \leq C.$$

The bound (5.34) can, however, be made sharper, and we show (cf. (5.39) below) that $|\nabla_{\tilde{x}} X - \text{Id}| \leq \varepsilon^{1/20}$ uniformly on $[-\varepsilon, T_*)$. In order to prove this, we appeal to (5.33), from which we subtract Id_{ij} , and then we contract with $\partial_{\tilde{x}_j} X_i - \text{Id}_{ij}$ to obtain that

$$(5.35) \quad \begin{aligned} \frac{d}{2dt} |\nabla_{\tilde{x}} X - \text{Id}|^2 &= (\partial_{\tilde{x}_j} X_i - \text{Id}_{ij}) \mathcal{S}_{ik} (\partial_{\tilde{x}_j} X_k - \text{Id}_{kj}) \\ &\quad + \mathcal{S}_{ij} (\partial_{\tilde{x}_j} X_i - \text{Id}_{ij}). \end{aligned}$$

We have introduced the notation

$$\mathcal{S}_{ik} = 2\beta_1 (\partial_{\tilde{x}_k} (\tilde{v}_i + \tilde{u}_i) \circ X),$$

and for a matrix A_{ij} we denote the Euclidean norm as $|A|^2 = A_{ij} A_{ij}$. Because of (2.16), which implies that for a vector field b we have $b \cdot \nabla_{\tilde{x}} \tilde{u}_j = \tilde{b} \cdot \tilde{\nabla}_x \tilde{u}_j + Jb \cdot N \partial_{x_1} \tilde{u}_j$, using the relation (5.31) between the \tilde{x} and x Lagrangian trajectories X and respectively φ , and appealing to (2.26)–(2.27), we note that the following identities hold

$$(5.36a) \quad \partial_N \tilde{u} \cdot N \circ X = J \partial_{x_1} \tilde{u} \cdot N \circ \varphi - \frac{1}{2} N_\gamma \partial_{x_\gamma} (w + z) \circ \varphi + N_\gamma \tilde{u} \cdot N_{,\gamma} \circ \varphi,$$

$$(5.36b) \quad \partial_N \tilde{u} \cdot T^\nu \circ X = J \partial_{x_1} a_\nu \circ \varphi - N_\gamma \partial_{x_\gamma} a_\nu \circ \varphi + N_\gamma \tilde{u} \cdot T_{,\gamma}^\nu \circ \varphi,$$

$$(5.36c) \quad \partial_{T^\mu} \tilde{u} \cdot N \circ X = \frac{1}{2} T_\gamma^\mu \partial_{x_\gamma} (w + z) \circ \varphi - T_\gamma^\mu \tilde{u} \cdot N_{,\gamma} \circ \varphi,$$

$$(5.36d) \quad \partial_{T^\mu} \tilde{u} \cdot T^\nu \circ X = T_\gamma^\mu \partial_{x_\gamma} a_\nu \circ \varphi - T_\gamma^\mu \tilde{u} \cdot T_{,\gamma}^\nu \circ \varphi.$$

The first term on the right side of the first line of the above list has the worst estimate when time integrated, cf. (5.28). Indeed, for all the other terms in the above list, by appealing to the bootstrap assumptions (4.4)–(4.13) and the estimate (9.1), we may deduce that their time integrals are $\mathcal{O}(\varepsilon)$. Combining these estimates we deduce that

$$(5.37) \quad \int_{-\varepsilon}^{T_*} |(\nabla_{\tilde{x}} \tilde{u}) \circ X| dt' \lesssim \varepsilon^{\frac{1}{18}}.$$

Similarly, using the relations (2.19), (2.30e), (5.31), and the estimate (9.2), we obtain that the time integral of $|(\nabla_{\tilde{x}} \tilde{v}) \circ X|$ is $\mathcal{O}(\varepsilon)$. Summarizing, we have that the matrix appearing on the right side of (5.35) satisfies

$$(5.38) \quad \int_{-\varepsilon}^{T_*} |\mathcal{S}| dt' \lesssim \varepsilon^{\frac{1}{18}}.$$

Using that $\nabla_{\tilde{x}} X|_{t=-\varepsilon} = \text{Id}$, from (5.35), (5.38), and ODE type bounds, we deduce that

$$(5.39) \quad \sup_{t \in [-\varepsilon, T_*]} |\nabla_{\tilde{x}} X(t) - \text{Id}| \lesssim e^{\int_{-\varepsilon}^t |\mathcal{S}| dt'} - 1 \lesssim \varepsilon^{\frac{1}{18}} e^{\varepsilon^{\frac{1}{18}}} \leq \varepsilon^{\frac{1}{20}}.$$

The above bound is merely a quantitative version of (5.34); it will be used in the proof of Theorem 7.4.

6 L^∞ Bounds for Specific Vorticity

We now establish bounds to solutions $\overset{\circ}{\zeta}$ of the specific vorticity equation (6.1)

From (2.24) and (2.25), we deduce that the normal and tangential components of the vorticity satisfy

$$(6.1a) \quad \partial_t(\overset{\circ}{\zeta} \cdot \mathbf{T}^2) + \mathbf{v} \cdot \nabla_x(\overset{\circ}{\zeta} \cdot \mathbf{T}^2) = \mathcal{F}_{21}(\overset{\circ}{\zeta} \cdot \mathbf{N}) + \mathcal{F}_{2\mu}(\overset{\circ}{\zeta} \cdot \mathbf{T}^\mu) + \mathcal{G}_2,$$

$$(6.1b) \quad \partial_t(\overset{\circ}{\zeta} \cdot \mathbf{T}^3) + \mathbf{v} \cdot \nabla_x(\overset{\circ}{\zeta} \cdot \mathbf{T}^3) = \mathcal{F}_{31}(\overset{\circ}{\zeta} \cdot \mathbf{N}) + \mathcal{F}_{3\mu}(\overset{\circ}{\zeta} \cdot \mathbf{T}^\mu) + \mathcal{G}_3,$$

where the transport velocity \mathbf{v} is defined by (5.10), and

$$(6.2a) \quad \begin{aligned} \mathcal{F}_{21} &= \mathbf{N} \cdot \partial_t \mathbf{T}^2 + 2\beta_1 \dot{Q}_{ij} \mathbf{T}_i^2 \mathbf{N}_j + \mathbf{v}_v(\mathbf{N} \cdot \mathbf{T}_{,v}^2) + 2\beta_1 \mathbf{N}_v \partial_{x_v} a_2 \\ &\quad - 2\beta_1 \mathbf{N}_v \overset{\circ}{u} \cdot \mathbf{T}_{,v}^2 \end{aligned}$$

$$(6.2b) \quad \begin{aligned} \mathcal{F}_{22} &= 2\beta_1 \mathbf{T}_v^2 \partial_{x_v} a_2 - 2\beta_1 \mathbf{T}_v^2 \overset{\circ}{u} \cdot \mathbf{T}_{,v}^2 \\ \mathcal{F}_{23} &= \mathbf{T}^3 \cdot \partial_t \mathbf{T}^2 + 2\beta_1 \dot{Q}_{ij} \mathbf{T}_i^2 \mathbf{T}_j^3 \mathbf{v}_v(\mathbf{T}^3 \cdot \mathbf{T}_{,v}^2) + 2\beta_1 \mathbf{T}_v^3 \partial_{x_v} a_2 \\ &\quad - 2\beta_1 \mathbf{T}_v^3 \overset{\circ}{u} \cdot \mathbf{T}_{,v}^2 \end{aligned}$$

$$(6.2c) \quad \begin{aligned} \mathcal{F}_{31} &= \mathbf{N} \cdot \partial_t \mathbf{T}^3 + 2\beta_1 \dot{Q}_{ij} \mathbf{T}_i^3 \mathbf{N}_j + \mathbf{v}_v(\mathbf{N} \cdot \mathbf{T}_{,v}^3) + 2\beta_1 \mathbf{N}_v \partial_{x_v} a_3 \\ &\quad - 2\beta_1 \mathbf{N}_v \overset{\circ}{u} \cdot \mathbf{T}_{,v}^3 \end{aligned}$$

$$(6.2d) \quad \begin{aligned} \mathcal{F}_{32} &= \mathbf{T}^2 \cdot \partial_t \mathbf{T}^3 + 2\beta_1 \dot{Q}_{ij} \mathbf{T}_i^3 \mathbf{T}_j^2 + \mathbf{v}_v(\mathbf{T}^2 \cdot \mathbf{T}_{,v}^3) + 2\beta_1 \mathbf{T}_v^2 \partial_{x_v} a_3 \\ &\quad - 2\beta_1 \mathbf{T}_v^2 \overset{\circ}{u} \cdot \mathbf{T}_{,v}^3 \end{aligned}$$

$$(6.2e)$$

$$(6.2f) \quad \mathcal{F}_{33} = 2\beta_1 T_v^3 \partial_{x_v} a_3 - 2\beta_1 T_v^3 \hat{u} \cdot T_{,v}^3,$$

and

$$(6.3a) \quad \mathcal{G}_2 = \frac{\alpha}{\gamma} \frac{\partial}{\partial \rho} (\partial_{T_3} \hat{\sigma} \partial_N \hat{k} - \partial_N \hat{\sigma} \partial_{T_3} \hat{k}) + \frac{\alpha}{\gamma} \frac{\partial}{\partial \rho} T_1^2 f_{,v} (\nabla_x \hat{\sigma} \times \nabla_x \hat{k})_v$$

$$(6.3b) \quad \mathcal{G}_3 = \frac{\alpha}{\gamma} \frac{\partial}{\partial \rho} (\partial_N \hat{\sigma} \partial_{T_2} \hat{k} - \partial_{T_2} \hat{\sigma} \partial_N \hat{k}) + \frac{\alpha}{\gamma} \frac{\partial}{\partial \rho} T_1^3 f_{,v} (\nabla_{\tilde{x}} \hat{\sigma} \times \nabla_{\tilde{x}} \hat{k})_v,$$

and from (2.14), $T_1^\mu = \frac{f_{, \mu}}{j}$.

PROPOSITION 6.1 (Bounds on specific vorticity). *For $-\varepsilon \leq t < \tau(T_*)$,*

$$(6.4) \quad \|\hat{\zeta} \circ \varphi(\cdot, t) - \hat{\zeta}(\cdot, -\varepsilon)\|_{L^\infty} \leq \varepsilon^{\frac{1}{20}}.$$

PROOF OF PROPOSITION 6.1. By the transformations (2.26), (2.30c), and (2.36a) together with the bootstrap bounds (4.13), (4.24), Lemma 9.1 we have that

$$(6.5) \quad \|\hat{u}\|_{L^\infty} \lesssim M^{\frac{1}{4}}, \quad \|\partial_{x_v}(\hat{u} \cdot N)\|_{L^\infty} \lesssim 1, \quad \|\partial_{x_v} a\|_{L^\infty} \leq M\varepsilon^{\frac{1}{2}}, \quad \|v\|_{L^\infty} \lesssim M^{\frac{1}{4}}.$$

Hence, these bounds, together with (4.2) and Lemma 9.1, yield the following bounds on the forcing functions: defined in (6.2)

$$(6.6) \quad \|\mathcal{F}_{ij}\|_{L^\infty} \lesssim 1 \quad \text{for } i, j \in \{1, 2, 3\}.$$

where we have used powers of ε to absorb powers of M .

Now, from the definitions (2.16) and (2.23), we have that

$$(6.7) \quad \begin{aligned} \hat{\rho}(x, t) \hat{\sigma}(x, t)^{1/\alpha} \hat{\zeta}(x, t) &= \tilde{\rho}(\tilde{x}, t) \tilde{\zeta}(\tilde{x}, t) = \tilde{\omega}(\tilde{x}, t) = \text{curl}_{\tilde{x}} \tilde{u}(\tilde{x}, t) \\ &= \text{curl}_{\tilde{x}} \hat{u}(x, t), \end{aligned}$$

and

$$(6.8) \quad \begin{aligned} \text{curl}_{\tilde{x}} \hat{u} \cdot N &= T_j^2 \partial_{\tilde{x}_j} \hat{u} \cdot T^3 - T_j^3 \partial_{\tilde{x}_j} \hat{u} \cdot T^2 \\ &= T_v^2 \partial_{x_v} \hat{u} \cdot T^3 - T_v^3 \partial_{x_v} \hat{u} \cdot T^2 \\ &= T_v^2 \partial_{x_v} a_3 - T_v^2 \hat{u} \cdot T_{,v}^3 - T_v^3 \partial_{x_v} a_2 + T_v^3 \hat{u} \cdot T_{,v}^2. \end{aligned}$$

from which it follows that

$$(6.9) \quad \hat{\zeta} \cdot N = \frac{T_v^2 \partial_{x_v} a_3 - T_v^2 \hat{u} \cdot T_{,v}^3 - T_v^3 \partial_{x_v} a_2 + T_v^3 \hat{u} \cdot T_{,v}^2}{\hat{\rho}}.$$

It follows from (3.4), Lemma 9.1, (5.21), (6.5), and (6.9), we have that

$$(6.10) \quad |\hat{\zeta} \cdot N| \lesssim M^{\frac{1}{4}} \varepsilon + M \varepsilon^{\frac{1}{2}} \lesssim \varepsilon^{\frac{1}{3}},$$

assuming ε is taken sufficiently small.

We define

$$\overline{\mathcal{F}}_{ij} = \mathcal{F}_{ij} \circ \varphi_{x_0}, \quad \overline{\mathcal{G}}_\mu = \mathcal{G}_\mu \circ \varphi_{x_0},$$

$$\mathcal{Q}_1 = (\hat{\zeta} \cdot N) \circ \varphi_{x_0}, \quad \mathcal{Q}_2 = (\hat{\zeta} \cdot T^2) \circ \varphi_{x_0}, \quad \mathcal{Q}_3 = (\hat{\zeta} \cdot T^3) \circ \varphi_{x_0}.$$

Then, (6.1) is written as the following system of ODEs:

$$\partial_t \mathcal{Q}_2 = \overline{\mathcal{F}}_{2j} \mathcal{Q}_j + \overline{\mathcal{G}}_2, \quad \partial_t \mathcal{Q}_3 = \overline{\mathcal{F}}_{3j} \mathcal{Q}_j + \overline{\mathcal{G}}_3.$$

Hence,

$$(6.11) \quad \frac{1}{2} \frac{d}{dt} (\mathcal{Q}_2^2 + \mathcal{Q}_3^2) = \bar{\mathcal{F}}_{\nu\mu} \mathcal{Q}_\nu \mathcal{Q}_\mu + \bar{\mathcal{F}}_{\mu 1} \mathcal{Q}_\mu \mathcal{Q}_1 + \mathcal{Q}_\mu \bar{\mathcal{G}}_\mu.$$

Now, we set $\mathcal{Y} = (\mathcal{Q}_2^2 + \mathcal{Q}_3^2)^{1/2}$. Using (6.6) and (6.10), we see from (6.11) that

$$\frac{d}{dt} \mathcal{Y} \lesssim \mathcal{Y} + \varepsilon^{\frac{1}{3}} + |\bar{\mathcal{G}}_2| + |\bar{\mathcal{G}}_3|,$$

and hence by Gronwall's inequality,

$$(6.12) \quad \begin{aligned} & |\mathcal{Y}(t) - \mathcal{Y}(-\varepsilon)| \\ & \lesssim (e^{\int_{-\varepsilon}^t C dt'} - 1) \mathcal{Y}(-\varepsilon) + e^{\int_{-\varepsilon}^t C dt'} \int_{-\varepsilon}^t (\varepsilon^{\frac{1}{3}} + |\bar{\mathcal{G}}_2| + |\bar{\mathcal{G}}_3|) dr \\ & \lesssim \varepsilon \mathcal{Y}(-\varepsilon) + \int_{-\varepsilon}^t (\varepsilon^{\frac{1}{3}} + |\bar{\mathcal{G}}_2| + |\bar{\mathcal{G}}_3|) dr, \end{aligned}$$

where we used the bound $t - \varepsilon \leq \tau(T_*) \leq 2\varepsilon$ from (4.1a).

We now prove that $\int_{-\varepsilon}^t \bar{\mathcal{G}}_\mu(r) dr$ is bounded for all $t \geq -\varepsilon$ such that $t < \tau(t)$. First note that by (2.30d) and (4.14), we see that

$$(6.13) \quad \|\nabla_x \mathring{\mathbf{k}}(\cdot, t)\|_{L^\infty} \lesssim \varepsilon^{\frac{1}{8}},$$

so it remains for us to bound $\exp \int_{-\varepsilon}^t |\partial_{\mathbf{T}\mu} \mathring{\sigma} \circ \varphi| dt'$ and $\exp \int_{-\varepsilon}^t |\partial_{\mathbf{N}} \mathring{\sigma} \circ \varphi| dt'$. Using the identities

$$(\mathbf{N} \cdot \nabla_{\tilde{x}}) \mathring{\sigma} = \partial_{x_1} \mathring{\sigma} \mathbf{J} + \mathbf{N}_\mu \partial_{x_\mu} \mathring{\sigma} \quad \text{and} \quad (\mathbf{T}^\nu \cdot \nabla_{\tilde{x}}) \mathring{\sigma} = \mathbf{T}_\mu^\nu \partial_{x_\mu} \mathring{\sigma},$$

and (2.26), we see that

$$\begin{aligned} \partial_{\mathbf{N}} \mathring{\sigma} &= \partial_{x_1} \mathring{u} \cdot \mathbf{N} \mathbf{J} - \partial_{x_1} z \mathbf{J} + \mathbf{N}_\mu \partial_{x_\mu} (\mathring{u} \cdot \mathbf{N}) - \mathbf{N}_\mu \partial_{x_\mu} z, \\ \partial_{\mathbf{T}^\nu} \mathring{\sigma} &= \mathbf{T}_\mu^\nu \partial_{x_\mu} (\mathring{u} \cdot \mathbf{N}) - \mathbf{T}_\mu^\nu \partial_{x_\mu} z. \end{aligned}$$

From (2.30b), (2.36a), (4.12), and (4.24), we find that

$$(6.14) \quad \|\partial_{\mathbf{T}^\nu} \mathring{\sigma}\|_{L^\infty} \lesssim 1,$$

and additionally with (5.28), we see that

$$(6.15) \quad \int_{-\varepsilon}^t |\partial_{\mathbf{N}} \mathring{\sigma} \circ \varphi| dt' \lesssim \varepsilon^{\frac{1}{18}}.$$

The estimates (6.13), (6.14), and (6.15) together with (4.25) and (5.21) show that

$$(6.16) \quad \int_{-\varepsilon}^t |\bar{\mathcal{G}}_\mu(s)| ds \lesssim \varepsilon^{\frac{1}{18}}.$$

From (6.12) and (6.16), we have that

$$|\mathcal{Q}_2(t) - \mathcal{Q}_2(-\varepsilon)| + |\mathcal{Q}_3(t) - \mathcal{Q}_3(-\varepsilon)| \lesssim \varepsilon (|\mathcal{Q}_2(-\varepsilon)| + |\mathcal{Q}_3(-\varepsilon)|) + \varepsilon^{\frac{1}{18}}$$

uniformly for all labels x_0 . Since $\mathbf{N}, \mathbf{T}^2, \mathbf{T}^3$ form an orthonormal basis, the above estimate and (6.10) imply that (6.4) holds. \square

7 Vorticity Creation

We analyze vorticity creation (see Theorem 7.4) through the evolution of the specific vorticity vector $\tilde{\zeta}$ in \tilde{x} -variables,

7.1 The blowup trajectory and a bound on the amplification factor

We obtain an estimate for the position of the particle \tilde{x}_0 , which is carried by the flow $X(\cdot, t)$ to the blowup location $\tilde{x} = 0$ as $t \rightarrow T_*$.

LEMMA 7.1 (Initial location of particle trajectory leading to blowup). *With the flow X defined by (5.30), let $X_{\tilde{x}_0}(t)$ denote the trajectory that emanates from the point \tilde{x}_0 . If $\lim_{t \rightarrow T_*} X_{\tilde{x}_0}(t) = 0$, then*

$$(7.1) \quad |(\tilde{x}_0)_1 - \beta_3 \kappa_0 \varepsilon| \leq 5\varepsilon^{\frac{7}{6}}, \quad |\tilde{x}_0| \leq 5\varepsilon^{\frac{7}{6}}.$$

PROOF. We consider the trajectory $X_{\tilde{x}_0}(t)$ for which $X_{\tilde{x}_0}(T_*) = 0$ and for notational simplicity, we drop the subscript \tilde{x}_0 and use $X(t)$ to denote this trajectory. The main idea is that the initial position of the particle $X(t)$, i.e., \tilde{x}_0 , may be computed by passing $t \rightarrow T_*$ in the identity $X(t) - \tilde{x}_0 = \int_{-\varepsilon}^t \partial_t X(t') dt'$, leading to

$$(7.2) \quad \tilde{x}_0 = - \int_{-\varepsilon}^{T_*} \partial_t X(t') dt'.$$

By revisiting the right side of (5.30), we obtain a sharp estimate for the right side of the above identity.

For convenience, in analogy to (2.26) we define

$$(7.3) \quad \tilde{w} = \tilde{u} \cdot \mathbf{N} + \tilde{\sigma}, \quad \tilde{z} = \tilde{u} \cdot \mathbf{N} - \tilde{\sigma}, \quad \tilde{a}_\nu = \tilde{u} \cdot \mathbf{T}^\nu.$$

We note that $\partial_{\tilde{x}_1} \tilde{w}(\tilde{x}, t) = \partial_{x_1} w(x, t)$. Furthermore, using (2.6) we have that

$$(7.4) \quad \begin{aligned} \partial_t X &= 2\beta_1(\tilde{v} + \tilde{u} \cdot \mathbf{N} \mathbf{N} + \tilde{u} \cdot \mathbf{T}^\nu \mathbf{T}^\nu) \circ X \\ &= 2\beta_1 \dot{Q} X - 2\beta_1 R^T \dot{\xi} + \beta_1(\tilde{w} \mathbf{N} + \tilde{z} \mathbf{N} + 2\tilde{a}_\nu \mathbf{T}^\nu) \circ X. \end{aligned}$$

First we note that using that \dot{Q} is skew-symmetric, that $X(T_*) = 0$, appealing to the bounds (4.1b), (4.13), and (4.24), together with (4.27), from the Grönwall inequality on $[t, T_*]$, we obtain that

$$(7.5) \quad |X(t)| \lesssim M^{\frac{1}{4}} \varepsilon.$$

This estimate is however not sharp enough; to do better, we need to carefully bound the term $2\beta_1 R^T \dot{\xi}$ on the right side of (7.4). Note (cf. (2.31)) that we have $(R^T \dot{\xi})_i = -V_i(0, s)$. Then, evaluating (2.33a) and (2.34a) at $y = 0$, and using the definition of the function f and our constraints (2.52), we deduce

$$\begin{aligned} 2\beta_1 (R^T \dot{\xi})_1 &= \kappa + \beta_2 Z^0 - \frac{1}{\beta_\tau} e^{-\frac{s}{2}} G_W^0, \\ 2\beta_1 (R^T \dot{\xi})_\mu &= 2\beta_1 A_\mu^0 - \frac{1}{\beta_\tau} e^{\frac{s}{2}} h_W^{\mu, 0}, \end{aligned}$$

in analogy to (2.61) and (A.7a). Using the $\dot{\kappa}$ estimate in (4.1b), the Z and A estimates in (4.12) and (4.13), and the bound (12.17) for G_W^0 and $h_W^{\mu,0}$, which is a consequence of the bootstrap assumptions, we deduce that

$$(7.6) \quad |2\beta_1(R^T \dot{\xi})_1 - \kappa_0| \lesssim M\varepsilon \quad \text{and} \quad |2\beta_1(R^T \dot{\xi})_\mu| \lesssim M\varepsilon^{\frac{4}{5}}$$

since $1 - \frac{5}{2m-7} > \frac{4}{5}$ for $m \geq 18$. Returning to (7.4), from (4.2), (4.12), (4.13), and (7.5), we have that

$$(7.7) \quad |2\beta_1 \dot{Q}X + \beta_1(\tilde{z}N + 2\tilde{a}_v \mathbb{T}^v) \circ X| \lesssim M^{\frac{9}{4}} \varepsilon^{\frac{3}{2}} + M\varepsilon \lesssim M\varepsilon.$$

Lastly, by (5.31) we have $\tilde{w} \circ X = w \circ \varphi$, and by (2.30a) we have $w = \kappa + e^{-\frac{s}{2}} W$. Thus, by also appealing to (4.1b), (4.5), (4.7), (9.1a), and the fact that $|\phi_{v\mu}(-\varepsilon)| \leq \varepsilon$ implies $|N(-\varepsilon) - e_1| \lesssim \varepsilon$, we obtain

$$(7.8) \quad |\tilde{w}N \circ X - \kappa_0 e_1| \leq |\kappa N - \kappa_0 e_1| + e^{-\frac{s}{2}} \|W\|_{L^\infty(\mathcal{X}(s))} \leq 3\varepsilon^{\frac{1}{6}}.$$

By inserting the estimates (7.6)–(7.8) into the right side of (7.4), we obtain that

$$(7.9) \quad |\partial_t X_1 + \beta_3 \kappa_0| \leq 4\varepsilon^{\frac{1}{6}} \quad \text{and} \quad |\partial_t X_v| \leq 4\varepsilon^{\frac{1}{6}}$$

upon taking ε to be sufficiently small in terms of M , and recalling that $\beta_1 - 1 = -\beta_3$. To conclude the proof of the lemma, we simply combine (7.2) with (7.9) and the estimate $|T_*| \leq \varepsilon^{3/2}$, as given by (4.27). \square

Remark 7.2. For the particle trajectory from Lemma 7.1, integrating (7.9) from on $[t, T_*]$, as opposed to $[-\varepsilon, T_*]$ as was done in (7.2), we obtain that

$$(7.10) \quad |X_1(t) - \beta_3 \kappa_0 e^{-s}| \leq 5\varepsilon^{\frac{1}{6}} e^{-s} \quad \text{and} \quad |X_v(t)| \leq 5\varepsilon^{\frac{1}{6}} e^{-s}.$$

Here we have again used $|e^s(T_* - t) - 1| \leq 2M\varepsilon$, which holds in view of (4.1b), (4.3), and (4.26).

The second preliminary estimate is a lower bound on $-\int_{-\varepsilon}^{T_*} \partial_{\tilde{x}_1} \tilde{w} \circ X$, as this quantity plays a key role in our proof of vorticity creation (cf. the estimate for the term I_1 in Theorem 7.4).

LEMMA 7.3. *With the flow X defined by (5.30), let $X_{\tilde{x}_0}(t)$ denote the trajectory that emanates from the point \tilde{x}_0 . If $X_{\tilde{x}_0}(T_*) = 0$ and the initial condition satisfies (3.24), then*

$$(7.11) \quad -\int_{-\varepsilon}^{T_*} \partial_{\tilde{x}_1} \tilde{w}(X_{\tilde{x}_0}(t), t) dt \geq \frac{1}{9} \kappa_0^{-\frac{2}{3}} \varepsilon^{\frac{1}{3}}.$$

PROOF. The proof of the lemma is based on two ideas: first, the time integral in (7.11) is dominated by values of t which are very close to $-\varepsilon$, where we can relate $\partial_{\tilde{x}_1} \tilde{w}$ to its initial datum; second, the flow $X(t)$ is related to the self-similar flow Φ_U via the relation (5.32), which allows us to appeal to sharp bounds for $\partial_1 W$ in estimating the contribution to (7.11) for $t \gg -\varepsilon$. We implement these ideas as follows.

We consider the trajectory $X_{\tilde{x}_0}(t)$ for which $X_{\tilde{x}_0}(T_*) = 0$, and for notational simplicity, we drop the subscript \tilde{x}_0 and use $X(t)$ to denote this trajectory. The associated self-similar initial datum variable y_0 is given via (2.15) and (2.29) as

$$(7.12) \quad y_0 = (\varepsilon^{-\frac{3}{2}}((\tilde{x}_0)_1 - f(\tilde{x}_0)), \varepsilon^{-\frac{1}{2}}\tilde{x}_0).$$

Due to Lemma 7.1 we know that \tilde{x}_0 satisfies (7.1), and since $|\phi_{\mu\nu}(-\varepsilon)| \leq \varepsilon$, we deduce that

$$(7.13) \quad |(y_0)_1 - \beta_3 \kappa_0 \varepsilon^{-\frac{1}{2}}| \leq 6\varepsilon^{-\frac{1}{3}} \quad \text{and} \quad |(y_0)_v| \leq 5\varepsilon^{\frac{2}{3}}.$$

Note that these bounds are set up precisely to account for the region specified in (3.24). In view of the precise estimates on the trajectory $X_{\tilde{x}_0}(t)$, we directly obtain sharp bounds on the self-similar Lagrangian flow $\Phi_U^{y_0}(s)$ emanating from y_0 . Indeed, by the ϕ bound in (4.1a), the relation between Φ_U and X in (5.32), and the bounds (7.10), we have that

$$(7.14) \quad (\beta_3 \kappa_0 - \varepsilon^{\frac{1}{7}})e^{\frac{s}{2}} \leq (\Phi_U^{y_0})_1(s) \leq (\beta_3 \kappa_0 + \varepsilon^{\frac{1}{7}})e^{\frac{s}{2}} \\ \text{and} \quad |(\Phi_U^{y_0})_v(s)| \leq \varepsilon^{\frac{1}{7}}e^{-\frac{s}{2}}.$$

Next, due to (5.32) and (7.3) we have that

$$(7.15) \quad \partial_{\tilde{x}_1} \tilde{w} \circ X_{\tilde{x}_0}(t) = e^s \partial_1 W \circ \Phi_U^{y_0}(s)$$

with the usual relation between t and s from (2.29). Since $dt = \beta_\tau e^{-s} ds$, we thus have that the integral we need to estimate in (7.11) may be rewritten as

$$(7.16) \quad - \int_{-\varepsilon}^{T_*} \partial_{\tilde{x}_1} \tilde{w}(X_{\tilde{x}_0}(t), t) dt = - \int_{-\log \varepsilon}^{\infty} \beta_\tau \partial_1 W \circ \Phi_U^{y_0}(s) ds.$$

Recall (cf. (4.3)) that $1 - 2Me^{-s} \leq \beta_\tau \leq 1 + 2Me^{-s}$, so that we just need to bound from below the integral of $-\partial_1 W \circ \Phi_U^{y_0}$. The remainder of the argument mimics the proof of Lemma 5.6

Fix y_0 as in (7.13), $s \in [-\log \varepsilon, \infty)$, and thus fix a value of $\Phi_U^{y_0}(s)$. We trace the particle trajectory of the flow \mathcal{V}_W (not \mathcal{V}_U !) backwards in time, and write $\Phi_W^{y'_0}(s) = \Phi_U^{y_0}(s)$, where the initial datum $\Phi_W^{y'_0}(s_0) = y'_0$ is given by the property that $|y'_0| = \mathcal{L}$ if $s_0 \geq -\log \varepsilon$, and $|y'_0| > \mathcal{L}$ if $s_0 = -\log \varepsilon$. We then appeal to Lemma 5.3 with y'_0 replacing y_0 . The lemma is applicable on the interval $[s_0, s]$ since $|y'_0| \geq \mathcal{L}$ and by (7.14) we have $|\check{\Phi}_W^{y'_0}(s)| = |\check{\Phi}_U^{y_0}(s)| \leq \varepsilon^{1/7}e^{-s/2} \leq \varepsilon^{1/2}$. By (5.5), we thus obtain that for any $s' \in [-\log \varepsilon, s]$ we have the estimates

$$(7.17) \quad |(\Phi_W^{y'_0})_1(s')| \geq \frac{3}{4}|(y_0)_1|e^{\frac{3(s'-s_0)}{2}} \quad \text{and} \quad |\check{\Phi}_W^{y'_0}(s')| \leq M\varepsilon^{\frac{1}{2}}.$$

Let us first consider the case that $|y'_0| > \mathcal{L}$ and $s_0 = -\log \varepsilon$. Based on (7.17) we now claim that $|(y'_0)_1| \leq 2\kappa_0 \varepsilon^{-1/2}$. If not, then by appealing to the first

estimate in (7.14), we thus deduce that

$$\begin{aligned} \frac{3}{2}\beta_3\kappa_0e^{\frac{s}{2}} &\geq |(\Phi_U^{y_0})_1(s)| = |(\Phi_W^{y'_0})_1(s)| \geq \frac{3}{4}|(y_0)_1|e^{\frac{3(s-s_0)}{2}} \\ &> \frac{3}{2}\kappa_0\varepsilon^{-\frac{1}{2}}e^{s-s_0}e^{\frac{s}{2}}\varepsilon^{\frac{1}{2}} \geq \frac{3}{2}\kappa_0e^{\frac{s}{2}}, \end{aligned}$$

which is a contradiction, since $\beta_3 = \frac{\alpha}{1+\alpha} < 1$. Therefore, from the above argument and the second bound in (7.14) evaluated at $s' = s_0$, we have that $\mathcal{L} = \varepsilon^{-1/10} < |(y'_0)_1| \leq 2\kappa_0\varepsilon^{-1/2}$ and $|(y'_0)_v| \leq M\varepsilon^{1/2} \leq \varepsilon^{\frac{1}{3}}$. Therefore, the point y'_0 exactly lies in the region stipulated in (3.24), and so by Lemma 3.1 in this case we have that

$$(7.18) \quad \partial_1 W(\Phi_W^{y'_0}(s_0), s_0) = \partial_1 W(y'_0, -\log \varepsilon) \in \left[-\frac{1}{2}|(y'_0)_1|^{-\frac{2}{3}}, -\frac{1}{4}|(y'_0)_1|^{-\frac{2}{3}}\right]$$

Next, let us first consider the case that $|y'_0| = \mathcal{L}$ and $s_0 > -\log \varepsilon$. In this case, instead of appealing to (3.24) we use the bootstrap (4.8b) and as shown earlier in (5.13) we deduce

$$(7.19) \quad \partial_1 W(\Phi_W^{y'_0}(s_0), s_0) = \partial_1 W(y'_0, s_0) \leq -\frac{1}{2}\tilde{\eta}^{-\frac{1}{3}}(y) \leq -\frac{1}{4}|(y'_0)_1|^{-\frac{2}{3}},$$

where we used (7.17) with $s' = s_0$ in the last inequality.

