Finite time blow-up for the nematic liquid crystal flow in dimension two

CHEN-CHIH LAI
University of British Columbia

FANGHUA LIN
New York University

CHANGYOU WANG
Purdue University

JUNCHENG WEI
University of British Columbia

AND

YIFU ZHOU
University of British Columbia

Abstract

We consider the initial-boundary value problem of a simplified nematic liquid crystal flow in a bounded, smooth domain $\Omega \subset \mathbb{R}^2$. Given any k distinct points in the domain, we develop a new *inner–outer gluing method* to construct solutions which blow up exactly at those k points as t goes to a finite time T. Moreover, we obtain a precise description of the blow-up.

Contents

Communications on Pure and Applied Mathematics, Vol. 000, 0001–0067 (2000)

1 Introduction

In this paper, we consider the following initial-boundary value problem of nematic liquid crystal flow in a bounded, smooth domain Ω in $\mathbb{R}^2$, and $T > 0$

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla P = \Delta v - \varepsilon_0 \nabla\cdot\left(\nabla u \odot \nabla u - \frac{1}{2}|\nabla u|^2 \mathbb{I}_2\right) & \text{in } \Omega\times(0,T), \\ \nabla\cdot v = 0 & \text{in } \Omega\times(0,T), \\ \partial_t u + v\cdot\nabla u = \Delta u + |\nabla u|^2 u & \text{in } \Omega\times(0,T), \end{cases} \tag{1.1}$$

with initial condition

$$(v,u)\big|_{t=0} = (v_0,u_0) \text{ in } \Omega, \tag{1.2}$$

and boundary condition

$$\begin{aligned} v &= 0 \text{ on } \partial\Omega\times(0,T), \\ u &= u_0 \text{ on } \partial\Omega\times(0,T), \end{aligned} \tag{1.3}$$

where $v : \Omega\times[0,T) \to \mathbb{R}^2$ is the fluid velocity field, $P : \Omega\times[0,T) \to \mathbb{R}$ is the fluid pressure function, $u : \Omega\times[0,T) \to \mathbb{S}^2$ stands for the orientation field of nematic liquid crystal molecules, $\nabla\cdot$ denotes the divergence operator, $\nabla u \odot \nabla u$ denotes the 2×2 matrix given by $(\nabla u \odot \nabla u)_{ij} = \nabla_i u \cdot \nabla_j u$, and $\mathbb{I}_2$ is the identity matrix on $\mathbb{R}^2$. The parameter $\varepsilon_0 > 0$ represents the competition between kinetic energy and elastic energy. $(v_0,u_0) : \Omega \to \mathbb{R}^2\times\mathbb{S}^2$ is a given initial data such that $\nabla\cdot v_0 = 0$.

The system (1.1) can be viewed as a coupling between the incompressible Navier–Stokes equations and the equations of heat flow of harmonic maps. Both the incompressible Navier–Stokes equations and the equations of harmonic map heat flow have been studied extensively. For the incompressible Navier–Stokes equations, the existence of global weak solutions to the initial value problem has been well-known since the fundamental works of Leray [42] and Hopf [35]. A more comprehensive theory on the Navier–Stokes equation can be found in classical books of Temam [67], Lions [53], see also [41], [26], [58], [68] and the references therein. The fundamental solution of the Stokes system, which is a linearized Navier–Stokes equation, was established by Solonnikov in [61], together with estimates of weak solutions to the Cauchy problem. Solonnikov also derived similar estimates of the initial-boundary value problem of the Stokes system in [62, 65, 64], and these sharp estimates would be very important in our construction. For the harmonic map heat flow, Struwe [66] and Chang [4] established the existence of a unique global weak solution in dimension two, which has at most finitely many singular points. In higher dimensions, the existence of a global weak solution has been proved by Chen and Struwe in [9] (see also Chen and Lin [8]). Examples of finite time blow-up solutions have been constructed in dimension $n \geq 3$ in [11] and [7], see also [27, 28]. In dimension two, Chang, Ding and Ye [5] established the first example of finite time singularities by a sub-super solution method for axially symmetric maps into the standard sphere. Angenent, Hulshof and Matano [2]

analyzed a 1-corotational blow-up solution in a disk with profile

$$u(x,t) = W\left(\frac{x}{\lambda(t)}\right) + O(1),$$

where W is the least energy harmonic map (of degree one)

$$W(y) = \frac{1}{1+|y|^2}\begin{bmatrix} 2y \\ |y|^2 - 1 \end{bmatrix},\ y \in \mathbb{R}^2,$$

$O(1)$ denotes a term that is bounded in H^1-norm, and $0 < \lambda(t) \to 0$ as $t \to T$. They obtained an estimation of the blow-up rate as $\lambda(t) = o(T-t)$. Using matched asymptotics formal analysis, van den Berg, Hulshof and King [69] showed that this rate should be given by

$$\lambda(t) \sim \kappa \frac{T-t}{|\log(T-t)|^2}$$

for some $\kappa > 0$. Raphaël and Schweyer succeeded in constructing an entire 1-corotational solution with this blow-up rate rigorously [57]. Recently, Davila, del Pino and Wei [15] constructed a non-symmetric solution that exhibits finite time blow-up at multiple points and studied its stability by using the *inner–outer gluing method*. More precisely, for any given finite set of points in Ω, they constructed solution blowing up exactly at those points simultaneously under suitable initial and boundary conditions. In another aspect, for higher-degree corotational harmonic map heat flow, global existence and blow-up have been investigated in a series of works [30, 29, 31, 32] and the references therein. For the general analysis of the bubbling phenomena and regularity results of the harmonic map heat flow, we refer the readers to the book [49].

The model equations for the nematic liquid crystal flow (1.1) that will be studied in this article are proposed in [45], and it is a simplified version of the Ericksen–Leslie system for the hydrodynamics flow of nematic liquid crystal material established by Ericksen [25] and Leslie [43]. The existence and uniqueness of solutions to (1.1) has attracted a lot of interests in recent years. In an earlier work [46], Lin and Liu considered the Ericksen-Leslie system with variable degree of orientations, and established a global existence of weak and classical solutions in dimensions three and two. There is also a partial regularity theorem for suitable weak solutions of approximate systems for (1.1), see [47], similar to those for the Navier–Stokes equation established by Caffarelli–Kohn–Nirenberg in [3]. Later in [48], a global existence of Leray–Hopf–Struwe type weak solutions of (1.1) in two dimensions is proved (see also [33], [34], [71], [36], [40] and [70]). More importantly, the uniqueness of such weak solution in dimension two can also be shown [50]. For the case of dimension three, much less is known. Lin and Wang [52] proved a global existence of (suitable) weak solutions satisfying the global energy inequality under a restrictive assumption that the initial orientation field $u_0(\Omega) \subset \mathbb{S}^2_+$. There are also blow up criteria for finite time singularities for local

strong solutions of (1.1) in both dimensions two and three, for instance, Huang and Wang [38]. We should also point out a recent interesting work by Chen and Yu [6]. They constructed global m-equivariant solutions in $\mathbb{R}^2$ where the orientation field blows up logarithmically as $t \to +\infty$. For a survey of some recent important developments of mathematical analysis of nematic liquid crystals we refer to [51].

The main concern of this paper is the existence of classical solutions to the nematic liquid crystal flow (1.1), that develop finite time singularities. In dimension three, the work [37] has provided two examples of finite time singularity of (1.1). The first example is an axisymmetric finite time blow-up solution constructed in a cylindrical domain (as remarked in [37] Remark 1.2(c), this blow-up example does not satisfy the no-slip boundary condition). The second example is constructed in a ball for any generic initial data (v_0, u_0) that has small enough energy, and u_0 has a non-zero Hopf-degree.

In this paper, we consider the two-dimensional nematic liquid crystal flow (1.1), where the velocity field satisfies no-slip boundary condition, i.e., $v = 0$ on $\partial\Omega$. We wish to point out that if $v \equiv 0$ in (1.1), then u is not only a solution of the harmonic map heat flow, it also satisfies the compatibility condition $\nabla \cdot (\nabla u \odot \nabla u - \frac{1}{2}|\nabla u|^2 \mathbb{I}_2) = \nabla P$ for a scalar function P. In fact, one can check that for the blow-up solution u to the harmonic map heat flow constructed by [5], as it is axisymmetric, $(u, 0)$ is also a blow-up solution to (1.1). On the other hand, the blow-up solutions u to the harmonic map heat flow in [15] can not satisfy (1.1) with $v \equiv 0$, whenever the number of blow up points $k > 1$.

Using the *inner–outer gluing method* for both u and v, we construct a solution (v, u) to problem (1.1) exhibiting finite time singularity when the parameter ε_0 is sufficiently small. More precisely, we have

Theorem 1.1. *There exists a sufficiently small $\varepsilon_0 > 0$ such that given k distinct points $q_1, \cdots, q_k \in \Omega$, if $T > 0$ is sufficiently small, then there exists a smooth initial data (v_0, u_0) such that the short time smooth solution (v, u) to the system (1.1) blows up exactly at those k points as $t \to T$. More precisely, there exist numbers $\kappa_j^* > 0$, ω_j^* and $u_* \in H^1(\Omega) \cap C(\bar{\Omega})$ such that*

$$u(x,t) - u_*(x) - \sum_{j=1}^{k} Q^1_{\omega_j} Q^2_{\alpha_j} Q^3_{\beta_j} \left[W\left(\frac{x - q_j}{\lambda_j(t)} \right) - W(\infty) \right] \to 0 \ \ as \ \ t \to T,$$

in $H^1(\Omega) \cap L^\infty(\Omega)$, where the blow-up rate and angles satisfy

$$\lambda_j(t) = \kappa_j^* \frac{T - t}{|\log(T - t)|^2} (1 + o(1)) \ \ as \ \ t \to T,$$

$$\omega_j \to \omega_j^*, \ \alpha_j \to 0, \ \beta_j \to 0, \quad as \ \ t \to T,$$

and Q^1_ω, Q^2_α and Q^3_β are rotation matrices defined in (2.2)*. In particular, it holds that*

$$|\nabla u(\cdot,t)|^2\,dx \rightharpoonup |\nabla u_*|^2\,dx + 8\pi\sum_{j=1}^{k}\delta_{q_j} \ \text{ as } \ t\to T,$$

as convergence of Radon measures. Furthermore, the velocity field satisfies

$$|v(x,t)| \le c\sum_{j=1}^{k}\frac{\lambda_j^{\nu_j-1}(t)}{1+\left|\frac{x-q_j}{\lambda_j(t)}\right|},\ 0<t<T,$$

for some $c>0$ and $0<\nu_j<1$, $j=1,\cdots,k$.

Concerning Theorem 1.1, we would like to make two remarks.

Remark 1.2.

- At each blow-up point $q_j\in\Omega$, $1\le j\le k$, the behavior of the velocity field v is precisely

  $$|v(x,t)| \le c\lambda_j^{\nu_j-1}(t) + o(1) \ \text{ for } \ \nu_j\in(0,1).$$

  Theorem 1.1 suggests that v might also blow up in finite time. In fact we conjecture that $\|v(\cdot,t)\|_{L^\infty}\sim|\log(T-t)|$ as $t\to T$. The singularity formation of the velocity field is driven by the Ericksen stress tensor $\nabla\cdot(\nabla u\odot\nabla u-\frac{1}{2}|\nabla u|^2\mathbb{I}_2)$, which is induced by the liquid crystal orientation field $u(x,t)$. Namely, $u(x,t)$ plays a role on generating the singular forcing in the incompressible Navier–Stokes equation. For results of the Navier–Stokes equation with singular forcing in dimension two, we refer to [10].

- It is well-known that the pressure P can be recovered from the velocity field v and the forcing. See for instance [26] and [68].

- The proof of Theorem 1.1 actually yields, on one hand, that the small constant ε_0 can be chosen to be a universal constant, that is independent of the domain Ω, blow-up points $q_1,\cdots,q_k$, and time T. On the other hand, no matter how small ε_0 would be, the two systems are fully coupled, because of the following scaling invariance:

  $$(v_\lambda(x,t),P_\lambda(x,t),u_\lambda(x,t)) = (\lambda v(\lambda x,\lambda^2 t),\lambda^2 P(\lambda x,\lambda^2 t),u(\lambda x,\lambda^2 t)).$$

  In addition, this nonlinear coupling property is also preserved in the linearized inner problem:

  $$\begin{cases} v_\tau+\nabla P=\Delta v-\varepsilon_0\nabla\cdot(\nabla W\odot\nabla\phi),\\ \nabla\cdot v=0,\\ \phi_\tau+v\cdot\nabla\phi=\Delta\phi+|\nabla W|^2\phi+2(\nabla W\cdot\nabla\phi)W. \end{cases}$$

Remark 1.3. While, in order to carry out fixed point argument in the inner–outer gluing procedure, we need to assume $\varepsilon_0 > 0$ in (1.1) to be sufficiently small, Theorem 1.1 does cover the relevant physical cases of the hydrodynamics of nematic liquid crystals where the fluid tends to have a large viscous effect. More precisely, instead of (1.1), if we consider

(1.4)
$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla P = \mu\Delta v - \tilde{\lambda}\nabla\cdot\big(\nabla u\odot\nabla u - \frac{1}{2}|\nabla u|^2\mathbb{I}_2\big) & \text{in } \Omega\times(0,T),\\ \nabla\cdot v = 0 & \text{in } \Omega\times(0,T),\\ \partial_t u + v\cdot\nabla u = \tilde{\gamma}(\Delta u + |\nabla u|^2 u) & \text{in } \Omega\times(0,T), \end{cases}$$

where $\mu > 0$, $\tilde{\lambda} > 0$, and $\tilde{\gamma} > 0$ represents the fluid viscosity, the competition parameter between the kinetic energy of fluid and the elastic energy of the liquid crystal orientation field, and the macroscopic relaxation time parameter respectively. Assume that $\frac{\mu}{\tilde{\lambda}} \gg 1$ and $\frac{\tilde{\gamma}}{\mu} \approx 1$. If we set $(\widetilde{v},\widetilde{u},\widetilde{P})(x,t) = \big(\frac{1}{\mu}v, u, \frac{1}{\mu^2}P\big)(x,\frac{t}{\mu})$, then it follows from direct calculations that $(\widetilde{v},\widetilde{u},\widetilde{P})$ solves (1.1) with the parameter $\varepsilon_0 = \frac{\tilde{\lambda}}{\mu} \ll 1$.

The proof of Theorem 1.1 is based on the *inner–outer gluing method*, which has been a very powerful tool in constructing solutions in many elliptic problems, see for instance [18, 19, 20, 16] and the references therein. Also, this method has been successfully applied to various parabolic flows recently, such as the infinite time and finite time blow-ups in energy critical heat equations [12, 23, 22, 24, 21], singularity formation for the 2-dimensional harmonic map heat flow [15], vortex dynamics in Euler flows [14], and others arising from geometry and fractional context [13, 59, 55, 60]. We refer the interested readers to a survey by del Pino [17] for more results in parabolic settings.

The nematic liquid crystal flow (1.1) is a strongly coupled system of the incompressible Navier–Stokes equation and the transported harmonic map heat flow. In this paper, the construction of the finite time blow-up solution is close in spirit to the singularity formation of the standard two dimensional harmonic map heat flow

(1.5)
$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u, & \text{in } \Omega\times(0,T),\\ u = u_0, & \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0) = u_0, & \text{in } \Omega. \end{cases}$$

In [15], by the *inner–outer gluing method*, Davila, del Pino and Wei successfully constructed type II finite time blow-up for the harmonic map heat flow (1.5). More precisely, the solution constructed in [15] takes the bubbling form

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 8\pi\sum_{j=1}^{k}\delta_{q_j}, \text{ as } t\to T,$$

where $u_* \in H^1(\Omega)\cap C(\bar{\Omega})$, $(q_1,\dots,q_k)\in\Omega^k$ are given k points, and δ_{q_j} denotes the unit Dirac mass at q_j for $j = 1,\cdots,k$. The construction in [15] consists of finding

a good approximate solution based on the 1-corotational harmonic maps and then looking for the inner and outer profiles of the small perturbations. Basically, the inner problem is the linearization around the harmonic map which captures the heart of the singularity formation, while the outer problem is a heat equation coupled with the inner problem.

Our construction of a finite time blow-up solution to the nematic liquid crystal flow (1.1)–(1.3) relies crucially on the delicate analysis carried out in [15]. However, because of the strong coupling between the Navier–Stokes equation with forcing for v and the transported harmonic map heat flow equation for u, we have to develop several new ingredients in our inner–outer gluing procedure for the system (1.1)–(1.3):

- Although the advection term $v\cdot\nabla v$ can be realized as a small perturbation in the Stokes system with forcing, the transported term $v\cdot\nabla u$ in the equation for the orientation field u can only be realized as a small perturbation of the outer problem for u, but not of the inner problem for u where the singularity occurs. In fact, since the system (1.1) is invariant under the following parabolic scalings:

$$(v_\lambda(x,t),P_\lambda(x,t),u_\lambda(x,t)) = (\lambda v(\lambda x,\lambda^2 t),\lambda^2 P(\lambda x,\lambda^2 t),u(\lambda x,\lambda^2 t)),\ \forall\lambda>0,$$

  in the self-similar variable (y,τ) near a singular point (q,T), roughly speaking, the order of $v[\phi]\cdot\nabla U$ is the same as that of h in the inner-linearized equation:

$$\partial_\tau\phi + v[\phi]\cdot\nabla_y W = L_W[\phi]+h,$$

  where L_W is the linearization of harmonic map equation around W given by (2.3). See Section 4 for more details.

- In [15], the parameter functions $\lambda(t)$, $\xi(t)$, $\omega(t)$, which correspond to the dilation, translation in the domain, and rotation about z-axis in the target space, respectively, were introduced to adjust certain orthogonality conditions to guarantee the existence of desired solutions to the harmonic map heat flow. However, to find the desired solution of the nematic liquid crystal flow (1.1) as stated in Theorem 1.1, we need to introduce two new parameter functions $\alpha(t)$ and $\beta(t)$ associated to the rotations about x and y axes in the target space, respectively. The reasons behind this are:

  i) Heuristically, in the inner problem of u, the velocity v may exhibit a logarithmic singularity induced by the off-diagonal effect of the Oseen-kernel S_{ij} (see (3.8)). The addition of these new parameter functions $\alpha(t)$ and $\beta(t)$ can balance such a logarithmic singularity off.

  ii) We need to solve the inner linearized problem of u to get a solution with space and time decay rates faster than that by [15], since we need to control the stress-tensor $\nabla\cdot(\nabla u\odot\nabla u-\frac{1}{2}|\nabla u|^2\mathbb{I}_2)$ appearing in the

equation for velocity field v of (1.1). For this purpose, we introduce two new parameter functions $\alpha(t)$ and $\beta(t)$ associated to the rotations about x and y axes in the target space, respectively, to adjust the orthogonality conditions at mode -1. See Section 2 for details.

iii) After the adjustment by suitable $\alpha(t)$ and $\beta(t)$, the smallness of parameter ε_0 can reduce $v\cdot\nabla u$ into a truly small perturbation in the inner problem of u.

- We also need to develop a new linear theory for the Stokes system with some novel weighted L^∞ estimates, which shall have its own interest. The construction of desired velocity field v shall be carried out by another new inner–outer gluing procedure, since the forcing term $\nabla\cdot(\nabla u\odot\nabla u-\frac{1}{2}|\nabla u|^2\mathbb{I}_2)$ in $(1.1)_1$ is concentrated near the blow-up points. See Section 3 for details.

The following picture roughly describes the above process.

$$\begin{array}{ccc} \text{Forcing in } (1.1)_1\colon -\varepsilon_0\nabla\cdot\left(\nabla u\odot\nabla u-\frac{1}{2}|\nabla u|^2\mathbb{I}_2\right) & \xrightarrow{(1)} & v \text{ in } (1.1)_3 \\ \Big\uparrow{\scriptstyle (4)} & & \Big\downarrow{\scriptstyle (2)} \\ \text{Mode } k \text{ in the inner problem of } u\colon \phi_k & \xleftarrow{(3)} & v\cdot\nabla u \text{ in } (1.1)_3 \end{array}$$

(1) Solve the incompressible Navier–Stokes equation with forcing coupled from the orientation u.

(2) The velocity v provides transported effect in the harmonic map heat flow.

(3) The transported term $v\cdot\nabla u$ is coupled in a nontrivial way through the inner problem at mode k since the velocity $v=v[\phi_k]$ carries the information of ϕ_k in step (1).

(4) Faster spatial and time decay of ϕ_k yields better forcing term in $(1.1)_1$, ensuring the implementation of the whole loop.

The paper is organized as follows. In Section 2, we will develop a new inner–outer gluing method for the harmonic map heat flow in order to handle the difficulties arising from the coupling effects of (1.1). In Section 3, we develop the linear theory for the Stokes system. In Section 4, using the newly developed *inner–outer gluing method*, we construct a finite time blow-up solution to the nematic liquid crystal flow by the fixed point argument.

Notation. Throughout the paper, we shall use the symbol " $\lesssim$ " to denote " $\le C$" for a positive constant C independent of t and T. Here C might be different from line to line.

2 Singularity formation for the harmonic map heat flow in dimension two

Closely related to the harmonic map heat flow in dimension two, the equation for the orientation field u can be regarded as a transported version with drift term. In this Section, we consider the two dimensional harmonic map heat flow $u : \Omega \times [0,T) \to \mathbb{S}^2$:

$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u, & \text{in } \Omega \times (0,T), \\ u = u_0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0, & \text{in } \Omega. \end{cases}$$

While following closely the general strategy of the construction developed by [15], we will establish several new estimates that are needed for the system (1.1). More precisely,

- A new linear theory at mode -1: This procedure consists of the following steps
  - Step 1: New corrections are added at mode -1 to cancel out the leading order of slow decaying error corresponding to the rotations around x and y axes in the target space (see Section 2.2).
  - Step 2: New orthogonality conditions are imposed at mode -1 which determine the dynamics of the new parameters $\alpha(t)$ and $\beta(t)$ (see Section 2.4).
  - Step 3: Under the orthogonality conditions at mode -1, the new linear theory at mode -1 is developed (see Section 2.5).
- Higher order estimates for inner and outer solutions are obtained in order to handle the forcing $-\varepsilon_0 \nabla \cdot \left(\nabla u \odot \nabla u - \frac{1}{2}|\nabla u|^2 \mathbb{I}_2\right)$ in the equation for velocity $(1.1)_1$ (see Sections 2.5–2.6).

We first introduce some notations and preliminaries.

2.1 Stationary problem: the equation of harmonic maps and its linearization

The equation of harmonic maps for $U : \mathbb{R}^2 \to \mathbb{S}^2$ is the quasilinear elliptic system

$$\Delta U + |\nabla U|^2 U = 0 \text{ in } \mathbb{R}^2. \tag{2.1}$$

For $\lambda > 0$, $\xi \in \mathbb{R}^2$, $\omega, \alpha, \beta \in \mathbb{R}$, we consider the family of solutions to (2.1) given by the following 1-corotational harmonic maps

$$U_{\lambda,\xi,\omega,\alpha,\beta}(x) = Q^1_\omega Q^2_\alpha Q^3_\beta W\left(\frac{x-\xi}{\lambda}\right),\ x \in \mathbb{R}^2,$$

where

$$Q^1_\omega := \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix},\ Q^2_\alpha := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

$$Q^3_\beta := \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.2}$$

are the rotation matrices about z, x and y axes in the target space, respectively, and W is the least energy harmonic map

$$W(y) = \frac{1}{1+|y|^2}\begin{bmatrix} 2y \\ |y|^2-1 \end{bmatrix},\ y \in \mathbb{R}^2.$$

In the polar coordinates $y = \rho e^{i\theta}$, $W(y)$ can be represented as

$$W(y) = \begin{bmatrix} e^{i\theta}\sin w(\rho) \\ \cos w(\rho) \end{bmatrix},\ w(\rho) = \pi - 2\arctan(\rho),$$

and we have

$$w_\rho = -\frac{2}{\rho^2+1},\ \sin w = -\rho w_\rho = \frac{2\rho}{\rho^2+1},\ \cos w = \frac{\rho^2-1}{\rho^2+1}.$$

For simplicity, we write

$$Q_{\omega,\alpha,\beta} := Q^1_\omega Q^2_\alpha Q^3_\beta.$$

The linearization of the harmonic map operator around W is the elliptic operator

$$L_W[\phi] = \Delta_y\phi + |\nabla W(y)|^2\phi + 2(\nabla W(y)\cdot\nabla\phi)W(y), \tag{2.3}$$

whose kernel functions are given by

$$\begin{cases} Z_{0,1}(y) = \rho w_\rho(\rho)E_1(y), \\ Z_{0,2}(y) = \rho w_\rho(\rho)E_2(y), \\ Z_{1,1}(y) = w_\rho(\rho)[\cos\theta E_1(y) + \sin\theta E_2(y)], \\ Z_{1,2}(y) = w_\rho(\rho)[\sin\theta E_1(y) - \cos\theta E_2(y)], \\ Z_{-1,1}(y) = \rho^2 w_\rho(\rho)[\cos\theta E_1(y) - \sin\theta E_2(y)], \\ Z_{-1,2}(y) = \rho^2 w_\rho(\rho)[\sin\theta E_1(y) + \cos\theta E_2(y)], \end{cases} \tag{2.4}$$

where the vectors

$$E_1(y) = \begin{bmatrix} e^{i\theta}\cos w(\rho) \\ -\sin w(\rho) \end{bmatrix},\ E_2(y) = \begin{bmatrix} ie^{i\theta} \\ 0 \end{bmatrix}$$

form an orthonormal basis of the tangent space $T_{W(y)}\mathbb{S}^2$. We see that

$$L_W[Z_{i,j}] = 0 \ \text{ for } i = \pm 1, 0,\ j = 1, 2.$$

Note that

$$L_U[\varphi] = \lambda^{-2} Q_{\omega,\alpha,\beta} L_W[\phi],\ \varphi(x) = \phi(y),\ y = \frac{x-\xi}{\lambda}.$$

In the sequel, it is of significance to compute the action of L_U on functions whose value is orthogonal to U pointwisely. Define

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi \cdot U)U.$$

We invoke several useful formulas proved in [15, Section 3]:

$$L_U[\Pi_{U^\perp}\Phi] = \Pi_{U^\perp}\Delta\Phi + \tilde{L}_U[\Phi],$$

where

$$\tilde{L}_U[\Phi] := |\nabla U|^2 \Pi_{U^\perp}\Phi - 2\nabla(\Phi\cdot U)\nabla U, \tag{2.5}$$

with

$$\nabla(\Phi\cdot U)\nabla U = \partial_{x_j}(\Phi\cdot U)\partial_{x_j}U.$$

In the polar coordinates

$$\Phi(x) = \Phi(r,\theta),\ x = \xi + re^{i\theta},$$

(2.5) can be expressed as (see [15, Section 3])

$$\tilde{L}_U[\Phi] = -\frac{2}{\lambda} w_\rho(\rho)\left[(\Phi_r\cdot U)Q_{\omega,\alpha,\beta}E_1 - \frac{1}{r}(\Phi_\theta\cdot U)Q_{\omega,\alpha,\beta}E_2\right],\ r = \lambda\rho.$$

