

Exact boundary controllability for the ideal magneto-hydrodynamic equations

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Abstract

We address the problem of controllability of the MHD system in a rectangular domain with a control prescribed on the side boundary. We identify a necessary and sufficient condition on the data to be null-controllable, i.e., can be driven to the zero state. We also show that the validity of this condition allows the states to be stirred to each other. If the condition is not satisfied, one can move from one state to another with the help of a simple shear external magnetic force.

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1. Introduction

We consider the two- or three-dimensional ideal magneto-hydrodynamic (MHD) equations for the unknown velocity $u: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, magnetic field $B: \Omega \times [0, T] \rightarrow \mathbb{R}^d$, and pressure $p: \Omega \times [0, T] \rightarrow \mathbb{R}$:

$$\partial_t u + u \cdot \nabla u - B \cdot \nabla B + \nabla p = 0 \quad (1.1a)$$

$$\partial_t B + u \cdot \nabla B - B \cdot \nabla u = 0 \quad (1.1b)$$

$$\operatorname{div} u = \operatorname{div} B = 0. \quad (1.1c)$$

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In general, $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3 is an arbitrary set with Lipschitz boundary $\partial\Omega$ and outward pointing unit normal vector $n = n(x)$ for $x \in \partial\Omega$, in which case (1.1) is classically supplemented with the boundary conditions

$$u \cdot n = B \cdot n = 0 \quad \text{on} \quad \partial\Omega.$$

For the existence theory of classical solutions to the ideal MHD system (1.1), see e.g. [2].

1.1. Main result and comments

To set the notation, let Γ be an open and non-empty subset of $\partial\Omega$ which has non-empty intersection with every connected component of $\partial\Omega$. The question of exact boundary controllability of (1.1) may be stated as follows. Given $T > 0$, initial data (u_0, B_0) , and terminal data (u_1, B_1) such that

$$\operatorname{div} u_0 = \operatorname{div} u_1 = \operatorname{div} B_0 = \operatorname{div} B_1 = 0 \quad \text{in} \quad \Omega \quad (1.2a)$$

$$u_0 \cdot n = u_1 \cdot n = B_0 \cdot n = B_1 \cdot n = 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma, \quad (1.2b)$$

does there exist a solution (u, B) of the MHD equations (1.1) such that

$$(u, B)|_{t=0} = (u_0, B_0) \quad (1.3a)$$

$$(u, B)|_{t=T} = (u_1, B_1) \quad (1.3b)$$

$$u \cdot n = B \cdot n = 0, \quad t \in [0, T], \quad x \in \partial\Omega \setminus \Gamma? \quad (1.3c)$$

In full generality, the answer to this question is no, as we demonstrate here. This contrasts sharply with the case of the incompressible Euler equations, in which the boundary control problem was first addressed in the two-dimensional case by Coron [3,4] and then by Glass in the fully general three-dimensional setting [6]. In this paper, we prove the exact boundary controllability for the MHD equations posed in a simple type of domain given that certain extra conditions are satisfied. After the statement of the main theorem, we provide some simple calculations indicating that in many scenarios, these conditions are necessary and in fact sharp.

Throughout the paper, we set $d = 2$ and $\Omega = [0, 1]^2$. The controlled portion Γ of the boundary is the set $\{x = 0, 1\} \times (0, 1)$, and we impose impermeability boundary conditions for $u = (u^1, u^2)$ and $B = (B^1, B^2)$ on $(0, 1) \times \{y = 0, 1\}$. This is the setting of MHD in a planar duct, as considered in a recent preprint by Rissel and Wang [7]. We prove the following theorem.

Theorem 1.1. *Suppose (u_0, B_0) and (u_1, B_1) are H^r regular divergence-free vector fields, where $r \geq 3$ is an integer, with vanishing normal components on $(0, 1) \times \{0, 1\}$, and assume that B_0 and B_1 satisfy*

$$\int_{[0,1]^2} B_0^1 dx dy = \int_{[0,1]^2} B_1^1 dx dy = 0. \quad (1.4)$$

Then there exists a solution $(u, B) \in C([0, T]; H^r([0, 1]^2))$ to (1.1) such that (1.2) and (1.3) hold. If (1.4) is not satisfied, then the same theorem holds but with a forcing term $h(t)e_x$ in the equation (1.1b) for the magnetic field B ; that is, (1.1b) is modified as

$$\partial_t B + u \cdot \nabla B - B \cdot \nabla u = h(t)e_x.$$

From here on, we fix $r \in \{3, 4, 5, \dots\}$ and note that non-integer values $r > 2$ can be covered using the same method.

The implementation of the condition (1.4) is new and provides the first instance of a solution to the MHD control problem *without* a bulk forcing term in the equation for the magnetic field, and it characterizes some scenarios where a bulk forcing term is necessary. We note that Rissel and Wang [7] used a forcing term in the equation for B which is a harmonic function but which is not as simple or as explicit as the forcing term $h(t)e_x$. Also, in [7] the forcing term is present regardless of the validity of the condition (1.4). Additive scalar controls such as $h(t)e_x$ have been introduced in several other control problems; see for example an application to the viscous Burgers equation in [5] and references therein.

The necessity of (1.4) may be seen from the following reasoning. Suppose that the pair (u, B) solves (1.1) on the set $\Omega = [0, 1]^2$, and $u^2|_{y=0,1} = B^2|_{y=0,1} = 0$. Then letting n^k denote the k -th component of the outward pointing normal vector n on the boundary $\partial([0, 1]^2)$, we may write

$$\frac{d}{dt} \int_{[0,1]^2} B^1 = \int_{[0,1]^2} \partial_k (B^k u^1 - u^k B^1) = \int_{\partial[0,1]^2} (B^k u^1 - u^k B^1) n^k. \quad (1.5)$$

When $k = 2$, i.e., on the top and bottom portions of the boundary, the integrand vanishes due to the assumptions on u^2 and B^2 at $y = 0, 1$. However, the integrand also vanishes when $k = 1$ since $B^1 u^1 - u^1 B^1 \equiv 0$. Thus we deduce that the mean of B^1 over the square is constant in time.

A substantial difficulty arises in the construction of the solution to an MHD-type system in (2.1). Construction requires changing to Elsässer variables $(u + B, u - B)$ and taking the curl of the new equations. In order to show that one can “undo” the curl and go back to the original (u, B) variables, one must show that the two Elsässer pressures agree, or equivalently that the solution to a certain elliptic equation vanishes; cf. Lemma 2.4 below. If the solution of the elliptic equation does not vanish, then the two Elsässer pressures do not agree, and returning to the original variables leads to an artificial forcing term in the equation for the magnetic field. Ensuring that the solution to the elliptic equation vanishes does not seem to mesh easily with the fact that u and B may penetrate the boundary. Rissel and Wang [7] comment further on this important issue in the introduction of their paper.

These aspects of the control problem are unique to the MHD equations; indeed, consider what happens to the mean of u^1 in the control problem for the Euler equations. If $u_0 = (\mathcal{U}, 0)$ is constant, then one may construct an exact solution to the Euler equations by

$$u(t, x) = g(t)e_x, \quad p(t, x) = xg'(t),$$

where $g(t)$ is any function satisfying $g(0) = \mathcal{U}$. So in order to drive a constant horizontal shear to zero, one may use the pressure as a forcing term to extinguish the shear. Of course such a construction is impossible in the equation for the magnetic field in MHD, leading to the condition

(1.4), as well as the modified statement of the theorem with a bulk forcing term $h(t)e_x$ in case (1.4) is not satisfied.