Having established (7.18) and (7.19), we use the $\partial_1 W$ evolution given in (2.44a) with $\gamma = (1, 0, 0)$ and deduce that

$$\partial_s(\partial_1 W \circ \Phi_W^{y'_0}) + (1 + \beta_\tau J \partial_1 W \circ \Phi_W^{y'_0})(\partial_1 W \circ \Phi_W^{y'_0}) = F_W^{(1,0,0)} \circ \Phi_W^{y'_0}.$$

Integrating this expression on $[s_0, s]$, recalling that by definition we have $\Phi_W^{y'_0}(s) = \Phi_U^{y_0}(s)$, using that by (7.18) and (7.19) we have that $-\partial_1 W(y'_0, s_0) > 0$ by appealing to the $F_W^{(1,0,0)}$ estimate in (9.19) and to the $\partial_1 W$ bootstrap in (4.7), we deduce

$$\begin{aligned} (7.20) \quad & -\partial_1 W(\Phi_U^{y_0}(s), s) \\ &= -\partial_1 W(y'_0, s_0) \exp\left(-\int_{s_0}^s 1 + \beta_\tau(J\partial_1 W) \circ \Phi_W^{y'_0}(s') ds'\right) \\ &\quad - \int_{s_0}^s F_W^{(1,0,0)} \circ \Phi_W^{y'_0}(s') \exp\left(-\int_{s'}^s 1 + \beta_\tau(J\partial_1 W) \circ \Phi_W^{y'_0}(s'') ds''\right) ds' \\ &\geq \frac{1}{4}|(y'_0)_1|^{-\frac{2}{3}}e^{-(s-s_0)} \exp\left(-3\int_{s_0}^s \eta^{-\frac{1}{3}} \circ \Phi_W^{y'_0}(s') ds'\right) \\ &\quad - \int_{s_0}^s e^{-\frac{s'}{2}} \eta^{-\frac{1}{3}} \circ \Phi_W^{y'_0}(s') e^{-(s-s')} \exp\left(3\int_{s'}^s \eta^{-\frac{1}{3}} \circ \Phi_W^{y_0}(s'') ds''\right) ds' \end{aligned}$$

Since $|y'_0| \geq \mathcal{L}$, by (7.17) we have

$$3\int_{s_0}^s \eta^{-\frac{1}{3}} \circ \Phi_W^{y'_0}(s') ds' \leq 4|(y'_0)_1|^{-\frac{2}{3}} \int_{s_0}^s e^{-(s'-s_0)} ds' \leq \varepsilon^{\frac{1}{16}}$$

and

$$\begin{aligned} \int_{s_0}^s e^{-\frac{s'}{5}} \eta^{-\frac{1}{3}} \circ \Phi_W^{y'_0}(s') ds' e^{-(s-s')} &\leq 2e^{-s} |(y'_0)_1|^{-\frac{2}{3}} \int_{s_0}^s e^{\frac{4s'}{5}} e^{-(s'-s_0)} ds' \\ &\leq 10\varepsilon^{\frac{1}{5}} e^{-(s-s_0)} |(y'_0)_1|^{-\frac{2}{3}}. \end{aligned}$$

Inserting these estimates into (7.20), we deduce

$$(7.21) \quad -\partial_1 W(\Phi_U^{y_0}(s), s) \geq \frac{1}{5} |(y'_0)_1|^{-\frac{2}{3}} e^{-(s-s_0)}.$$

The bound (7.21) holds both in the case that $s_0 > -\log \varepsilon$ and $|y'_0| = \mathcal{L}$, and also in the case that $s_0 = -\log \varepsilon$ and $|y'_0| > \mathcal{L}$ and $|(y'_0)_1| \leq 2\kappa_0 \varepsilon^{-1/2}$. The last observation is that in either case, the bound (7.21) implies

$$(7.22) \quad -\partial_1 W(\Phi_U^{y_0}(s), s) \geq \frac{1}{5} (2\kappa_0 \varepsilon^{-\frac{1}{2}})^{-\frac{2}{3}} e^{-(s-s_0)} \geq \frac{1}{8} \kappa_0^{-\frac{2}{3}} \varepsilon^{\frac{1}{3}} e^{-(s+\log \varepsilon)}.$$

Lastly, using (7.22) we bound from below the right side of (7.16) and obtain

$$-\int_{-\log \varepsilon}^{\infty} \beta_\tau \partial_1 W \circ \Phi_U^{y_0}(s) ds \geq \frac{1-2M\varepsilon}{8} \kappa_0^{-\frac{2}{3}} \varepsilon^{\frac{1}{3}} \int_{-\log \varepsilon}^{\infty} e^{-(s+\log \varepsilon)} ds \geq \frac{1}{9} \kappa_0^{-\frac{2}{3}} \varepsilon^{\frac{1}{3}},$$

which completes the proof. \square

7.2 Vorticity creation from irrotational data

We now return to the specific vorticity equation (2.11), which we shall now write as

$$(7.23) \quad \partial_t \tilde{\zeta} - 2\beta_1 \dot{Q} \tilde{\zeta} + 2\beta_1 (\tilde{v} + \tilde{u}) \cdot \nabla_{\tilde{x}} \tilde{\zeta} = 2\beta_1 \text{Def}_{\tilde{x}} \tilde{u} \cdot \tilde{\zeta} + \tilde{b} \quad \text{for } t \in [-\varepsilon, T_*),$$

where we use \tilde{b} to denote the baroclinic term in (\tilde{x}, t) -variables:

$$(7.24) \quad \tilde{b} = 2\beta_1 \frac{\alpha}{\rho} \tilde{\sigma} \nabla_{\tilde{x}} \tilde{\sigma} \times \nabla_{\tilde{x}} \tilde{k},$$

and the (rate of) deformation tensor is defined by

$$\text{Def}_{\tilde{x}} \tilde{u} = \frac{1}{2} (\nabla_{\tilde{x}} \tilde{u} + \nabla_{\tilde{x}} \tilde{u}^T),$$

which is the symmetric part of the velocity gradient. In components, $(\text{Def}_{\tilde{x}} \tilde{u} \cdot \tilde{\zeta})_i = \frac{1}{2} (\partial_{\tilde{x}_j} \tilde{u}_i + \partial_{\tilde{x}_i} \tilde{u}_j) \tilde{\zeta}_j$.

By definition of the $X_{\tilde{x}_0}(t)$ flow in (5.30), so that $X_{\tilde{x}_0}(-\varepsilon) = \tilde{x}_0$ upon composing (7.23) with $X_{\tilde{x}_0}(t)$ and denoting

$$(7.25) \quad \begin{aligned} \zeta(\tilde{x}_0, t) &= \tilde{\zeta} \circ X_{\tilde{x}_0}(t), \quad D(\tilde{x}_0, t) = 2\beta_1 \text{Def}_{\tilde{x}} \tilde{u} \circ X_{\tilde{x}_0}(t), \\ b(\tilde{x}_0, t) &= \tilde{b} \circ X_{\tilde{x}_0}(t), \end{aligned}$$

we have

$$(7.26) \quad \frac{d}{dt} \zeta = (2\beta_1 \dot{Q} + D) \cdot \zeta + b.$$

At this stage two observations are in order. First, due to (5.31) we have that $\zeta = \tilde{\zeta} \circ X = \tilde{\zeta} \circ \varphi$, so that the bound (6.4) translates into

$$(7.27) \quad |\zeta(\tilde{x}_0, t) - \tilde{\zeta}(\tilde{x}_0, -\varepsilon)| \leq \varepsilon^{\frac{1}{21}}.$$

Second, we note that by (5.36), (5.37), (9.1), and (3.2), for any $(i, j) \neq (1, 1)$ we have

$$(7.28) \quad \int_{-\varepsilon}^{T_*} |D_{ij}(t')| dt' \lesssim M\varepsilon,$$

while for $(i, j) = (1, 1)$ we have

$$(7.29) \quad \int_{-\varepsilon}^{T_*} |D_{11}(t')| dt' \lesssim \varepsilon^{\frac{1}{18}}.$$

We omit the detailed proofs of (7.28) and (7.29) but note that as already discussed in the paragraph below (5.36), only the time integral of $|\partial_N \tilde{u} \cdot N \circ X|$ is not $\mathcal{O}(\varepsilon)$; and since $|N - e_1| \lesssim \varepsilon$, this corresponds to only the $(1, 1)$ component of the D matrix as having a time integral which may be larger than $\mathcal{O}(\varepsilon)$. Taking into account also the \dot{Q} estimate in (4.2), we rewrite

$$(7.30) \quad 2\beta_1 \dot{Q} + D =: \text{diag}(D_{11}, 0, 0) + D_{\text{small}} =: D_{\text{main}} + D_{\text{small}}$$

with

$$(7.31) \quad \int_{-\varepsilon}^{T_*} |D_{\text{small}}(t')| dt' \lesssim M\varepsilon.$$

With this information, since D_{main} is a diagonal matrix, we may write the solution of ODE (7.26) pointwise in \tilde{x}_0 as

$$(7.32) \quad \begin{aligned} \zeta(\cdot, t) &= e^{\int_{-\varepsilon}^t D_{\text{main}}(\cdot, t') dt'} \tilde{\zeta}(\cdot, -\varepsilon) \\ &+ \int_{-\varepsilon}^t e^{\int_{t'}^t D_{\text{main}}(\cdot, t'') dt''} (b(\cdot, t') + D_{\text{small}}(\cdot, t') \cdot \zeta(\cdot, t')) dt', \end{aligned}$$

where in view of (7.29)

$$(7.33) \quad |e^{\int_{t'}^t D_{\text{main}}(\cdot, t'') dt''} - \text{Id}| = |\text{diag}(e^{\int_{t'}^t D_{11}(\cdot, t'') dt''}, 1, 1) - \text{Id}| \lesssim \varepsilon^{\frac{1}{18}}.$$

The solution formula (7.32), along with the bounds (7.27), (7.31), and (7.33), show that vorticity creation is essentially implied by (lower) bounds on $\int_{-\varepsilon}^t b(\cdot, t') dt'$. This is indeed the main idea in the proof of vorticity creation, which we establish next.

In the following theorem, we show that when the initial vorticity is zero, the Euler dynamics instantaneously creates vorticity, and that for appropriately chosen initial data, the vorticity remains nontrivial at the formation of the shock.

THEOREM 7.4 (Vorticity creation). *Consider \tilde{x}_0 such that the flow $X_{\tilde{x}_0}(t)$ converges to the blowup point 0 as $t \rightarrow T_*$. More generally, consider any \tilde{x}_0 satisfying (7.1). Suppose that the initial datum verifies (3.24), and that the initial baroclinic*

torque at this point, $\tilde{b}(\tilde{x}_0, -\varepsilon)$, is nontrivial. For example, this may be ensured by choosing

$$(7.34) \quad \partial_{\tilde{x}_1} \tilde{k}(\tilde{x}_0, -\varepsilon) = 0 \quad \partial_{\tilde{x}_3} \tilde{k}(\tilde{x}_0, -\varepsilon) = 0, \quad \partial_{\tilde{x}_2} \tilde{k}(\tilde{x}_0, -\varepsilon) < 0.$$

If the initial datum is irrotational, i.e. $\tilde{\zeta}(\tilde{x}, -\varepsilon) = 0$ for all $\tilde{x} \in \mathbb{R}^3$, then vorticity is instantaneously created and remains nonvanishing in the neighborhood of the shock location $(\tilde{x}, t) = (0, T_*)$. Quantitatively, with the choice (7.34) we have that

$$(7.35) \quad |\tilde{\zeta}(\tilde{x}, t)| \geq c_\alpha \kappa_0^{\frac{1}{3} - \frac{1}{\alpha}} \varepsilon^{\frac{1}{3}} |\partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}_0)|$$

for all (\tilde{x}, t) in a small neighborhood of the shock location $(0, T_*)$, where $c_\alpha > 0$ is a constant that only depends on α .

PROOF. As alluded to in the discussion preceding the theorem, the proof is based on following the Lagrangian flow $X_{\tilde{x}_0}(t)$, which arrives at the shock location as $t \rightarrow T_*$, and study the vorticity production caused by the baroclinic torque term \mathbf{b} . We note that (7.35) is proven by establishing this bound at $\tilde{x} = X_{\tilde{x}_0}(t)$ with $t \rightarrow T_*$ for one component of the vorticity vector, and arguing by continuity, the fact that the vorticity remains continuous all the way up to the blowup time ensures that the lower bound holds for (\tilde{x}, t) in a neighborhood of $(0, T_*)$.

For simplicity of the presentation we provide a lower bound on the third component of the vorticity; this is why in assumption (7.34) we have chosen very specific gradient components for \tilde{k} and $\tilde{\sigma}$. Recall the notation (7.25). Using that the initial datum is irrotational, from the solution formula (7.32), the bounds (7.27), (7.31), (7.33), and the fact that the matrix \mathbf{D}_{main} only has a nontrivial $(1, 1)$ entry, we obtain

$$(7.36) \quad |\zeta_3(\tilde{x}_0, t) - \int_{-\varepsilon}^t \mathbf{b}_3(\tilde{x}_0, t') dt'| \lesssim (1 + \varepsilon^{\frac{1}{18}}) \varepsilon^{\frac{1}{21}} M \varepsilon \lesssim \varepsilon.$$

The remainder of the proof consists of analyzing the time integral of $\mathbf{b}_3(\tilde{x}_0, t) = \tilde{b}(X_{\tilde{x}_0}(t), t)$.

Let us denote the cofactor matrix associated to $\nabla_{\tilde{x}} X$ and its Jacobian determinant, respectively, by

$$\mathcal{B}(\tilde{x}, t) = \text{Cof}(\nabla_{\tilde{x}} X), \quad J(\tilde{x}, t) = \det(\nabla_{\tilde{x}} X),$$

so that

$$(\nabla_{\tilde{x}} X)^{-1} = J^{-1} \mathcal{B}.$$

Two components of the cofactor matrix that we shall make use of are given by

$$\begin{aligned} \mathcal{B}_2^2 &= \partial_{\tilde{x}_2} X_2 (\partial_{\tilde{x}_1} X_1 \partial_{\tilde{x}_3} X_3 - \partial_{\tilde{x}_1} X_3 \partial_{\tilde{x}_3} X_1), \\ \mathcal{B}_1^2 &= \partial_{\tilde{x}_1} X_2 (\partial_{\tilde{x}_3} X_1 \partial_{\tilde{x}_2} X_3 - \partial_{\tilde{x}_2} X_1 \partial_{\tilde{x}_3} X_3). \end{aligned}$$

From (5.39), we see that

$$(7.37) \quad |J - 1| \lesssim \varepsilon^{\frac{1}{20}}, \quad |\mathcal{B}_2^2 - 1| \lesssim \varepsilon^{\frac{1}{20}}, \quad \text{and} \quad |\mathcal{B}_1^2| \lesssim \varepsilon^{\frac{1}{10}}.$$

Then, transport equation (2.5c) shows that

$$(7.38) \quad \tilde{k} \circ X_{\tilde{x}}(t) = \tilde{k}(\tilde{x}, -\varepsilon) =: \tilde{k}_0(\tilde{x})$$

so that

$$(7.39) \quad \partial_{\tilde{x}_j} \tilde{k} \circ X_{\tilde{x}}(t) = J^{-1}(\tilde{x}, t) \partial_{\tilde{x}_\ell} \tilde{k}_0(\tilde{x}) \mathcal{B}_j^\ell(\tilde{x}, t).$$

The point of the first two assumptions in (7.34) is to single out one of the three elements in the sum over ℓ in (7.39), which now reduces to

$$(7.40) \quad \partial_{\tilde{x}_j} \tilde{k} \circ X_{\tilde{x}}(t) = J^{-1}(\tilde{x}, t) \partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}) \mathcal{B}_j^2(\tilde{x}, t).$$

For the remainder of the proof, we fix X to denote the trajectory which collides with the blowup at time $t = T_*$ so that $X(T_*) = 0$. Using (7.40) and recalling (7.3), we return to (7.24) and obtain that

$$(7.41) \quad \begin{aligned} b_3 &= \tilde{b} \circ X \\ &= 2\beta_1 \frac{\alpha}{\gamma} \frac{\tilde{\sigma}}{\tilde{\rho}} \circ X (\partial_{\tilde{x}_1} \tilde{\sigma} \circ X \partial_{\tilde{x}_2} \tilde{k} \circ X - \partial_{\tilde{x}_2} \tilde{\sigma} \circ X \partial_{\tilde{x}_1} \tilde{k} \circ X) \\ &= 2\beta_1 \frac{\alpha}{\gamma} \frac{\tilde{\sigma}}{\tilde{\rho}} \circ X J^{-1} \partial_{\tilde{x}_2} \tilde{k}_0 (\mathcal{B}_2^2 \partial_{\tilde{x}_1} \tilde{\sigma} \circ X - \mathcal{B}_1^2 \partial_{\tilde{x}_2} \tilde{\sigma} \circ X) \\ &= \beta_1 \frac{\alpha}{\gamma} \frac{\tilde{\sigma}}{\tilde{\rho}} \circ X J^{-1} \partial_{\tilde{x}_2} \tilde{k}_0 (\mathcal{B}_2^2 \partial_{\tilde{x}_1} \tilde{w} \circ X - \mathcal{B}_2^2 \partial_{\tilde{x}_1} \tilde{z} \circ X - 2\mathcal{B}_1^2 \partial_{\tilde{x}_2} \tilde{\sigma} \circ X) \\ &=: b_3^{(1)} - b_3^{(2)} - b_3^{(3)}. \end{aligned}$$

We first note that by the relation of σ and ρ , in view of (2.5c) we have

$$(7.42) \quad \frac{\beta_1 \alpha}{\gamma} \frac{\tilde{\sigma}}{\tilde{\rho}} \circ X = \frac{\beta_1}{\gamma} e^{\frac{\tilde{k}_0}{2}} (\tilde{\rho} \circ X)^{\alpha-1}$$

so that by (5.21) and the initial L^∞ assumption on $k(\cdot, -\varepsilon)$ we have

$$(7.43) \quad \left| \frac{\beta_1 \alpha}{\gamma} \frac{\tilde{\sigma}}{\tilde{\rho}} \circ X - \frac{\beta_1}{\gamma} \left(\frac{\alpha \kappa_0}{2} \right)^{\frac{\alpha-1}{\alpha}} \right| \lesssim \varepsilon^{\frac{1}{10}}.$$

Combined with (7.37), our bootstrap assumptions derivatives of Z in (4.12) and on $U \cdot N$ and S in (4.24), similarly to (7.31) we obtain that the last two terms in (7.41) have time integrals bounded as

$$(7.44) \quad \int_{-\varepsilon}^{T_*} |b_3^{(2)}(\tilde{x}_0, t)| + |b_3^{(3)}(\tilde{x}_0, t)| dt \lesssim M\varepsilon.$$

In order to conclude the proof, we need to estimate the time integral of the first term in (7.41), namely, $b_3^{(1)}$. This is precisely the reason that Lemma 7.3 was created. First, we note that by (7.15) and (7.21), we have that $\partial_{\tilde{x}_1} \tilde{w} \circ X(t) < 0$ for all $t \in [-\varepsilon, T_*]$; that is, this term is signed. Taking into account (7.37), (7.43), and the third assumption in (7.34), we obtain the pointwise-in-time bound

$$(7.45) \quad b_3^{(1)}(\tilde{x}_0, t) \geq \frac{\beta_1}{2\gamma} \left(\frac{\alpha \kappa_0}{2} \right)^{\frac{\alpha-1}{\alpha}} \partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}_0) \partial_{\tilde{x}_1} \tilde{w} \circ X_{\tilde{x}_0}(t).$$

To conclude the proof we combine the assumption $\partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}_0) < 0$ with (7.45) and (7.11) to deduce

$$(7.46) \quad \begin{aligned} \int_{-\varepsilon}^{T_*} b_3^{(1)}(\tilde{x}_0, t) &\geq \frac{\beta_1}{2\gamma} \left(\frac{\alpha\kappa_0}{2}\right)^{\frac{\alpha-1}{\alpha}} |\partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}_0)| \frac{1}{9} \kappa_0^{-\frac{2}{3}} \varepsilon^{\frac{1}{3}} \\ &= 2c_\alpha \kappa_0^{\frac{1}{3} - \frac{1}{\alpha}} \varepsilon^{\frac{1}{3}} |\partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}_0)|, \end{aligned}$$

where $c_\alpha > 0$ is a constant that depends only on α . The point here is that the lower bound is $\mathcal{O}(\varepsilon^{1/3})$, while the error terms in both (7.36) and (7.44) are $\mathcal{O}(\varepsilon)$. Combining these estimates we deduce that

$$(7.47) \quad \zeta_3(\tilde{x}_0, t) \geq \int_{-\varepsilon}^t b_3^{(1)}(\tilde{x}_0, t') dt' - M^2 \varepsilon \geq \frac{3}{2} c_\alpha \kappa_0^{\frac{1}{3} - \frac{1}{\alpha}} \varepsilon^{\frac{1}{3}} |\partial_{\tilde{x}_2} \tilde{k}_0(\tilde{x}_0)|$$

upon taking ε to be sufficiently small. \square

8 \dot{H}^m Bounds

DEFINITION 8.1 (Modified \dot{H}^m -norm). For $m \geq 18$ we introduce the seminorm

$$(8.1) \quad \begin{aligned} E_m^2(s) &= E_m^2[U, \mathcal{P}, \mathcal{H}](s) \\ &:= \sum_{|\gamma|=m} \lambda^{|\gamma|} (\|\partial^\gamma U(\cdot, s)\|_{L^2}^2 + \|\mathcal{H} \partial^\gamma \mathcal{P}(\cdot, s)\|_{L^2}^2 + \kappa_0^2 \|\partial^\gamma \mathcal{H}(\cdot, s)\|_{L^2}^2) \end{aligned}$$

where $\lambda = \lambda(m) \in (0, 1)$ is to be made precise below (cf. Lemma 8.3).

Clearly, E_m^2 is equivalent to the homogenous Sobolev norm \dot{H}^m for U , \mathcal{P} , and \mathcal{H} , and since $\kappa_0 \geq 2$, we have the quantitative inequalities

$$(8.2) \quad \begin{aligned} \frac{\lambda^m}{2} (\|U\|_{\dot{H}^m}^2 + \|\mathcal{P}\|_{\dot{H}^m}^2 + \|\mathcal{H}\|_{\dot{H}^m}^2) \\ \leq E_m^2 \leq \kappa_0^2 (\|U\|_{\dot{H}^m}^2 + \|\mathcal{P}\|_{\dot{H}^m}^2 + \|\mathcal{H}\|_{\dot{H}^m}^2). \end{aligned}$$

The bound (8.2) follows from

$$(8.3) \quad |\mathcal{H}(y, s) - 1| \leq \frac{2\varepsilon}{\gamma}$$

and the triangle inequality, upon taking ε sufficiently small. In turn, (8.3) is a consequence of the definition (2.41) and of the bootstrap (4.14).

Additionally, in order to apply the interpolation inequalities from Appendix A.2, we need to establish a quantitative equivalence between the E_m seminorm defined in (8.1) and the classical homogenous \dot{H}^m norm of the quantities U , S , and K (recall that these are related to U , \mathcal{P} , and \mathcal{H} via the nonlinear transformation given in (2.41)). In this direction we have:

LEMMA 8.2 (Asymptotic equivalence of norms). *For $\kappa_0 \geq 1$ sufficiently large in terms of γ , and for ε sufficiently small in terms of κ_0 , M , and m , we have the estimate*

$$(8.4) \quad \begin{aligned} & \lambda^m (\|U\|_{\dot{H}^m}^2 + \|S\|_{\dot{H}^m}^2 + \|K\|_{\dot{H}^m}^2 - e^{-2s}) \\ & \leq E_m^2 \leq \kappa_0^2 (\|U\|_{\dot{H}^m}^2 + \|S\|_{\dot{H}^m}^2 + \|K\|_{\dot{H}^m}^2 + e^{-2s}) \end{aligned}$$

for all $s \geq -\log \varepsilon$. As a consequence, we also have the estimate

$$(8.5) \quad \begin{aligned} \kappa_0^{-2} E_m^2 - e^{-2s} & \leq e^{-s} \|W\|_{\dot{H}^m}^2 + \|Z\|_{\dot{H}^m}^2 + \|A\|_{\dot{H}^m}^2 + \|K\|_{\dot{H}^m}^2 \\ & \leq 4\lambda^{-m} E_m^2 + 4e^{-2s}. \end{aligned}$$

PROOF OF LEMMA 8.2. We directly have

$$(8.6) \quad \lambda^m \|U\|_{\dot{H}^m}^2 \leq \sum_{|\gamma|=m} \lambda^{|\gamma|} \|\partial^\gamma U\|_{L^2}^2 \leq \|U\|_{\dot{H}^m}^2,$$

which gives a direct comparison between the \dot{H}^m norm of U and the U -part of E_m .

Next, we turn to the \mathcal{H} -part of E_m . The chain rule yields $\mathcal{H}^{-1} \nabla \mathcal{H} = \frac{1}{2\gamma} \nabla K$. Applying $m-1$ more derivatives, by the Faà di Bruno formula, we have that there exists a constant C_m which only depends on m , such that pointwise we have the bound

$$(8.7) \quad \left| \mathcal{H}^{-1} \partial^\gamma \mathcal{H} - \frac{1}{2\gamma} \partial^\gamma K \right| \leq C_m \sum_{(i_1, \dots, i_{m-1}) \in I_m} \prod_{j=1}^{m-1} |D^{j_i} K|^{i_j}$$

where the index set I_m is given by

$$I_m = \left\{ (i_1, \dots, i_{m-1}) : i_j \geq 0, \sum_{j=1}^{m-1} j i_j = m \right\}.$$

In particular, note that whenever $(i_1, \dots, i_{m-1}) \in I_m$, we must have $\sum_{j=1}^{m-1} i_j \geq 2$. This fact is crucial for the argument below and has to do with the fact that we have already accounted on the left side for the term with the highest order of derivatives. In (8.7) as usual we have written $D^j K$ to denote $D^\beta K$ for some multi-index β with $|\beta| = j$. Using the interpolation inequality (A.3), for all $1 \leq j \leq m-1$ we next estimate

$$(8.8) \quad \| |D^j K|^{i_j} \|_{L^{\frac{2m}{j i_j}}} = \| D^j K \|_{L^{\frac{2m}{j}}}^{i_j} \lesssim \| K \|_{L^\infty}^{i_j (1 - \frac{j}{m})} \| K \|_{\dot{H}^m}^{\frac{j i_j}{m}}.$$

Moreover, note that for $(i_1, \dots, i_{m-1}) \in I_m$ we have that $\sum_{j=1}^{m-1} \frac{j i_j}{2m} = \frac{1}{2}$, so that these are Hölder conjugate exponents corresponding to an L^2 norm. Thus, applying the L^2 norm to (8.7), using the Hölder inequality and the interpolation

bound (8.8), we obtain

$$\begin{aligned}
 \|\mathcal{H}^{-1}\partial^\gamma \mathcal{H} - \frac{1}{2\gamma}\partial^\gamma K\|_{L^2} &\leq C_m \sum_{(i_1, \dots, i_{m-1}) \in I_m} \prod_{j=1}^{m-1} \|K\|_{L^\infty}^{i_j(1-\frac{j}{m})} \|K\|_{\dot{H}^m}^{\frac{j}{m}} \\
 (8.9) \qquad \qquad \qquad &\leq C_m \sum_{(i_1, \dots, i_{m-1}) \in I_m} \|K\|_{L^\infty}^{-1+\sum_{j=1}^{m-1} i_j} \|K\|_{\dot{H}^m}
 \end{aligned}$$

for some m -dependent constant C_m (which may increase from line to line), whenever $|\gamma| = m$. At this point we use that $(i_1, \dots, i_{m-1}) \in I_m$, we must have $\sum_{j=1}^{m-1} i_j \geq 2$, which is combined with the bootstrap (4.14) to conclude

$$(8.10) \qquad \qquad \qquad \|\mathcal{H}^{-1}\partial^\gamma \mathcal{H} - \frac{1}{2\gamma}\partial^\gamma K\|_{L^2} \leq C_m \varepsilon \|K\|_{\dot{H}^m}.$$

We next appeal to the pointwise estimate on \mathcal{H} in (8.3), and since $\kappa_0 \geq 1$, we deduce that

$$(8.11) \qquad \frac{\kappa_0^2}{2\gamma^2} \lambda^m \|K\|_{\dot{H}^m}^2 \leq \sum_{|\gamma|=m} \lambda^{|\gamma|} \kappa_0^2 \|\partial^\gamma \mathcal{H}\|_{L^2}^2 \leq \frac{\kappa_0^2}{\gamma^2} \|K\|_{\dot{H}^m}^2$$

where we have used that $\lambda \in (0, 1)$, and that ε is sufficiently small to absorb the C_m constant in (8.10).

Lastly, we turn to the \mathcal{P} -part of E_m . From (2.41) we obtain $S = \mathcal{P}\mathcal{H}$, and thus, by the binomial formula and the Moser estimate (A.4), we have

$$\begin{aligned}
 &\|\partial^\gamma S - \mathcal{H}\partial^\gamma \mathcal{P} - \mathcal{P}\partial^\gamma \mathcal{H}\|_{L^2} \\
 &\leq C_m (\|\nabla \mathcal{H}\|_{L^\infty} \|\mathcal{P}\|_{\dot{H}^{m-1}} + \|\nabla \mathcal{P}\|_{L^\infty} \|\mathcal{H}\|_{\dot{H}^{m-1}}).
 \end{aligned}$$

Furthermore, using the interpolation bound (A.5) applied to $\nabla \mathcal{P}$ and $\nabla \mathcal{H}$, and the ε -Young inequality, we obtain that for any $\delta \in (0, 1)$ we have

$$\begin{aligned}
 &\|\partial^\gamma S - \mathcal{H}\partial^\gamma \mathcal{P} - \mathcal{P}\partial^\gamma \mathcal{H}\|_{L^2} \\
 &\leq C_m \left(\|\nabla \mathcal{H}\|_{L^\infty} \|\nabla \mathcal{P}\|_{L^\infty}^{\frac{2}{2m-5}} \|\mathcal{P}\|_{\dot{H}^m}^{1-\frac{2}{2m-5}} \right. \\
 (8.12) \qquad &\quad \left. + \|\nabla \mathcal{P}\|_{L^\infty} \|\nabla \mathcal{H}\|_{L^\infty}^{\frac{2}{2m-5}} \|\mathcal{H}\|_{\dot{H}^m}^{1-\frac{2}{2m-5}} \right) \\
 &\leq \delta \|\mathcal{P}\|_{\dot{H}^m} + \delta \|\mathcal{H}\|_{\dot{H}^m} \\
 &\quad + C_m \delta^{-\frac{2m-7}{2}} \left(\|\nabla \mathcal{H}\|_{L^\infty}^{\frac{2m-5}{2}} \|\nabla \mathcal{P}\|_{L^\infty} + \|\nabla \mathcal{P}\|_{L^\infty}^{\frac{2m-5}{2}} \|\nabla \mathcal{H}\|_{L^\infty} \right)
 \end{aligned}$$

where the m -dependent constant C_m may change from line to line. From the definitions (2.41), the K estimates in (4.14), the W and Z bounds in (4.7) and (4.12), the relations $\mathcal{H}\nabla \mathcal{P} = \nabla S - S\mathcal{H}^{-1}\nabla \mathcal{H}$, and $2\nabla S = e^{-\frac{\delta}{2}}\nabla W - \nabla Z$, we deduce

$$(8.13) \qquad \qquad \qquad \|\nabla \mathcal{H}\|_{L^\infty} \leq \varepsilon^{\frac{1}{3}} e^{-\frac{\delta}{2}} \quad \text{and} \quad \|\nabla \mathcal{P}\|_{L^\infty} \leq \left(\frac{1}{2} + \varepsilon^{\frac{1}{4}}\right) e^{-\frac{\delta}{2}}.$$

Taking ε to be sufficiently small to absorb the m and M dependent constants, we obtain from (8.12) and (8.13) that

$$(8.14) \quad \begin{aligned} & \|\partial^\gamma S - \mathcal{H} \partial^\gamma \mathcal{P} - \mathcal{P} \partial^\gamma \mathcal{H}\|_{L^2} \\ & \leq \delta \|\mathcal{P}\|_{\dot{H}^m} + \delta \|\mathcal{H}\|_{\dot{H}^m} + \delta^{-\frac{2m-7}{2}} e^{-\frac{2m-3}{4}s} \end{aligned}$$

for any constant $\delta \in (0, 1)$. Using that $|S - \kappa_0/2| \leq 5\varepsilon^{1/6}$ (which follows from the bootstrap assumptions on $\dot{\kappa}$, W , and Z), and appealing to (8.3), we obtain

$$(8.15) \quad |\mathcal{P}(y, s) - \frac{\kappa_0}{2}| \leq 6\varepsilon^{\frac{1}{6}}$$

upon taking ε to be sufficiently small in terms of M and κ_0 . At last, we combine (8.14)–(8.15), use the \mathcal{P} and \mathcal{H} part of the comparison (8.2), choose δ sufficiently small depending on κ_0 and λ , and then ε sufficiently small in terms of κ_0 , λ , δ , and m to deduce that

$$(8.16) \quad \lambda^m \|S\|_{\dot{H}^m}^2 \leq \sum_{|\gamma|=m} \lambda^{|\gamma|} (\|\mathcal{H} \partial^\gamma \mathcal{P}\|_{L^2}^2 + \kappa_0^2 \|\partial^\gamma \mathcal{H}\|_{L^2}^2) + e^{-2s},$$

and taking $\kappa_0 \geq 2$, we also have

$$(8.17) \quad \|S\|_{\dot{H}^m}^2 \geq \frac{1}{8} \sum_{|\gamma|=m} \lambda^{|\gamma|} (\|\mathcal{H} \partial^\gamma \mathcal{P}\|_{L^2}^2 + \kappa_0^2 \|\partial^\gamma \mathcal{H}\|_{L^2}^2) - e^{-2s}.$$

Combining (8.6), (8.11), (8.16), and (8.17), we arrive at the proof of (8.4).