Assume that $\Phi(x) : \Omega \to \mathbb{C}\times\mathbb{R}$ is a C^1 function in the form

$$\Phi(x) = \begin{bmatrix} \varphi_1(x) + i\varphi_2(x) \\ \varphi_3(x) \end{bmatrix}. \tag{2.6}$$

If we write

$$\varphi = \varphi_1 + i\varphi_2,\ \bar{\varphi} = \varphi_1 - i\varphi_2$$

and

$$\mathrm{div}\varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2,\ \mathrm{curl}\varphi = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1,$$

then we have the following formula (see [15, Section 3])

$$\tilde{L}_U[\Phi] = [\tilde{L}_U]_0[\Phi] + [\tilde{L}_U]_1[\Phi] + [\tilde{L}_U]_2[\Phi], \tag{2.7}$$

where

$$\begin{cases} [\tilde{L}_U]_0[\Phi] = \lambda^{-1}\rho w_\rho^2[\mathrm{div}(e^{-i\omega}\varphi)Q_{\omega,\alpha,\beta}E_1 + \mathrm{curl}(e^{-i\omega}\varphi)Q_{\omega,\alpha,\beta}E_2], \\ [\tilde{L}_U]_1[\Phi] = -2\lambda^{-1}w_\rho\cos w[(\partial_{x_1}\varphi_3)\cos\theta + (\partial_{x_2}\varphi_3)\sin\theta]Q_{\omega,\alpha,\beta}E_1 \\ \qquad\qquad -2\lambda^{-1}w_\rho\cos w[(\partial_{x_1}\varphi_3)\sin\theta - (\partial_{x_2}\varphi_3)\cos\theta]Q_{\omega,\alpha,\beta}E_2, \\ [\tilde{L}_U]_2[\Phi] = \lambda^{-1}\rho w_\rho^2[\mathrm{div}(e^{i\omega}\bar{\varphi})\cos 2\theta - \mathrm{curl}(e^{i\omega}\bar{\varphi})\sin 2\theta]Q_{\omega,\alpha,\beta}E_1 \\ \qquad\qquad +\lambda^{-1}\rho w_\rho^2[\mathrm{div}(e^{i\omega}\bar{\varphi})\sin 2\theta + \mathrm{curl}(e^{i\omega}\bar{\varphi})\cos 2\theta]Q_{\omega,\alpha,\beta}E_2. \end{cases} \tag{2.8}$$

If we assume

$$\Phi(x) = \begin{bmatrix} \phi(r)e^{i\theta} \\ 0 \end{bmatrix},\ x = \xi + re^{i\theta},\ r = \lambda\rho,$$

where $\phi(r)$ is complex-valued, then we have the following formula

$$\tilde{L}_U[\Phi] = \frac{2}{\lambda} w_\rho^2(\rho) \left[\mathrm{Re}(e^{-i\omega}\partial_r\phi(r)) Q_{\omega,\alpha,\beta} E_1 + \frac{1}{r}\mathrm{Im}(e^{-i\omega}\phi(r)) Q_{\omega,\alpha,\beta} E_2 \right].$$

If Φ is of the form

$$\Phi(x) = \varphi_1(\rho,\theta) Q_{\omega,\alpha,\beta} E_1 + \varphi_2(\rho,\theta) Q_{\omega,\alpha,\beta} E_2,\ x = \xi + \lambda\rho e^{i\theta}$$

in the polar coordinates, then the linearized operator L_U acting on Φ can be expressed as (see [15, Section 3])

$$\begin{aligned} &L_U[\Phi] \\ &= \lambda^{-2}\left(\partial_{\rho\rho}\varphi_1 + \frac{\partial_\rho\varphi_1}{\rho} + \frac{\partial_{\theta\theta}\varphi_1}{\rho^2} + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_1 - \frac{2}{\rho^2}\partial_\theta\varphi_2 \cos w \right) Q_{\omega,\alpha,\beta} E_1 \\ &+ \lambda^{-2}\left(\partial_{\rho\rho}\varphi_2 + \frac{\partial_\rho\varphi_2}{\rho} + \frac{\partial_{\theta\theta}\varphi_2}{\rho^2} + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_2 + \frac{2}{\rho^2}\partial_\theta\varphi_1 \cos w \right) Q_{\omega,\alpha,\beta} E_2. \end{aligned}$$

In next section, we shall find proper approximate solutions to the harmonic map heat flow based on the 1-corotational harmonic maps, and evaluate the error.

2.2 Approximate solution and error estimates

We now consider the harmonic map heat flow

$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u, & \text{in } \Omega\times(0,T), \\ u = u_0, & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_0, & \text{in } \Omega, \end{cases} \tag{2.9}$$

where $u : \bar{\Omega}\times(0,T) \to \mathbb{S}^2$, and $u_0 : \bar{\Omega} \to \mathbb{S}^2$ is a given smooth map. For notational simplicity, we shall only carry out the construction in the single bubble case $k = 1$ and mention the minor changes for the general case when needed. We define the error operator

$$S[u] = -\partial_t u + \Delta u + |\nabla u|^2 u.$$

We shall look for solution $u(x,t)$ to problem (2.9) which at leading order takes the form

$$U(x,t) := U_{\lambda(t),\xi(t),\omega(t),\alpha(t),\beta(t)} = Q_{\omega(t),\alpha(t),\beta(t)} W\left(\frac{x-\xi(t)}{\lambda(t)}\right). \tag{2.10}$$

Here $\lambda(t)$, $\xi(t)$, $\omega(t)$, $\alpha(t)$ and $\beta(t)$ are parameter functions of class $C^1((0,T))$ to be determined later. To get a desired blow-up solution, we assume

$$\lambda(t) \to 0,\ \xi(t) \to q \ \text{ as } \ t \to T,$$

where q is a given point in Ω.

A useful observation is that as long as the constraint $|u| = 1$ is kept for all $t \in (0,T)$ and $u = U + \tilde{w}$ where the perturbation $\tilde{w}$ is uniformly small, say, $|\tilde{w}| \le \frac{1}{2}$, then for u to solve (2.9), it suffices that

$$S(U+\tilde{w}) = b(x,t)U \tag{2.11}$$

for some scalar function b. Indeed, since $|u| = 1$, we get

$$b(U\cdot u) = S(u)\cdot u = -\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2}\Delta|u|^2 = 0.$$

Thus $b \equiv 0$ follows from $U \cdot u \ge \frac{1}{2}$.

We look for the small perturbation $\tilde{w}(x,t)$ with $|U+\tilde{w}| = 1$ in the form

$$\tilde{w} = \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U,$$

where φ is an arbitrarily small perturbation with values in $\mathbb{R}^3$, and

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi\cdot U)U, \quad a(\zeta) = \sqrt{1-|\zeta|^2} - 1.$$

By $\Delta U + |\nabla U|^2 U = 0$, we compute

$$S(U+\Pi_{U^\perp}\varphi + aU) = -U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + c(\Pi_{U^\perp}\varphi)U,$$

where for $\zeta = \Pi_{U^\perp}\varphi$, $a = a(\zeta)$,

$$\begin{aligned}
L_U(\zeta) &= \Delta\zeta + |\nabla U|^2\zeta + 2(\nabla U\cdot\nabla\zeta)U,\\
N_U(\zeta) &\\
&= \big[2\nabla(aU)\cdot\nabla(U+\zeta) + 2\nabla U\cdot\nabla\zeta + |\nabla\zeta|^2 + |\nabla(aU)|^2\big]\zeta - aU_t + 2\nabla a\cdot\nabla U,\\
c(\zeta) &= \Delta a - a_t + (|\nabla(U+\zeta+aU)|^2 - |\nabla U|^2)(1+a) - 2\nabla U\cdot\nabla\zeta.
\end{aligned}$$

Since we just need to have an equation in the form (2.11) satisfied, we obtain that

$$u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U \tag{2.12}$$

solves (2.9) if φ satisfies

$$-U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + b(x,t)U = 0 \tag{2.13}$$

for some scalar function $b(x,t)$. The strategy for constructing φ is based on the *inner–outer gluing method*. We decompose φ in (2.12) into inner and outer profiles

$$\varphi = \varphi_{in} + \varphi_{out},$$

where φ_{in}, φ_{out} solve the inner and outer problems we shall describe below. In terms of φ_{in} and φ_{out}, equation (2.13) is reduced to

$$\begin{aligned}
&-\partial_t\varphi_{in} + L_U[\varphi_{in}] + \tilde{L}_U[\varphi_{out}] - \Pi_{U^\perp}[\partial_t\varphi_{out} - \Delta\varphi_{out} + U_t]\\
&+N_U(\varphi_{in} + \Pi_{U^\perp}\varphi_{out}) + (\varphi_{out}\cdot U)U_t + bU = 0.
\end{aligned} \tag{2.14}$$

The inner solution φ_{in} will be assumed to be supported only near $x = \xi(t)$ and better expressed in the scaled variable $y = \frac{x-\xi(t)}{\lambda(t)}$ with zero initial condition and $\varphi_{in}\cdot U = 0$ so that $\Pi_{U^\perp}\varphi_{in} = \varphi_{in}$, while the outer solution φ_{out} will consist of several parts

whose role is essentially to satisfy (2.14) in the region away from the concentration point $x = \xi(t)$.

For the outer problem, since we want the size of the error to be small, we shall add three corrections Φ^0, Φ^α and Φ^β which depend on the parameter functions $\lambda(t)$, $\xi(t)$, $\omega(t)$, $\alpha(t)$, $\beta(t)$ such that

$$\Pi_{U^\perp}[\partial_t(\Phi^0+\Phi^\alpha+\Phi^\beta)-\Delta(\Phi^0+\Phi^\alpha+\Phi^\beta)+U_t]$$

gets concentrated near $x = \xi(t)$ by eliminating the leading orders in the first error U_t associated to the dilation and rotations about x, y and z axes. We write

$$\varphi_{out}(x,t) = \Psi^*(x,t)+\Phi^0(x,t)+\Phi^\alpha(x,t)+\Phi^\beta(x,t),$$

where

$$\Psi^* = \psi + Z^*$$

with $Z^* : \Omega\times(0,\infty)\to\mathbb{R}^3$ satisfying

$$\begin{cases} \partial_t Z^* = \Delta Z^*, & \text{in } \Omega\times(0,\infty),\\ Z^*(\cdot,t) = 0, & \text{on } \partial\Omega\times(0,\infty),\\ Z^*(\cdot,0) = Z_0^*, & \text{in } \Omega. \end{cases}$$

For the inner problem, we define

$$\varphi_{in}(x,t) = \eta_R Q_{\omega,\alpha,\beta}\phi(y,t)$$

with

$$\eta_R(x,t) = \eta\left(\frac{|x-\xi(t)|}{\lambda(t)R(t)}\right),\ y = \frac{x-\xi(t)}{\lambda(t)},\ \eta(s) = \begin{cases} 1, & \text{for } s<1,\\ 0, & \text{for } s>2, \end{cases}$$

where $\phi(y,t)$ satisfies $\phi(\cdot,0) = 0$ and $\phi(\cdot,t)\cdot W = 0$, and $R(t) > 0$ is determined later. Then equation (2.13) becomes

(2.15)

$$\begin{aligned} 0 &= \lambda^{-2}\eta_R Q_{\omega,\alpha,\beta}[-\lambda^2\phi_t + L_W[\phi] + \lambda^2 Q^{-1}_{\omega,\alpha,\beta}\tilde{L}_U[\Psi^*]]\\ &\quad + \eta_R Q_{\omega,\alpha,\beta}(\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi + \lambda^{-1}\dot\xi\cdot\nabla_y\phi - (Q^{-1}_{\omega,\alpha,\beta}\frac{d}{dt}Q_{\omega,\alpha,\beta})\phi)\\ &\quad + \tilde{L}_U[\Phi^0+\Phi^\alpha+\Phi^\beta] - \Pi_{U^\perp}[\partial_t(\Phi^0+\Phi^\alpha+\Phi^\beta)-\Delta_x(\Phi^0+\Phi^\alpha+\Phi^\beta)+U_t]\\ &\quad - \partial_t\Psi^* + \Delta\Psi^* + (1-\eta_R)\tilde{L}_U[\Psi^*] + Q_{\omega,\alpha,\beta}[(\Delta_x\eta_R)\phi + 2\nabla_x\eta_R\nabla_x\phi - (\partial_t\eta_R)\phi]\\ &\quad + N_U(\eta_R Q_{\omega,\alpha,\beta}\phi + \Pi_{U^\perp}(\Phi^0+\Phi^\alpha+\Phi^\beta+\Psi^*))\\ &\quad + ((\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta)\cdot U)U_t + bU. \end{aligned}$$

We now give the precise definitions of Φ^0, Φ^α, Φ^β, and estimate the error

$$\tilde{L}_U[\Phi^0+\Phi^\alpha+\Phi^\beta]-\Pi_{U^\perp}[\partial_t(\Phi^0+\Phi^\alpha+\Phi^\beta)-\Delta_x(\Phi^0+\Phi^\alpha+\Phi^\beta)+U_t].$$

We shall choose Φ^0, Φ^α, Φ^β in a way such that

$$\partial_t(\Phi^0+\Phi^\alpha+\Phi^\beta)-\Delta_x(\Phi^0+\Phi^\alpha+\Phi^\beta)+U_t \approx 0 \text{ for } |x-\xi|\gg\lambda$$

so that the error in the outer problem is of smaller order.

The error of the approximate solution defined in (2.10) is

$$\mathscr{S}[U] = -\partial_t U = -\big[\underbrace{\dot\lambda\partial_\lambda U + \dot\omega\partial_\omega U}_{:=\mathscr{E}_0} + \underbrace{\dot\xi\cdot\partial_\xi U}_{:=\mathscr{E}_1} + \underbrace{\dot\alpha\partial_\alpha U + \dot\beta\partial_\beta U}_{:=\mathscr{E}_{-1}}\big]$$

where

$$\begin{cases} \partial_\lambda U(x) = \lambda^{-1} Q_{\omega,\alpha,\beta} Z_{0,1}(y) \\ \partial_\omega U(x) = Q_{\omega,\alpha,\beta} Z_{0,2}(y) + Q_{\omega,\alpha,\beta}(A_{\alpha,\beta} - J_1) W(y) \\ \partial_{\xi_1} U(x) = \lambda^{-1} Q_{\omega,\alpha,\beta} Z_{1,1}(y) \\ \partial_{\xi_2} U(x) = \lambda^{-1} Q_{\omega,\alpha,\beta} Z_{1,2}(y) \\ \partial_\alpha U(x) = \dfrac{1}{2} Q_{\omega,\alpha,\beta}\left[Z_{-1,2}(y) + Z_{1,2}(y)\right] + Q_{\omega,\alpha,\beta}(A_\beta - J_2) W(y) \\ \partial_\beta U(x) = -\dfrac{1}{2} Q_{\omega,\alpha,\beta}\left[Z_{-1,1}(y) + Z_{1,1}(y)\right] \end{cases}$$

with $Z_{i,j}$ defined in (2.4) for $i = 0, \pm 1$, $j = 1, 2$,

$$A_{\alpha,\beta} = \begin{bmatrix} 0 & -\cos\alpha\cos\beta & \sin\alpha \\ \cos\alpha\cos\beta & 0 & \cos\alpha\sin\beta \\ -\sin\alpha & -\cos\alpha\sin\beta & 0 \end{bmatrix}, \; J_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.16}$$

and

$$A_\beta = \begin{bmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & -\cos\beta \\ 0 & \cos\beta & 0 \end{bmatrix}, \; J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is worth mentioning that $A_{\alpha,\beta} - J_1 = o(1)$ and $A_\beta - J_2 = o(1)$ as $\alpha, \beta \ll 1$. Writing $y = \frac{x-\xi}{\lambda} = \rho e^{i\theta}$, we have

$$\begin{aligned} \mathscr{E}_0(x,t) =& -Q_{\omega,\alpha,\beta}\left[\dot\lambda\lambda^{-1}\rho w_\rho(\rho) E_1(y) + \dot\omega\rho w_\rho(\rho) E_2(y)\right], \\ \mathscr{E}_1(x,t) =& -\dot\xi_1\lambda^{-1} w_\rho(\rho) Q_{\omega,\alpha,\beta}\left[\cos\theta E_1(y) + \sin\theta E_2(y)\right] \\ & -\dot\xi_2\lambda^{-1} w_\rho(\rho) Q_{\omega,\alpha,\beta}\left[\sin\theta E_1(y) - \cos\theta E_2(y)\right]. \end{aligned}$$

Notice that the slow decaying part of the error $\mathscr{S}[U]$ consists of

$$\begin{aligned} \mathscr{E}_0(x,t) =& -\frac{2r}{r^2+\lambda^2}\left(\dot\lambda Q_{\omega,\alpha,\beta} E_1 + \lambda\dot\omega Q_{\omega,\alpha,\beta} E_2\right) \\ \approx& -\frac{2r}{r^2+\lambda^2}\begin{bmatrix} (\dot\lambda + i\lambda\dot\omega) e^{i(\theta+\omega)} \\ 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}&\mathscr{E}_{-1}(x,t)\\ &= Q_{\omega,\alpha,\beta}\left[\frac{\dot\alpha}{2}[Z_{-1,2}(y)+Z_{1,2}(y)]+\dot\alpha(A_\beta-J_2)W-\frac{\dot\beta}{2}[Z_{-1,1}(y)+Z_{1,1}(y)]\right]\\ &:= \mathscr{E}_{-1,2}+\mathscr{E}_{-1,1},\end{aligned}$$

where

$$\mathscr{E}_{-1,2}=Q_{\omega,\alpha,\beta}\frac{\dot\alpha}{1+\rho^2}\begin{bmatrix}-2\rho\sin\beta\sin\theta\\ 2\rho\sin\beta\cos\theta-(\rho^2-1)\cos\beta\\ 2\rho\cos\beta\sin\theta\end{bmatrix}$$

and

$$\mathscr{E}_{-1,1}=Q_{\omega,\alpha,\beta}\frac{\dot\beta}{1+\rho^2}\begin{bmatrix}\rho^2-1\\ 0\\ -2\rho\cos\theta\end{bmatrix}.$$

In the sequel, we write

$$p(t)=\lambda(t)e^{i\omega(t)}.$$

Then

$$-\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda+i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix}=-\frac{2r}{r^2+\lambda^2}\begin{bmatrix}\dot p(t)e^{i\theta}\\ 0\end{bmatrix}:=\tilde{\mathscr{E}}_0(x,t).$$

To reduce the size of $\mathscr{S}[U]$, we add corrections

$$\Phi^0[p,\xi]:=\begin{bmatrix}\varphi^0(r,t)e^{i\theta}\\ 0\end{bmatrix},\ \Phi^\alpha=Q_{\omega,\alpha,\beta}\begin{bmatrix}0\\ \alpha(t)\\ 0\end{bmatrix},\ \Phi^\beta=Q_{\omega,\alpha,\beta}\begin{bmatrix}-\beta(t)\\ 0\\ 1\end{bmatrix},\tag{2.17}$$

where

$$\varphi^0(r,t)=-\int_{-T}^t r\dot p(s)k(z(r),t-s)ds$$

with

$$z(r)=\sqrt{r^2+\lambda^2},\ k(z,t)=2\frac{1-e^{-\frac{z^2}{4t}}}{z^2}.$$

By direct computations, the new error produced by Φ^0 is

$$\Phi^0_t-\Delta_x\Phi^0+\tilde{\mathscr{E}}_0=\tilde{\mathscr{R}}_0+\tilde{\mathscr{R}}_1,\quad \tilde{\mathscr{R}}_0=\begin{bmatrix}\mathscr{R}_0\\ 0\end{bmatrix},\quad \tilde{\mathscr{R}}_1=\begin{bmatrix}\mathscr{R}_1\\ 0\end{bmatrix}$$

where

$$\mathscr{R}_0:=-re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t\dot p(s)(zk_z-z^2k_{zz})(z(r),t-s)\,ds$$

and

$$\mathscr{R}_1 := -e^{i\theta}\mathrm{Re}\,(e^{-i\theta}\dot{\xi}(t))\int_{-T}^{t}\dot{p}(s)\,k(z(r),t-s)\,ds + \frac{r}{z^2}e^{i\theta}\,(\lambda\dot{\lambda}(t)-\mathrm{Re}\,(re^{i\theta}\dot{\xi}(t)))\int_{-T}^{t}\dot{p}(s)\;zk_z(z(r),t-s)\,ds.$$

Observe that $\mathscr{R}_1$ is of smaller order. Moreover, we can evaluate

$$\begin{aligned}&\tilde{L}_U[\Phi^0]+\Pi_{U^\perp}[-U_t+\Delta\Phi^0-\Phi^0_t]\\ &=\tilde{L}_U[\Phi^0]-\mathscr{E}_1+\Pi_{U^\perp}[\tilde{\mathscr{E}}_0]-\mathscr{E}_0-\Pi_{U^\perp}[\tilde{\mathscr{R}}_0]-\Pi_{U^\perp}[\tilde{\mathscr{R}}_1]-\mathscr{E}_{-1}\\ &=\mathscr{K}_0[p,\xi]+\mathscr{K}_1[p,\xi]-\Pi_{U^\perp}[\tilde{\mathscr{R}}_1]-\mathscr{E}_{-1}\end{aligned}$$

where

$$\mathscr{K}_0[p,\xi]=\mathscr{K}_{01}[p,\xi]+\mathscr{K}_{02}[p,\xi]$$

with

(2.18)

$$\mathscr{K}_{01}[p,\xi]:=-\frac{2}{\lambda}\rho w_\rho^2\int_{-T}^{t}\Big[\mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)})Q_{\omega,\alpha,\beta}E_1+\mathrm{Im}\,(\dot{p}(s)e^{-i\omega(t)})Q_{\omega,\alpha,\beta}E_2\Big]\cdot k(z,t-s)\,ds$$

$$\begin{aligned}&\mathscr{K}_{02}[p,\xi]\\ &:=\frac{1}{\lambda}\rho w_\rho^2\left[\dot{\lambda}-\int_{-T}^{t}\mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)})rk_z(z,t-s)z_r\,ds\right]Q_{\omega,\alpha,\beta}E_1\\ &\quad-\frac{1}{4\lambda}\rho w_\rho^2\cos w\left[\int_{-T}^{t}\mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)})\,(zk_z-z^2k_{zz})(z,t-s)\,ds\right]Q_{\omega,\alpha,\beta}E_1\end{aligned}$$

(2.19)

$$-\frac{1}{4\lambda}\rho w_\rho^2\left[\int_{-T}^{t}\mathrm{Im}\,(\dot{p}(s)e^{-i\omega(t)})\,(zk_z-z^2k_{zz})(z,t-s)\,ds\right]Q_{\omega,\alpha,\beta}E_2,$$

(2.20)

$$\mathscr{K}_1[p,\xi]:=\frac{1}{\lambda}w_\rho\left[\mathrm{Re}\left((\dot{\xi}_1-i\dot{\xi}_2)e^{i\theta}\right)Q_{\omega,\alpha,\beta}E_1+\mathrm{Im}\left((\dot{\xi}_1-i\dot{\xi}_2)e^{i\theta}\right)Q_{\omega,\alpha,\beta}E_2\right].$$

Next we consider the new error estimates produced by Φ^α and Φ^β. It is obvious that $\tilde{L}_U[\Phi^\alpha]=0$ and $\tilde{L}_U[\Phi^\beta]=0$. Direct computations show that

$$Q^{-1}_{\omega,\alpha,\beta}\left(\frac{d}{dt}Q_{\omega,\alpha,\beta}\right)\begin{bmatrix}0\\ \alpha\\ 0\end{bmatrix}=\begin{bmatrix}-\dot{\omega}\alpha\cos\alpha\cos\beta-\alpha\dot{\alpha}\sin\beta\\ 0\\ \dot{\alpha}\alpha\cos\beta-\dot{\omega}\alpha\cos\alpha\sin\beta\end{bmatrix},$$

$$
\begin{aligned}
&Q^{-1}_{\omega,\alpha,\beta}\left(\frac{d}{dt}Q_{\omega,\alpha,\beta}\right)\begin{bmatrix}-\beta\\0\\1\end{bmatrix}\\
&=\begin{bmatrix}\dot\omega\sin\alpha+\dot\beta\\ \dot\omega(\cos\alpha\sin\beta-\beta\cos\alpha\cos\beta)-\dot\alpha(\beta\sin\beta+\cos\beta)\\ \dot\omega\beta\sin\alpha+\dot\beta\beta\end{bmatrix},
\end{aligned}
$$

and thus

(2.21)

$$
\begin{aligned}
-\partial_t\Phi^\alpha+\Delta\Phi^\alpha-\mathscr{E}_{-1,2}&=Q_{\omega,\alpha,\beta}\begin{bmatrix}\dot\omega\alpha\cos\alpha\cos\beta+\dot\alpha\sin\beta\left(\alpha+\frac{2\rho}{1+\rho^2}\sin\theta\right)\\ -\dot\alpha\left(1-\frac{\rho^2-1}{1+\rho^2}\cos\beta+\frac{2\rho}{1+\rho^2}\sin\beta\cos\theta\right)\\ \dot\omega\alpha\cos\alpha\sin\beta-\dot\alpha\cos\beta\left(\alpha+\frac{2\rho}{1+\rho^2}\sin\theta\right)\end{bmatrix}\\
&:=\mathscr{R}_{-1,2}[\alpha,\beta]
\end{aligned}
$$

and

$$
\begin{aligned}
&-\partial_t\Phi^\beta+\Delta\Phi^\beta-\mathscr{E}_{-1,1}\\
&=Q_{\omega,\alpha,\beta}\begin{bmatrix}\frac{2}{1+\rho^2}\dot\beta-\dot\omega\sin\alpha-\dot\beta\\ -\dot\omega(\cos\alpha\sin\beta-\beta\cos\alpha\cos\beta)+\dot\alpha(\beta\sin\beta+\cos\beta)\\ -\dot\omega\beta\sin\alpha-\dot\beta\left(\beta-\frac{2\rho}{1+\rho^2}\cos\theta\right)\end{bmatrix}\\
&:=\mathscr{R}_{-1,1}[\alpha,\beta].
\end{aligned}
$$

Consequently, we obtain

$$
-\partial_t(\Phi^\alpha+\Phi^\beta)+\Delta(\Phi^\alpha+\Phi^\beta)-\mathscr{E}_{-1}=\mathscr{R}_{-1}[\alpha,\beta],
$$

where

$$
\mathscr{R}_{-1}[\alpha,\beta]:=\mathscr{R}_{-1,1}[\alpha,\beta]+\mathscr{R}_{-1,2}[\alpha,\beta]. \tag{2.22}
$$

2.3 Inner–outer gluing system

Collecting the error estimates in the previous section, we will get a solution solving (2.15) if the pair (ϕ,Ψ^*) solves the *inner–outer gluing system*

(2.23)

$$
\begin{cases}
\lambda^2\partial_t\phi=L_W[\phi]+\lambda^2Q^{-1}_{\omega,\alpha,\beta}\left[\tilde L_U[\Psi^*]+\mathscr{K}_0[p,\xi]+\mathscr{K}_1[p,\xi]+\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\right],\\
\quad\text{in }\ \mathscr{D}_{2R}\\
\phi(\cdot,0)=0,\ \text{ in }\ B_{2R(0)}\\
\phi\cdot W=0,\ \text{ in }\ \mathscr{D}_{2R}
\end{cases}
$$

$$
\partial_t\Psi^*=\Delta_x\Psi^*+\mathscr{G}[p,\xi,\Psi^*,\alpha,\beta,\phi]\ \text{ in }\ \Omega\times(0,T), \tag{2.24}
$$

where

$$
\begin{aligned}
&\mathscr{G}[p,\xi,\Psi^*,\alpha,\beta,\phi]\\
&:= (1-\eta_R)\tilde{L}_U[\Psi^*]+(\Psi^*\cdot U)U_t+\mathcal{Q}_{\omega,\alpha,\beta}(\phi\Delta_x\eta_R+2\nabla_x\eta_R\cdot\nabla_x\phi-\phi\partial_t\eta_R)\\
&+\eta_R\mathcal{Q}_{\omega,\alpha,\beta}(-(\mathcal{Q}^{-1}_{\omega,\alpha,\beta}\frac{d}{dt}\mathcal{Q}_{\omega,\alpha,\beta})\phi+\lambda^{-1}\dot{\lambda}y\cdot\nabla_y\phi+\lambda^{-1}\dot{\xi}\cdot\nabla_y\phi)\\
&+(1-\eta_R)\left(\mathscr{K}_0[p,\xi]+\mathscr{K}_1[p,\xi]+\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\right)-\Pi_{U^\perp}[\tilde{\mathscr{R}}_1]\\
&+N_U[\eta_R\mathcal{Q}_{\omega,\alpha,\beta}\phi+\Pi_{U^\perp}(\Phi^0+\Phi^\alpha+\Phi^\beta+\Psi^*)]\\
&+\Big((\Phi^0+\Phi^\alpha+\Phi^\beta)\cdot U\Big)U_t,
\end{aligned}
$$

the linearization $L_W[\phi]$ is defined in (2.3), and

$$\mathscr{D}_{2R}:=\big\{(y,t):y\in B_{2R(t)},t\in(0,T)\big\}$$

with the radius

$$R=R(t)=\lambda_*(t)^{-\gamma_*},\ \text{ with }\ \lambda_*(t)=\frac{|\log T|(T-t)}{|\log(T-t)|^2}\ \text{ and }\ \gamma_*\in(0,1/2). \tag{2.25}$$

The reason for choosing such $R(t)$ and $\lambda_*(t)$ will be made clear later on. If the pair (ϕ,Ψ^*) solves the inner–outer gluing system (2.23)–(2.24), then we get a desired solution

$$
\begin{aligned}
u(x,t)=U&+\Pi_{U^\perp}[\eta_R\mathcal{Q}_{\omega,\alpha,\beta}\phi+\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta]\\
&+a(\Pi_{U^\perp}[\eta_R\mathcal{Q}_{\omega,\alpha,\beta}\phi+\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta])U
\end{aligned}
$$

which solves problem (2.9). We take the boundary condition $u\big|_{\partial\Omega}=\mathbf{e_3}:=\begin{bmatrix}0\\0\\1\end{bmatrix}$, which amounts to

$$
\begin{aligned}
&\Pi_{U^\perp}[\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta]+a(\Pi_{U^\perp}[U+\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta])U\\
&=\mathbf{e_3}-U\ \text{ on }\ \partial\Omega\times(0,T).
\end{aligned}
$$

So it suffices to take the boundary condition for the outer problem (2.24) as

$$\Psi^*\big|_{\partial\Omega}=\mathbf{e_3}-U-\Phi^0-\Phi^\alpha-\Phi^\beta.$$

2.4 Reduced equations for parameter functions

In this section, we will derive the parameter functions $\lambda(t)$, $\xi(t)$, $\omega(t)$, $\alpha(t)$ and $\beta(t)$ at leading order as $t\to T$.