Since the mean of B must be constant in time, it seems plausible that (1.4) may be replaced with the slightly weaker condition

$$\int_{[0,1]^2} B_0^1 dx dy = \int_{[0,1]^2} B_1^1 dx dy. \quad (1.6)$$

Such a strengthening would be optimal, and we pose it as an open question whether Theorem 1.1 can be obtained in this way; cf. Section 2.2 for further comments.

Extensions of our results to other domains or to higher dimensions would also be interesting.

1.2. Simplifications and setup

We claim that we can reduce the problem to

$$T = 1 \quad (1.7a)$$

$$\|u_0\|_{H^r} + \|B_0\|_{H^r} \ll 1 \quad (1.7b)$$

$$(u_1, B_1) = (0, 0) \quad (1.7c)$$

$$h(t) = H'(t), \quad (1.7d)$$

where $H(t): [0, 1] \rightarrow \mathbb{R}$ is any smooth function satisfying

$$H(0) = \int_{[0,1]^2} B_0^1, \quad H(1) = 0, \quad H \ll 1.$$

To see that these simplifications still imply Theorem 1.1 in full generality, first note that the MHD equations are invariant under the rescaling

$$u(t, x) \rightarrow \lambda u(\lambda t, x), \quad B(t, x) \rightarrow \lambda B(\lambda t, x), \quad p \rightarrow \lambda^2 p(\lambda t, x). \quad (1.8)$$

In the case that (1.4) is satisfied, we choose $H \equiv 0$. Note that without loss of generality, we may assume that T is sufficiently small as after reaching zero, the solution can be continued as zero until the end of the interval. The smallness of T is related to the implicit constant in (1.7b). Then for $\lambda = T/2$, we rescale $(u_0, B_0) \rightarrow \lambda(u_0, B_0)$ and $(u_1, B_1) \rightarrow \lambda(u_1, B_1)$, and send both to $(0, 0)$ in time 1 using solutions $(\tilde{u}_0, \tilde{B}_0)$ and $(\tilde{u}_1, \tilde{B}_1)$ to MHD, respectively. Then we reverse the direction of time and change the signs of $(\tilde{u}_1, \tilde{B}_1)$, due to the scaling (1.8), and glue it together with $(\tilde{u}_0, \tilde{B}_0)$ to produce

$$(\tilde{u}, \tilde{B}): [0, 2] \times [0, 1]^2 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3, \quad (\tilde{u}, \tilde{B})|_{t=0} = \lambda(u_0, B_0), \quad (\tilde{u}, \tilde{B})|_{t=2} = \lambda(u_1, B_1).$$

Then defining

$$(u, B)(t, x) = \lambda^{-1}(\tilde{u}, \tilde{B})(\lambda^{-1}t, x),$$

we obtain a solution $(u, B): \Omega \times [0, T]$ to (1.1) satisfying (1.2) and (1.3).

In the case that (1.4) is *not* satisfied, we may set H to be a suitable non-constant function. Note that from (1.5) and assuming that (1.4) is not satisfied, it is not possible for H to be a constant function unless

$$\int_{[0,1]^2} B_0^1 dx dy = \int_{[0,1]^2} B_1^1 dx dy.$$

In any case, proceeding as before, we obtain that (u, B) solves the control problem, but with a forcing term $h(t)e_x = H'(t)e_x$ in the equation for the magnetic field. Therefore, we work under the assumptions (1.7a)–(1.7d) from here on.

1.3. Outline

The argument is structured as follows.

1. Show that the domain and the initial data (u_0, B_0) may be extended to yield functions which are periodic in x on a larger domain and still satisfy the appropriate divergence-free and impermeability conditions. This is achieved in Lemma 2.1. Then we show that we can drive the system to a state in which the mean of $B^1(t)$ vanishes at some time t . This is achieved in Proposition 2.2.
2. Show that for divergence-free vector fields on the square $[0, 1]^2$ for which the mean of the first component vanishes (such as $B(t)$ after the application of the previous step), there is a divergence-free extension which vanishes on a large portion of $[-1, 5] \times [0, 1]$. This is achieved in Lemma 2.9. Then we show that compactly supported magnetic fields B may be expelled from the domain $[0, 1]^2$ using a strong, background, horizontal shear in u . We carry out this step on the periodic domain $\mathbb{T} \times [0, 1]$, where $\mathbb{T} = [0, 6]$, extended periodically. This is achieved in Section 2.2.
3. Now that the magnetic field vanishes on $[0, 1]^2$, the MHD on this domain reduces to the Euler equations, and we may appeal to known control results for the Euler equations. This is achieved in Section 2.3.

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2. Proof of Theorem 1.1

2.1. First step: extensions and local existence near background shears

The first step consists of a lemma on extension of divergence-free vector fields and a local existence-type theorem for an MHD-type equation in the presence of a background shear. In the remainder of this section, we denote

$$\mathbb{T} = [0, 6],$$

extended periodically.

Lemma 2.1 (Extending to periodic data). *Given an H^r divergence-free vector field $u: [0, 1]^2 \rightarrow \mathbb{R}^2$ with $u^2|_{y=0,1} = 0$, there exists $u_E = (u_E^1, u_E^2): \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2$ in $H^r(\mathbb{T} \times [0, 1])$ such that*

$$u_E|_{0 \leq x \leq 1} = u, \quad u_E^2|_{y=0,1} = 0, \quad \int_{\mathbb{T} \times [0,1]} u_E^2 = 0;$$

in particular, u_E is periodic in x with period 6, and u_E satisfies the inequality

$$\|u_E\|_{H^r(\mathbb{T} \times [0,1])} \lesssim \|u\|_{H^r([0,1]^2)}.$$

Next, we state the local existence theorem, which is the workhorse of the paper. This proposition is stated on the set $\mathbb{T} \times [0, 1]$ and demonstrates the local existence of smooth solutions near background shears.

Proposition 2.2 (Local existence near background shears). *Let $u_0, B_0: \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2$ be divergence-free vector fields with sufficiently small H^r norm, where $r \geq 3$ is an integer, and assume that the means of u_0^2 and B_0^2 vanish. Let $H: [0, 1] \rightarrow \mathbb{R}$ be a smooth function depending on B_0 , and $H_u: [0, 1] \rightarrow \mathbb{R}$ a smooth function depending on u_0 , as in (2.6)–(2.7). Then there exists a $\mathbb{T}$ -periodic solution $(u, B, \nabla p)$, defined for $t \in [0, 1]$, to the following MHD-type system which is close to the background shear profiles $u_{\text{shear}} = H_u(t)e_x$ and $B_{\text{shear}} = H(t)e_x$ and solves*

$$\partial_t u + u \cdot \nabla u + \nabla p = B \cdot \nabla B \quad (2.1a)$$

$$\partial_t B + u \cdot \nabla B - B \cdot \nabla u = H'(t)e_x \quad (2.1b)$$

$$\operatorname{div} u = \operatorname{div} B = 0 \quad (2.1c)$$

$$u^2|_{y=0,1} = B^2|_{y=0,1} = 0 \quad (2.1d)$$

$$\int_{\mathbb{T} \times [0,1]} u^2(t) \equiv \int_{\mathbb{T} \times [0,1]} B^2(t) \equiv 0 \quad (2.1e)$$

$$u|_{t=0} = u_0 \quad (2.1f)$$

$$B|_{t=0} = B_0 \quad (2.1g)$$

$$\int_{\mathbb{T} \times [0,1]} B^1|_{t=1} = 0. \quad (2.1h)$$

Furthermore, if $\int_{[0,1]^2} B_0^1 = 0$, then we may take $H \equiv 0$.