The proof of (8.5) follows once we recall the identities $W = e^{\frac{s}{2}}(U \cdot \mathbf{N} + S - \kappa)$, $Z = U \cdot \mathbf{N} - S$, which follow from (2.37), and the definition $A_\nu = U \cdot \mathbf{T}^\nu$. Therefore, by (9.1a), (A.3), using the Poincaré inequality in the \check{y} -direction, and the fact that the diameter of $\mathcal{X}(s)$ in the \check{y} -directions is $4\varepsilon^{1/6}e^{s/2}$, for any γ with $|\gamma| = m$, we obtain

$$\begin{aligned} & \|e^{-\frac{s}{2}} \partial^\gamma W - \mathbf{N} \cdot \partial^\gamma U - \partial^\gamma S\|_{L^2} \\ & + \|\partial^\gamma Z - \mathbf{N} \cdot \partial^\gamma U + \partial^\gamma S\|_{L^2} + \|\partial^\gamma A_\nu - \mathbf{T}^\nu \cdot \partial^\gamma U\|_{L^2} \\ & \leq 2\|[\partial^\gamma, \mathbf{N}] \cdot U\|_{L^2} + \|[\partial^\gamma, \mathbf{T}^\nu] \cdot U\|_{L^2} \\ & \lesssim \sum_{j=1}^m \left(\|D^j \mathbf{N}\|_{L^\infty} + \|D^j \mathbf{T}^\nu\|_{L^\infty} \right) \|D^{m-j} U\|_{L^2(\mathcal{X}(s))} \\ & \lesssim \varepsilon \sum_{j=1}^m e^{-\frac{js}{2}} (4\varepsilon^{\frac{1}{6}} e^{\frac{s}{2}})^j \|U\|_{\dot{H}^m} \\ & \lesssim \varepsilon \|U\|_{\dot{H}^m}. \end{aligned}$$

Summing over $|\gamma| = m$, and appealing to (8.4), the estimate (8.5) follows. \square

8.1 Higher-order derivatives for the $(U, \mathcal{P}, \mathcal{H})$ -system

In order to estimate $E_m(s)$ we need the differentiated form of the $(U, \mathcal{P}, \mathcal{H})$ -system (2.42). For this purpose, fix $\gamma \in \mathbb{N}_0^3$ with $|\gamma| = m$ and apply ∂^γ to (2.42) to obtain

$$(8.18a) \quad \begin{aligned} & \partial_s(\partial^\gamma U_i) + (\mathcal{V}_U \cdot \nabla)(\partial^\gamma U_i) + \mathcal{D}_\gamma(\partial^\gamma U_i) - 2\beta_1\beta_\tau e^{-s} \dot{Q}_{ij}(\partial^\gamma U_j) \\ & + 2\beta_\tau\beta_3 \mathcal{H}^2(\partial^\gamma \mathcal{P}) \mathbf{JN}_i e^{\frac{s}{2}} \partial_1 \mathcal{P} + 2\gamma_1\beta_\tau\beta_3 \mathcal{H}^2 e^{\frac{s}{2}} \partial_1 \mathcal{P} \mathbf{JN}_i(\partial^\gamma \mathcal{P}) \\ & + 2\beta_\tau\beta_3 \mathcal{H}^2 \mathcal{P}(\mathbf{JN}_i e^{\frac{s}{2}} \partial_1(\partial^\gamma \mathcal{P}) + e^{-\frac{s}{2}} \delta^{iv} \partial_v(\partial^\gamma \mathcal{P})) = \mathcal{F}_{U_i}^{(\gamma)}, \end{aligned}$$

$$(8.18b) \quad \begin{aligned} & \partial_s(\partial^\gamma \mathcal{P}) + (\mathcal{V}_U \cdot \nabla)(\partial^\gamma \mathcal{P}) + \mathcal{D}_\gamma(\partial^\gamma \mathcal{P}) \\ & + 2\beta_\tau\beta_3 e^{\frac{s}{2}} \mathbf{J} \partial_1(U \cdot \mathbf{N})(\partial^\gamma \mathcal{P}) + 2\gamma_1\beta_\tau\beta_3 e^{\frac{s}{2}} \partial_1 \mathcal{P} \mathbf{JN}_j(\partial^\gamma U_j) \\ & + 2\beta_\tau\beta_3 \mathcal{P}(e^{\frac{s}{2}} \mathbf{JN}_j \partial_1(\partial^\gamma U_j) + e^{-\frac{s}{2}} \partial_v(\partial^\gamma U_v)) = \mathcal{F}_{\mathcal{P}}^{(\gamma)}, \end{aligned}$$

$$(8.18c) \quad \partial_s(\partial^\gamma \mathcal{H}) + (\mathcal{V}_U \cdot \nabla)(\partial^\gamma \mathcal{H}) + \mathcal{D}_\gamma(\partial^\gamma \mathcal{H}) = \mathcal{F}_{\mathcal{H}}^{(\gamma)},$$

where the damping function \mathcal{D}_γ is defined as

$$(8.19) \quad \mathcal{D}_\gamma = \gamma_1(1 + \partial_1 g_U) + \frac{1}{2}|\gamma|,$$

the transport velocity \mathcal{V}_U is given in (2.39c), and since $|\gamma| \geq 3$ the forcing functions in (8.18) are given by

$$(8.20a) \quad \begin{aligned} \mathcal{F}_{U_i}^{(\gamma)} &= D_\gamma(\partial^\gamma U_i) - \llbracket \partial^\gamma, \mathcal{V}_U \cdot \nabla \rrbracket U_i - 2\beta_\tau\beta_3 e^{-\frac{s}{2}} \delta^{iv} \llbracket \partial^\gamma, \mathcal{H}^2 \mathcal{P} \rrbracket \partial_v \mathcal{P} \\ &+ 2\beta_\tau\beta_3 e^{\frac{s}{2}} ((\gamma_1 + 1) \mathcal{H}^2 \partial_1 \mathcal{P} \mathbf{JN}_i(\partial^\gamma \mathcal{P}) - \llbracket \partial^\gamma, \mathcal{H}^2 \mathcal{P} \mathbf{JN}_i \rrbracket \partial_1 \mathcal{P}), \end{aligned}$$

$$(8.20b) \quad \begin{aligned} \mathcal{F}_{\mathcal{P}}^{(\gamma)} &= D_\gamma(\partial^\gamma \mathcal{P}) - \llbracket \partial^\gamma, \mathcal{V}_U \cdot \nabla \rrbracket \mathcal{P} - 2\beta_\tau\beta_3 e^{-\frac{s}{2}} \llbracket \partial^\gamma, \mathcal{P} \rrbracket \partial_v U_v \\ &+ 2\beta_\tau\beta_3 e^{\frac{s}{2}} ((\partial^\gamma \mathcal{P}) \mathbf{JN}_j \partial_1 U_j + \gamma_1 \partial_1 \mathcal{P} \mathbf{JN}_i(\partial^\gamma U_i) - \llbracket \partial^\gamma, \mathcal{P} \mathbf{JN}_i \rrbracket \partial_1 U_i), \end{aligned}$$

$$(8.20c)$$

$$\mathcal{F}_{\mathcal{H}}^{(\gamma)} = D_\gamma(\partial^\gamma \mathcal{H}) - \llbracket \partial^\gamma, \mathcal{V}_U \cdot \nabla \rrbracket \mathcal{H}.$$

In (8.20) we have used the notation $\llbracket a, b \rrbracket$ to denote the commutator $ab - ba$. Note that two additional forcing terms are singled out on the left side of (8.18b); this is because these terms provide a contribution that has to be absorbed in the damping term.

The E_m energy estimate is obtained by testing (8.18a) with $\partial^\gamma U_i$, (8.18b) with $\mathcal{H}^2 \partial^\gamma \mathcal{P}$, and (8.18c) with $\kappa_0^2 \partial^\gamma \mathcal{H}$. Adding the resulting differential equations produces the cancellation of all terms involving $m + 1$ derivatives, which upon integrating by parts allows us to close the energy estimate. This computation is detailed in Section 8.3 below. Prior to this, in the next subsection we give estimates for the forcing terms defined in (8.20).

8.2 Forcing estimates

In order to analyze (8.18) we first estimate the forcing terms defined in (8.20). This is achieved next:

LEMMA 8.3. *Consider the forcing functions defined in (8.20). Let $m \geq 18$, fix $0 < \delta \leq \frac{1}{32}$, and define the parameter $\lambda = \lambda(\delta, m)$ from (8.1) to equal $\lambda = \frac{\delta^2}{16m^2}$. Then, we have that*

$$(8.21a) \quad 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{F}_{U_i}^{(\gamma)} \partial^\gamma U_i| \leq (5 + 9\delta) E_m^2 + e^{-s} M^{4m-1},$$

$$(8.21b) \quad 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{F}_{\mathcal{P}}^{(\gamma)} \mathcal{H}^2 \partial^\gamma \mathcal{P}| \leq (2 + 8\delta) E_m^2 + e^{-s} M^{4m-1},$$

$$(8.21c) \quad 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \kappa_0^2 \int_{\mathbb{R}^3} |\mathcal{F}_{\mathcal{H}}^{(\gamma)} \partial^\gamma \mathcal{H}| \leq (2 + 4\delta) E_m^2 + e^{-s} M^{4m-1},$$

for ε taken sufficiently small in terms of m , δ , λ , M , and κ_0 .

PROOF OF LEMMA 8.3. Throughout this proof, when there is no need to keep track of the binomial coefficients from the product rule we denote a partial derivative ∂^γ with $|\gamma| = m$ simply as D^m .

Upon expanding the commutator terms in (8.20), the forcing functions defined here may be written as

$$(8.22a) \quad \mathcal{F}_{U_i}^{(\gamma)} = \mathcal{F}_{U_i}^{(m)} + \mathcal{F}_{U_i}^{(<m)},$$

$$(8.22b) \quad \mathcal{F}_{\mathcal{P}}^{(\gamma)} = \mathcal{F}_{\mathcal{P}}^{(m)} + \mathcal{F}_{\mathcal{P}}^{(<m)},$$

$$(8.22c) \quad \mathcal{F}_{\mathcal{H}}^{(\gamma)} = \mathcal{F}_{\mathcal{H}}^{(m)} + \mathcal{F}_{\mathcal{H}}^{(<m)},$$

where the upper index (m) indicates that terms with exactly m derivatives are present, while the upper index $(< m)$ indicates that all terms have at most $m - 1$ derivatives on them. These terms are defined by

$$(8.23a) \quad \begin{aligned} \mathcal{F}_{U_i}^{(m)} = & -(\gamma_\mu \partial_\mu g_U \partial_1 \partial^{\gamma-e_\mu} U_i + \gamma_j \partial_j h_U^\nu \partial_\nu \partial^{\gamma-e_j} U_i + \partial^\gamma g_U \partial_1 U_i + \partial^\gamma h_U^\nu \partial_\nu U_i) \\ & - 2\beta_\tau \beta_3 (\gamma_\mu e^{\frac{s}{2}} \partial_\mu (\mathcal{H}^2 \mathcal{P} \mathbf{JN}_i) \partial^{\gamma-e_\mu} \partial_1 \mathcal{P} + 2\gamma_1 \mathcal{P} \mathbf{JN}_i \mathcal{H} e^{\frac{s}{2}} \partial_1 \mathcal{H} \partial^\gamma \mathcal{P} \\ & + e^{-\frac{s}{2}} \delta^{iv} \gamma_j \partial_j (\mathcal{H}^2 \mathcal{P}) \partial^{\gamma-e_j} \partial_\nu \mathcal{P}) \\ & - 2\beta_\tau \beta_3 (e^{-\frac{s}{2}} \delta^{iv} \partial_\nu \mathcal{P} \partial^\gamma (\mathcal{H}^2 \mathcal{P}) + e^{\frac{s}{2}} \mathcal{P} \partial_1 \mathcal{P} \partial^\gamma (\mathcal{H}^2 \mathbf{JN}_i)) \end{aligned}$$

$$(8.23a) \quad =: \mathcal{F}_{U_i,(1)}^{(m)} + \mathcal{F}_{U_i,(2)}^{(m)} + \mathcal{F}_{U_i,(3)}^{(m)}$$

$$\mathcal{F}_{U_i}^{(<m)} = - \sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} g_U \partial^\beta \partial_1 U_i + \partial^{\gamma-\beta} h_U^\nu \partial^\beta \partial_\nu U_i) -$$

$$\begin{aligned}
& -2\beta_\tau\beta_3 \sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} \left(e^{\frac{s}{2}} \partial^{\gamma-\beta} (\mathcal{H}^2 \mathcal{P} \mathbf{JN}_i) \partial^\beta \partial_1 \mathcal{P} \right. \\
& \quad \left. + e^{-\frac{s}{2}} \delta^{iv} \partial^{\gamma-\beta} (\mathcal{H}^2 \mathcal{P}) \partial^\beta \partial_v \mathcal{P} \right) \\
& -2\beta_\tau\beta_3 e^{\frac{s}{2}} \partial_1 \mathcal{P} \sum_{j=1}^{m-1} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} (\mathcal{H}^2 \mathbf{JN}_i) \partial^\beta \mathcal{P} \\
(8.23b) \quad & =: \mathcal{F}_{U_i, (1)}^{(<m)} + \mathcal{F}_{U_i, (2)}^{(<m)} + \mathcal{F}_{U_i, (3)}^{(<m)}
\end{aligned}$$

for the $\partial^\gamma U$ evolution, by

$$\begin{aligned}
\mathcal{F}_{\mathcal{P}}^{(m)} &= -(\gamma_\mu \partial_\mu g_U \partial_1 \partial^{\gamma-e_\mu} \mathcal{P} + \gamma_j \partial_j h_U^\nu \partial_v \partial^{\gamma-e_j} \mathcal{P} + \partial^\gamma g_U \partial_1 \mathcal{P} + \partial^\gamma h_U^\nu \partial_v \mathcal{P}) \\
& -2\beta_\tau\beta_3 (\gamma_\mu e^{\frac{s}{2}} \partial_\mu (\mathcal{P} \mathbf{JN}_i) \partial^{\gamma-e_\mu} \partial_1 U_i + e^{-\frac{s}{2}} \partial_v U_v \partial^\gamma \mathcal{P} \\
& \quad + e^{-\frac{s}{2}} \gamma_j \partial_j \mathcal{P} \partial^{\gamma-e_j} \partial_v U_v) \\
(8.24) \quad & =: \mathcal{F}_{\mathcal{P}, (1)}^{(m)} + \mathcal{F}_{\mathcal{P}, (2)}^{(m)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{\mathcal{P}}^{(<m)} &= -\sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} g_U \partial^\beta \partial_1 \mathcal{P} + \partial^{\gamma-\beta} h_U^\nu \partial^\beta \partial_v \mathcal{P}) \\
& -2\beta_\tau\beta_3 \sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} \left(e^{\frac{s}{2}} \partial^{\gamma-\beta} (\mathcal{P} \mathbf{JN}_i) \partial^\beta \partial_1 U_i \right. \\
& \quad \left. + e^{-\frac{s}{2}} \partial^{\gamma-\beta} \mathcal{P} \partial^\beta \partial_v U_v \right) \\
& -2\beta_\tau\beta_3 \partial_1 U_i \gamma_\mu (\mathbf{JN}_i)_{,\mu} \partial^{\gamma-e_\mu} \mathcal{P}
\end{aligned}$$

$$\begin{aligned}
(8.25) \quad & =: \mathcal{F}_{\mathcal{P}, (1)}^{(<m)} + \mathcal{F}_{\mathcal{P}, (2)}^{(<m)} + \mathcal{F}_{\mathcal{P}, (3)}^{(<m)}
\end{aligned}$$

for the $\partial^\gamma \mathcal{P}$ equation, and by

$$\begin{aligned}
\mathcal{F}_{\mathcal{H}}^{(m)} &= -(\gamma_\mu \partial_\mu g_U \partial_1 \partial^{\gamma-e_\mu} \mathcal{H} + \gamma_j \partial_j h_U^\nu \partial_v \partial^{\gamma-e_j} \mathcal{H} + \partial^\gamma g_U \partial_1 \mathcal{H} \\
(8.26a) \quad & \quad + \partial^\gamma h_U^\nu \partial_v \mathcal{H})
\end{aligned}$$

$$\begin{aligned}
(8.26b) \quad \mathcal{F}_{\mathcal{H}}^{(<m)} &= -\sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} (\partial^{\gamma-\beta} g_U \partial^\beta \partial_1 \mathcal{H} + \partial^{\gamma-\beta} h_U^\nu \partial^\beta \partial_v \mathcal{H})
\end{aligned}$$

for the $\partial^\gamma \mathcal{H}$ equation.

Proof of (8.21a). We shall first prove (8.21a), and to do so, we estimate separately the terms in the sum (8.22a). Let us treat the term which contains the

highest-order derivatives, namely, $\mathcal{F}_{U_i}^{(m)}$. This term is decomposed in three pieces, cf. (8.23a), and we estimate each piece separately.

Recall that g_U and h_U^ν are defined in (2.33c) and (2.34c) and that

$$(8.27) \quad U_i = U \cdot \mathbf{N} \mathbf{N}_i + A_\nu \mathbf{T}_i^\nu = \frac{1}{2}(e^{-\frac{s}{2}} W + \kappa + Z) \mathbf{N}_i + A_\nu \mathbf{T}_i^\nu.$$

Also, note that f and V are quadratic functions of \check{y} , whereas $\mathbf{J} \mathbf{N}$ is an affine function of \check{y} ; therefore ∂^γ annihilates these terms and we have¹⁰

$$(8.28) \quad \frac{1}{\beta_1 \beta_\tau} \partial^\gamma g_U = 2e^{\frac{s}{2}} \partial^\gamma (\mathbf{J} \mathbf{N} \cdot U) = 2e^{\frac{s}{2}} \mathbf{J} \mathbf{N} \cdot \partial^\gamma U + 2\gamma_\mu (\mathbf{J} \mathbf{N}_\rho)_\mu \partial^{\gamma-e_\mu} U_\rho,$$

$$(8.29) \quad \frac{1}{\beta_1 \beta_\tau} \partial^\gamma h_U^\mu = 2e^{-\frac{s}{2}} \partial^\gamma U_\mu.$$

In view of these definitions, using that $\lambda \leq 1$, $\beta_\tau \beta_1 \leq 1$, and that $\partial_1 \partial^{\gamma-e_\mu} U_i$ produces a favorable imbalance of $\lambda^{1/2}$, for the first term in (8.23a) we have that

$$(8.30) \quad \begin{aligned} & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (1)}^{(m)} \partial^\gamma U_i| \\ & \leq 2E_m^2 (m\lambda^{\frac{1}{2}} \|\check{\nabla} g_U\|_{L^\infty} + m \|\nabla h_U\|_{L^\infty} + 2e^{\frac{s}{2}} \|\mathbf{J}\| \|\partial_1 U\|_{L^\infty} + 2e^{-\frac{s}{2}} \|\check{\nabla} U\|_{L^\infty}) \\ & \quad + 4m E_m \|(\mathbf{J} \mathbf{N})_\mu\|_{L^\infty} \|\partial_1 U\|_{L^\infty} \|U\|_{\dot{H}^{m-1}}. \end{aligned}$$

Estimate (8.30) is the perfect example of the usage of the parameter λ appearing in the definition of the energy E_m : it yields a factor of $\lambda^{1/2}$ next to the term $m \|\check{\nabla} g_U\|_{L^\infty} \approx m$ in the first term of (8.30). Without this factor, the resulting coefficient of E_m^2 appearing on the right side of (8.21a) would be larger than $2m$, which would not allow us to close the energy estimate. But by choosing $\lambda = \frac{\delta^2}{12m^2}$, we have that $2m\lambda^{1/2} < \delta$. Using the definitions of g_U , h_U^ν , and U , the bounds (4.3), (4.5), (4.7), (4.11), (4.12), (4.13), (9.1a), (5.12), the norm equivalence (8.2), and the interpolation inequality (A.5) applied to ∇U , we estimate

$$\begin{aligned} \|\check{\nabla} g_U\|_{L^\infty} & \leq \|\check{\nabla} (\mathbf{J} W)\|_{L^\infty} + \|\check{\nabla} G_U\|_{L^\infty} \leq 1 + \varepsilon^{\frac{1}{4}}, \\ \|\nabla h_U\|_{L^\infty} & \leq \varepsilon^{\frac{1}{4}}, \\ \|\mathbf{J} \partial_1 U\|_{L^\infty} & \leq \frac{1}{2} e^{-\frac{s}{2}} \|\mathbf{J} \partial_1 W\|_{L^\infty} + \|\partial_1 Z\|_{L^\infty} + 2\|\partial_1 A\|_{L^\infty} \leq \frac{1}{2} (1 + \varepsilon^{\frac{1}{4}}) e^{-\frac{s}{2}}, \\ \|\check{\nabla} U\|_{L^\infty} & \leq \varepsilon^{\frac{1}{4}}, \\ \|(\mathbf{J} \mathbf{N})_\mu\|_{L^\infty} & \leq \varepsilon^{\frac{1}{4}}, \\ \|\partial_1 U\|_{L^\infty} \|U\|_{\dot{H}^{m-1}} & \leq C_m \|\partial_1 U\|_{L^\infty}^{\frac{2m-3}{2m-5}} \|U\|_{\dot{H}^m}^{\frac{2m-7}{2m-5}} \\ & \leq \|U\|_{\dot{H}^m} + C'_m \|\partial_1 U\|_{L^\infty}^{\frac{2m-3}{2}} \leq 2\lambda^{-\frac{m}{2}} E_m + e^{-s} \end{aligned}$$

¹⁰ Note that (8.28) holds whenever $|\gamma| \geq 4$. This is because $g_U = 2\beta_1 \beta_\tau e^{\frac{s}{2}} (U \cdot \mathbf{N} \mathbf{J} + V \cdot \mathbf{N} \mathbf{J} - \dot{f})$, with $V \cdot \mathbf{N} \mathbf{J}$ being a cubic polynomial in y and \dot{f} a quadratic polynomial in \check{y} .

for an arbitrary $\delta \in (0, 1)$, upon choosing ε to be sufficiently small to absorb the stray powers of M and all implicit, δ -dependent, and m -dependent constants. Combining the above estimates with (8.30), we obtain

$$\begin{aligned}
 (8.31) \quad & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (1)}^{(m)} \partial^\gamma U_i| \\
 & \leq 2E_m^2 \left(\frac{\delta}{4} (1 + \varepsilon^{\frac{1}{4}}) + m\varepsilon^{\frac{1}{4}} + 1 + \varepsilon^{\frac{1}{4}} + 2\varepsilon^{\frac{3}{4}} \right) \\
 & \quad + 4mE_m \varepsilon^{\frac{1}{4}} (2\lambda^{-\frac{m}{2}} E_m + e^{-s}) \\
 & \leq (2 + \delta) E_m^2 + e^{-2s}.
 \end{aligned}$$

Quite similarly, using that $\lambda \leq 1$, that $\beta_\tau \beta_3 \leq 1$, and that $\partial_1 \partial^\gamma e^{-\mu} \mathcal{P}$ produces a favorable imbalance of $\lambda^{1/2}$, for the second term in (8.23a), we have

$$\begin{aligned}
 (8.32) \quad & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (2)}^{(m)} \partial^\gamma U_i| \\
 & \leq 4E_m^2 (m\lambda^{\frac{1}{2}} e^{\frac{s}{2}} \|\mathcal{H}^{-1} \check{\nabla}(\mathcal{H}^2 \mathcal{P} \mathbf{JN})\|_{L^\infty} \\
 & \quad + 2me^{\frac{s}{2}} \|\mathcal{P} \mathbf{JN} \partial_1 \mathcal{H}\|_{L^\infty} + me^{-\frac{s}{2}} \|\mathcal{H}^{-1} \nabla(\mathcal{H}^2 \mathcal{P})\|_{L^\infty}).
 \end{aligned}$$

Using the estimates (9.1a), (8.3), (8.13), and (8.15) we obtain that

$$\begin{aligned}
 \|\mathcal{H}^{-1} \check{\nabla}(\mathcal{H}^2 \mathcal{P} \mathbf{JN})\|_{L^\infty} & \leq \left(\frac{1}{2} + \varepsilon^{\frac{1}{6}}\right) e^{-\frac{s}{2}}, \\
 \|\mathcal{P} \mathbf{JN} \partial_1 \mathcal{H}\|_{L^\infty} & \leq \varepsilon^{\frac{1}{4}} e^{-\frac{s}{2}}, \\
 \|\mathcal{H}^{-1} \nabla(\mathcal{H}^2 \mathcal{P})\|_{L^\infty} & \leq \left(\frac{1}{2} + \varepsilon^{\frac{1}{6}}\right) e^{-\frac{s}{2}}.
 \end{aligned}$$

Using the above estimates, and recalling our choice of $\lambda = \frac{\delta^2}{16m^2}$, the bound (8.32) becomes

$$\begin{aligned}
 (8.33) \quad & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (2)}^{(m)} \partial^\gamma U_i| \\
 & \leq 4E_m^2 \left(\frac{\delta}{4} \left(\frac{1}{2} + \varepsilon^{\frac{1}{6}}\right) + 2m\varepsilon^{\frac{1}{4}} + m\varepsilon \left(\frac{1}{2} + \varepsilon^{\frac{1}{6}}\right) \right) \\
 & \leq \delta E_m^2,
 \end{aligned}$$

upon taking ε to be sufficiently small. Lastly, for the third term in (8.23a), we similarly have

$$\begin{aligned}
 (8.34) \quad & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (3)}^{(m)} \partial^\gamma U_i| \\
 & \leq 4E_m e^{-\frac{s}{2}} \|\check{\nabla} \mathcal{P}\|_{L^\infty} \|\mathcal{H}^2 \mathcal{P}\|_{\dot{H}^m} \\
 & \quad + 4e^{\frac{s}{2}} \|\mathcal{P}\|_{L^\infty} \|\partial_1 \mathcal{P}\|_{L^\infty} \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\partial^\gamma (\mathcal{H}^2 \mathbf{JN}) \cdot \partial^\gamma U|.
 \end{aligned}$$

For the second term in (8.34) we recall that \mathbf{JN} is an affine function, and thus $D^2(\mathbf{JN}) = 0$. From the Leibniz rule, the Moser inequality (A.4), the estimates (4.1a), (9.1a), (8.3), (8.13), the interpolation bound (A.5), and the norm comparison (8.2), we moreover have that

$$\begin{aligned}
 & \|\partial^\gamma (\mathcal{H}^2 \mathbf{JN}) - 2\mathcal{H} \mathbf{JN} \partial^\gamma \mathcal{H}\|_{L^2} \\
 & \leq e^{-\frac{s}{2}} \|(\mathbf{JN})_{,\mu}\|_{L^\infty} \gamma_\mu \|\partial^{\gamma-e\mu} (\mathcal{H}^2)\|_{L^2} + C_m \|\mathbf{JN}\|_{L^\infty} \sum_{j=1}^{m-1} \|D^j \mathcal{H} D^{m-j} \mathcal{H}\|_{L^2} \\
 (8.35) \quad & \leq C_m \varepsilon^{\frac{1}{2}} e^{-\frac{s}{2}} \|\mathcal{H}\|_{L^\infty} \|\mathcal{H}\|_{\dot{H}^{m-1}} + C_m \|\nabla \mathcal{H}\|_{L^\infty} \|\mathcal{H}\|_{\dot{H}^{m-1}} \\
 & \leq C_m \varepsilon^{\frac{1}{2}} e^{-\frac{s}{2}} \|\mathcal{H}\|_{L^\infty} \|\nabla \mathcal{H}\|_{L^\infty}^{\frac{2}{2m-5}} \|\mathcal{H}\|_{\dot{H}^m}^{\frac{2m-7}{2m-5}} + C_m \|\nabla \mathcal{H}\|_{L^\infty}^{\frac{2m-3}{2m-5}} \|\mathcal{H}\|_{\dot{H}^m}^{\frac{2m-7}{2m-5}} \\
 & \leq C_m (\varepsilon^{\frac{1}{3}} e^{-\frac{s}{2}})^{\frac{2m-3}{2m-5}} (\lambda^{-\frac{m}{2}} E_m)^{\frac{2m-7}{2m-5}} \\
 & \leq \varepsilon^{\frac{1}{3}} e^{-\frac{s}{2}} E_m + \varepsilon^{\frac{1}{2}} e^{-s}
 \end{aligned}$$

by taking ε to be sufficiently small in terms of m and λ . From (9.1a), (8.3), (8.35), the definition of the E_m norm in (8.1), and the Cauchy–Bunyakovsky inequality, we deduce that

$$(8.36) \quad \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\partial^\gamma (\mathcal{H}^2 \mathbf{JN}) \cdot \partial^\gamma U| \leq 2\varepsilon^{\frac{1}{3}} e^{-\frac{s}{2}} E_m^2 + \varepsilon e^{-s} + 3\kappa_0^{-1} E_m^2.$$

The above estimate is combined with the bound

$$\|\mathcal{P}\|_{L^\infty} \|\partial_1 \mathcal{P}\|_{L^\infty} \leq \left(\frac{\kappa_0}{4} + \varepsilon^{\frac{1}{8}}\right) e^{-\frac{s}{2}},$$

which follows from (8.13) and (8.15), and with the estimate

$$\begin{aligned}
 \|\check{\nabla} \mathcal{P}\|_{L^\infty} \|\mathcal{H}^2 \mathcal{P}\|_{\dot{H}^m} & \leq C_m e^{-\frac{s}{2}} (\|\mathcal{P}\|_{\dot{H}^m} + \kappa_0 \|\mathcal{H}\|_{\dot{H}^m}) \\
 & \leq C_m \kappa_0 e^{-\frac{s}{2}} \lambda^{-\frac{m}{2}} E_m,
 \end{aligned}$$

which follows from the fact that $\kappa_0 \geq 1$, the Moser inequality, (8.2), (8.3), (8.13), and (8.15) to imply that the right side of (8.34) is further estimated as

$$\begin{aligned}
 & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (3)}^{(m)} \partial^\gamma U_i| \\
 & \leq C_m \kappa_0 \lambda^{-\frac{m}{2}} E_m^2 e^{-s} + 4 \left(\frac{\kappa_0}{4} + \varepsilon^{\frac{1}{8}}\right) \left(2\varepsilon^{\frac{1}{3}} e^{-\frac{s}{2}} E_m^2 + \varepsilon e^{-s} + 3\kappa_0^{-1} E_m^2\right) \\
 (8.37) \quad & \leq (3 + \delta) E_m^2 + \varepsilon^{\frac{1}{2}} e^{-s},
 \end{aligned}$$

after taking ε to be sufficiently small, in terms of δ, κ_0 , and m .

The bounds (8.31), (8.33), and (8.37) provide the needed estimate for the contribution of the $\mathcal{F}_{U_i}^{(m)}$ term in (8.22a) to (8.21a). It remains to bound the contribution from the lower order term $\mathcal{F}_{U_i}^{(<m)}$, which we recall is decomposed in three pieces, according to (8.23b). Next, we estimate these three contributions.

The difficulty in addressing the $\mathcal{F}_{U_i, (1)}^{(<m)}$ term defined in (8.23b) arises due to the fact that the bootstrap assumption for A in (4.13) does not include bounds on the full Hessian $\nabla^2 A$. Therefore, we need to split off the A_ν (i.e., $U \cdot \mathbb{T}^\nu$) contributions from the W and Z contributions (i.e., $U \cdot \mathbb{N}$) to this term. Using (8.27) we write the first term in (8.23b) as

$$(8.38) \quad \mathcal{F}_{U_i, (1)}^{(<m)} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

where

$$\begin{aligned} \mathcal{I}_1 &= - \sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} g_U \partial^\beta \partial_1 (U \cdot \mathbb{N} \mathbb{N}_i), \\ \mathcal{I}_2 &= - \sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma-\beta} g_U \partial^\beta (\partial_1 A_\nu \mathbb{T}_i^\nu), \\ \mathcal{I}_3 &= - \sum_{j=1}^{m-2} \sum_{|\beta|=j, \beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\alpha-\beta} h_U^\nu \partial^\beta \partial_\nu U_i. \end{aligned}$$

We estimate the contributions of the three terms in (8.38) individually.

First, for the \mathcal{I}_1 term in (8.38), by Lemma A.3 with $q = \frac{6(2m-3)}{2m-1}$, we have that

$$(8.39) \quad \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{I}_1 \partial^\gamma U_i| \lesssim \|D^m g_U\|_{L^2}^a \|D^m U\|_{L^2}^b \|D^2 g_U\|_{L^q}^{1-a} \|D^2(U \cdot \mathbb{N} \mathbb{N})\|_{L^q}^{1-b} \|D^m U\|_{L^2},$$

where a and b obey $a + b = 1 - \frac{1}{2m-4}$. Note by (2.33c) that g_U does not include any A term. Thus, using the bootstrap bounds (4.1)–(4.12), or alternatively by appealing directly to (4.7), (9.1a), and the last bound in (9.4), and the definition of $\mathcal{X}(s)$ in (4.4), we deduce that

$$(8.40) \quad \|D^2 g_U\|_{L^q(\mathcal{X}(s))} \lesssim M \|\eta^{-\frac{1}{6}}\|_{L^q(\mathcal{X}(s))} + M^2 e^{-\frac{s}{2}} |\mathcal{X}(s)|^{\frac{1}{q}} \lesssim M$$

since $q \in [\frac{11}{2}, 6)$ for $m \geq 18$. Similarly, from the first four bounds in (4.24) (bounds which do not rely on any A estimates) and from (9.1a) (which only uses (4.1a) and (4.5)), we deduce that

$$(8.41) \quad \|D^2((U \cdot \mathbb{N})\mathbb{N})\|_{L^q(\mathcal{X}(s))} \lesssim M e^{-\frac{s}{2}} \|\eta^{-\frac{1}{6}}\|_{L^q(\mathcal{X}(s))} + M e^{-s} |\mathcal{X}(s)|^{\frac{1}{q}} \lesssim M e^{-\frac{s}{2}}.$$

Moreover, from (8.28), the bounds listed above (8.31), the Poincaré inequality in the \check{y} -direction, and the fact that the diameter of $\mathcal{X}(s)$ in the e_μ -directions is $4\varepsilon^{\frac{1}{6}} e^{\frac{s}{2}}$ we have that

$$(8.42) \quad \|D^m g_U\|_{L^2} \lesssim e^{\frac{s}{2}} \|U\|_{\dot{H}^m} + \varepsilon^{\frac{1}{4}} \|U\|_{\dot{H}^{m-1}} \lesssim e^{\frac{s}{2}} \|U\|_{\dot{H}^m}.$$

By combining (8.40)–(8.42) we obtain that the right side of (8.39) is bounded from above as

$$\begin{aligned} & \|D^m g_U\|_{L^2}^a \|D^m U\|_{L^2}^b \|D^2 g_U\|_{L^q}^{1-a} \|D^2(U \cdot \mathbf{N})\|_{L^q}^{1-b} \|D^m U\|_{L^2} \\ & \lesssim (e^{\frac{s}{2}} \|U\|_{\dot{H}^m})^a \|U\|_{\dot{H}^m}^b M^{1-a} (Me^{-\frac{s}{2}})^{1-b} \|U\|_{\dot{H}^m} \\ & \lesssim M^{2-a-b} e^{\frac{(a+b-1)s}{2}} \|U\|_{\dot{H}^m}^{1+a+b}. \end{aligned}$$

Recalling from Lemma A.3 that $1 - a - b = \frac{1}{2m-4} \in (0, 1)$, using the norm equivalence (8.2), by Young's inequality with a small parameter $\delta > 0$, we have that the left side of (8.39) is bounded as

$$\begin{aligned} 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{I}_1 \partial^\gamma U_i| & \leq C_m M^{2-a-b} e^{\frac{(a+b-1)s}{2}} \lambda^{\frac{-m(1+a+b)}{2}} E_m^{1+a+b} \\ & \leq \delta E_m^2 + e^{-s} M^{4m-3}. \end{aligned}$$

In the last inequality we have used that by definition $\lambda = \lambda(m, \delta)$, $\delta \in (0, \frac{1}{32}]$ is a fixed universal constant, and C_m is a constant that only depends on m ; thus, we may use a power of M (which is taken to be sufficiently large) to absorb all the m and δ dependent constants.