The inner problem (2.23) has the form

$$
\begin{cases}
\lambda^2\phi_t=L_W[\phi]+h[p,\xi,\alpha,\beta,\Psi^*](y,t) & \text{in } \mathscr{D}_{2R},\\
\phi\cdot W=0 & \text{in } \mathscr{D}_{2R},\\
\phi(\cdot,0)=0 & \text{in } B_{2R(0)}.
\end{cases} \tag{2.26}
$$

Here we recall that we write $p(t) = \lambda(t)e^{i\omega(t)}$. For convenience, we assume that $h(y,t)$ is defined for all $y \in \mathbb{R}^2$ extending outside $\mathscr{D}_{2R}$ as

$$\begin{aligned} &h[p,\xi,\alpha,\beta,\Psi^*] \\ &= \lambda^2 Q^{-1}_{\omega,\alpha,\beta}\chi_{\mathscr{D}_{2R}}\left[\tilde{L}_U[\Psi^*] + \mathscr{K}_0[p,\xi] + \mathscr{K}_1[p,\xi] + \Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\right], \end{aligned}$$

where χ_A denotes the characteristic function of a set A, $\mathscr{K}_0$ is defined in (2.18), (2.19), $\mathscr{K}_1$ in (2.20) and $\mathscr{R}_{-1}$ in (2.22). If $\lambda(t)$ has a relatively smooth vanishing as $t \to T$, it is then natural that the term $\lambda^2\phi_t$ is of smaller order and the equation (2.26) is approximated by the elliptic problem

$$L_W[\phi] + h[p,\xi,\alpha,\beta,\Psi^*] = 0, \quad \phi \cdot W = 0 \quad \text{in } B_{2R}. \tag{2.27}$$

We consider the kernel functions $Z_{l,j}(y)$ defined in (2.4), which satisfy $L_W[Z_{l,j}] = 0$ for $l = 0, \pm 1$, $j = 1,2$. If there is a solution $\phi(y,t)$ to (2.27) with sufficient decay, then necessarily

$$\int_{B_{2R}} h[p,\xi,\alpha,\beta,\Psi^*](y,t) \cdot Z_{l,j}(y)\,dy = 0 \quad \text{for all } t \in (0,T), \tag{2.28}$$

for $l = 0, \pm 1$, $j = 1,2$. These orthogonality conditions (2.28) amount to an integro-differential system of equations for $p(t)$, $\xi(t)$, $\alpha(t)$, $\beta(t)$, which, as a matter of fact, *determine* the correct values of the parameter functions so that the solution pair (ϕ, Ψ^*) with appropriate asymptotics exists.

For the reduced equations of $p(t)$ and $\xi(t)$ which correspond to mode $l = 0$ and mode $l = 1$, respectively, we invoke some useful expressions and results in [15, Section 5]. Let

$$\mathscr{B}_{0j}[p](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q^{-1}_{\omega,\alpha,\beta}[\mathscr{K}_0[p,\xi] + \mathscr{K}_1[p,\xi] + \Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]] \cdot Z_{0,j}(y)\,dy,$$

for $j = 1,2$. From (2.22), (2.22) and (2.21), direct computations yield

$$\begin{aligned} &\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]] \cdot Z_{0,1}(y)\,dy \\ &= \pi\left(-\frac{16R^2}{4R^2+1} + 4\log(4R^2+1)\right) \\ &\quad \cdot (\dot{\omega}\alpha\cos\alpha\sin\beta - \alpha\dot{\alpha}\cos\beta - \dot{\omega}\beta\sin\alpha - \beta\dot{\beta}), \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]] \cdot Z_{0,2}(y)\,dy \\ &= \pi\left(-\frac{16R^2}{4R^2+1} + 4\log(4R^2+1)\right)\dot{\alpha}\sin\beta. \end{aligned} \tag{2.30}$$

Combining (2.18), (2.19) with (2.29) and (2.30), the following expressions for $\mathscr{B}_{01}$, $\mathscr{B}_{02}$ are readily obtained by similar computations as in [15, Section 5]

$$\mathscr{B}_{01}[p](t) = \int_{-T}^{t} \mathrm{Re}\,(\dot{p}(s)e^{-i\omega(t)})\,\Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right)\frac{ds}{t-s} - 2\dot{\lambda}(t) + o(1)$$
$$\mathscr{B}_{02}[p](t) = \int_{-T}^{t} \mathrm{Im}\,(\dot{p}(s)e^{-i\omega(t)})\,\Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right)\frac{ds}{t-s},$$

where $o(1) \to 0$ as $t \to T$, and $\Gamma_j(\tau)$ are smooth functions defined as

$$\Gamma_1(\tau) = -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta) + 2\zeta K_\zeta(\zeta)\frac{\rho^2}{1+\rho^2} - 4\cos(w)\zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho,$$
$$\Gamma_2(\tau) = -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta) - \zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho\,,$$

where

$$K(\zeta) = 2\frac{1-e^{-\frac{\zeta}{4}}}{\zeta}.$$

Using the expressions of $\Gamma_j(\tau)$, we get

$$\begin{cases} |\Gamma_j(\tau) - 1| \le C\tau(1+|\log\tau|) & \text{for } \tau < 1, \\ |\Gamma_j(\tau)| \le \dfrac{C}{\tau} & \text{for } \tau > 1. \end{cases}$$

Define

$$\mathscr{B}_0[p] := \frac{1}{2}e^{i\omega(t)}\left(\mathscr{B}_{01}[p] + i\mathscr{B}_{02}[p]\right) \tag{2.31}$$

and

$$a_{0j}[p,\xi,\alpha,\beta,\Psi^*] := -\frac{\lambda}{2\pi}\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\tilde{L}_U[\Psi^*]\cdot Z_{0,j}(y)\,dy,\ j=1,2,$$
$$a_0[p,\xi,\alpha,\beta,\Psi^*] := \frac{1}{2}e^{i\omega(t)}\left(a_{01}[p,\xi,\alpha,\beta,\Psi^*] + ia_{02}[p,\xi,\alpha,\beta,\Psi^*]\right).$$

Similarly, we let

$$\mathscr{B}_{1j}[\xi](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q^{-1}_{\omega,\alpha,\beta}[\mathscr{K}_0[p,\xi] + \mathscr{K}_1[p,\xi] + \mathscr{R}_{-1}[\alpha,\beta]]\cdot Z_{1,j}(y)\,dy,$$

for $j = 1,2$, and

$$\mathscr{B}_1[\xi](t) := \mathscr{B}_{11}[\xi](t) + i\mathscr{B}_{12}[\xi](t).$$

Directly using the expressions (2.22), (2.22) and (2.21), we have

$$\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\cdot Z_{1,1}(y)\,dy$$
$$= \frac{8\pi R^2}{4R^2+1}(\dot{\omega}\alpha\cos\alpha\cos\beta + \dot{\alpha}\alpha\sin\beta - \dot{\omega}\sin\alpha + \dot{\beta}),$$

and

$$\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\cdot Z_{1,2}(y)\,dy$$
$$= -\frac{8\pi R^2}{4R^2+1}(\dot\alpha - \dot\omega\beta\cos\alpha\cos\beta - \dot\alpha\beta\sin\beta + \dot\omega\cos\alpha\sin\beta).$$

Therefore, by (2.20), (2.4) and the fact that $\int_0^\infty \rho w_\rho^2 d\rho = 2$, we obtain

$$\mathscr{B}_1[\xi](t) = 2[\dot\xi_1(t) + i\dot\xi_2(t) + o(1)] \text{ as } t\to T.$$

At last, we let

$$a_{1j}[p,\xi,\alpha,\beta,\Psi^*] := \frac{\lambda}{2\pi}\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\tilde L_U[\Psi^*]\cdot Z_{1,j}(y)\,dy,\ j=1,2,$$
$$a_1[p,\xi,\alpha,\beta,\Psi^*] := -e^{i\omega(t)}(a_{11}[p,\xi,\alpha,\beta,\Psi^*] + ia_{12}[p,\xi,\alpha,\beta,\Psi^*]).$$

We thus obtain that the four conditions (2.28) for $l=0,1$ are reduced to the system of two complex equations

$$\mathscr{B}_0[p] = a_0[p,\xi,\alpha,\beta,\Psi^*], \tag{2.32}$$
$$\mathscr{B}_1[\xi] = a_1[p,\xi,\alpha,\beta,\Psi^*]. \tag{2.33}$$

We observe that

$$\mathscr{B}_0[p] = \int_{-T}^{t-\lambda^2}\frac{\dot p(s)}{t-s}ds + O\big(\|\dot p\|_\infty\big) + o(1) \text{ as } t\to T.$$

To get an approximation for a_0, we need to analyze the operator $\tilde L_U$ in a_0. To this end, we write

$$\Psi^* = \begin{bmatrix}\psi^*\\ \psi_3^*\end{bmatrix},\quad \psi^* = \psi_1^* + i\psi_2^*.$$

From (2.7) and (2.8), we have

$$\tilde L_U[\Psi^*](y,t) = [\tilde L_U]_0[\Psi^*] + [\tilde L_U]_1[\Psi^*] + [\tilde L_U]_2[\Psi^*],$$

where

$$\begin{cases}
[\tilde L_U]_0[\Psi^*] = \lambda^{-1}Q_{\omega,\alpha,\beta}\rho w_\rho^2[\mathrm{div}(e^{-i\omega}\psi^*)E_1 + \mathrm{curl}(e^{-i\omega}\psi^*)E_2],\\
[\tilde L_U]_1[\Psi^*] = -2\lambda^{-1}Q_{\omega,\alpha,\beta}w_\rho\cos w[(\partial_{x_1}\psi_3^*)\cos\theta + (\partial_{x_2}\psi_3^*)\sin\theta]E_1\\
\qquad\qquad -2\lambda^{-1}Q_{\omega,\alpha,\beta}w_\rho\cos w[(\partial_{x_1}\psi_3^*)\sin\theta - (\partial_{x_2}\psi_3^*)\cos\theta]E_2,\\
[\tilde L_U]_2[\Psi^*] = \lambda^{-1}Q_{\omega,\alpha,\beta}\rho w_\rho^2[\mathrm{div}(e^{i\omega}\bar\psi^*)\cos2\theta - \mathrm{curl}(e^{i\omega}\bar\psi^*)\sin2\theta]E_1\\
\qquad\qquad +\lambda^{-1}Q_{\omega,\alpha,\beta}\rho w_\rho^2[\mathrm{div}(e^{i\omega}\bar\psi^*)\sin2\theta + \mathrm{curl}(e^{i\omega}\bar\psi^*)\cos2\theta]E_2,
\end{cases}$$

and the differential operators in Ψ^* on the right hand sides are evaluated at (x,t) with $x=\xi(t)+\lambda(t)y$, $y=\rho e^{i\theta}$ while $E_j = E_j(y)$ for $j=1,2$. From the above decomposition, assuming that Ψ^* is of class C^1 in the space variable, we then get

$$a_0[p,\xi,\alpha,\beta,\Psi^*] = [\mathrm{div}\,\psi^* + i\,\mathrm{curl}\,\psi^*](\xi,t) + o(1) \text{ as } t\to T.$$

Similarly, since $\int_0^\infty w_\rho^2 \cos w\rho \, d\rho = 0$, we get

$$\begin{aligned} a_1[p,\xi,\alpha,\beta,\Psi^*] &= 2(\partial_{x_1}\psi_3^* + i\partial_{x_2}\psi_3^*)(\xi,t)\int_0^\infty \cos w\, w_\rho^2 \rho \, d\rho + o(1) \\ &= o(1) \ \text{ as } \ t \to T. \end{aligned}$$

We now simplify the system (2.32)–(2.33) in the form

$$\int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s} ds = [\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](\xi(t),t) + o(1) + O(\|\dot p\|_\infty)$$

$$\dot\xi(t) = o(1) \ \text{ as } \ t \to T. \tag{2.34}$$

For the moment, we assume that the function $\Psi^*(x,t)$ is fixed and sufficiently regular, and we regard T as a parameter that will always be taken smaller if necessary. We recall that we need $\xi(T) = q$ where $q \in \Omega$ is given, and $\lambda(T) = 0$. Equation (2.34) immediately suggests us to take $\xi(t) \equiv q$ as the first approximation. Neglecting lower order terms, $p(t) = \lambda(t)e^{i\omega(t)}$ satisfies the following integro-differential system

$$\int_{-T}^{t-\lambda^2(t)} \frac{\dot p(s)}{t-s} ds = \operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0) =: a_0^*. \tag{2.35}$$

At this point, we make the following assumption

$$\operatorname{div}\psi^*(q,0) < 0, \tag{2.36}$$

which implies that $a_0^* = -|a_0^*|e^{i\omega_0}$ for a unique $\omega_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let us take

$$\omega(t) \equiv \omega_0.$$

Equation (2.35) then becomes

$$\int_{-T}^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s} ds = -|a_0^*|. \tag{2.37}$$

We claim that a good approximate solution to (2.37) as $t \to T$ is given by

$$\dot\lambda(t) = -\frac{\kappa}{\log^2(T-t)}$$

for a suitable $\kappa > 0$. Indeed, we have

$$\begin{aligned} \int_{-T}^{t-\lambda^2(t)} \frac{\dot\lambda(s)}{t-s} ds &= \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s} ds + \dot\lambda(t)\left[\log(T-t) - 2\log(\lambda(t))\right] \\ &\quad + \int_{t-(T-t)}^{t-\lambda^2(t)} \frac{\dot\lambda(s) - \dot\lambda(t)}{t-s} ds \\ &\approx \int_{-T}^{t} \frac{\dot\lambda(s)}{T-s} ds - \dot\lambda(t)\log(T-t) := \Upsilon(t) \end{aligned}$$

as $t \to T$. We see that

$$\log(T-t)\frac{d\Upsilon(t)}{dt} = \frac{d}{dt}(\log^2(T-t)\,\dot{\lambda}(t)) = 0$$

from the explicit form of $\dot{\lambda}(t)$. Thus $\Upsilon(t)$ is a constant. As a consequence, equation (2.37) is approximately satisfied if κ is such that

$$\kappa \int_{-T}^{T} \frac{\dot{\lambda}(s)}{T-s}\,ds \;=\; -|a_0^*|,$$

which finally gives us the approximate expression

$$\dot{\lambda}(t) = -|\operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0)|\,\dot{\lambda}_*(t),$$

where

$$\dot{\lambda}_*(t) = -\frac{|\log T|}{\log^2(T-t)}.$$

Naturally, imposing $\lambda_*(T) = 0$, we then have

$$\lambda_*(t) = \frac{|\log T|}{\log^2(T-t)}(T-t)\,(1+o(1)) \;\text{ as }\; t \to T. \tag{2.38}$$

Next, we consider (2.28) for the case of mode $l = -1$, which gives the reduced equations of $\alpha(t)$ and $\beta(t)$. By (2.22), (2.22) and (2.21), we evaluate

$$\begin{aligned}
&\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\cdot Z_{-1,1}(y)\,dy \\
&= 4\pi\left(-\frac{4R^2(2R^2+1)}{4R^2+1} + \log(4R^2+1)\right) \\
&\quad\cdot(-\dot{\beta} - \dot{\omega}\sin\alpha + \dot{\omega}\alpha\cos\alpha\cos\beta + \dot{\alpha}\alpha\sin\beta) \\
&= 8\pi\left[(R^2 - \log R)\dot{\beta}(1+o(1))\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_{2R}} Q^{-1}_{\omega,\alpha,\beta}\Pi_{U^\perp}[\mathscr{R}_{-1}[\alpha,\beta]]\cdot Z_{-1,2}(y)\,dy \\
&= 4\pi\left(\frac{4R^2(2R^2+1)}{4R^2+1} - \log(4R^2+1)\right) \\
&\quad\cdot(\dot{\alpha}(1-\beta\sin\beta - 2\cos\beta) + \dot{\omega}\cos\alpha(\sin\beta - \beta\cos\beta)) \\
&= 8\pi\left[(-R^2 + \log R)\dot{\alpha}(1+o(1))\right],
\end{aligned}$$

where we recall that $\omega(t) \equiv \omega_0$. Since

$$\int_{B_{2R}} \lambda^2 Q^{-1}_{\omega,\alpha,\beta}\left[\tilde{L}_U[\Psi^*] + \mathscr{K}_0 + \mathscr{K}_1\right]\cdot Z_{-1,j}(y)\,dy = c_j\lambda$$

for some $c_j \in \mathbb{R}$, for $j = 1,2$, the orthogonality condition (2.28) with $l = -1$ gives

$$8\pi\lambda^2(-R^2+\log R)\,\dot\beta(1+o(1)) = c_1\lambda,$$
$$8\pi\lambda^2(R^2-\log R)\,\dot\alpha(1+o(1)) = c_2\lambda.$$

Thus, by (2.38) and the definition of $R = R(t)$ in (2.25), good choices for $\alpha(t)$ and $\beta(t)$ at leading orders are

$$\alpha(t) = c_\alpha(T-t)^{\delta_1}(1+o(1)),\ \beta(t) = c_\beta(T-t)^{\delta_2}(1+o(1)),\ \text{ as } t\to T$$

for some $\delta_1,\ \delta_2 > 0$ and $c_\alpha,\ c_\beta \in \mathbb{R}$.

2.5 Linear theory for the inner problem

To capture the heart of the singularity formation, a linear theory of the inner problem (2.23) is required. In contrast with that by [15], it turns out that we will have to establish a decay estimate of second order derivative of ϕ in order to handle the coupling effects between the inner–outer problem of u and that of v below. We consider

$$\begin{cases} \lambda^2\partial_t\phi = L_W[\phi] + h(y,t), & \text{in } \mathscr{D}_{2R}, \\ \phi(\cdot,0) = 0, & \text{in } B_{2R(0)}, \\ \phi\cdot W = 0, & \text{in } \mathscr{D}_{2R}, \end{cases} \tag{2.39}$$

where we recall from (2.25) that

$$R = R(t) = \lambda_*(t)^{-\gamma_*},\ \text{ with }\ \lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}\ \text{ and }\ \gamma_* \in (0,1/2).$$

We regard $h(y,t)$ as a function defined in $\mathbb{R}^2\times(0,T)$ with compact support, and assume that $h(y,t)$ has the space-time decay of the following type

$$|h(y,t)| \lesssim \frac{\lambda_*^\nu(t)}{1+|y|^a},\quad h\cdot W = 0,$$

where $\nu > 0$ and $a \in (2,3)$. Define the norm

$$\|h\|_{\nu,a} := \sup_{(y,t)\in\mathbb{R}^2\times(0,T)} \lambda_*^{-\nu}(t)(1+|y|^a)|h(y,t)|.$$

In the polar coordinates, $h(y,t)$ can be written as

$$h(y,t) = h^1(\rho,\theta,t)E_1(y) + h^2(\rho,\theta,t)E_2(y),\ y = \rho e^{i\theta}$$

since $h\cdot W = 0$. Expanding in the Fourier series, we write

$$\tilde h(\rho,\theta,t) := h^1 + ih^2 = \sum_{k=-\infty}^{\infty}\tilde h_k(\rho,t)e^{ik\theta},\quad \tilde h_k = \tilde h_{k1} + i\tilde h_{k2} \tag{2.40}$$

such that

$$h(y,t) = \sum_{k=-\infty}^{\infty} h_k(y,t) := h_0(y,t) + h_1(y,t) + h_{-1}(y,t) + h_\perp(y,t) \tag{2.41}$$

with

$$h_k(y,t) = \mathrm{Re}(\tilde{h}_k(\rho,t)e^{ik\theta})E_1 + \mathrm{Im}(\tilde{h}_k(\rho,t)e^{ik\theta})E_2,\ k\in\mathbb{Z}. \tag{2.42}$$

We consider the kernel functions $Z_{k,j}$ defined in (2.4), and define

$$\bar{h}_k(y,t) := \sum_{j=1}^{2} \frac{\chi Z_{k,j}(y)}{\int_{\mathbb{R}^2}\chi|Z_{k,j}|^2}\int_{\mathbb{R}^2} h(z,t)\cdot Z_{k,j}(z)\,dz,\ k=0,\pm1,\ j=1,2, \tag{2.43}$$

where

$$\chi(y,t) = \begin{cases} w_\rho^2(|y|) & \text{if } |y|<2R(t),\\ 0 & \text{if } |y|\geq 2R(t).\end{cases}$$

The main result of this section is stated as follows.

Proposition 2.1. Assume that $a\in(2,3)$, $\nu>0$, $\delta\in(0,1)$ and $\|h\|_{\nu,a}<+\infty$. Let us write

$$h = h_0 + h_1 + h_{-1} + h_\perp \ \text{ with } \ h_\perp = \sum_{k\neq 0,\pm1} h_k.$$

Then there exists a solution $\phi[h]$ of problem (2.39), which defines a linear operator of h, and satisfies the following estimate in $\mathscr{D}_{2R}$

$$\begin{aligned}
&|\phi(y,t)| + (1+|y|)\,|\nabla_y\phi(y,t)| + (1+|y|)^2\left|\nabla_y^2\phi(y,t)\right|\\
\lesssim\ & \lambda_*^\nu(t)\min\left\{\frac{R^{\delta(5-a)}(t)}{1+|y|^3}, \frac{1}{1+|y|^{a-2}}\right\}\|h_0-\bar{h}_0\|_{\nu,a} + \frac{\lambda_*^\nu(t)R^2(t)}{1+|y|}\|\bar{h}_0\|_{\nu,a}\\
&+\frac{\lambda_*^\nu(t)}{1+|y|^{a-2}}\left\|h_1-\bar{h}_1\right\|_{\nu,a} + \frac{\lambda_*^\nu(t)R^4(t)}{1+|y|^2}\left\|\bar{h}_1\right\|_{\nu,a}\\
&+\lambda_*^\nu(t)\,\|h_{-1}-\bar{h}_{-1}\|_{\nu,a} + \lambda_*^\nu(t)\log R(t)\,\|\bar{h}_{-1}\|_{\nu,a}\\
&+\frac{\lambda_*^\nu(t)}{1+|y|^{a-2}}\,\|h_\perp\|_{\nu,a}.
\end{aligned}$$

The construction of the solution ϕ to problem (2.39) will be carried out in each Fourier mode. Write

$$\phi = \sum_{k=-\infty}^{\infty}\phi_k, \qquad \phi_k(y,t) = \mathrm{Re}(\varphi_k(\rho,t)e^{ik\theta})E_1 + \mathrm{Im}(\varphi_k(\rho,t)e^{ik\theta})E_2.$$

In each mode k, the pair (ϕ_k,h_k) satisfies

$$\begin{cases}\lambda^2\partial_t\phi_k = L_W[\phi_k] + h_k(y,t), & \text{in } \mathscr{D}_{4R},\\ \phi_k(y,0)=0, & \text{in } B_{4R(0)},\end{cases} \tag{2.44}$$

which is equivalent to the following problem

$$\begin{cases}\lambda^2\partial_t\varphi_k = \mathscr{L}_k[\varphi_k] + \tilde{h}_k(\rho,t), & \text{in } \tilde{\mathscr{D}}_{4R},\\ \varphi_k(\rho,0)=0, & \text{in } (0,4R(0)),\end{cases}$$

where $\tilde{\mathscr{D}}_{4R} = \{(\rho,t):\ t\in(0,T),\ \rho\in(0,4R(t))\}$, and

$$\mathscr{L}_k[\varphi_k] := \partial_{\rho\rho}\varphi_k + \frac{\partial_\rho\varphi_k}{\rho} - (k^2+2k\cos w+\cos(2w))\frac{\varphi_k}{\rho^2}.$$

It is direct to see that the kernel functions for $\mathscr{L}_k$ such that $\mathscr{L}_k[Z_k]=0$ at modes $k=0,\pm1$ are given by

$$Z_0(\rho)=\frac{\rho}{1+\rho^2},\ Z_1(\rho)=\frac{1}{1+\rho^2},\ Z_{-1}(\rho)=\frac{2\rho^2}{1+\rho^2}. \tag{2.45}$$

We have the following lemma proved in [15, Section 7].