For the precise quantification of how close the solution needs to be to the background shear, cf. the statement of Lemma 2.6.

Proof of Lemma 2.1. Introduce the stream function

$$\psi(x, y) = - \int_{\ell} u^{\perp}(\tilde{x}, \tilde{y}) \cdot (d\tilde{x}, d\tilde{y}), \quad (2.2)$$

where ℓ denotes a sufficiently regular path from $(0, 0)$ to (x, y) , which satisfies $\nabla^\perp \psi = u$, denoting $u^\perp = (-u^2, u^1)$ and $\nabla^\perp = (-\partial_2, \partial_1)$. Note that the integral is independent of a chosen path since $\operatorname{div} u = 0$. Clearly, the condition $u^2|_{y=0,1} = 0$ implies $\psi|_{y=0} = 0$ and ψ is constant on the upper boundary $\{y = 1\}$, whose value we denote by $\mathcal{C}_\psi$. Since it is needed below, note that, in particular,

$$\int_0^1 u^1(x, y) dy = -\mathcal{C}_\psi \quad \text{for } x \in \mathbb{T}, \quad (2.3)$$

i.e., the integral $\int_0^1 u^1(x, y) dy$ is independent of x . Now consider the standard H^r extension operator (cf. Theorem 5.19 from [1]) which takes in a function ϕ defined for $x \in [0, 1]$ and returns

$$\tilde{\phi}(x, y) = \begin{cases} \phi(x, y) & x > 0 \\ \sum_{j=1}^{r+1} \lambda_j \phi(-jx, y) & -\frac{1}{r+1} < x \leq 0, \end{cases} \quad (2.4)$$

where λ_j are unique solutions to a system of linear equations, one of which imposes that

$$\sum_{j=1}^{r+1} \lambda_j = 1. \quad (2.5)$$

Let $\theta: [-1, 2] \rightarrow [0, 1]$ be a function depending only on $x \in [-1, 2]$ which satisfies

$$\theta(x) = 0 \quad \text{for } x \in [-1/(r+2), 1 + 1/(r+2)]$$

and

$$\theta(x) = 1 \quad \text{for } x \in [-1, -3/4] \cup [7/4, 2].$$

Now, extend ψ to $\psi_{\mathbb{T}}: [-1, 5] \times [0, 1] \rightarrow \mathbb{R}$ as follows. For $x \in [-1, 0]$, we define $\psi_{\mathbb{T}}(x, y) = \mathcal{C}_\psi \theta(x)y + (1 - \theta(x))\tilde{\psi}(x, y)$, while for $x \in [1, 2]$ we use $\psi_{\mathbb{T}}(x, y) = \mathcal{C}_\psi \theta(x)y + (1 - \theta(x))\tilde{\psi}(2 - x, y)$. Here we have used the notation that $\tilde{\psi}$ means “apply the H^r extension operator defined above to ψ .” For $2 \leq x \leq 5$, we set $\psi_{\mathbb{T}}(x, y) = \mathcal{C}_\psi y$ (this extra room is convenient below to move around the support of the magnetic field). Then define $\tilde{u}_E = \nabla^\perp \tilde{\psi}_{\mathbb{T}}$. Finally,

$$\int_{\mathbb{T} \times [0, 1]} u_E^2 = \int_{[-1, 5] \times [0, 1]} \partial_x \tilde{\psi}_{\mathbb{T}} = 0,$$

by x -periodicity of $\tilde{\psi}_{\mathbb{T}}$. $\square$

Now, we turn to the proof of Proposition 2.2. We look for a solution of this system which satisfies

$$u = \tilde{u} + H_u(t)e_x = \tilde{u} + u_{\text{shear}} \quad (2.6a)$$

$$H_u(0) = \int_{[0,1]^2} u_0^1, \quad H_u(1) = 0 \quad (2.6b)$$

$$\int_{[0,1]^2} \tilde{u}(t) = 0 \quad \forall t \in [0, 1] \quad (2.6c)$$

and

$$B = \tilde{B} + H(t)e_x = \tilde{B} + B_{\text{shear}} \quad (2.7a)$$

$$H(0) = \int_{[0,1]^2} B_0^1, \quad H(1) = 0 \quad (2.7b)$$

$$\int_{[0,1]^2} \tilde{B}(t) = 0 \quad \forall t \in [0, 1]. \quad (2.7c)$$

With this ansatz in mind and noting that $\nabla u_{\text{shear}} = \nabla B_{\text{shear}} = 0$, and that $\partial_t u_{\text{shear}}$ is equal to $\partial_x p_{\text{shear}}$ which is periodic (although p_{shear} itself may *not* be periodic), (2.1) now reads

$$\partial_t \tilde{u} + (\tilde{u} + u_{\text{shear}}) \cdot \nabla \tilde{u} + \nabla q = (\tilde{B} + B_{\text{shear}}) \cdot \nabla \tilde{B} \quad (2.8a)$$

$$\partial_t \tilde{B} + (\tilde{u} + u_{\text{shear}}) \cdot \nabla \tilde{B} - (\tilde{B} + B_{\text{shear}}) \cdot \nabla \tilde{u} = 0 \quad (2.8b)$$

$$\operatorname{div} \tilde{u} = \operatorname{div} \tilde{B} = 0 \quad (2.8c)$$

$$\tilde{u}^2|_{y=0,1} = \tilde{B}^2|_{y=0,1} = 0 \quad (2.8d)$$

$$\int_{\mathbb{T} \times [0,1]} \tilde{u}^2(t) \equiv \int_{\mathbb{T} \times [0,1]} \tilde{B}^2(t) \equiv 0 \quad (2.8e)$$

$$\tilde{u}|_{t=0} = u_0 - u_{\text{shear}}|_{t=0} \quad (2.8f)$$

$$\int_{[0,1]^2} \tilde{u}^1(t) \equiv 0 \quad (2.8g)$$

$$\tilde{B}|_{t=0} = B_0 - B_{\text{shear}}|_{t=0} \quad (2.8h)$$

$$\int_{[0,1]^2} \tilde{B}^1(t) \equiv 0. \quad (2.8i)$$

Observe that $\int_{[0,1]^2} \tilde{u}^1(t) = 0$ is equivalent to $\int_{\mathbb{T} \times [0,1]} \tilde{u}^1(t) = 0$ by (2.3) resulting from the divergence-free condition. Similarly, $\int_{[0,1]^2} \tilde{B}^1(t) = 0$ is equivalent to $\int_{\mathbb{T} \times [0,1]} \tilde{B}^1(t) = 0$. We shall prove that one can solve this system for $\tilde{u}$, $\tilde{B}$, and q which are $\mathbb{T}$ -periodic in x and that $q: \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}$ solves an elliptic problem that enforces $\operatorname{div} \tilde{u} = 0$ and $\tilde{u}^2|_{y=0,1} = 0$:

$$\begin{aligned}\Delta q &= \operatorname{div}(-\tilde{u} \cdot \nabla \tilde{u} - u_{\text{shear}} \cdot \nabla \tilde{u} + (\tilde{B} + B_{\text{shear}}) \cdot \nabla \tilde{B}) \\ \partial_y q|_{y=0,1} &= 0 \\ \int_{\mathbb{T} \times [0,1]} q &= 0.\end{aligned}$$