Next, we estimate the \mathcal{I}_2 term in (8.38). First, we note that by (A.3) we have

$$\begin{aligned} \|\mathcal{I}_2\|_{L^2} & \lesssim \sum_{j=1}^{m-2} \|D^{m-1-j} D g_U\|_{L^{\frac{2(m-1)}{m-1-j}}} \|D^j (\partial_1 A_v \mathbf{T}^v)\|_{L^{\frac{2(m-1)}{j}}} \\ & \lesssim \sum_{j=1}^{m-2} \|g_U\|_{\dot{H}^m}^{\frac{m-1-j}{m-1}} \|D g_U\|_{L^\infty}^{\frac{j}{m-1}} \|\partial_1 A_v \mathbf{T}^v\|_{\dot{H}^{m-1}}^{\frac{j}{m-1}} \|\partial_1 A_v \mathbf{T}^v\|_{L^\infty}^{\frac{m-1-j}{m-1}}. \end{aligned}$$

Then, by appealing to (2.33c), (4.7), (4.13), (9.1a), (9.4), (8.2), (8.42), and (A.4), we deduce

$$\begin{aligned} \|\mathcal{I}_2\|_{L^2} & \lesssim \sum_{j=1}^{m-2} (e^{\frac{s}{2}} \|U\|_{\dot{H}^m})^{\frac{m-1-j}{m-1}} (\|A\|_{\dot{H}^m} + M \varepsilon e^{-\frac{m+2}{2}s})^{\frac{j}{m-1}} (Me^{-\frac{3s}{2}})^{\frac{m-1-j}{m-1}} \\ & \lesssim \sum_{j=1}^{m-2} (\lambda^{-\frac{k}{2}} E_m)^{\frac{m-1-j}{m-1}} (\lambda^{-\frac{k}{2}} E_m + M \varepsilon e^{-\frac{m+2}{2}s})^{\frac{j}{m-1}} (Me^{-s})^{\frac{m-1-j}{m-1}} \\ & \lesssim (M \varepsilon)^{\frac{1}{m-1}} \lambda^{-\frac{m}{2}} E_m + Me^{-s} \end{aligned}$$

since $\|D g_U\|_{L^\infty} \lesssim 1$. By taking ε sufficiently small, in terms of M , $\lambda = \lambda(m, \delta)$, δ , and m , we obtain from the above estimate that

$$(8.43) \quad 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{I}_2 \partial^\gamma U_i| \leq \delta E_m^2 + e^{-s}$$

for all $s \geq -\log \varepsilon$.

At last, we estimate the \mathcal{J}_3 term in (8.38), which is estimated similarly to the \mathcal{J}_2 term as

$$\|\mathcal{J}_3\|_{L^2} \lesssim \sum_{j=1}^{m-2} \|h_U\|_{\dot{H}^m}^{\frac{m-1-j}{m-1}} \|Dh_U\|_{L^\infty}^{\frac{j}{m-1}} \|\partial_\nu U_i\|_{\dot{H}^{m-1}}^{\frac{j}{m-1}} \|\partial_\nu U_i\|_{L^\infty}^{\frac{m-1-j}{m-1}}.$$

From (8.29), the bounds (4.7), (4.12), (4.13), (9.1a), and the Moser inequality (A.4), we have

$$\begin{aligned} \|h_U\|_{\dot{H}^m} &\lesssim e^{-\frac{s}{2}} \|\mathbf{N}U \cdot \mathbf{N}\|_{\dot{H}^m} + \kappa e^{-\frac{s}{2}} \|A_\gamma \mathbf{T}^\gamma\|_{\dot{H}^m}, \\ &\lesssim M e^{-\frac{s}{2}} \|U\|_{\dot{H}^m} + M \varepsilon e^{-\frac{m+1}{2}s}. \end{aligned}$$

On the other hand, by (9.5) we have $\|Dh_U\|_{L^\infty} \lesssim e^{-s}$, while from (4.7), (4.12), (4.13), and (8.27) we obtain $\|\check{\nabla} U\|_{L^\infty} \lesssim e^{-s/2}$. Combining the above three estimates, we deduce that

$$\begin{aligned} \|\mathcal{J}_3\|_{L^2} &\lesssim \sum_{j=1}^{m-2} \left(M e^{-\frac{s}{2}} \|U\|_{\dot{H}^m} + e^{-2s} \right)^{\frac{m-1-j}{m-1}} e^{-\frac{j}{m-1}s} \|U\|_{\dot{H}^m}^{\frac{j}{m-1}} e^{-\frac{m-1-j}{2(m-1)}s} \\ &\lesssim M e^{-s} \|U\|_{\dot{H}^m} + e^{-s} \end{aligned}$$

from which we deduce

$$(8.44) \quad 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{J}_3 \partial^\gamma U_i| \leq \varepsilon^{\frac{1}{2}} E_m^2 + e^{-s}$$

upon taking M to be sufficiently large in terms of m , and ε sufficiently large in terms of M . Combining (8.43), (8.43), and (8.44), we have thus shown that

$$(8.45) \quad 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |F_{U_i, (1)}^{(<m)} \partial^\gamma U_i| \leq (2\delta + \varepsilon^{\frac{1}{2}}) E_m^2 + M^{4m-2} e^{-s}.$$

We next turn to the second term in (8.23b), which involves only derivatives of \mathcal{P} , \mathcal{H} , and \mathbf{JN} . For the first term (the one with an $e^{s/2}$ prefactor) we apply the same bound as in (8.39), while for the second term we use (A.3) to obtain

$$\begin{aligned} &2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{F}_{U_i, (2)}^{(<m)} \partial^\gamma U_i| \\ &\lesssim (e^{\frac{s}{2}} \|D^m(\mathcal{H}^2 \mathcal{P} \mathbf{JN})\|_{L^2})^a \|D^m \mathcal{P}\|_{L^2}^b \\ &\quad \times (e^{\frac{s}{2}} \|D^2(\mathcal{H}^2 \mathcal{P} \mathbf{JN})\|_{L^q})^{1-a} \|D^2 \mathcal{P}\|_{L^q}^{1-b} \|D^m U\|_{L^2} \\ &\quad + e^{-\frac{s}{2}} \sum_{j=1}^{m-2} \|\mathcal{H}^2 \mathcal{P}\|_{\dot{H}^m}^{\frac{m-1-j}{m-1}} \|D(\mathcal{H}^2 \mathcal{P})\|_{L^\infty}^{\frac{j}{m-1}} \|\mathcal{P}\|_{\dot{H}^m}^{\frac{j}{m-1}} \|\check{\nabla} \mathcal{P}\|_{L^\infty}^{\frac{m-1-j}{m-1}} \\ &\quad \|D^m U\|_{L^2} \end{aligned}$$

(8.46)

$$=: \mathcal{T}_1 + \mathcal{T}_2,$$

with $q = \frac{6(2m-3)}{2m-1}$, and $a + b = 1 - \frac{1}{2m-4}$. Recalling that $\mathcal{P} = S\mathcal{H}^{-1} = (U \cdot N - Z)\mathcal{H}^{-1}$, the definition of \mathcal{H} , our bootstrap assumptions on Z and K , exactly as in (8.41) we have the estimate

$$\begin{aligned} \|D^2 \mathcal{P}\|_{L^q(\mathcal{X}(s))} &\lesssim \|D^2(U \cdot N)\|_{L^q(\mathcal{X}(s))} \\ &\quad + (\|D^2 Z\|_{L^\infty} + \|D\mathcal{H}\|_{L^\infty} \|D(U \cdot N - Z)\|_{L^\infty}) |\mathcal{X}(s)|^{\frac{1}{q}} \\ &\quad + \|U \cdot N - Z\|_{L^\infty} (\|D^2 \mathcal{H}\|_{L^\infty} + \|D\mathcal{H}\|_{L^\infty}^2) |\mathcal{X}(s)|^{\frac{1}{q}} \\ &\lesssim M e^{-\frac{s}{2}}. \end{aligned}$$

Thus, the Hessian of \mathcal{P} obeys the same estimate as the Hessian of $(U \cdot N)N$ in (8.41). Similarly, by using (9.1a), (8.3), (8.13), and (8.15), as in (8.40) and (8.41), we have

$$\begin{aligned} e^{\frac{s}{2}} \|D^2(\mathcal{H}^2 \mathcal{P} JN)\|_{L^q(\mathcal{X}(s))} \\ \lesssim e^{\frac{s}{2}} \|D^2(\mathcal{H}^2 \mathcal{P})\|_{L^q(\mathcal{X}(s))} + \|D(\mathcal{H}^2 \mathcal{P})\|_{L^\infty} |\mathcal{X}(s)|^{\frac{1}{q}} \lesssim M. \end{aligned}$$

The above estimate is exactly the same as the Hessian of g_U bound in (8.40). Clearly we have that $\|\mathcal{P}\|_{\dot{H}^m} \lesssim \lambda^{-\frac{m}{2}} E_m$, and additionally, from the Moser inequality (9.1a), (8.3), (8.13), and (8.15) we have that

$$e^{\frac{s}{2}} \|\mathcal{H}^2 \mathcal{P} JN\|_{\dot{H}^m} \lesssim e^{\frac{s}{2}} (\kappa_0 \|\mathcal{H}\|_{\dot{H}^m} + \|\mathcal{P}\|_{\dot{H}^m}) \lesssim e^{\frac{s}{2}} \lambda^{-\frac{m}{2}} E_m,$$

which is the same as the bound on the \dot{H}^m norm of g_U obtained in (8.42). In view of these analogies, proceeding in exactly the same way as in (8.43), we obtain that the first term in (8.46) is estimated as

$$(8.47) \quad \mathcal{T}_1 \leq \delta E_m^2 + e^{-s} M^{4m-3}.$$

For the second term in (8.43) we recall that by the Moser inequality, (8.3), and (8.15) we have $\|\mathcal{H}^2 \mathcal{P}\|_{\dot{H}^m} \lesssim \|\mathcal{P}\|_{\dot{H}^m} + \kappa_0 \|\mathcal{H}\|_{\dot{H}^m} \lesssim \lambda^{-\frac{m}{2}} E_m$, and by also appealing to (8.13) we obtain

$$\begin{aligned} (8.48) \quad \mathcal{T}_2 &\lesssim \lambda^{-m} e^{-\frac{s}{2}} E_m^2 \sum_{j=1}^{m-2} \|D(\mathcal{H}^2 \mathcal{P})\|_{L^\infty}^{\frac{j}{m-1}} \|\check{\nabla} \mathcal{P}\|_{L^\infty}^{\frac{m-1-j}{m-1}} \\ &\lesssim \lambda^{-m} e^{-s} E_m^2 \leq \delta E_m^2 \end{aligned}$$

after taking ε to be sufficiently small to absorb the m , λ , and δ -dependent constants. By combining (8.46), (8.47), and (8.48), we obtain that

$$(8.49) \quad 2 \sum_{|\gamma|=m} \lambda^{|\check{\gamma}|} \int_{\mathbb{R}^3} |\mathcal{F}_{U_i, (2)}^{(<m)} \partial^\gamma U_i| \leq 2\delta E_m^2 + e^{-s} M^{4m-3}.$$

At last, we consider the third term in (8.23b). Recall that from (8.13) that $e^{\frac{s}{2}} \|\partial_1 \mathcal{P}\|_{L^\infty} \leq 1$, and that since JN is linear in \check{y} , by the Poincaré inequality

in the \check{y} -direction and the fact that the diameter of $\mathcal{X}(s)$ in the \check{e} -directions is $4\varepsilon^{\frac{1}{6}}e^{\frac{s}{2}}$, we obtain that $\|\mathcal{H}^2 \mathbf{JN}\|_{\dot{H}^m} \lesssim \|\mathcal{H}\|_{\dot{H}^m}$. Thus, by appealing to (9.1a), (8.2), (8.13), (A.3), and the Poincaré inequality in the \check{y} -direction we arrive at

$$\begin{aligned}
 & 2 \sum_{|\gamma|=m} \lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{F}_{U_i,(3)}^{(<m)} \partial^\gamma U_i| \\
 & \lesssim E_m \sum_{j=1}^{m-1} \|D(\mathcal{H}^2 \mathbf{JN})\|_{L^\infty}^{\frac{j}{m-1}} \|\mathcal{H}^2 \mathbf{JN}\|_{\dot{H}^m}^{\frac{m-j-1}{m-1}} \|\mathcal{P}\|_{L^\infty}^{1-\frac{j}{m-1}} \|\mathcal{P}\|_{\dot{H}^{m-1}}^{\frac{j}{m-1}} \\
 & \lesssim E_m \sum_{j=1}^{m-1} (\varepsilon e^{-\frac{s}{2}})^{\frac{j}{m-1}} \|\mathcal{H}\|_{\dot{H}^m}^{\frac{m-j-1}{m-1}} \kappa_0^{1-\frac{j}{m-1}} (\varepsilon^{\frac{1}{6}} e^{\frac{s}{2}} \|\mathcal{P}\|_{\dot{H}^m})^{\frac{j}{m-1}} \\
 & \leq \delta E_m^2
 \end{aligned} \tag{8.50}$$

upon taking ε to be sufficiently small, in terms of λ , m , and κ_0 .

The bounds (8.45), (8.49), and (8.50) provide the needed estimate for the contribution of the $\mathcal{F}_{U_i}^{(<m)}$ term in (8.22a) to (8.21a), thereby completing the proof of (8.21a). \square

Proof of (8.21b). The proof is extremely similar to that of (8.21a). Comparing the forcing terms in (8.24) with those in (8.23a), and those in (8.25) with those in (8.23b), we see that they only differ by exchanging U with \mathcal{P} in several places; in fact, here we have fewer terms to bound. The contribution from $\mathcal{F}_{\mathcal{P},(1)}^{(m)}$ is estimated in precisely the same way as the one from $\mathcal{F}_{U_i,(1)}^{(m)}$ in (8.31). Similarly, the contribution from $\mathcal{F}_{\mathcal{P},(2)}^{(m)}$ is estimated in precisely the same way as the one from $\mathcal{F}_{U_i,(2)}^{(m)}$ in (8.33). Note that there is no third term in the definition of $\mathcal{F}_{\mathcal{P}}^{(m)}$, and thus we do not need to add a $(3 + \delta)$ to our error estimate, as we had to do for the U forcing in view of (8.37). Next, $\mathcal{F}_{\mathcal{P},(1)}^{(<m)}$, $\mathcal{F}_{\mathcal{P},(2)}^{(<m)}$, and $\mathcal{F}_{\mathcal{P},(3)}^{(<m)}$ are bounded in precisely the same way as $\mathcal{F}_{U_i,(1)}^{(<m)}$, $\mathcal{F}_{U_i,(2)}^{(<m)}$, and $\mathcal{F}_{U_i,(3)}^{(<m)}$ in (8.45), (8.49), and (8.50), respectively. To avoid redundancy, we omit these details. \square

Proof of (8.21c). Again, the proof is similar to that of (8.21a), except that in (8.26a) and (8.26b) we have much fewer terms. We need to be slightly careful here, as the $\partial^\gamma \mathcal{H}$ evolution is tested with $\kappa_0^2 \partial^\gamma \mathcal{H}$, rather than just $\partial^\gamma \mathcal{H}$, and we need to ensure that our damping bounds are independent of κ_0 ! The reason this is achieved is as follows. For the terms that contain a $D^m \mathcal{H}$, such as the first two terms in (8.26a), there is no issue as each of the two powers of κ_0 are paired with an $\|\mathcal{H}\|_{\dot{H}^m}$. An issue may arise in terms which contain $D^m U$, such as the last two terms in (8.26a). The important thing to notice here is that each such term is paired with $\|\nabla \mathcal{H}\|_{L^\infty}$. As opposed to $\nabla \mathcal{P}$, which satisfies $\|\nabla \mathcal{P}\|_{L^\infty} \approx \frac{1}{2} e^{-s/2}$, by (8.13) we have that $\|\nabla \mathcal{H}\|_{L^\infty} \leq \varepsilon^{1/3} e^{-s/2}$. This additional factor of $\varepsilon^{1/3}$ is

able to absorb all the stray powers of κ_0 . A similar argument applies to the terms in (8.26b), showing that the resulting bounds are independent of κ_0 . The contribution from $\mathcal{F}_{\mathcal{H}}^{(m)}$ is estimated in precisely the same way as the one from $\mathcal{F}_{U_i,(1)}^{(m)}$ in (8.31), while the contribution of $\mathcal{F}_{\mathcal{H}}^{(<m)}$ is bounded in precisely the same way as $\mathcal{F}_{U_i,(1)}^{(<m)}$ in (8.45). To avoid redundancy, we omit further details.

8.3 The E_m energy estimate

We now turn to the main energy estimate for the differentiated system (8.18).

PROPOSITION 8.4 (\dot{H}^m estimate for U , \mathcal{P} , and \mathcal{H}). *For any integer $m \geq 18$, with δ and $\lambda = \lambda(m, \delta)$ as specified in Lemma 8.3 we have the estimate*

$$(8.51) \quad E_m^2(s) \leq e^{-2(s-s_0)} E_m^2(s_0) + 3e^{-s} M^{4m-1} (1 - e^{-(s-s_0)})$$

for all $s \geq s_0 \geq -\log \varepsilon$.

PROOF OF PROPOSITION 8.4. We fix $\gamma \in \mathbb{N}_0^3$ with $|\gamma| = m$, and consider the sum of the L^2 inner product of (8.18a) with $2\lambda^{|\gamma|} \partial^\gamma U^i$, the L^2 inner product of (8.18b) with $2\lambda^{|\gamma|} \mathcal{H}^2 \partial^\gamma P$, and the L^2 inner product of (8.18c) with $2\kappa_0^2 \lambda^{|\gamma|} \partial^\gamma \mathcal{H}$. With the damping function \mathcal{D}_γ from (8.19) and the transport velocity \mathcal{V}_U defined in (2.39c), using the fact that \dot{Q} is skew-symmetric and that $(\partial_s + \mathcal{V}_U \cdot \nabla) \mathcal{H} = 0$, we find that

$$\begin{aligned}
 & \frac{d}{ds} \lambda^{|\gamma|} \int_{\mathbb{R}^3} (|\partial^\gamma U|^2 + \mathcal{H}^2 |\partial^\gamma \mathcal{P}|^2 + \kappa_0^2 |\partial^\gamma \mathcal{H}|^2) \\
 & + \lambda^{|\gamma|} \int_{\mathbb{R}^3} (2\mathcal{D}_\gamma - \operatorname{div} \mathcal{V}_U) (|\partial^\gamma U|^2 + \mathcal{H}^2 |\partial^\gamma \mathcal{P}|^2 + \kappa_0^2 |\partial^\gamma \mathcal{H}|^2) \\
 & + 8\gamma_1 \beta_\tau \beta_3 \lambda^{|\gamma|} \int_{\mathbb{R}^3} \mathcal{H}^2 (\partial^\gamma \mathcal{P}) \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) e^{\frac{s}{2}} \partial_1 \mathcal{P} \\
 & + 4\beta_\tau \beta_3 \lambda^{|\gamma|} \int_{\mathbb{R}^3} (\partial^\gamma \mathcal{P}) \mathcal{H}^2 \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) e^{\frac{s}{2}} \partial_1 \mathcal{P} \\
 & \quad + (\partial^\gamma \mathcal{P})^2 \mathcal{H}^2 \mathbf{J} e^{\frac{s}{2}} \partial_1 (U \cdot \mathbf{N}) \\
 & + 4\beta_\tau \beta_3 \lambda^{|\gamma|} \int_{\mathbb{R}^3} \mathcal{H}^2 \mathcal{P} (e^{\frac{s}{2}} \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) \partial_1 (\partial^\gamma \mathcal{P})) \\
 & \quad + e^{\frac{s}{2}} \mathbf{J} \partial_1 (\mathbf{N} \cdot \partial^\gamma U) (\partial^\gamma \mathcal{P}) \\
 & + 4\beta_\tau \beta_3 \lambda^{|\gamma|} \int_{\mathbb{R}^3} \mathcal{H}^2 \mathcal{P} (e^{-\frac{s}{2}} (\partial^\gamma U_v) \partial_v (\partial^\gamma \mathcal{P})) \\
 & \quad + e^{-\frac{s}{2}} (\partial^\gamma \mathcal{P}) \partial_v (\partial^\gamma U_v) \\
 & = 2\lambda^{|\gamma|} \int_{\mathbb{R}^3} (\mathcal{F}_{U^i}^{(\gamma)} \partial^\gamma U_i + \mathcal{H}^2 \mathcal{F}_{\mathcal{P}}^{(\gamma)} \partial^\gamma \mathcal{P} + \kappa_0^2 \mathcal{F}_{\mathcal{H}}^{(\gamma)} \partial^\gamma \mathcal{H}).
 \end{aligned}
 \tag{8.52}$$

Integrating by parts in the last two integrals on the left-hand side of (8.52), we get

$$\begin{aligned}
& 4\beta_\tau\beta_3\lambda^{|\gamma|} \int_{\mathbb{R}^3} \mathcal{H}^2 \mathcal{P} (e^{\frac{s}{2}} \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) \partial_1 (\partial^\gamma \mathcal{P}) + e^{\frac{s}{2}} \mathbf{J} \partial_1 (\mathbf{N} \cdot \partial^\gamma U) (\partial^\gamma \mathcal{P})) \\
& + 4\beta_\tau\beta_3\lambda^{|\gamma|} \int_{\mathbb{R}^3} \mathcal{H}^2 \mathcal{P} (e^{-\frac{s}{2}} (\partial^\gamma U_\nu) \partial_\nu (\partial^\gamma \mathcal{P}) + e^{-\frac{s}{2}} (\partial^\gamma \mathcal{P}) \partial_\nu (\partial^\gamma U_\nu)) \\
& = -4\beta_\tau\beta_3\lambda^{|\gamma|} \int_{\mathbb{R}^3} (e^{\frac{s}{2}} \partial_1 (\mathcal{H}^2 \mathcal{P}) \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) \\
& \quad + e^{-\frac{s}{2}} \partial_\nu (\mathcal{H}^2 \mathcal{P}) (\partial^\gamma U_\nu)) (\partial^\gamma \mathcal{P})
\end{aligned}$$

where we have used that $\partial_1 \mathbf{J} = 0$. Therefore, upon rearranging, the energy equality (8.52) becomes

$$\begin{aligned}
(8.53) \quad & \frac{d}{ds} \lambda^{|\gamma|} \int_{\mathbb{R}^3} (|\partial^\gamma U|^2 + \mathcal{H}^2 |\partial^\gamma \mathcal{P}|^2 + \kappa_0^2 |\partial^\gamma \mathcal{H}|^2) \\
& + \lambda^{|\gamma|} \int_{\mathbb{R}^3} (2\mathcal{D}_\gamma - \operatorname{div} \mathcal{V}_U) (|\partial^\gamma U|^2 + \mathcal{H}^2 |\partial^\gamma \mathcal{P}|^2 + \kappa_0^2 |\partial^\gamma \mathcal{H}|^2) \\
& + 8\gamma_1\beta_\tau\beta_3\lambda^{|\gamma|} \int_{\mathbb{R}^3} \mathcal{H}^2 (\partial^\gamma \mathcal{P}) \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) e^{\frac{s}{2}} \partial_1 \mathcal{P} \\
& + 4\beta_\tau\beta_3\lambda^{|\gamma|} \int_{\mathbb{R}^3} (\partial^\gamma \mathcal{P})^2 \mathcal{H}^2 \mathbf{J} e^{\frac{s}{2}} \partial_1 (U \cdot \mathbf{N}) \\
& \quad - 2(\partial^\gamma \mathcal{P}) \mathcal{H} \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) e^{\frac{s}{2}} \mathcal{P} \partial_1 \mathcal{H} \\
& - 4\beta_\tau\beta_3\lambda^{|\gamma|} \int_{\mathbb{R}^3} (\partial^\gamma U_\nu) (\partial^\gamma \mathcal{P}) e^{-\frac{s}{2}} \partial_\nu (\mathcal{H}^2 \mathcal{P}) \\
& = 2\lambda^{|\gamma|} \int_{\mathbb{R}^3} (\mathcal{F}_{U_i}^{(\gamma)} \partial^\gamma U_i + \mathcal{H}^2 \mathcal{F}_{\mathcal{P}}^{(\gamma)} \partial^\gamma \mathcal{P} + \kappa_0^2 \mathcal{F}_{\mathcal{H}}^{(\gamma)} \partial^\gamma \mathcal{H}).
\end{aligned}$$

We shall next obtain a lower bound for the second, third, and fourth integrals on the right side of (8.53).

For the second integral, we recall (8.19), use (2.39c), and the bounds (4.11), (9.4), (9.5), and (5.12) to obtain the lower bound

$$\begin{aligned}
(8.54) \quad & 2\mathcal{D}_\gamma - \operatorname{div} \mathcal{V}_U \\
& = |\gamma| - \frac{5}{2} + 2\gamma_1 + (2\gamma_1 - 1)(\beta_\tau\beta_1 \mathbf{J} \partial_1 W + \partial_1 G_U) - \partial_\nu h_U^\nu \\
& \geq |\gamma| - \frac{5}{2} + 2\gamma_1 - \beta_\tau\beta_1(2\gamma_1 - 1)_+ - \varepsilon^{\frac{1}{16}}.
\end{aligned}$$

For the third integral, we note that by the definitions (2.37) and (2.41)

$$(8.55) \quad 2\mathcal{H} \nabla \mathcal{P} = e^{-\frac{s}{2}} \nabla W - \nabla Z - \frac{1}{\gamma} S \nabla K$$

and thus, from (4.11), (4.7), (4.12), (4.24), (5.12), the third integral on the left-hand side of (8.53) has an integrand which is bounded as

$$\begin{aligned}
(8.56) \quad & \frac{8\gamma_1\beta_\tau\beta_3\mathcal{H}^2 |(\partial^\gamma \mathcal{P}) \mathbf{J}(\mathbf{N} \cdot \partial^\gamma U) e^{\frac{s}{2}} \partial_1 \mathcal{P}|}{|\partial^\gamma U|^2 + \mathcal{H}^2 |\partial^\gamma \mathcal{P}|^2} \\
& \leq 4\gamma_1\beta_\tau\beta_3 \mathbf{J} \mathcal{H} e^{\frac{s}{2}} |\partial_1 \mathcal{P}| \leq
\end{aligned}$$

$$\begin{aligned}
&\leq 2\gamma_1\beta_\tau\beta_3\mathcal{J}|\partial_1 W - e^{\frac{s}{2}}\partial_1 Z - \gamma^{-1}Se^{\frac{s}{2}}\partial_1 K| \\
&\leq 2\gamma_1\beta_\tau\beta_3(1 + 2M^{\frac{1}{2}}e^{-s}) \\
&\leq 2\gamma_1\beta_\tau\beta_3 + \varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Lastly, we compute $\partial_1(U \cdot \mathbf{N})$ from (2.37), $\partial_\nu \mathcal{P}$ from (8.55), and by using (4.11), (4.7), (4.12), (4.14), (4.24), (5.12), (8.3), and (8.15), the integrand in the fourth integral on the left-hand side of (8.53) may be estimated as

$$\begin{aligned}
&\frac{4\beta_\tau\beta_3|(\partial^\nu \mathcal{P})^2 \mathcal{H}^2 \mathcal{J} e^{\frac{s}{2}} \partial_1(U \cdot \mathbf{N}) - 2(\partial^\nu \mathcal{P}) \mathcal{H} \mathcal{J} (\mathbf{N} \cdot \partial^\nu U) e^{\frac{s}{2}} \mathcal{P} \partial_1 \mathcal{H}|}{|\partial^\nu U|^2 + \mathcal{H}^2 |\partial^\nu \mathcal{P}|^2} \\
&+ \frac{4\beta_\tau\beta_3|(\partial^\nu U_\nu)(\partial^\nu \mathcal{P}) e^{-\frac{s}{2}} \partial_\nu (\mathcal{H}^2 \mathcal{P})|}{|\partial^\nu U|^2 + \mathcal{H}^2 |\partial^\nu \mathcal{P}|^2} \\
(8.57) \quad &\leq 4\beta_\tau\beta_3 \left(\mathcal{J} e^{\frac{s}{2}} |\partial_1(U \cdot \mathbf{N})| + \mathcal{J} e^{\frac{s}{2}} |\mathcal{P} \partial_1 \mathcal{H}| + \frac{1}{2} \mathcal{H}^{-1} e^{-\frac{s}{2}} |\partial_\nu (\mathcal{H}^2 \mathcal{P})| \right) \\
&\leq 4\beta_\tau\beta_3 \left(\frac{1}{2} + M e^{-s} \right) \\
&\leq 2\beta_\tau\beta_3 + \varepsilon^{\frac{1}{2}}.
\end{aligned}$$

Combining the bounds (8.54), (8.56), and (8.57), with the energy equality (8.53), we arrive at

$$\begin{aligned}
&\frac{d}{ds} \int_{\mathbb{R}^3} \lambda^{|\gamma|} (|\partial^\nu U|^2 + \mathcal{H}^2 |\partial^\nu \mathcal{P}|^2 + \kappa_0^2 |\partial^\nu \mathcal{H}|^2) \\
(8.58) \quad &+ \mathcal{D}_{\text{total}} \int_{\mathbb{R}^3} \lambda^{|\gamma|} (|\partial^\nu U|^2 + \mathcal{H}^2 |\partial^\nu \mathcal{P}|^2 + \kappa_0^2 |\partial^\nu \mathcal{H}|^2) \\
&\leq 2\lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{F}_{U^i}^{(\gamma)} \partial^\nu U_i + \mathcal{H}^2 \mathcal{F}_{\mathcal{P}}^{(\gamma)} \partial^\nu \mathcal{P} + \kappa_0^2 \mathcal{F}_{\mathcal{H}}^{(\gamma)} \partial^\nu \mathcal{H}|,
\end{aligned}$$

where we have denoted

$$\mathcal{D}_{\text{total}} = |\gamma| - \frac{5}{2} + 2\gamma_1 - \beta_\tau\beta_1(2\gamma_1 - 1)_+ - \varepsilon^{\frac{1}{16}} - 2\gamma_1\beta_\tau\beta_3 - 2\beta_\tau\beta_3 - 2\varepsilon^{\frac{1}{2}}.$$

The crucial observation here is that because $\beta_1 + \beta_3 = 1$ (cf. (2.17)), and appealing to (4.3), the damping term $\mathcal{D}_{\text{total}}$ has the lower bound

$$(8.59) \quad \mathcal{D}_{\text{total}} \geq |\gamma| - \frac{5}{2} + 2\gamma_1(1 - \beta_\tau) - 2\beta_\tau\beta_3 - \varepsilon^{\frac{1}{16}} - 2\varepsilon^{\frac{1}{2}} \geq m - \frac{9}{2}$$

for ε taken sufficiently small in terms of α and m . Upon summing over $|\gamma| = m$, the energy inequality and (8.58) and the damping lower bound (8.59) thus yield

$$\begin{aligned}
&\frac{d}{ds} E_m^2 + \left(m - \frac{9}{2}\right) E_m^2 \\
(8.60) \quad &\leq \sum_{|\gamma|=m} 2\lambda^{|\gamma|} \int_{\mathbb{R}^3} |\mathcal{F}_{U^i}^{(\gamma)} \partial^\nu U_i + \mathcal{H}^2 \mathcal{F}_{\mathcal{P}}^{(\gamma)} \partial^\nu \mathcal{P} + \kappa_0^2 \mathcal{F}_{\mathcal{H}}^{(\gamma)} \partial^\nu \mathcal{H}|.
\end{aligned}$$

We are left with estimating the right-hand side of (8.60), which is the content of Lemma 8.3 above. By Lemma 8.3, for $0 < \delta \leq \frac{1}{32}$,

$$\frac{d}{ds} E_m^2(s) + (m-6)E_m^2(s) \leq (9+21\delta)E_m^2 + 3e^{-s}M^{4m-1},$$

and hence, by since $m \geq 18$, we have that

$$\frac{d}{ds} E_m^2 + 2E_m^2 \leq 3e^{-s}M^{4m-1},$$

and so we obtain that

$$E_m^2(s) \leq e^{-2(s-s_0)} E_m^2(s_0) + 3e^{-s}M^{4m-1}(1 - e^{-(s-s_0)}),$$

for all $s \geq s_0 \geq -\log \varepsilon$. This concludes the proof of Proposition 8.4. \square

In conclusion of this section, we mention that Proposition 8.4 applied with $s_0 = -\log \varepsilon$, in conjunction with Lemma 8.2, yields the proof of Proposition 4.3.

PROOF OF PROPOSITION 4.3. The initial datum assumption (3.23) together with the first bound in (8.5) implies that

$$E_m^2(-\log \varepsilon) \leq 2\kappa_0^2 \varepsilon.$$

Thus, from (8.51) the second bound in (8.5), we obtain

$$\begin{aligned} e^{-s} \|W\|_{\dot{H}^m}^2 + \|Z\|_{\dot{H}^m}^2 + \|A\|_{\dot{H}^m}^2 + \|K\|_{\dot{H}^m}^2 \\ \leq 4\lambda^{-m} E_m^2(s) + 4e^{-2s} \\ \leq 8\kappa_0^2 \lambda^{-m} \varepsilon^{-1} e^{-2s} + 12\lambda^{-m} e^{-s} M^{4m-1} (1 - \varepsilon^{-1} e^{-s}) + 4e^{-2s} \\ \leq 16\kappa_0^2 \lambda^{-m} \varepsilon^{-1} e^{-2s} + e^{-s} M^{4m} (1 - \varepsilon^{-1} e^{-s}) \end{aligned}$$

by taking M sufficiently slow. The inequalities (4.16a)–(4.16b) then follow. \square

9 Auxiliary Lemmas and Bounds on Forcing Functions

We record some useful bounds that will be used throughout the section.