Lemma 2.1 ([15]). *Suppose $\nu>0$, $0<a<3$, $a\neq 1,2$ and*

$$\|h_k(y,t)\|_{\nu,a}<+\infty.$$

Then problem (2.44) *has a unique solution which takes the form*

$$\phi_k(y,t)=\mathrm{Re}(\varphi_k(\rho,t)e^{ik\theta})E_1+\mathrm{Im}(\varphi_k(\rho,t)e^{ik\theta})E_2$$

and satisfies the boundary condition

$$\phi_k(y,t)=0,\ y\in\partial B_{4R(t)}(0),\ \forall\, t\in(0,T).$$

Moreover, the following estimates hold

$$|\phi_k(y,t)|\lesssim\lambda_*^\nu k^{-2}\|h\|_{\nu,a}\begin{cases}R^{2-a}, & \text{for } a<2\\ (1+\rho)^{2-a}, & \text{for } a>2\end{cases}\quad \text{for } k\geq 2,$$

$$|\phi_{-1}(y,t)|\lesssim\lambda_*^\nu\|h\|_{\nu,a}\begin{cases}R^{2-a}, & \text{for } a<2\\ \log R, & \text{for } a>2,\end{cases}$$

$$|\phi_0(y,t)|\lesssim\frac{\lambda_*^\nu\|h\|_{\nu,a}}{1+\rho}\begin{cases}R^{3-a}, & \text{for } a<1\\ R^2, & \text{for } a>1,\end{cases}$$

$$|\phi_1(y,t)|\lesssim\frac{\lambda_*^\nu R^4\|h\|_{\nu,a}}{(1+\rho)^2}.$$

The higher regularity estimates for solutions constructed in Lemma 2.1 are given by the following lemma. Before we state the lemma, we first introduce the Hölder semi-norm, which is better expressed in the (y,τ)-variable. Define

$$\tau(t)=\int_0^t\frac{ds}{\lambda^2(s)} \tag{2.46}$$

so that

$$\begin{cases}\partial_\tau\phi = L_W[\phi]+h(y,\tau) & \text{in } \mathscr{D}_{4\gamma R},\\ \phi(\cdot,0)=0 & \text{in } B_{4\gamma R(0)}.\end{cases}$$

We denote the parabolic ball

$$\mathscr{B}_\ell(y,\tau)=\{(y',\tau'):|y-y'|^2+|\tau-\tau'|<\ell^2\},$$

and also introduce the Hölder semi-norm

$$[g]_{\alpha,A} := \sup_{(y,\tau),(y',\tau')\in A} \frac{|g(y,\tau)-g(y',\tau')|}{|y-y'|^\alpha + |\tau-\tau'|^{\alpha/2}}$$

for $\alpha \in (0,1)$ and a set A. We denote $C^{\alpha,\alpha/2}(A)$ by the set of functions on A such that $[g]_{\alpha,A} < +\infty$, endowed with the norm

$$\|g\|_{C^{\alpha,\alpha/2}(A)} = \|g\|_{L^\infty(A)} + [g]_{\alpha,A}.$$

Lemma 2.2. *Let ϕ be a solution to*

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h(y,t), & \text{in } \mathscr{D}_{4\gamma R}, \\ \phi(\cdot,0) = 0, & \text{in } B_{4\gamma R(0)}, \end{cases} \tag{2.47}$$

where $h(y,t) \in C^{\alpha,\alpha/2}(\mathscr{B}_\ell(y,\tau)\cap \mathscr{D}_{4\gamma R})$ for some $\alpha > 0$ and $\ell = \frac{|y|}{4}+1$. If for some $a,b,\gamma,M > 0$ we have

$$\begin{aligned} &|\phi(y,t)| + (1+|y|)^2 |h(y,t)| + (1+|y|)^{2+\alpha}[h(y,t)]_{\alpha,\mathscr{B}_\ell(y,\tau)\cap\mathscr{D}_{4\gamma R}} \\ &\le M\frac{\lambda_*^b(t)}{(1+|y|)^a} \quad \text{in } \mathscr{D}_{4\gamma R}, \end{aligned} \tag{2.48}$$

then there exists a constant C such that

$$(1+|y|)|\nabla_y \phi(y,t)| + (1+|y|)^2 |\nabla_y^2 \phi(y,t)| \le CM\frac{\lambda_*^b(t)}{(1+|y|)^a} \quad \text{in } \mathscr{D}_{3\gamma R}. \tag{2.49}$$

Here

$$\mathscr{D}_{\gamma R} = \{(y,t) : |y| < \gamma R(t),\ t \in (0,T)\}.$$

Moreover, if ϕ satisfies the Dirichlet boundary condition $\phi(\cdot,t) = 0$ on $\partial B_{4\gamma R(t)}$ for all $t \in (0,T)$, then the estimate (2.49) *is valid in the entire region $\mathscr{D}_{4\gamma R}$.*

Proof. In the (y,τ)-variable with τ given by (2.46), problem (2.47) reads as

$$\begin{cases} \partial_\tau \phi = L_W[\phi] + h(y,\tau) & \text{in } \mathscr{D}_{4\gamma R}, \\ \phi(\cdot,0) = 0 & \text{in } B_{4\gamma R(0)}. \end{cases}$$

Let $\tau_1 > 0$ and $y_1 \in B_{3\gamma R(\tau_1)}(0)$. Let $\rho = \frac{|y_1|}{4}+1$ so that $B_\rho(y_1) \subset B_{4\gamma R(\tau_1)(0)}$. We prove (2.49) by the scaling argument. Define

$$\tilde\phi(z,s) = \phi(y_1+\rho z, \tau_1+\rho^2 s),\ z \in B_1(0),\ s > -\frac{\tau_1}{\rho^2}.$$

For the case $\tau_1 < \rho^2$, $\tilde\phi(z,s)$ satisfies the following equation

$$\partial_s \tilde\phi = \Delta_z \tilde\phi + A(z,s)\cdot\nabla_z\tilde\phi + B(z,s)\tilde\phi + \tilde h(z,s) \ \text{ in } \ B_1(0)\times(-1,0],$$

where the coefficients $A(z,s)$ and $B(z,s)$ are uniformly bounded by $O((1+\rho)^{-2})$ in $B_1(0)\times(-1,0]$ and

$$\tilde h(z,s) = \rho^2 h(y_1+\rho z, \tau_1+\rho^2 s).$$

Let $b' > 0$ such that $\tau^{-b'} \sim \lambda_*^b(t)$ from (2.46). By the facts $\rho \le CR(\tau_1)$ and $R^2(\tau_1) \ll \tau_1$ for τ_1 large, we have

$$C_1\tau_1^{-b'} \le (\tau_1+\rho^2 s)^{-b'} \le C_2\tau_1^{-b'}$$

for some positive constants C_1, C_2 independent of τ_1. Then standard interior gradient estimates together with the assumption (2.48) imply

$$\begin{aligned}\|\nabla_z\tilde{\phi}\|_{L^\infty(B_{1/4}(0)\times(1,2))} &\lesssim \|\tilde{\phi}\|_{L^\infty(B_{1/2}(0)\times(0,2))} + \|\tilde{h}\|_{L^\infty(B_{1/2}(0)\times(0,2))}\\ &\lesssim \tau_1^{-b'}\rho^{2-a},\end{aligned}$$

which in particular gives

$$\rho|\nabla_y\phi(y_1,\tau_1)| = |\nabla_z\tilde{\phi}(0,1)| \lesssim \tau_1^{-b'}\rho^{2-a}.$$

On the other hand, from interior parabolic Schauder estimates and (2.48), we have

$$\begin{aligned}\|\nabla_z^2\tilde{\phi}\|_{L^\infty(B_{1/4}(0)\times(1,2))} &\lesssim \|\tilde{\phi}\|_{L^\infty(B_{1/2}(0)\times(0,2))} + \|\tilde{h}\|_{C^{\alpha,\alpha/2}(B_{1/2}(0)\times(0,2))}\\ &\lesssim \tau_1^{-b'}\rho^{2-a},\end{aligned}$$

and in particular

$$\rho^2|\nabla_y^2\phi(y_1,\tau_1)| = |\nabla_z^2\tilde{\phi}(0,1)| \lesssim \tau_1^{-b'}\rho^{2-a}.$$

For the case $\tau_1 \ge \rho^2$ the argument is similar. In this case $\tilde{\phi}$ satisfies the equation in $B_1(0)\times(-\frac{\tau_1}{\rho^2},0]$ and it has initial condition 0 at $s=-\frac{\tau_1}{\rho^2}$. Then similarly by the standard boundary estimate, we get the desired bound. Finally, translating the above bounds into (y,t)-variable, we conclude the validity of (2.49). □

As we can see from Lemma 2.1, the estimates at modes $k=0,\pm1$ are worse than high modes $k\ge 2$. In fact, if certain orthogonality conditions are imposed on $h(y,t)$, better estimates of ϕ can be obtained at modes $k=0,\pm1$. In the sequel, we omit the subscript for each mode if there is no confusion.

Mode $k=0$

We consider

$$\begin{cases}\lambda^2\partial_t\varphi = L_W[\varphi] + h(y,t) + \sum_{j=1,2}\tilde{c}_{0j}Z_{0,j}w_\rho^2 & \text{in } \mathscr{D}_{2R}\\ \varphi\cdot W = 0 & \text{in } \mathscr{D}_{2R}\\ \varphi = 0 & \text{on } \partial B_{2R}\times(0,T)\\ \varphi(\cdot,0) = 0 & \text{in } B_{2R(0)}\end{cases} \tag{2.50}$$

at mode 0. By carrying out another inner–gluing scheme for mode 0, the following Lemma was proved in [15, Proposition 7.2].

Lemma 2.3 ([15]). *Let $\delta \in (0,1)$, $\nu > 0$ and $a \in (2,3)$. Assume $\|h\|_{\nu,a} < +\infty$. Then there exists a solution $(\phi, \tilde{c}_{0j})$ of problem* (2.50) *which defines a linear operator in $h(y,t)$ such that*

$$|\varphi(y,t)| + (1+|y|)|\nabla_y \varphi(y,t)| \lesssim \lambda_*^{\nu}(t)\|h\|_{\nu,a} \begin{cases} \dfrac{R^{\delta(5-a)}}{(1+|y|)^3}, & |y| \le 2R^{\delta} \\ \dfrac{1}{(1+|y|)^{a-2}}, & 2R^{\delta} \le |y| \le 2R \end{cases}$$

and

$$\tilde{c}_{0j} = -\frac{\int_{\mathbb{R}^2} hZ_{0,j}}{\int_{\mathbb{R}^2} w_{\rho}^2 |Z_{0,j}|^2} - G[h],$$

where G is linear in h satisfying

$$|G[h]| \lesssim \lambda_*^{\nu}(t) R^{-\delta\sigma'} \|h\|_{\nu,a}$$

for $\sigma' \in (0, a-2)$.

Mode $k = -1$

We consider problem (2.44) for $k = -1$ and the kernel functions defined in (2.4). We first state a result proved in [15, Lemma 7.5].

Lemma 2.4 ([15]). *Let $a \in (2,3)$, $\nu > 0$ and $k = -1$. If h_{-1} in* (2.44) *satisfies $\|h_{-1}\|_{\nu,a} < \infty$ and*

$$\int_{\mathbb{R}^2} h_{-1}(y,t) Z_{-1,j}(y)\,dy = 0 \ \text{ for } \ j = 1,2, \ \forall\, t \in (0,T),$$

then there exists a solution ϕ_{-1} to problem (2.44) *at mode -1 which defines a linear operator of h_{-1}, and ϕ_{-1} satisfies*

$$|\phi_{-1}(y,t)| \lesssim \lambda_*^{\nu}(t)\|h_{-1}\|_{\nu,a} \min\left\{\log R, \frac{R^{4-a}}{1+|y|^2}\right\}.$$

Since the incompressible Navier–Stokes equation is essentially coupled with the transported harmonic map heat flow through the inner problem, the linear theory required for mode $k = -1$ should be very refined, and Lemma 2.4 cannot be applied to gain contraction when we finally show the existence of desired blow-up solution. Instead, we shall develop a new linear theory at mode -1. The main result for mode -1 is stated as follows.

Lemma 2.5. *Let $a \in (2,3)$, $\nu > 0$ and $k = -1$. If h_{-1} in* (2.44) *satisfies $\|h_{-1}\|_{\nu,a} < \infty$ and*

$$\int_{\mathbb{R}^2} h_{-1}(y,t) Z_{-1,j}(y)\,dy = 0 \ \text{ for } \ j = 1,2, \ \forall\, t \in (0,T),$$

then there exists a solution ϕ_{-1} to problem (2.44) *at mode -1 which defines a linear operator of h_{-1}, and ϕ_{-1} satisfies*

$$|\phi_{-1}(y,t)| \lesssim \lambda_*^{\nu}(t)\|h_{-1}\|_{\nu,a}.$$

Proof. For convenience, we change variable (2.46) and consider

$$\partial_\tau \varphi_{-1} = \mathscr{L}_{-1}[\varphi_{-1}] + \tilde{h}_{-1}.$$

By letting $\varphi_{-1}(\rho,\tau) = Z_{-1}(\rho) f_{-1}(\rho,\tau)$ and using $\mathscr{L}_{-1}[Z_{-1}] = 0$, we obtain

$$\partial_\tau f_{-1} = \frac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2 \nabla f_{-1}) + \frac{\tilde{h}_{-1}}{Z_{-1}}, \tag{2.51}$$

where $Z_{-1}(\rho)$ is defined in (2.45). We first solve

$$\mathrm{div}(Z_{-1}^2 \nabla f_0) = \tilde{h}_{-1} Z_{-1}. \tag{2.52}$$

By the orthogonality condition $\int_{\mathbb{R}^2} h_{-1}(y,t) Z_{-1,j}(y) dy = 0$, we get

$$|\nabla f_0| \lesssim \frac{\tau^{-\nu'}}{1+|y|^{a-1}} \|h_{-1}\|_{\nu,a}, \tag{2.53}$$

where $\nu' > 0$ is the number such that $\lambda_*^\nu \sim \tau^{-\nu'}$ under the change of variable (2.46). Thus, by (2.52), the problem (2.51) becomes

$$\partial_\tau f_{-1} = \frac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2 \nabla f_{-1}) + \frac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2 \nabla f_0).$$

In order to estimate f_{-1}, we need to estimate the fundamental solution S to the problem

$$\begin{cases} \partial_\tau S = \dfrac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2 \nabla S), \\ S\big|_{\tau=0} = \delta_0, \end{cases}$$

where δ_0 is the Dirac delta function at the origin. We consider

$$\begin{cases} \partial_\tau S^\varepsilon = \dfrac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2 \nabla S^\varepsilon), \\ S^\varepsilon\big|_{\tau=0} = \dfrac{1}{2\pi\varepsilon^2} e^{-\frac{|x|^2}{2\varepsilon^2}}. \end{cases}$$

We note that as $\varepsilon \to 0$, $S^\varepsilon\big|_{\tau=0} dx \rightharpoonup \delta_0$. Let $V^\varepsilon = S^\varepsilon_\rho$. Then differentiating the above equation with respect to ρ, we obtain

$$\begin{cases} \partial_\tau V^\varepsilon = \dfrac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2 \nabla V^\varepsilon) + \partial_{\rho\rho}(\log Z_{-1}^2) V^\varepsilon, \\ V^\varepsilon\big|_{\tau=0} = -\dfrac{|x|}{2\pi\varepsilon^4} e^{-\frac{|x|^2}{2\varepsilon^2}}. \end{cases} \tag{2.54}$$

We claim that $V^\varepsilon < 0$. Indeed, we can easily check that $\partial_{\rho\rho}(\log Z_{-1}^2) < 0$. Therefore, by $V^\varepsilon\big|_{\tau=0} = -\frac{|x|}{2\pi\varepsilon^4} e^{-\frac{|x|^2}{2\varepsilon^2}} < 0$ and the maximum principle, we have $V^\varepsilon < 0$. Then we can write

$$\int_0^\infty |S^\varepsilon_\rho(s,\rho)| ds = -\int_0^\infty V^\varepsilon(s,\rho) ds := -M^\varepsilon(\rho).$$

Integrating equation (2.54) over τ from 0 to ∞, we get

$$\frac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2\nabla M^\varepsilon)+\partial_{\rho\rho}(\log Z_{-1}^2)M^\varepsilon=-\frac{|x|}{2\pi\varepsilon^4}e^{-\frac{|x|^2}{2\varepsilon^2}}.$$

Let $M^\varepsilon=\partial_\rho G^\varepsilon$, where G^ε satisfies

$$\frac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2\nabla G^\varepsilon)=\frac{1}{2\pi\varepsilon^2}e^{-\frac{|x|^2}{2\varepsilon^2}}. \tag{2.55}$$

By $Z_{-1}(\rho)=\frac{2\rho^2}{\rho^2+1}$, we write

$$\begin{aligned}\frac{1}{Z_{-1}^2}\mathrm{div}(Z_{-1}^2\nabla G^\varepsilon)&=\frac{1}{Z_{-1}^2(\rho)\rho}\partial_\rho(Z_{-1}^2(\rho)\rho\partial_\rho G^\varepsilon)\\&=\partial_{\rho\rho}G^\varepsilon+\frac{\rho^2+5}{\rho(\rho^2+1)}\partial_\rho G^\varepsilon.\end{aligned} \tag{2.56}$$

From (2.55) and (2.56), we obtain

$$\begin{aligned}\int_0^\infty|S_\rho^\varepsilon(s,\rho)|ds=&-M^\varepsilon(\rho)=-\partial_\rho G^\varepsilon(\rho)\\&=\frac{1}{2\pi\varepsilon^2}\frac{(1+\rho^2)^2}{\rho^5}\int_\rho^\infty\frac{r^5}{(1+r^2)^2}e^{-\frac{r^2}{2\varepsilon^2}}dr\\&\le\frac{1}{2\pi\varepsilon^2}\frac{(1+\rho^2)^2}{\rho^5}\int_\rho^\infty re^{-\frac{r^2}{2\varepsilon^2}}dr\\&\le\frac{1}{2\pi}\frac{1+\rho^4}{\rho^5}.\end{aligned}$$

Therefore, by letting $\varepsilon\to 0$, we obtain

$$\int_0^\infty|S_\rho(s,\rho)|ds\lesssim\frac{1+\rho^4}{\rho^5}. \tag{2.57}$$

Duhamel's formula gives

$$\begin{aligned}f_{-1}(0,\tau)=&\int_\tau^\infty\int_0^\infty S_\rho(s-\tau,\rho)\nabla f_0 Z_{-1}^2(\rho)\rho d\rho ds\\\lesssim&\int_0^\infty\left(\int_\tau^\infty|S_\rho(s-\tau,\rho)|ds\right)|\nabla f_0|Z_{-1}^2(\rho)\rho d\rho.\end{aligned}$$

By (2.53) and (2.57), we conclude

$$|f_{-1}(0,\tau)|\lesssim\tau^{-\nu_1}.$$

In the original time variable t, we get

$$|f_{-1}(0,t)|\lesssim\lambda_*^\nu(t),$$

and parabolic regularity theory readily yields

$$|f_{-1}(\rho,t)|\lesssim\lambda_*^\nu(t).$$

Therefore, we obtain

$$|\phi_{-1}(y,t)| \lesssim \lambda_*^{\nu}(t)\|h_{-1}\|_{\nu,a}$$

as desired. □

Mode $k=1$

We assume that $h_1(y,t)$ is defined in the entire space $\mathbb{R}^2\times(0,T)$ such that

$$h_1(y,t) = \mathrm{div}_y G(y,t) \tag{2.58}$$

with

$$|G(y,t)| \lesssim \frac{\lambda_*^{\nu}(t)}{1+|y|^{a-1}},\ (y,t)\in\mathbb{R}^2\times(0,T) \tag{2.59}$$

for $\nu>0$ and $a\in(2,3)$. By the blow-up argument, the following lemma was proved in [15, Lemma 7.6].

Lemma 2.6 ([15])**.** *Assume that $\nu>0$, $a\in(2,3)$ and h_1 takes the form* (2.58) *such that* (2.59) *holds and*

$$\int_{\mathbb{R}^2} h_1(y,t)Z_{1,j}(y)dy = 0 \ \text{ for all } \ t\in(0,T)$$

for $j=1,2$. Then there exists a solution $\phi_1(y,t)$ to problem (2.44) *for $k=1$ which defines a linear operator of $h_1(y,t)$, and $\phi_1(y,t)$ satisfies*

$$|\phi_1(y,t)| \lesssim \frac{\lambda_*^{\nu}(t)\|h_1\|_{\nu,a}}{1+|y|^{a-2}} \ \text{ in } \ \mathscr{D}_{3R}.$$

A direct consequence of Lemma 2.6 is the following

Lemma 2.7 ([15])**.** *Assume $\nu>0$, $a\in(2,3)$ and*

$$\int_{B_{2R}} h_1(y,t)Z_{1,j}(y)dy = 0 \ \text{ for all } \ t\in(0,T)$$

for $j=1,2$. Then there exists a solution $\phi_1(y,t)$ to problem (2.44) *with $k=1$ which defines a linear operator of $h_1(y,t)$, and $\phi_1(y,t)$ satisfies*

$$|\phi_1(y,t)| \lesssim \frac{\lambda_*^{\nu}(t)\|h_1\|_{\nu,a}}{1+|y|^{a-2}}.$$

By the construction in each mode, now we prove Proposition 2.1.

Proof of Proposition 2.1. Let h be defined in $\mathscr{D}_{2R}$ with $\|h\|_{\nu,a}<+\infty$. We consider

$$\begin{cases} \lambda^2\partial_t\phi = L_W[\phi]+h & \text{in } \mathscr{D}_{4R},\\ \phi(\cdot,0)=0 & \text{in } B_{4R(0)}. \end{cases}$$

Let ϕ_k be the solution estimated in Lemma 2.1 to

$$\begin{cases} \lambda^2 \partial_t \phi_k = L_W[\phi_k] + h_k & \text{in } \mathscr{D}_{4R}, \\ \phi_k(\cdot,t) = 0 & \text{on } \partial B_{4R} \times (0,T), \\ \phi_k(\cdot,0) = 0 & \text{in } B_{4R(0)}. \end{cases}$$

In addition, we let $\phi_{0,1}$, $\phi_{1,1}$, $\phi_{-1,1}$ solve

$$\begin{cases} \lambda^2 \partial_t \phi_{k,1} = L_W[\phi_{k,1}] + \bar{h}_k & \text{in } \mathscr{D}_{4R}, \\ \phi_{k,1}(\cdot,t) = 0 & \text{on } \partial B_{4R} \times (0,T), \\ \phi_{k,1}(\cdot,0) = 0 & \text{in } B_{4R(0)}, \end{cases}$$

for $k = 0, \pm 1$, where $\bar{h}_k$ is defined in (2.43). Consider the functions $\phi_{0,2}$ constructed in Lemma 2.3, $\phi_{-1,2}$ constructed in Lemma 2.5, and $\phi_{1,2}$ constructed in Lemma 2.6, that solve for $k = 0, \pm 1$

$$\begin{cases} \lambda^2 \partial_t \phi_{k,2} = L_W[\phi_{k,2}] + h_k - \bar{h}_k & \text{in } \mathscr{D}_{3R}, \\ \phi_{k,2}(\cdot,0) = 0 & \text{in } B_{3R(0)}. \end{cases}$$

Define

$$\phi := \sum_{k=0,\pm 1} (\phi_{k,1} + \phi_{k,2}) + \sum_{k \neq 0,\pm 1} \phi_k$$

which is a bounded solution to the following equation

$$\lambda^2 \partial_t \phi = L_W[\phi] + h(y,t) \quad \text{in } \mathscr{D}_{3R}.$$

Moreover, it defines a linear operator of h. Applying the estimates for the components in Lemmas 2.1, 2.3, 2.5, and 2.6, we obtain

$$\begin{aligned} |\phi(y,t)| \lesssim\ & \lambda_*^{\nu}(t) \min\left\{ \frac{R^{\delta(5-a)}(t)}{1+|y|^3}, \frac{1}{1+|y|^{a-2}} \right\} \|h_0 - \bar{h}_0\|_{\nu,a} + \frac{\lambda_*^{\nu}(t) R^2(t)}{1+|y|} \|\bar{h}_0\|_{\nu,a} \\ & + \frac{\lambda_*^{\nu}(t)}{1+|y|^{a-2}} \left\|h_1 - \bar{h}_1\right\|_{\nu,a} + \frac{\lambda_*^{\nu}(t) R^4(t)}{1+|y|^2} \left\|\bar{h}_1\right\|_{\nu,a} \\ & + \lambda_*^{\nu}(t) \|h_{-1} - \bar{h}_{-1}\|_{\nu,a} + \lambda_*^{\nu}(t) \log R(t) \|\bar{h}_{-1}\|_{\nu,a} \\ & + \frac{\lambda_*^{\nu}(t)}{1+|y|^{a-2}} \|h_\perp\|_{\nu,a}. \end{aligned}$$

in $\mathscr{D}_{3R}$. Finally, Lemma 2.2 yields that the same bound holds for $(1+|y|)|\nabla_y \phi|$ and $(1+|y|)^2 |\nabla_y^2 \phi|$ in $\mathscr{D}_{2R}$. The function $\phi\big|_{\mathscr{D}_{2R}}$ solves equation (2.39), and it defines a linear operator of h satisfying the desired estimates. The proof is complete. □

2.6 Linear theory for the outer problem

In order to solve the outer problem (2.24), we need to develop a linear theory to the associated linear problem of (2.24), which is basically a heat equation. However, we will have to establish a decay estimate of second order derivative of ψ in order to handle the coupling effects between the inner–outer problem of u and that of v below.