Note that the MHD system exhibits a loss of derivatives; thus in order to solve this system, we need to switch to the Elsässer variables

$$z_1 = \tilde{u} + \tilde{B}, \quad z_2 = \tilde{u} - \tilde{B}.$$

In these variables, the equations in (2.8) become

$$\partial_t z_1 + (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla z_1 + \nabla q = 0 \quad (2.10a)$$

$$\partial_t z_2 + (z_1 + u_{\text{shear}} + B_{\text{shear}}) \cdot \nabla z_2 + \nabla q = 0 \quad (2.10b)$$

$$\operatorname{div} z_1 = \operatorname{div} z_2 = 0 \quad (2.10c)$$

$$z_1^2|_{y=0,1} = z_2^2|_{y=0,1} = 0 \quad (2.10d)$$

$$\int_{\mathbb{T} \times [0,1]} z_1(t) \equiv \int_{\mathbb{T} \times [0,1]} z_2(t) \equiv 0 \quad (2.10e)$$

$$z_1|_{t=0} = u_0 - u_{\text{shear}}|_{t=0} + B_0 - B_{\text{shear}}|_{t=0} \quad (2.10f)$$

$$z_2|_{t=0} = u_0 - u_{\text{shear}}|_{t=0} - B_0 + B_{\text{shear}}|_{t=0}. \quad (2.10g)$$

Note that the conditions on the means of $\tilde{u}^1$, $\tilde{u}^2$, $\tilde{B}^1$, and $\tilde{B}^2$ have been consolidated into (2.10e), asserting that the means of *both* components of z_1 and z_2 vanish. Taking the curl of the first two equations in (2.10), using $\epsilon_{\ell j}$ to denote the classical Levi-Civita symbol, and denoting $\omega_1 := \nabla^\perp \cdot z_1$ and $\omega_2 := \nabla^\perp \cdot z_2$ yields

$$\partial_t \omega_1 + (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla \omega_1 = -\partial_k z_1^\ell \epsilon_{\ell j} \partial_j z_2^k \quad (2.11a)$$

$$\partial_t \omega_2 + (z_1 + u_{\text{shear}} + B_{\text{shear}}) \cdot \nabla \omega_2 = -\partial_k z_2^\ell \epsilon_{\ell j} \partial_j z_1^k. \quad (2.11b)$$

Lemma 2.4 below shows that if we have solved this “vorticity-Elsässer-MHD” system, where we have substituted (2.11) for the first two equations in (2.10), then in fact we have solved (2.8). We first need the following De Rham-type result.

Lemma 2.3 (Periodic De Rham’s theorem). Assume that $v \in L_{loc}^2(\mathbb{R} \times [0, 1])$ is L -periodic in the x variable, where $L > 0$, and suppose that it satisfies $\nabla^\perp \cdot v = 0$ and $\int_{[0,L] \times [0,1]} v^1 = 0$. Then there exists a function $q \in H_{loc}^1(\mathbb{R} \times [0, 1])$, which is L -periodic in the x variable, and satisfies

$$v = \nabla q \quad (2.12)$$

on $\mathbb{R} \times (0, 1)$.

Proof of Lemma 2.3. By the classical De Rham’s theorem [8, Proposition I.1.1], there exists a distribution $q \in \mathcal{D}'(\mathbb{R} \times (0, 1))$ such that (2.12) holds. Using [8, Proposition I.1.2(i)], we have $q \in H_{\text{loc}}^1(\mathbb{R} \times [0, 1])$, so we only need to establish periodicity. By the periodicity of v , we have $\nabla(q(x + L, y) - q(x, y)) = 0$, for $(x, y) \in \mathbb{R} \times (0, 1)$, which implies $q(x + L, y) - q(x, y) = a$, for all $(x, y) \in \mathbb{R} \times (0, 1)$, where $a \in \mathbb{R}$ is a constant. Since $0 = \int_{[0, L] \times [0, 1]} v^1 = \int_{[0, L] \times [0, 1]} \partial_1 q = \int_{[0, 1]} (q(L, y) - q(0, y)) = a$, we get $a = 0$, implying the L -periodicity of q . $\square$

We note in passing that any smooth vector field $v: \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2$ which satisfies $\int_{\mathbb{T} \times [0, 1]} v^1 = 0$ allows a unique $L^2(\mathbb{T} \times [0, 1])$ -orthogonal decomposition reminiscent of the Helmholtz-Hodge decomposition of the form

$$v = \nabla p + \nabla^\perp q, \quad \partial_y p|_{y=0,1} = v^2|_{y=0,1}, \quad q|_{y=0,1} = 0, \quad \int_{\mathbb{T} \times [0, 1]} p = 0,$$

where $p, q: \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}$ are smooth and periodic. We construct q as the periodic-in- x solution to the elliptic problem

$$\begin{aligned} \Delta q &= \nabla^\perp \cdot v \\ q|_{y=0,1} &= 0. \end{aligned}$$

Now, considering $v - \nabla^\perp q$, we have $\nabla^\perp \cdot (v - \nabla^\perp q) = 0$ and $\int_{\mathbb{T} \times [0, 1]} (v^1 + \partial_2 q) = 0$. Applying Lemma 2.3 to $v - \nabla^\perp q$, we may write it as the gradient of a periodic function p , which without loss of generality may be taken to have zero mean. The L^2 -orthogonality is immediate from integration by parts, the fact $q|_{y=0,1} = 0$ by construction, and the periodicity in x of v , p , and q . Uniqueness follows from the construction, in particular the imposition of the mean-zero conditions.

Lemma 2.4 (Solving vorticity-Elsässer MHD). *Solving (2.10) but with (2.11) taking the place of the first two equations in (2.10) is equivalent to solving (2.8). Consequently, solving either provides a solution to (2.1).*

In the above lemma it is understood that z_1 and z_2 are related to ω_1 and ω_2 as in Lemma 2.8, which shows how to reconstruct the appropriate divergence-free velocity field from a given vorticity in our context.