LEMMA 9.1. *For $y \in \mathcal{X}(s)$ and for $m \geq 0$ we have*

$$(9.1a) \quad |\check{\nabla}^m f| + |\check{\nabla}^m(\mathbf{N} - \mathbf{N}_0)| + |\check{\nabla}^m(\mathbf{T}^\nu - \mathbf{T}_0^\nu)| \\ + |\check{\nabla}^m(\mathbf{J} - 1)| + |\check{\nabla}^m(\mathbf{J}^{-1} - 1)| \lesssim \varepsilon M^2 e^{-\frac{m+2}{2}s} |\check{y}|^2 \lesssim \varepsilon e^{-\frac{m}{2}s},$$

$$(9.1b) \quad |\check{\nabla}^m \dot{f}| + |\check{\nabla}^m \dot{\mathbf{N}}| \lesssim M^2 e^{-\frac{m+2}{2}s} |\check{y}|^2 \lesssim \varepsilon^{\frac{1}{4}} e^{-\frac{m}{2}s}.$$

Moreover, we have the following estimates on V

$$(9.2) \quad |\partial^\gamma V| \lesssim \begin{cases} M^{\frac{1}{4}} & \text{if } |\gamma| = 0, \\ M^2 \varepsilon^{\frac{1}{2}} e^{-\frac{3}{2}s} & \text{if } |\gamma| = 1 \text{ and } \gamma_1 = 1, \\ M^2 \varepsilon^{\frac{1}{2}} e^{-\frac{s}{2}} & \text{if } |\gamma| = 1 \text{ and } \gamma_1 = 0, \\ M^4 \varepsilon^{\frac{3}{2}} e^{-s} & \text{if } |\gamma| = 2 \text{ and } \gamma_1 = 0, \\ 0 & \text{else.} \end{cases}$$

for all $y \in \mathcal{X}(s)$.

PROOF OF LEMMA 9.1. The estimates (9.1a) follow directly from the definitions of f , N , T , and J , together with the bounds on ϕ given in (4.1a) and the inequality (4.5). Similarly, (9.1b) follows by using the ϕ estimate in (4.1b). To obtain the bound (9.2), we recall that V is defined in (2.31), employ the bounds on $\dot{\xi}$ and \dot{Q} given by (4.1b) and (4.2), and the fact that $|R - \text{Id}| \leq 1$, which follows from (4.1a) and the definition of R in (2.2) of [4]. \square

9.1 Transport estimates

LEMMA 9.2 (Estimates for G_W , G_Z , G_U , h_W , h_Z , and h_U). For $\varepsilon > 0$ sufficiently small and $y \in \mathcal{X}(s)$, the function G_W satisfies

$$(9.3) \quad |\partial^\gamma G_W| \lesssim \begin{cases} Me^{-\frac{s}{2}} + M^{\frac{1}{2}}|\gamma_1|e^{-s} + \varepsilon^{\frac{1}{3}}|\check{\gamma}| & \text{if } |\gamma| = 0, \\ M^2\varepsilon^{\frac{1}{2}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ Me^{-\frac{s}{2}} & \text{if } \gamma = (1, 0, 0) \text{ or } |\gamma| = 2, \\ M^{\frac{1}{2}}e^{-s} & \text{if } \gamma = (2, 0, 0). \end{cases}$$

the functions G_Z and G_U satisfy

$$(9.4) \quad \begin{aligned} & |\partial^\gamma(G_Z + (1 - \beta_2)e^{\frac{s}{2}}\kappa_0)| + |\partial^\gamma(G_U + (1 - \beta_1)e^{\frac{s}{2}}\kappa_0)| \\ & \lesssim \begin{cases} \varepsilon^{\frac{1}{2}}e^{\frac{s}{2}} & \text{if } |\gamma| = 0, \\ M^2\varepsilon^{\frac{1}{2}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ Me^{-\frac{s}{2}} & \text{if } \gamma = (1, 0, 0) \text{ or } |\gamma| = 2, \\ M^{\frac{1}{2}}e^{-s} & \text{if } \gamma = (2, 0, 0). \end{cases} \end{aligned}$$

and finally, the functions h_W , h_Z , and h_U satisfy the estimates

$$(9.5) \quad |\partial^\gamma h_W| + |\partial^\gamma h_Z| + |\partial^\gamma h_U| \lesssim \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-s}\eta^{-\frac{1}{6}} & \text{if } \gamma = (1, 0, 0) \text{ or } (|\gamma| = 2 \text{ and } |\check{\gamma}| = 1, 2), \\ e^{-(2 - \frac{3}{2m-5})s} & \text{if } \gamma = (2, 0, 0). \end{cases}$$

Furthermore, for $|\gamma| \in \{3, 4\}$ we have the lossy global estimates

$$(9.6) \quad |\partial^\gamma G_W| \lesssim e^{-(\frac{1}{2} - \frac{|\gamma|-1}{2m-7})s},$$

$$(9.7) \quad |\partial^\gamma h_W| \lesssim e^{-s},$$

for all $y \in \mathcal{X}(s)$.

PROOF OF LEMMA 9.2. The bounds for the first three cases in (9.3) and (9.4) are the same as in lemma 7.2 in [4]. It remains to consider the case $\gamma = (2, 0, 0)$. By (2.33), we have that

$$|\partial_1^2 G_W| + |\partial_1^2 G_Z| + |\partial_1^2 G_U| \lesssim e^{\frac{s}{2}} |\partial_1^2 Z|,$$

so that an application of (4.12) provides the bounds for both (9.3) and (9.4).

For the estimates (9.5), the proof of the first three cases is given in lemma 7.2 in [4]. For the case $\gamma = (2, 0, 0)$, by (2.34), we have that

$$|\partial_1^2 h_W| + |\partial_1^2 h_Z| + |\partial_1^2 h_U| \lesssim e^{-\frac{s}{2}} (|\partial_1^2 Z| + |\partial_1^2 A|) \lesssim M^{\frac{1}{4}} e^{-2s} + e^{-(2-\frac{3}{2m-5})s},$$

where we have applied (4.12) and (4.17) to attain the desired estimate. \square

9.2 Forcing estimates

LEMMA 9.3 (Estimates on $\partial^\gamma F_W$, $\partial^\gamma F_Z$, and $\partial^\gamma F_A$). *For $y \in \mathcal{X}(s)$ we have the force bounds*

$$(9.8) \quad |\partial^\gamma F_W| + e^{\frac{s}{2}} |\partial^\gamma F_Z| \lesssim \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-s} \eta^{-\frac{1}{15}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ e^{-\frac{5}{8}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{4}{2m-7})s} \eta^{-\frac{1}{15}} & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 2, \\ e^{-(\frac{5}{8}-\frac{9}{4(2m-7)})s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \end{cases}$$

$$(9.9) \quad |\partial^\gamma F_{Av}| \lesssim \begin{cases} M^{\frac{1}{2}} e^{-s} & \text{if } |\gamma| = 0, \\ (M^{\frac{1}{2}} + M^2 \eta^{-\frac{1}{6}}) e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{3}{2m-7})s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2. \end{cases}$$

Moreover, we have the following higher-order estimate at $y = 0$,

$$(9.10) \quad |(\partial^\gamma \tilde{F}_W)^0| \lesssim e^{-(\frac{1}{2}-\frac{4}{2m-7})s} \quad \text{for } |\gamma| = 3,$$

and the bound on \tilde{F}_W given by

$$(9.11) \quad |\partial^\gamma \tilde{F}_W| \lesssim M \varepsilon^{\frac{1}{6}} \begin{cases} \eta^{-\frac{1}{6}} & \text{if } |\gamma| = 0, \\ \eta^{-\frac{2}{5}} & \text{if } \gamma = (1, 0, 0), \\ \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ 1 & \text{if } |\gamma| = 4 \text{ and } |y| \leq \ell, \end{cases}$$

holds for all $|y| \leq \mathcal{L}$.

PROOF OF LEMMA 9.3. By the definition (2.35a) we have

$$\begin{aligned} |\partial^\gamma F_W| &\lesssim |\partial^\gamma (S T_\mu^\nu \partial_\mu A_\nu)| + e^{-\frac{s}{2}} |\partial^\gamma (A_\nu T_i^\nu \dot{N}_i)| + e^{-\frac{s}{2}} |\partial^\gamma (A_\nu T_j^\nu N_i)| \\ &\quad + e^{-\frac{s}{2}} |\partial^\gamma ((V_\mu + N_\mu U \cdot N + A_\nu T_\mu^\nu) A_\gamma T_i^\gamma N_{i,\mu})| \\ &\quad + e^{-\frac{s}{2}} |\partial^\gamma (S (A_\nu T_{\mu,\mu}^\nu + U \cdot N N_{\mu,\mu}))| \\ &\quad + \underbrace{|e^s \partial^\gamma (J S^2 \partial_1 K)| + |\partial^\gamma (N_\mu S^2 \partial_\mu K)|}_{\mathcal{J}_{W,\gamma}}. \end{aligned}$$

The bounds for the first five terms on the right side follow as in the proof of Lemma 7.3 in [4], and we have that

$$|\partial^\gamma F_W| \lesssim |\mathcal{J}_{W,\gamma}| + \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-s} \eta^{-\frac{1}{6} + \frac{2|\gamma|+1}{3(2m-5)}}(y) & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 1, 2, \\ M^2 e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{3}{2m-7})s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2 \end{cases}$$

Invoking (3.4), (4.14), (4.20), (9.1a) and Lemma 4.5, we obtain that

$$(9.12) \quad |\mathcal{J}_{W,\gamma}| \lesssim \sum_{\beta \leq \gamma, \beta_1=0} e^{-\frac{|\beta|}{2}s} (e^s |\partial^{\gamma-\beta} (S^2 \partial_1 K)| + \varepsilon |\partial^{\gamma-\beta} (S^2 \check{\nabla} K)|),$$

$$(9.13) \quad \lesssim \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-s} (\eta^{-\frac{1}{15}} + e^{-\frac{5}{8}s}) & \text{if } \gamma = (1, 0, 0), \\ e^{-\frac{5}{8}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{4}{2m-7})s} \eta^{-\frac{1}{15}} + e^{-(\frac{13}{8}-\frac{3}{2m-7})s} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{4}{2m-7})s} \eta^{-\frac{1}{15}} & \text{if } \gamma = (2, 0, 0), \\ e^{-(\frac{5}{8}+\frac{1-2\beta}{2m-7})s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2. \end{cases}$$

Using the same set of estimates we also obtain the lossy bound

$$(9.14) \quad |\mathcal{J}_{W,\gamma}| \lesssim e^{-\frac{s}{2}}$$

for $|\gamma| = 3$, which we shall need later in order to prove (9.10), and

$$(9.15) \quad |\mathcal{J}_{W,\gamma}| \lesssim \varepsilon^{\frac{1}{6}}$$

for $|\gamma| = 4$ and $|y| \leq \ell$, which we shall need later in order to prove the last case of (9.11).

Then, additionally using (4.5), we obtain the stated bounds claimed in (9.8) for $\partial^\gamma F_W$. Comparing (2.35b) and (2.35a), we note that the estimates on $\partial^\gamma F_Z$ claimed in (9.8) are completely analogous to the estimates one $\partial^\gamma F_W$ up to a factor of $e^{-\frac{s}{2}}$.

Now we consider the estimates on F_A . By definition (2.35c), we have that

$$\begin{aligned} |\partial^\gamma F_{A_v}| &\lesssim e^{-\frac{s}{2}} |\partial^\gamma (S T_\mu^\nu \partial_\mu S)| + e^{-s} |\partial^\gamma ((U \cdot \mathbf{N} \mathbf{N}_i + A_\gamma \mathbf{T}_i^\gamma) \dot{\mathbf{T}}_i^\nu)| \\ &\quad + e^{-s} |\partial^\gamma ((V_\mu + U \cdot \mathbf{N} \mathbf{N}_\mu + A_\gamma \mathbf{T}_\mu^\gamma) (U \cdot \mathbf{N} \mathbf{N}_i + A_\gamma \mathbf{T}_i^\gamma) \mathbf{T}_{i,\mu}^\nu)| \\ &\quad + e^{-s} |\partial^\gamma ((U \cdot \mathbf{N} \mathbf{N}_j + A_\gamma \mathbf{T}_j^\gamma) \mathbf{T}_i^\nu)| + \underbrace{|e^{-\frac{s}{2}} \partial^\gamma (S^2 \mathbf{T}_\mu^\nu \partial_\mu K)|}_{\mathcal{J}_{A,\gamma}}. \end{aligned}$$

Applying the bounds for the first four terms on the right side from lemma 7.3 of [4], we see that

$$(9.16) \quad |\partial^\gamma F_{Av}| \lesssim |\mathcal{J}_{A,\gamma}| + \begin{cases} M^{\frac{1}{2}} e^{-s} & \text{if } |\gamma| = 0, \\ (M^{\frac{1}{2}} + M^2 \eta^{-\frac{1}{6}}) e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{3}{2m-7})s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2. \end{cases}$$

Thus, combining the above estimates, we obtain (9.9).

Again, using the same argument as in lemma 7.3 in [4] for $|\gamma| = 3$, and using (9.14) yields

$$(9.17) \quad |(\partial^\gamma \tilde{F}_W)^0| \lesssim |(\mathcal{J}_{W,\gamma})^0| + e^{-(\frac{1}{2}-\frac{4}{2m-7})s} \lesssim e^{-(\frac{1}{2}-\frac{4}{2m-7})s},$$

and also for all $|y| \leq \mathcal{L}$,

$$(9.18) \quad |\partial^\gamma \tilde{F}_W| \lesssim |\mathcal{J}_{W,\gamma}| + M \varepsilon^{\frac{1}{6}} \begin{cases} \eta^{-\frac{1}{6}}(y) & \text{if } |\gamma| = 0, \\ \eta^{-\frac{1}{2}+\frac{3}{2m-5}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ 1 & \text{if } |\gamma| = 4 \text{ and } |y| \leq \ell. \end{cases}$$

The estimate (9.17) verifies (9.10), while combining (9.18) with (4.5), (9.13) and (9.15) verifies (9.11). \square

COROLLARY 9.4 (Estimates on the forcing terms). *Assume that $m \geq 18$. Then, we have*

$$(9.19) \quad |F_W^{(\gamma)}| \lesssim \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-\frac{s}{15}} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ \varepsilon^{\frac{5}{24}} \eta^{-\frac{5}{24}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ \eta^{-(\frac{29}{60}-\frac{8}{3(2m-7)})} \psi^{\frac{1}{4}} & \text{if } \gamma_1 = 2 \text{ and } |\check{\gamma}| = 0, \\ M^{\frac{1}{3}} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ M^{\frac{2}{3}} \eta^{-(\frac{5}{24}-\frac{1}{2m-7})} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \end{cases}$$

$$(9.20) \quad |F_Z^{(\gamma)}| \lesssim \begin{cases} e^{-s} & \text{if } |\gamma| = 0, \\ M^2 e^{-\frac{3}{2}s} \eta^{-\frac{1}{15}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, \\ e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-\frac{3s}{2}} (M^{\frac{1}{2}} + e^{\frac{4s}{2m-7}} \eta^{-\frac{1}{15}}) & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 2, \\ e^{-(\frac{9}{8}-\frac{9}{4(2m-7)})s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2 \end{cases}$$

$$(9.21) \quad |F_{Av}^{(\gamma)}| \lesssim \begin{cases} M^{\frac{1}{2}} e^{-s} & \text{if } |\gamma| = 0, \\ (M^{\frac{1}{2}} + M^2 \eta^{-\frac{1}{6}}) e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{3}{2m-7})s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \end{cases}$$

$$(9.22) \quad |F_K^{(\gamma)}| \lesssim \begin{cases} M^2 e^{-\frac{3}{2}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, 2, \\ \varepsilon^{\frac{1}{8}} e^{-\frac{3}{2}s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, 1, \\ \varepsilon^{\frac{1}{8}} e^{-\frac{3}{2}s} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} & \text{if } \gamma_1 = 2 \text{ and } |\gamma| = 0 \end{cases}$$

Moreover, we have the following higher-order estimate:

$$(9.23) \quad |\tilde{F}_W^{(\gamma),0}| \lesssim e^{-(\frac{1}{2}-\frac{4}{2m-7})s} \quad \text{for } |\gamma| = 3,$$

and the following estimates on $\tilde{F}_W^{(\gamma)}$:

$$(9.24) \quad |\tilde{F}_W^{(\gamma)}| \lesssim \varepsilon^{\frac{1}{11}} \eta^{-\frac{2}{5}} \quad \text{for } \gamma = (1, 0, 0) \text{ and } |y| \leq \mathcal{L},$$

$$(9.25) \quad |\tilde{F}_W^{(\gamma)}| \lesssim \varepsilon^{\frac{1}{12}} \eta^{-\frac{1}{3}} \quad \text{for } \gamma_1 = 0, |\check{\gamma}| = 1 \text{ and } |y| \leq \mathcal{L},$$

$$(9.26) \quad |\tilde{F}_W^{(\gamma)}| \lesssim \varepsilon^{\frac{1}{8}} + \varepsilon^{\frac{1}{10}} (\log M)^{|\check{\gamma}|-1} \quad \text{for } |\gamma| = 4 \text{ and } |y| \leq \ell.$$

PROOF OF COROLLARY 9.4. First we establish (9.19). Note that in this estimate $|\gamma| \leq 2$, and thus by definition (2.45), we have

$$\begin{aligned} |F_W^{(\gamma)}| &\lesssim |\partial^\gamma F_W| + \underbrace{\sum_{0 \leq \beta < \gamma} \left(|\partial^{\gamma-\beta} G_W \partial_1 \partial^\beta W| + |\partial^{\gamma-\beta} h_W^\mu \partial_\mu \partial^\beta W| \right)}_{=:\mathcal{I}_1} \\ &\quad + \underbrace{\mathbf{1}_{|\gamma|=2} \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1=\gamma_1}} |\partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta W|}_{=:\mathcal{I}_2}. \end{aligned}$$

We will first consider the case $\gamma \neq (2, 0, 0)$, since the estimates are analogous to the estimates in the previous paper. We have from corollary 7.4 of [4] that

$$|\mathcal{I}_1| \lesssim M \eta^{-\frac{1}{3}} (e^{-\frac{s}{2}} + \varepsilon^{\frac{1}{3}} (\mathbf{1}_{|\gamma|=2} + \mathbf{1}_{|\gamma|=|\check{\gamma}|=1})) \quad \text{and} \quad |\mathcal{I}_2| \lesssim \mathbf{1}_{|\gamma|=2} M^{\frac{|\check{\gamma}|}{3}} \eta^{-\frac{1}{3}}.$$

Thus combining these estimates with (9.8), we obtain that

$$(9.27) \quad |F_W^{(\gamma)}| \lesssim M \eta^{-\frac{1}{3}} e^{-\frac{s}{2}}, \quad \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-s} \eta^{-\frac{1}{15}} & \text{if } \gamma = (1, 0, 0), \\ e^{-\frac{5}{8}s} + M \varepsilon^{\frac{1}{3}} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{4}{2m-7})s} \eta^{-\frac{1}{15}} + (M \varepsilon^{\frac{1}{3}} + M^{\frac{1}{3}}) \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ e^{-(\frac{5}{8}-\frac{3}{2m-7})s} + (M \varepsilon^{\frac{1}{3}} + M^{\frac{2}{3}}) \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2. \end{cases}$$

Then applying (4.5) we obtain (9.19) for all cases except $\gamma = (2, 0, 0)$.

For the special case $\gamma = (2, 0, 0)$, we have from (4.5), (4.6) (with $q = 2$), (4.7), (9.3), and (9.5),

$$\begin{aligned} |\mathcal{J}_1| &\lesssim M^{\frac{1}{2}} e^{-s} \eta^{-\frac{1}{3}} + M^{\frac{4}{3}} e^{-\frac{s}{2}} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} + e^{-(2-\frac{3}{2m-5})s} + M^{\frac{2}{3}} e^{-s} \eta^{-\frac{1}{2}}(y) \\ &\lesssim M^{\frac{4}{3}} e^{-\frac{s}{2}} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}}. \end{aligned}$$

From (9.8) and (4.6) (with $q = \frac{3}{4} \frac{7-2m}{11-2m}$), we have that

$$|\partial^\gamma F_W| \lesssim e^{-(1-\frac{4}{2m-7})s} \eta^{-\frac{1}{15}} \lesssim \psi^{\frac{1}{4}} \eta^{-(\frac{29}{60}-\frac{8}{3(2m-7)})}.$$

Thus since $\mathcal{J}_2 = 0$ for $\gamma = (2, 0, 0)$, we obtain (9.8) for this case.

Similarly, for $|\gamma| \leq 2$, from (2.46) we have that

$$\begin{aligned} |F_Z^{(\gamma)}| &\lesssim |\partial^\gamma F_Z| + \sum_{0 \leq \beta < \gamma} (|\partial^{\gamma-\beta} G_Z \partial_1 \partial^\beta Z| + |\partial^{\gamma-\beta} h_Z^\mu \partial_\mu \partial^\beta Z|) \\ (9.28) \quad &+ \mathbf{1}_{|\gamma|=2} |\partial_1 Z \partial^\gamma (JW)| + \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1=\gamma_1}} |\partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta Z| \\ &=: |\partial^\gamma F_Z| + \mathcal{J}_1 + \mathbf{1}_{|\gamma|=2} |\partial_1 Z \partial^\gamma (JW)| + \mathcal{J}_2. \end{aligned}$$

Utilizing the bounds obtained in corollary 7.4 of [4], we have that Utilizing the bounds obtained in Corollary 7.4 of [4], we have that

$$\begin{aligned} |\partial_1 Z \partial^\gamma (JW)| &\lesssim M^{\frac{1}{2}} e^{-\frac{3}{2}s} (M \eta^{-\frac{1}{6}} \mathbf{1}_{\gamma_1=0} + M^{\frac{2}{3}} \eta^{-\frac{1}{3}} \mathbf{1}_{\gamma_1 \geq 1} + \varepsilon e^{-\frac{s}{2}}) \quad \text{for } |\gamma| = 2, \\ \mathcal{J}_1 &\lesssim e^{-\frac{3}{2}s} (M^2 e^{-\frac{s}{2}} + M^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} \mathbf{1}_{|\check{\gamma}| \geq 1} + M \varepsilon^{\frac{1}{2}} \eta^{-\frac{1}{6}}) \quad \text{for } |\gamma| \leq 2, \\ \mathcal{J}_2 &\lesssim (\mathbf{1}_{|\check{\gamma}|=1} M^{\frac{1}{2}} + \mathbf{1}_{|\check{\gamma}|=2} M) e^{-\frac{3}{2}s}, \quad \text{for } |\gamma| = 2, |\check{\gamma}| \geq 1. \end{aligned}$$

Thus combining the above estimates with (4.5) and (9.8), we obtain that

$$|F_Z^{(\gamma)}| \lesssim M^2 e^{-\frac{3}{2}s} \eta^{-\frac{1}{6}} + e^{-\frac{s}{2}} \begin{cases} e^{-\frac{s}{2}} & \text{if } |\gamma| = 0, \\ e^{-s} \eta^{\frac{1}{15}} & \text{if } \gamma = (1, 0, 0), \\ e^{-\frac{s}{8}} + M^{\frac{1}{2}} e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-(1-\frac{4}{2m-7})s} \eta^{-\frac{1}{15}} + M^{\frac{1}{2}} e^{-s} & \text{if } \gamma_1 \geq 1 \text{ and } |\gamma| = 2, \\ e^{-(\frac{5}{8}+\frac{1-2\beta}{2m-7})s} + M \varepsilon^{\frac{1}{2}} e^{-s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2. \end{cases}$$

The bounds for $|F_A^{(\gamma)}|$ are obtained in the identical fashion as the bounds for (7.20) in [4].

To prove the $|F_K^{(\gamma)}|$ estimate for $|\gamma| \leq 2$, from (2.46), we have that

$$\begin{aligned} |F_K^{(\gamma)}| &\lesssim \sum_{0 \leq \beta < \gamma} (|\partial^{\gamma-\beta} G_U \partial_1 \partial^\beta K| + |\partial^{\gamma-\beta} h_U^\mu \partial_\mu \partial^\beta K|) + \mathbf{1}_{|\gamma|=2} |\partial_1 K \partial^\gamma (JW)| \\ &+ \sum_{\substack{|\beta|=|\gamma|-1 \\ \beta \leq \gamma, \beta_1=\gamma_1}} |\partial^{\gamma-\beta} (JW) \partial_1 \partial^\beta K| \end{aligned}$$

(9.29)

$$=: \mathcal{J}_1 + \mathbf{1}_{|\gamma|=2} |\partial_1 K \partial^\gamma (JW)| + \mathcal{J}_2.$$

Let us further split \mathcal{J}_1 as

$$\mathcal{J}_1 = \underbrace{\sum_{0 \leq \beta < \gamma} |\partial^{\gamma-\beta} G_U \partial_1 \partial^\beta K|}_{\mathcal{J}_{1,1}} + \underbrace{\sum_{0 \leq \beta < \gamma} |\partial^{\gamma-\beta} h_U^\mu \partial_\mu \partial^\beta K|}_{\mathcal{J}_{1,2}}.$$

Estimating $\mathcal{J}_{1,1}$, using (4.14) and (9.4), we have that

$$\begin{aligned} |\mathcal{J}_{1,1}| &\lesssim M^2 \begin{cases} \varepsilon^{\frac{1}{2}} |\partial_1 K| & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-\frac{s}{2}} |\partial_1 K| + \varepsilon^{\frac{1}{2}} |\check{\nabla} \partial_1 K| & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \\ e^{-\frac{s}{2}} |\partial_1 K| & \text{if } \gamma = (1, 0, 0), \\ e^{-\frac{s}{2}} (|\partial_1 K| + |\check{\nabla} \partial_1 K|) + \varepsilon^{\frac{1}{2}} |\partial_1^2 K| & \text{if } \gamma_1 = 1 \text{ and } |\gamma| = 1, \\ e^{-s} |\partial_1 K| + e^{-\frac{s}{2}} |\partial_1^2 K| & \text{if } \gamma_1 = 2 \text{ and } |\gamma| = 0, \end{cases} \\ &\lesssim \begin{cases} e^{-\frac{3}{2}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, 2, \\ e^{-2s} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, 1, \\ e^{-\frac{5}{2}s} & \text{if } \gamma_1 = 2 \text{ and } |\gamma| = 0. \end{cases} \end{aligned}$$

Similarly, estimating $\mathcal{J}_{1,2}$, using (4.14) and (9.5), we have that

$$\begin{aligned} |\mathcal{J}_{1,2}| &\lesssim \begin{cases} e^{-s} |\check{\nabla} K| & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-s} (\eta^{-\frac{1}{6}} |\check{\nabla} K| + |\check{\nabla}^2 K|) & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \\ e^{-s} \eta^{-\frac{1}{6}} |\check{\nabla} K| & \text{if } \gamma = (1, 0, 0), \\ e^{-s} (\eta^{-\frac{1}{6}} (|\check{\nabla} K| + |\check{\nabla}^2 K|) + |\check{\nabla} \partial_1 K|) & \text{if } \gamma_1 = 1 \text{ and } |\gamma| = 1, \\ e^{-s} (e^{-(1-\frac{3}{2m-5})s} |\check{\nabla} K| + \eta^{-\frac{1}{6}} |\check{\nabla} \partial_1 K|) & \text{if } \gamma_1 = 2 \text{ and } |\gamma| = 0, \end{cases} \\ &\lesssim \varepsilon^{\frac{1}{8}} \begin{cases} e^{-\frac{3}{2}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, 2, \\ e^{-\frac{3}{2}s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, 1, \\ e^{-(\frac{5}{2}-\frac{3}{2m-5})s} & \text{if } \gamma_1 = 2 \text{ and } |\gamma| = 0. \end{cases} \end{aligned}$$

For $\mathbf{1}_{|\gamma|=2} |\partial_1 K \partial^\gamma (JW)|$, using (4.7) and (4.14) yields

$$|\partial_1 K \partial^\gamma (JW)| \lesssim \begin{cases} e^{-\frac{3}{2}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 2, \\ M^{\frac{2}{3}} \varepsilon^{\frac{1}{4}} e^{-\frac{3}{2}s} \eta^{-\frac{1}{3}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ M^{\frac{1}{3}} \varepsilon^{\frac{1}{4}} e^{-\frac{3}{2}s} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} & \text{if } \gamma_1 = 2 \text{ and } |\check{\gamma}| = 0. \end{cases}$$

Next, for \mathcal{J}_2 , we have that

$$|\mathcal{J}_2| \lesssim \begin{cases} e^{-\frac{3}{2}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, \\ e^{-2s} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus combining the above estimates, we attain

$$|F_K^{(\gamma)}| \lesssim \begin{cases} M^2 e^{-\frac{3}{2}s} & \text{if } \gamma_1 = 0 \text{ and } |\check{\gamma}| = 1, 2, \\ \varepsilon^{\frac{1}{8}} e^{-\frac{3}{2}s} \eta^{-\frac{1}{6}} & \text{if } \gamma_1 = 1 \text{ and } |\check{\gamma}| = 0, 1, \\ \varepsilon^{\frac{1}{8}} e^{-\frac{3}{2}s} \eta^{-\frac{1}{3}} \psi^{\frac{1}{4}} & \text{if } \gamma_1 = 2 \text{ and } |\check{\gamma}| = 0, \end{cases}$$

where we used (4.6) (with $q = \frac{3}{4} \frac{5-2m}{8-2m}$).

The proof of the bounds (9.23)–(9.26) is exactly the same as the proof of (7.21)–(7.24) in [4], with the caveat that we have changed the exponent of η in (9.24), which reflects the change in exponent of η in the estimate (9.11) for $\gamma = (1, 0, 0)$ relative to the corresponding estimate in our previous paper. \square

10 Closure of L^∞ -Based Bootstrap for Z , A , and K

Having established bounds on trajectories as well as on the vorticity, we now improve the bootstrap assumptions for $\partial^\gamma Z$ and $\partial^\gamma A$ stated in (4.12) and (4.13). We shall obtain estimates for $\partial^\gamma Z \circ \Phi_Z^{y_0}$ and $\partial^\gamma A \circ \Phi_U^{y_0}$ which are weighted by an appropriate exponential factor $e^{\mu s}$.

From (2.44b) we obtain that $e^{\mu s} \partial^\gamma Z$ is a solution of

$$\partial_s(e^{\mu s} \partial^\gamma Z) + D_Z^{(\gamma, \mu)}(e^{\mu s} \partial^\gamma Z) + (\mathcal{V}_Z \cdot \nabla)(e^{\mu s} \partial^\gamma Z) = e^{\mu s} F_Z^{(\gamma)},$$

where the damping function is given by

$$D_Z^{(\gamma, \mu)} := -\mu + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_2 \beta_\tau \gamma_1 \mathcal{J} \partial_1 W.$$

Upon composing with the flow of \mathcal{V}_Z , from Grönwall's inequality it follows that

$$\begin{aligned} (10.1) \quad & e^{\mu s} |\partial^\gamma Z \circ \Phi_Z^{y_0}(s)| \\ & \leq \varepsilon^{-\mu} |\partial^\gamma Z(y_0, -\log \varepsilon)| \exp \left(- \int_{-\log \varepsilon}^s D_Z^{(\gamma, \mu)} \circ \Phi_Z^{y_0}(s') ds' \right) \\ & \quad + \int_{-\log \varepsilon}^s e^{\mu s'} |F_Z^{(\gamma)} \circ \Phi_Z^{y_0}(s')| \exp \left(- \int_{s'}^s D_Z^{(\gamma, \mu)} \circ \Phi_Z^{y_0}(s'') ds'' \right) ds'. \end{aligned}$$

Similarly, from (2.44c) we have that $e^{\mu s} \partial^\gamma A$ and $e^{\mu s} \partial^\gamma K$ are solutions of

$$\partial_s(e^{\mu s} \partial^\gamma K) + D_K^{(\gamma, \mu)}(e^{\mu s} \partial^\gamma K) + (\mathcal{V}_U \cdot \nabla)(e^{\mu s} \partial^\gamma K) = e^{\mu s} F_K^{(\gamma)},$$

where

$$D_K^{(\gamma, \mu)} := -\mu + \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} + \beta_1 \beta_\tau \gamma_1 \mathcal{J} \partial_1 W,$$

and hence, again by Gronwall's inequality, we have that

$$\begin{aligned} (10.2) \quad & e^{\mu s} |\partial^\gamma K \circ \Phi_U^{y_0}(s)| \\ & \leq \varepsilon^{-\mu} |\partial^\gamma K(y_0, -\log \varepsilon)| \exp \left(- \int_{-\log \varepsilon}^s D_K^{(\gamma, \mu)} \circ \Phi_U^{y_0}(s') ds' \right) \\ & \quad + \int_{-\log \varepsilon}^s e^{\mu s'} |F_K^{(\gamma)} \circ \Phi_U^{y_0}(s')| \exp \left(- \int_{s'}^s D_K^{(\gamma, \mu)} \circ \Phi_U^{y_0}(s'') ds'' \right) ds'. \end{aligned}$$

For each choice of $\gamma \in \mathbb{N}_0^3$ present in (4.12) and (4.13), we shall require that the exponential factor μ satisfies

$$(10.3) \quad \mu \leq \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2},$$

which, in turn, shows that

$$(10.4) \quad D_Z^{(\gamma, \mu)} \leq 2\beta_2\gamma_1|\partial_1 W|.$$

For the last inequality, we have used the bound $|\beta_\tau \mathbf{J}| \leq 2$, which follows from (4.3) and (9.1a). Combining (10.3), (10.4), and (5.9), for $s \geq s' \geq -\log \varepsilon$, we find that

$$(10.5) \quad \begin{aligned} & \exp\left(-\int_{s'}^s D_Z^{(\gamma, \mu)} \circ \Phi_Z^{y_0}(s') ds'\right) \\ & \lesssim \exp\left(\left(\mu - \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2}\right)(s - s')\right) \lesssim 1. \end{aligned}$$

Replacing β_2 with β_1 in (10.4), we similarly obtain that for $s \geq s' \geq -\log \varepsilon$,

$$(10.6) \quad \exp\left(-\int_{s'}^s D_K^{(\gamma, \mu)} \circ \Phi_U^{y_0}(s') ds'\right) \lesssim 1.$$

Then as a consequence of (10.1), (10.3), (10.5), and (10.6),

$$(10.7) \quad \begin{aligned} & e^{\mu s} |\partial^\gamma Z \circ \Phi_Z^{y_0}(s)| \\ & \lesssim \varepsilon^{-\mu} |\partial^\gamma Z(y_0, -\log \varepsilon)| \\ & \quad + \int_{-\log \varepsilon}^s e^{\mu s'} |F_Z^{(\gamma)} \circ \Phi_Z^{y_0}(s')| \exp\left(\left(\mu - \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2}\right)(s - s')\right) ds' \end{aligned}$$

$$(10.8) \quad \begin{aligned} & e^{\mu s} |\partial^\gamma K \circ \Phi_Z^{y_0}(s)| \\ & \lesssim \varepsilon^{-\mu} |\partial^\gamma K(y_0, -\log \varepsilon)| \\ & \quad + \int_{-\log \varepsilon}^s e^{\mu s'} |F_S^{(\gamma)} \circ \Phi_Z^{y_0}(s')| \exp\left(\left(\mu - \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2}\right)(s - s')\right) ds' \end{aligned}$$

and

$$(10.9) \quad \begin{aligned} & e^{\mu s} |\partial^\gamma Z \circ \Phi_Z^{y_0}(s)| \\ & \lesssim \varepsilon^{-\mu} |\partial^\gamma Z(y_0, -\log \varepsilon)| + \int_{-\log \varepsilon}^s e^{\mu s'} |F_Z^{(\gamma)} \circ \Phi_Z^{y_0}(s')| ds', \end{aligned}$$

$$(10.10) \quad \begin{aligned} & e^{\mu s} |\partial^\gamma K \circ \Phi_U^{y_0}(s)| \\ & \lesssim \varepsilon^{-\mu} |\partial^\gamma K(y_0, -\log \varepsilon)| + \int_{-\log \varepsilon}^s e^{\mu s'} |F_S^{(\gamma)} \circ \Phi_U^{y_0}(s')| ds'. \end{aligned}$$

10.1 Estimates on Z

For convenience of notation, in this section we set $\Phi = \Phi_Z^{y_0}$. We start with the case $\gamma = 0$, for which we set $\mu = 0$. Then, the first line of (9.20) combined with (10.9) and our initial datum assumption (3.18) show that

$$|Z \circ \Phi(s)| \lesssim |Z(y_0, -\log \varepsilon)| + \int_{-\log \varepsilon}^s e^{-s'} ds' \lesssim \varepsilon.$$

This improves the bootstrap assumption (4.12) for $\gamma = 0$, upon taking M to be sufficiently large to absorb the implicit universal constant in the above inequality.