For $q \in \Omega$ and $T > 0$ sufficiently small, we consider the problem

$$\begin{cases} \psi_t = \Delta_x \psi + f(x,t) & \text{in } \Omega \times (0,T), \\ \psi = 0 & \text{on } \partial\Omega \times (0,T), \\ \psi(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{2.60}$$

The right hand side of (2.60) is assumed to be bounded with respect to some weights that appear in the outer problem (2.24). Thus we define the weights

$$\begin{cases} \rho_1 := \lambda_*^{\Theta} (\lambda_* R)^{-1} \chi_{\{r \le 3\lambda_* R\}}, \\ \rho_2 := T^{-\sigma_0} \dfrac{\lambda_*^{1-\sigma_0}}{r^2} \chi_{\{r \ge \lambda_* R\}}, \\ \rho_3 := T^{-\sigma_0}, \end{cases} \tag{2.61}$$

where $r = |x-q|$, $\Theta > 0$ and $\sigma_0 > 0$ is small. For a function $f(x,t)$ we define the L^∞-weighted norm

$$\|f\|_{**} := \sup_{\Omega \times (0,T)} \Big(1 + \sum_{i=1}^{3} \rho_i(x,t)\Big)^{-1} |f(x,t)|. \tag{2.62}$$

The factor T^{σ_0} in front of ρ_2 and ρ_3 is a simple way to have parts of the error small in the outer problem. Also, we define the L^∞-weighted norm for ψ

$$\begin{aligned} \|\psi\|_{\sharp,\Theta,\gamma} := {} & \lambda_*^{-\Theta}(0) \frac{1}{|\log T| \lambda_*(0) R(0)} \|\psi\|_{L^\infty(\Omega\times(0,T))} + \lambda_*^{-\Theta}(0) \|\nabla_x \psi\|_{L^\infty(\Omega\times(0,T))} \\ & + \sup_{\Omega\times(0,T)} \lambda_*^{-\Theta-1}(t) R^{-1}(t) \frac{1}{|\log(T-t)|} |\psi(x,t) - \psi(x,T)| \\ & + \sup_{\Omega\times(0,T)} \lambda_*^{-\Theta}(t) |\nabla_x \psi(x,t) - \nabla_x \psi(x,T)| + \|\nabla_x^2 \psi\|_{L^\infty(\Omega\times(0,T))} \\ & + \sup \lambda_*^{-\Theta}(t) (\lambda_*(t) R(t))^{2\gamma} \frac{|\nabla_x \psi(x,t) - \nabla_x \psi(x',t')|}{(|x-x'|^2 + |t-t'|)^\gamma}, \end{aligned} \tag{2.63}$$

where $\Theta > 0$, $\gamma \in (0, \frac{1}{2})$, and the last supremum is taken in the region

$$x, x' \in \Omega, \quad t, t' \in (0,T), \quad |x-x'| \le 2\lambda_*(t) R(t), \quad |t-t'| < \frac{1}{4}(T-t).$$

We shall measure the solution ψ to the problem (2.60) in the norm $\| \ \|_{\sharp,\Theta,\gamma}$ defined in (2.63) where $\gamma \in \left(0, \frac{1}{2}\right)$, and we require that Θ and γ_* (recall that $R =$

$\lambda_*^{-\gamma_*}$ in (2.25)) satisfy

$$\gamma_* \in \left(0, \frac{1}{2}\right), \quad \Theta \in (0, \gamma_*). \tag{2.64}$$

The condition $\gamma_* \in (0, \frac{1}{2})$ is a basic assumption to have the singularity appear inside the self-similar region. The condition $\Theta > 0$ is needed for Lemma 2.8. The assumption $\Theta < \gamma_*$ is made so that the estimates provided by Lemma 2.9 are stronger than that of Lemma 2.8.

We invoke some useful estimates proved in [15, Appendix A] as follows.

Proposition 2.2 ([15]). Assume (2.64) holds. For $T > 0$ sufficiently small, there is a linear operator that maps a function $f : \Omega \times (0,T) \to \mathbb{R}^3$ with $\|f\|_{**} < \infty$ into ψ which solves problem (2.60). Moreover, the following estimate holds

$$\|\psi\|_{\sharp,\Theta,\gamma} \le C\|f\|_{**},$$

where $\gamma \in (0, \frac{1}{2})$.

The proof of Proposition 2.2 was achieved in [15] by considering

$$\begin{cases} \psi_t = \Delta\psi + f \quad \text{in } \Omega \times (0,T), \\ \psi(x,0) = 0, \quad x \in \Omega, \\ \psi(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \end{cases} \tag{2.65}$$

and decomposing the equation into three parts corresponding to the weights of the right hand side defined in (2.61).

Lemma 2.8 ([15])**.** *Assume $\gamma_* \in (0, \frac{1}{2})$ and $\Theta > 0$. Let ψ solve* (2.65) *with f satisfying*

$$|f(x,t)| \le \lambda_*^{\Theta}(t)(\lambda_*(t)R(t))^{-1}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}.$$

Then the following estimates hold

$$\begin{aligned} |\psi(x,t)| &\le C\lambda_*^{\Theta}(0)\lambda_*(0)R(0)|\log T|, \\ |\psi(x,t) - \psi(x,T)| &\le C\lambda_*^{\Theta}(t)\lambda_*(t)R(t)|\log(T-t)|, \\ |\nabla\psi(x,t)| &\le C\lambda_*^{\Theta}(0), \\ |\nabla\psi(x,t) - \psi(x,T)| &\le C\lambda_*^{\Theta}(t), \\ |\nabla_x^2\psi(x,t)| &\le C, \end{aligned}$$

and for any $\gamma \in (0, \frac{1}{2})$,

$$\frac{|\nabla\psi(x,t) - \nabla\psi(x,t')|}{|t-t'|^{\gamma}} \le C\frac{\lambda_*^{\Theta}(t)}{(\lambda_*(t)R(t))^{2\gamma}}$$

for any x, and $0 \le t' \le t \le T$ such that $t - t' \le \frac{1}{10}(T-t)$,

$$\frac{|\nabla\psi(x,t) - \nabla\psi(x',t')|}{|x-x'|^{2\gamma}} \le C \frac{\lambda_*^{\Theta}(t)}{(\lambda_*(t)R(t))^{2\gamma}}$$

for any $|x-x'| \le 2\lambda_*(t)R(t)$ and $0 \le t \le T$.

Lemma 2.9 ([15]). *Assume $\gamma_* \in (0, \frac{1}{2})$ and $m \in (\frac{1}{2}, 1)$. Let ψ solve* (2.65) *with f satisfying*

$$|f(x,t)| \le \frac{\lambda_*^m(t)}{|z-q|^2} \chi_{\{|x-q| \ge \lambda_*(t)R(t)\}}.$$

Then the following estimates hold

$$\begin{aligned}
|\psi(x,t)| &\le CT^m |\log T|^{2-m}, \\
|\psi(x,t) - \psi(x,T)| &\le C|\log T|^m (T-t)^m |\log(T-t)|^{2-2m}, \\
|\nabla\psi(x,t)| &\le C\frac{T^{m-1}|\log T|^{2-m}}{R(T)}, \\
|\nabla\psi(x,t) - \nabla\psi(x,T)| &\le C\frac{\lambda_*^{m-1}(t)|\log(T-t)|}{R(t)}, \\
|\nabla_x^2\psi(x,t)| &\le C,
\end{aligned}$$

and for any $\gamma \in (0, \frac{1}{2})$,

$$\frac{|\nabla\psi(x,t) - \nabla\psi(x',t')|}{(|x-x'|^2 + |t-t'|)^{\gamma}} \le C \frac{1}{(\lambda_*(t)R(t))^{2\gamma}} \frac{\lambda_*^{m-1}(t)|\log(T-t)|}{R(t)}$$

for any $|x-x'| \le 2\lambda_*(t)R(t)$ and $0 \le t' \le t \le T$ such that $t - t' \le \frac{1}{10}(T-t)$.

Lemma 2.10 ([15]). *Let ψ solve* (2.65) *with f such that*

$$|f(x,t)| \le 1,$$

Then the following estimates hold

$$\begin{aligned}
|\psi(x,t)| &\le Ct, \\
|\psi(x,t) - \psi(x,T)| &\le C(T-t)|\log(T-t)|, \\
|\nabla\psi(x,t)| &\le CT^{1/2}, \\
|\nabla\psi(x,t) - \nabla\psi(x,T)| &\le C(T-t)^{1/2}, \\
|\nabla_x^2\psi(x,t)| &\le C, \\
|\nabla\psi(x,t_2) - \nabla\psi(x,t_1)| &\le C|t_2 - t_1|^{1/2}, \\
|\nabla\psi(x_1,t) - \nabla\psi(x_2,t)| &\le C|x_1 - x_2||\log(|x_1 - x_2|)|.
\end{aligned}$$

Remark 2.11. We note that the estimates for $|\nabla_x^2\psi(x,t)|$ in Lemmas 2.8–2.10 are achieved by writing the original equation (2.65) in the self-similar variables (y,τ):

$$\psi(x,t)=\tilde{\psi}\left(\frac{x-\xi}{\lambda},\tau(t)\right),$$

where $y=\frac{x-\xi}{\lambda}$ and τ is defined in (2.46). Then $\tilde{\psi}(y,\tau)$ satisfies the equation

$$\partial_\tau\tilde{\psi}=\Delta_y\tilde{\psi}+(\lambda\dot{\xi}+\dot{\lambda}\lambda y)\cdot\nabla_y\tilde{\psi}+\lambda^2 f(\lambda y+\xi,t(\tau)).$$

By similar argument as in the proof of Lemma 2.2, we can show the boundedness of $|\nabla_x^2\psi(x,t)|$ by the scaling argument and parabolic regularity estimates, which is sufficient for the final gluing procedure in Section 4 to work.

3 Model problem: Stokes system

In order to solve the incompressible Navier–Stokes equation in (1.1), a linear theory of certain linearized problem is required. In this section, we consider the Stokes system

$$\begin{cases}\partial_t v+\nabla P=\Delta v+\nabla\cdot F, & \text{in } \Omega\times(0,T),\\ \nabla\cdot v=0, & \text{in } \Omega\times(0,T),\\ v=0, & \text{on } \partial\Omega\times(0,T),\\ v(\cdot,0)=v_0, & \text{in } \Omega,\end{cases}\tag{3.1}$$

which is the linearized problem of the incompressible Navier–Stokes equation in (1.1). The idea is the following. Apriori we assume that the nonlinearity $v\cdot\nabla v$ is a perturbation under certain topology. Then we develop a linear theory for the Stokes system under which we shall see that $v\cdot\nabla v$ is indeed a smaller perturbation under some assumptions in Section 4.

Our aim is to find a velocity field v solving (3.1) with proper decay ensuring the inner–outer gluing scheme to be carried out. Suppose that $F(x,t)$ in (3.1) has the space-time decay of the type

$$|F(x,t)|\le C\frac{\lambda_*^{\nu-2}(t)}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}},\ |\nabla_x F(x,t)|\le C\frac{\lambda_*^{\nu-3}(t)}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+2}}\tag{3.2}$$

for $\nu>0$ and $a>1$. Here $q\in\Omega$ is the singular point for the orientation field $u(x,t)$ and

$$\lambda_*(t)=\frac{|\log T|(T-t)}{|\log(T-t)|^2}.$$

We define the norm

$$\begin{aligned}\|F\|_{S,\nu-2,a+1} := &\sup_{(x,t)\in\Omega\times(0,T)} \lambda_*^{2-\nu}(t)\left(1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}\right)|F(x,t)| \\ &+ \sup_{(x,t)\in\Omega\times(0,T)} \lambda_*^{3-\nu}(t)\left(1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+2}\right)|\nabla_x F(x,t)|.\end{aligned} \tag{3.3}$$

The main result of this section is stated as follows.

Proposition 3.1. Assume $\|F\|_{S,\nu-2,a+1} < +\infty$ with $\nu > 0$, $a > 1$, and $\|v_0\|_{B^{2-2/p}_{p,p}} < +\infty$, where the Besov norm $\|\cdot\|_{B^{2-2/p}_{p,p}}$ is defined by (3.38). Then there exists a solution (v,P) to the Stokes system (3.1) satisfying

- in the region near q: $B_{2\delta}(q) = \{x \in \Omega : |x-q| < 2\delta\}$ for $\delta > 0$ fixed and small,
$$|v(x,t)| \lesssim \|F\|_{S,\nu-2,a+1}\frac{\lambda_*^{\nu-1}(t)}{1+\left|\frac{x-q}{\lambda_*(t)}\right|},$$
and
$$|P(x,t)| \lesssim \|F\|_{S,\nu-2,a+1}\left(\frac{\lambda_*^{\nu}(t)}{|x-q|^2} + \frac{\lambda_*^{\nu-2}(t)}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}}\right).$$
- in the region away from q: $\Omega \setminus B_\delta(q)$
$$\|v\|_{W^{2,1}_p((\Omega\setminus B_\delta(q))\times(0,T))} + \|\nabla P\|_{L^p((\Omega\setminus B_\delta(q))\times(0,T))} \lesssim \|F\|_{S,\nu-2,a+1} + \|v_0\|_{B^{2-2/p}_{p,p}}$$
for $(\nu-1)p+1 > 0$. Moreover, if $\nu > 1/2$, then
$$\|v\|_{C^{\alpha,\alpha/2}((\Omega\setminus B_\delta(q))\times(0,T))} \lesssim \|F\|_{S,\nu-2,a+1} + \|v_0\|_{B^{2-2/p}_{p,p}}$$
for $0 < \alpha \le 2-4/p$.

To prove Proposition 3.1, we decompose the solution $v(x,t)$ to problem (3.1) into inner and outer profiles

$$v(x,t) = \eta_\delta v_{in}(x,t) + v_{out}(x,t),$$

where the smooth cut-off function

$$\eta_\delta(x) = \begin{cases} 1, & \text{for } |x-q| < \delta \\ 0, & \text{for } |x-q| > 2\delta \end{cases}$$

with $\delta > 0$ fixed and sufficiently small such that $dist(q,\partial\Omega) > 2\delta$. We denote

$$B_{2\delta}(q) = \{x \in \Omega : |x-q| < 2\delta\}.$$

It is direct to see that a solution to problem (3.1) is found if v_{in} and v_{out} satisfy

$$\begin{cases} \partial_t v_{in} + \nabla P_1 = \Delta v_{in} + \nabla \cdot F_{in}, & \text{in } \mathbb{R}^2 \times (0,T), \\ \nabla \cdot v_{in} = 0, & \text{in } \mathbb{R}^2 \times (0,T), \\ v_{in}(\cdot,0) = 0, & \text{in } \mathbb{R}^2, \end{cases} \tag{3.4}$$

$$\begin{cases} \partial_t v_{out} + \nabla(P - \eta_\delta P_1) = \Delta v_{out} + (1-\eta_\delta)\nabla \cdot F + 2\nabla \eta_\delta \cdot \nabla v_{in} \\ \qquad\qquad + (\Delta \eta_\delta) v_{in} - P_1 \nabla \eta_\delta, \ \text{ in } \ \Omega \times (0,T), \\ \nabla \cdot v_{out} = -\nabla \eta_\delta \cdot v_{in}, \ \text{ in } \ \Omega \times (0,T), \\ v_{out} = 0, \ \text{ on } \ \partial\Omega \times (0,T), \\ v_{out}(\cdot,0) = v_0, \ \text{ in } \ \Omega, \end{cases} \tag{3.5}$$

where $F_{in} = F\chi_{\{B_{2\delta}(q)\times(0,T)\}}$. The estimate of the inner part (3.4) is achieved by the representation formula in the entire space, while the outer part (3.5) is done by $W^{2,1}_p$-theory of the Stokes system.

Lemma 3.1. *For $\|F\|_{S,\nu-2,a+1} < +\infty$, the solution (v_{in}, P_1) of the system* (3.4) *satisfies*

$$|v_{in}(x,t)| \lesssim \|F\|_{S,\nu-2,a+1} \frac{\lambda_*^{\nu-1}(t)}{1 + \left|\frac{x-q}{\lambda_*(t)}\right|}, \tag{3.6}$$

and

$$|P_1(x,t)| \lesssim \|F\|_{S,\nu-2,a+1} \left(\frac{\lambda_*^{\nu}(t)}{|x-q|^2} + \frac{\lambda_*^{\nu-2}(t)}{1 + \left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}} \right). \tag{3.7}$$

Proof. For simplicity, we shall write v_{in} as v in the following proof. Denote $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. The estimate (3.6) is obtained by the well-known representation formula in the entire space

$$v_i(x,t) = \int_{\mathbb{R}^2} S_{ij}(x-z,t)(v(\cdot,0))_j(z)\,dz - \int_0^t \int_{\mathbb{R}^2} \partial_{z_k} S_{ij}(x-z,t-s) F_{jk}(z,s)\,dzds,$$

where S_{ij} is the Oseen tensor, which is the fundamental solution of the Stokes system derived by Oseen [56], defined by

$$S_{ij}(x,t) = G(x,t)\delta_{ij} - \frac{1}{2\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^2} G(y,t) \log|x-y|\,dy \tag{3.8}$$

with $G(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{4\pi t}$, and $F = (F_{jk})_{2\times 2}$. It is well known (see [61] for instance) that

$$|D_x^l D_t^k S(x,t)| \le C_{k,l} \frac{1}{(|x|^2 + t)^{k + \frac{2+l}{2}}}. \tag{3.9}$$

Since $v(\cdot,0)=0$, we then get for $i=1,2$,

$$
\begin{aligned}
|v_i(x,t)| &\lesssim \|F\|_{S,\nu-2,a+1}\int_0^t\int_{\mathbb{R}^2}\frac{1}{(|x-z|+\sqrt{t-s})^3}\frac{\lambda_*^{\nu-2}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+1}}\,dzds \\
&:= \|F\|_{S,\nu-2,a+1}(I_1+I_2),
\end{aligned}
\tag{3.10}
$$

where we decompose

$$
I_1=\int_0^{t-(T-t)^2}\int_{\mathbb{R}^2}\frac{1}{(|x-z|+\sqrt{t-s})^3}\frac{\lambda_*^{\nu-2}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+1}}\,dzds,
$$

and

$$
I_2=\int_{t-(T-t)^2}^t\int_{\mathbb{R}^2}\frac{1}{(|x-z|+\sqrt{t-s})^3}\frac{\lambda_*^{\nu-2}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+1}}\,dzds.
$$

Estimate of I_1.

To estimate I_1, we evaluate

$$
\begin{aligned}
&I_1 \\
&\lesssim \int_0^{t-(T-t)^2}\int_{\mathbb{R}^2}\frac{1}{(|x-z|^2+(t-s))^{3/2}}\frac{\lambda_*^{\nu+a-1}(s)}{\lambda_*^{a+1}(s)+|z-q|^{a+1}}\,dzds \\
&\lesssim \lambda_*^{\nu+a-1}(t)\int_{\mathbb{R}^2}\frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}}\frac{1}{(|x-z|^2+(T-t)^2)^{1/2}}\,dz \\
&\lesssim \lambda_*^{\nu+a-1}(t) \\
&\cdot\left(\int_{D_1(x)}+\int_{D_2(x)}+\int_{D_3(x)}\right)\frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}}\frac{1}{(|x-z|^2+\lambda_*^2(t))^{1/2}}\,dz,
\end{aligned}
\tag{3.11}
$$

where

$$
D_1(x):=\left\{z\in\mathbb{R}^2:|z-q|\le\frac{|x-q|}{2}\right\},
\tag{3.12}
$$

$$
D_2(x):=\left\{z\in\mathbb{R}^2:\frac{|x-q|}{2}\le|z-q|\le 2|x-q|\right\},
\tag{3.13}
$$

$$
D_3(x):=\left\{z\in\mathbb{R}^2:|z-q|\ge 2|x-q|\right\}.
\tag{3.14}
$$

We first compute

$$
\begin{aligned}
&\int_{D_1(x)}\frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}}\frac{1}{(|x-z|^2+\lambda_*^2(t))^{1/2}}\,dz \\
&\lesssim \frac{1}{|x-q|+\lambda_*(t)}\int_0^{\frac{|x-q|}{2}}\frac{r}{\lambda_*^{a+1}(t)+r^{a+1}}dr \\
&\lesssim \frac{\lambda_*^{1-a}(t)}{|x-q|+\lambda_*(t)}.
\end{aligned}
\tag{3.15}
$$

Similarly, we have

$$\begin{aligned}
&\int_{D_2(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{(|x-z|^2+\lambda_*^2(t))^{1/2}}\,dz \\
&\lesssim \frac{1}{\lambda_*^{a+1}(t)+|x-q|^{a+1}} \int_0^{3|x-q|} \frac{r}{r+\lambda_*(t)}\,dr \qquad (3.16)\\
&\lesssim \frac{1}{|x-q|^a+\lambda_*^a(t)},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{D_3(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{(|x-z|^2+\lambda_*^2(t))^{1/2}}\,dz \\
&\lesssim \frac{1}{|x-q|+\lambda_*(t)} \int_{2|x-q|}^{\infty} \frac{r}{\lambda_*^{a+1}(t)+r^{a+1}}\,dr \qquad (3.17)\\
&\lesssim \frac{1}{|x-q|^a+\lambda_*^a(t)}.
\end{aligned}$$

Collecting (3.11), (3.15), (3.16) and (3.17), we obtain

$$I_1 \lesssim \frac{\lambda_*^{\nu-1}(t)}{1+|y|}, \qquad (3.18)$$

where we write $y=\frac{x-q}{\lambda_*(t)}$ for simplicity.

Estimate of I_2.

To estimate I_2, we change variable

$$\tilde{s}=\frac{|x-z|}{(t-s)^{1/2}},$$

and thus

$$\begin{aligned}
I_2 &\lesssim \int_{\mathbb{R}^2}\int_{\frac{|x-z|}{T-t}}^{\infty} \frac{1}{(1+\tilde{s})^3|x-z|} \frac{\lambda_*^{\nu+a-1}(t)}{\lambda_*^{a+1}(t)+|z-q|^{a+1}}\,d\tilde{s}dz \\
&\lesssim \lambda_*^{\nu+a+1}(t) \int_{\mathbb{R}^2} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\,dz \qquad (3.19)\\
&\lesssim \lambda_*^{\nu+a+1}(t) \left(\int_{D_1(x)}+\int_{D_2(x)}+\int_{D_3(x)}\right) \\
&\qquad \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\,dz,
\end{aligned}$$

where $D_1(x)$, $D_2(x)$ and $D_3(x)$ are defined in (3.12), (3.13) and (3.14), respectively. For the above integral, we consider the following two cases.

- Case 1: $|x-q| \le \lambda_*(t)$. We have

$$\begin{aligned}
&\int_{D_1(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\, dz \\
&\lesssim \frac{1}{|x-q|(\lambda_*^2(t)+|x-q|^2)} \int_0^{\frac{|x-q|}{2}} \frac{r}{\lambda_*^{a+1}(t)+r^{a+1}}\, dr \\
&\lesssim \frac{\lambda_*^{-a-1}(t)}{\lambda_*(t)|x-q|^2}|x-q|^2 \\
&\lesssim \lambda_*^{-a-2}(t),
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\int_{D_2(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\, dz \\
&\lesssim \frac{1}{\lambda_*^{a+1}(t)+|x-q|^{a+1}} \int_0^{3|x-q|} \frac{1}{\lambda_*^2(t)+r^2}\, dr \\
&\lesssim \lambda_*^{-a-2}(t),
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\int_{D_3(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\, dz \\
&\lesssim \int_{2|x-q|}^{\infty} \frac{1}{\lambda_*^{a+1}(t)+r^{a+1}} \frac{1}{r-|x-q|} \frac{1}{\lambda_*^2(t)+(r-|x-q|)^2}\, r\, dr \\
&\lesssim \int_{|x-q|}^{\infty} \frac{1}{\lambda_*^{a+1}(t)+(\tilde{r}+|x-q|)^{a+1}} \frac{1}{\tilde{r}} \frac{1}{\lambda_*^2(t)+\tilde{r}^2} (\tilde{r}+|x-q|)\, d\tilde{r} \\
&\lesssim \lambda_*^{-a-2}(t).
\end{aligned} \tag{3.22}$$

Observe that in this case $|x-q| \le \lambda_*(t)$ we have $1 \lesssim \frac{1}{1+|y|^\varepsilon}$ for $\varepsilon > 0$, where $y = \frac{x-q}{\lambda_*(t)}$. Therefore, for the case $|x-q| \le \lambda_*(t)$, we conclude

$$I_2 \lesssim \frac{\lambda_*^{\nu-1}(t)}{1+|y|^2} \tag{3.23}$$

by (3.19)–(3.22).

- Case 2: $|x-q| \ge \lambda_*(t)$. In this case, we compute

$$\begin{aligned}
&\int_{D_1(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\, dz \\
&\lesssim \frac{1}{\lambda_*^2(t)+|x-q|^2} \frac{1}{|x-q|} \int_0^{\frac{|x-q|}{2}} \frac{r}{\lambda_*^{a+1}(t)+r^{a+1}}\, dr \\
&\lesssim \frac{\lambda_*^{-a-2}(t)}{1+|y|^2},
\end{aligned} \tag{3.24}$$

$$
\begin{aligned}
&\int_{D_2(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\, dz \\
&\lesssim \frac{1}{\lambda_*^{a+1}(t)+|x-q|^{a+1}} \int_0^{3|x-q|} \frac{1}{\lambda_*^2(t)+r^2}\, dr \\
&\lesssim \frac{\lambda_*^{-a-2}(t)}{1+|y|^{a+1}},
\end{aligned} \tag{3.25}
$$

and

$$
\begin{aligned}
&\int_{D_3(x)} \frac{1}{\lambda_*^{a+1}(t)+|z-q|^{a+1}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\, dz \\
&\lesssim \frac{1}{\lambda_*^2(t)+|x-q|^2} \frac{1}{|x-q|} \int_{2|x-q|}^{\infty} \frac{r}{\lambda_*^{a+1}(t)+r^{a+1}}\, dr \\
&\lesssim \frac{1}{\lambda_*^2(t)+|x-q|^2} \frac{1}{\lambda_*(t)} \frac{1}{\lambda_*^{a-1}(t)+|x-q|^{a-1}} \\
&\lesssim \frac{\lambda_*^{-a-2}(t)}{1+|y|^{a+1}}.
\end{aligned} \tag{3.26}
$$

From (3.19), (3.24), (3.25) and (3.26), one has

$$
I_2 \lesssim \frac{\lambda_*^{-a-2}(t)}{1+|y|^2} \tag{3.27}
$$

for the case $|x-q| \geq \lambda_*(t)$.

In conclusion, we get

$$
|v_{in}(x,t)| \lesssim \|F\|_{S,\nu-2,a+1} \frac{\lambda_*^{\nu-1}(t)}{1+|y|}
$$

from (3.10), (3.18), (3.23) and (3.27).

We now derive the estimate (3.7) for P_1. Recall the representation formula for P_1:

$$
P_1(x,t) = \int_0^t \int_{\mathbb{R}^2} Q_j(x-z,t-s)\partial_{z_k} F_{jk}(z,s)\, dzds,
$$

where Q_j is given by

$$
Q_j(x,t) = \frac{\delta(t)}{2\pi} \frac{x_j}{|x|^2}.
$$

Thus,

$$
\begin{aligned}
P_1(x,t) &= \int_{\mathbb{R}^2} \frac{1}{2\pi} \frac{x_j-z_j}{|x-z|^2}\, \partial_{z_k} F_{jk}(z,t)\, dz \\
&= \left(\int_{D_1(x)} + \int_{D_2(x)} + \int_{D_3(x)} \right) \frac{1}{2\pi} \frac{x_j-z_j}{|x-z|^2}\, \partial_{z_k} F_{jk}(z,t)\, dz \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}
$$

where $D_1(x)$, $D_2(x)$, and $D_3(x)$ are defined in (3.12), (3.13), and (3.14), respectively.