Proof of Lemma 2.4. Assume that we have a solution of (2.11). It is easy to check that

$$\begin{aligned} &\nabla^\perp \cdot (\partial_t z_1 + (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla z_1) \\ &= \partial_t \omega_1 + (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla \omega_1 + \partial_k z_1^\ell \epsilon_{\ell j} \partial_j z_2^k = 0. \end{aligned}$$

In order to apply Lemma 2.3, we need to verify that the integral over $\mathbb{T} \times [0, 1]$ of the first component of $\partial_t z_1 + (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla z_1$ vanishes. For the first term, $\partial_t z_1$, this is clear, while for the second we have

$$\int_{\mathbb{T} \times [0,1]} (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla z_1^1 = \int_{\mathbb{T} \times [0,1]} \partial_i ((z_2^i + u_{\text{shear}}^i - B_{\text{shear}}^i) z_1^1) = 0,$$

where in last equality we separately integrate for $i = 1$ and $i = 2$. When $i = 1$, we use periodicity, while when $i = 2$ it is important that the expression inside the parentheses vanishes for $y = 0$ and $y = 1$. By Lemma 2.3, there exists a $\mathbb{T}$ -periodic function q_1 such that

$$\partial_t z_1 + (z_2 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla z_1 = -\nabla q_1,$$

where $q_1 : \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}$ solves

$$\begin{aligned} -\Delta q_1 &= \partial_k ((z_2 + u_{\text{shear}} - B_{\text{shear}})^\ell \partial_\ell z_1^k) \\ \partial_2 q_1|_{y=0,1} &= 0 \\ \int_{\mathbb{T} \times [0,1]} q_1 &= 0. \end{aligned}$$

We similarly have

$$\partial_t z_2 + (z_1 + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla z_2 = -\nabla q_2,$$

where

$$\begin{aligned} -\Delta q_2 &= \partial_k ((z_1 + u_{\text{shear}} + B_{\text{shear}})^\ell \partial_\ell z_2^k) \\ \partial_2 q_2|_{y=0,1} &= 0 \\ \int_{\mathbb{T} \times [0,1]} q_2 &= 0. \end{aligned}$$

We find that $q_1 - q_2$ is $\mathbb{T}$ -periodic in x and solves

$$\begin{aligned} -\Delta(q_1 - q_2) &= 0 \\ \partial_2(q_1 - q_2)|_{y=0,1} &= 0 \\ \int_{\mathbb{T} \times [0,1]} (q_1 - q_2) &= 0, \end{aligned}$$

from where $q_1 = q_2$ and we have a solution to (2.10); to obtain that $q_1 - q_2$ is harmonic, we write

$$\begin{aligned} -\Delta(q_1 - q_2) &= \partial_k \partial_\ell ((z_2 + u_{\text{shear}} - B_{\text{shear}})^\ell z_1^k) - \partial_k \partial_\ell ((z_1 + u_{\text{shear}} + B_{\text{shear}})^\ell z_2^k) \\ &= \partial_k \partial_\ell ((u_{\text{shear}} - B_{\text{shear}})^\ell z_1^k) - \partial_k \partial_\ell ((u_{\text{shear}} + B_{\text{shear}})^\ell z_2^k) \\ &= \partial_\ell ((u_{\text{shear}} - B_{\text{shear}})^\ell \partial_k z_1^k) - \partial_\ell ((u_{\text{shear}} + B_{\text{shear}})^\ell \partial_k z_2^k) = 0, \end{aligned}$$

where we used $\nabla u_{\text{shear}} = \nabla B_{\text{shear}} = 0$ in the third equality and the divergence-free condition in the last. Reconstructing the equations for $\tilde{u}$ and $\tilde{B}$ from z_1 and z_2 as usual and using that $q_1 = q_2$

then shows that we have a solution to (2.8). To conclude the proof, we must demonstrate the other direction of the equivalence, but this only amounts to taking the curl of the first two equations in (2.10). $\square$

Returning to the proof of Proposition 2.2, we will be done if we can set up and solve the fixed point iteration:

$$\omega_{1,n} = \nabla^\perp \cdot z_{1,n} \quad (2.17a)$$

$$\omega_{2,n} = \nabla^\perp \cdot z_{2,n} \quad (2.17b)$$

$$\partial_t \omega_{1,n} + (z_{2,n-1} + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla \omega_{1,n} = -\partial_k z_{1,n-1}^\ell \epsilon_{\ell j} \partial_j z_{2,n-1}^k \quad (2.17c)$$

$$\partial_t \omega_{2,n} + (z_{1,n-1} + u_{\text{shear}} + B_{\text{shear}}) \cdot \nabla \omega_{2,n} = -\partial_k z_{2,n-1}^\ell \epsilon_{\ell j} \partial_j z_{1,n-1}^k \quad (2.17d)$$

$$\operatorname{div} z_{1,n} = \operatorname{div} z_{2,n} = 0 \quad (2.17e)$$

$$z_{1,n}^2|_{y=0,1} = z_{2,n}^2|_{y=0,1} = 0 \quad (2.17f)$$

$$z_{1,n}|_{t=0} = u_0 - u_{\text{shear}}|_{t=0} + B_0 - B_{\text{shear}}|_{t=0} \quad (2.17g)$$

$$z_{2,n}|_{t=0} = u_0 - u_{\text{shear}}|_{t=0} - B_0 + B_{\text{shear}}|_{t=0} \quad (2.17h)$$

$$\int_{\mathbb{T} \times [0,1]} z_{1,n}(t) \equiv \int_{\mathbb{T} \times [0,1]} z_{2,n}(t) \equiv 0. \quad (2.17i)$$

We split the proof into the following three lemmas.

Lemma 2.5 (*Constructing the iterates*). *If u_0 and B_0 belong to H^r , then there exists a sequence $\{(z_{1,n}, z_{2,n})\}_{n=0}^\infty$ which is well-defined, satisfies the equations in (2.17), and $z_{1,n}, z_{2,n} \in C([0, 1]; H^r(\mathbb{T} \times [0, 1]))$.*

Lemma 2.6 (*Uniform bound in high regularity*). *The sequence $\{(z_{1,n}, z_{2,n})\}_{n=0}^\infty$ belongs to the ball $B_{C\varepsilon}(0) \subset C([0, 1]; H^r(\mathbb{T} \times [0, 1]))$ provided the initial data for $z_{1,n}$ and $z_{2,n}$ from (2.17) are smaller than ε in $H^r(\mathbb{T} \times [0, 1])$, where $\varepsilon > 0$ is sufficiently small and C is a constant.*

The smallness of data can always be ensured by a sufficiently powerful ε -dependent rescaling of the original problem; cf. (1.7). Note that rescaling the initial data by ε shrinks the values of $H_u(0)$ and $H(0)$ from (2.6) and (2.7), although the values at later times t are still unconstrained. Thus we may first choose ε sufficiently small as determined by Lemma 2.6 and Lemma 2.7, and then choose H_u and H appropriately.

Lemma 2.7 (*Contraction in H^1*). *The sequence $\{(z_{1,n}, z_{2,n})\}_{n=1}^\infty$ satisfies the contraction inequality*

$$\begin{aligned} & \|z_{1,n} - z_{1,n+1}\|_{L^\infty([0,1]; H^1(\mathbb{T} \times [0,1]))} + \|z_{2,n} - z_{2,n+1}\|_{L^\infty([0,1]; H^1(\mathbb{T} \times [0,1]))} \\ & \leq \frac{1}{2} (\|z_{1,n} - z_{1,n-1}\|_{L^\infty([0,1]; H^1(\mathbb{T} \times [0,1]))} + \|z_{2,n} - z_{2,n-1}\|_{L^\infty([0,1]; H^1(\mathbb{T} \times [0,1]))}) \end{aligned} \quad (2.18)$$

provided that ε , $z_{1,0}$, and $z_{2,0}$ are sufficiently small.

Proof of Proposition 2.2. Assuming the conclusions of the preceding three lemmas, the contraction mapping principle provides a unique limit point of the sequence $(z_{1,n}, z_{2,n})$, that is, we have a unique solution to the system of equations (2.10), and thus (2.1), concluding the proof of Proposition 2.2. $\square$

Thus it remains to prove Lemmas 2.5–2.7. To prove Lemma 2.5, we first need to state an existence and uniqueness theorem for a periodic div-curl problem.