For the case $\gamma = (1, 0, 0)$, we set $\mu = \frac{3}{2}$ so that (10.3) is verified, and hence from (3.18), the second case in (9.20), and (10.9), we find that

$$\begin{aligned} e^{\frac{3}{2}s} |\partial_1 Z \circ \Phi(s)| &\lesssim \varepsilon^{-\frac{3}{2}} |\partial_1 Z(y_0, -\log \varepsilon)| + \int_{-\log \varepsilon}^s e^{\frac{3}{2}s'} |F_Z^{(\gamma)} \circ \Phi_Z^{y_0}(s')| ds' \\ &\lesssim 1 + M^2 \int_{-\log \varepsilon}^s (1 + |\Phi_1(s')|^2)^{-\frac{1}{15}} ds' \\ &\lesssim 1 + \varepsilon^{\frac{1}{30}} M^2 \int_{-\log \varepsilon}^s e^{\frac{s}{30}} (1 + |\Phi_1(s')|^2)^{-\frac{1}{15}} ds'. \end{aligned}$$

Now, applying (5.8) with $\sigma_1 = \frac{1}{30}$ and $\sigma_2 = \frac{2}{15}$, we deduce that by taking ε sufficiently small,

$$(10.11) \quad M e^{\frac{3}{2}s} |\partial_1 Z \circ \Phi(s)| \lesssim 1,$$

which improves the bootstrap assumption (4.12) for M taken sufficiently large.

For the case that $\gamma_1 = 1$ and $|\check{\gamma}| = 1$, we set $\mu = \frac{3}{2}$, so that

$$\mu - \frac{3\gamma_1 + \gamma_2 + \gamma_3}{2} = \frac{1}{2} - \gamma_1 \leq -\frac{1}{2}.$$

We deduce from (10.7), the fourth case in (9.20), the initial datum assumption (3.18), and Lemma 5.4 with $\sigma_1 = \frac{5s}{2m-7}$, $m \geq 18$, and $\sigma_2 = \frac{2}{15}$, that

$$\begin{aligned} (10.12) \quad e^{\frac{3}{2}s} |\partial^\gamma Z \circ \Phi(s)| &\lesssim \varepsilon^{-\frac{3}{2}} |\partial^\gamma Z(y_0, -\log \varepsilon)| \\ &\quad + \int_{-\log \varepsilon}^s (M^{\frac{1}{2}} + M \varepsilon^{\frac{1}{2m-7}} e^{\frac{5s}{2m-7}} (1 + |\Phi_1(s')|^2)^{-\frac{1}{15}}) e^{-\frac{s-s'}{2}} ds' \\ &\lesssim 1 + M^{\frac{1}{2}} + M \varepsilon^{\frac{1}{2m-7}} \lesssim M^{\frac{1}{2}}. \end{aligned}$$

This improves the bootstrap stated in (4.12) by using the factor $M^{1/2}$ to absorb the implicit constant in the above inequality.

We are left to consider γ for which $\gamma_1 = 0$ and $1 \leq |\check{\gamma}| \leq 2$. For $|\gamma| = |\check{\gamma}| = 1$, setting $\mu = \frac{1}{2}$ (which satisfies (10.3)), we obtain from (10.9), the forcing bound

(9.20), and the initial datum assumption (3.18) that

$$(10.13) \quad e^{\frac{s}{2}} |\check{\nabla} Z \circ \Phi(s)| \lesssim \varepsilon^{-\frac{1}{2}} |\check{\nabla} Z(y_0, -\log \varepsilon)| + M^{\frac{1}{2}} \int_{-\log \varepsilon}^s e^{-s'} ds' \lesssim \varepsilon^{1/2}.$$

Finally, for $|\gamma| = |\check{\gamma}| = 2$ we set $\mu = 1$. As a consequence of (9.20), (3.18), and (10.9), we obtain

$$(10.14) \quad \begin{aligned} e^s |\check{\nabla}^2 Z \circ \Phi(s)| \\ \lesssim \varepsilon^{-1} |\check{\nabla}^2 Z(y_0, -\log \varepsilon)| + \int_{-\log \varepsilon}^s e^{-(\frac{1}{8} - \frac{3}{2m-7})s} ds' \lesssim 1, \end{aligned}$$

Together, the estimates (10.11)–(10.14) improve the bootstrap bound (4.12) by taking M sufficiently large.

10.2 Estimates on K

We shall now set $\Phi = \Phi_U^{y_0}$. For the case $\gamma = (1, 0, 0)$, we set $\mu = \frac{3}{2}$ so that (10.3) is verified, and hence from (3.20), the second case in (9.22), and (10.10), we find that

$$\begin{aligned} e^{\frac{3}{2}s} |\partial_1 K \circ \Phi(s)| &\lesssim \varepsilon^{-\frac{3}{2}} |\partial_1 K(y_0, -\log \varepsilon)| + \int_{-\log \varepsilon}^s e^{\frac{3}{2}s'} |F_K^{(\gamma)}(s')| ds' \\ &\lesssim \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{8}} \int_{-\log \varepsilon}^s (1 + |\Phi_1(s')|^2)^{-\frac{1}{6}} ds', \end{aligned}$$

so that applying (5.8) with $\sigma_1 = 0$ and $\sigma_2 = \frac{1}{3}$, and taking ε sufficiently small, we deduce that

$$(10.15) \quad e^{\frac{3}{2}s} |\partial_1 K \circ \Phi(s)| \leq \varepsilon^{\frac{1}{4}},$$

which improves the second bootstrap assumption in (4.14).

Next, we study the case that $\gamma_1 = 0$ and $1 \leq |\check{\gamma}| \leq 2$. For $|\gamma| = |\check{\gamma}| = 1$, setting $\mu = \frac{1}{2}$ (which satisfies (10.3)), we obtain from (10.10), the forcing bound (9.22), and the initial datum assumption (3.20) that

$$(10.16) \quad \begin{aligned} e^{\frac{s}{2}} |\check{\nabla} K \circ \Phi(s)| &\lesssim \varepsilon^{-\frac{1}{2}} |\check{\nabla} K(y_0, -\log \varepsilon)| \\ &\quad + M^2 \int_{-\log \varepsilon}^s e^{-s'} ds' \lesssim \varepsilon^{1/2}. \end{aligned}$$

For $|\gamma| = |\check{\gamma}| = 2$ we set $\mu = 1$. As a consequence of (9.22), (3.20), and (10.10), we obtain

$$(10.17) \quad \begin{aligned} e^s |\check{\nabla}^2 K \circ \Phi(s)| &\lesssim \varepsilon^{-1} |\check{\nabla}^2 K(y_0, -\log \varepsilon)| \\ &\quad + M^2 \int_{-\log \varepsilon}^s e^{-\frac{s'}{2}} ds' \lesssim \varepsilon^{\frac{1}{4}}, \end{aligned}$$

For $|\gamma_1| = |\check{\gamma}| = 1$ we set $\mu = \frac{13}{8}$ so that (10.3) is verified. From (9.22), (3.20), and (10.8), we apply (5.8) with $\sigma_1 = \frac{1}{4}$ and $\sigma_2 = \frac{2}{3}$ to obtain that

$$\begin{aligned} e^{\frac{13}{8}s} |\partial_1 \check{\nabla} K \circ \Phi(s)| &\lesssim \varepsilon^{-\frac{13}{8}} |\partial_1 \check{\nabla} K(y_0, -\log \varepsilon)| + \varepsilon^{\frac{1}{8}} \int_{-\log \varepsilon}^s e^{\frac{s'}{8}} (1 + |\Phi_1(s')|^2)^{-\frac{1}{3}} ds' \\ &\lesssim \varepsilon^{-\frac{13}{8}} |\partial_1 \check{\nabla} K(y_0, -\log \varepsilon)| + \varepsilon^{\frac{1}{4}} \int_{-\log \varepsilon}^s e^{\frac{s'}{4}} (1 + |\Phi_1(s')|)^{-\frac{2}{3}} ds' \\ &\lesssim \varepsilon^{\frac{3}{8}} + \varepsilon^{\frac{1}{4}} \leq \varepsilon^{\frac{1}{8}}. \end{aligned}$$

We next consider the case that $\gamma = (2, 0, 0)$. From (2.44d), we have that

$$\partial_s \partial_{11} K + (3 + \beta_1 \beta_\tau \mathcal{J} \partial_1 W) \partial_{11} K + (\mathcal{V}_U \cdot \nabla) \partial_{11} K = F_S^{(2,0,0)},$$

and hence

$$\begin{aligned} (10.18) \quad &\partial_s (e^{2s} \eta^{\frac{1}{15}} \partial_{11} K) + D_K^{(2,0,0)} (e^{2s} \eta^{\frac{1}{15}} \partial_{11} K) + \mathcal{V}_U \cdot \nabla (e^{2s} \eta^{\frac{1}{15}} \partial_{11} K) \\ &= e^{2s} \eta^{\frac{1}{15}} F_S^{(2,0,0)} \end{aligned}$$

where

$$\begin{aligned} D_K^{(2,0,0)} &= \frac{4}{5} + \beta_1 \beta_\tau \mathcal{J} \partial_1 W \\ &\quad + \frac{1}{15} \eta^{-1} - \frac{2}{15} \eta^{-1} (y_1 (\beta_1 \beta_\tau \mathcal{J} W + G_U) + 3h_U^\nu y_\nu |\check{\gamma}|^4). \end{aligned}$$

Composing with Φ , we find that

$$\begin{aligned} &|e^{2s} \eta^{\frac{1}{15}} \partial_{11} K(s)| \\ &\leq |\varepsilon^{-2} \eta^{\frac{1}{15}} \partial_{11} K(-\log \varepsilon)| \exp \left(- \int_{s_0}^s D_K^{(2,0,0)} \circ \Phi(s') ds' \right) \\ &\quad + \int_{s_0}^s |e^{2s} \eta^{\frac{1}{15}} F_S^{(2,0,0)} \circ \Phi(s')| \exp \left(- \int_{s'}^s D_K^{(2,0,0)} \circ \Phi(s'') ds'' \right) ds'. \end{aligned}$$

Thanks to (5.16) and (11.8), we have that

$$\exp \left(- \int_{s_0}^s D_K^{(2,0,0)} \circ \Phi(s') ds' \right) \lesssim 1,$$

and thus using the third case in (9.22), and the initial datum assumption (3.20), it follows that

$$\begin{aligned} (10.19) \quad &\eta^{\frac{1}{15}} e^{2s} |\partial_{11} K \circ \Phi(s)| \\ &\lesssim \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{8}} \int_{-\log \varepsilon}^s e^{\frac{s'}{2}} \eta(\Phi(s'))^{-\frac{4}{15}} \psi^{\frac{1}{4}}(\Phi(s'), s') ds'. \end{aligned}$$

Now by definition of the weight ψ , we have that

$$\begin{aligned} e^{\frac{s}{2}} \eta^{-\frac{4}{15}} \psi^{\frac{1}{4}} \circ \Phi &\lesssim (e^{\frac{s}{2}} \eta^{-\frac{31}{60}} + e^{-\frac{s}{4}} \eta^{-\frac{1}{60}}) \circ \Phi \\ &\lesssim e^{\frac{s}{2}} \eta^{-\frac{31}{60}} \circ \Phi + \varepsilon^{\frac{1}{60}} e^{-\frac{3}{10}s} \\ &\lesssim e^{\frac{s}{2}} (1 + |\Phi|)^{-\frac{31}{30}} + \varepsilon^{\frac{1}{60}} e^{-\frac{3}{10}s} \end{aligned}$$

where we used (4.5) for the second inequality. It follows that

$$\begin{aligned} &\int_{-\log \varepsilon}^s e^{\frac{s}{2}} \eta(\Phi(s'))^{-\frac{4}{15}} \psi^{\frac{1}{4}}(\Phi(s'), s') ds' \\ &\lesssim \int_{-\log \varepsilon}^s (e^{\frac{s'}{2}} (1 + |\Phi|)^{-\frac{31}{30}} + \varepsilon^{\frac{1}{60}} e^{-\frac{3}{10}s'}) ds' \lesssim 1, \end{aligned}$$

where we have used the fact that $\int_{-\log \varepsilon}^s \varepsilon^{\frac{1}{60}} e^{-\frac{3}{10}s'} ds' \lesssim \varepsilon^{\frac{19}{60}}$ as well as (5.8) with $\sigma_1 = 1/2$ and $\sigma_2 = \frac{31}{30}$. Hence,

$$\eta^\alpha e^{2s} |\partial_{11} K \circ \Phi(s)| \leq \varepsilon^{\frac{1}{6}},$$

which improves the fourth bootstrap assumption stated in (4.14).

10.3 Estimates on A

We can now close the bootstrap bounds (4.13) for $\partial^\gamma A$. The bounds for the case that $\gamma_1 = 0$ and $|\gamma| = 0, 1, 2$ follow the same argument as given in (10.14) in [4], whereas the estimate for $\partial_1 A$ makes use of estimates for the vorticity.

LEMMA 10.1 (Relating A and Ω). *With the self-similar specific vorticity Ω given by (2.43),*

$$\begin{aligned} (10.20a) \quad e^{\frac{3s}{2}} J \partial_1 A_2 &= (\alpha e^{-\frac{K}{2}} S)^{\frac{1}{\alpha}} \Omega \cdot T^3 + \frac{1}{2} T_\mu^2 (\partial_\mu W + e^{\frac{s}{2}} \partial_\mu Z) - e^{\frac{s}{2}} N_\mu \partial_\mu A_2 \\ &\quad - \frac{1}{2} (\kappa + e^{-\frac{s}{2}} W + Z) (\text{curl}_{\mathbb{F}} N) \cdot T^3 - A_2 (\text{curl}_{\mathbb{F}} T^2) \cdot T^3 \end{aligned}$$

$$\begin{aligned} (10.20b) \quad e^{\frac{3s}{2}} J \partial_1 A_3 &= -(\alpha e^{-\frac{K}{2}} S)^{\frac{1}{\alpha}} \Omega \cdot T^2 + \frac{1}{2} T_\mu^3 (\partial_\mu W + e^{\frac{s}{2}} \partial_\mu Z) - e^{\frac{s}{2}} N_\mu \partial_\mu A_3 \\ &\quad + \frac{1}{2} (\kappa + e^{-\frac{s}{2}} W + Z) (\text{curl}_{\mathbb{F}} N) \cdot T^2 - A_3 (\text{curl}_{\mathbb{F}} T^3) \cdot T^2. \end{aligned}$$

Propositions 4.6 and 6.1 together with the estimates (4.7), (4.12), (4.13), (4.5), and (9.1a), and Lemma 10.1 show that

$$\begin{aligned} (10.21) \quad e^{\frac{3s}{2}} |\partial_1 A_\nu| &\lesssim \kappa_0^\alpha \varepsilon^{\frac{1}{21}} + (1 + \varepsilon^{1/2} M^{\frac{1}{2}}) + (\kappa_0 + \varepsilon^{\frac{1}{6}} + M\varepsilon) + M\varepsilon \\ &\leq M^{\frac{1}{4}}, \end{aligned}$$

for M taken sufficiently large with respect to $\kappa_0^{1/\alpha} C_{\kappa_0, \alpha}$.

PROOF OF LEMMA 10.1 We note that for the velocity \mathring{u} and with respect to the orthonormal basis (N, T^2, T^3) we have that

$$\begin{aligned} \operatorname{curl}_{\tilde{x}} \mathring{u} &= (\partial_{T^3} \mathring{u} \cdot N - \partial_N \mathring{u} \cdot T^3) T^2 - (\partial_{T^2} \mathring{u} \cdot N - \partial_N \mathring{u} \cdot T^2) T^3 \\ &\quad + (\partial_{T^2} \mathring{u} \cdot T^3 - \partial_{T^3} \mathring{u} \cdot T^2) N. \end{aligned}$$

Now, from the definitions (2.4), (2.10), (2.16), (2.23), (2.30d), (2.36c), and (2.43), we have that

$$\begin{aligned} (\alpha e^{-\frac{K}{2} S})^{1/\alpha} (y, s) \Omega(y, s) &= (\alpha e^{-\frac{K}{2} \tilde{\sigma}} (x, t))^{1/\alpha} \tilde{\zeta}(x, t) = \tilde{\rho}(\tilde{x}, t) \tilde{\zeta}(\tilde{x}, t) \\ &= \tilde{\omega}(\tilde{x}, t) = \operatorname{curl}_{\tilde{x}} \tilde{u}(\tilde{x}, t) = \operatorname{curl}_{\tilde{x}} \mathring{u}(x, t), \end{aligned}$$

In particular,

$$\begin{aligned} (10.22) \quad (\alpha e^{-\frac{K}{2} S})^{1/\alpha} (y, s) \Omega(y, s) &= \operatorname{curl}_{\tilde{x}} \mathring{u}(x, t) \\ &= \operatorname{curl}_{\tilde{x}} (\mathring{u}(\tilde{x}_1 - f(\tilde{x}, t), \tilde{x}_2, \tilde{x}_3, t)). \end{aligned}$$

We only establish the formula for $\partial_1 A_3$, as the one for $\partial_1 A_2$ is obtained identically. To this end, we write

$$\operatorname{curl}_{\tilde{x}} \mathring{u} \cdot T^2 = T_j^3 \partial_{\tilde{x}_j} \mathring{u}(x, t) \cdot N - N_j \partial_{\tilde{x}_j} \mathring{u}(x, t) \cdot T^3.$$

By the chain rule and the fact that N is orthogonal to T^3 , we have that

$$\partial_{\tilde{x}_j} \mathring{u}(x, t) T_j^3 = \partial_{x_1} \mathring{u} T_1^3 - f_{,v} \partial_{x_1} \mathring{u} T_v^3 + \partial_{x_v} \mathring{u} T_v^3 = \partial_{x_v} \mathring{u}(x, t) T_v^3.$$

The important fact to notice here is that no x_1 -derivatives of \mathring{u} remain. Similarly,

$$\partial_{\tilde{x}_j} \mathring{u}(x, t) N_j = \partial_{x_1} \mathring{u} N_1 - f_{,v} \partial_{x_1} \mathring{u} N_v + \partial_{x_v} \mathring{u} N_v = J \partial_{x_1} \mathring{u} + \partial_{x_v} \mathring{u}(x, t) N_v.$$

Hence, it follows that

$$\begin{aligned} (10.23) \quad \operatorname{curl}_{\tilde{x}} \mathring{u} \cdot T^2 &= T_v^3 \partial_{x_v} (\mathring{u}(x, t) \cdot N) - J \partial_{x_1} a_3 - N_v \partial_{x_v} (\mathring{u}(x, t) \cdot T^3) \\ &\quad + \mathring{u}(x, t) \cdot (\partial_{x_v} T^3 N_v - \partial_{x_v} N T_v^3) \\ &= \frac{1}{2} T_v^3 \partial_{x_v} (w + z) - J \partial_{x_1} a_3 - N_v \partial_{x_v} a_3 \\ &\quad + \left(\frac{1}{2} (w + z) N + a_v T^v \right) \cdot (\partial_N T^3 - \partial_{T^3} N) \end{aligned}$$

where we have used (2.27), (2.26), and (7.3). The identities (10.22) and (10.23) and the definition of the self-similar transformation in (2.29) and (2.30) yield the desired formula for $\partial_1 A_3$. \square

11 Closure of L^∞ -Based Bootstrap for W

The goal of this section is to close the bootstrap assumptions which involve W , \tilde{W} , and their derivatives, stated in (4.7) and (4.8a)–(4.10).

11.1 Estimates for $\partial^\nu W(y, s)$ for $|y| \leq \ell$

The estimates in this section closely mirror those given in Section 11.1 of [4], as such will we simply summarize the argument.

The fourth derivative

Composing with the flow $\Phi_W^{y_0}(s)$, we have that for $|\gamma| = 4$ that

$$\frac{d}{ds} \left(\partial^\gamma \tilde{W} \circ \Phi_W^{y_0} \right) + \left(D_{\tilde{W}}^{(\gamma)} \circ \Phi_W^{y_0} \right) \left(\partial^\gamma \tilde{W} \circ \Phi_W^{y_0} \right) = \tilde{F}_W^{(\gamma)} \circ \Phi_W^{y_0},$$

where

$$(11.1) \quad D_{\tilde{W}}^{(\gamma)} := \frac{3\gamma_1 + \gamma_2 + \gamma_3 - 1}{2} + \beta_\tau J (\partial_1 \bar{W} + \gamma_1 \partial_1 W) \geq \frac{1}{3},$$

which is a consequence of (4.3) and (4.11). Then as a consequence of (9.26), (11.1), and (3.14) and the Grönwall inequality we have that for all $|y_0| \leq \ell$ and all $s \geq -\log \varepsilon$ such that $|\Phi_W^{y_0}(s)| \leq \ell$ the following estimate

$$(11.2) \quad |\partial^\gamma \tilde{W} \circ \Phi_W^{y_0}| \lesssim \varepsilon^{\frac{1}{8}} + \varepsilon^{\frac{1}{10}} (\log M)^{|\gamma|-1}.$$

Hence the bootstrap assumption (4.9b) closes assuming the ε is chosen sufficiently small relative to M .

Estimates for $\partial^\gamma W$ with $|\gamma| \leq 3$ and $|y| \leq \ell$

We first consider the estimate on $(\partial^\gamma \tilde{W})^0$ for $|\gamma| = 3$. Evaluating (2.50) at $y = 0$ and applying (4.9b), (4.10), (12.17), (9.23), and (4.3) yields the estimate

$$(11.3) \quad \begin{aligned} |\partial_s (\partial^\gamma \tilde{W})^0| &\lesssim e^{-(\frac{1}{2} - \frac{4}{2m-7})s} + M(\log M)^4 \varepsilon^{\frac{1}{10}} e^{-s + \frac{4s}{2m-7}} + M \varepsilon^{\frac{1}{4}} e^{-s} \\ &\lesssim e^{-(\frac{1}{2} - \frac{4}{2m-7})s}. \end{aligned}$$

Using the initial datum assumption (3.15) and integrating in time, we may show

$$(11.4) \quad |\partial^\gamma \tilde{W}(0, s)| \leq \frac{1}{10} \varepsilon^{\frac{1}{4}}$$

for all $|\gamma| \leq 3$, and all $s \geq -\log \varepsilon$, closing the bootstrap bound (4.10).

The bootstraps (4.9a) corresponding to $0 \leq |y| \leq \ell$, then follow as a consequence of constraints (2.52) which imply

$$\tilde{W}(0, s) = \nabla \tilde{W}(0, s) = \nabla^2 \tilde{W}(0, s) = 0,$$

together with the estimates (4.9b), (11.4), and the fundamental theorem of calculus, integrating from $y = 0$.

Note that the bootstraps (4.8a), (4.8b) and (4.8c), for the case $|y| \leq \ell$, follows as a consequence of (4.9a), assuming ε is sufficiently small.

11.2 A framework for weighted estimates

Let us briefly recall the framework for weighted estimates introduced in Section 11.2 of [4]. For brevity will drop some intermediary calculations. Suppose some quantity \mathcal{R} , satisfies an evolution equation of the form

$$(11.5) \quad \partial_s \mathcal{R} + D_{\mathcal{R}} \mathcal{R} + \mathcal{V}_W \cdot \nabla \mathcal{R} = F_{\mathcal{R}}.$$

Weighting \mathcal{R} by η^μ ,

$$q := \eta^\mu \mathcal{R},$$

then q satisfies the evolution equation

$$(11.6) \quad \partial_s q + \underbrace{(D_{\mathcal{R}} - \eta^{-\mu} \mathcal{V}_W \cdot \nabla \eta^\mu)}_{=: D_q} q + \mathcal{V}_W \cdot \nabla q = \underbrace{\eta^\mu F_{\mathcal{R}}}_{=: F_q}.$$

where \mathcal{D}_q may be expanded as

$$(11.7) \quad D_q = D_{\mathcal{R}} - 3\mu + 3\mu\eta^{-1} - 2\mu\eta^{-1} \underbrace{(y_1(\beta_\tau \mathbf{J}W + G_W) + 3h_W^\nu y_\nu |\dot{y}|^4)}_{=: \mathcal{D}_\eta} \dots$$

As a consequence of (4.7), (4.5), (9.1b), (4.3), (9.3), and (9.5) we have for all $s \geq -\log \varepsilon$

$$(11.8) \quad |\mathcal{D}_\eta| \leq 5\eta^{-\frac{1}{3}} + e^{-\frac{s}{3}},$$

assuming ε to be sufficiently small in order to absorb powers of M .

Using the evolution equation (11.6), composing with the trajectories $\Phi_W^{y_0}(s)$ such that $\Phi_W^{y_0}(s_0) = y_0$ for some $s_0 \geq -\log \varepsilon$ with $|y_0| \geq \ell$ and applying Grönwall's inequality yields

$$(11.9) \quad |q \circ \Phi_W^{y_0}(s)| \leq |q(y_0)| \exp \left(- \int_{s_0}^s D_q \circ \Phi_W^{y_0}(s') ds' \right) + \int_{s_0}^s |F_q \circ \Phi_W^{y_0}(s')| \exp \left(- \int_{s'}^s D_q \circ \Phi_W^{y_0}(s'') ds'' \right) ds'.$$

For the special case $\ell \leq |y_0| \leq \mathcal{L}$, we may apply (11.8), (5.4), and the inequality $2\eta(y) \geq 1 + |y|^2$ to conclude

$$(11.10) \quad 2\mu \int_{s_0}^s |\mathcal{D}_\eta \circ \Phi_W^{y_0}(s')| ds' \leq 70 \log \frac{1}{\ell},$$

for all $|\mu| \leq \frac{1}{2}$. Consequently, the estimates (11.9) and (11.10) yield

$$(11.11) \quad |q \circ \Phi_W^{y_0}(s)| \leq \ell^{-70} |q(y_0)| \exp \left(\int_{s_0}^s (3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s') ds' \right) + \ell^{-70} \int_{s_0}^s |F_q \circ \Phi_W^{y_0}(s')| \exp \left(\int_{s'}^s (3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s'') ds'' \right) ds'.$$

We will need to consider two scenarios for the initial trajectory: either $s_0 > -\log \varepsilon$ and $|y_0| = 0$ or $s_0 = -\log \varepsilon$ and $|y_0| \geq \ell$. We note that as long as $|y_0| \geq \ell$, then $|\Phi_W^{y_0}(s)| \geq \ell$ for all $s > s_0$ as a consequence of Lemma 5.2.

Now consider the case $|y_0| \geq \mathcal{L}$. In place of (11.10) for the case $\ell \leq |y_0| \leq \mathcal{L}$, we have the stronger estimate

$$(11.12) \quad 2\mu \int_{s_0}^s |\mathcal{D}_\eta \circ \Phi_W^{y_0}(s')| ds' \leq \varepsilon^{\frac{1}{16}},$$

for $s_0 \geq -\log \varepsilon$, and $|\mu| \leq \frac{1}{2}$. Hence (11.9) and (11.12) yield

$$\begin{aligned}
 & |q \circ \Phi_W^{y_0}(s)| \\
 & \leq e^{\varepsilon^{\frac{1}{16}}} |q(y_0)| \exp \left(\int_{s_0}^s (3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s') ds' \right) \\
 & (11.13) \\
 & + e^{\varepsilon^{\frac{1}{16}}} \int_{s_0}^s |F_q \circ \Phi_W^{y_0}(s')| \exp \left(\int_{s'}^s (3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s'') ds'' \right) ds'.
 \end{aligned}$$

11.3 Estimates of $W(y, s)$, $\partial_1 W(y, s)$ and $\nabla W(y, s)$ for $\ell \leq |y| \leq \mathcal{L}$

The estimates of $\tilde{W}(y, s)$, $\partial_1 \tilde{W}(y, s)$ and $\tilde{\nabla} \tilde{W}(y, s)$ for $\ell \leq |y| \leq \mathcal{L}$ mimic those given in Section 11.3 - 11.4 in [4]. As such, we prove only an abridged summary of the arguments.

In order to close the bootstrap bound (4.8a) on $\tilde{W}(y, s)$ for $|y| \geq \ell$, we will use the framework in Section 11.2 with $\mathcal{R} = \tilde{W}$, $\mu = -\frac{1}{6}$. With these choices, the weighted quantity $q = \eta^{-1/6} \tilde{W}$, the quantity $3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1}$ present in (11.11) is $-\beta_\tau \mathcal{J} \partial_1 W + \frac{1}{2} \eta^{-1}$ and $F_q = \eta^{-\frac{1}{6}} \tilde{F}_W$.

Applying (4.3), (9.1a), (5.4), and (4.7), we have

$$(11.14) \quad \int_{s_0}^s \beta_\tau |\mathcal{J} \partial_1 W| \circ \Phi_W^{y_0}(s') + \frac{1}{2} \eta^{-1} \circ \Phi_W^{y_0}(s') ds' \leq 40 \log \frac{1}{\ell}$$

for all $s \geq s_0 \geq -\log \varepsilon$. The estimate (5.4) and (9.11) yield the forcing estimate

$$(11.15) \quad \int_{s_0}^s |\eta^{-\frac{1}{6}} \tilde{F}_W| \circ \Phi_W^{y_0}(s') ds' \lesssim \varepsilon^{\frac{1}{8}} \log \frac{1}{\ell}$$

for all $s \geq s_0 \geq -\log \varepsilon$, and $\ell \in (0, 1/10]$.

Combining the bounds (11.14) and (11.15) into (11.11), and using the initial data assumption (3.13a) if $s_0 = -\log \varepsilon$, or alternatively (4.9a) if $s_0 > -\log \varepsilon$, we obtain

$$(11.16) \quad \eta^{-\frac{1}{6}}(y) |\tilde{W}(y, s)| \leq \frac{1}{10} \varepsilon^{\frac{1}{11}}$$

for all $\ell \leq |y| \leq \mathcal{L}$ and all $s \geq -\log \varepsilon$. Where we have employed small powers of ε to absorb all the ℓ and M factors. The above estimate (11.16) closes the bootstrap (4.8a).

We now aim to close the bootstrap bound (4.8b) on $\partial_1 \tilde{W}(y, s)$ for $\ell \leq |y| \leq \mathcal{L}$. For this case, we set $\mathcal{R} = \partial_1 \tilde{W}$, $\mu = \frac{1}{3}$ and hence $q = \eta^{\frac{1}{3}} \partial_1 \tilde{W}$. By (2.50) with $\gamma = (1, 0, 0)$, we have $3\mu - D_{\mathcal{R}} = -\beta_\tau \mathcal{J}(\partial_1 W + \partial_1 \bar{W})$, and $F_q = \eta^{\frac{1}{3}} \tilde{F}_W^{(1,0,0)}$.

Similar to the estimate (11.14), we may bound the contributions to (11.11) due to the damping term $3\mu - D_{\mathcal{R}}$ by

$$(11.17) \quad \int_{s_0}^s \beta_\tau |\mathcal{J}(\partial_1 W + \partial_1 \bar{W})| \circ \Phi_W^{y_0}(s') ds' \leq 80 \log \frac{1}{\ell}.$$

The contribution due to the forcing $F_q = \eta^{1/3} \tilde{F}_W^{(1,0,0)}$ is bounded using (5.4) and (9.24) in order to attain

$$(11.18) \quad \int_{s_0}^s |F_q| \circ \Phi_W^{y_0}(s') ds' \lesssim \varepsilon^{\frac{1}{11}} \log \frac{1}{\ell}.$$

Inserting (11.17) and (11.18) into (11.11), and using our initial datum assumption (3.13b) when $s_0 = -\log \varepsilon$, respectively (4.9b) for $s_0 > -\log \varepsilon$, yields

$$(11.19) \quad \eta^{\frac{1}{6}}(y) |\partial_1 \tilde{W}(y, s)| \leq \frac{1}{10} \varepsilon^{\frac{1}{12}}$$

for all $\ell \leq |y| \leq \mathcal{L}$ and all $s \geq -\log \varepsilon$, where we again have used small powers of ε to absorb all the ℓ and M factors. The above estimate closes the bootstrap (4.8b).