We perform integration by parts to estimate I. In fact, one has

$$
\begin{aligned}
\mathrm{I} &\lesssim \|F\|_{S,\nu-2,a+1}\Big(\int_{D_1(x)} \frac{1}{|x-z|^2}\,\frac{\lambda_*^{\nu-2}(t)}{1+\left|\frac{z-q}{\lambda_*(t)}\right|^{a+1}}\,dz \\
&\qquad + \int_{\partial D_1(x)} \frac{1}{|x-z|}\,\frac{\lambda_*^{\nu-2}(t)}{1+\left|\frac{z-q}{\lambda_*(t)}\right|^{a+1}}\,dz\Big) \\
&\lesssim \|F\|_{S,\nu-2,a+1}\Big(\frac{\lambda_*^{\nu-2}(t)}{|x-q|^2}\int_0^{\frac{|x-q|}{2}} \frac{\lambda_*^{a+1}(t)}{\lambda_*^{a+1}+r^{a+1}}\,r\,dr \\
&\qquad + \frac{1}{|x-q|}\,\frac{\lambda_*^{\nu-2}}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}}\,|x-q|\Big) \\
&\lesssim \|F\|_{S,\nu-2,a+1}\Big(\frac{\lambda_*^{\nu}(t)}{|x-q|^2}+\frac{\lambda_*^{\nu-2}}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}}\Big).
\end{aligned}
\tag{3.28}
$$

The way to estimate II and III is straightforward. More specifically, we have

$$
\begin{aligned}
\mathrm{II} &= \frac{1}{2\pi}\int_{D_2(x)} \frac{x_j-z_j}{|x-z|^2}\,\partial_{z_k}F_{jk}(z,t)\,dz \\
&\lesssim \|F\|_{S,\nu-2,a+1}\int_{D_2(x)} \frac{1}{|x-z|}\,\frac{\lambda_*^{\nu-3}(t)}{1+\left|\frac{z-q}{\lambda_*(t)}\right|^{a+2}}\,dz \\
&\lesssim \|F\|_{S,\nu-2,a+1}\,\frac{\lambda_*^{\nu-3}(t)}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+2}}\int_0^{3|x-q|}\frac{1}{r}\,r\,dr \\
&\lesssim \|F\|_{S,\nu-2,a+1}\frac{\lambda_*^{\nu-2}}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}},
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
\mathrm{III} &= \frac{1}{2\pi}\int_{D_3(x)} \frac{x_j-z_j}{|x-z|^2}\,\partial_{z_k}F_{jk}(z,t)\,dz \\
&\lesssim \|F\|_{S,\nu-2,a+1}\int_{D_3(x)} \frac{1}{|x-z|}\,\frac{\lambda_*^{\nu-3}(t)}{1+\left|\frac{z-q}{\lambda_*(t)}\right|^{a+2}}\,dz \\
&\lesssim \|F\|_{S,\nu-2,a+1}\,\lambda_*^{\nu+a-1}(t)\int_{2|x-q|}^{\infty}\frac{1}{r-|x-q|}\,\frac{1}{\lambda_*^{a+2}(t)+r^{a+2}}\,r\,dr \\
&= \|F\|_{S,\nu-2,a+1}\,\lambda_*^{\nu+a-1}(t)\int_{|x-q|}^{\infty}\frac{1}{u}\,\frac{1}{\lambda_*^{a+2}(t)+(u+|x-q|)^{a+2}}\,(u+|x-q|)\,du,
\end{aligned}
$$

where we changed the variables $u = r - |x-q|$. Hence $u \geq |x-q|$ implies that

$$\begin{aligned}
\mathrm{III} &\lesssim \|F\|_{S,\nu-2,a+1}\, \lambda_*^{\nu+a-1}(t) \int_{|x-q|}^{\infty} \frac{1}{\lambda_*^{a+2}(t) + (u+|x-q|)^{a+2}}\, du \\
&\lesssim \|F\|_{S,\nu-2,a+1}\, \lambda_*^{\nu+a-1}(t)\, \frac{1}{\lambda_*^{a+1}(t) + |x-q|^{a+1}} \qquad (3.30) \\
&= \|F\|_{S,\nu-2,a+1} \frac{\lambda_*^{\nu-2}}{1 + \left|\frac{x-q}{\lambda_*(t)}\right|^{a+1}}.
\end{aligned}$$

Collecting (3.28), (3.29), and (3.30), we obtain the estimate (3.7), and the proof is complete. □

In order to apply $W_p^{2,1}$-theory of the Stokes system to the outer part (3.5), the estimates for ∇v_{in} and $\partial_t(v_{in}\cdot\nabla\eta_\delta)$ are further needed. We have the following lemma.

Lemma 3.2. *Under the assumptions of Lemma 3.1, the following estimates hold*

$$|\nabla_x v_{in}(x,t)| \lesssim \|F\|_{S,\nu-2,a+1} \frac{\lambda_*^{\nu-2}(t)}{1 + \left|\frac{x-q}{\lambda_*(t)}\right|}, \qquad (3.31)$$

and

$$\|\partial_t(v_{in}\cdot\nabla\eta_\delta)\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))} \lesssim \|F\|_{S,\nu-2,a+1} \qquad (3.32)$$

for $(\nu-1)p+1>0$.

Proof. Since we impose zero initial condition on v_{in}, we have

$$|\partial_{x_l} v_i(x,t)| \lesssim \|F\|_{S,\nu-2,a+1} \int_0^t \int_{\mathbb{R}^2} \frac{1}{(|x-z|^2+(t-s))^{3/2}} \frac{\lambda_*^{\nu-3}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+2}}\, dzds$$

where we have used (3.9). We decompose the above integral and first estimate

$$\begin{aligned}
&\int_0^{t-(T-t)^2} \int_{\mathbb{R}^2} \frac{1}{(|x-z|^2+(t-s))^{3/2}} \frac{\lambda_*^{\nu-3}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+2}}\, dzds \\
&\lesssim \lambda_*^{\nu+a-1}(t) \left(\int_{D_1(x)} + \int_{D_2(x)} + \int_{D_3(x)}\right) \frac{1}{\lambda_*^{a+2}(t)+|z-q|^{a+2}} \frac{1}{|x-z|+\lambda_*(t)}\, dz,
\end{aligned}$$

where $D_1(x)$, $D_2(x)$ and $D_3(x)$ are defined in (3.12), (3.13) and (3.14), respectively. Then we can easily check the following

$$\begin{aligned}
&\int_{D_1(x)} \frac{1}{\lambda_*^{a+2}(t)+|z-q|^{a+2}} \frac{1}{|x-z|+\lambda_*(t)}\,dz \lesssim \frac{\lambda_*^{-a}(t)}{|x-q|+\lambda_*(t)}\\
&\int_{D_2(x)} \frac{1}{\lambda_*^{a+2}(t)+|z-q|^{a+2}} \frac{1}{|x-z|+\lambda_*(t)}\,dz \lesssim \frac{1}{|x-q|^{a+1}+\lambda_*^{a+1}(t)}\\
&\int_{D_3(x)} \frac{1}{\lambda_*^{a+2}(t)+|z-q|^{a+2}} \frac{1}{|x-z|+\lambda_*(t)}\,dz \lesssim \frac{1}{|x-q|^{a+1}+\lambda_*^{a+1}(t)}
\end{aligned}$$

and thus

$$\int_0^{t-(T-t)^2}\int_{\mathbb{R}^2} \frac{1}{\left(|x-z|^2+(t-s)\right)^{3/2}} \frac{\lambda_*^{\nu-3}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+2}}\,dzds \lesssim \frac{\lambda_*^{\nu-2}(t)}{1+|y|},$$

where we write $y=\frac{x-q}{\lambda_*}$. For the other part, we have

$$\begin{aligned}
&\int_{t-(T-t)^2}^{t}\int_{\mathbb{R}^2} \frac{1}{\left(|x-z|^2+(t-s)\right)^{3/2}} \frac{\lambda_*^{\nu-3}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+2}}\,dzds\\
\lesssim &\int_{\mathbb{R}^2}\int_{\frac{|x-z|}{T-t}}^{\infty} \frac{1}{(1+\tilde{s})^3|x-z|} \frac{\lambda_*^{\nu+a-1}(t)}{\lambda_*^{a+2}(t)+|z-q|^{a+2}}\,d\tilde{s}dz\\
\lesssim &\lambda_*^{\nu+a+1}(t)\int_{\mathbb{R}^2} \frac{1}{\lambda_*^{a+2}(t)+|z-q|^{a+2}} \frac{1}{|x-z|} \frac{1}{\lambda_*^2(t)+|x-z|^2}\,dz,
\end{aligned}$$

where we have changed variable $\tilde{s}=\frac{|x-z|}{\sqrt{t-s}}$. Similar to the proof of Lemma 3.1, the following bound holds

$$\int_{t-(T-t)^2}^{t}\int_{\mathbb{R}^2} \frac{1}{\left(|x-z|^2+(t-s)\right)^{3/2}} \frac{\lambda_*^{\nu-3}(s)}{1+\left|\frac{z-q}{\lambda_*(s)}\right|^{a+2}}\,dzds \lesssim \frac{\lambda_*^{\nu-2}(t)}{1+|y|^2}.$$

Collecting the above estimates, we conclude the validity of (3.31).

Next we prove (3.32). Multiplying equation (3.4) by $\nabla\eta_\delta$, we obtain that $v_{in}\cdot\nabla\eta_\delta$ satisfies the equation

$$\begin{aligned}
&\partial_t(v_{in}\cdot\nabla\eta_\delta)\\
&=\Delta(v_{in}\cdot\nabla\eta_\delta)-\Delta(\nabla\eta_\delta)\cdot v_{in}-2\nabla^2\eta_\delta\cdot\nabla v_{in}-\nabla P_1\cdot\nabla\eta_\delta+(\nabla\cdot F_{in})\cdot\nabla\eta_\delta.
\end{aligned}$$

Thanks to the cut-off function η_δ, standard $W_p^{2,1}$-theory for parabolic equation yields

$$\begin{aligned}
&\|\partial_t(v_{in}\cdot\nabla\eta_\delta)\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))}\\
\lesssim\ &\|v_{in}\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))}+\|\nabla v_{in}\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))}\\
&+\|\nabla P_1\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))}+\|\nabla\cdot F\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))}.
\end{aligned}\tag{3.33}$$

Using the $W_p^{2,1}$-theory for the Stokes system (see [63] for instance), we readily see that

$$\|\nabla P_1\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))} \lesssim \|\nabla\cdot F\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))}. \tag{3.34}$$

From (3.33), (3.34), (3.6), (3.31) and the assumption $\|F\|_{S,\nu-2,a+1}<+\infty$, we obtain

$$\|\partial_t(v_{in}\cdot\nabla\eta_\delta)\|_{L^p((B_{2\delta}(q)\setminus B_\delta(q))\times(0,T))} \lesssim \|F\|_{S,\nu-2,a+1}$$

provided $(\nu-1)p+1>0$. The proof is complete. □

We are ready to estimate the outer part (3.5).

Lemma 3.3. *For $\|F\|_{S,\nu-2,a+1}<+\infty$ and $\|v_0\|_{B^{2-2/p}_{p,p}}<+\infty$, the solution (v_{out},P) of the system* (3.5) *satisfies*

(3.35)
$$\|v_{out}\|_{W^{2,1}_p(\Omega\times(0,T))}+\|\nabla(P-\eta_\delta P_1)\|_{L^p(\Omega\times(0,T))} \lesssim \|F\|_{S,\nu-2,a+1}+\|v_0\|_{B^{2-2/p}_{p,p}}$$

for $(\nu-1)p+1>0$. If we further assume $\nu\in(1/2,1)$, then we have

$$\|v_{out}\|_{C^{\alpha,\alpha/2}(\Omega\times(0,T))} \lesssim \|F\|_{S,\nu-2,a+1}+\|v_0\|_{B^{2-2/p}_{p,p}} \tag{3.36}$$

for $0<\alpha\le 2-4/p$.

Proof. The $W^{2,1}_p$ estimate of solutions to Stokes system with non-zero divergence derived in [63, Theorem 3.1] shows that

$$\begin{aligned}&\|v_{out}\|_{W^{2,1}_p(\Omega\times(0,T))}+\|\nabla(P-\eta_\delta P_1)\|_{L^p(\Omega\times(0,T))}\\ &\lesssim \|(1-\eta_\delta)\nabla\cdot F+2\nabla\eta_\delta\cdot\nabla v_{in}+(\Delta\eta_\delta)v_{in}-P_1\nabla\eta_\delta\|_{L^p(\Omega\times(0,T))}\\ &\quad+\|\nabla\eta_\delta\cdot v_{in}\|_{L^p(0,T;W^1_p(\Omega))}+\|\partial_t(\nabla\eta_\delta\cdot v_{in})\|_{L^p(0,T;W^{-1}_p(\Omega))}+\|v_0\|_{B^{2-2/p}_{p,p}},\end{aligned} \tag{3.37}$$

where $\|\cdot\|_{B^{2-2/p}_{p,p}}$ is the Besov norm defined in (3.38). Thanks to the cut-off function η_δ, we get

$$|(1-\eta_\delta)\nabla\cdot F|\lesssim\|F\|_{S,\nu-2,a+1}\lambda_*^{\nu+a-1},$$

and from (3.6), (3.7), (3.31) and (3.32), one has

$$|\nabla\eta_\delta\cdot\nabla v_{in}|+|(\Delta\eta_\delta)v_{in}|+|P_1\nabla\eta_\delta|\lesssim\|F\|_{S,\nu-2,a+1}\lambda_*^{\nu-1},$$

and also

$$\begin{aligned}|\nabla\eta_\delta\cdot v_{in}|&\lesssim\|F\|_{S,\nu-2,a+1}\lambda_*^{\nu},\\ \|\partial_t(\nabla\eta_\delta\cdot v_{in})\|_{L^p(0,T;W^{-1}_p(\Omega))}&\lesssim\|F\|_{S,\nu-2,a+1}.\end{aligned}$$

It is worth noting that $\|\cdot\|_{L^p(0,T;W^{-1}_p(\Omega))}\le\|\cdot\|_{L^p(0,T;L^p(\Omega))}$ (see [1] for instance). Therefore, estimate (3.37) together with the above bounds imply (3.35) for $(\nu-1)p+1>0$. The Hölder estimate (3.36) then follows from a standard parabolic version of Morrey type inequality (see [44] for instance). The proof is complete. □

The proof of Proposition 3.1 is a direct consequence of Lemma 3.1 and Lemma 3.3.

For the behavior of the velocity field v, we further make several remarks:

Remark 3.4.

- From (3.3), Proposition 3.1 implies
$$\|v\|_{S,\nu-1,1} \lesssim \|F\|_{S,\nu-2,a+1}.$$
- Since v is divergence-free, we can write $v\cdot\nabla v = \nabla\cdot(v\otimes v)$, where $\otimes$ is the tensor product defined by $(v\otimes w)_{ij} = v_i w_j$. If we solve v in the class $\|v\|_{S,\nu-1,1}<\infty$, then the nonlinearity in the Navier–Stokes equation
$$|v\cdot\nabla v| \lesssim \frac{\lambda_*^{2\nu-3}(t)}{1+\left|\frac{x-q}{\lambda_*(t)}\right|^3}$$
is indeed a perturbation compared to $\nabla\cdot F$, which enables us to solve v by the fixed point argument in Section 4.
- The initial velocity v_0 in the outer problem (3.5) can be chosen arbitrarily in the Besov space $B^{2-2/p}_{p,p}$, with $(\nu-1)p+1>0$, in which the norm is defined by

(3.38)
$$\|v_0\|_{B^{2-2/p}_{p,p}} := \left(\int_{|z|<1}|z|^{-2p}\int_{\Omega(z)}|v_0(x+2z)-2v_0(x+z)+v_0(x)|^p dxdz\right)^{1/p} + \|v_0\|_{L^p(\Omega)},$$
where $\Omega(z)=\{x\in\Omega : x+tz\in\Omega, t\in[0,1]\}$, as long as it agrees with zero at the boundary and satisfies the condition
$$\nabla\cdot v_0 = -\nabla\eta_\delta\cdot v_{in}(x,0) = 0.$$

4 Solving the nematic liquid crystal flow

In this section, we shall apply the linear theories developed in Section 2 and Section 3 to show the existence of the desired blow-up solution to (1.1)–(1.3) by means of the fixed point argument. Apriori we need some assumptions on the behavior of the parameter functions $p(t)=\lambda(t)e^{i\omega(t)}$ and $\xi(t)$
$$c_1|\dot\lambda_*(t)| \le |\dot p(t)| \le c_2|\dot\lambda_*(t)| \ \text{ for all } \ t\in(0,T),$$
$$|\dot\xi(t)| \le \lambda_*^{\sigma}(t) \ \text{ for all } \ t\in(0,T),$$
where c_1, c_2 and σ are some positive constants independent of T. We recall that
$$R = R(t) = \lambda_*^{-\gamma_*}(t) \ \text{ with } \ \lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2} \ \text{ and } \ \gamma_*\in(0,1/2).$$

Similar to the harmonic map heat flow, we look for solution u solving problem (1.1) in the form

$$u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U,$$

with

$$\varphi = \eta_R Q_{\omega,\alpha,\beta}\phi(y,t) + \Psi^*(x,t) + \Phi^0(x,t) + \Phi^\alpha(x,t) + \Phi^\beta(x,t),$$

where we decompose Ψ^* into

$$\Psi^* = Z^* + \psi.$$

Here Z^* satisfies

$$\begin{cases} \partial_t Z^* = \Delta Z^*, & \text{in } \Omega\times(0,\infty) \\ Z^*(\cdot,t) = 0, & \text{on } \partial\Omega\times(0,\infty) \\ Z^*(\cdot,0) = Z_0^*, & \text{in } \Omega \end{cases}$$

For the same technical reasons as shown in [15], we make some assumptions on $Z_0^*(x)$ as follows. Let us write

$$Z_0^*(x) = \begin{bmatrix} z_0^*(x) \\ z_{03}^*(x) \end{bmatrix}, \quad z_0^*(x) = z_{01}^*(x) + iz_{02}^*(x).$$

Consistent with (2.36), the first condition that we need is

$$\operatorname{div} z_0^*(q) < 0.$$

In addition, we require that $Z_0^*(q) \approx 0$ in a non-degenerate way.

We will get a desired solution (v,u) to problem (1.1) if $(v,\phi,\Psi^*,p,\xi,\alpha,\beta)$ solves the following *inner–outer gluing system*

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla P = \Delta v - \varepsilon_0\nabla\cdot\mathscr{F}[p,\xi,\alpha,\beta,\Psi^*,\phi,v], \ \text{ in } \ \Omega\times(0,T), \\ \nabla\cdot v = 0, \ \text{ in } \ \Omega\times(0,T), \\ v = 0, \ \text{ on } \ \partial\Omega\times(0,T), \\ v(\cdot,0) = v_0, \ \text{ in } \ \Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} \lambda^2\partial_t\phi = L_W[\phi] + \mathscr{H}[p,\xi,\alpha,\beta,\Psi^*,\phi,v], \ \text{ in } \ \mathscr{D}_{2R}, \\ \phi(\cdot,0) = 0, \ \text{ in } \ B_{2R(0)}, \\ \phi\cdot W = 0, \ \text{ in } \ \mathscr{D}_{2R}, \end{cases} \tag{4.2}$$

$$\begin{cases} \partial_t\Psi^* = \Delta_x\Psi^* + \mathscr{G}[p,\xi,\alpha,\beta,\Psi^*,\phi,v] \ \text{ in } \ \Omega\times(0,T), \\ \Psi^* = \mathbf{e_3} - U - \Phi^0 - \Phi^\alpha - \Phi^\beta \ \text{ on } \ \partial\Omega\times(0,T), \\ \Psi^*(\cdot,0) = (1-\chi)\left(\mathbf{e_3} - U - \Phi^0 - \Phi^\alpha - \Phi^\beta\right) \ \text{ in } \ \Omega, \end{cases} \tag{4.3}$$

where

$$\mathscr{F}[p,\xi,\alpha,\beta,\Psi^*,\phi,v] = \left(\nabla u\odot\nabla u - \frac{1}{2}|\nabla u|^2\mathbb{I}_2\right), \tag{4.4}$$

with

$$\begin{aligned} u &= U + \Pi_{U^\perp}[\eta_R Q_{\omega,\alpha,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta] \\ &\quad + a(\Pi_{U^\perp}[\eta_R Q_{\omega,\alpha,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta])U, \end{aligned}$$

$$\begin{aligned} &\mathscr{H}[p,\xi,\alpha,\beta,\Psi^*,\phi,v] \\ &= \lambda^2 Q^{-1}_{\omega,\alpha,\beta}\Bigg[\tilde{L}_U[\Psi^*] + \mathscr{K}_0[p,\xi] + \mathscr{K}_1[p,\xi] + \Pi_{U^\perp}[\mathscr{R}_{-1}] - \lambda^{-1}\Pi_{U^\perp}(v\cdot\nabla_y U) \\ &\quad - \lambda^{-1}\Pi_{U^\perp}\Big(v\cdot\nabla_y\Big(\Pi_{U^\perp}[\eta_R Q_{\omega,\alpha,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta]\Big)\Big) \\ &\quad - \lambda^{-1}\Pi_{U^\perp}\Big(v\cdot\nabla_y\Big(a(\Pi_{U^\perp}[\eta_R Q_{\omega,\alpha,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta])U\Big)\Big)\Bigg], \end{aligned}$$

and

$$\begin{aligned} &\mathscr{G}[p,\xi,\alpha,\beta,\Psi^*,\phi,v] \\ &:= (1-\eta_R)\tilde{L}_U[\Psi^*] + (\Psi^*\cdot U)U_t + Q_{\omega,\alpha,\beta}(\phi\Delta_x\eta_R + 2\nabla_x\eta_R\cdot\nabla_x\phi - \phi\partial_t\eta_R) \\ &\quad + \eta_R Q_{\omega,\alpha,\beta}\left(-\left(Q^{-1}_{\omega,\alpha,\beta}\frac{d}{dt}Q_{\omega,\alpha,\beta}\right)\phi + \lambda^{-1}\dot{\lambda}y\cdot\nabla_y\phi + \lambda^{-1}\dot{\xi}\cdot\nabla_y\phi\right) \\ &\quad + (1-\eta_R)(\mathscr{K}_0[p,\xi] + \mathscr{K}_1[p,\xi] + \Pi_{U^\perp}[\mathscr{R}_{-1}]) - \Pi_{U^\perp}[\tilde{\mathscr{R}}_1] \\ &\quad + N_U[\eta_R Q_{\omega,\alpha,\beta}\phi + \Pi_{U^\perp}(\Phi^0 + \Phi^\alpha + \Phi^\beta + \Psi^*)] \\ &\quad + \Big((\Phi^0 + \Phi^\alpha + \Phi^\beta)\cdot U\Big)U_t - (1-\eta_R)v\cdot\nabla U \\ &\quad - (1-\eta_R)v\cdot\nabla\left(\Pi_{U^\perp}\left[\eta_R Q_{\omega,\alpha,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta\right]\right) \\ &\quad - (1-\eta_R)v\cdot\nabla\left(a\left(\Pi_{U^\perp}\left[\eta_R Q_{\omega,\alpha,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta\right]\right)U\right). \end{aligned}$$

Here χ in (4.3) is a smooth cut-off function which is supported near a fixed neighborhood of q independent of T.