Lemma 2.8 (Periodic div-curl problem). *Consider the system*

$$\operatorname{div} z = 0, \quad \operatorname{curl} z = \omega, \quad z^2|_{y=0,1} = 0, \quad \int_{\mathbb{T} \times [0,1]} z^1 = 0. \quad (2.19)$$

For every $\omega \in H^s(\mathbb{T} \times [0, 1])$, where $s \geq 1$, there exists a unique solution $z \in H^{s+1}(\mathbb{T} \times [0, 1])$. Also, the mapping $\omega \mapsto z$ is continuous from $H^s(\mathbb{T} \times [0, 1])$ to $H^{s+1}(\mathbb{T} \times [0, 1])$.

Proof of Lemma 2.8. To obtain the existence, first solve $\Delta\psi = \omega$ for $\psi : \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}$ with the boundary conditions $\psi|_{y=0,1} = 0$ on top and bottom and periodic boundary conditions in x . Then set $z = \nabla^\perp \psi$. It is easy to check that z satisfies the first three conditions in (2.19). For the fourth condition in (2.19), we have $\int_{\mathbb{T} \times [0,1]} z^1 = -\int_{\mathbb{T} \times [0,1]} \partial_2 \psi = \int_{\mathbb{T}} (\psi(x, 0) - \psi(x, 1)) = 0$, since $\psi|_{y=0,1} = 0$.

For uniqueness, assume that z is periodic and satisfies (2.19) with $\omega = 0$. By $\operatorname{curl} z = 0$ and $\int_{\mathbb{T} \times [0,1]} z^1 = 0$, due to Lemma 2.3, there exists a periodic function ϕ such that $z = \nabla \phi$. The divergence condition and $z^2|_{y=0,1} = 0$ then imply that ϕ solves the homogeneous Neumann problem $\Delta\phi = 0$ and $\partial_2 \phi|_{y=0,1} = 0$. Thus ϕ is constant, from where $z = 0$. $\square$

Note that, by the proof of Lemma 2.8, a consequence of $\operatorname{div} z = 0$, $\int_{\mathbb{T} \times [0,1]} z^1 = 0$, and periodicity is that $\int_{\mathbb{T} \times [0,1]} z^2 = 0$.

Proof of Lemma 2.5. We begin by defining the time-independent first iterates

$$z_{1,0} = u_0 - u_{\text{shear}}|_{t=0} + B_0 - B_{\text{shear}}|_{t=0}, \quad z_{2,0} = u_0 - u_{\text{shear}}|_{t=0} - B_0 + B_{\text{shear}}|_{t=0}. \quad (2.20)$$

From (2.6)–(2.7) we have that $z_{1,0}^1$ and $z_{2,0}^1$ have vanishing averages over $\mathbb{T} \times [0, 1]$ for all times, as desired in (2.17i). From the assumptions of Proposition 2.2, we have that the averages of $z_{1,0}^2$ and $z_{2,0}^2$ also vanish for all times. Both first iterates are also clearly divergence free, satisfy (2.17f), and belong to $C([0, 1]; H^r(\mathbb{T} \times [0, 1]))$.

Now assume that the pair $(z_{1,n-1}, z_{2,n-1})$ is given for $n \geq 1$ and satisfies (2.17a)–(2.17i). To construct $(z_{1,n}, z_{2,n})$, we first solve (2.17c) and (2.17d) for $\omega_{1,n}$ and $\omega_{2,n}$, respectively, using the method of characteristics on the set $\mathbb{T} \times [0, 1]$. This is possible because the velocity fields and forcing terms for both equations are periodic in x , and the velocity fields do not penetrate the boundaries at $y = 0$ and $y = 1$. We then solve the div-curl problem

$$\begin{aligned} \operatorname{div} z_{i,n}(t) &= 0 \\ \operatorname{curl} z_{i,n}(t) &= \omega_{i,n}(t) \end{aligned}$$

$$\begin{aligned} z_{i,n}^2(t)|_{y=0,1} &= 0 \\ \int_{\mathbb{T} \times [0,1]} z_{i,n}(t) &= 0, \end{aligned}$$

using Lemma 2.8 for $z_{i,n}(t): \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2$ with $i = 1, 2$ and $t \in [0, 1]$, obtaining the estimate

$$\|z_{i,n}\|_{H^r(\mathbb{T} \times [0,1])} \leq C \|\omega_{i,n}\|_{H^{r-1}(\mathbb{T} \times [0,1])}.$$

From the inductive assumption that $z_{i,n-1}$ belongs to $C([0, 1]; H^r(\mathbb{T} \times [0, 1]))$ and standard estimates for transport equations,¹ we deduce that $z_{i,n} \in C([0, 1]; H^r(\mathbb{T} \times [0, 1]))$. Finally, in order to obtain $z_{i,n}(0) = z_0$, we simply use $\operatorname{curl} z_{i,n}(0) = \operatorname{curl} z_0$, the continuity of $\operatorname{curl} z_{i,n}$ in t , and the continuity of the map $\omega \mapsto z$ in Lemma 2.8. $\square$

Proof of Lemma 2.6. The proof proceeds by induction on n and a standard energy argument for small data. The estimates for $n = 0$ follow from the fact that we have defined the first iterates to be time-independent; cf. (2.20). Next, assume that we have

$$\|z_{1,n-1}\|_{H^r}, \|z_{2,n-1}\|_{H^r} \leq 2\epsilon, \quad (2.22)$$

for $t \in [0, 1]$. Applying the standard H^{r-1} estimates on (2.17c) and (2.17d), while observing that $\nabla u_{\text{shear}} = \nabla B_{\text{shear}} = 0$, we get

$$\begin{aligned} \frac{d}{dt} \|\omega_{i,n}\|_{H^{r-1}(\mathbb{T} \times [0,1])} \\ \leq C \|\nabla z_{1-i,n-1} \otimes \omega_{i,n-1}\|_{H^{r-1}(\mathbb{T} \times [0,1])} + C \|\nabla z_{1,n-1} \otimes \nabla z_{2,n-1}\|_{H^{r-1}(\mathbb{T} \times [0,1])} \leq C\epsilon^2, \end{aligned} \quad (2.23)$$

for $i = 1, 2$, where we used that H^{r-1} is an algebra in the first inequality and (2.22) in the second. Next, we integrate (2.23), use the assumption on the size of the initial data, and apply Lemma 2.8, obtaining

$$\|z_{1,n}\|_{H^r}, \|z_{2,n}\|_{H^r} \leq \epsilon + C\epsilon^2. \quad (2.24)$$

Choosing ϵ sufficiently small, we conclude that (2.22) remains valid for $\|z_{1,n}\|_{H^r}$ and $\|z_{2,n}\|_{H^r}$, completing the induction step. $\square$