Finally, we aim to close the bootstrap (4.8c) on $\tilde{\nabla} \tilde{W}(y, s)$ for $|y| \geq \ell$. We set $\mathcal{R} = \tilde{\nabla} W$ and $\mu = 0$, so that $q = \tilde{\nabla} \tilde{W}$. From (2.50) with $\gamma \in \{(0, 1, 0), (0, 0, 1)\}$, we have $3\mu - D_{\mathcal{R}} = -\beta_{\tau} J \partial_1 \bar{W}$ and $F_q = \tilde{F}_W^{(\gamma)}$.

The integral of the damping term arising in (11.11) is bounded using (11.14) by $40 \log \ell^{-1}$. The contribution due to the forcing F_q is bounded using (5.4) and (9.25) in order to attain

$$(11.20) \quad \int_{s_0}^s |F_q| \circ \Phi_W^{y_0}(s') ds' \lesssim \varepsilon^{\frac{1}{12}} \log \frac{1}{\ell}.$$

Inserting (11.14) and (11.20) into (11.11), and using our initial datum assumption (3.13c) and (4.9b), we arrive at

$$(11.21) \quad |\tilde{\nabla} \tilde{W}(y, s)| \leq \frac{1}{10} \varepsilon^{\frac{1}{13}}$$

for all $\ell \leq |y| \leq \mathcal{L}$ and all $s \geq -\log \varepsilon$, thereby closing the bootstrap bound (4.8c). We also note that the bootstrap bound (4.7) for the cases that $|\gamma| = 0, 1$ and $\ell \leq |y| \leq \mathcal{L}$ follow as a consequence of (4.8) together with the \bar{W} bound (2.48) in [4].

11.4 Estimate for $\partial^{\gamma} W(y, s)$ with $|\gamma| = 2$ for $|y| \geq \ell$

We now consider the case $|\gamma| = 2$, and establish the third and fifth bounds of (4.7). Unlike the bounds given in Section 11.6 of [4], the bound for $\partial_{11} W$ makes use of two weight functions, and requires a new type of analysis. As such, we now consider the case that $\gamma_1 = 2$ and $|\check{\gamma}| = 0$. We have that

$$\begin{aligned} \partial_s(\eta^{\frac{1}{3}} \partial_{11} W) + \bar{\mathcal{D}}_W^{(2,0,0)}(\eta^{\frac{1}{3}} \partial_{11} W) + \mathcal{V}_W \cdot \nabla(\eta^{\frac{1}{3}} \partial_{11} W) &= \eta^{\frac{1}{3}} F_W^{(2,0,0)}, \\ \bar{\mathcal{D}}_W^{(2,0,0)} &= \frac{3}{2} + \eta^{-1} - \frac{2}{3} \underbrace{\eta^{-1} (y_1(\beta_{\tau} J W + G_W) + 3h_W^{\nu} y_{\nu} |\check{y}|^4)}_{=: \mathcal{D}_{\eta}}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \partial_s(\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W) + \mathcal{D}_W^{(2,0,0)}(\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W) \\ + \mathcal{V}_W \cdot \nabla(\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W) &= \eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} F_W^{(2,0,0)}, \end{aligned}$$

where

$$\begin{aligned}\mathcal{D}_W^{(2,0,0)} &= \frac{3}{2} + \eta^{-1} - \frac{2}{3}D_\eta - \frac{3}{4}e^{-3s}\psi^{-1}\eta + \frac{1}{2}\psi^{-1}y_1(e^{-3s} - \eta^{-2})\mathcal{V}_W^1 \\ &\quad + \frac{3}{2}\psi^{-1}|\check{y}|^4 y_\mu(e^{-3s} - \eta^{-2})\mathcal{V}_W^\mu \\ &= \frac{3}{2} + \eta^{-1} - \frac{2}{3}D_\eta - \frac{3}{4}e^{-3s}\psi^{-1} - \frac{3}{4}\psi^{-1}\frac{y_1^2 + |\check{y}|^6}{\eta^2} \\ &\quad + \frac{1}{2}\psi^{-1}\underbrace{(e^{-3s} - \eta^{-2})(y_1(\beta_\tau JW + G_W) + 3|\check{y}|^4 y_\mu h_W^\mu)}_{=:\mathcal{D}_\psi},\end{aligned}$$

and therefore

$$\begin{aligned}(11.22) \quad & \left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}\partial_{11}W \circ \Phi_W^{y_0}(s) \right| \\ & \leq \left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}\partial_{11}W(y_0) \right| \exp\left(-\int_{s_0}^s \mathcal{D}_W^{(2,0,0)} \circ \Phi_W^{y_0}(s') ds'\right) \\ & \quad + \int_{s_0}^s \left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}F_W^{(2,0,0)} \circ \Phi_W^{y_0}(s') \right| \exp\left(-\int_{s'}^s \mathcal{D}_W^{(2,0,0)} \circ \Phi_W^{y_0}(s'') ds''\right) ds'.\end{aligned}$$

Since $\psi^{-1} \leq \eta$, we then have that $\psi^{-1}\eta^{-2}(y_1^2 + |\check{y}|^6) \leq 1$. Moreover, using (4.5), we see that

$$e^{-3s}\psi^{-1} \leq e^{-3s}\eta \leq 40\varepsilon,$$

and thus, we have that

$$(11.23) \quad \frac{3}{2} - \frac{3}{4}\psi^{-1}\frac{y_1^2 + |\check{y}|^6}{\eta^2} - e^{-3s}\psi^{-1} \geq 0.$$

Again, since $\psi^{-1} \leq \eta$, then (4.5) yields

$$|\psi^{-1}(e^{-3s} - \eta^{-2})| \leq \frac{4}{3}\eta^{-1},$$

Therefore, we see from the definition (11.7) of $|\mathcal{D}_\eta|$ that $|\mathcal{D}_\psi| \leq \frac{4}{3}|\mathcal{D}_\eta|$. It follows from (11.10) that

$$(11.24) \quad \int_{s_0}^s \left| \left(\frac{2}{3}\mathcal{D}_\eta + \frac{1}{2}\mathcal{D}_\psi \right) \circ \Phi_W^{y_0}(s') \right| ds' \leq \frac{4}{3} \int_{s_0}^s |\mathcal{D}_\eta \circ \Phi_W^{y_0}(s')| ds' \leq 140 \log \frac{1}{\ell},$$

for all $|y_0| \geq \ell$. By (11.23) and (11.24), we see that (11.22) is bounded as

$$(11.25) \quad \begin{aligned} \left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}\partial_{11}W \circ \Phi_W^{y_0}(s) \right| &\leq \ell^{-140} \left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}\partial_{11}W(y_0) \right| \\ &\quad + \ell^{-140} \int_{s_0}^s \left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}F_W^{(2,0,0)} \circ \Phi_W^{y_0}(s') \right| ds'. \end{aligned}$$

With the estimate (9.19) for $F_W^{(2,0,0)}$, we obtain that

$$\left| \eta^{\frac{1}{3}}\psi^{-\frac{1}{4}}F_W^{(2,0,0)} \right| \lesssim \eta^{-\frac{3}{20} + \frac{8}{3(2m-7)}} \lesssim \eta^{-\frac{1}{10}}.$$

Hence, following (11.12), we see that for $|y_0| \geq \ell$,

$$\int_{s_0}^s |\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} F_W^{(2,0,0)} \circ \Phi_W^{y_0}(s')| ds' \leq \int_{s_0}^{\infty} (1 + \ell^2 e^{\frac{2}{5}(s'-s_0)})^{-\frac{1}{10}} ds' \lesssim \ell^{-\frac{1}{5}}.$$

By appealing to our initial datum assumption (3.17b) if $s_0 = -\log \varepsilon$, and to (4.9a) when $s > -\log \varepsilon$, the bound (11.25) shows that

$$(11.26) \quad |\eta^{\frac{1}{3}} \psi^{-\frac{1}{4}} \partial_{11} W \circ \Phi_W^{y_0}(s)| \lesssim \ell^{-141} \lesssim M^{\frac{1}{4}}.$$

By choosing first M sufficiently large, the bootstrap assumption (4.7) is then improved by (11.26).

It remains to consider the case $|\gamma| = 2$ and $|\check{\gamma}| = 1, 2$. The arguments will mimic those given in section 11.6 of [4], and as such, we provide an abridged version of those arguments. For the case $|\check{\gamma}| = 1$ and $\gamma_1 = 1$, we set $\mu = \frac{1}{3}$, whereas, for the case $|\check{\gamma}| = 2$ and $\gamma_1 = 0$, we set $\mu = \frac{1}{6}$. Consequently, the damping term $3\mu - D_{\mathcal{R}}$ present in (11.11) is given by

$$(11.27) \quad 3\mu - D_{\mathcal{R}} = \begin{cases} -\frac{1}{2} - \beta_{\tau} J \partial_1 W & \text{for } |\check{\gamma}| = 1 \text{ and } \gamma_1 = 1, \\ -\beta_{\tau} J \partial_1 W & \text{for } |\gamma| = 2 \text{ and } \gamma_1 = 0. \end{cases}$$

Let us first restrict to the case $\gamma_1 = 0$ and $|\check{\gamma}| = 2$. Analogous to (11.14), we have

$$(11.28) \quad \int_{s_0}^s \beta_{\tau} |J \partial_1 W| \circ \Phi_W^{y_0}(s') ds' \leq 40 \log \frac{1}{\ell},$$

and analogously to (11.15), applying (9.19), we have

$$(11.29) \quad \int_{s_0}^s |\eta^{\frac{1}{6}} F_W^{(\gamma)}| \circ \Phi_W^{y_0}(s') ds' \leq M^{\frac{5}{6}} \log \frac{1}{\ell}.$$

Substituting the bounds (11.28) and (11.29) into (11.11), and utilizing our initial datum assumption (3.17c) when $s_0 = -\log \varepsilon$, and to (4.9a) when $s > -\log \varepsilon$, we deduce

$$\eta^{\frac{1}{7}}(y) |\check{\nabla}^2 W(y, s)| \leq \ell^{-110} \eta^{\frac{1}{6}}(y_0) |\check{\nabla}^2 W(y_0, s_0)| + M^{\frac{5}{6}} \ell^{-110} \log \frac{1}{\ell} \leq \frac{1}{10} M^{\frac{1}{6}}$$

where we have assumed that M is sufficiently large, used our choice $\ell = (\log M)^{-5}$, and assumed ε is sufficiently small relative to M . Thus we close the bootstrap (4.7) for the case $\gamma_1 = 0$ and $|\check{\gamma}| = 2$.

We now turn our attention to the case $|\check{\gamma}| = 1$, with $\gamma_1 = 1$. Applying (11.27) and (11.28) yields the damping bound

$$(11.30) \quad \exp \left(\int_{s'}^s (3\mu - D_{\mathcal{R}} \circ \Phi_W^{y_0}(s'')) ds'' \right) \leq \ell^{-120} e^{\frac{s'-s}{2}}$$

for any $s > s' > s_0 \geq -\log \varepsilon$. Substituting (11.30), together with the forcing estimate (9.19) into (11.11), and appealing to our initial datum assumption (3.17a) if $s_0 = -\log \varepsilon$, and to (4.9a) when $s > -\log \varepsilon$, we deduce

$$(11.31) \quad \eta^{\frac{1}{3}}(y) |\partial^{\gamma} W(y, s)| \leq \frac{1}{10} M^{\frac{1}{6}}$$

where we have assumed that M is sufficiently large, used our choice $\ell = (\log M)^{-5}$, and assumed ε is sufficiently small relative to M . Thus we close the bootstrap (4.7) for the case $|\check{\gamma}| = 1$, with $\gamma_1 = 1$.

11.5 Estimate of $W(y, s)$, $\partial_1 W(y, s)$, and $\nabla W(y, s)$ for $|y| \geq \mathcal{L}$

The estimates of $W(y, s)$, $\partial_1 W(y, s)$, and $\check{\nabla} W(y, s)$ for $|y| \geq \mathcal{L}$ are nearly identical to those given in sections 11.7, 11.8, and 11.9 of [4]. As such, we prove only an abridged summary of the arguments.

Consider first the estimate on $W(y, s)$. We set $\mu = -\frac{1}{6}$ and $\mathcal{R} = W$, so that $q = \eta^{-\frac{1}{6}} \check{W}$. We have $3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1} = \frac{1}{2}\eta^{-1}$ and $F_q = \eta^{-\frac{1}{6}}(F_W - e^{-\frac{s}{2}}\beta_\tau \dot{\kappa})$. The contribution of the damping in (11.13) gives us

$$\int_{s_0}^s \frac{1}{2}\eta^{-1} \circ \Phi_W^{y_0}(s') ds' \leq \mathcal{L}^{-\frac{2}{3}} = \varepsilon^{\frac{1}{16}},$$

and we have from (9.8) and (4.1b) the forcing bound

$$\int_{s_0}^s |F_q \circ \Phi_W^{y_0}(s')| ds' \lesssim \varepsilon^{\frac{1}{2}}.$$

Substituting the above two estimates into (11.13), we obtain

$$|\eta^{-\frac{1}{6}} W \circ \Phi_W^{y_0}(s)| \leq 1 + \varepsilon^{\frac{1}{19}},$$

where for the case $s_0 > -\log \varepsilon$ we used (4.8a) and \bar{W} bound (2.48) in [4], and for the case $s_0 = -\log \varepsilon$, we used the initial data assumption (3.16a). Thus we close the bootstrap bound in the first line of (4.7).

For the case $\partial_1 W(y, s)$ we set $q = \eta^{\frac{1}{3}} \partial_1 W$ so that $3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1} \leq -\beta_\tau \mathcal{J} \partial_1 W$ and $F_q \eta^{\frac{1}{3}} F_W^{(1,0,0)}$. Applying (4.7) and Lemma 5.2 yields

$$(11.32) \quad \int_{s_0}^s (3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1}) \circ \Phi_W^{y_0}(s') ds' \leq \varepsilon^{\frac{1}{16}}.$$

As a consequence of (9.19) and the fact that $|y_0| \geq \mathcal{L}$, we obtain

$$\int_{s_0}^s |F_q \circ \Phi_W^{y_0}(s')| ds' \lesssim \varepsilon^{3\alpha}.$$

Substituting the above two estimates into (11.13), we obtain

$$|\eta^{\frac{1}{3}} \partial_1 W \circ \Phi_W^{y_0}(s)| \leq \frac{3}{2}.$$

where for the case $s_0 > -\log \varepsilon$ we used (4.8b) and the \bar{W} bound (2.48) in [4], and for the case $s_0 = -\log \varepsilon$, we used the initial data assumption (3.16b). Thus we close the bootstrap bound in the second line of (4.7).

Finally, we consider the estimate of $\check{\nabla} W(y, s)$ for $|y| \geq \mathcal{L}$. We set $\mu = 0$ and $q = \check{\nabla} W$. The damping term is $3\mu - D_{\mathcal{R}} - 3\mu\eta^{-1} = -\beta_\tau \mathcal{J} \partial_1 W$, and so we may

reuse the estimate (11.32). The forcing term F_q may be bounded directly using the third case in (9.19), which yields

$$\int_{s_0}^s |F_q \circ \Phi_W^{y_0}(s')| ds' \leq \varepsilon^{\frac{1}{8}}.$$

We deduce from (11.13) that

$$|\tilde{\nabla} W \circ \Phi_W^{y_0}(s)| \leq \frac{5}{6}.$$

where for the case $s_0 > -\log \varepsilon$, we used (4.8c) and the \bar{W} bound (2.48) in [4], and for the case $s_0 = -\log \varepsilon$, we used the initial data assumption (3.16c). Thus we close the bootstrap bound in the second line of (4.7).

12 Constraints and Evolution of Modulation Variables

12.1 Solving for the dynamic modulation parameters

In Section (2.7) we have used the evolution equations for W , ∇W , and $\nabla^2 W$ at $y = 0$ to derive implicit equations for the time derivatives our modulation parameters. The goal of this subsection is to show that these implicit equations are indeed solvable with the initial conditions (2.53). For this purpose it convenient to introduce the notation

$$\mathcal{P}_\diamond(b_1, \dots, b_n | c_1, \dots, c_n) \quad \text{and} \quad \mathcal{R}_\diamond(b_1, \dots, b_n | c_1, \dots, c_n)$$

to denote a linear function in the parameters c_1, \dots, c_n with coefficients that depend on b_1, \dots, b_n through smooth polynomials (for \mathcal{P}_\diamond) and rational functions (for \mathcal{R}_\diamond), and on the derivatives of Z , A , and K evaluated at $y = 0$. The subscript \diamond denotes a label, used to distinguish the various functions \mathcal{P}_\diamond and \mathcal{R}_\diamond . We note that all of the denominators in \mathcal{R}_\diamond are bounded from below by a universal constant. It is important to note that the notation \mathcal{P}_\diamond and \mathcal{R}_\diamond is never used when explicit bounds are required. Throughout this section, we will use the bootstrap assumptions in Section 4 to establish uniform bounds on the coefficients, which in turn, yields local well-posedness of the coupled system of ODE for the modulation variables.

The definition of $\dot{\kappa}$ in (2.55) may be written schematically using the notation introduced above as

$$(12.1) \quad \dot{\kappa} = \mathcal{P}_\kappa(\kappa, \phi | \dot{Q}, \frac{1}{\beta_\tau} e^{\frac{s}{2}} h_W^0, \frac{1}{\beta_\tau} e^{\frac{s}{2}} G_W^0),$$

where we have used the explicit formula (A.8) to determine the dependence of \mathcal{P}_κ . Once we compute h_W^0 and G_W^0 (cf. (2.60a)–(2.60b) below) we will return to the formula (12.1). We point out at this stage that in (12.17) below we will show that both h_W^0 and G_W^0 decay at a rate which is strictly faster than $e^{-\frac{s}{2}}$, which shows that their contribution to $\dot{\kappa}$ will be under control.

Similarly, the definition of $\dot{\tau}$ in (2.56) may be written schematically as

$$(12.2) \quad \dot{\tau} = \mathcal{P}_\tau(\kappa, \phi | e^{-2s} \dot{Q}, \frac{1}{\beta_\tau} h_W^0),$$

where we have used the explicit formulae (A.7b) and (A.9) to determine the dependence of \mathcal{P}_τ .

The schematic dependence of $\dot{Q}_{1,v}$ is determined from (2.57). Using (A.7c) and (A.10) and placing the leading-order term in \dot{Q} on one side, we obtain

$$\begin{aligned}
 \dot{Q}_{1v} = & -e^{-\frac{s}{2}} \dot{Q}_{1\mu} \partial_v A_\mu^0 + e^{-s} \dot{Q}_{\mu\xi} A_\xi^0 \phi_{\mu\nu} + e^{-s} \dot{Q}_{\mu\nu} A_\xi^0 \phi_{\xi\mu} - \frac{\beta_2}{2\beta_1} e^{\frac{s}{2}} \partial_v Z^0 \\
 & + e^{-s} A_\mu^0 \dot{\phi}_{\mu\nu} + \frac{\beta_3}{2\beta_1} ((\kappa - Z^0) \partial_{v\mu} A_\mu^0 - \partial_v Z^0 \partial_\mu A_\mu^0) \\
 & + \frac{\beta_3}{\beta_1} e^{-\frac{s}{2}} Z^0 \partial_v Z^0 (\phi_{22} + \phi_{33}) \\
 & + \frac{\beta_3}{2\beta_1} e^{-s} (\kappa - Z^0) A_\xi^0 T_{\mu,\nu}^{\xi,0} + e^{-\frac{s}{2}} ((\partial_v A_\mu^0 - \frac{1}{2} e^{-\frac{s}{2}} (\kappa + Z^0) \phi_{\mu\nu}) A_\gamma^0) \phi_{\gamma\mu} \\
 & + \frac{1}{2\beta_1 \beta_\tau} h_W^{\mu,0} \partial_v A_\gamma^0 \phi_{\gamma\mu} + (\frac{1}{2\beta_1 \beta_\tau} e^{\frac{s}{2}} h_W^{\gamma,0} - A_\gamma^0) \phi_{\gamma v} \\
 & - \frac{1}{4} \beta_4 (\kappa - Z^0) ((\kappa - Z^0) (e^s \partial_{1v} K^0 - e^{-\frac{s}{2}} \partial_\mu K \phi_{\mu\nu}) - 2 \partial_v Z^0 e^s \partial_1 K),
 \end{aligned}
 \tag{12.3}$$

which may be written schematically as

$$\dot{Q}_{1v} = \mathcal{P}_{Q,v} \left(\kappa, \phi \mid \frac{1}{\beta_\tau} e^{\frac{s}{2}} h_W^0, e^{-s} \dot{\phi}, e^{-s} \dot{Q} \right).
 \tag{12.4}$$

Note that once \dot{Q}_{1v} is known, we can determine \dot{n}_2 and \dot{n}_3 by recalling from [4, eqs. (A.4)–(A.5)] that

$$\begin{aligned}
 \begin{bmatrix} 1 + \frac{n_2^2}{n_1(1+n_1)} & \frac{n_2 n_3}{n_1(1+n_1)} \\ \frac{n_2 n_3}{n_1(1+n_1)} & 1 + \frac{n_3^2}{n_1(1+n_1)} \end{bmatrix} \begin{bmatrix} \dot{n}_2 \\ \dot{n}_3 \end{bmatrix} = \left(\text{Id} + \frac{\check{n} \otimes \check{n}}{n_1(1+n_1)} \right) \dot{\check{n}} = \begin{bmatrix} \dot{Q}_{12} \\ \dot{Q}_{13} \end{bmatrix},
 \end{aligned}
 \tag{12.5}$$

where $n_1 = \sqrt{1 - n_2^2 - n_3^2}$. Since the vector \check{n} is small (see (4.1a) below), and the matrix on the left side is an $\mathcal{O}(|\check{n}|^2)$ perturbation of the identity matrix, we obtain from (12.5) a definition of $\dot{\check{n}}$, as desired.

Next, we determine the dependence of $h_W^{\mu,0}$ and G_W^0 . Inspecting (A.7d)–(A.7f) and (A.10)–(A.11) and inserting them into (2.60b), we obtain the dependence

$$\frac{1}{\beta_\tau} h_W^{\mu,0} = e^{-\frac{s}{2}} \mathcal{R}_{h,\mu} (\kappa, \phi \mid e^{-s} \dot{Q}, e^{-2s} \dot{\phi}) - \frac{1}{\beta_\tau} h_W^{\gamma,0} (\mathcal{H}^0)^{-1}_{\mu i} \phi_{\xi\gamma} \partial_{1i} A_\xi^0.$$

Note that although h_W^0 appears on both sides of the above, in view of (4.17) the dependence on the right side is paired with a factor less than $e^{-s} \leq \varepsilon$, and the functions $\phi_{\xi\gamma}$ are themselves expected to be $\leq \varepsilon$ for all $s \geq -\log \varepsilon$ (cf. (4.1a) below). This allows us to solve for $h_W^{\mu,0}$ and schematically write

$$\frac{1}{\beta_\tau} h_W^{\mu,0} = e^{-\frac{s}{2}} \mathcal{R}_{h,\mu} (\kappa, \phi \mid e^{-s} \dot{Q}, e^{-2s} \dot{\phi}).
 \tag{12.6}$$

Returning to (2.60a), inspecting (A.7d)–(A.7f) and (A.10)–(A.11), and using (12.6), we also obtain the dependence

$$\frac{1}{\beta_\tau} G_W^0 = e^{-\frac{s}{2}} \mathcal{R}_{h,\mu} (\kappa, \phi \mid e^{-s} \dot{Q}, e^{-2s} \dot{\phi}).
 \tag{12.7}$$

Next, we determine the dependence of $\dot{\xi}_j$. From (2.60a)–(2.60b), (2.61), (A.7a), and the fact that $RR^T = \text{Id}$ we deduce that

$$(12.8) \quad \begin{aligned} \dot{\xi}_j &= R_{ji}(R^T \dot{\xi})_i = R_{j1} \left(\frac{1}{2\beta_1} (\kappa + \beta_2 Z^0) - \frac{1}{2\beta_1 \beta_\tau} e^{-\frac{s}{2}} G_W^0 \right) \\ &\quad + R_{j\mu} \left(A_\mu^0 - \frac{1}{2\beta_1 \beta_\tau} e^{\frac{s}{2}} h_W^{\mu,0} \right) \end{aligned}$$

for $j \in \{1, 2, 3\}$. Using (12.6) and (12.7), we may then schematically write

$$(12.9) \quad \dot{\xi}_j = \mathcal{R}_{\xi,j}(\kappa, \phi \mid e^{-s} \dot{Q}, e^{-2s} \dot{\phi}).$$

Lastly, note that $\dot{\phi}_{\gamma\gamma}$ is determined in terms of $e^{\frac{s}{2}} \partial_{\gamma\gamma} G_W^0$ (which we rewrite in terms of G_W^0 , $h_W^{\mu,0}$ and $\partial_{\gamma\gamma} F_W^0$ via (2.63)) through the first term on the right-side of (A.7e)

$$(12.10) \quad \begin{aligned} \dot{\phi}_{\gamma\gamma} &= -\frac{1}{\beta_\tau} e^{\frac{s}{2}} (G_W^0 \partial_{1\gamma\gamma} W^0 + h_W^{\mu,0} \partial_{\mu\gamma\gamma} W^0 - \partial_{\gamma\gamma} F_W^0) + \beta_2 e^s \partial_{\gamma\gamma} Z^0 \\ &\quad - 2\beta_1 (\dot{Q}_{\xi\gamma} \phi_{\xi\gamma} + \dot{Q}_{\xi v} \phi_{\xi\gamma}) + \left(\frac{1}{\beta_\tau} e^{-\frac{s}{2}} G_W^0 - \kappa - \beta_2 Z^0 \right) \mathbf{N}_{1,\gamma\gamma}^0 \\ &\quad + \mathbf{J}_{\gamma\gamma}^0 \frac{1}{\beta_\tau} e^{-\frac{s}{2}} G_W^0, \end{aligned}$$

and (2.60a) is used to determine G_W^0 . In light of (A.11), (12.7), and (12.10), we may schematically write

$$\dot{\phi}_{\gamma\gamma} = \mathcal{R}_{\phi,\gamma\gamma}(\kappa, \phi \mid e^{-s} \dot{Q}, e^{-s} \dot{\phi}) - \dot{Q}_{\xi\gamma} \phi_{\xi\gamma} - \dot{Q}_{\xi v} \phi_{\xi\gamma},$$

which may be then combined with (12.4) and (12.6) to yield

$$(12.11) \quad \dot{\phi}_{\gamma\gamma} = \mathcal{R}_{\phi,\gamma\gamma}(\kappa, \phi \mid e^{-s} \dot{Q}, e^{-s} \dot{\phi}),$$

thus spelling out the dependences of $\dot{\phi}$ on the other dynamic variables.

The equations (12.1), (12.2), (12.4), (12.9), and (12.11) only *implicitly* define $\dot{\kappa}$, $\dot{\tau}$, \dot{Q}_{1v} , $\dot{\xi}_j$, and $\dot{\phi}_{\gamma\gamma}$. We may, however, spell out this implicit dependence and arrive at an autonomous system of ODEs for all 10 of our modulation parameters, cf. (12.12)–(12.13) below.

By combining (12.4) and (12.6) with (12.5) and recalling (12.11), we obtain that

$$\dot{\phi}_{\gamma\gamma} = \mathcal{R}_{\phi,\gamma\gamma}(\kappa, \phi, \check{n} \mid e^{-s} \check{n}, e^{-s} \dot{\phi}) \quad \text{and} \quad \dot{n}_v = \mathcal{R}_{n,v}(\kappa, \phi, \check{n} \mid e^{-s} \check{n}, e^{-s} \dot{\phi}).$$

Therefore, since $e^{-s} \leq \varepsilon$ and the functions $\mathcal{P}_{\phi,\gamma\gamma}$ and $\mathcal{P}_{n,v}$ are linear in $e^{-s} \check{n}$ and $e^{-s} \dot{\phi}$, then as long as κ , ϕ , and \check{n} remain bounded, and ε is taken to be sufficiently small (in particular, for a short time after $t = -\log \varepsilon$), we may analytically solve for $\dot{\phi}$ and \dot{n} as rational functions (with bounded denominators) of κ , ϕ , and \check{n} , with coefficients which only depend on the derivatives of Z , A , K at $y = 0$. We write this schematically as

$$(12.12) \quad \dot{\phi}_{\gamma\gamma} = \mathcal{E}_{\phi,\gamma\gamma}(\kappa, \phi, \check{n}) \quad \text{and} \quad \dot{n}_v = \mathcal{E}_{n,v}(\kappa, \phi, \check{n}).$$

Here the $\mathcal{E}_{\phi,\gamma v}(\kappa, \phi, \check{n})$ and $\mathcal{E}_{n,v}(\kappa, \phi, \check{n})$ are suitable smooth functions of their arguments, as described above. With (12.12) in hand, we return to (12.1) and (12.2), which are to be combined with (12.6) and (12.9) to obtain that

$$(12.13) \quad \dot{\kappa} = \mathcal{E}_{\kappa}(\kappa, \phi, \check{n}), \quad \dot{\tau} = \mathcal{E}_{\tau}(\kappa, \phi, \check{n}), \quad \dot{\xi}_j = \mathcal{E}_{\xi,j}(\kappa, \phi, \check{n}),$$

for suitable smooth functions \mathcal{E}_{κ} , \mathcal{E}_{τ} , and $\mathcal{E}_{\xi,j}$ of $(\kappa, \phi, \check{n})$, with coefficients which depend on the derivatives of Z , A , and K at $y = 0$.

Remark 12.1 (Local solvability). The system of ten nonlinear ODEs described in (12.12) and (12.13) are used to determine the time evolutions of our 10 dynamic modulation variables. The local-in-time solvability of this system is ensured by the fact that $\mathcal{E}_{\phi,\gamma v}$, $\mathcal{E}_{n,v}$, \mathcal{E}_{κ} , \mathcal{E}_{τ} , and $\mathcal{E}_{\xi,j}$ are rational functions of κ, ϕ, n_2 and n_3 , with coefficients that only depend on $\partial^\gamma Z^0$, $\partial^\gamma A^0$, and $\partial^\gamma K^0$ with $|\gamma| \leq 3$, and moreover that these functions are smooth in the neighborhood of the initial values given by (2.53); hence, unique C^1 solutions exist for a sufficiently small time. We emphasize that these functions are explicit.

12.2 Closure of Bootstrap Estimates for the Dynamic Variables

Once one traces back the identities in Section 12.1 and Appendix A.3 we may close the bootstrap assumptions for the modulation parameters, (4.1).

The starting point is to obtain bounds for G_W^0 and $h_W^{\mu,0}$ by appealing to (2.60a)–(2.60b). The matrix \mathcal{H}^0 defined in (2.59) can be rewritten as

$$\begin{aligned} \mathcal{H}^0(s) &= (\partial_1 \nabla^2 W)^0(s) = (\partial_1 \nabla^2 \bar{W})^0 + (\partial_1 \nabla^2 \tilde{W})^0(s) \\ &= \text{diag}(6, 2, 2) + (\partial_1 \nabla^2 \tilde{W})^0(s). \end{aligned}$$

From the bootstrap assumption (4.10) we have that $|(\partial_1 \nabla^2 \tilde{W})^0(s)| \leq \varepsilon^{1/4}$ for all $s \geq -\log \varepsilon$, and thus

$$(12.14) \quad |(\mathcal{H}^0)^{-1}(s)| \leq 1$$

for all $s \geq -\log \varepsilon$. Next, we estimate $\partial_1 \nabla F_W^0$. Using (A.10), (A.11), the bootstrap assumptions (4.1a)–(4.3), the bounds (4.12)–(4.20), and the fact that $|\mathbf{T}_{\mu,\mu\nu}^{\zeta,0}| \leq |\phi|^2$, after a computation we arrive at

$$\begin{aligned} |\partial_1 \nabla F_W^0| &\lesssim M \varepsilon^{\frac{1}{2}} e^{-s} + M^2 e^{-\frac{3}{2}(1-\frac{4}{2m-5})s} + |h_W^{\cdot,0}| M^3 \varepsilon e^{-\frac{3}{2}(1-\frac{4}{2m-5})s} \\ (12.15) \quad &+ M e^{-(1-\frac{5}{2m-7})s} \\ &\lesssim \varepsilon^2 |h_W^{\cdot,0}| + M e^{-(1-\frac{5}{2m-7})s}. \end{aligned}$$

Moreover, from (A.7d), (A.7f), (4.1a), and (4.1b), the first line in (4.12), and the previously established bound (12.15) we establish that

$$\begin{aligned} |\partial_1 \nabla G_W^0| + |\partial_1 \nabla F_W^0| \\ (12.16) \quad &\lesssim e^{\frac{s}{2}} |\partial_1 \nabla Z^0| + M^4 \varepsilon^{\frac{3}{2}} e^{-\frac{3s}{2}} + \varepsilon^2 |h_W^{\cdot,0}| + M e^{-(1-\frac{5}{2m-7})s} \\ &\lesssim \varepsilon^2 |h_W^{\cdot,0}| + M e^{-(1-\frac{5}{2m-7})s}. \end{aligned}$$

The bounds (12.14) and (12.16), are then inserted into (2.60a)–(2.60b). After absorbing the $\varepsilon^2 |h_W^{0,0}|$ term into the left side, we obtain to estimate

$$(12.17) \quad |G_W^0(s)| + |h_W^{\mu,0}(s)| \lesssim M e^{-(1-\frac{5}{2m-7})s}.$$

The bound (12.17) plays a crucial role in the following subsections. We note that for $m \geq 18$ we have $1 - \frac{5}{2m-7} > \frac{4}{5}$, and hence so the bound (12.17) implies that $|G_W^0(s)| + |h_W^{\mu,0}(s)| \lesssim M e^{-4s/5}$.