As discussed in Section 2.5, suitable inner solution with space-time decay can be obtained under certain orthogonality conditions, which will be achieved by adjusting the parameter functions $p(t)$, $\xi(t)$, $\alpha(t)$ and $\beta(t)$. In order to solve the inner problem (4.2), we further decompose it based on the Fourier modes

$$\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 + \mathscr{H}_3 + \mathscr{H}_4,$$

with

$$
\begin{aligned}
&\mathscr{H}_1[p,\xi,\alpha,\beta,\Psi^*,\phi,v]\\
&=\left(\lambda^2 Q^{-1}_{\omega,\alpha,\beta}\left(\tilde{L}_U[\Psi^*]_0+\tilde{L}_U[\Psi^*]_2+\mathscr{K}_0[p,\xi]\right)+\lambda Q^{-1}_{\omega,\alpha,\beta}[\Pi_{U^\perp}(v\cdot\nabla u)]_0\right)\chi_{\mathscr{D}_{2R}},\\
&\mathscr{H}_2[p,\xi,\alpha,\beta,\Psi^*,\phi,v]\\
&=\Big(\lambda^2 Q^{-1}_{\omega,\alpha,\beta}\left(\tilde{L}_U[\Psi^*]^{(0)}_1+\mathscr{K}_1[p,\xi]\right)\\
&\qquad+\lambda Q^{-1}_{\omega,\alpha,\beta}\left([\Pi_{U^\perp}(v\cdot\nabla u)]_1+[\Pi_{U^\perp}(v\cdot\nabla u)]_\perp\right)\Big)\chi_{\mathscr{D}_{2R}},\\
&\mathscr{H}_3[p,\xi,\alpha,\beta,\Psi^*,\phi,v]\\
&=\lambda^2 Q^{-1}_{\omega,\alpha,\beta}\left(\tilde{L}_U[\Psi^*]_1-\tilde{L}_U[\Psi^*]^{(0)}_1\right)\chi_{\mathscr{D}_{2R}},\\
&\mathscr{H}_4[p,\xi,\alpha,\beta,\Psi^*,\phi,v]\\
&=\left(\lambda^2 Q^{-1}_{\omega,\alpha,\beta}(\Pi_{U^\perp}[\mathscr{R}_{-1,1}]+\Pi_{U^\perp}[\mathscr{R}_{-1,2}])+\lambda Q^{-1}_{\omega,\alpha,\beta}[\Pi_{U^\perp}(v\cdot\nabla u)]_{-1}\right)\chi_{\mathscr{D}_{2R}},
\end{aligned}
$$

where $[\Pi_{U^\perp}(v\cdot\nabla u)]_0$, $[\Pi_{U^\perp}(v\cdot\nabla u)]_{-1}$, $[\Pi_{U^\perp}(v\cdot\nabla u)]_1$ and $[\Pi_{U^\perp}(v\cdot\nabla u)]_\perp$ correspond respectively to modes 0, -1, 1 and higher modes $k\geq 2$ defined in (2.40)–(2.42), and

$$
\begin{aligned}
&\tilde{L}_U[\Phi]^{(0)}_1\\
&\quad=-2\lambda^{-1}w_\rho\cos w\left[(\partial_{x_1}\varphi_3(\xi(t),t))\cos\theta+(\partial_{x_2}\varphi_3(\xi(t),t)))\sin\theta\right]Q_{\omega,\alpha,\beta}E_1\\
&\qquad-2\lambda^{-1}w_\rho\cos w\left[(\partial_{x_1}\varphi_3(\xi(t),t)))\sin\theta-(\partial_{x_2}\varphi_3(\xi(t),t)))\cos\theta\right]Q_{\omega,\alpha,\beta}E_2
\end{aligned}
$$

in the notation (2.6). Then by decomposing $\phi=\phi_1+\phi_2+\phi_3+\phi_4$ in a similar manner as $\mathscr{H}_i$'s, the inner problem (4.2) becomes

$$
\begin{cases}
\lambda^2\partial_t\phi_1=L_W[\phi_1]+\mathscr{H}_1[p,\xi,\alpha,\beta,\Psi^*,\phi,v]\\
\qquad\qquad-\displaystyle\sum_{j=1,2}\tilde{c}_{0j}[\mathscr{H}_1[p,\xi,\alpha,\beta,\Psi^*,\phi,v]]w_\rho^2 Z_{0,j}\\
\qquad\qquad-\displaystyle\sum_{j=1,2}c_{1j}[\mathscr{H}_1[p,\xi,\alpha,\beta,\Psi^*,\phi,v]]w_\rho^2 Z_{1,j}\quad\text{in }\mathscr{D}_{2R}\\
\phi_1\cdot W=0\quad\text{in }\mathscr{D}_{2R}\\
\phi_1(\cdot,0)=0\quad\text{in }B_{2R(0)}
\end{cases}
\tag{4.5}
$$

$$
\begin{cases}
\lambda^2\partial_t\phi_2=L_W[\phi_2]+\mathscr{H}_2[p,\xi,\alpha,\beta,\Psi^*,\phi,v]\\
\qquad\qquad-\displaystyle\sum_{j=1,2}c_{1j}[\mathscr{H}_2[p,\xi,\alpha,\beta,\Psi^*,\phi,v]]w_\rho^2 Z_{1,j}\quad\text{in }\mathscr{D}_{2R}\\
\phi_2\cdot W=0\quad\text{in }\mathscr{D}_{2R}\\
\phi_2(\cdot,0)=0\quad\text{in }B_{2R(0)}
\end{cases}
\tag{4.6}
$$

$$
\begin{cases}
\lambda^2\partial_t\phi_3 = L_W[\phi_3] + \mathscr{H}_3[p,\xi,\alpha,\beta,\Psi^*,\phi,v] \\
\qquad\qquad - \sum_{j=1,2} c_{1j}[\mathscr{H}_3[p,\xi,\alpha,\beta,\Psi^*,\phi,v]]w_\rho^2 Z_{1,j} \\
\qquad\qquad + \sum_{j=1,2} c_{0j}^*[p,\xi,\alpha,\beta,\Psi^*,\phi,v]w_\rho^2 Z_{0,j} \quad \text{in } \mathscr{D}_{2R} \\
\phi_3\cdot W = 0 \quad \text{in } \mathscr{D}_{2R} \\
\phi_3(\cdot,0) = 0 \quad \text{in } B_{2R(0)}
\end{cases} \tag{4.7}
$$

$$
\begin{cases}
\lambda^2\partial_t\phi_4 = L_W[\phi_4] + \mathscr{H}_4[p,\xi,\alpha,\beta,\Psi^*,\phi,v] \\
\qquad\qquad - \sum_{j=1,2} c_{-1,j}[\mathscr{H}_4[p,\xi,\alpha,\beta,\Psi^*,\phi,v]]w_\rho^2 Z_{-1,j} \\
\phi_4\cdot W = 0 \quad \text{in } \mathscr{D}_{2R} \\
\phi_4(\cdot,0) = 0 \quad \text{in } B_{2R(0)}
\end{cases} \tag{4.8}
$$

$$
c_{0j}^*(t) - \tilde{c}_{0j}(t) = 0 \ \text{ for all } \ t\in(0,T), \quad j = 1,2, \tag{4.9}
$$

$$
c_{1j}(t) = 0 \ \text{ for all } \ t\in(0,T), \quad j = 1,2, \tag{4.10}
$$

$$
c_{-1,j}(t) = 0 \ \text{ for all } \ t\in(0,T), \quad j = 1,2. \tag{4.11}
$$

Based on the linear theory developed in Section 2.5, we shall solve the inner problems (4.5)–(4.8) in the norms below.

- We use the norm $\|\cdot\|_{\nu_i,a_i}$ to measure the right hand side $\mathscr{H}_i$ with $i = 1,\cdots,4$, where

$$
\|h\|_{\nu_i,a_i} = \sup_{\mathbb{R}^2\times(0,T)} \frac{|h(y,t)|}{\lambda_*^{\nu_i}(t)(1+|y|)^{-a_i}} \tag{4.12}
$$

  with $\nu_i > 0$, $a_i \in (2,3)$ for $i = 1, 2, 4$, and $a_3 \in (1,3)$.
- We use the norm $\|\cdot\|_{*,\nu_1,a_1,\delta}$ to measure the solution ϕ_1 solving (4.5), where

$$
\|\phi\|_{*,\nu_1,a_1,\delta} = \sup_{\mathscr{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)|\nabla_y\phi(y,t)| + (1+|y|)^2|\nabla_y^2\phi(y,t)|}{\lambda_*^{\nu_1}(t)\max\left\{\frac{R^{\delta(5-a_1)}}{(1+|y|)^3}, \frac{1}{(1+|y|)^{a_1-2}}\right\}}
$$

  with $\nu_1 \in (0,1)$, $a_1 \in (2,3)$, $\delta > 0$ fixed small.
- We use the norm $\|\cdot\|_{\text{in},\nu_2,a_2-2}$ to measure the solution ϕ_2 solving (4.6), where

$$
\|\phi\|_{\text{in},\nu_2,a_2-2} = \sup_{\mathscr{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)|\nabla_y\phi(y,t)| + (1+|y|)^2|\nabla_y^2\phi(y,t)|}{\lambda_*^{\nu_2}(t)(1+|y|)^{2-a_2}}
$$

  with $\nu_2 \in (0,1)$, $a_2 \in (2,3)$.

- We use the norm $\|\cdot\|_{**,\nu_3}$ to measure the solution ϕ_3 solving (4.7), where
$$\|\phi\|_{**,\nu_3} = \sup_{\mathscr{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y\phi(y,t)| + (1+|y|)^2|\nabla_y^2\phi(y,t)|}{\lambda_*^{\nu_3}(t)R^2(t)(1+|y|)^{-1}}$$
with $\nu_3 > 0$.
- We use the norm $\|\cdot\|_{***,\nu_4}$ to measure the solution ϕ_4 solving (4.8), where
$$\|\phi\|_{***,\nu_4} = \sup_{\mathscr{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y\phi(y,t)| + (1+|y|)^2|\nabla_y^2\phi(y,t)|}{\lambda_*^{\nu_4}(t)}$$
with $\nu_4 > 0$.

Based on the linear theory in Section 2.6, we shall solve the outer problem (4.3) in the following norms.

- We use the norm $\|\cdot\|_{**}$ defined in (2.62) to measure the right hand side $\mathscr{G}$ in the outer problem (4.3).
- We use the norm $\|\cdot\|_{\sharp,\Theta,\gamma}$ defined in (2.63) to measure the solution ψ solving the outer problem (4.3), where $\Theta > 0$ and $\gamma \in (0,1/2)$.

Based on the linear theory developed in Section 3, we shall solve the incompressible Navier–Stokes equation (4.1) in the following norms.

- We use the norm $\|\cdot\|_{S,\nu-2,a+1}$ defined in (3.3) to measure the forcing $\mathscr{F}$, where $\nu > 0$ and $a \in (1,2)$.
- We use the norm $\|\cdot\|_{S,\nu-1,1}$ defined in (3.3) to measure the velocity field v solving problem (4.1), where $\nu > 0$.

We then define
$$\tilde{E}_1 = \{\phi_1 \in L^\infty(\mathscr{D}_{2R}) : \nabla_y\phi_1 \in L^\infty(\mathscr{D}_{2R}),\ \|\phi_1\|_{*,\nu_1,a_1,\delta} < \infty\}$$
$$\tilde{E}_2 = \{\phi_2 \in L^\infty(\mathscr{D}_{2R}) : \nabla_y\phi_2 \in L^\infty(\mathscr{D}_{2R}),\ \|\phi_2\|_{\mathrm{in},\nu_2,a_2-2} < \infty\}$$
$$\tilde{E}_3 = \{\phi_3 \in L^\infty(\mathscr{D}_{2R}) : \nabla_y\phi_3 \in L^\infty(\mathscr{D}_{2R}),\ \|\phi_3\|_{**,\nu_3} < \infty\}$$
$$\tilde{E}_4 = \{\phi_4 \in L^\infty(\mathscr{D}_{2R}) : \nabla_y\phi_4 \in L^\infty(\mathscr{D}_{2R}),\ \|\phi_4\|_{***,\nu_4} < \infty\}$$
and use the notation
$$E_\phi = \tilde{E}_1 \times \tilde{E}_2 \times \tilde{E}_3 \times \tilde{E}_4, \quad \Phi = (\phi_1,\phi_2,\phi_3,\phi_4) \in E_\phi$$
$$\|\Phi\|_{E_\phi} = \|\phi_1\|_{*,\nu_1,a_1,\delta} + \|\phi_2\|_{\mathrm{in},\nu_2,a_2-2} + \|\phi_3\|_{**,\nu_3} + \|\phi_4\|_{***,\nu_4}.$$
We define the closed ball
$$\mathscr{B} = \{\Phi \in E_\phi : \|\Phi\|_{E_\phi} \le 1\}.$$
For the outer problem (4.3), we shall solve ψ in the space
$$E_\psi = \left\{\psi \in L^\infty(\Omega \times (0,T)) : \|\psi\|_{\sharp,\Theta,\gamma} < \infty\right\}.$$

For the incompressible Navier–Stokes equation (4.1), we shall solve the velocity field v in the space

$$E_v = \left\{ v \in L^2(\Omega;\mathbb{R}^2) : \nabla\cdot v = 0,\ \|v\|_{S,\nu-1,1} < M\varepsilon_0 \right\}, \tag{4.13}$$

where $\varepsilon_0 > 0$ is the number in (1.1) which is fixed sufficiently small, and $M > 0$ is some fixed number.

To introduce the space for the parameter function $p(t)$, we recall the integral operator $\mathscr{B}_0$ defined in (2.31) of the approximate form

$$\mathscr{B}_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s} ds + O\big(\|\dot p\|_\infty\big).$$

For $\Theta \in (0,1)$, $l \in \mathbb{R}$ and a continuous function $g : I \to \mathbb{C}$, we define the norm

$$\|g\|_{\Theta,l} = \sup_{t\in[-T,T]} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|,$$

and for $\gamma \in (0,1)$, $m \in (0,\infty)$, $l \in \mathbb{R}$, we define the semi-norm

$$[g]_{\gamma,m,l} = \sup (T-t)^{-m} |\log(T-t)|^l \frac{|g(t)-g(s)|}{(t-s)^\gamma},$$

where the supremum is taken over $s \le t$ in $[-T,T]$ such that $t-s \le \frac{1}{10}(T-t)$.

The following result was proved in [15, Section 8].

Proposition 4.1. Let $\alpha,\gamma \in (0,\frac12)$, $l \in \mathbb{R}$, $C_1 > 1$. If $\alpha_0 \in (0,1]$, $\Theta \in (0,\alpha_0)$, $m \in (0,\Theta-\gamma]$, and $a(t) : [0,T] \to \mathbb{C}$ satisfies

$$\begin{cases} \dfrac{1}{C_1} \le |a(T)| \le C_1, \\ T^\Theta |\log T|^{1+\sigma-l} \|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1, \end{cases} \tag{4.14}$$

for some $\sigma > 0$, then for $T > 0$ sufficiently small there exist two operators $\mathscr{P}$ and $\mathscr{R}_0$ so that $p = \mathscr{P}[a] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathscr{B}_0[p](t) = a(t) + \mathscr{R}_0[a](t), \quad t \in [0,T]$$

with

$$\begin{aligned} &|\mathscr{R}_0[a](t)| \\ &\le C\Big(T^\sigma + T^\Theta \frac{\log|\log T|}{|\log T|} \|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \Big) \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}, \end{aligned}$$

for some $\sigma > 0$.

Proposition 4.1 gives an approximate inverse $\mathscr{P}$ of the operator $\mathscr{B}_0$, so that given $a(t)$ satisfying (4.14), $p := \mathscr{P}[a]$ satisfies

$$\mathscr{B}_0[p] = a + \mathscr{R}_0[a], \quad \text{in } [0,T],$$

for a small remainder $\mathscr{R}_0[a]$. Moreover, the proof of Proposition 4.1 in [15] gives the decomposition

$$\mathscr{P}[a] = p_{0,\kappa} + \mathscr{P}_1[a],$$

with

$$p_{0,\kappa}(t) = \kappa |\log T| \int_t^T \frac{1}{|\log(T-s)|^2}\, ds, \quad t \le T,$$

$\kappa = \kappa[a] \in \mathbb{C}$, and the function $p_1 = \mathscr{P}_1[a]$ has the estimate

$$\|p_1\|_{*,3-\sigma} \le C |\log T|^{1-\sigma} \log^2(|\log T|).$$

Here the semi-norm $\| \ \|_{*,3-\sigma}$ is defined by

$$\|g\|_{*,3-\sigma} = \sup_{t\in[-T,T]} |\log(T-t)|^{3-\sigma} |\dot{g}(t)|,$$

and $\sigma \in (0,1)$. This leads us to define the space

$$X_p := \{p_1 \in C([-T,T;\mathbb{C}]) \cap C^1([-T,T;\mathbb{C}]) \ : \ p_1(T) = 0, \ \|p_1\|_{*,3-\sigma} < \infty\},$$

where we represent p by the pair (κ, p_1) in the form $p = p_{0,\kappa} + p_1$.

We define the space for $\xi(t)$ as

$$X_\xi = \left\{ \xi \in C^1((0,T);\mathbb{R}^2) : \dot{\xi}(T) = 0, \|\xi\|_{X_\xi} < \infty \right\}$$

where

$$\|\xi\|_{X_\xi} = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*^{-\sigma}(t) |\dot{\xi}(t)|$$

for some $\sigma \in (0,1)$, and we define the spaces for $\alpha(t)$, $\beta(t)$ as follows

$$X_\alpha = \left\{ \xi \in C^1((0,T)) : \alpha(T) = 0, \|\alpha\|_{X_\alpha} < \infty \right\}$$

where

$$\|\alpha\|_{X_\alpha} = \sup_{t\in(0,T)} \lambda_*^{-\delta_1}(t) |\alpha(t)| + \sup_{t\in(0,T)} \lambda_*^{1-\delta_1}(t) |\dot{\alpha}(t)|$$

and

$$X_\beta = \left\{ \beta \in C^1((0,T)) : \beta(T) = 0, \|\beta\|_{X_\beta} < \infty \right\}$$

where

$$\|\beta\|_{X_\beta} = \sup_{t\in(0,T)} \lambda_*^{-\delta_2}(t) |\beta(t)| + \sup_{t\in(0,T)} \lambda_*^{1-\delta_2}(t) |\dot{\beta}(t)|.$$

Here $\delta_1, \delta_2 \in (0,1)$.

In conclusion, we will solve the inner–outer gluing system (4.1), (4.3), (4.5), (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11) in the space

$$\mathscr{X} = E_v \times E_\psi \times E_\phi \times X_p \times X_\xi \times X_\alpha \times X_\beta \tag{4.15}$$

by means of fixed point argument.

4.1 Estimates of the orientation field u

The equation for the orientation field u is close in spirit to the harmonic map heat flow (2.9). To get the desired blow-up, we only need to show the drift term $v\cdot\nabla u$ is a small perturbation in the topology chosen above. Then the construction of the orientation field u is a direct consequence of [15] with slight modifications.

Effect of the drift term $v\cdot\nabla u$ in the outer problem

In the outer problem (4.3), it is direct to see that the main contribution in the drift term $v\cdot\nabla u$ comes from $v\cdot\nabla U$ since all the other terms are of smaller orders. We get that for some positive constant ε,

(4.16)
$$\begin{aligned}
&|(1-\eta_R)v\cdot\nabla u|\\
=&\Big|(1-\eta_R)v\cdot\nabla U+(1-\eta_R)v\cdot\nabla\Big(\Pi_{U^\perp}\Big[\eta_R\mathcal{Q}_{\omega,\alpha,\beta}\phi+\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta\Big]\Big)\\
&+(1-\eta_R)v\cdot\nabla\Big(a\Big(\Pi_{U^\perp}\Big[\eta_R\mathcal{Q}_{\omega,\alpha,\beta}\phi+\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta\Big]\Big)U\Big)\Big|\\
\lesssim&|(1-\eta_R)v\cdot\nabla U|\\
\lesssim&\frac{\lambda_*^{\nu-1}(t)\|v\|_{S,\nu-1,1}}{1+\left|\frac{x-q}{\lambda_*(t)}\right|}\frac{\lambda_*(t)}{|x-q|^2+\lambda_*^2(t)}\chi_{\{|x-q|\geq\lambda_*(t)R(t)\}}\\
\lesssim&T^\varepsilon\rho_2
\end{aligned}$$

provided $\nu>m$ with $m\in(1/2,1)$ obtained in Lemma 2.9, where ρ_2 is the weight of the $\|\cdot\|_{**}$-norm (see (2.61)) for the right hand side of the outer problem. Therefore, as long as ν is chosen sufficiently close to 1, the influence of the drift term $v\cdot\nabla u$ in the outer problem is negligible, and it is indeed a perturbation compared to the rest terms already estimated in the harmonic map heat flow [15, Section 6.6].

Effect of the drift term $v\cdot\nabla u$ in the inner problem

Since the inner problem is decomposed into different modes (4.5)–(4.8), a key observation is that the drift term $v\cdot\nabla u$ will get coupled in each mode. In other words, the mode k solved from the velocity equation with forcing $-\varepsilon_0\nabla\cdot(\nabla U\odot\nabla\phi_k)$ enters mode k of the inner problem via the drift term $v\cdot\nabla u$. We now analyze the projections of $v\cdot\nabla u$ on different modes. Recall that

$$v\cdot\nabla u=v\cdot\nabla[U+\varphi_{in}+\Pi_{U^\perp}\varphi_{out}+a(\Pi_{U^\perp}(\varphi_{in}+\varphi_{out}))U]$$

where

$$\varphi_{in}=\eta_R\mathcal{Q}_{\omega,\alpha,\beta}(\phi_1+\phi_2+\phi_3+\phi_4),\quad \varphi_{out}=\Psi^*+\Phi^0+\Phi^\alpha+\Phi^\beta.$$

Notice that the leading term in $v\cdot\nabla u$ is $v\cdot\nabla U$. Since $(v\cdot\nabla U,U)=0$, we have

$$\Pi_{U^\perp}(v\cdot\nabla U)=v\cdot\nabla U.$$

Denote $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. We write U in the polar coordinates

$$\nabla U = \lambda^{-1} \begin{bmatrix} \cos\theta w_\rho E_1 - \frac{\sin\theta}{\rho} \sin w E_2 \\ \sin\theta w_\rho E_1 + \frac{\cos\theta}{\rho} \sin w E_2 \end{bmatrix}.$$

Therefore, the projection of $v \cdot \nabla u$ on mode k ($k \in \mathbb{Z}$) is of the following size

$$\begin{aligned} &\left|[\Pi_{U^\perp}(v \cdot \nabla u)]_k\right| \lesssim \left|[\Pi_{U^\perp}(v \cdot \nabla U)]_k\right| \lesssim \\ &\left| \int_0^{2\pi} \big(v_1 \cos\theta \cos(k\theta) w_\rho + v_2 \sin\theta \cos(k\theta) w_\rho \right. \\ &\qquad + v_1 \frac{\sin\theta \sin(k\theta)}{\rho} \sin w - v_2 \frac{\cos\theta \sin(k\theta)}{\rho} \sin w \big) d\theta \\ &+ i \int_0^{2\pi} \big(v_1 \cos\theta \sin(k\theta) w_\rho + v_2 \sin\theta \sin(k\theta) w_\rho \\ &\left. - v_1 \frac{\sin\theta \cos(k\theta)}{\rho} \sin w + v_2 \frac{\cos\theta \cos(k\theta)}{\rho} \sin w \big) d\theta \right| \end{aligned}$$

from which we obtain

$$\left|\lambda [\Pi_{U^\perp}(v \cdot \nabla u)]_k\right| \leq \frac{M \varepsilon_0 \lambda_*^\nu}{1 + |y|^3} \tag{4.17}$$

where M and ε_0 are given in (4.13). Thus, it holds that

$$\|\lambda [\Pi_{U^\perp}(v \cdot \nabla u)]_k\|_{\nu,a} \leq M \varepsilon_0.$$

Since ε_0 is a sufficiently small number, we find that the projection $[\Pi_{U^\perp}(v \cdot \nabla u)]_k$ can be regarded as a perturbation compared to the rest terms in the right hand sides of the inner problems (4.5)–(4.8).

In summary, the coupling of the drift term $v \cdot \nabla u$ in the inner and outer problems of the harmonic map heat flow is essentially negligible under the topology chosen above. Therefore, with slight modifications, the fixed point formulation for

$$\partial_t u + v \cdot \nabla u = \Delta u + |\nabla u|^2 u$$

can be carried out in a similar manner as in [15].

For the outer problem (4.3), it was already estimated in [15] that in the space $\mathscr{X}$ defined in (4.15), it holds that for some $\varepsilon > 0$

$$\begin{aligned} &\|\mathscr{G}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] - (1 - \eta_R) v \cdot \nabla u\|_{**} \\ &\lesssim T^\varepsilon (\|\Phi\|_{E_\phi} + \|\psi\|_{\sharp,\Theta,\gamma} + \|p\|_{X_p} + \|\xi\|_{X_\xi} + \|\alpha\|_{X_\alpha} + \|\beta\|_{X_\beta} + 1) \end{aligned}$$

provided

$$\begin{cases} 0<\Theta<\min\Big\{\gamma_*,\dfrac{1}{2}-\gamma_*,\nu_1-1+\gamma_*(a_1-1),\\ \qquad\qquad \nu_2-1+\gamma_*(a_2-1),\nu_3-1,\nu_4-1+\gamma_*\Big\},\\ \Theta<\min\{\nu_1-\delta\gamma_*(5-a_1)-\gamma_*,\nu_2-\gamma_*,\nu_3-3\gamma_*,\nu_4-\gamma_*\},\\ \delta\ll 1. \end{cases} \tag{4.18}$$

On the other hand, from (4.16), we find that

$$\begin{aligned}&\|(1-\eta_R)v\cdot\nabla u\|_{**}\\ &\lesssim T^{\varepsilon}(\|v\|_{S,\nu-1,1}+\|\Phi\|_{E_\phi}+\|\psi\|_{\sharp,\Theta,\gamma}+\|p\|_{X_p}+\|\xi\|_{X_\xi}+\|\alpha\|_{X_\alpha}+\|\beta\|_{X_\beta}+1)\end{aligned}$$

provided

$$\nu>\frac{1}{2}. \tag{4.19}$$

Therefore, we conclude the validity of the following proposition by Proposition 2.2.

Proposition 4.2. Assume (4.18) and (4.19) hold. If $T>0$ is sufficiently small, then there exists a solution $\psi=\Psi(v,\Phi,p,\xi,\alpha,\beta)$ to problem (4.3) with

$$\begin{aligned}&\|\Psi(v,\Phi,p,\xi,\alpha,\beta)\|_{\sharp,\Theta,\gamma}\\ &\lesssim T^{\varepsilon}(\|v\|_{S,\nu-1,1}+\|\Phi\|_{E_\phi}+\|p\|_{X_p}+\|\xi\|_{X_\xi}+\|\alpha\|_{X_\alpha}+\|\beta\|_{X_\beta}+1),\end{aligned}$$

for some $\varepsilon>0$.

We denote $\mathscr{T}_\psi$ by the operator which returns ψ given in Proposition 4.2.

For the inner problems (4.5)–(4.8), our next step is to take $\Phi\in E_\phi$ and substitute

$$\Psi^*(v,\Phi,p,\xi,\alpha,\beta)=Z^*+\Psi(v,\Phi,p,\xi,\alpha,\beta)$$

into (4.2). We can then write equations (4.5)–(4.8) as the fixed point problem

$$\Phi=\mathscr{A}(\Phi) \tag{4.20}$$

where

$$\mathscr{A}(\Phi)=(\mathscr{A}_1(\Phi),\mathscr{A}_2(\Phi),\mathscr{A}_3(\Phi),\mathscr{A}_4(\Phi)),\quad \mathscr{A}:\bar{\mathscr{B}}_1\subset E_\phi\to E_\phi$$

with

$$\begin{aligned}\mathscr{A}_1(\Phi)&=\mathscr{T}_1(\mathscr{H}_1[v,\Psi^*(v,\Phi,p,\xi,\alpha,\beta),p,\xi,\alpha,\beta]),\\ \mathscr{A}_2(\Phi)&=\mathscr{T}_2(\mathscr{H}_2[v,\Psi^*(v,\Phi,p,\xi,\alpha,\beta),p,\xi,\alpha,\beta]),\\ \mathscr{A}_3(\Phi)&=\mathscr{T}_3\Big(\mathscr{H}_3[v,\Psi^*(v,\Phi,p,\xi,\alpha,\beta),p,\xi,\alpha,\beta]\\ &\qquad+\sum_{j=1}^{2}c_{0j}^*[v,\Psi^*(v,\Phi,p,\xi,\alpha,\beta),p,\xi,\alpha,\beta]w_\rho^2Z_{0,j}\Big),\\ \mathscr{A}_4(\Phi)&=\mathscr{T}_4\Big(\mathscr{H}_4[v,\Psi^*(v,\Phi,p,\xi,\alpha,\beta),p,\xi,\alpha,\beta]\Big).\end{aligned}$$

Here $\mathscr{T}_1(\cdot)$, $\mathscr{T}_2(\cdot)$, $\mathscr{T}_3(\cdot)$, $\mathscr{T}_4(\cdot)$ denotes the inner solution that solves (4.5), (4.6), (4.7), (4.8), respectively. Neglecting $\Pi_{U^\perp}(v\cdot\nabla u)$, the contraction for the inner problem was shown in [15, Section 6.7] under the conditions

$$\begin{cases} \nu_1 < 1 \\ \nu_2 < 1-\gamma_*(a_2-2) \\ \nu_3 < \min\left\{1+\Theta+2\gamma_*\gamma, \nu_1+\frac{1}{2}\delta\gamma_*(a_1-2)\right\} \\ \nu_4 < 1 \end{cases} \tag{4.21}$$

On the other hand, from (4.17), we obtain

$$\begin{aligned} &\left\| \lambda Q^{-1}_{\omega,\alpha,\beta}[\Pi_{U^\perp}(v\cdot\nabla u)]_0 \right\|_{\nu_1,a_1} \le M\varepsilon_0\lambda_*^{\nu-\nu_1}(t) \\ &\left\| \lambda Q^{-1}_{\omega,\alpha,\beta}([\Pi_{U^\perp}(v\cdot\nabla u)]_1 + [\Pi_{U^\perp}(v\cdot\nabla u)]_\perp) \right\|_{\nu_2,a_2} \le M\varepsilon_0\lambda_*^{\nu-\nu_2}(t) \\ &\left\| \lambda Q^{-1}_{\omega,\alpha,\beta}[\Pi_{U^\perp}(v\cdot\nabla u)]_{-1} \right\|_{\nu_4,a} \le M\varepsilon_0\lambda_*^{\nu-\nu_4}(t) \end{aligned} \tag{4.22}$$

Recall that the parameter $\varepsilon_0 > 0$ in (1.1) is fixed and sufficiently small. Therefore, by letting

$$\begin{cases} \nu = \nu_1 = \nu_2 = \nu_4 \\ 1 < a < 2 \end{cases} \tag{4.23}$$

the smallness in (4.22) comes from $\varepsilon_0 \ll 1$. Applying the linear theory developed in Section 2.5 for the inner problems (4.5)–(4.8), we then conclude the following proposition.