Proof of Lemma 2.7. We set $\omega_1 := \omega_{1,n+1} - \omega_{1,n}$. From (2.17c), we find that ω_1 satisfies the equation

$$\begin{aligned} \partial_t \omega_1 + (z_{2,n} + u_{\text{shear}} - B_{\text{shear}}) \cdot \nabla \omega_1 \\ = (z_{2,n-1} - z_{2,n}) \cdot \nabla \omega_{1,n} - (\partial_k z_{1,n}^\ell - \partial_k z_{1,n-1}^\ell) \epsilon_{\ell j} \partial_j z_{2,n}^k - \partial_k z_{1,n-1}^\ell \epsilon_{\ell j} (\partial_j z_{2,n}^k - \partial_j z_{2,n-1}^k), \end{aligned}$$

¹ A slightly more sophisticated version of such estimates is used in the proof of Lemma 2.6 to demonstrate the uniform bound in higher regularity.

with $\omega_1|_{t=0} = 0$. Multiplying by ω_1 , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega_1\|_{L^2(\mathbb{T} \times [0,1])}^2 &\lesssim \|z_{2,n-1} - z_{2,n}\|_{L^2(\mathbb{T} \times [0,1])} \|\omega_1 \nabla \omega_{1,n}\|_{L^2(\mathbb{T} \times [0,1])} \\ &\quad + \|\nabla(z_{1,n-1} - z_{1,n})\|_{L^2(\mathbb{T} \times [0,1])} \|\nabla z_{2,n} \omega_1\|_{L^2(\mathbb{T} \times [0,1])} \\ &\quad + \|\nabla(z_{2,n-1} - z_{2,n})\|_{L^2(\mathbb{T} \times [0,1])} \|\nabla z_{1,n-1} \omega_1\|_{L^2(\mathbb{T} \times [0,1])} \\ &\lesssim \varepsilon \|\omega_1\|_{L^2(\mathbb{T} \times [0,1])} (\|z_{1,n-1} - z_{1,n}\|_{H^1(\mathbb{T} \times [0,1])} \\ &\quad + \|z_{2,n-1} - z_{2,n}\|_{H^1(\mathbb{T} \times [0,1])}). \end{aligned}$$

Using a Grönwall argument and choosing $\varepsilon > 0$ sufficiently small, we deduce that

$$\begin{aligned} &\|\omega_{1,n} - \omega_{1,n+1}\|_{L^\infty([0,1]; L^2(\mathbb{T} \times [0,1]))} \\ &\leq \frac{1}{4K} (\|z_{1,n} - z_{1,n-1}\|_{L^\infty([0,1]; H^1(\mathbb{T} \times [0,1]))} + \|z_{2,n} - z_{2,n-1}\|_{L^\infty([0,1]; H^1(\mathbb{T} \times [0,1]))}), \end{aligned} \quad (2.25)$$

where K is sufficiently large. Utilizing Lemma 2.8, with $s = 1$ while assuming the constant K in (2.25) is sufficiently large, we get

$$\begin{aligned} &\|\omega_{1,n} - \omega_{1,n+1}\|_{L^\infty([0,1]; L^2(\mathbb{T} \times [0,1]))} \\ &\leq \frac{1}{4} (\|\omega_{1,n} - \omega_{1,n+1}\|_{L^\infty([0,1]; L^2(\mathbb{T} \times [0,1]))} + \|\omega_{2,n} - \omega_{2,n+1}\|_{L^\infty([0,1]; L^2(\mathbb{T} \times [0,1]))}). \end{aligned}$$

Establishing the analogous estimate for $\omega_{2,n} - \omega_{2,n+1}$ and summing concludes the proof of (2.18). $\square$

The first step is now as follows. The initial data u_0 and B_0 are extended using Lemma 2.1 to data which are divergence-free and $\mathbb{T}$ -periodic with the means of u^2 and B^2 vanishing over $\mathbb{T} \times [0, 1]$. Choosing suitable $u_{\text{shear}} = H_u(t)e_x$ and $B_{\text{shear}} = H(t)e_x$, we can drive the system to the state such that the means of u and B over $[0, 6] \times [0, 1]$ vanish at time $t = 1$.

2.2. Second step: expelling the magnetic field

Recall that in the first step, we solved (2.8) using the ansatz (2.6)–(2.7), which set $u_{\text{shear}} = H_u(t)e_x$. However, at no point did we impose any restrictions on H_u . The purpose of the second step is to show that with an application of Lemma 2.9, stated next, on the set $\mathbb{T} \times [0, 1]$ and a smart choice of H_u in another, we may apply the methodology from Proposition 2.2 to control the support of B at later times.

Lemma 2.9 (Truncating to a compactly supported data). *Given an H^r divergence-free and $\mathbb{T}$ -periodic vector field $B: \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2$ with*

$$\int_{\mathbb{T} \times [0,1]} B^1 = 0, \quad B^2|_{y=0,1} = 0,$$

there exists an H^r -regular divergence-free $B_T: \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2$ which satisfies $B_T^2|_{y=0,1} = 0$, with $B_T \equiv 0$ for $x \in [7/4, 5]$ or $x \in [-1, -3/4]$, and $B_T = B$ for $x \in [0, 1]$. Moreover, the mapping $B \mapsto B_T$ is linear and we have the inequality

$$\|B_T\|_{H^r(\mathbb{T} \times [0,1])} \lesssim \|B\|_{H^r(\mathbb{T} \times [0,1])}.$$

Proof of Lemma 2.9. From (2.3) and the assumption $\int_{\mathbb{T} \times [0,1]} B^1 = 0$, we have that the stream function ψ for B , as defined in (2.2) with u replaced by B , satisfies $\psi(x, 0) = \psi(x, 1) = 0$ for all $x \in [0, 1]$. Define $B_T(x, t) = \nabla^\perp(\psi(x, y)\theta(x))$ for a $\mathbb{T}$ -periodic cutoff function θ which satisfies $\theta \equiv 1$ for $-1/4 \leq x \leq 5/4$ and $\theta \equiv 0$ for $-1 \leq x \leq -3/4$ or $7/4 \leq x \leq 5$. Since $\psi(x, 0) = \psi(x, 1) = 0$, we have that $\psi\theta$ is constant on $y = 0, 1$. The rest of the assertions of Lemma 2.9 follow immediately. $\square$

Lemma 2.10 (Background shears and the support of B). Let $(u_0, B_0): \mathbb{T} \times [0, 1] \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$ be vector fields satisfying the same assumptions as in Proposition 2.2, with the additional assumptions

$$B|_{[-1,5] \setminus [-3/4, 7/4]} \equiv 0, \quad \int_{\mathbb{T} \times [0,1]} B_0 = \int_{\mathbb{T} \times [0,1]} u_0 = 0. \quad (2.26)$$

Then there exists a $\mathbb{T}$ -periodic solution $(u, B, \nabla p)$ defined for $t \in [0, 1]$ to the MHD-type system

$$\partial_t u + u \cdot \nabla u + \nabla p = B \cdot \nabla B \quad (2.27a)$$

$$\partial_t B + u \cdot \nabla B - B \cdot \nabla u = 0 \quad (2.27b)$$

$$\operatorname{div} u = \operatorname{div} B = 0 \quad (2.27c)$$

$$u^2|_{y=0,1} = B^2|_{y=0,1} = 0 \quad (2.27d)$$

$$\int_{\mathbb{T} \times [0,1]} u^2(t) \equiv \int_{\mathbb{T} \times [0,1]} B^2(t) \equiv 0 \quad (2.27e)$$

$$u|_{t=0} = u_0 \quad (2.27f)$$

$$B|_{t=0} = B_0, \quad (2.27g)$$

which in addition is such that u is close to the background shear profile $u_{\text{shear}} = H_u(t)e_x$, and

$$\operatorname{supp} B|_{t=1} \subset \cup_{m \in \mathbb{Z}} [1.5 + 6m, 4.5 + 6m]. \quad (2.28)$$

As above, the closeness to the background shear profile is quantified as in the statement of Lemma 2.6.