The $\dot{\tau}$ estimate

From (2.56), the definition of $\partial_1 G_W^0$ in (A.7b), the definition of $\partial_1 F_W^0$ in (A.9), the bootstrap estimates (4.1a)–(4.3), (4.12)–(4.14), and the previously established bound (12.17), we obtain that

$$(12.18) \quad \begin{aligned} |\dot{\tau}| &\lesssim |\partial_1 G_W^0| + |\partial_1 F_W^0| \\ &\lesssim e^{\frac{s}{2}} |\partial_1 Z^0| + e^{-\frac{s}{2}} |\check{\nabla} A^0| + M |\check{\nabla} \partial_+ M^2 \varepsilon e^{-2s} |A^0| 1A^0| \\ &\quad + M^2 \varepsilon^{\frac{1}{2}} e^{-\frac{s}{2}} |\partial_1 A^0| + M^3 \varepsilon e^{-s} + M e^s |\partial_{11} K^0| + M e^{\frac{s}{2}} |\partial_1 S^0| \\ &\lesssim M^{\frac{1}{2}} e^{-s} + M \varepsilon^{\frac{1}{2}} e^{-s} + M e^{-\frac{3}{2}(1-\frac{2}{2m-5})s} + M^3 \varepsilon e^s + M \varepsilon^{\frac{1}{8}} e^{-s} \\ &\leq \frac{M}{4} e^{-s}, \end{aligned}$$

where we have used a power of M to absorb the implicit constant in the first inequality above. This improves the bootstrap bound for $\dot{\tau}$ in (4.1b) by a factor of 4. Integrating in time from $-\varepsilon$ to T_* where $|T_*| \leq \varepsilon$, we also improve the τ bound in (4.1a) by a factor of 2, thereby closing the τ bootstrap.

The $\dot{\kappa}$ estimate

From (2.55)–(4.3), the bound (12.17), the definition of F_W^0 in (A.8), the estimates (4.12)–(4.14), and the fact that $\frac{5}{2m-7} < \frac{1}{5}$, we deduce that

$$\begin{aligned} |\dot{\kappa}| &\lesssim e^{\frac{s}{2}} |G_W^0| + e^{\frac{s}{2}} |F_W^0| \\ &\lesssim M e^{-\frac{s}{2} + \frac{5s}{2m-7}} + (\kappa_0 + M\varepsilon) M \varepsilon^{\frac{1}{2}} e^{-\frac{s}{2}} + M^3 \varepsilon^{\frac{3}{2}} e^{-\frac{s}{2}} + M^4 \varepsilon^2 e^{-\frac{s}{2}} \\ &\quad + e^{-\frac{s}{2}} (\kappa_0^2 + M^2 \varepsilon^2) M^2 \varepsilon + (\kappa_0 + M\varepsilon) \varepsilon^{\frac{1}{4}} e^{-\frac{s}{2}} \\ &\leq \frac{1}{2} e^{-\frac{3s}{10}}. \end{aligned}$$

Here we have used a small (m -dependent) power of ε to absorb the implicit constant in the second estimate above, thereby improving the $\dot{\kappa}$ bootstrap bound in (4.1b) by a factor of 2. Integrating in time, we furthermore deduce that

$$(12.19) \quad |\kappa(t) - \kappa_0| \leq \varepsilon^{\frac{13}{10}}$$

since $|T_*| \leq \varepsilon$. Upon taking ε to be sufficiently small in terms of κ_0 , we improve the κ bound in (4.1a).

The $\dot{\xi}$ estimate

In order to bound the $\dot{\xi}$ vector, we appeal to (12.8), to (12.17), to the $|\gamma| = 0$ cases in (4.12) and (4.13), to the bound $|R - \text{Id}| \leq \varepsilon$, and to the $|\check{n}|$ estimate in (4.1a) to deduce that

$$(12.20) \quad \begin{aligned} |\dot{\xi}_j| &\lesssim \kappa_0 + |Z^0| + e^{-\frac{s}{2}} |G_W^0| + |A_\mu^0| + e^{\frac{s}{2}} |h_W^{\mu,0}| \\ &\lesssim \kappa_0 + M\varepsilon + Me^{-\frac{s}{2} + \frac{5s}{2m-7}} \lesssim \kappa_0, \end{aligned}$$

upon taking ε sufficiently small in terms of M and κ_0 . The bootstrap estimate for $\dot{\xi}$ in (4.1b) is then improved by taking M sufficiently large, in terms of κ , while the bound on ξ in (4.1a) follows by integration in time.

The $\dot{\phi}$ estimate

Using (12.10), the fact that $|N_{1,\mu\nu}^0| + |J_{,\mu\nu}^0| \lesssim |\phi|^2$, the bootstrap assumptions (4.1a), (4.1b), (4.10), the bounds (4.2), and the previously established estimate (12.17), we obtain

$$(12.21) \quad \begin{aligned} |\dot{\phi}_{\gamma\nu}| &\lesssim e^{\frac{s}{2}} (Me^{-s(1-\frac{5}{2m-7})} + |\partial_{\gamma\nu} F_W^0|) + e^s |\partial_{\gamma\nu} Z^0| + M^4 \varepsilon^{\frac{3}{2}} \\ &\quad + (Me^{-\frac{3s}{2} + \frac{5s}{2m-7}} + \kappa_0 + |Z^0|) M^4 \varepsilon^2 + M^5 \varepsilon^2 e^{-\frac{3s}{2} + \frac{5s}{2m-7}}. \end{aligned}$$

Using the definition of $\check{\nabla}^2 F_W^0$ in (A.11), appealing to the bootstrap assumptions (and their consequences) from Section 4, the previously established estimate (12.17), and the fact that $|T_{\mu,\gamma\nu}^{\xi,0}| + |N_{1,\mu\nu}^0| + |J_{,\mu\nu}^0| + |N_{\xi,\mu\nu\gamma}^0| \lesssim |\phi|^2$, after a lengthy computation one may show that

$$|\partial_{\gamma\nu} F_W^0| \lesssim e^{-\frac{s}{2}},$$

which shows that the term $e^{s/2} |\partial_{\gamma\nu} F_W^0|$ in (12.21) is subdominant when compared to $e^s |\partial_{\gamma\nu} Z^0| \lesssim M$ present in (12.21). In establishing the above estimate it was crucial that $e^s |\partial_{1\gamma\nu} K^0| \lesssim e^{-\frac{s}{2}}$, which from (4.20) since $m \geq 18$. Combining the above two estimates with the Z bounds in (4.12), we derive

$$(12.22) \quad \begin{aligned} |\dot{\phi}_{\gamma\nu}| &\lesssim e^{\frac{s}{2}} (Me^{-\frac{4s}{5}} + e^{-\frac{s}{2}}) + M + M^4 \varepsilon^{\frac{3}{2}} \\ &\quad + (Me^{-s} + \kappa_0 + \varepsilon M) M^4 \varepsilon^2 + M^5 \varepsilon^2 e^{-s} \\ &\lesssim M. \end{aligned}$$

Taking M sufficiently large to absorb the implicit constant, we deduce $|\dot{\phi}| \leq \frac{1}{4} M^2$, which improves the $\dot{\phi}$ bootstrap in (4.1b) by a factor of 4. Integrating in time on $[-\varepsilon, T_*)$, an interval of length $\leq 2\varepsilon$, and using that $|\phi(-\log \varepsilon)| \leq \varepsilon$, we improve the ϕ bootstrap in (4.1a) by a factor of 2.

The \dot{n} estimate

First we obtain estimates on $|\dot{Q}_{1\nu}|$, by appealing to the identity (12.3). Using the bootstrap assumptions (4.1a), (4.1b), (4.12)–(4.14), the estimates (4.2) and (12.17), and the fact that $|\mathbf{T}_{\mu,\mu\nu}^{\zeta,0}| \lesssim |\phi|^2$, we obtain

$$\begin{aligned}
 (12.23) \quad |\dot{Q}_{1\nu}| &\lesssim M^2 \varepsilon^{\frac{1}{2}} e^{-\frac{s}{2}} |\partial_\nu A_\mu^0| + M^4 \varepsilon^{\frac{3}{2}} e^{-s} |A^0| + e^{\frac{s}{2}} |\check{\nabla} Z^0| + M^2 e^{-s} |A^0| \\
 &\quad (M |\check{\nabla}^2 A^0| + |\check{\nabla} Z^0| |\check{\nabla} A^0|) + M^2 \varepsilon e^{-\frac{s}{2}} |Z^0| |\check{\nabla} Z^0| + M^5 \varepsilon^2 e^{-s} |A^0| \\
 &\quad + e^{-\frac{s}{2}} ((|\check{\nabla} A^0| + M^3 \varepsilon e^{-\frac{s}{2}}) |A^0|) M^2 \varepsilon + M^3 \varepsilon e^{-s} |\check{\nabla} A^0| + M^2 \varepsilon (M e^{-\frac{s}{2}} + |A^0|) \\
 &\quad (\kappa_0 + M \varepsilon) ((\kappa_0 + M \varepsilon) (e^s |\partial_1 \check{\nabla} K^0| + M^2 \varepsilon e^{-\frac{s}{2}} |\check{\nabla} K^0|) - 2 |\check{\nabla} Z^0| e^s |\partial_1 K^0|) \\
 &\lesssim M \varepsilon^{\frac{1}{2}},
 \end{aligned}$$

upon taking ε sufficiently small in terms of M . Moreover, using the bootstrap assumption $|\check{n}| \leq M \varepsilon^{\frac{3}{2}}$, we deduce that the matrix on the left side of (12.5) is within ε of the identity matrix, and thus so is its inverse. We deduce from (12.5) and (12.23) that

$$(12.24) \quad |\dot{n}| \leq \frac{M^2 \varepsilon^{\frac{1}{2}}}{4}.$$

upon taking M to be sufficiently large to absorb the implicit constant. The closure of the \check{n} bootstrap is then achieved by integrating in time on $[-\varepsilon, T_*)$.

13 Conclusion of the Proof: Theorems 3.2 and 3.3

We first note that the system (2.32) for the unknowns (W, Z, A, K) , with initial data (W_0, Z_0, Z_0, K_0) chosen to satisfy the conditions of the theorem, is locally well-posed. To see this, we note that the transformations from (1.3) to (2.32) are smooth for sufficiently short time, and that (1.3) is locally well-posed in the Sobolev space H^k for $k \geq 3$. Here we have implicitly used that the system of 10 nonlinear ODEs (12.12) and (12.13), which specify the modulation functions have local-in-time existence and uniqueness as discussed in Remark 12.1. Moreover, solutions to (1.3) satisfy the following continuation principle (see, for example, [19]): Suppose $(u, \sigma, k) \in C([-\varepsilon, T), H^k)$ is a solution to (1.3) satisfying the uniform bound $\|u(\cdot, t)\|_{C^1} + \|\sigma(\cdot, t)\|_{C^1} + \|k(\cdot, t)\|_{C^1} \leq K < \infty$, then if in addition σ is uniformly bounded from below on the interval $[-\varepsilon, T)$, there exists $T_1 > T$ such that (u, σ, k) extends to a unique solution of (1.3) on $[0, T_1)$. Consequently, the solution (W, Z, A, K) in self-similar variables may be continued so long as (W, Z, A, K) remain uniformly bounded in H^k , the modulation functions remain bounded, and the density remains bounded from below.

In Sections 5–12, we close the bootstrap assumptions on W, Z, A, K and on the modulation functions. By Proposition 4.6, the density remains uniformly strictly positive and bounded. Thus, as a consequence of the continuation principle stated above, we obtain a global-in-self-similar-time solution $(W, Z, A, K) \in C([-\log \varepsilon, +\infty); H^m) \cap C^1([-\log \varepsilon, +\infty); H^{m-1})$ to (2.32) for $m \geq 18$. This

solution satisfies the bounds stated in Sections 4.2–4.6. The asymptotic stability of $W(y, s)$ follows from:

THEOREM 13.1 (Convergence to stationary solution). *There exists a 10-dimensional symmetric 3-tensor \mathcal{A} such that, with $\bar{W}_{\mathcal{A}}$ defined in Appendix A.1, we have that the solution $W(\cdot, s)$ of (2.32a) satisfies*

$$\lim_{s \rightarrow \infty} W(y, s) = \bar{W}_{\mathcal{A}}(y)$$

for any fixed $y \in \mathbb{R}^3$.

We note that the proof of Theorem 13.1 is the same as the proof of Theorem 13.4 in [4] once we include the contributions of the entropy function K , which can be estimated using (4.14). The limiting profile $\bar{W}_{\mathcal{A}}$ satisfies the conditions stated in Theorem 3.2 due to Proposition A.1.

The remaining conclusions of Theorem 3.2 follow from the statements given in Sections 4.7 and 4.8 (for the time and location of the singularity, and the regularity of the solution at this time), Proposition 4.3 (for the vanishing of derivatives of A , Z , and K as $s \rightarrow \infty$), Proposition 6.1 (for the vorticity upper bounds), and Theorem 7.4 (for the vorticity creation estimates).

The proof of Theorem 3.3 is the same as the proof of Theorem 3.2 in [4]. The addition of entropy does not necessitate modifications to that proof as the assumptions on the initial entropy in Theorem 3.2 (see (3.20) and (3.23)) are stable with respect to small perturbations.

Appendix A

A.1 A family of self-similar solutions to the 3D Burgers equation

PROPOSITION A.1 (Stationary solutions for self-similar 3D Burgers). *Let \mathcal{A} be a symmetric 3-tensor such that $\mathcal{A}_{1jk} = \mathcal{M}_{jk}$ with \mathcal{M} a positive definite symmetric matrix. Then, there exists a C^∞ solution $\bar{W}_{\mathcal{A}}$ to*

$$(A.1) \quad -\frac{1}{2} \bar{W}_{\mathcal{A}} + \left(\frac{3y_1}{2} + \bar{W}_{\mathcal{A}} \right) \partial_1 \bar{W}_{\mathcal{A}} + \frac{\check{y}}{2} \cdot \check{\nabla} \bar{W}_{\mathcal{A}} = 0,$$

which has the following properties:

- $\bar{W}_{\mathcal{A}}(0) = 0$, $\partial_1 \bar{W}_{\mathcal{A}}(0) = -1$, $\partial_2 \bar{W}_{\mathcal{A}}(0) = 0$,
- $\partial^\alpha \bar{W}_{\mathcal{A}}(0) = 0$ for $|\alpha|$ even,
- $\partial^\alpha \bar{W}_{\mathcal{A}}(0) = \mathcal{A}_\alpha$ for $|\alpha| = 3$.

See appendix A.1 in [4] for the proof of Proposition A.1.

A.2 Interpolation

The following is taken from [4, appendix A.3]. We include the inequalities here for convenience to the reader.

LEMMA A.2 (Gagliardo-Nirenberg-Sobolev). *Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$. Fix $1 \leq q, r \leq \infty$ and $j, m \in \mathbb{N}$, and $\frac{j}{m} \leq \alpha \leq 1$. Then, if*

$$\frac{1}{p} = \frac{j}{d} + \alpha \left(\frac{1}{r} - \frac{m}{d} \right) + \frac{1-\alpha}{q},$$

then

$$(A.2) \quad \|D^j u\|_{L^p} \leq C \|D^m u\|_{L^r}^\alpha \|u\|_{L^q}^{1-\alpha}.$$

We shall make use of (A.2) for the case that $p = \frac{2m}{j}$, $r = 2$, $q = \infty$, which yields

$$(A.3) \quad \|D^j \varphi\|_{L^{\frac{2m}{j}}} \lesssim \|\varphi\|_{\dot{H}^m}^{\frac{j}{m}} \|\varphi\|_{L^\infty}^{1-\frac{j}{m}},$$

whenever $\varphi \in H^m(\mathbb{R}^3)$ has compact support. The above estimate and the Leibniz rule classically imply the Moser inequality

$$(A.4) \quad \|\phi \varphi\|_{\dot{H}^m} \lesssim \|\phi\|_{L^\infty} \|\varphi\|_{\dot{H}^m} + \|\phi\|_{\dot{H}^m} \|\varphi\|_{L^\infty}$$

for all $\phi, \varphi \in H^m(\mathbb{R}^3)$ with compact support. At various stages in the proof we also appeal to the following special case of (A.2)

$$(A.5) \quad \|\varphi\|_{\dot{H}^{m-2}} \lesssim \|\varphi\|_{\dot{H}^{m-1}}^{\frac{2m-7}{2m-5}} \|\varphi\|_{L^\infty}^{\frac{2}{2m-5}},$$

for $\varphi \in H^{m-1}(\mathbb{R}^3)$ with compact support. Lastly, in Section 8 we make use of the following:

LEMMA A.3. *Let $m \geq 4$ and $0 \leq l \leq m-3$. Then for $a + b = 1 - \frac{1}{2m-4} \in (0, 1)$ and $q = \frac{6(2m-3)}{2m-1}$,*

$$(A.6) \quad \|D^{2+l} \phi D^{m-1-l} \varphi\|_{L^2} \lesssim \|D^m \phi\|_{L^2}^a \|D^m \varphi\|_{L^2}^b \|D^2 \phi\|_{L^q}^{1-a} \|D^2 \varphi\|_{L^q}^{1-b}.$$

See [4] for the proof.

A.3 The functions G_W , F_W , and their derivatives at $y = 0$

Using (2.13), the definition of G_W in (2.33a), and the constraints in (2.52), we deduce that the first and second derivatives of G_W evaluate at $y = 0$ are given by¹¹

$$(A.7a) \quad \frac{1}{\beta_\tau} \partial_1 G_W^0 = \beta_2 e^{\frac{s}{2}} \partial_1 Z^0,$$

$$(A.7b) \quad \frac{1}{\beta_\tau} \partial_v G_W^0 = \beta_2 e^{\frac{s}{2}} \partial_v Z^0 + 2\beta_1 (\dot{Q}_{1v} + A_\gamma^0 \phi_{\gamma v}) - e^{\frac{s}{2}} \frac{1}{\beta_\tau} h_W^{\gamma 0} \phi_{\gamma v},$$

$$(A.7c) \quad \frac{1}{\beta_\tau} \partial_{11} G_W^0 = \beta_2 e^{\frac{s}{2}} \partial_{11} Z^0,$$

$$(A.7d) \quad \frac{1}{\beta_\tau} \partial_{1v} G_W^0 = \beta_2 e^{\frac{s}{2}} \partial_{1v} Z^0 - 2\beta_1 e^{-\frac{3s}{2}} \dot{Q}_{\gamma 1} \phi_{\gamma v},$$

¹¹ Here we have used the identities: $T_{\mu, \nu}^{\gamma 0} = 0$, $N_{\mu, \nu \gamma}^0 = 0$, and $T_{1, v \gamma}^{\zeta, 0} = 0$, $N_{1, v}^0 = 0$, and $N_{\mu, v}^0 = -\phi_{\mu v}$, $N_{\zeta, \mu v}^0 = 0$.

$$\begin{aligned}
\text{(A.7e)} \quad \frac{1}{\beta_\tau} \partial_{\gamma\nu} G_W^0 &= e^{-\frac{s}{2}} \left(-\dot{\phi}_{\gamma\nu} + \beta_2 e^s \partial_{\gamma\nu} Z^0 \right. \\
&\quad \left. - 2\beta_1 (\dot{Q}_{\xi\gamma} \phi_{\xi\nu} + \dot{Q}_{\xi\nu} \phi_{\xi\gamma} + R_{j1} \dot{\xi}_j N_{1,\gamma\nu}^0 \right. \\
\text{(A.7f)} \quad &\quad \left. + e^{-\frac{s}{2}} \frac{G_W^0}{\beta_\tau} J_{,\gamma\nu}^0 \right).
\end{aligned}$$

Appealing to (2.13) and (2.35a), which is equivalent to

$$\begin{aligned}
\frac{1}{\beta_\tau} F_W &= -2\beta_3 S T_{\mu}^{\nu} \partial_{\mu} A_{\nu} + 2\beta_1 e^{-\frac{s}{2}} A_{\nu} T_i^{\nu} \dot{N}_i + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{ij} A_{\nu} T_j^{\nu} N_i \\
&\quad + \left(\frac{1}{\beta_\tau} h_W^{\mu} - \beta_3 e^{-\frac{s}{2}} N_{\mu} \left(\kappa + e^{-\frac{s}{2}} W - \frac{\beta_1 + \beta_2}{\beta_3} Z \right) \right) A_{\gamma} T_i^{\gamma} N_{i,\mu} \\
&\quad - 2\beta_3 e^{-\frac{s}{2}} S (A_{\nu} T_{\mu,\mu}^{\nu} + U \cdot N N_{\mu,\mu}) + \beta_4 S^2 (J e^s \partial_1 K + N_{\mu} \partial_{\mu} K),
\end{aligned}$$

we may derive the following explicit expressions for F_W and its derivatives up to order 2, evaluated at $y = 0$ ¹² for the case of no derivatives we have

$$\begin{aligned}
\text{(A.8)} \quad \frac{1}{\beta_\tau} F_W^0 &= -\beta_3 (\kappa - Z^0) \partial_{\mu} A_{\mu}^0 + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{1\mu} A_{\mu}^0 - \frac{1}{\beta_\tau} h_W^{\mu,0} A_{\xi}^0 \phi_{\xi\mu} \\
&\quad + \frac{1}{2} \beta_3 e^{-\frac{s}{2}} (\kappa - Z^0) (\kappa + Z^0) (\phi_{22} + \phi_{33}) + \frac{1}{4} \beta_4 (\kappa - Z^0)^2 e^s \partial_1 K^0;
\end{aligned}$$

the first ∂_1 derivative of F_W is given by

$$\begin{aligned}
\text{(A.9)} \quad \frac{1}{\beta_\tau} \partial_1 F_W^0 &= \beta_3 (e^{-\frac{s}{2}} + \partial_1 Z^0) \partial_{\mu} A_{\mu}^0 - \beta_3 (\kappa - Z^0) \partial_{1\mu} A_{\mu}^0 + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{1\mu} \partial_1 A_{\mu}^0 \\
&\quad - \left(\frac{1}{\beta_\tau} h_W^{\mu,0} \partial_1 A_{\xi}^0 + 2\beta_1 e^{-\frac{s}{2}} (\partial_1 A_{\mu}^0 + e^{-\frac{s}{2}} \dot{Q}_{\mu 1}) A_{\xi}^0 \right) \phi_{\xi\mu} \\
&\quad - \frac{1}{2} \beta_3 e^{-s} ((1 + e^{\frac{s}{2}} \partial_1 Z^0) (\kappa + Z^0) + (\kappa - Z^0) (1 - e^{\frac{s}{2}} \partial_1 Z^0)) (\phi_{22} + \phi_{33}) \\
&\quad + \frac{1}{4} \beta_4 (\kappa - Z^0) ((\kappa - Z^0) e^s \partial_{11} K^0 - 2(e^{-\frac{s}{2}} + \partial_1 Z^0) e^s \partial_1 K^0);
\end{aligned}$$

the first $\tilde{\nabla}$ is given by

$$\begin{aligned}
\frac{1}{\beta_\tau} \partial_{\nu} F_W^0 &= -\beta_3 ((\kappa - Z^0) \partial_{\nu\mu} A_{\mu}^0 - \partial_{\nu} Z^0 \partial_{\mu} A_{\mu}^0) - 2\beta_1 e^{-s} A_{\mu}^0 \dot{\phi}_{\mu\nu} \\
&\quad + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{1\mu} \partial_{\nu} A_{\mu}^0 - 2\beta_1 e^{-s} \dot{Q}_{\mu\xi} A_{\xi}^0 \phi_{\mu\nu} \\
&\quad - \beta_3 e^{-\frac{s}{2}} Z^0 \partial_{\nu} Z^0 (\phi_{22} + \phi_{33}) \\
&\quad - \beta_3 e^{-s} (\kappa - Z^0) A_{\xi}^0 T_{\mu,\mu\nu}^{\xi,0} - \frac{1}{\beta_\tau} h_W^{\mu,0} \partial_{\nu} A_{\gamma}^0 \phi_{\gamma\mu} \\
&\quad - 2\beta_1 e^{-\frac{s}{2}} ((e^{-\frac{s}{2}} \dot{Q}_{\mu\nu} + \partial_{\nu} A_{\mu}^0 - \frac{1}{2} e^{-\frac{s}{2}} (\kappa + Z^0) \phi_{\mu\nu}) A_{\gamma}^0) \phi_{\gamma\mu} \\
&\quad + \frac{1}{4} \beta_4 (\kappa - Z^0) ((\kappa - Z^0) (e^s \partial_{1\nu} K^0 - e^{-\frac{s}{2}} \partial_{\mu} K \phi_{\mu\nu}) - 2\partial_{\nu} Z^0 e^s \partial_1 K);
\end{aligned}$$

¹² Here we have used the identities: $N_{\mu,\mu}^0 = -\phi_{22} - \phi_{33}$, $T_{\mu,\mu}^{\nu,0} = 0$, $\dot{N}_i^0 = 0$, $\dot{N}_{1,\nu}^0 = 0$, $\dot{N}_{\mu,\nu}^0 = -\dot{\phi}_{\mu\nu}$, $T_{1,\nu}^{\gamma,0} = \phi_{\gamma\nu}$, $T_{i,\nu}^{\gamma,0} N_{i,\mu}^0 = 0$, $T_i^{\gamma,0} N_{i,\mu\nu}^0 = 0$, $N_{\mu,\mu\nu}^0 = 0$, $\dot{N}_{\xi,\nu\gamma}^0 = 0$, and $J_{,\nu\gamma}^0 = \phi_{2\nu} \phi_{2\gamma} + \phi_{3\nu} \phi_{3\gamma}$.

the second ∂_1 derivative of F_W at $y = 0$ is given by

$$\begin{aligned}
 & \frac{1}{\beta_\tau} \partial_{11} F_W^0 \\
 (A.10) \quad & = \beta_3(e^{-\frac{s}{2}} + \partial_1 Z^0) \partial_\mu A_\mu^0 - \beta_3(\kappa - Z^0) \partial_{1\mu} A_\mu^0 + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{1\mu} \partial_{11} A_\mu^0 \\
 & - (2\beta_1 e^{-\frac{s}{2}} + \frac{1}{\beta_\tau} h_W^{\mu,0}) \partial_{11} A_\xi^0 \phi_{\xi\mu} - 4\beta_1 e^{-\frac{s}{2}} (\partial_1 A_\mu^0 + e^{-\frac{3s}{2}} \dot{Q}_{\mu 1}) \partial_1 A_\xi^0 \phi_{\xi\mu} \\
 & - \beta_3 e^{-\frac{s}{2}} (Z^0 \partial_{11} Z^0 - e^{-s} (1 - e^s (\partial_1 Z^0)^2)) (\phi_{22} + \phi_{33}) + \frac{\beta_4}{2} (1 + e^{\frac{s}{2}} \partial_1 Z^0)^2 \partial_1 K^0 \\
 & + \beta_4 (\kappa - Z^0) (\frac{1}{4} (\kappa - Z^0) e^s \partial_{111} K^0 - (e^{-\frac{s}{2}} + \partial_1 Z^0) e^s \partial_{11} K^0 - \frac{1}{2} \partial_{11} Z^0 e^s \partial_1 K);
 \end{aligned}$$

a ∂_1 derivative combined with a $\tilde{\nabla}$ derivative is given by

$$\begin{aligned}
 & \frac{1}{\beta_\tau} \partial_{1v} F_W^0 \\
 & = -\beta_3 \left((\kappa - Z^0) \partial_{1v\mu} A_\mu^0 - \partial_{1v} Z^0 \partial_\mu A_\mu^0 - \partial_v Z^0 \partial_{1\mu} A_\mu^0 - (e^{-\frac{s}{2}} + \partial_1 Z^0) \partial_{v\mu} A_\mu^0 \right) \\
 & - 2\beta_1 e^{-s} \partial_1 A_\mu^0 \dot{\phi}_{\mu v} + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{1\mu} \partial_{1v} A_\mu^0 - 2\beta_1 e^{-s} \dot{Q}_{\mu\xi} \partial_1 A_\xi^0 \phi_{\mu v} \\
 & - \beta_3 e^{-\frac{s}{2}} (\partial_1 Z^0 \partial_v Z^0 + Z^0 \partial_{1v} Z^0) (\phi_{22} + \phi_{33}) \\
 & - \beta_3 e^{-s} \left((\kappa - Z^0) \partial_1 A_\xi^0 - (e^{-\frac{s}{2}} + \partial_1 Z^0) A_\xi^0 \right) \mathbb{T}_{\mu,\nu}^{\xi,0} \\
 & - 2\beta_1 e^{-\frac{s}{2}} \left((e^{-\frac{s}{2}} \dot{Q}_{\mu\nu} + \partial_v A_\mu^0) \partial_1 A_\nu^0 + (e^{-\frac{3s}{2}} \dot{Q}_{\mu 1} + \partial_1 A_\mu^0) \partial_v A_\nu^0 + A_\mu^0 \partial_{1v} A_\nu^0 \right) \phi_{\gamma\mu} \\
 & - \frac{1}{\beta_\tau} h_W^{\mu,0} \partial_{1v} A_\gamma^0 \phi_{\gamma\mu} + \beta_1 e^{-s} \left((\kappa + Z^0) \partial_1 A_\gamma^0 - (e^{-\frac{s}{2}} - \partial_1 Z^0) A_\gamma^0 \right) \phi_{\mu\nu} \phi_{\gamma\mu} \\
 & - \frac{1}{2} \beta_4 (\kappa - Z^0) (\partial_{1v} Z^0 e^s \partial_1 K^0 + \partial_v Z^0 e^s \partial_{11} K^0) \\
 & - \frac{1}{2} \beta_4 (\kappa - Z^0) (e^{-\frac{s}{2}} + \partial_1 Z^0) (e^s \partial_{1v} K^0 - \phi_{\mu\nu} e^{-\frac{s}{2}} \partial_\mu K^0) \\
 & + \frac{1}{2} \beta_4 (e^{-\frac{s}{2}} + \partial_1 Z^0) \partial_v Z^0 \partial_1 K^0 e^s + \frac{1}{4} \beta_3 \beta_4 (\kappa - Z^0)^2 (e^s \partial_{11v} K^0 - \phi_{\mu\nu} e^{-\frac{s}{2}} \partial_{1\mu} K^0)
 \end{aligned}$$

and lastly, the second $\tilde{\nabla}$ derivative is given by

$$\begin{aligned}
 & \frac{1}{\beta_\tau} \partial_{\gamma v} F_W^0 \\
 & = -2\beta_3 (\partial_{v\gamma} (K \partial_\mu A_\mu))^0 - \beta_3 e^{-s} (\kappa - Z^0) \partial_\mu A_\xi^0 \mathbb{T}_{\mu,\nu\gamma}^{\xi,0} \\
 & - 2\beta_1 e^{-s} \partial_v A_\mu^0 \dot{\phi}_{\mu\gamma} - 2\beta_1 e^{-s} \partial_\gamma A_\mu^0 \dot{\phi}_{\mu v} - \beta_3 e^{-\frac{s}{2}} \partial_\gamma Z^0 \partial_v Z^0 (\phi_{22} + \phi_{33}) \\
 & + 2\beta_1 e^{-\frac{s}{2}} \dot{Q}_{1\mu} \partial_{\gamma v} A_\mu^0 - 2\beta_1 e^{-s} \dot{Q}_{\xi\mu} \partial_v A_\mu^0 \phi_{\xi\gamma} - 2\beta_1 e^{-s} \dot{Q}_{\xi\mu} \partial_\gamma A_\mu^0 \phi_{\xi v} \\
 & + 2\beta_1 e^{-\frac{3s}{2}} A_\mu^0 (\dot{Q}_{1\xi} (\phi_{v\mu} \phi_{\xi\gamma} + \phi_{\mu\gamma} \phi_{\xi v} + \phi_{v\gamma} \phi_{\mu\xi} + \mathbb{T}_{\xi,\nu\gamma}^{\mu,0}) + \dot{Q}_{1\mu} \mathbb{N}_{1,\nu\gamma}^0) \\
 & - \beta_3 e^{-s} ((\kappa - Z^0) \partial_v A_\xi^0 - \partial_v Z^0 A_\xi^0) \mathbb{T}_{\mu,\mu\gamma}^{\xi,0} - \frac{1}{2} \beta_3 e^{-\frac{3s}{2}} (\kappa - Z^0) (\kappa + Z^0) \mathbb{N}_{\mu,\mu\nu\gamma}^0 \\
 & - 2\beta_1 e^{-\frac{s}{2}} (e^{-\frac{s}{2}} \dot{Q}_{\mu\nu} \partial_\gamma A_\xi^0 + e^{-\frac{s}{2}} \dot{Q}_{\mu\gamma} \partial_v A_\xi^0) \phi_{\xi\mu} - 2\beta_1 e^{-\frac{3s}{2}} A_\mu^0 A_\xi^0 \mathbb{T}_{\mu,\nu\gamma}^{\xi,0} \phi_{\mu} \\
 & - 2\beta_1 e^{-\frac{s}{2}} (\partial_{v\mu} A_\mu^0 A_\xi^0 + \partial_\mu A_\mu^0 \partial_v A_\nu^0 + \partial_v A_\mu^0 \partial_\mu A_\nu^0) \phi_{\xi\mu} \\
 & + 2\beta_1 e^{-s} (\partial_v ((U \cdot N) A_\xi)^0 \phi_{\mu\gamma} \phi_{\xi\mu} + \partial_\gamma ((U \cdot N) A_\xi)^0 \phi_{\mu\nu} \phi_{\xi\mu}) \\
 & - \frac{1}{\beta_\tau} h_W^{\mu,0} \partial_{v\gamma} A_\xi^0 \phi_{\xi\mu} + e^{-s} \frac{1}{\beta_\tau} h_W^{\mu,0} A_\mu^0 (\phi_{\mu\nu} \mathbb{N}_{1,\mu\gamma}^0 + \phi_{\mu\gamma} \mathbb{N}_{1,\mu\nu}^0 + \mathbb{N}_{\alpha,\mu\nu\gamma}^0) \\
 & + \frac{1}{4} \beta_4 (\kappa - Z^0)^2 (e^s \partial_{1\gamma v} K^0 + (\phi_{2v} \phi_{2\gamma} + \phi_{3v} \phi_{3\gamma}) \partial_1 K^0)
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}\beta_4(\kappa - Z^0)^2 e^{-\frac{\kappa}{2}} (\phi_{\mu\nu}\partial_{\mu\gamma}K^0 + \phi_{\mu\gamma}\partial_{\mu\nu}K^0) \\
& -\frac{1}{2}\beta_4(\kappa - Z^0)(\partial_\gamma Z^0(e^s\partial_{1\nu}K^0 - \phi_{\mu\nu}e^{-\frac{\kappa}{2}}\partial_\mu K^0) \\
& + \partial_\nu Z^0(e^s\partial_{1\gamma}K^0 - \phi_{\mu\gamma}e^{-\frac{\kappa}{2}}\partial_\mu K^0)) \\
& + \frac{1}{2}\beta_4(\partial_\gamma Z^0\partial_\nu Z^0 - (\kappa - Z^0)\partial_{\gamma\nu}Z^0)e^s\partial_1 S^0
\end{aligned}
\tag{A.11}$$

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