Proposition 4.3. Assume (4.21) and (4.23) hold. If $T > 0$ and $\varepsilon_0 > 0$ are sufficiently small, then the system of equations (4.20) for $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ has a solution $\Phi \in E_\phi$.

We denote by $\mathscr{T}_p$, $\mathscr{T}_\xi$, $\mathscr{T}_\alpha$ and $\mathscr{T}_\beta$ the operators which return the parameter functions $p(t)$, $\xi(t)$, $\alpha(t)$, $\beta(t)$, respectively. The argument for adjusting the parameter functions such that (4.9)–(4.11) hold is essentially similar to that of [15]. Note that the influence of the coupling $v\cdot\nabla u$ is negligible as shown in Section 4.1. Therefore, the leading orders for the parameter functions $p(t)$, $\xi(t)$, $\alpha(t)$, $\beta(t)$ are the same as in Section 2.4. The reduced problem (4.9) yields an integro-differential equation for $p(t)$ which can be solved by the same argument as in [15], while the reduced problems (4.10)–(4.11) give relatively simpler equations for $\xi(t)$, $\alpha(t)$, $\beta(t)$, which can be solved by the fixed point argument. We omit the details.

4.2 Estimates of the velocity field v

To solve the incompressible Navier–Stokes equation (4.1), we need to analyze the coupled forcing term

$$\varepsilon_0\nabla\cdot(\nabla u\odot\nabla u - 1/2|\nabla u|^2\mathbb{I}_2).$$

Observe that the main contribution in the forcing comes from $U+\eta_R Q_{\omega,\alpha,\beta}(\phi_0+\phi_1+\phi_{-1}+\phi_\perp)$, where ϕ_0, ϕ_1, ϕ_{-1}, $\phi_\perp$ are in mode 0, 1, -1 and higher modes, respectively. From the linear theory in Section 2.5, the dominant terms are U and ϕ_0. So we next need to evaluate

$$\nabla\cdot(\nabla U\odot\nabla U-1/2\,|\nabla U|^2\,\mathbb{I}_2)\ \text{ and }\ \nabla\cdot(\nabla U\odot\nabla\phi_0-1/2\,(\nabla U:\nabla\phi_0)\,\mathbb{I}_2),$$

where $\nabla U:\nabla\phi_0=\sum_{ij}\partial_i U_j\partial_i(\phi_0)_j$. Recall

$$U(y)=\begin{bmatrix}e^{i\theta}\sin w(\rho)\\ \cos w(\rho)\end{bmatrix},\ E_1(y)=\begin{bmatrix}e^{i\theta}\cos w(\rho)\\ -\sin w(\rho)\end{bmatrix},\ E_2(y)=\begin{bmatrix}ie^{i\theta}\\ 0\end{bmatrix}$$

so that

$$\partial_\rho U=w_\rho E_1,\ \partial_\theta U=\sin wE_2,$$
$$\partial_\rho E_1=-w_\rho U,\ \partial_\theta E_1=\cos wE_2.$$

Note that

$$\nabla\cdot(\nabla U\odot\nabla U-1/2\,|\nabla U|^2\,\mathbb{I}_2)=\Delta U\cdot\nabla U=-|\nabla U|U\cdot\nabla U=0.$$

For $\nabla\cdot(\nabla U\odot\nabla\phi_0-1/2\,(\nabla U:\nabla\phi_0)\,\mathbb{I}_2)$, we express the forcing in the polar coordinates. Since $\phi_0=\varphi_0E_1$ where $\varphi_0=\varphi_0(\rho)$, the first component

$$\begin{aligned}&(\lambda^3\nabla\cdot(\nabla U\odot\nabla\phi_0))_1\\ &=\nabla_y\cdot(\nabla_y U\odot\nabla_y\phi_0)_1\\ &=\partial_{y_1}\left(\cos^2\theta\,\partial_\rho\varphi_0\,w_\rho+\frac{\sin^2\theta}{\rho^2}\,\varphi_0\sin w\cos w\right)\\ &\quad+\partial_{y_2}\left(\sin\theta\cos\theta\,\partial_\rho\varphi_0\,w_\rho-\frac{\sin\theta\,\cos\theta}{\rho^2}\,\varphi_0\sin w\cos w\right).\end{aligned}$$

Changing ∂_{y_1} and ∂_{y_2} into ∂_ρ and ∂_θ, we obtain

$$\begin{aligned}&(\lambda^3\nabla\cdot(\nabla U\odot\nabla\phi_0))_1\\ &=\cos\theta\,\partial_\rho\left(\cos^2\theta\,\partial_\rho\varphi_0\,w_\rho+\frac{\sin^2\theta}{\rho^2}\,\varphi_0\sin w\cos w\right)\\ &\quad-\frac{\sin\theta}{\rho}\,\partial_\theta\left(\cos^2\theta\,\partial_\rho\varphi_0\,w_\rho+\frac{\sin^2\theta}{\rho^2}\,\varphi_0\sin w\cos w\right)\\ &\quad+\sin\theta\,\partial_\rho\left(\sin\theta\cos\theta\,\partial_\rho\varphi_0\,w_\rho-\frac{\sin\theta\cos\theta}{\rho^2}\,\varphi_0\sin w\cos w\right)\\ &\quad+\frac{\cos\theta}{\rho}\,\partial_\theta\left(\sin\theta\cos\theta\,\partial_\rho\varphi_0\,w_\rho-\frac{\sin\theta\cos\theta}{\rho^2}\,\varphi_0\sin w\cos w\right)\\ &=\cos\theta\left(\partial_\rho^2\varphi_0\,w_\rho+\partial_\rho\varphi_0\,w_{\rho\rho}+\frac{1}{\rho}\,\partial_\rho\varphi_0\,w_\rho-\frac{1}{\rho^3}\,\varphi_0\sin w\cos w\right)\\ &=\cos\theta\left[\partial_\rho\left(\partial_\rho\varphi_0\,w_\rho+\frac{\varphi_0\,w_\rho}{\rho}+\int\varphi_0w_\rho^2\right)\right].\end{aligned}$$

A similar calculation implies that the second component

$$(\lambda^3\nabla\cdot(\nabla U\odot\nabla\phi_0))_2=\sin\theta\left[\partial_\rho\left(\partial_\rho\varphi_0 w_\rho+\frac{\varphi_0 w_\rho}{\rho}+\int\varphi_0 w_\rho^2\right)\right].$$

So $\nabla\cdot(\nabla U\odot\nabla\phi_0)=\nabla\left[\lambda^{-3}(\partial_\rho\varphi_0 w_\rho+\frac{\varphi_0 w_\rho}{\rho}+\int\varphi_0 w_\rho^2)\right]$ is a potential. Moreover, it is obvious that $\nabla\cdot(|\nabla U|^2\mathbb{I}_2)$ is a potential. Therefore, $\nabla\cdot(\nabla U\odot\nabla\phi_0-1/2\,(\nabla U:\nabla\phi_0)\mathbb{I}_2)$ is a potential, which can be absorbed in the pressure P in problem (4.1).

On the other hand, the contribution from mode 1 and higher modes ($|k|\geq 2$) in $\mathscr{F}$ defined by (4.4) has the following bound

$$|\nabla U\odot\nabla\phi_k|\leq\frac{\lambda^{\nu_2-2}(t)}{1+|y|^{1+a_2}}\|\phi_k\|_{\mathrm{in},\nu_2,a_2-2},\quad k\neq-1,0.$$

Therefore, by (4.23) and $a_2\in(2,3)$, the leading term in $\mathscr{F}$ is

$$|\nabla U\odot\nabla\phi_{-1}|\leq\frac{\lambda^{\nu_4-2}(t)}{1+|y|^3}\|\phi_{-1}\|_{***,\nu_4},\tag{4.24}$$

from which we conclude that

$$\|\varepsilon_0\nabla U\odot\nabla\phi_{-1}\|_{S,\nu-2,a+1}\leq\varepsilon_0.$$

Remark 4.1. As we can see above, using the new linear theory at mode -1 (see Lemma 2.5), the size of the transported term turns out to be of the same order as the right hand side at mode -1. This is the motivation of introducing new parameters $\alpha(t)$, $\beta(t)$ and developing new linear theory at mode -1, otherwise the transported term would carry extra logarithm growth in time (see Lemma 2.4). On the other hand, the assumption $\varepsilon_0\ll 1$ is required here to guarantee the contraction in the fixed point argument.

On the other hand, as mentioned in Remark 3.4, the nonlinear term $v\cdot\nabla v$ in (4.1) is of smaller order compared to the forcing $\varepsilon_0\nabla\cdot\mathscr{F}$ if we look for a solution v in the function space E_v defined in (4.13). Indeed, since $v\in E_v$, we have

$$|v\cdot\nabla v|\lesssim\frac{\lambda_*^{2\nu-2}(t)}{1+|y|^3}$$

so that

$$\|v\cdot\nabla v\|_{S,\nu-2,a+1}\lesssim\lambda_*^{\nu}(t)\ll 1\ \text{ as }\ t\to T.$$

Thus, the incompressible Navier–Stokes equation (4.1) can be regarded as a perturbed Stokes system

$$\partial_t v+\nabla P=\Delta v-\varepsilon_0\nabla\cdot\mathscr{F}_1[p,\xi,\alpha,\beta,\Psi^*,\phi,v]$$

with

$$\mathscr{F}_1[p,\xi,\alpha,\beta,\Psi^*,\phi,v]=\mathscr{F}[p,\xi,\alpha,\beta,\Psi^*,\phi,v]+v\otimes v,$$

where we have used the fact that v is divergence-free so that we can write $v\cdot\nabla v=\nabla\cdot(v\otimes v)$. We denote $\mathscr{T}_v$ by the operator which returns the solution v, namely

$$\mathscr{T}_v:E_v\to E_v$$

$$v \mapsto \mathscr{T}_v(v).$$

By (4.24) and the linear theory for the Stokes system developed in Section 3, we obtain

(4.25)
$$\begin{aligned}&\|\mathscr{T}_v(v)\|_{S,\nu-1,1}\\ &\le C\varepsilon_0\big(\|v\|_{S,\nu-1,1}+\|\Phi\|_{E_\phi}+\|\psi\|_{\sharp,\Theta,\gamma}+\|p\|_{X_p}+\|\xi\|_{X_\xi}+\|\alpha\|_{X_\alpha}+\|\beta\|_{X_\beta}+1\big).\end{aligned}$$

4.3 Proof of Theorem 1.1

Consider the operator

$$\mathscr{T} = (\mathscr{A}, \mathscr{T}_\psi, \mathscr{T}_v, \mathscr{T}_p, \mathscr{T}_\xi, \mathscr{T}_\alpha, \mathscr{T}_\beta) \tag{4.26}$$

defined in Section 4.1 and Section 4.2. To prove Theorem 1.1, our strategy is to show that the operator $\mathscr{T}$ has a fixed point in $\mathscr{X}$ by the Schauder fixed point theorem. Here the function space $\mathscr{X}$ is defined in (4.15). The existence of a fixed point in the desired space $\mathscr{X}$ follows from a similar manner as in [15].

By collecting Proposition 4.1, Proposition 4.2, Proposition 4.3 and (4.25), we conclude that the operator maps $\mathscr{X}$ to itself. On the other hand, the compactness of the operator $\mathscr{T}$ can be proved by suitable variants of the estimates. Indeed, if we vary the parameters γ_*, Θ, ν, a, ν_1, a_1, ν_2, a_2, ν_3, ν_4, δ slightly such that all the restrictions in (4.18), (4.19), (4.21) and (4.23) are satisfied, then one can show that the operator $\mathscr{T}$ has a compact embedding in the sense that if a sequence is bounded in the new variant norms, then there exists a subsequence which converges in the original norms used in $\mathscr{X}$. Thus, the compactness follows directly from a standard diagonal argument by Arzelà–Ascoli's theorem. Therefore, the existence of the desired solution for the single bubble case $k = 1$ follows from the Schauder fixed point theorem.

The general case of multiple-bubble blow-up is essentially identical. The ansatz is modified as follows: we look for solution u of the form

$$u(x,t) = \sum_{j=1}^{k} U_j + \Pi_{U_j^\perp}\varphi_j + a(\Pi_{U_j^\perp}\varphi_j)U_j,$$

where

$$\begin{aligned}U_j &= U_{\lambda_j(t),\xi_j(t),\omega_j(t),\alpha_j(t),\beta_j(t)}, \quad \varphi_j = \varphi_{in}^j + \varphi_{out}^j,\\ \varphi_{in}^j &= \eta_{R(t)}(y_j)Q_{\omega_j(t),\alpha_j(t),\beta_j(t)}\phi(y_j,t), \quad y_j = \frac{x-\xi_j(t)}{\lambda_j(t)},\\ \varphi_{out}^j &= \psi(x,t) + Z^*(x,t) + \Phi_j^0 + \Phi_j^\alpha + \Phi_j^\beta.\end{aligned}$$

Here Φ_j^0, Φ_j^α and Φ_j^β are corrections defined in a similar way as in (2.17) with λ, ξ, ω, α, β replaced by λ_j, ξ_j, ω_j, α_j, β_j. We are then led to one outer problem and k inner problems for u together with one Navier–Stokes equation for v with exactly

analogous estimates. A string of fixed point problems can be solved in the same manner. We omit the details. □

Acknowledgment. F. Lin is partially supported by NSF grant 1501000. C. Wang is partially supported by NSF grant 1764417. J. Wei is partially supported by NSERC of Canada. We are grateful to Professor Tai-Peng Tsai for useful discussions. Y. Zhou would like to thank Dr. Mingfeng Qiu for useful discussions. The authors wish to thank the anonymous referee for very helpful comments.

Bibliography

[1] Adams, R. A.; Fournier, J. J. F. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] Angenent, S. B.; Hulshof, J.; Matano, H. The radius of vanishing bubbles in equivariant harmonic map flow from D^2 to S^2. *SIAM J. Math. Anal.*, **41** (2009), no. 3, 1121–1137.

[3] Caffarelli, L.; Kohn, R.; Nirenberg, L. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, **35** (1982), no. 6, 771–831.

[4] Chang, K.-C. Heat flow and boundary value problem for harmonic maps. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **6** (1989), no. 5, 363–395.

[5] Chang, K.-C.; Ding, W. Y.;, Ye, R. G. Finite-time blow-up of the heat flow of harmonic maps from surfaces. *J. Differential Geom.*, **36** (1992), no. 2, 507–515.

[6] Chen, Y.; Yu, Y. Global m-equivariant solutions of nematic liquid crystal flows in dimension two. *Arch. Ration. Mech. Anal.*, **226** (2017), no. 2, 767–808.

[7] Chen, Y. M.; Ding, W. Y. Blow-up and global existence for heat flows of harmonic maps. *Invent. Math.*, **99** (1990), no. 3, 567–578.

[8] Chen, Y. M.; Lin, F.-H. Evolution of harmonic maps with Dirichlet boundary conditions. *Comm. Anal. Geom.*, **1** (1993), no. 3-4, 327–346.

[9] Chen, Y. M.; Struwe, M. Existence and partial regularity results for the heat flow for harmonic maps. *Math. Z.*, **201** (1989), no. 1, 83–103.

[10] Constantin, P.; Seregin, G. Hölder continuity of solutions of 2D Navier-Stokes equations with singular forcing. In *Nonlinear partial differential equations and related topics*, volume 229 of *Amer. Math. Soc. Transl. Ser. 2*, 87–95. Amer. Math. Soc., Providence, RI, 2010.

[11] Coron, J.-M.; Ghidaglia, J. M. Explosion en temps fini pour le flot des applications harmoniques. *C. R. Acad. Sci. Paris Sér. I Math.*, **308** (1989), no. 12, 339–344.

[12] Cortázar, C.; del Pino, M.; Musso, M. Green's function and infinite-time bubbling in the critical nonlinear heat equation. *J. Eur. Math. Soc. (JEMS)*, **22** (2020), no. 1, 283–344.

[13] Daskalopoulos, P.; Del Pino, M.; Sesum, N. Type II ancient compact solutions to the Yamabe flow. *J. Reine Angew. Math.*, **738** (2018), 1–71.

[14] Dávila, J.; Del Pino, M.; Musso, M.; Wei, J. C. Gluing Methods for Vortex Dynamics in Euler Flows. *Arch. Ration. Mech. Anal.*, **235** (2020), no. 3, 1467–1530.

[15] Dávila, J.; Del Pino, M.; Wei., J. C. Singularity formation for the two-dimensional harmonic map flow into S^2. *Invent. Math.*, **219** (2020), no. 2, 345–466.

[16] Dávila, M.; Pistoia,A.; Vaira, G. Bubbling solutions for supercritical problems on manifolds. *J. Math. Pures Appl. (9)*, **103** (2015), no. 6, 1410–1440.

[17] Del Pino, M. Bubbling blow-up in critical parabolic problems. In *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions*, 73–116. Springer, 2017.

[18] Del Pino, M.; Kowalczyk, M.; Wei, J. C. Concentration on curves for nonlinear Schrödinger equations. *Comm. Pure Appl. Math.*, **60** (2007), no. 1, 113–146.

[19] Del Pino, M.; Kowalczyk, M.; Wei, J. C. On De Giorgi's conjecture in dimension $N \geq 9$. *Ann. of Math. (2)*, **174** (2011), no. 3, 1485–1569.

[20] Del Pino, M.; Musso, M.; Pacard, F. Bubbling along boundary geodesics near the second critical exponent. *J. Eur. Math. Soc. (JEMS)*, **12** (2010), no. 6, 1553–1605.

[21] Del Pino, M.; Musso, M.; Wei, J. C. Existence and stability of infinite time bubble towers in the energy critical heat equation. *arXiv preprint arXiv:1905.13500. Anal. PDE*, to appear.

[22] Del Pino, M.; Musso, M.; Wei, J. C. Geometry driven type II higher dimensional blow-up for the critical heat equation. *J. Funct. Anal.*, **280** (2021), no. 1, 108788, 49.

[23] Del Pino, M.; Musso, M.; Wei, J. C. Infinite-time blow-up for the 3-dimensional energy-critical heat equation. *Anal. PDE*, **13** (2020), no. 1, 215–274.

[24] Del Pino, M.; Musso, M.; Wei, J. C.; Zheng, Y. Q. Sign-changing blowing-up solutions for the critical nonlinear heat equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, to appear. *arXiv preprint arXiv:1811.00039*, 2018.

[25] Ericksen, J. L. Hydrostatic theory of liquid crystals. *Arch. Rational Mech. Anal.*, **9** (1962), 371–378.

[26] Galdi, G. P. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.

[27] Grotowski, J. F. Harmonic map heat flow for axially symmetric data. *Manuscripta Math.*, **73** (1991), no. 2, 207–228.

[28] Grotowski, J. F. Finite time blow-up for the harmonic map heat flow. *Calc. Var. Partial Differential Equations*, **1** (1993), no. 2, 231–236.

[29] Guan, M.J.; Gustafson, S.; Tsai, T. P. Global existence and blow-up for harmonic map heat flow. *J. Differential Equations*, **246** (2009), no. 1, 1–20.

[30] Gustafson,S.; Kang, K. K.; Tsai, T.-P. Asymptotic stability of harmonic maps under the Schrödinger flow. *Duke Math. J.*, **145** (2008), no. 3, 537–583.

[31] Gustafson, S.; Nakanishi, K. J.; Tsai, T. P. Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on $\mathbb{R}^2$. *Comm. Math. Phys.*, **300** (2010), no. 1, 205–242.

[32] Gustafson, S.; Roxanas, D. Global solutions for the critical, higher-degree corotational harmonic map heat flow to $\mathbb{S}^2$. *arXiv preprint arXiv:1711.06476*, 2017.

[33] Hong, M. C. Global existence of solutions of the simplified Ericksen-Leslie system in dimension two. *Calc. Var. Partial Differential Equations*, **40** (2011), no. 1-2, 15–36.

[34] Hong, M. C.; Xin, Z. P. Global existence of solutions of the liquid crystal flow for the Oseen-Frank model in $\mathbb{R}^2$. *Adv. Math.*, **231** (2012), 3-4, 1364–1400.

[35] Hopf, E. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, **4** (1951), 213–231.

[36] Huang, J. R.; Lin, F. H.; Wang, C. Y. Regularity and existence of global solutions to the Ericksen-Leslie system in $\mathbb{R}^2$. *Comm. Math. Phys.*, **331** (2014), no. 2, 805–850.

[37] Huang, T.; Lin, F. H.; Liu, C.; Wang, C. Y. Finite time singularity of the nematic liquid crystal flow in dimension three. *Arch. Ration. Mech. Anal.*, **221** (2016), no. 3, 1223–1254.

[38] Huang, T.; Wang, C. Y. Blow up criterion for nematic liquid crystal flows. *Comm. Partial Differential Equations*, **37** (2012), no. 5, 875–884.

[39] Huang, T.; Wang, C. Y.; Wen, H. Y. Blow up criterion for compressible nematic liquid crystal flows in dimension three. *Arch. Ration. Mech. Anal.*, **204** (2012), no. 1, 285–311.

[40] Lei, Z.; Li, D.; Zhang, X. Y. Remarks of global wellposedness of liquid crystal flows and heat flows of harmonic maps in two dimensions. *Proc. Amer. Math. Soc.*, **142** (2014), no. 11, 3801–3810.

[41] Lemarié-Rieusset, P. G. *Recent developments in the Navier-Stokes problem*, Vol. 431, *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[42] Leray, J. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, **63** (1934), no. 1, 193–248.

[43] Leslie, F. M. Some constitutive equations for liquid crystals. *Arch. Rational Mech. Anal.*, **28** (1968), no. 4, 265–283.

[44] Lieberman, G. M. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[45] Lin, F. H. Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena. *Comm. Pure Appl. Math.*, **42** (1989), no. 6, 789–814.

[46] Lin, F. H.; Liu, C. Nonparabolic dissipative systems modeling the flow of liquid crystals. *Comm. Pure Appl. Math.*, **48** (1995), no. 5, 501–537.

[47] Lin, F. H.; Liu, C. Partial regularity of the dynamic system modeling the flow of liquid crystals. *Discrete Contin. Dynam. Systems*, **2** (1996), no. 1, 1–22.

[48] Lin, F. H.; Lin, J. Y.; Wang, C. Y. Liquid crystal flows in two dimensions. *Arch. Ration. Mech. Anal.*, **197** (2010), no. 1, 297–336.

[49] Lin, F. H.; Wang, C. Y. *The analysis of harmonic maps and their heat flows*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[50] Lin, F. H.; Wang, C. Y. On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals. *Chin. Ann. Math. Ser. B*, **31** (2010), no. 6, 921–938.

[51] Lin, F. H.; Wang, C. Y. Recent developments of analysis for hydrodynamic flow of nematic liquid crystals. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **372** (2014), vol. 2029, 20130361, 18.

[52] Lin, F. H.; Wang, C. Y. Global existence of weak solutions of the nematic liquid crystal flow in dimension three. *Comm. Pure Appl. Math.*, **69** (2016), no. 8, 1532–1571.

[53] Lions, P. L. *Mathematical topics in fluid mechanics, Vol. 1*, *Oxford Lecture Series in Mathematics and its Applications*, Vol. 3. The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[54] Ma, W. Y.; Gong, H. J.; Li, J. K. Global strong solutions to incompressible Ericksen-Leslie system in $\mathbb{R}^3$. *Nonlinear Anal.*, **109** (2014), 230–235.

[55] Musso, M.; Sire, Y.; Wei, J. C.; Zheng, Y. Q; Zhou, Y. F. Infinite time blow-up for the fractional heat equation with critical exponent. *Math. Ann.*, **375** (2019), no. 1-2, 361–424.

[56] Oseen, C. W. Neuere methoden und ergebnisse in der hydrodynamik. *Leipzig: Akademische Verlagsgesellschaft mb H.*, 1927.

[57] Raphaël, P.; Schweyer, R. Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow. *Comm. Pure Appl. Math.*, **66** (2013), no. 3, 414–480.

[58] Seregin, G. *Lecture notes on regularity theory for the Navier-Stokes equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[59] Sire, Y.; Wei, J. C.; Zheng, Y. Q. Infinite time blow-up for half-harmonic map flow from $\mathbb{R}$ into $\mathbb{S}^1$. *Amer. J. Math.*, to appear. *arXiv preprint arXiv:1711.05387*, 2017.

[60] Sire, Y.; Wei, J. C.; Zheng, Y. Q. Singularity formation in the harmonic map flow with free boundary. *arXiv preprint arXiv:1905.05937*, 2019.

[61] Solonnikov, V. A. Estimates for solutions of a non-stationary linearized system of Navier-Stokes equations. *Trudy Mat. Inst. Steklov.*, **70** (1964), 213–317.

[62] Solonnikov, V. A. L_p-estimates for solutions to the initial boundary-value problem for the generalized Stokes system in a bounded domain. *J. Math. Sci. (New York)*, **105** (2001), no. 5, 2448–2484. Function theory and partial differential equations.

[63] Solonnikov, V. A. Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator. *Uspekhi Mat. Nauk*, **58** (2003), no. 2(350), 123–156.

[64] Solonnikov, V. A. On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity. *J. Math. Sci. (N. Y.)*, **114** (2003), no. 5, 1726–1740. Function theory and applications.

[65] Solonnikov, V. A. Potential theory for the nonstationary Stokes problem in nonconvex domains. In *Nonlinear problems in mathematical physics and related topics, I*, *Int. Math. Ser. (N. Y.)*, Vol. 1, 349–372. Kluwer/Plenum, New York, 2002.

[66] Struwe, M. On the evolution of harmonic mappings of Riemannian surfaces. *Comment. Math. Helv.*, **60** (1985), no. 4, 558–581.

[67] Temam, R. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[68] Tsai, T. P. *Lectures on Navier-Stokes equations*, *Graduate Studies in Mathematics*, Vol. 192. American Mathematical Society, Providence, RI, 2018.

[69] Van den Berg, J. B.; Hulshof, J.; King, J. B. Formal asymptotics of bubbling in the harmonic map heat flow. *SIAM J. Appl. Math.*, **63** (2003), no. 5, 1682–1717.

[70] Wang, M.; Wang, W. D. Global existence of weak solution for the 2-D Ericksen-Leslie system. *Calc. Var. Partial Differential Equations*, **51** (2014), no. 3-4, 915–962.

[71] Xu, X.; Zhang, Z. F. Global regularity and uniqueness of weak solution for the 2-D liquid crystal flows. *J. Differential Equations*, **252** (2012), no. 2, 1169–1181.

Received Month 200X.