Proof of Lemma 2.10. We begin as in the proof of Proposition 2.2 with an ansatz $u = \tilde{u} + H_u(t)e_x = \tilde{u} + u_{\text{shear}}$, only with no background shear in B this time. Note that this is possible by (2.26) and the final sentence of the statement of Proposition 2.2. We choose $H_u(t): [0, 1] \rightarrow \mathbb{R}$ to be any smooth function such that

$$H(0) = H(1) = 0, \quad \int_0^1 H_u(t) dt = 2.5.$$

A consequence of this choice is that if $X(t, x): [0, 1] \times \mathbb{T} \rightarrow \mathbb{T}$ is the solution to the ordinary differential equation

$$\partial_t X(t, x) = H_u(t), \quad X|_{t=0} = x,$$

then

$$X(1, x) = x + \int_0^1 H_u(t) dx = x + 2.5.$$

Now applying the same methodology as in the proof of Proposition 2.2, we may construct solutions to the MHD-type system such that

$$\|u - u_{\text{shear}}\|_{C([0,1]; H^r(\mathbb{T} \times [0,1]))} \leq C\varepsilon, \quad (2.29)$$

as in Lemma 2.6. Then define $\tilde{X}: [0, 1] \times \mathbb{T} \rightarrow \mathbb{T}$ as the solution to the ordinary differential equation

$$\partial_t \tilde{X}(t, x) = u\left(t, \tilde{X}(t, x)\right), \quad \tilde{X}|_{t=0} = x.$$

Then since $B_{\text{shear}} = 0$, the vector transport equation for $\tilde{B} = B$ implies

$$\text{supp } B|_{t=1} = \tilde{X}(1, \text{supp } B_0) := \left\{x \in \mathbb{T} : \tilde{X}^{-1}(x, 1) \in \text{supp } B_0\right\}.$$

Assuming that ε is chosen sufficiently small, (2.29) implies

$$|\tilde{X}(t, x) - X(t, x)| \leq 0.25.$$

Then we have that

$$\begin{aligned} \text{supp } B|_{t=1} &\subset \cup_{m \in \mathbb{Z}} [-0.75 + 2.25 + 6m, 1.75 + 2.75 + 6m] \\ &= \cup_{m \in \mathbb{Z}} [1.5 + 6m, 4.5 + 6m], \end{aligned}$$

concluding the proof of (2.28). $\square$

Now, we carry out the following.

1. **Compactly supported extensions of u and B :** Consider $(u|_{t=1}, B|_{t=1})$ as given by the conclusions of Proposition 2.2. From (2.1h) and (2.6)–(2.7), we have that the means of u^1 and B^1 at time $t = 1$ vanish, which is, by (2.3), equivalent to the averages of u^1 and B^1 over $\mathbb{T} \times [0, 1]$ vanishing. Applying Lemma 2.9, we can modify $B|_{t=1}$ to produce B_T , which stays the same in a neighborhood of $[0, 1] \times [0, 1]$ but satisfies $B = 0$ for $x \in [7/4, 5]$ or $x \in [-1, -3/4]$.
2. **Application of Lemma 2.10:** We apply Lemma 2.10 to $(u_0, B_0) = (u|_{t=1}, B_T)$. This ensures that $\text{supp } B \subset \cup_{m \in \mathbb{Z}} (1.5 + 6m, 4.5 + 6m)$ at $t = 1$. Thus the magnetic field B now vanishes inside the set $[0, 1]^2$ at time $t = 1$.

We now provide some commentary explaining why (1.6) is not sufficient for our method of proof. Note that from the assumption that B^1 has vanishing average in Lemma 2.10, we have that $B = \tilde{B}$ in the language of the proof of Proposition 2.2. But B^1 would *not* have vanishing average if either the initial or ending data for the control problem does not have vanishing average, and we set $B_{\text{shear}} = 0$ in Proposition 2.2. In these cases, a compactly supported divergence free extension which does not penetrate the upper and lower boundaries is in general not possible. Therefore, the vector transport equation for $B = \tilde{B}$ would now read

$$\partial_t \tilde{B} + u \cdot \nabla \tilde{B} - (B_E e_x + \tilde{B}) \cdot \nabla u = 0,$$

where B_E is the average of B^1 , which is non-zero and preserved in time. This new equation does not transport the support of $\tilde{B}$, and so it is not clear how to ensure that $\tilde{B}$ leaves the domain $[0, 1]^2$. Even if one could ensure that $\tilde{B}$ leaves this domain so that $B = B_E e_x$ at some later time, this property would not be preserved upon application of a control method to the remaining part of the velocity. Control methods for Euler connect the desired initial and terminal data through some common state halfway through the time interval, usually the 0 state; a reasonable guess for the MHD analogue would be to connect both states through $(u, B) = (0, B_E e_x)$. Since we cannot send the initial or terminal data to this state, and there is no obvious alternative, we instead connect the initial and terminal data through $(0, 0)$, thus necessitating a forcing term in the equation for B .

2.3. Third step: control for Euler

Now that the magnetic field vanishes inside of the domain $[0, 1]^2$, we truncate the extended domain $[-1, 5] \times [0, 1]$ back to $[0, 1]^2$. On $[0, 1]^2$, we now have a vector field u which does not necessarily vanish, but a magnetic field B which vanishes. Solving the MHD equations on $[0, 1]^2$ with vanishing data for the magnetic field is clearly equivalent to solving the Euler equations on $[0, 1]^2$. So applying any control method for Euler ([3,4,6]) finishes the proof.

References

- [1] R.A. Adams, J.J.F. Fournier, *Sobolev Spaces, Second, Pure and Applied Mathematics (Amsterdam)*, vol. 140, Elsevier/Academic Press, Amsterdam, 2003, MR2424078.
- [2] R.E. Caflisch, I. Klapper, G. Steele, Remarks on singularities, dimension and energy dissipation for dynamics and MHD, *Commun. Math. Phys.* 184 (2) (1997) 443–455, MR1462753.
- [3] J.-M. Coron, Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels, *C. R. Acad. Sci., Sér. I Math.* 317 (3) (1993) 271–276, MR1233425.

- [4] J.-M. Coron, On the controllability of 2-D incompressible perfect fluids, *J. Math. Pures Appl.* (9) 75 (1996) 155–188, MR1380673.
- [5] J.-M. Coron, S. Xiang, Small-time global stabilization of the viscous Burgers equation with three scalar controls, *J. Math. Pures Appl.* (9) 151 (2021) 212–256, MR4265693.
- [6] O. Glass, Exact boundary controllability of 3-D Euler equation, *ESAIM Control Optim. Calc. Var.* 5 (2000) 1–44, MR1745685.
- [7] M. Rissel, Y.-G. Wang, Global exact controllability of ideal incompressible hydrodynamic flows through a planar duct, *arXiv preprint*, arXiv:2105.12321, 2021, available at 2105.12321.
- [8] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1984 edition, MR1846644.